

Probabilistic bounds and applications to signal detection

↓
Markov, Chebyshev, Chernoff

(See more details in B&V 7.9)

↓
$$P(X \geq a) \leq \frac{E(X)}{a} \text{ for } X \in \mathbb{R}_+$$

Simple proof for distributions with densities:



$$E(X) = \int_0^a x dp + \int_a^\infty x dp \geq 0 + \int_a^\infty a dp = a P(X \geq a)$$

Chebyshev: $P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$
 $E(X) = \mu, \quad E(X - \mu) = 0$

Chernoff: gives more precise tail estimates for certain distributions (Bernoulli)

Goal: define more generic framework for convex optimization problems yielding such bounds

Key problem:

Let X be a random variable on $S \subseteq \mathbb{R}^n$

$$\begin{aligned} &\max \text{Prob}(X \in C) - ? \\ &\text{subject to } E f_i(X) = a_i \end{aligned}$$

 $\leftarrow \begin{matrix} f_i: \mathbb{R}^n \rightarrow \mathbb{R}, i=1 \dots N \\ \text{e.g., moments of } X \end{matrix}$

$$P(X \in C) = E(\uparrow \mathbb{1}_C(X))$$

indicator function

$$\mathbb{1}_C(z) = \begin{cases} 1 & \text{if } z \in C \\ 0 & \text{otherwise} \end{cases}$$

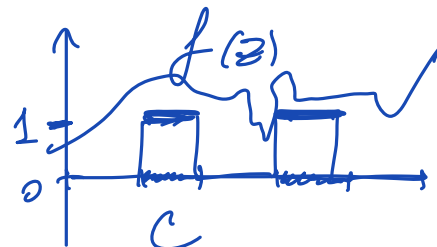
① Finite case: $X = x_i$ with probability $p_i, i=1 \dots N$

$$E f(X) = \sum_{i=1}^N p_i f(x_i)$$

$$\left[\begin{array}{l} \max \sum p_i \\ i: x_i \in C \\ \sum_{i=1}^N p_i f_j(x_i) = a_j \\ \sum p_i = 1 \end{array} \right] \quad \textcircled{LP}$$

② General case: consider $f(z) = \sum_{i=0}^N x_i f_i(z)$, $\mathbb{E} f_0(z) = 1$

Key observation: if $f(z) \geq 1_C(z)$, $\mathbb{E} f(z) \geq \mathbb{P}(X \in C)$



$(x_0 \dots x_{N+1})$ are decision variables

$$\begin{aligned} \min \quad & x_0 + a_1 x_1 + \dots + a_N x_N \\ \text{s.t.} \quad & f(z) = \sum_{i=0}^N x_i f_i(z) \geq 1 \quad z \in C \\ & f(z) = \sum_{i=0}^N x_i f_i(z) \geq 0 \quad z \in S \setminus C \end{aligned}$$

or,

$$\begin{aligned} \min \quad & \mathbb{E} f(X) \\ \text{s.t.} \quad & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \end{aligned}$$

where

$$\mathbb{E} f(z) = \langle a, x \rangle, \text{ where } a = \begin{pmatrix} 1 \\ a_1 \\ \vdots \\ a_N \end{pmatrix}$$

It is a convex problem:

$$\begin{cases} g_1(x) := 1 - \inf_{z \in C} f(z) \leq 0 \\ g_2(x) := - \inf_{z \in S \setminus C} f(z) \leq 0 \end{cases}$$

convex, as \inf is a concave function

By key observation, solution to the problem \mathbb{P} upper bounds $\mathbb{P}(X \in C)$

When is it easy to solve?

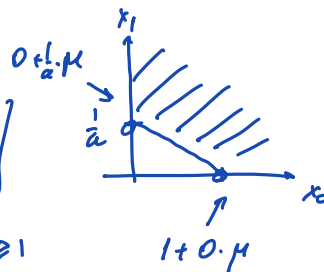
① Analytically: some simple settings

Assume $S = \mathbb{R}_+$, $C = [a, \infty)$
 $\mathbb{E} X = \mu \leq a$
 $(f_i(x) = x)$

$$\min x_0 + \mu x_1$$

$$\text{s.t.} \begin{cases} x_0 + z x_1 \geq 1 & z \geq a \\ x_0 + z x_1 \geq 0 & \forall z \end{cases}$$

$$x_0, x_1 \geq 0, \quad x_0 + a x_1 \geq 1$$



$$\min \begin{cases} \frac{\mu}{a} & \text{if } \mu \leq a \\ 1 & \text{if } \mu \geq a \end{cases}$$

$$\text{So } \mathbb{P}(X \geq a) \leq \frac{\mu}{a}$$

Case 2: with second moments

⑥ Assume $S = \mathbb{R}^n$

$$\begin{aligned} \left[\begin{array}{l} \mathbb{E} X = \mu \in \mathbb{R}^n \\ \mathbb{E} X X^T = \Sigma \in \mathbb{S}^n \end{array} \right] & \leftarrow n \text{ functions with exp's } \mu_1 \dots \mu_n \\ & \leftarrow n^2 \text{ functions } f_{jk} = x_j x_k \\ & \Sigma_{ij} = \mathbb{E} x_i x_j \end{aligned}$$

