Ege Erdoğan

EE 314 HW 3

1) $V = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{U}_1}, \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\vec{U}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{U}_3} \right\}$

a)
$\vec{U}_1 . \vec{U}_2 = 1 - 1 + 0 = 0$ ✓
$\vec{U}_1 . \vec{U}_3 = 0 + 0 + 0 = 0$ ✓
$\vec{U}_2 . \vec{U}_3 = 0 + 0 + 0 = 0$ ✓

$\boxed{\text{Yes}}$, This set is orthogonal

b) $\vec{U}_1 . \vec{U}_1 = 1 + 1 = 2 \neq 1 \Rightarrow$ It is not orthonormal set.

To obtain orthonormal set.

$$\vec{N}_1 = \frac{\vec{U}_1}{|\vec{U}_1|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \rightarrow \text{magnitude } 1 \ ✓$$

$$\vec{N}_2 = \frac{\vec{U}_2}{|\vec{U}_2|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \rightarrow \text{magnitude } 1 \ ✓$$

$$\vec{N}_3 = \vec{U}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \text{magnitude } 1. \ ✓$$

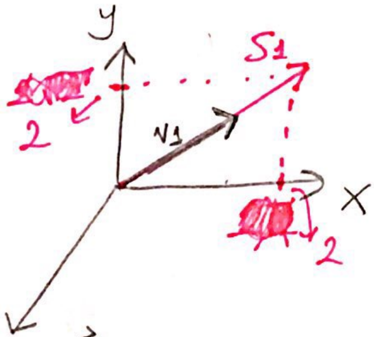
$$\left\{ \underbrace{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}}_{N_1}, \underbrace{\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}}_{N_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{N_3} \right\} \rightarrow \text{orthonormal set}$$

c)
$\vec{S}_1 = 2\sqrt{2} . \vec{N}_1$
$\vec{S}_2 = \vec{N}_3 - \sqrt{2} \vec{N}_2$
$\vec{S}_3 = 2\sqrt{2} \vec{N}_2$
$\vec{S}_4 = \sqrt{2} \vec{N}_1 + \vec{N}_3$

1)

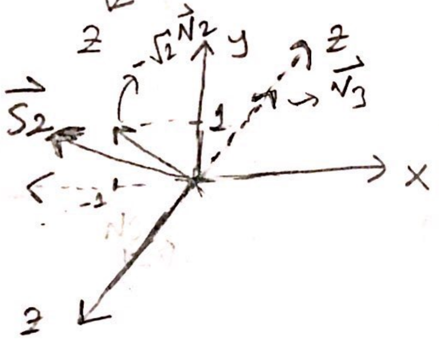
d)

S_1:



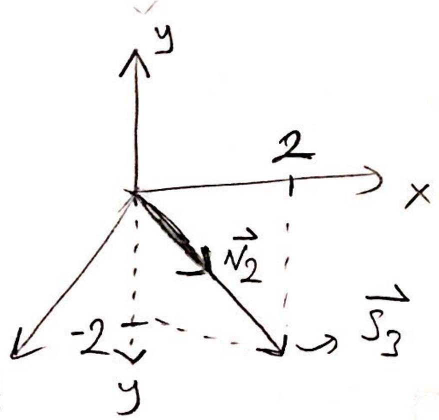
$$\vec{S_1} = 2\sqrt{2}\,\vec{v_1} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

S_2:



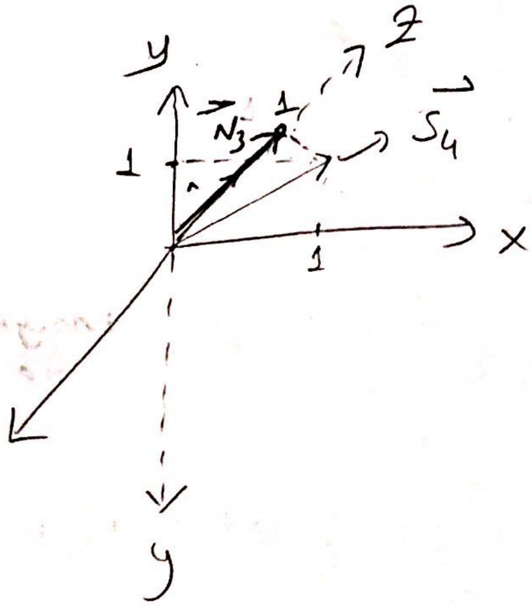
$$\vec{S_2} = \vec{v_3} - \sqrt{2}\,\vec{v_2} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

S_3:



