STRANGE ATTRACTORS: 2D MAPPINGS OF CHAOTIC SYSTEMS

A PREPRINT

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June 6, 2020

1 Introduction

Using the definition given in the text for a strange attractor:

"...we define an attractor to be a closed set A with the following properties:

1. A is an *invariant set*: any trajectory $x(t)$ that starts in A stays in A for all time.
2. A *attracts an open set of initial conditions*: there is an open set U containing A such that if $x(0) \in U$, then the distance from $x(t)$ to A tends to zero as $t \to \infty$. This means that A attracts all trajectories that start sufficiently close to it. The largest such U is called the *basin of attraction* of A.
3. A is *minimal*: there is no proper subset of A that satisfies conditions 1 and 2.

...we define a *strange attractor* to be an attractor that exhibits sensitive dependence on initial conditions." [3]

In this paper, we explore strange attractors by providing detailed solutions to selected problems from the text Nonlinear Dynamics by Steven Strogatz [3], a summary and reproduction of the salient results from Hénon's 1976 paper (A two-dimensional mapping with a strange attractor [2]), and the relevant code used in our calculations. We specifically examine 2D iterated maps that exhibit chaotic behavior over some parameter range for two reasons; these transformations are often significantly easier to analyze than a system of differential equations, and given the map has an inverse, guarantees a unique trajectory through each point in phase space such that it is equivalent to the original differential system.

2 Selected Problems from Strogatz [3] Chapter 9

2.1 Strogatz 12.1.8: Hénon's Area-Preserving Quadratic Map

The map

$$x_{n+1} = x_n \cos(\alpha) - (y_n - x_n^2)\sin(\alpha) \tag{1}$$
$$y_{n+1} = x_n \sin(\alpha) + (y_n - x_n^2)\cos(\alpha) \tag{2}$$

illustrates many of the remarkable properties of area-preserving maps (Hénon 1969, 1983). Here $0 \leq \alpha \leq \pi$ is a parameter.

a) Verify that the map is area-preserving.

Solution

A map is area-preserving given that $|\det(J)|=1$. An arbitrary 2-D map *maps* a rectangle of area $dxdy$ to a parallelogram of area $|\det\big(J_{(x,y)}|\big)dxdy$, therefore the above statement can be seen to be true.

Given the Hénon Quadratic Map, solving for components of the Jacobian with $x_{n+1} = f(x, y)$ and $y_{n+1} = g(x, y)$:

$$\frac{\partial f}{\partial x} = \cos\alpha + 2x_n \sin\alpha, \quad \frac{\partial f}{\partial y} = -\sin\alpha, \quad \frac{\partial g}{\partial x} = \sin\alpha - 2x_n\cos\alpha, \quad \frac{\partial g}{\partial y} = \cos\alpha$$

$$J = \begin{bmatrix} \cos\alpha + 2x_n\sin\alpha & -\sin\alpha \\ \sin\alpha - 2x_n\cos\alpha & \cos\alpha \end{bmatrix}$$

And the determinant as:

$$\det\{J\} = \cos^2\alpha + sin^2\alpha = 1$$

b) Find the inverse mapping.

Solution

Solving eq. (1) for y_n as a function of x_n and x_{n+1}:

$$y_n = \frac{x_n\cos\alpha - x_{n+1} + x_n^2\sin\alpha}{\sin\alpha}$$

substituting into eq. (2), solving for x_n as a function of x_{n+1} and y_{n+1}, and substituting this into our equation for y_n to get the inverse mapping:

$$x_n = y_{n+1}\sin\alpha + x_{n+1}\cos\alpha \tag{3}$$
$$y_n = x_{n+1}^2\cos^2\alpha + y_{n+1}^2\sin^2\alpha + 2x_{n+1}y_{n+1}\sin\alpha\cos\alpha + x_{n+1}\cos^2\alpha + y_{n+1}\sin\alpha\cos\alpha \tag{4}$$

c) Explore the map on the computer for various a. For instance, try $\cos(\alpha)$ = 0.24, and use initial conditions in the square -1 $\leq x, y \leq 1$. You should be able to find a lovely chain of five islands surrounding the five points of a period-5 cycle. Then zoom in on the neighborhood of the point $x = 0.57, y = 0.16$. You'll see smaller islands, and maybe even smaller islands around them! The complexity extends all the way down to finer and finer scales. If you modify the parameter to $\cos(\alpha)$ = 0.22 , you'll still see a prominent chain of five islands, but it's now surrounded by a noticeable ***chaotic sea***. This mixture of regularity and chaos is typical for area-preserving maps (and for Hamiltonian systems, their continuous-time counterpart).

