

Approximate optimization

Recall an example from the 1st lecture (see slides 17-18)

Max cut problem (NP-hard)

Approximate solution: a cut that contains $\geq \frac{1}{2} \cdot \text{size of max cut / edges}$ (using probabilistic arguments)

Another example: integer optimization problem (also NP-hard)

$$\text{maximize } \sum_{i,j} A_{ij} x_i x_j \quad x_i = \pm 1 \quad A \in \text{Sym}(n) \quad (INT) \\ i=1 \dots n$$

↓
Approximate relaxation

$$\max_{\substack{i,j=1 \\ (x_i^T x_i = 1)}} \sum_{i,j} A_{ij} \langle x_i, x_j \rangle \quad x_i \in \mathbb{R}^n \quad i=1 \dots n \quad (SDP-INT)$$

Why is this an SDP program?

$$W \rightarrow W_{ij} = \langle x_i, x_j \rangle \quad \text{How to find } x_i, x_j? \\ \boxed{\begin{matrix} & \\ & \\ & \\ & \end{matrix}} \quad W = VV^T \quad (\text{Cholesky})$$

$$\max_{\substack{W \succeq 0 \\ W_{ii}=1}} \langle A^T W \rangle \quad W := (w_{ij}) \in \mathbb{R}^{n \times n} \quad w_{ij} = \langle x_i, x_j \rangle \quad \text{Gram matrix} \quad \text{Exercise: check that } X \succeq 0.$$

(Thm) INT(A) - maximum in integer optimization, $A \succeq 0$

$$\text{INT}(A) \stackrel{(1)}{\leq} \text{SDP-INT}(A) \stackrel{(2)}{\leq} 2K \cdot \text{INT}(A)$$

(Pf) ① Let x_1, \dots, x_n give max to INT(A)

$$\text{Define } X := (x_i, 0, \dots, 0)$$

$$\text{then } \sum_{i,j} a_{ij} \langle x_i, x_j \rangle = \sum_{i,j} a_{ij} x_i x_j$$

② Follows from symmetric Grothendieck's inequality

$$A \in \text{Sym}(n), |\sum_{i,j} a_{ij} x_i x_j| \leq 1 \text{ for } x_i \in \{-1, 1\}$$

$$\text{Then for any } u_i, v_i \in \mathbb{R}^n, \|u_i\| = \|v_i\| = 1: |\sum_{i,j} a_{ij} \langle u_i, v_j \rangle| \leq 2K$$

TODOS: prove ② and learn how to translate solutions rather than optimal values

We will discuss a more general statement:

Grothendieck's inequality

$A \in \mathbb{R}^{m \times n}$, for any $x_i, y_j \in \{-1, 1\}^n$ and $|\sum_{ij} a_{ij} x_i y_j| \leq 1$

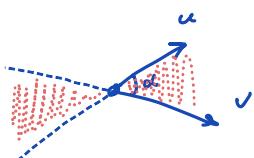
Then $\forall u_i, v_j \in \mathbb{R}^n / \mathbb{R}^m \quad \|u_i\| = \|v_j\| = 1 : |\sum_{ij} a_{ij} \langle u_i, v_j \rangle| \leq k$

Pf idea:

Grothendieck's identity: $g \sim N(0, I_n)$, then for any fixed vectors $u, v \in S^{n-1}$

$$\mathbb{E} \operatorname{sgn} \langle g, u \rangle \cdot \operatorname{sgn} \langle g, v \rangle = \frac{2}{\pi} \arcsin \langle u, v \rangle$$