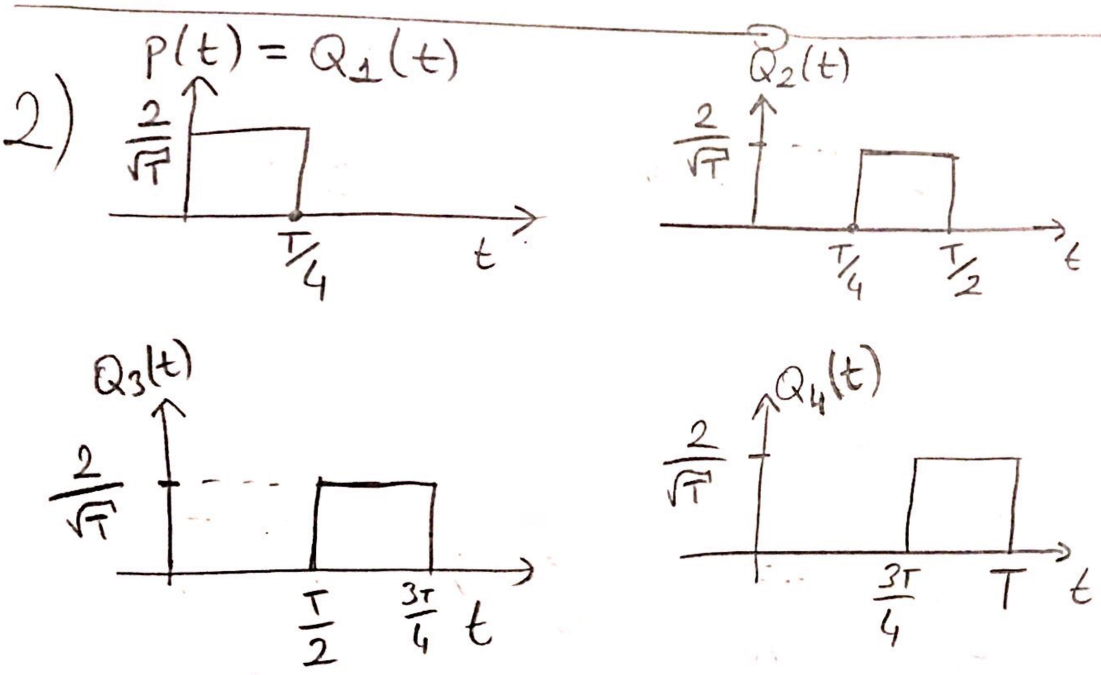
$$\vec{S_3} = 2\sqrt{2}\,\vec{v_2} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

S_4:



$$\vec{S_4} = \sqrt{2}\,\vec{v_1} + \vec{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

2)



a) Look dot product pairwise;

$$\int_{\langle T\rangle} Q_1(t).Q_2(t)\,dt = \int_0^{T/4} \frac{2}{\sqrt{T}} \cdot 0 \cdot dt + \int_{T/4}^{T/2} 0 \cdot \frac{2}{\sqrt{T}}\,dt = \boxed{0}$$ ✓

$$\int_{\langle T\rangle} Q_1(t) \cdot Q_3(t)\,dt = \int_0^{T/4} \frac{2}{\sqrt{T}} \cdot 0\,dt + \int_{T/2}^{3T/4} 0 \cdot \frac{2}{\sqrt{T}}\,dt = \boxed{0}$$ ✓

⋮

Same operation for each pair. All of them is equal to zero. Therefore, this signal set is $\boxed{\text{orthogonal}}$.

CamScanner ile tarandı

2)

b) $\int_0^{T/4} \frac{4}{T}\, dt = \frac{4t}{T}\Big|_0^{T/4} = \frac{4}{T} \cdot \frac{T}{4} = \boxed{1}$

$\int_{T/4}^{T/2} \frac{4}{T}\, dt = \frac{4}{T} \cdot t \Big|_{T/4}^{T/2} = 2 - 1 = \boxed{1}$

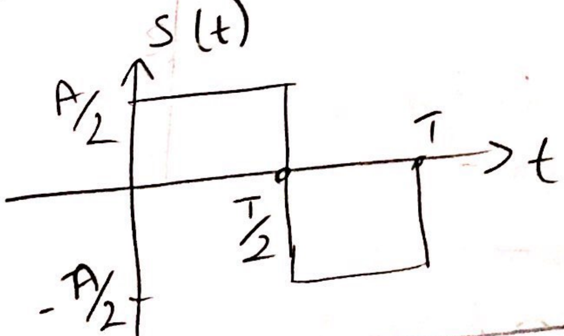
$\int_{T/2}^{3T/4} \frac{4}{T}\, dt = \frac{4}{T} \cdot t \Big|_{T/2}^{3T/4} = 3 - 2 = \boxed{1}$

$\int_{3T/4}^{T} \frac{4}{T}\, dt = \frac{4}{T}\, t \Big|_{3T/4}^{T} = 4 - 3 = \boxed{1}$

This signal set is $\boxed{\text{orthonormal}}$.

