

# Semidefinite programming.

$$\left[ \begin{array}{l} \min \operatorname{Tr}(Cx) \\ x \in \text{Sym}^{n \times n} \\ \text{such that } \operatorname{Tr}(A_i x) = b_i \\ x \succeq 0 \end{array} \right]$$

## 1. Reductions

$A_i$  and  $C$  can be assumed symmetric, since

$$\operatorname{Tr}(Ax) = \operatorname{Tr}(Ax^T) = \operatorname{Tr}(XA^T) = \operatorname{Tr}(A^T X)$$

$$\text{so } \operatorname{Tr}\left(\frac{A+A^T}{2}x\right) = \operatorname{Tr}(Ax)$$

## 2. Is it a convex optimization problem?

- a) Feasible set is convex:
  - PSD matrices form a convex set
  - Intersection with the other convex set (affine constraints)  $\cap$  spectrahedron
- b)  $\circ$  (can have infinitely many vertices)
  - Does not satisfy the definition of convex opt. problem as is.  
(see discussion below)
  - $X \succeq 0 \Leftrightarrow y^T X y \geq 0$  is a convex constraint for every  $y \in \mathbb{R}^n$
  - infinitely many convex conditions
  - finitely many polynomial constraints (equivalent to the original problem) — from Sylvester criterion.

## 3. Existence of optimal value $\not\Rightarrow$ existence of optimal solution

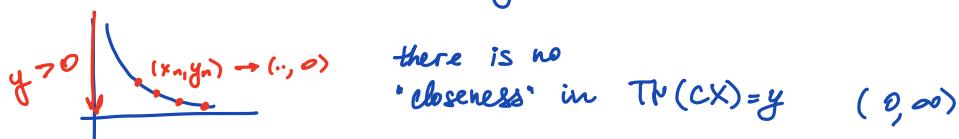
Example:  $\begin{cases} \min y \\ \frac{1}{x} \leq y \\ x, y \geq 0 \end{cases}$

This is an SDP problem

- $\left\{ \begin{array}{l} xy \geq 1 \\ x, y \geq 0 \end{array} \right\} \Leftrightarrow \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \succeq 0$  (Sylvester)
- $\operatorname{Tr}((0 \ 0) \cdot (\begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix})) = \operatorname{Tr}(0 \ 0) = y$
- $\operatorname{Tr}((0 \ 1) \begin{pmatrix} x & z \\ z & y \end{pmatrix}) = 2z = 2$

$y_n \rightarrow 0$  as  $x_n \rightarrow 0$ ; 0-optimal value that is not attained at a feasible point

Convexity of the constraint set is not enough



Compare with LP:

$$\begin{cases} \min c^T x \\ Ax \leq b \\ x \geq 0 \end{cases} \quad \text{if } c^T x_n \rightarrow p_*$$

$\exists x : p_* = c^T x$  The set  $\{c^T x \mid x \text{ is feasible}\}$  is closed

#### 4. Duality

Let  $p_*$  be an optimal value

$$L(X, \lambda, \mu) = \text{Tr}(Cx) + \sum \lambda_i (b_i - \text{Tr}(Ax)) - \text{Tr}(x\mu) \quad \text{- Lagrangian}$$

$$D(\lambda, \mu) = \min_X L(X, \lambda, \mu) \quad \text{- dual function}$$

Lemma  $\forall \lambda, \mu \geq 0 \quad D(\lambda, \mu) \leq p_*$

Note: we prove it for the case when optimal solution  $x^*$  exists

(pf)

$$L(x^*, \lambda, \mu) = \text{Tr}(Cx^*) - \text{Tr}(x^*\mu) = p^* - \text{Tr}(x^*\mu) \leq p$$

[since  $\lambda A \geq 0 \Rightarrow \text{Tr}(AB) \geq 0$  (HW)]

So,  $\min L \leq p^*$  as well.

Dual problem should provide the best lower bound for  $p_*$ .