Why? Rotation invariance of g : let $g \in \operatorname{span}\{u, v\}$



$$\mathbb{P} \{ \operatorname{sgn} \langle g, u \rangle \cdot \operatorname{sgn} \langle g, v \rangle = -1 \} = \frac{\alpha}{\pi}$$

$$\mathbb{E}_{g \sim \dots} = (-1) \cdot \frac{\alpha}{\pi} + (1) \cdot \frac{\pi - \alpha}{\pi} = -\frac{2}{\pi} \left(\frac{\pi}{2} - \alpha \right) = \frac{2}{\pi} \arcsin \langle u, v \rangle \text{ since}$$

$$\begin{cases} \cos \alpha = \langle u, v \rangle & (\|u\| \cdot \|v\| \cos \alpha = \langle u, v \rangle) \\ \arcsin \alpha = \frac{\pi}{2} - \arccos \alpha \end{cases}$$

Now, $\sin \alpha \approx \alpha$ for small α

If we would have $\mathbb{E} \operatorname{sgn} \langle g, u \rangle \operatorname{sgn} \langle g, v \rangle = \frac{2}{\pi} \langle u, v \rangle$, then

$$\frac{2}{\pi} \sum_{ij} a_{ij} \langle u_i, v_j \rangle = \sum_{ij} a_{ij} \underbrace{\mathbb{E} \operatorname{sgn} \langle g, u_i \rangle \operatorname{sgn} \langle g, v_j \rangle}_{\approx 1} \leq 1 \Rightarrow k \approx 1.67 \left(\frac{\pi}{2}\right)$$

(is wrong!)

Kernel trick: represent $\frac{2}{\pi} \arcsin \langle u, v \rangle = \langle u', v' \rangle$ in a higher-dimensional space

Namely, we define $u' = \Phi(u)$, $v' = \Phi(v)$ tensor spaces

How to rectify non-linearities in scalar products?

Rank-one tensor: $u \in \mathbb{R}^n$

$$u \otimes u \otimes \dots \otimes u = \underbrace{u \otimes u \otimes \dots \otimes u}_{k \text{ times}} \in \mathbb{R}^{n \times \dots \times n}$$

$$(u) \quad \underbrace{u}_{i_1, i_2, \dots, i_k} = u_{i_1} \cdot u_{i_2} \cdots u_{i_k}$$

$$\langle u, v \rangle := \sum_i u_i v_i$$

$$\text{Claim: } \langle u^{\otimes k}, v^{\otimes k} \rangle = \langle u, v \rangle^k \text{ (check!)}$$

in $\mathbb{R}^{n \times \dots \times n}$

linear nonlinear

Idea:

$$\text{Idea 2: } \langle u, v \rangle + \langle u \otimes u, v \otimes v \rangle = \langle (u, u \otimes u), (v, v \otimes v) \rangle \quad \textcircled{D}$$

$$\text{Indeed, } \langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = \sum x_i y_i \rightarrow$$

Overall, we have

Lemma $f(x) = \sum_{k=0}^{\infty} a_k x^k$ real analytic function $\mathbb{R} \rightarrow \mathbb{R}$

Then $\exists \Phi: \mathbb{R}^n \rightarrow K$: $\langle \Phi(u), \Psi(v) \rangle = f(u, v)$
some higher-dimensional space.

$$\text{Example } \langle \Phi(u), \Phi(v) \rangle = \underbrace{2 \langle u, v \rangle^2 + 5 \langle u, v \rangle^3}_{\textcircled{D}}$$

$\Phi: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n} \oplus \mathbb{R}^{n \times n}$
cartesian product

$$\textcircled{D} = 2 \langle u^{\otimes 2}, v^{\otimes 2} \rangle + 5 \langle u^{\otimes 3}, v^{\otimes 3} \rangle = \langle (2u^{\otimes 2}, 5u), (2v^{\otimes 2}, 5v) \rangle$$

Note: if we need to subtract a term, we have $\Psi \neq \Phi$.

$$\text{Now, this can be applied to } f(x) = \sin\left(\frac{\beta\pi}{2} \langle u, v \rangle\right)$$