Since set is orthonormal, it forms a <u>basis</u>.

c)



$$\boxed{s(t) = \frac{A\sqrt{T}}{4}\left(Q_1(t) + Q_2(t)\right) - \frac{A\sqrt{T}}{4}\left(Q_3(t) + Q_4(t)\right)}$$

3) $f_{xy}(x,y) = \begin{cases} x^2 + \frac{xy}{3} & 0 \le x \le 1 \text{ and } 0 \le y \le 2 \\ 0 & \text{else} \end{cases}$

a) $f_x(x) = \int_0^2 x^2 + \frac{xy}{3}\, dy = x^2 y + \frac{xy^2}{6} \Big|_0^2 = 2x^2 + \frac{4x}{6}$

$$\boxed{f_x(x) = \begin{cases} 2x^2 + \frac{4x}{6}, & 0 \le y \le 2 \text{ and } 0 \le x \le 1 \\ 0, & \text{else} \end{cases}}$$

$f_y(y) = \int_0^1 x^2 + \frac{xy}{3}\, dx = \frac{x^3}{3} + \frac{x^2 y}{6} \Big|_0^1 = \frac{y}{6} + \frac{1}{3}$

$$\boxed{f_y(y) = \begin{cases} \frac{y+2}{6}, & 0 \le x \le 1 \text{ and } 0 \le y \le 2 \\ 0, & \text{else} \end{cases}}$$

b) $E[X] = \int_0^1 2x^3 + \frac{4x^2}{6}\, dx = \frac{x^4}{2} + \frac{4x^3}{18} \Big|_0^1 = \frac{1}{2} + \frac{4}{18} = \boxed{\frac{13}{18}}$

$E[Y] = \int_0^2 \frac{y^2}{6} + \frac{2y}{6}\, dy = \frac{y^3}{18} + \frac{y^2}{6} \Big|_0^2 = \frac{8}{18} + \frac{4}{6} = \boxed{\frac{20}{18}}$

$E[X^2] = \int_0^1 2x^4 + \frac{4x^3}{6}\, dx = \frac{2x^5}{5} + \frac{x^4}{6} \Big|_0^1 = \frac{2}{5} + \frac{1}{6} = \boxed{\frac{17}{30}}$

$E[Y^2] = \int_0^2 \frac{y^3}{6} + \frac{2y^2}{6}\, dy = \frac{y^4}{24} + \frac{2y^3}{18} \Big|_0^2 = \underset{\frac{2}{3}}{\frac{16}{24}} + \underset{\frac{8}{9}}{\frac{16}{18}} = \boxed{\frac{14}{9}}$

$\sigma_x^2 = E[X^2] - E[X]^2 = \frac{17}{30} - \frac{169}{324} = \boxed{\frac{438}{9720} = \frac{73}{1620}}$

$\sigma_y^2 = E[Y^2] - E[Y]^2 = \frac{14}{9} - \frac{400}{324} = \boxed{\frac{936}{2916} = \frac{26}{81}}$

3) continue:

c) $P(X>0.5)=\int_{0.5}^{1} f_X(x)\,dx = \frac{2x^3}{3}+\frac{2x^2}{6}\Big|_{0.5}^{1} = \frac{2}{3}+\frac{1}{3}-\frac{1}{12}-\frac{1}{12}$

$= \boxed{\frac{5}{6}}$

d) $P(Y<X)=\int_0^1\int_0^x x^2+\frac{xy}{3}\,dy\,dx = \int_0^1\left(x^2y+\frac{xy^2}{6}\Big|_0^x\right)dx$

$=\int_0^1 x^3+\frac{x^3}{6}\,dx = \frac{x^4}{4}+\frac{x^4}{24}\Big|_0^1 = \boxed{\frac{7}{24}}$

e) $P(Y<0.5 \mid X<0.5) = \frac{P(Y<0.5,\ X<0.5)}{P(X<0.5)}$

$P(Y<0.5,\ X<0.5) = \int_0^{0.5}\int_0^{0.5} x^2+\frac{xy}{3}\,dy\,dx = \int_0^{0.5}\left(x^2y+\frac{xy^2}{6}\Big|_0^{0.5}\right)dx$