Solution

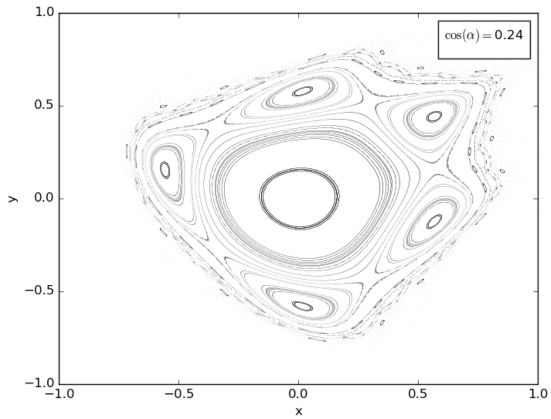


Figure 1: The mapping for the value $\cos(\alpha) = 0.24$

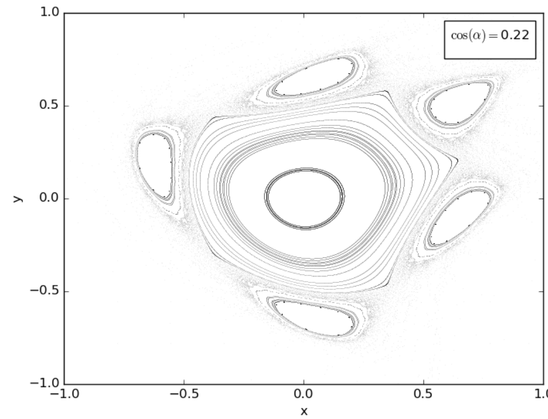


Figure 2: The mapping for the value $\cos(\alpha) = 0.22$

2.2 Strogatz 12.1.9: The Standard Map

The map

$$x_{n+1} = x_n + y_{n+1} \tag{5}$$
$$y_{n+1} = y_n + k \sin x_n \tag{6}$$

is called the standard map because it arises in many different physical contexts, ranging from the dynamics of periodically kicked oscillators to the motion of charged particles perturbed by a broad spectrum of oscillating fields (Jensen 1987, Lichtenberg and Lieberman 1992). The variables x, y, and the governing equations are all to be evaluated modulo 2π. The nonlinearity parameter $k \geq 0$ is a measure of how hard the system is being driven.

a) Show that the map is area-preserving for all k.

Solution

Solving for the components of the Jacobian with $x_{n+1} = f(x, y)$ and $y_{n+1} = g(x, y)$:

$$\frac{\partial f}{\partial x} = 1 + k\cos x_n, \quad \frac{\partial f}{\partial y} = 1, \quad \frac{\partial g}{\partial x} = k\cos x_n, \quad \frac{\partial g}{\partial y} = 1$$

$$J = \begin{bmatrix} 1 + k\cos x_n & 1 \\ k\cos x_n & 1 \end{bmatrix}$$

And the determinant as:

$$\det\{J\} = 1 + k\cos x - k\cos x = 1$$

Therefore as stated in 2.1.a, the map is area-preserving regardless of the value of k, as the determinant is independent of k.

b) Plot various orbits for $k = 0$. (This corresponds to the integrable limit of the system.)

Solution

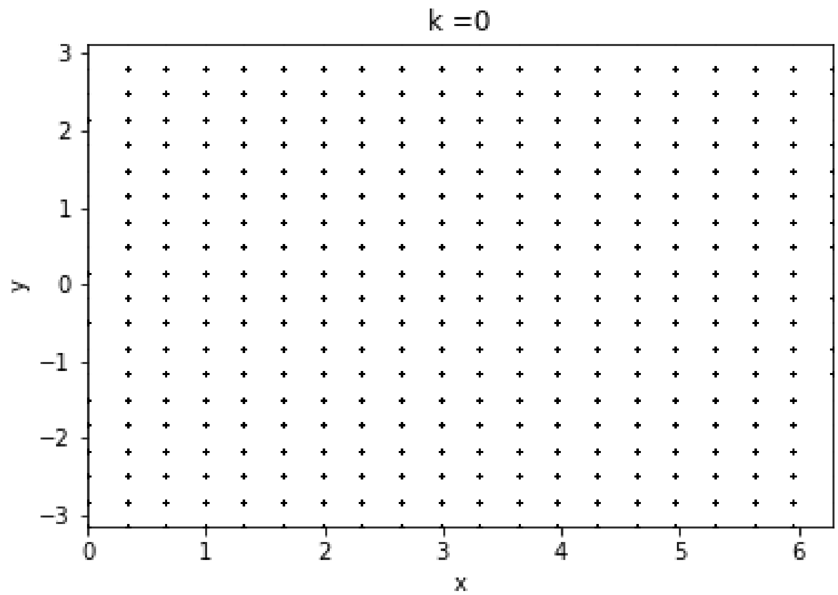


Figure 3: Standard map for $k = 0$. Trajectories are confined to discrete orbits of $(x_0 + ny_0, y_0)$ for $n = 1, 2, 3...$ and any initial conditions.

c) Using a computer, plot the phase portrait for $k = 0.5$. Most orbits should still look regular.

Solution

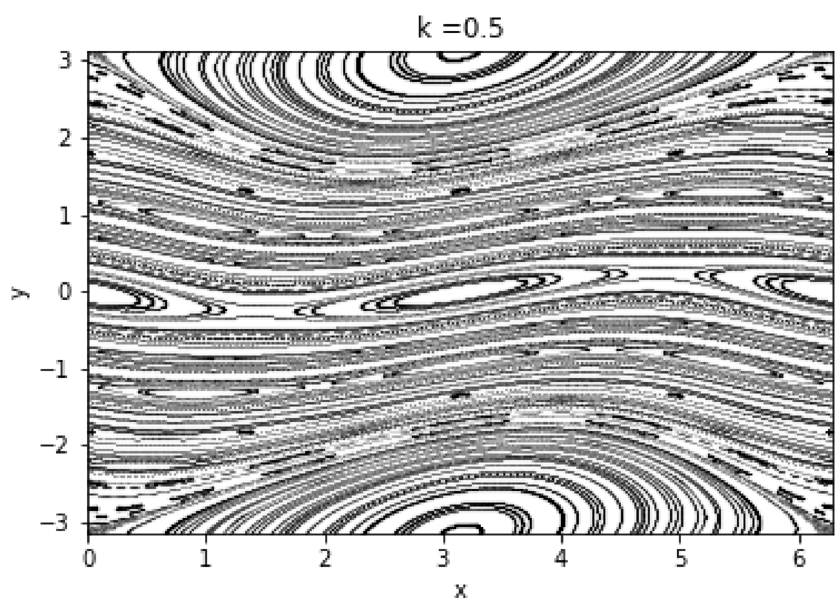


Figure 4: Standard map for $k = 0.5$. No apparent chaotic orbits at this parameter value, but quasiperiodic orbits exist.

d) Show that for $k = 1$, the phase portrait contains both islands and chaos.

