

$$1) T(n) = \begin{cases} 9T(\frac{n}{3}) + O(n^2), & n > 1 \\ 1, & n = 1 \end{cases}$$

$$\hookrightarrow T(n) = 9T(\frac{n}{3}) + n^2$$

For better guess:

$$T(3) = 9T(1) + 9 = 18 = 3^2 \cdot 2$$

$$T(9) = 9T(3) + 81 = 9 \cdot 18 + 81 = 243 = 3^5 = 3^4 \cdot 3$$

$$T(27) = 9T(9) + 27^2 = 9(243) + 729 = 2187 = 3^6 \cdot 4 \rightarrow \log_3 27$$

$$T(81) = 9T(27) + 81^2 = 3^8 \cdot 5 \rightarrow \log_3 81$$

It follows the rule  $n^2 \log_3 n \rightarrow$  Since it just affect the rate of increase, there is no difference between  $\log$  and  $\log_3$  in the big O notation

i. Make a guess:  $T(n) = O(n^2 \log n)$

ii. Claim a hypothesis:  $T(n) \leq cn^2 \log n$

Assume that:  $T(k) \leq ck^2 \log k, \forall k < n$

iii. Show that hypothesis is true:

$$\frac{n}{3} < n \Rightarrow T(\frac{n}{3}) \leq c \frac{n^2}{9} \log \frac{n}{3}$$

$$T(n) = 9T(\frac{n}{3}) + n^2 \leq 9c \frac{n^2}{9} \log \frac{n}{3} + n^2$$

$$= cn^2(\log n - \log 3) + n^2 = cn^2 \log n + (1-c)n^2$$

$$T(n) \leq cn^2 \log n + (1-c)n^2 \leq cn^2 \log n$$

$$T(n) \leq (cn^2 \log n) - (c-1)n^2 \rightarrow \text{we are free to pick any } c \text{ value. Pick } c \geq 1. \text{ Then,}$$

$$\boxed{T(n) \leq cn^2 \log n}$$

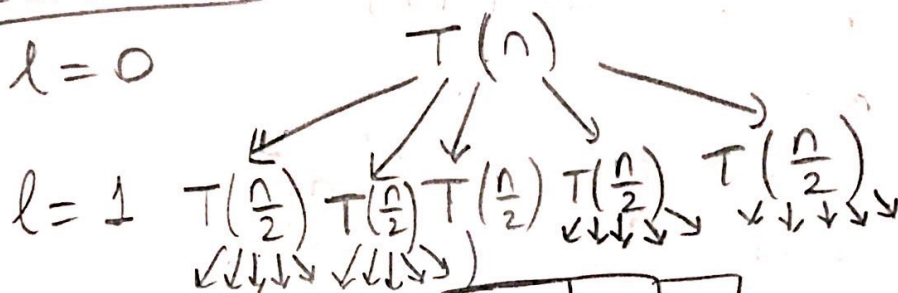
Therefore, our hypothesis is true.

2) a)  $T(n) = 5T\left(\frac{n}{2}\right) + O(n) = \boxed{5T\left(\frac{n}{2}\right) + n}$

Total Cost of the level

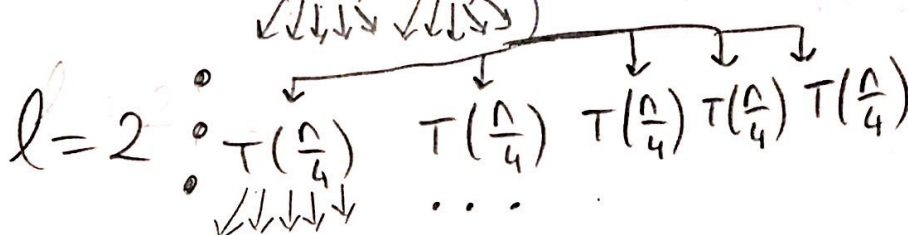
b) Recursion level

$l=0$



$n$

$\frac{5n}{2}$



$\frac{25n}{4} = \left(\frac{5}{2}\right)^2 \cdot n$

$l=3 \quad T\left(\frac{n}{8}\right) T\left(\frac{n}{8}\right) \dots$

$\frac{125n}{8} = \left(\frac{5}{2}\right)^3 \cdot n$

$\boxed{\text{Rule} = \left(\frac{5}{2}\right)^l \cdot n}$

$l = \log n \quad T(1) T(1) \dots$

$\left(\frac{5}{2}\right)^{\log n} \cdot n$

$T(n) = n + \frac{5n}{2} + \left(\frac{5}{2}\right)^2 n + \left(\frac{5}{2}\right)^3 n + \dots + \left(\frac{5}{2}\right)^{\log n} \cdot n$

$= \sum_{l=0}^{\log n} \left(\frac{5}{2}\right)^l \cdot n = n \sum_{l=0}^{\log n} \left(\frac{5}{2}\right)^l = n \cdot \frac{\left(\frac{5}{2}\right)^{\log n + 1} - 1}{\left(\frac{5}{2}\right) - 1}$

$\boxed{\sum_{k=0}^n x^k = \frac{x^{n+1} - 1}{x - 1}}$

Simplify;  
"ignore denominator"