Proof $u_i' := \Phi(u_i)$
 $v_j' := \Psi(v_j)$: $\frac{2}{\pi} \arcsin \langle u_i', v_j' \rangle = \beta \langle u_i, v_j \rangle$

$$\beta \sum A_{ij} \langle u_i, v_j \rangle = \sum A_{ij} \cdot \frac{2}{\pi} \arcsin \langle u_i', v_j' \rangle = \sum A_{ij} \cdot \# \operatorname{sgn} \langle g, u_i' \rangle \cdot \operatorname{sgn} \langle g, v_j' \rangle \leq 1$$

if u_i', v_j' have unit norms

defines selection of β :

$$\|u_i'\|^2 = \|\Phi(u_i)\|^2 = \sum_{k=0}^{\infty} |a_k| \|u_i\|_2^{2k} \quad (\text{from Lemma})$$

$$\Phi = \sin\left(\frac{\beta\pi}{2} x\right) \quad \sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots$$

$$\|\Phi(u)\|^2 = \sinh\left(\frac{\beta\pi}{2}\right) = 1 \quad \sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots$$

$\Downarrow \quad \|x\|=1$

$$\frac{\beta\pi}{2} = \sinh^{-1}(t) = \ln(t + \sqrt{t^2 + 1})$$

$$\beta \approx \frac{1}{1.783} \quad \blacksquare$$

Notes: ① Grothendieck's inequality can be proved without kernel tricks with worse constant K (e288)

$$\text{Recall: } \sum A_{ij} \langle u_i, v_j \rangle \leq K$$

General plan: $u_i = \langle g, u_i \rangle$

$$v_j = \langle g, v_j \rangle \quad g \sim N(0, I_N)$$

Then $u_i, v_j \sim N(0, 1)$ rotation invariance

$$\mathbb{E} u_i \cdot v_j = \langle u_i, v_j \rangle \quad \text{define correlations}$$

$$K \geq \sum A_{ij} \langle u_i, v_j \rangle = \mathbb{E} (\sum A_{ij} u_i v_j)$$

$$\text{Truncation } u_i = \frac{u_i \cdot \mathbf{1}_{\{|u_i| \leq R\}}}{u_i^-} + \frac{u_i \cdot \mathbf{1}_{\{|u_i| > R\}}}{u_i^+}$$

Tails are small and

$$\mathbb{E} |\sum_{ij} A_{ij} u_i^- v_i^+| \leq R^2 \quad (\text{so } K \text{ can be taken as } R^2)$$

② Kernel trick is useful beyond Grothendieck's inequality:

$K: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ let it be some function. When is there a

$$\text{transformation } \Phi: \mathcal{X} \rightarrow \mathbb{R}^M: \quad \langle \Phi(u), \Phi(v) \rangle = K(u, v)$$

(Generalization of a statement about $K(u, v) = f(\langle u, v \rangle)$)

Mercer's thm: It happens if and only if K is a positive-semidefinite kernel: for any finite collection of points $u_1, \dots, u_n \in \mathcal{X}$, the matrix

$$\begin{pmatrix} K(u_1, u_1) & K(u_1, u_2) & \dots & K(u_1, u_n) \\ K(u_2, u_1) & \ddots & & \\ \vdots & & \ddots & \\ K(u_n, u_1) & \dots & \dots & K(u_n, u_n) \end{pmatrix} \succeq 0$$

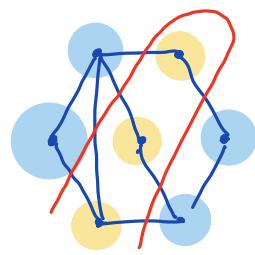
Φ -feature map

$$K(u, v) = \exp \left(-\frac{\|u - v\|_2^2}{2\sigma^2} \right), \quad K(u, v) = (\langle u, v \rangle + \gamma)^k \dots$$

Φ gives separability conditions, and in many applications not need to be computed

Application to MAX-CUT

and solution approximation



0.878-approximation algorithm (Goemans, Williamson)

$$G = (V, E)$$

Adjacency matrix A (binary)

$x \in \{\pm 1\}^n$ - partition onto two sets of edges.