Solution

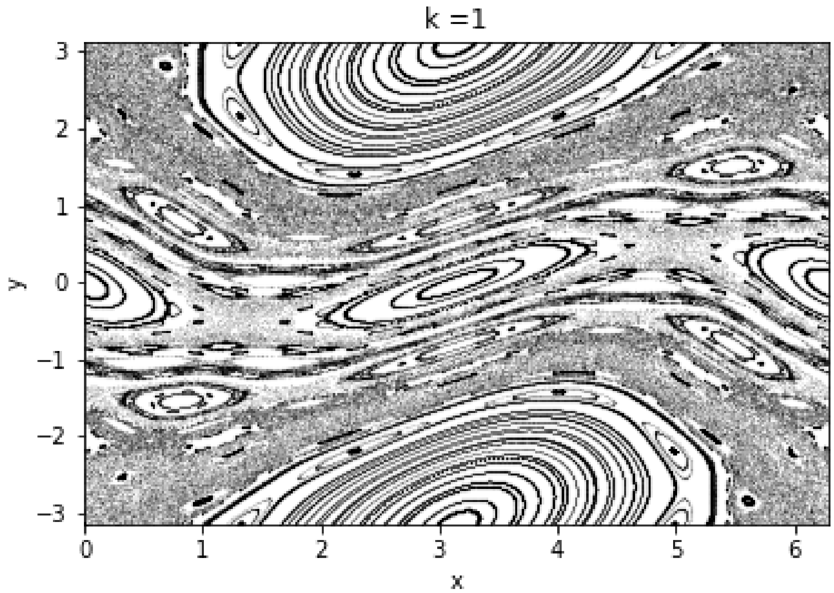


Figure 5: Standard map for $k = 1$. Portrait contains islands where quasiperiodic solutions exist, and a chaotic sea as expected.

e) Show that at $k = 2$, the chaotic sea has engulfed almost all the islands.

Solution

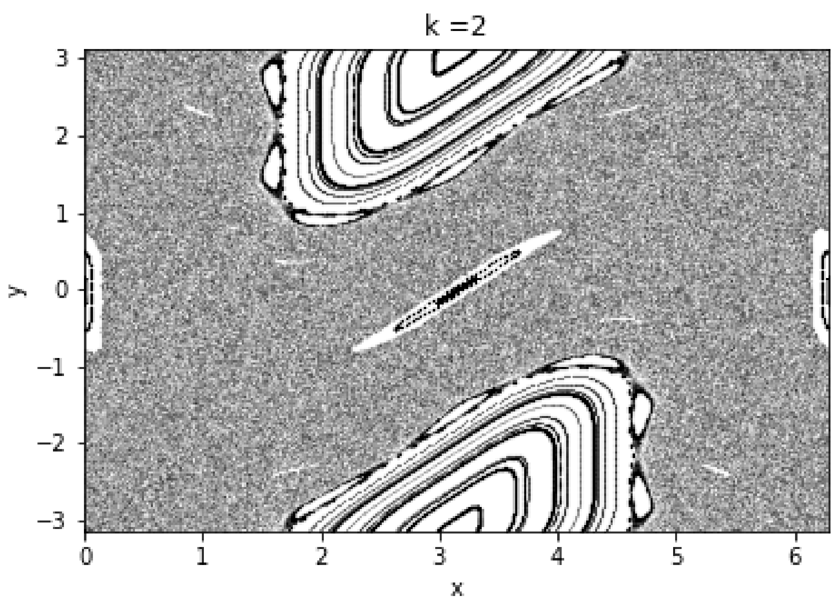


Figure 6: Standard map for $k = 2$. Chaotic orbits have taken up most of the phase portrait for this parameter value.

2.3 Strogatz 12.2.14-12.2.18: The Lozi Map

The following exercises deal with the Lozi map

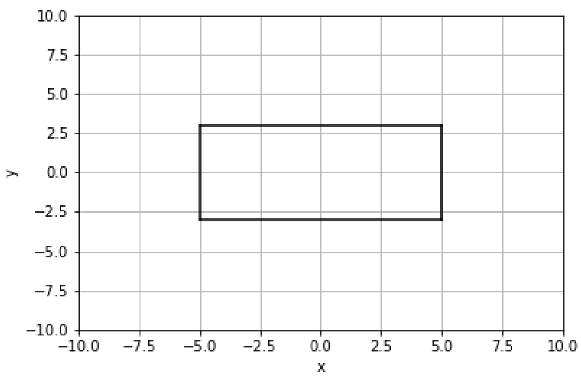
$$x_{n+1} = 1 + y_n - a|x_n| \tag{7}$$
$$y_{n+1} = bx_n \tag{8}$$

where a, b are real parameters, with $-1 < b < 1$ (Lozi 1978). Note its similarity to the Hénon map. The Lozi map is notable for being one of the first systems proven to have a strange attractor (Misiurewicz 1980). This has only recently been achieved for the Hénon map (Benedicks and Carleson 1991) and is still an unsolved problem for the Lorenz equations.