$T(n) = n \cdot \left(\frac{5}{2} \cdot \left(\frac{5}{2}\right)^{\log n}\right) - n = \frac{5}{2} n^{\log 5} - n = \boxed{\Theta(n^{\log 5})}$

dominates



3) a)  $f(n) = 3n^2$ ,  $g(n) = n^2$

$\rightarrow 3n^2 \leq cn^2$ , for  $c \geq 3$ ,  $n \geq n_0$   $\Rightarrow$   $f(n) = O(g(n))$

$\rightarrow 3n^2 \geq cn^2$ , for  $c \leq 3$ ,  $n \geq n_0$   $\Rightarrow$   $f(n) = \Omega(g(n))$

Thus, both holds  $f(n) = \Theta(g(n))$

b)  $f(n) = 2n^4 - 3n^2 + 7$ ,  $g(n) = n^5$

$\rightarrow 2n^4 - 3n^2 + 7 \leq cn^5$ ,  $c = 2$ ,  $n \geq 2$   $\Rightarrow$   $f(n) = O(g(n))$

c)  $f(n) = \frac{\log n}{n}$ ,  $g(n) = \frac{1}{n}$

$\rightarrow \frac{\log n}{n} \geq \frac{c}{n}$ ,  $c = 1$ ,  $n \geq 2$   $\Rightarrow$   $f(n) = \Omega(g(n))$

d)  $f(n) = \log n$ ,  $g(n) = \log n + \frac{1}{n}$

$\rightarrow \log n \leq c \log n + \frac{c}{n}$ ,  $c = 1$ ,  $n \geq 1$   $\Rightarrow$   $f(n) = O(g(n))$

e)  $f(n) = 2^{k \log n}$ ,  $g(n) = n^k$

$\rightarrow \frac{2^{k \log n}}{n^k} \leq c n^k$ ,  $c=1$ ,  $n \geq n_0$   $\Rightarrow f(n) = O(g(n))$   
 $n^k \leq c n^k$ ,  $c=1$ ,  $n \geq n_0$   $\Rightarrow f(n) = \Omega(g(n))$

$\rightarrow n^k \geq c n^k$ ,  $c=1$ ,  $n \geq n_0$   $\Rightarrow f(n) = \Theta(g(n))$

Thus, both  $\Theta(g(n))$

f)  $f(n) = 2^n$ ,  $g(n) = 2^{2^n}$

$\rightarrow 2^n \leq c 2^{2^n}$ ,  $c=1$ ,  $n \geq 0$   $\Rightarrow f(n) = O(g(n))$

g)  $f(n) = 2^{\sqrt{\log n}}$ ,  $g(n) = (\log n)^{100}$

$2^{\sqrt{\log n}} \leq 100 c \log n$   
 $2^{\sqrt{\log n}} \leq 2^{\log n} \leq 100 c \log n$   
 $2^{\sqrt{\log n}} \leq n \leq 100 c \log n$

$C=1$ ,  $n \geq 0$   $\Rightarrow f(n) = O(g(n))$

$$3) \text{ h) } f(n) = \begin{cases} 4^n, & n < 2^{1000} \\ 2^{1000} \cdot n^2, & n \geq 2^{1000} \end{cases}, \quad g(n) = \frac{n^2}{2^{1000}}$$

For  $n \geq 2^{1000}$ ;

$$2^{1000} \cdot n^2 \leq c \cdot \frac{n^2}{2^{1000}}, \quad c \geq 2^{2000}, \quad n \geq 2^{1000} \Rightarrow \boxed{f(n) = O(g(n))}$$

$$2^{2000} \leq c$$

$$c \leq 2^{2000}, \quad n \geq 2^{1000} \Rightarrow \boxed{f(n) = \Omega(g(n))}$$

Thus, both holds

$$\boxed{f(n) = \Theta(g(n))}$$



$$4) a) T(n) = 3T\left(\frac{n}{4}\right) + O(n)$$

$$a=3 \geq 1, \quad b=4 > 1, \quad O(n)=f(n)=n \quad \text{asymptotically positive}$$

$$\log_b a = \log_4 3 \approx 0.792$$

$$n^{\log_b a} = n^{0.792} \quad \downarrow \text{case 3 may apply ;}$$

$$f(n)=n = \Omega(n^{0.792+\epsilon})$$

$$a f\left(\frac{n}{b}\right) \leq c f(n)$$

$$3 f\left(\frac{n}{4}\right) \leq c n$$

$$\frac{3n}{4} \leq c n \rightarrow \text{this holds for } \frac{3}{4} \leq c < 1$$

$$\text{Thus, } \boxed{T(n) = \Theta(f(n)) = \Theta(n)}$$

$$b) T(n) = 2^n T\left(\frac{n}{2}\right) + O(n)$$

$$a=2^n \geq 1, \quad b=2 > 1, \quad O(n)=f(n)=n \quad \text{asymptotically pos.}$$

$$\log_b a = \log_2 2^n = n \Rightarrow n^{\log_b a} = n^n$$

There is no way we can compare  $f(n)$  and  $n^n$ ,

Thus, We can not solve this recursion by Master Theorem