$$\text{CUT}(G, x) = \frac{1}{2} \sum_{x_i \cdot x_j = -1} A_{ij} = \frac{1}{4} \sum_{i,j=1}^n A_{ij} (1 - x_i x_j)$$

$$\text{MAX-CUT}(G) = \frac{1}{4} \max \left\{ \sum_{i,j} A_{ij} (1 - x_i x_j) : x_i = \pm 1 \quad \forall i \right\}$$

$$\text{SDP}(G) := \frac{1}{4} \max \left\{ \sum_{i,j} A_{ij} (1 - \langle x_i, x_j \rangle) : \|x\|_2 = 1 \quad \forall i \right\}$$

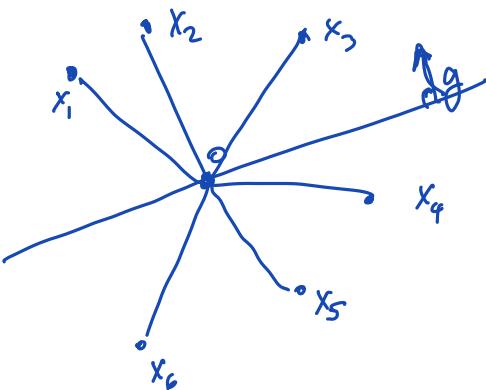
How to translate the solution (x_i) into labels $x_i = \pm 1$?

Randomized rounding step: choose a random hyperplane in \mathbb{R}^n

Assign $x_i = 1$ depending on a subspace defined by a hyperplane.

Equivalently, we choose $g \sim N(0, I_n)$ and define $x_i = \text{sgn } \langle x_i, g \rangle$

(random hyperplanes are defined by their random normals)

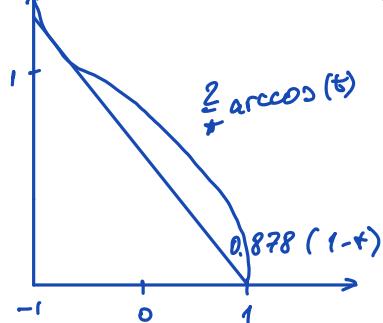


Theorem:

$$\mathbb{E} \text{CUT}(G, x) \stackrel{(1)}{\geq} 0.878 \text{SDP}(G) \stackrel{(2)}{\geq} 0.878 \text{MAX-CUT}(G)$$

where x is the result of randomized rounding of (x_i)

Recall Grothendieck's identity $\mathbb{E} \operatorname{sgn} \langle g, u \rangle \operatorname{sgn} \langle g, v \rangle = \frac{2}{\pi} \arccos \langle u, v \rangle$



$$\mathbb{E} \operatorname{sgn} \langle g, u \rangle \operatorname{sgn} \langle g, v \rangle = \frac{2}{\pi} \arccos \langle u, v \rangle$$

$$1 - \frac{2}{\pi} \arccos t = \frac{2}{\pi} \arccos t \geq 0.878(1-t)$$

$$t \in [-1, 1]$$

Proof $\mathbb{E} \text{CUT}(G, x) = \frac{1}{4} \sum_{i,j} A_{ij} (1 - \mathbb{E} x_i x_j)$

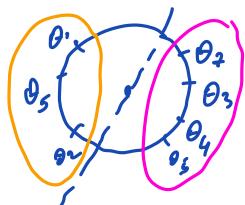
$$1 - \mathbb{E} x_i x_j = 1 - \mathbb{E} \operatorname{sgn} \langle X_i, g \rangle \operatorname{sgn} \langle X_j, g \rangle = 1 - \frac{2}{\pi} \arccos \langle X_i, X_j \rangle \geq 0.878 (1 - \langle X_i, X_j \rangle)$$

$$\text{So, } \mathbb{E} \text{CUT}(G, x) \geq 0.878 \cdot \frac{1}{4} \sum_{i,j} A_{ij} (1 - \langle X_i, X_j \rangle) = 0.878 \text{ SDP}(G)$$