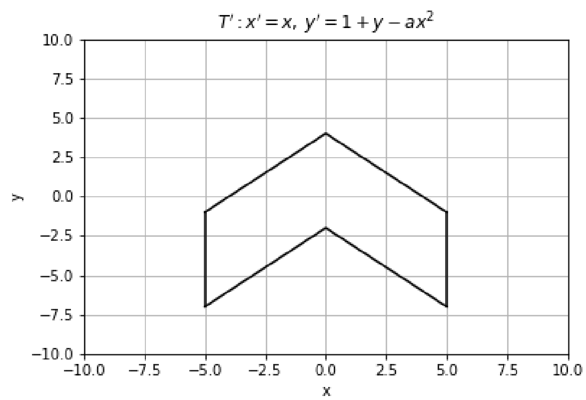
2.3.1 Strogatz 12.2.14

In the style of Figure 12.2.1, plot the image of a rectangle under the Lozi map.

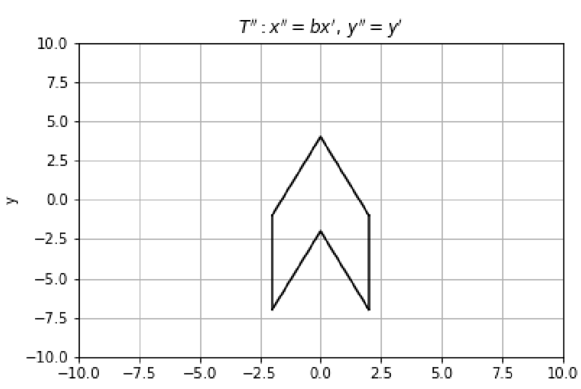
Solution



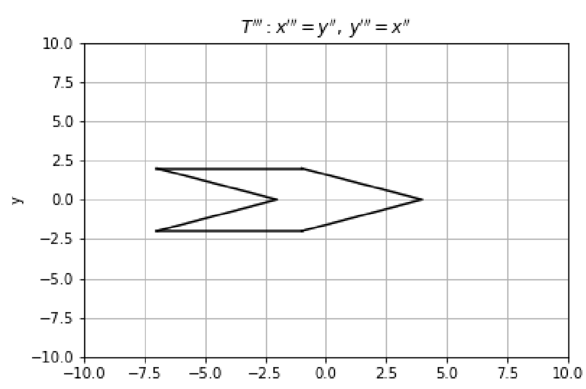
(a)



(b)



(c)



(d)

Figure 7: The transformation $T = T'T''T'''$ of a 10x6 rectangle centered at the origin under the Lozi map with parameters $a = 1, b = 0.4$. Comparing (a) and (d), we can see that the Lozi map is dissipative if $|b| < 1$ as proven in §2.3.2

2.3.2 Strogatz 12.2.15

Show that the Lozi map contracts areas if $-1 < b < 1$.

Solution

Solving for the components of the Jacobian with $x_{n+1} = f(x, y)$ and $y_{n+1} = g(x, y)$:

$$\frac{\partial f}{\partial x} = -a \cdot \mathrm{sgn}(x), \quad \frac{\partial f}{\partial y} = 1, \quad \frac{\partial g}{\partial x} = b, \quad \frac{\partial g}{\partial y} = 0$$

$$J = \begin{bmatrix} -a \cdot \mathrm{sgn}(x) & 1 \\ b & 0 \end{bmatrix}$$

$$\det\{J\} = -b$$

A map will contract areas $\iff$ $|\det(J)| < 1$ as shown in 2.1.a. Thus, for $-1 < b < 1$ we have $|-b| < 1$ and this condition is satisfied for all (x, y). QED

2.3.3 Strogatz 12.2.16

Find and classify the fixed points of the Lozi map.

Solution

Fixed points on an iterative map can be found where $x_{n+1} = x_n$ and $y_{n+1} = y_n$. Thus our equations become:

$$\begin{aligned} x_n &= 1 + y_n - a|x_n| \\ y_n &= bx_n \end{aligned}$$

Solving for x_n

$$x_n = 1 + bx_n - a|x_n|$$

For $x_n > 0$:

$$\begin{aligned} x_n(1 + a - b) - 1 &= 0 \\ x_n &= \frac{1}{1 + a - b} \\ y_n &= \frac{b}{1 + a - b} \end{aligned}$$

For $x_n < 0$:

$$\begin{aligned} x_n(1 - a - b) - 1 &= 0 \\ x_n &= \frac{1}{1 - a - b} \\ y_n &= \frac{b}{1 - a - b} \end{aligned}$$

The fixed points are:

$$P_+ = (\tfrac{1}{1+a-b}, \tfrac{b}{1+a-b}), \quad P_- = (\tfrac{1}{1-a-b}, \tfrac{b}{1-a-b})$$

P_+ will only exist for $b < 1 + a$ given the condition $x_n > 0$. P_- will only exist for $b > 1 - a$ given the condition $x_n < 0$.

Stability analysis: evaluating the Jacobian J at the fixed points P_+, P_- and finding the eigenvalues we get

$$\det \begin{bmatrix} -a\,\mathrm{sgn}(x) - \lambda & 1 \\ b & -\lambda \end{bmatrix} = \lambda^2 + \lambda a\,\mathrm{sgn}(x) - b = 0$$

$$\Rightarrow \lambda_+ = \frac{-a \pm \sqrt{a^2 + 4b}}{2}, \quad \lambda_- = \frac{a \pm \sqrt{a^2 + 4b}}{2}$$

(where $_\pm$ denotes $x > 0, x < 0$, respectively). Both Jacobians have $\mathrm{Tr}(J_\pm) = a$ and $\det(J_\pm) = -b$. If we restrict $0 < b < 1$ or $\det(J_\pm) < 0$; both fixed points P_+, P_- will be saddle points for any a. For $-1 < b < 0$ or $\det(J_\pm) > 0$; P_+ is only defined for $a > b - 1$ given $x_n > 0$, and P_- is only defined for $a > 1 - b$ given $x_n < 0$. From here, we can see $\mathrm{Tr}(J_+) = a < 0$ and $\mathrm{Tr}(J_-) = a > 0$ for all time.