$=\int_0^{0.5}\frac{x^2}{2}+\frac{x}{24}\,dx = \frac{x^3}{6}+\frac{x^2}{48}\Big|_0^{0.5} = \frac{1}{48}+\frac{1}{192} = \boxed{\frac{5}{192}}$

$P(X<0.5) = 1-\underbrace{P(X>0.5)}_{\text{part c}} = 1-\frac{5}{6} = \boxed{\frac{1}{6}}$

$P(Y<0.5 \mid X<0.5) = \frac{\frac{5}{192}}{\frac{1}{6}} = \boxed{\frac{5}{32}}$

f) $cor(X,y) = \dfrac{cov(x,y)}{\sigma_x \sigma_y} = \dfrac{E(xy) - E[x].E[y]}{\sigma_x \sigma_y}$

$$E[XY] = \int_0^1\int_0^2 x^3y + \frac{x^2y^2}{3}\,dy\,dx = \int_0^1 \frac{x^3y^2}{2} + \frac{x^2y^3}{9}\Big|_0^2 dx$$

$$\int_0^1 2x^3 + \frac{8x^2}{9}\,dx = \frac{x^4}{2} + \frac{8x^3}{27}\Big|_0^1$$

$$E[XY] = \frac{1}{2} + \frac{8}{27} = \boxed{\frac{43}{54}}$$

$$cor(x,y) = \frac{\frac{43}{54} - \frac{13}{18}\cdot\frac{20}{18}}{\sqrt{\frac{73}{1620}}\cdot\sqrt{\frac{26}{81}}} = \frac{-0.006172}{0.120} = -0.05$$

Since $E[XY] \neq 0$, they are $\boxed{\text{not orthogonal}}$.

g) $cov(x,y) = \frac{43}{54} - \left(\frac{13}{18}\cdot\frac{20}{18}\right) = -0.006172$

They are <u>negatively correlated</u> (we found in part f)

4) a) $P(0|1) = ?$

when $A_p + n > \frac{A_p}{2} \xrightarrow{\text{output}} 1$.

$$P(0|1) = P\left(A_p + n < \frac{A_p}{2}\right) = \boxed{P\left(n < -\frac{A_p}{2}\right)}$$

<u>We know:</u> $P\left(n > -\frac{A_p}{2}\right) = Q\left(-\frac{A_p}{2\sigma}\right)$

Thus, $P\left(n < -\frac{A_p}{2}\right) = 1 - P\left(n > -\frac{A_p}{2}\right)$

$$= 1 - \underbrace{Q\left(-\frac{A_p}{2\sigma}\right)}$$

$$= \cancel{1} - \cancel{1} + Q\left(\frac{A_p}{2\sigma}\right)$$

$$\boxed{P\left(n < -\frac{A_p}{2}\right) = Q\left(\frac{A_p}{2\sigma}\right)}$$

Q table

b) $\boxed{P(1|0) = P\left(n > \frac{A_p}{2}\right) = Q\left(\frac{A_p}{2\sigma}\right)}$

c) $P(\text{Error}) = P(\underset{\text{input}}{X=0}) \cdot P(1|0) + P(X=1) \cdot P(0|1)$

$$P(\text{Error}) = 0.5\, Q\left(\frac{A_p}{2\sigma}\right) + 0.5\, Q\left(\frac{A_p}{2\sigma}\right)$$

$$\boxed{P(\text{Error}) = Q\left(\frac{A_p}{2\sigma}\right)}$$

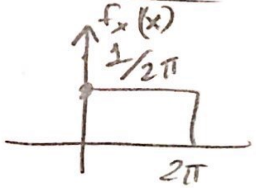
5) $Var(x) = 500 - 100 = 400$, $\boxed{\sigma = 20}$, $\boxed{\mu = 10}$

a) $P(10 \leq x \leq 20) = P\left(\frac{10-10}{20} \leq \frac{x-10}{20} \leq \frac{20-10}{20}\right) = P(0 \leq Z \leq 0.5)$

$= \boxed{0.19146}$

5) continue:

b)



$$E[Y] = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{A\cos(x)}_{0}\,dx = \boxed{0}$$

$$E[Y^2] = \frac{1}{2\pi}\int_0^{2\pi} A^2 \underbrace{\cos^2(x)}_{\left(\frac{1+\cos 2x}{2}\right)}\,dx = \frac{A^2}{2\pi}\left(\frac{x}{2} + \frac{\sin 2x}{4}\Big|_0^{2\pi}\right) = \boxed{\frac{A^2}{2}}$$

$$\sigma_y^2 = E[Y^2] - E[Y]^2 = \boxed{\frac{A^2}{2}}$$