1) $T(n) = \begin{cases} 9T\left(\frac{n}{3}\right) + O(n^2), & n>1 \\ 1, & n=1 \end{cases}$

$T(n) = 9T\left(\frac{n}{3}\right) + n^2$

For better guess;

$T(3) = 9T(1) + 9 = 18 = 3^2 \cdot 2$

$T(9) = 9T(3) + 81 = 9.18+81 = 3^5 = 3^4 \cdot 3$

$T(27) = 9T(9) + 27^2 = 9(\quad) + 27^2 = 3^6 \cdot 4 \to \log_3 27$

$T(81) = 9T(27) + 81^2 = 3^8 \cdot 5 \to \log_3 81$

It follows the rule $n^2 \log_3 n$ → Since it just affect the rate of increase, there is no difference between log and $\log_3$ in the bigO notation

i. Make a guess: $T(n) = O(n^2 \log n)$

ii. Claim a hypothesis: $T(n) \leq cn^2 \log n$

Assume that: $T(k) \leq ck^2 \log k, \ \forall k < n$

iii. Show that hypothesis is true:

$\frac{n}{3} < n \Rightarrow T\left(\frac{n}{3}\right) \leq c\frac{n^2}{9} \log \frac{n}{3}$

$$T(n) = 9T\left(\frac{n}{3}\right) + n^2$$

$$\leq 9c\frac{n^2}{9} \log \frac{n}{3} + n^2$$

$$= cn^2(\log n - \underbrace{\log 3}_{1}) + n^2 = cn^2 \log n + (1-c)n^2$$

$$T(n) \leq cn^2 \log n + (1-c)n^2 \leq cn^2 \log n$$

$$T(n) \leq cn^2 \log n - (c-1)n^2$$

we are free to pick any c value

pick $c \geq 1$. Then,

$$\boxed{T(n) \leq cn^2 \log n}$$

Therefore, our hypothesis is true.

2)

a) $T(n) = 5T\left(\frac{n}{2}\right) + O(n) = \boxed{5T\left(\frac{n}{2}\right) + n}$

b) $\underbrace{\text{Recursion level}}$ $\qquad\qquad$ $\underbrace{\text{Total Cost of the level}}$

$l = 0 \qquad T(n) \qquad\qquad\qquad n$

$l = 1 \qquad T\left(\frac{n}{2}\right)\ T\left(\frac{n}{2}\right)\ T\left(\frac{n}{2}\right)\ T\left(\frac{n}{2}\right)\ T\left(\frac{n}{2}\right) \qquad\qquad \frac{5n}{2}$

$l = 2 \qquad T\left(\frac{n}{4}\right)\ T\left(\frac{n}{4}\right)\ T\left(\frac{n}{4}\right)\ T\left(\frac{n}{4}\right)\ T\left(\frac{n}{4}\right) \ \cdots \qquad \frac{25n}{4} = \left(\frac{5}{2}\right)^2 \cdot n$

$l = 3 \qquad T\left(\frac{n}{8}\right)\ T\left(\frac{n}{8}\right) \cdots \qquad \frac{125n}{8} = \left(\frac{5}{2}\right)^3 \cdot n$

$\boxed{\text{Rule} = \left(\frac{5}{2}\right)^l \cdot n}$

$\vdots$

$l = \log n \qquad T(1)\ T(1)\ \cdots \qquad \left(\frac{5}{2}\right)^{\log n} \cdot n$

$$T(n) = n + \frac{5n}{2} + \left(\frac{5}{2}\right)^2 n + \left(\frac{5}{2}\right)^3 \cdot n + \dots + \left(\frac{5}{2}\right)^{\log n} \cdot n$$

$$= \sum_{l=0}^{\log n} \left(\frac{5}{2}\right)^l \cdot n = n \sum_{l=0}^{\log n} \left(\frac{5}{2}\right)^l = n \cdot \frac{\left(\frac{5}{2}\right)^{\log n + 1} - 1}{\left(\frac{5}{2}\right) - 1}$$

$$\boxed{\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}}$$

Simplfy ;
"ignore denominator"

$$T(n) = n \cdot \left(\frac{5}{2} \cdot \left(\frac{5}{2}\right)^{\log n}\right) - n = \underbrace{\frac{5}{2} n^{\log 5}}_{\text{dominates}} - n = \boxed{\Theta\left(n^{\log 5}\right)}$$

$$n^{\log 5 - 1} = \frac{n^{\log 5}}{n}$$

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3) a) $f(n) = 3n^2$, $g(n) = n^2$

$\rightarrow 3n^2 \leq cn^2$, for $c \geq 3$, $n \geq n_0$ (for any positive n_0) $\Rightarrow \boxed{f(n) = O(g(n))}$