Therefore, with $0 < b < 1$ the fixed points P_+ and P_- are both **saddle points**. When $-1 < b < 0$, the fixed point P_+ will become **stable** and P_- will **remain unstable** across the bifurcation at $b = 0$.

2.3.4 Strogatz 12.2.17

Find and classify the 2-cycles of the Lozi map.

Solution

Proof. Take the next steps of the Lozi map as,

$$\begin{aligned} x_{n+2} &= 1 + bx_n - |1 + y_n - a|x_n|| \\ y_{n+2} &= b(1 + y_n - a|x_n|) \end{aligned}$$

Let x and y be a fixed point then $x_n = x_{n+1} = x_{n+2} = x^*$ and $y_n = y_{n+1} + y_{n+2} = y^*$ This simplifies the above to,

$$x^* = 1 + bx_n - |1 + y^* - a|x^*|| \tag{1}$$
$$y^* = b(1 + y^* - a|x^*|). \tag{2}$$

Substituting eq. (2) into eq. (1) yields,

$$\begin{aligned} x^* &= 1 + bx^* - a|1 + b(1 + y^* - a|x^*|) - a|x^*|| \\ &= 1 + bx^* - a + ab + aby^* - a^2bx^* - ax^* \end{aligned}$$

Solving for x^* in the limited case $1 - b < a$ yields,

$$x^* = \frac{1 - a - b}{(b-1)^2 + a^2}$$

Solving for y^* in the limited case $1 - b < a$ yields,

$$y^* = \frac{b(1 + a - b)}{(b-1)^2 + a^2}$$

Solving for x^* in the limited case $1 - b > a$ yields,

$$x^* = \frac{1 + a - b}{(b-1)^2 + a^2}$$

Solving for y^* in the limited case $1 - b > a$ yields,

$$y^* = \frac{b(1 - a - b)}{(b-1)^2 + a^2}$$

Now we must classify the 2-cycles of the Lozi map. For this, take the Jacobian.

$$J = \begin{bmatrix} b + a^2\,\mathrm{sgn}\,x(1 + y - a|x| & -a) \\ -ab & b \end{bmatrix} = \begin{bmatrix} b - a^2 & -a \\ ab & b \end{bmatrix}$$

Find the eigenvalues of the matrix,

$$|\lambda| = \left| \frac{-a^2 + 2b\sqrt{a^2 - 4b}}{2} \right| < 1$$

This is a period-2 cycle. $\square$

2.3.5 Strogatz 12.2.18

Show numerically that the Lozi map has a strange attractor when $a = 1.7, b = 0.5$.

Solution

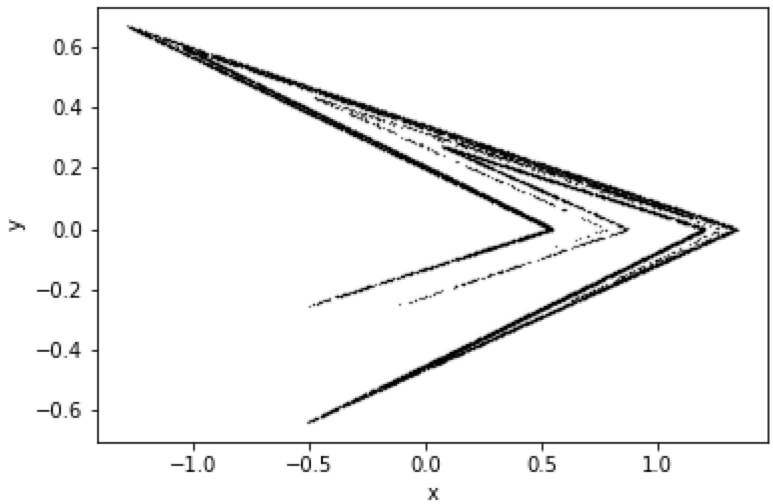


Figure 8: A numerical plot of the Lozi map for parameter values $a = 1.7, b = 0.5$. Starting from the initial condition $(x_0, y_0) = (0, 0)$ and taking 10,000 iterations, we get a strange attractor resembling the Hénon map. The transient regime has been omitted from the plot.

3 Analysis and Reproduction of Hénon (1976)

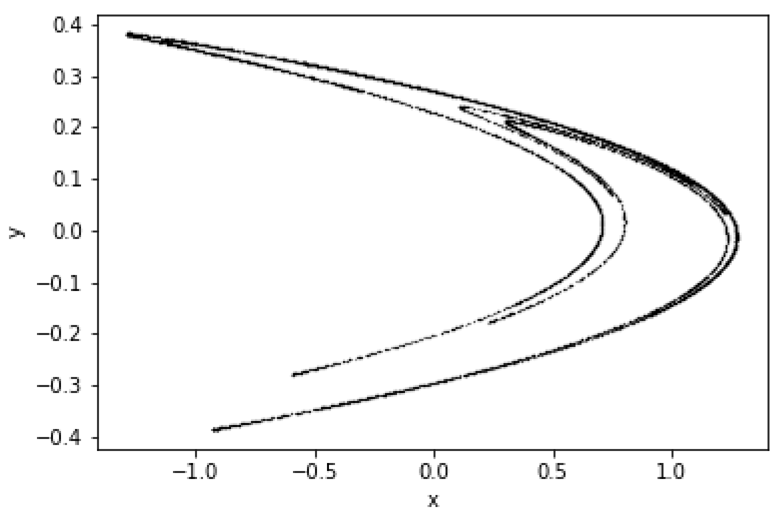


Figure 9: A reproduction of the strange attractor in Hénon's original paper [2], with parameter values $a = 1.4$ and $b = 0.3$ calculated over 10,000 iterations