This proves ①. ② follows from the theorem on page ④.
 (which also holds when A has zero diagonal)
 see KW4 problem 5

It is believed that it is NP-hard to get better approximation (NP-hard, reduces to Unique Games Conjecture)

Another way to visualize randomized rounding



For the directions $\theta_1, \dots, \theta_7$

How to avoid solving an SDP here?

in \mathbb{R}^2 :

$$\langle v_{\theta_i}, v_{\theta_j} \rangle = \cos(\theta_i - \theta_j) \quad \text{angle}$$

$$f(\theta_1, \dots, \theta_n) = \sum_{i,j=1}^n A_{ij} \cos(\theta_i - \theta_j) \quad \text{- energy function}$$

$$f: [0, 2\pi]^n \rightarrow \mathbb{R}, \quad A - \text{adjacency}$$

f is minimized with gradient descent, ...

Then randomized rounding is used to separate θ_i 's

"Rank 2 relaxation heuristics..." (Burer, Monteiro, Zhang) 2001

Steinerberger '21 - another ways to find small values of energy functions and related graph partitions

Other applications of Grothendieck's inequality for polynomial time approximation e.g. [Khot, Naor '11]

- Szemerédi partitions of graphs
- Frieze-Kannan matrix decomposition
- Acyclic graphs

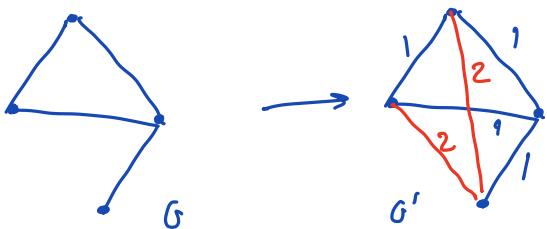
Other problems and proving inapproximability

① K-center problem: given a complete graph so that edges have "lengths" satisfying Δ inequality, find a K-element set minimizing max distance to it $\min_S (\max_{v \in V} d(v, S))$

• For any $\epsilon > 0$ it is not $(2-\epsilon)$ approximable unless $NP = P$

Proof idea: reduction to Dominating Set (is there a set with K vertices so that each vertex is either in the set or is adjacent to 1 in the set?)

Exercise: This reduces to STABLE SET



Any $2-\epsilon$ approximation algorithm on G' can decide whether there are K centers with distance 1 or only 2.

This means deciding whether the same K -element set will work as K -dominating set in G .

② MAX-3-SAT - find an assignment that maximizes the number of satisfied clauses

$$(\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \dots$$

NP-hard - reduces to satisfiability

T/F at random, for each clause $P(\text{satisfied}) = \frac{7}{8}$

$$\mathbb{E} [\# \text{ of satisfied clauses}] = \frac{7k}{8}$$

\exists assignment $\geq \frac{7k}{8}$ true clauses

$P(\text{random assignment satisfies } \geq \frac{7k}{8} \text{ clauses}) =$

$$K P(\geq \frac{7k}{8} \text{ are satisfied}) \geq \sum_{j \geq \frac{7k}{8}} P(j \text{ clauses are satisfied}) = \frac{7}{8}^k - \sum_{j < \frac{7k}{8}} j P_j \geq \frac{7}{8}^k - (\frac{7}{8}^k \cdot \frac{1}{8}) \cdot 1 \geq \frac{1}{8}$$

(Thm) [Karpinski, 87] No better approximation is possible unless P=NP

③ MAX CTR \rightarrow both rely on

PCP - probabilistically checkable proofs (accepts correct proofs and rejects incorrect proofs with high probability)

(Thm) $NP = PCP(O(\log n), O(1))$

↑
probabilistic verifiers

in addition take a sequence of $O(\log n)$ random bits

can be simulated by deterministic algorithm at all $2^{O(\log n)} \approx n$ possible random inputs R