Idea: maximize  $D(\lambda, \mu)$

$$\begin{cases} \max D(\lambda, \mu) \\ \text{s.t. } \mu \geq 0 \end{cases}$$

$$D(\lambda, \mu) = \min_L L(X, \lambda, \mu) = \begin{cases} \lambda^T b & \text{if } C - \sum \lambda_i A_i - \mu I = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Solving for  $C - \sum \lambda_i A_i - \mu = 0$  for some  $\mu \geq 0$  we have

$$C \leq \sum \lambda_i A_i$$

Dual problem:

$$\begin{cases} \max_{\lambda \in \mathbb{R}^m} b^T \lambda \\ \text{s.t. } \sum_i \lambda_i A_i \leq C \end{cases}$$

decision variables  
 $\downarrow$   
 $A_0 + \lambda_1 A_1 + \dots + \lambda_n A_n \leq 0$

Linear Matrix Inequality (LMI)

Duality theorems:

Weak duality: Let  $X$  is feasible to SDP-primal,  $\lambda$ -feasible for SDP-dual. Then  $\text{Tr}(CX) \geq b^T \lambda$

Strong duality  $\times$

Examples: Only one of 2 optimal solutions are achieved

Dual to problem ①  $b = g$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} \max 2\lambda \\ \lambda A \leq C \end{cases} \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda = 0$$

optimal solution

② SDP duality gap:

$$\text{Let } X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{pmatrix}$$

$$\begin{cases} \min -x_{33} \\ \text{s.t. } X \succeq 0 \\ 2x_{12} + x_{33} = 1 \\ x_{22} \geq 0 \end{cases}$$

$$\min x \quad (\text{Lancaster example})$$

$$\begin{pmatrix} 0 & x & 0 \\ x & y & 0 \\ 0 & 0 & x+1 \end{pmatrix} \succeq 0$$

$$\rightarrow \max -x \quad (x=0)$$

$$\min -x_{33} = 1$$

$$x \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + y \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem: If primal and dual SDP are strictly feasible (exists positive definite feasible  $X$ )

then optimal values are achieved and  $\text{Tr}(CX^*) = b^T \lambda^*$ .

Remark: dual form can be convenient to formulate problems:

examples

1)  $\min y: \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y \geq 0$

2)  $\begin{pmatrix} 1+x & y \\ y & 1-x \end{pmatrix} \geq 0 \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + y \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \geq 0 \iff \begin{array}{l} x \in [-1, 1], y \geq 0 \\ x^2 + y^2 \leq 1 \end{array}$



## 5. LP and SOCP as special cases of SDP

$$\text{LP} \left[ \begin{array}{l} \min c^T x \\ a_i^T x = b_i \\ x \geq 0 \end{array} \right] \iff \left[ \begin{array}{l} \min \text{Tr}(\text{diag}(c)x) \\ \text{Tr}(\text{diag}(a)x) = b_i \\ x \geq 0 \end{array} \right]$$



non-diagonal entries  
do not participate in  
constraint and objective function

SOCPr: we will show inclusion  $\text{SOCPr} \subset \text{SDP}$

Lemma:  $X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$  is a block matrix with  $A \succeq 0$

Then  $X \succeq 0 \iff S := C - B^T A^{-1} B \succeq 0$   
(Schur complement)

$$\text{SOCPr} \left\{ \begin{array}{l} \min f^T x \\ \|A_i^T x + b_i\|_2 \leq c_i^T x + d_i \end{array} \right.$$

Without loss of generality  $c_i^T x + d_i > 0$

Then,  $(c_i^T x + d_i)^2 \geq \|A_i^T x + b_i\|_2^2$

$\iff (c_i^T x + d_i) - (A_i^T x + b_i)^T \frac{1}{c_i^T x + d_i} (A_i^T x + b_i) \geq 0$

Lemma

$$\begin{pmatrix} (c_i^T x + d_i) I & (A_i^T x + b_i) \\ (A_i^T x + b_i)^T & (c_i^T x + d_i) \end{pmatrix} \succeq 0$$

(Proof of Lemma)

$$f(u, v) = u^T A u + 2v^T B^T u + v^T C v = \begin{pmatrix} u \\ v \end{pmatrix}^T X \begin{pmatrix} u \\ v \end{pmatrix}$$

$f$  is strictly convex in  $u$ .