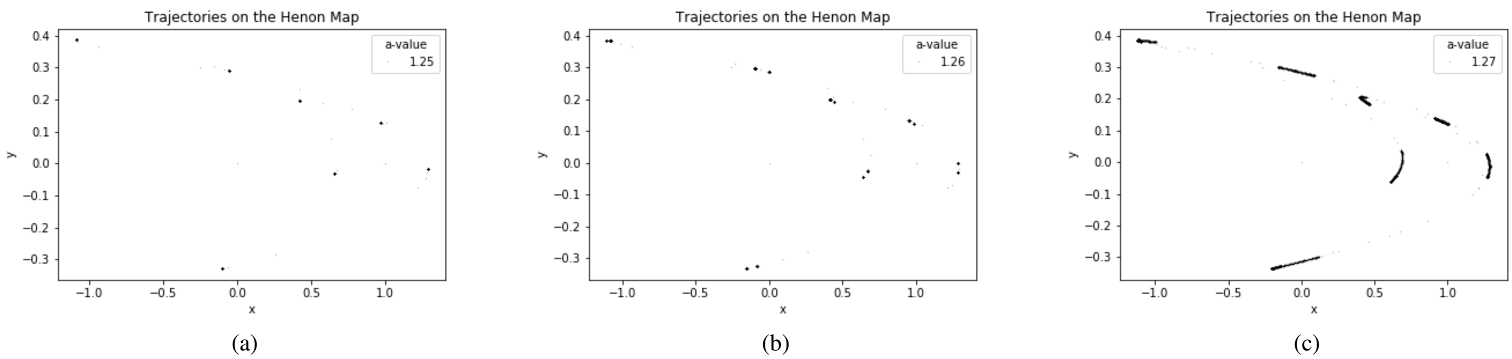


Figure 10: Plots of a trajectory starting at (0, 0) with fixed parameter $b = 0.3$ and a = 1.25 (a), 1.26 (b), and 1.27 (c). These successive plots show a period-doubling cascade towards chaos at $a = 1.4$

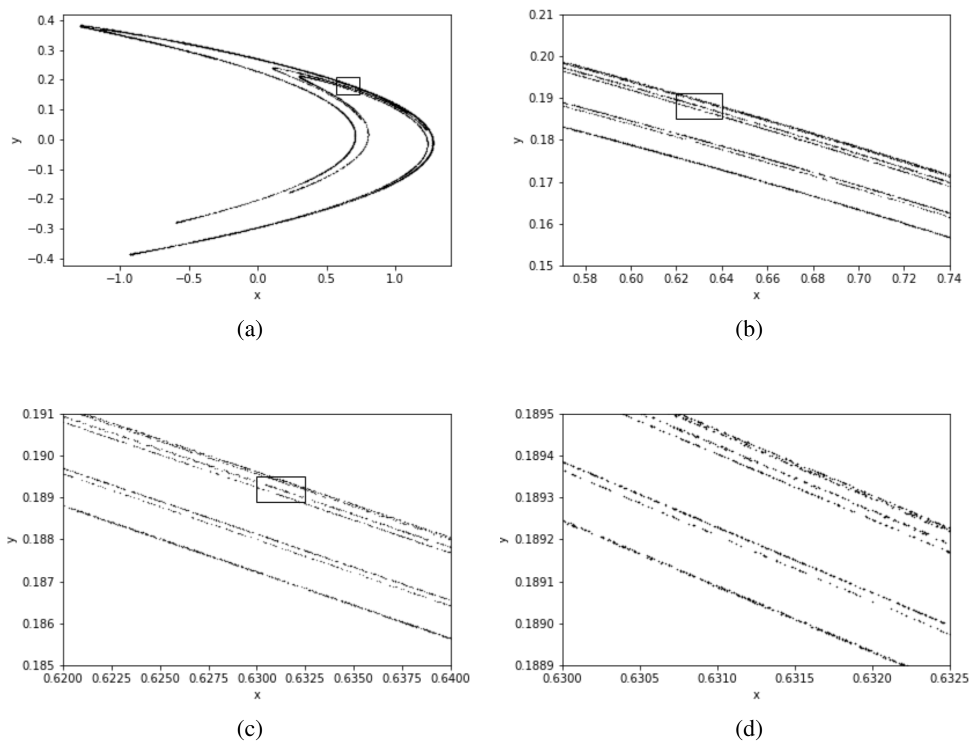
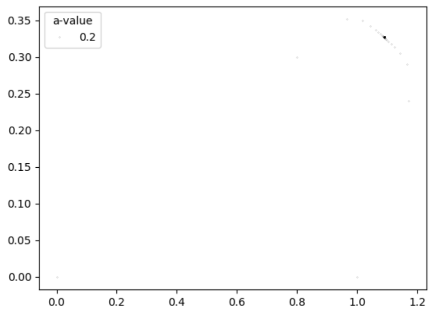


Figure 11: Plots for the Hénon map in Figure 9 seen at progressively smaller scales. Zooming in reveals the fractal nature of the strange attractor.