$$\text{So, } f(z) = x_0 + \sum_{i=1}^n x_i z_i + \sum_{i,j=1}^n x_{ij} z_i z_j$$

$$\text{or } f(z) = z^T P z + 2q^T z + r \quad (P \in \text{Sym}(n), q \in \mathbb{R}^n, r \in \mathbb{R} \text{ are decision variables})$$

$$\mathbb{E} f(x) = \mathbb{E} (x^T P x + 2q^T x + r) = \mathbb{E} (x^T P x x^T) + 2\mathbb{E} q^T x + r = \text{tr}(\Sigma P) + 2q^T \mu + r$$

objective function: $\min_{P, q, r} \mathbb{E} f(x)$

Now, constraints:

$$\bullet f(z) \geq 0 \quad \forall z \Leftrightarrow \begin{bmatrix} P & q \\ q^T & r \end{bmatrix} \succeq 0$$

$$\bullet f(z) \geq 1 \quad \forall z \in C.$$

Further, let us assume $C \subset \mathbb{R}^n \setminus P$, where $P := \{z \mid a_i^T z < b_i, i=1 \dots k\}$

$$z \in C \text{ means } \exists i: a_i^T z \geq b_i \quad (or, b_i - a_i^T x \leq 0).$$

So, for any $i=1 \dots k$, there is no x : $b_i - a_i^T x \leq 0$, but $f(z) < 1$ (or $f(z) - 1 = z^T P z + 2q^T z + r - 1 < 0$)

By Thm 4 below, $\exists \tau_i \geq 0$ such that $\begin{bmatrix} P & q \\ q^T & r-1 \end{bmatrix} \succeq \sum \tau_i \begin{bmatrix} 0 & a_i/2 \\ a_i^T/2 & -b_i \end{bmatrix}$.

Thm 4: Theorem of alternatives of a pair of quadratic inequalities:

Thm Suppose $\exists x: x^T A_1 x + 2b_1^T x + c_1 < 0$. Then

$$\exists x: x^T A_1 x + 2b_1^T x + c_1 < 0, \quad x^T A_2 x + 2b_2^T x + c_2 \leq 0$$

$$\Leftrightarrow \exists \lambda \geq 0 \quad \begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} + \lambda \begin{bmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{bmatrix} \succeq 0.$$

Proof: B&V
§ B2, B4

Remark: $0 \leq \begin{bmatrix} x \\ 1 \end{bmatrix}^T \left(\begin{bmatrix} A_1 & b_1 \\ b_1^T & c_1 \end{bmatrix} + \lambda \begin{bmatrix} A_2 & b_2 \\ b_2^T & c_2 \end{bmatrix} \right) \begin{bmatrix} x \\ 1 \end{bmatrix} = x^T A_1 x + 2b_1^T x + c_1 + \lambda (x^T A_2 x + 2b_2^T x + c_2) < 0$

(Weak alternative is obvious, together (1) and (2) lead to a contradiction)

As a conclusion, problem * is of the form:

$$\left[\begin{array}{l} \min \text{tr}(\Sigma P) + 2q^T \mu + r \\ \begin{bmatrix} P & q \\ q^T & r \end{bmatrix} \succeq 0 \\ \begin{bmatrix} P & q \\ q^T & r-1 \end{bmatrix} \succeq \sum \tau_i \begin{bmatrix} 0 & a_i/2 \\ a_i^T/2 & -b_i \end{bmatrix} \\ \tau_i \geq 0 \end{array} \right] = \alpha, \text{ then } 1-\alpha \text{ is a lower bound for the probability of a location inside the polytope}$$

It is an SDP

Example $S \in \{s_1, \dots, s_m\} \subseteq \mathbb{R}^n$ signal constellation

One of s_i 's is transmitted via a noisy channel

received $x = s + v$
 \uparrow random noise

$$\mathbb{E}v = 0 \quad \mathbb{E}vv^T = \sigma^2 I$$

Minimum distance estimator: s_k closest to x

Prob (correct detection)?

It is given by a polytope:

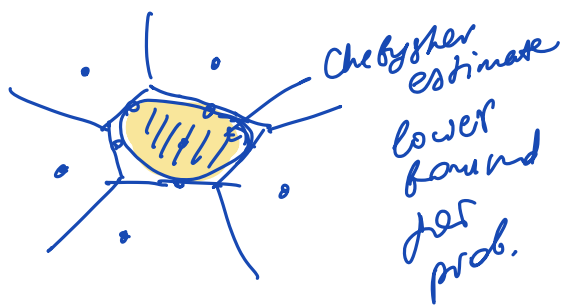
$$\|x - s_k\|_2 \leq \|x - s_j\|_2 \quad j \neq k$$

$$\|v\|_2^2 \leq \|v + s_k - s_j\|_2^2$$

\vdots

$$2 \langle s_j - s_k, v + s_k \rangle \leq \|s_j\|^2 - \|s_k\|^2 \quad \text{for each } j \neq k$$

Voronoi region V_k



(probability of correct detection of each of the signals depending on σ)

Fig 7-6 Boid & Van

$x^T P x + 2q^T x + r \leq 1$
 with optimal P, q, r
 define yellow ellipsoid