$$\frac{\partial f}{\partial u} = 2Au + 2Bv = 0 \quad u = -A^{-1}Bv \rightarrow \text{gives global minimum in } u.$$

$$f_v^* = \min_u f = v^T (C - B^T A^{-1} B)v = v^T S v$$

$\Rightarrow$  If  $S \succ 0$   $v^T S v < 0$  implies  $\min_u f < 0$  and  $z^T z < 0$  for  
 $z = \begin{pmatrix} -A^{-1}Bv \\ v \end{pmatrix}$

$$\Leftarrow \begin{pmatrix} u \\ v \end{pmatrix}^T X \begin{pmatrix} u \\ v \end{pmatrix} = f(u, v) \geq 0 \quad \text{if } S \text{ is PSD.}$$

Various considerations:

6. Spectahedron is a convex and closed set  $S$

Convex optimization problems

$$\textcircled{B} \left[ \begin{array}{l} \min f(x) \\ g_i(x) \leq 0 \text{ convex} \\ h_j(x) = 0 \text{ affine} \end{array} \right]$$

Consider  $d_S(x) = \inf_{s \in S} \|x - s\|$

This function is convex!

Indeed,

$$\begin{aligned} d_S((1-\lambda)x + \lambda y) &\leq \sqrt{s_1^2 + s_2^2} \quad \forall s_1, s_2 \in S \\ &\leq (1-\lambda)\|x - s_1\| + \lambda\|y - s_2\| \\ &= (1-\lambda)d_S(x) + \lambda d_S(y) \end{aligned}$$

for proper  $s_1, s_2$

Now,  $x \in S \Leftrightarrow d_S(x) = 0$

$\textcircled{S}$  take  $s \in x$

$\textcircled{E}$  If  $d_S(x) = 0$   $\exists$  sequence  $s_i \in S$ :  
 $\|x - s_i\| \rightarrow 0$   
 $\Rightarrow s_i \rightarrow x \Rightarrow x \in S$  (closedness)

So, one can take  $g_i(x) = d_g(x)$  in  $\oplus$ . But this does not make the description more constructive when  $x \geq 0$ .

Recall:  $x^3 \leq 0$  can be written as  $x \leq 0$   
 ↗ no!  
 ↗ convex constraint

## 7. Cone programs duality

$$\left[ \begin{array}{l} \min c^T x \\ Ax = b \\ x \in K \end{array} \right] \quad \begin{array}{l} K - \text{convex cone} \\ K^* = \{y \mid x^T y \geq 0 \} - \text{dual cone} \end{array}$$

- Examples
  - $K = \mathbb{R}^n$        $K^* = \{0\}$
  - $K = \mathbb{R}_{\geq 0}^n$        $K^* = \mathbb{R}_{\geq 0}^n$
  - $K = PSD_n$        $K^* = PSD_n$
  - $K$ -Lorentz cone       $K^* = K$

Dual program:  $\left[ \begin{array}{l} \max b^T y \\ \sum y_i a_i + s = c \\ s \in K^* \end{array} \right]$

Weak duality:

$$c^T x = (\sum y_i a_i + s)^T x = \sum y_i a_i^T x + s^T x \geq b^T y$$

$s^T x \geq 0$  since  $s \in K^*$ ,  $x \in K$

For  $\mathbb{R}_{\geq 0}^n$  cone:  $P: \left[ \begin{array}{l} \min c^T x \\ Ax = b \\ x \geq 0 \end{array} \right]$        $D: \left[ \begin{array}{l} \max b^T y \\ \sum y_i a_i + s = c \\ s \geq 0 \end{array} \right] \leftrightarrow \left[ \begin{array}{l} \max b^T y \\ \sum y_i a_i \leq c \end{array} \right]$

For SDP cone:  $P: \left[ \begin{array}{l} \min c^T X \\ AX = B \\ X \geq 0 \end{array} \right]$        $D: \left[ \begin{array}{l} \max b^T Y \\ \sum A_i Y_i + S = C \\ S \geq 0 \end{array} \right]$

$A \cdot X := \text{Tr}(AX)$

Exercise: Prove that Lorentz cone is self-dual. Derive SOCP dual form