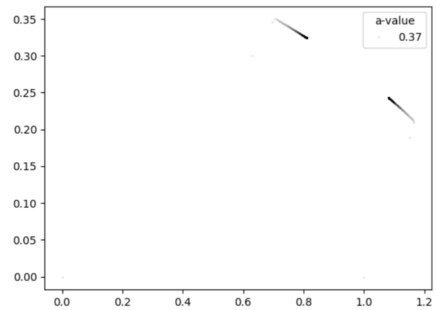
4 Summary of Hénon (1976)

In **A Two dimensional Mapping with a Strange Attractor** (Hénon 1976), Hénon seeks to find a model that shows the same properties as Lorenz's strange attractor, but is as simple as possible in order to allow for fast in-depth numerical analysis of the the system.Hénon proposes simplifying Lorenz's three-dimensional differential system of equations through three steps.

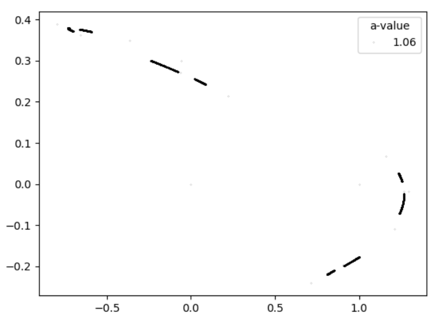
In the first step Hénon refers to the Poincaré map, a surface S is mapped into itself by T. In this process of mapping, one no longer views the trajectories in three-dimensional space given by the system of equations, but rather the set of



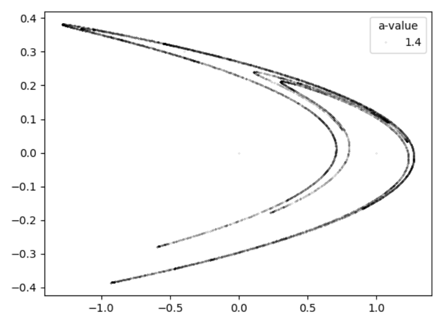
(a)



(b)



(c)



(d)

Figure 12: Critical values for the Henon map

points in S created via self intersection. In doing so the problem can be viewed as a two-dimensional mapping problem rather than a three-dimensional differential system of equations.

The first step still results in a system requiring integration, so it is proposed that the mapping problem be represented as explicit equations s.t. given a point A in S, mapping $T(A)$ in S may be directly found. It is acknowledged that such a set of explicit equations will not correspond directly to the Lorenz differential system, however Hénon references a past paper of his stating that the essential properties may be replicated.
The third step seeks to find a mapping T that replicates the folding and stretching nature of the Lorenz system, that allows for divergence of trajectories while still tending towards an attractor. This is accomplished by:

1. Folding over the y-axis via parameter a:
$$y' = y + 1 - ax^2, x' = x$$
2. Contraction in the x-axis via parameter b:
$$y'' = y', x'' = bx'$$
3. Orient back around the x-axis:
$$y''' = x'', x''' = y''$$

Viewing these transformations as an iterative process leads to the iterative system:

$$x_{n+1} = y_i + 1 - ax_i^2, \quad y_{i+1} = bx_i$$

It is important to note that this self-mapping processes is area-preserving if $|b| = 1$. This is evident geometrically looking at the mapping process in its step-by-step form, as the only step that does not preserve area is dependent on b. Thus using a value of $|b| < 1$ will result in a area diminishing map as is desired. This can be verified by evaluation of the Jacobian, which has a constant value of $-b$, as is expected. These properties match the negative divergence of the Lorenz system that make it an attractor. Additionally the mapping T is invertable, T^{-1} exists, thus T is a one-to-one mapping. This pairs with another property of the Lorenz system in that any given point produces a unique trajectory. At this point Hénon has proposed an iterative map that has key similarities to the Lorenz system, next parameters will be explored to attempt to produce a strange attractor.

In the process of finding appropriate parameters, the fixed points of the system are first evaluated:

$$x_{n+1} = x_n, y_{n+1} = y_n$$
$$\boxed{y^* = bx_n}$$
$$x_n = y_n + 1 - ax_n^2$$
$$ax_n^2 + x(1-b) - 1 = 0$$
$$\boxed{x^* = \frac{-(1-b) \pm \sqrt{(1-b)^2 + 4a}}{2a}}$$

These points are real for $a > (1-b)^2/4$, and that condition will mark the lower bound for evaluation of the parameter a. Hénon chooses 0.3 for b, as mentioned the value should be s.t. $|b| < 1$ to ensure negative divergence of the system, and that value of b was small enough that folding was adequate, but finer structures of the attractor were still visable upon inspection. Choosing an a smaller than the lower bound cited results in trajectories tending towards infinity, similarly an upper bound was found, this time numerically, which Hénon cites as $a = 1.55$. The attractor that exists between these bounds exhibits three phases. An initial stability created by the fixed points when a has a value between the lower bound, and a value a_1, where at a_1 a bifurcation occurs resulting in both fixed points being unstable. The attractor remains simple, until reaching a critical value of $a_2 = 1.06$, in which successive period double tends to infinity resulting in chaos, and the attractor becomes a strange attractor. These states can be observed in Figure 12, and the value of $a = 1.4$ is what Hénon chooses for his numerical analysis.

Figure 2 in [2] is obtained via numerical integration for the parameters $a = 1.4, b = 0.3$, which we have reproduced in Figure 9. Initial conditions that are not already on the attractor quickly approach it, and thus it is possible to safely eliminate the transient regime. We can also see that the longitudinal structure of the attractor along the curves is relatively simple, while the transverse structure is significantly more complicated. Most interestingly, its clear the underlying structure is exactly that of a Cantor set. Zooming in on the attractor, each curve in 9 is in fact made up of an infinity of quasi-parallel curves. Lorenz had inferred the Cantor set structure, but was unable to see it due to a very small contracting ratio after one circuit, $\approx 10^{-5}$. In his results, Hénon is able to view a number of successive levels in the Cantor set structure with a contracting ratio of 0.3. Finally, Hénon claims that these results infer the existence of a trapping region from which trajectories cannot escape.