$\rightarrow 3n^2 \geq cn^2$, for $c \leq 3$, $n \geq n_0$ (for any positive n_0) $\Rightarrow \boxed{f(n) = \Omega(g(n))}$

Thus, both holds $\boxed{f(n) = \Theta(g(n))}$

b) $f(n) = 2n^4 - 3n^2 + 7$, $g(n) = n^5$

$\rightarrow 2n^4 - 3n^2 + 7 \leq cn^5$, $c = 2$, $n \geq \underset{n_0}{2}$ $\Rightarrow \boxed{f(n) = O(g(n))}$

c) $f(n) = \frac{\log n}{n}$, $g(n) = \frac{1}{n}$

$\rightarrow \frac{\log n}{n} \geq \frac{c}{n}$, $c = 1$, $n \geq \underset{n_0}{2}$ $\Rightarrow \boxed{f(n) = \Omega(g(n))}$

d) $f(n) = \log n$, $g(n) = \log n + \frac{1}{n}$

$\rightarrow \log n \leq c \log n + \frac{c}{n}$, $c = 1$, $n \geq \underset{n_0}{1}$ $\Rightarrow \boxed{f(n) = O(g(n))}$

e) $f(n) = 2^{k \log n}$, $g(n) = n^k$

$\rightarrow \underbrace{2^{k \log n}}_{n^k} \leq cn^k$

$n^k \leq cn^k$, $c = 1$, $n \geq n_0$ ← for any positive number $\Rightarrow \boxed{f(n) = O(g(n))}$

$\rightarrow n^k \geq cn^k$, $c = 1$, $n \geq n_0$ ← for any positive number $\Rightarrow \boxed{f(n) = \Omega(g(n))}$

Thus, both $\boxed{\Theta(g(n))}$

f) $f(n) = 2^n$, $g(n) = 2^{2^n}$

$\rightarrow 2^n \leq c2^{2^n}$, $c = 1$, $n \geq 0$ ← n_0 $\Rightarrow \boxed{f(n) = O(g(n))}$

g) $f(n) = 2^{\sqrt{\log n}}$, $g(n) = (\log n)^{100}$

$$\left.\begin{array}{l} \underbrace{2^{\sqrt{\log n}}}_{\downarrow} \leq 100c\log n \\ 2^{\sqrt{\log n}} \leq \underbrace{2^{\log n}} \leq 100c\log n \\ 2^{\sqrt{\log n}} \leq n \leq 100c\log n \end{array}\right\} c = 1, n \geq 0 \leftarrow n_0$$

$\Downarrow$

$\boxed{f(n) = O(g(n))}$

3) h) $f(n) = \begin{cases} 4^n, & n < 2^{1000} \\ 2^{1000} \cdot n^2, & n \geq 2^{1000} \end{cases}$, $g(n) = \dfrac{n^2}{2^{1000}}$

For $n \geq 2^{1000}$;

$2^{1000} \cdot n^2 \leq c \cdot \dfrac{n^2}{2^{1000}}$, $c \geq 2^{2000}$, $\underset{\downarrow}{\overset{n_0}{}} n \geq 2^{1000}$ $\Rightarrow \boxed{f(n) = O(g(n))}$

$2^{2000} \leq c$

, $c \leq 2^{2000}$, $n \geq 2^{1000}$ $\Rightarrow \boxed{f(n) = \Omega(g(n))}$

Thus, both holds

$$\boxed{f(n) = \Theta(g(n))}$$

4) a) $T(n) = 3T\left(\frac{n}{4}\right) + O(n)$

$a = 3 \geq 1$, $b = 4 > 1$, $O(n) = f(n) = n$ asymptotically positive

$\log_b a = \log_4 3 \approx 0.792$

$n^{\log_b a} = n^{0.792}$ case 3 may apply;

$f(n) = n = \Omega\left(n^{0.79+\varepsilon}\right)$

$a f\left(\frac{n}{b}\right) \overset{?}{\leq} c f(n)$

$3 f\left(\frac{n}{4}\right) \leq c n$

$\frac{3n}{4} \leq cn \rightarrow$ this holds for $\frac{3}{4} \leq c < 1$

Thus, $\boxed{T(n) = \Theta(f(n)) = \Theta(n)}$

b) $T(n) = 2^n T\left(\frac{n}{2}\right) + O(n)$

$a = 2^n \geq 1$, $b = 2 > 1$, $O(n) = f(n) = n$ asymptotically pos.

$\log_b a = \log_2 2^n = n \Rightarrow n^{\log_b a} = n^n$

There is no way we can compare $f(n)$ and n^n,

Thus, $\boxed{\text{we can not solve}}$ this recursion by Master Theorem

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