5 Relevant Code

Hénon's Area-Preserving Quadratic Map

```
def HAPQMap(a, x0, y0, N): # Henon's Area-Preserving Quadratic Map
    x, y = np.zeros(N), np.zeros(N)
    x[0], y[0] = x0, y0
    for i in range(1, N, 1):
        x[i] = x[i - 1] * np.cos(a) - (y[i - 1] - x[i - 1] ** 2) * np.sin(a)
        y[i] = x[i - 1] * np.sin(a) + (y[i - 1] - x[i - 1] ** 2) * np.cos(a)
    return x, y

def plotHAPQMap(value, x1, x2, y1, y2, N):
    x0 = np.linspace(-1, 1, 10) # mapping with initial conditions -1 <= x, y <= 1
    y0 = np.linspace(-1, 1, 10)
    coords = np.zeros((len(x0), len(y0)), dtype=tuple) # initializing array [S] of initial conditions
    alpha = np.arccos(value) # angle given in problem
    plt.figure()
    for m in range(0, len(x0)): # [S]_{mn} matrix value
        for n in range(0, len(y0)):
            coords[m][n] = (x0[m], y0[n])
    for i in range(0, len(coords[0])):
        for j in range(0, len(coords[1])):
            p, q = coords[i][j] # unpacking x, y values
            x, y = HAPQMap(alpha, p, q, 5000) # calculating x, y arrays over 5000 iterations
            plt.plot(x, y, '.', color='k', markersize=N)
    plt.xlim((x1, x2))
    plt.ylim((y1, y2))
    plt.xlabel('x')
    plt.ylabel('y')
    plt.legend(title=r'$\cos(\alpha) = $'+str(round(value, 3)))
    plt.show()
```

The Standard Map

```
    def STNDmap(k, x0, y0, N): # Chirikov's Standard Map
        x, y = np.zeros(N), np.zeros(N)
        x[0], y[0] = x0, y0
        for i in range(1, N, 1):
            y[i] = (y[i - 1] + k * np.sin(x[i - 1])) % (2 * np.pi)
            x[i] = (x[i - 1] + y[i - 1] + k * np.sin(x[i - 1])) % (2 * np.pi)
    return x, y

    def plotSTNDMap(k):
        x0 = np.linspace(0, 2 * np.pi, 20) # mapping of a square with side length 2pi
        y0 = np.linspace(0, 2 * np.pi, 20)
        coords = np.zeros((len(x0), len(y0)), dtype=tuple) # initializing array [S]
        for m in range(0, len(x0)): # [S]_{mn} matrix value
            for n in range(0, len(y0)):
                coords[m][n] = (x0[m], y0[n])
        for i in range(0, len(coords[0])): #
            for j in range(0, len(coords[1])):
                p, q = coords[i][j] # unpacking coordinate values from tuple
                x, y = STNDmap(k, p, q, 1000) # calculating x, y arrays over 1000 iterations
                plt.plot(x, y - np.pi, '.', color='k', markersize=.05)
        plt.xlim((0, 2 * np.pi))
        plt.ylim((-np.pi, np.pi))
        plt.xlabel('x')
        plt.ylabel('y')
```

```
        plt.show()
```

The Lozi Map

```
def transform1(a, x, y):
    x = x
    y = 1 + y - a * abs(x)
    return x, y

def transform2(b, x, y):
    x = b * x
    y = y
    return x, y

def transform3(x, y):
    x = y
    y = x
    return x, y

def LoziMap(a, b, x0, y0, N):
    x, y = np.zeros(N), np.zeros(N)
    x[0], y[0] = x0, y0
    for i in range(1, N, 1):
        x[i] = 1 + y[i - 1] - a * abs(x[i - 1])
        y[i] = b * x[i - 1]
    return x, y

def plotLoziMap(a, b):
    x, y = LoziMap(a, b, 0, 0, 10000)
    plt.plot(x[5:], y[5:], '.', color='k', markersize=.5)
    plt.xlabel('x')
    plt.ylabel('y')
    # plt.savefig('loziattractor.png')
    plt.show()
```

Henon's Strange Attractor

```
def henonmap(a, b, x0, y0, N):
    x, y = np.zeros(N), np.zeros(N)
    x[0], y[0] = x0, y0
    for i in range(1, N, 1):
        x[i] = y[i - 1] + 1 - a * x[i - 1] ** 2
        y[i] = b * x[i - 1]
    return x, y

def plotHenonMap():
    a = 1.4
    b = 0.3
    x0 = 0
    y0 = 0
    N = 10000
    fig = plt.figure()
    ax = fig.add_subplot(1, 1, 1)
    x, y = henonmap(a, b, x0, y0, N)
    plt.plot(x[5:], y[5:], '.', color='k', markersize=.3, label='1.4')
    plt.xlabel('x')
    plt.ylabel('y')
    # plt.xlim((.57, .74))
    # plt.ylim((.15, .21))
    # rect = patches.Rectangle((.57, .15), 0.17, 0.06, fill=False)
```

```
# ax.add_patch(rect)
# plt.savefig('henona1_4.png')
plt.show()
```

6 Acknowledgments

Code/Plots: Evan Eastin

Selected Problems: Group effort, all separately did the problems and compared answers

Paper Summary: Noah Lamb, Evan Eastin

Creating/Formatting/Updating LaTeX doc: Group effort

References

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