Applications of SDP: combinatorial optimization.

Independent set problem

$G = (V,E)$, $|V| = n$

Independent set – no inner edges. Stability number – largest stable set.

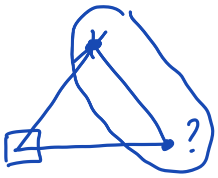
It is NP-hard to test if $\alpha(G)$ is NP-hard; SDP relaxation–?

$$\alpha(G) = \left[\begin{array}{ll} \max\limits_{x} \sum x_i & \\ x_i + x_j \le 1 & \leftarrow \{i,j\} \in E \\ x_i \in \{0,1\} & \end{array}\right] \xrightarrow{\text{LP relaxation}} \alpha_{LP}(G) = \left[\begin{array}{l} \max\limits_{x} \sum x_i \\ x_i + x_j \le 1 \\ 0 \le x_i \le 1 \end{array}\right]$$

$$\alpha \le \alpha_{LP}$$

Better relaxation?

We can add more valid inequalities to LP.



(C_2): For a 2-clique, $x_i + x_j \le 1$ (2-clique - edge between 2 vertices)

(C_3) For a 3-clique ($\triangle$), $x_i + x_j + x_k \le 1$

(C_4) For a 4-clique ($\boxtimes$), $x_i + x_j + x_k + x_l \le 1$ – Clique inequalities

$\vdots$

Number of cliques is exponential in the size of the graph...

$$\alpha_{LP}^{(k)} := \left[\begin{array}{l} \max \sum x_i \\ 0 \le x_i \le 1 \\ C_2, \dots C_k \end{array}\right]$$

SDP relaxation (Lovász, 1979)

$$\alpha_{SDP}(G) = \left[\begin{array}{ll} \max\limits_{X \in Sym(n)} \sum\limits_{i,j} X_{ij} & \\ Tr(X) = 1 & \\ X_{ij} = 0 & \{i,j\} \in E \\ X \succeq 0 & \end{array}\right] \quad (*)$$

$\leftarrow$ $Tr(JX)$, where J is a matrix of all 1s

$\boxed{\text{Theorem}}$ $\forall G$: $\alpha(G) \overset{①}{\leq} \alpha_{SDP}(G) \overset{②}{\leq} \alpha_{LP}^{(k)}(G)$ $\quad \forall k \geq 2$

Proof of ① Let S be a maximum stable set of G, $|S| = k$

Let $X = x \cdot x^T$, $x \in \mathbb{R}^n$ $\quad x_i = \begin{cases} \frac{1}{\sqrt{k}} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$

$X_{ij} = x_i x_j = 0$ if $\{i,j\} \in E$ (at least one of x_i and $x_j = 0$)

$y^T X y = \sum_{i,j} X_{ij} y_i y_j = \sum_{i,j} x_i x_j y_i y_j = \sum_{i,j} (x_i y_i) \cdot (x_j y_j) = (\sum x_i y_i)^2 \geq 0$

$Tr(X) = \sum X_{ii} = \sum_i x_i^2 = \sum_{i \in S} \frac{1}{k} = 1$

X is feasible for an SDP.

$\sum X_{ij} = \sum_{i,j} x_i x_j = (\sum x_i)^2 = \left(\frac{|S|}{\sqrt{k}}\right)^2 = \frac{|S|^2}{k} = |S|$

So, the value of the objective $\alpha_{SDP} \geq \alpha$.

Let's further study $\alpha_{SDP}(G)$, what is its dual?

$$(P) \left[\begin{array}{l} \min \ Tr(CX) \\ Tr(A_i X) = b_i \\ X \succeq 0 \end{array} \right] \qquad (D) \left[\begin{array}{l} \max_y \ b^T y \\ \sum_{i=1}^m y_i A_i \preceq C \end{array} \right]$$

$$(P\star) \left[\begin{array}{l} -\min \ Tr(-JX) \\ Tr X = 1 \\ X_{ij} = 0 \quad (i,j) \in E \\ X \succeq 0 \end{array} \right] \longleftrightarrow \left[\begin{array}{l} -\max_{t, y_{ij}} t \\ tI + \sum y_{ij} E_{ij} \preceq -J \end{array} \right]$$

$$\begin{array}{cc} & (i,j) \\ E_{ij} = & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ (j,i) & \end{array}$$

$\Updownarrow$

$$\left[\begin{array}{l} \min_{t, y} -t \\ -tI - \sum y_{ij} E_{ij} \succeq J \end{array} \right]$$

$\Updownarrow$

$$\left[\begin{array}{l} \min_{t, y_{ij}} \ t \\ tI + \sum y_{ij} E_{ij} \succeq J \end{array} \right]$$

$$
(D^*)\quad \left[\begin{array}{l}
\min\limits_{t\in\mathbb{R},\, Z\in \mathrm{Sym}(n)} t \\
tI + Z - J \succeq 0 \\
Z_{ij} = 0 \quad \text{if } i=j \text{ or } \{i,j\}\notin E
\end{array}\right]
$$

Both (P^*) and (D^*) are strictly feasible, so there is no duality gap! And they share the same optimal value.

Another equivalent form is

$$
\left[\begin{array}{ll}
\min\limits_{Z\in \mathrm{Sym}(n+1)} Z_{n+1,n+1} & \\
Z_{n+1,i} = Z_{ii} = 1 & i=1,\dots,n \\
Z_{ij} = 0 & \{i,j\}\in E \\
Z \succeq 0 &
\end{array}\right] \quad (E^*)
$$

If (t,A) is feasible for (D^*),

$$(tI + A - J)_{11} = t-1 \geq 0 \Rightarrow t>0$$

$$
Z := \left(\begin{array}{c|c}
I + \frac{1}{t}A & \vdots \\
\hline
1\,1\,\cdots\,1 & t
\end{array}\right)
$$

$Z \succeq 0 \iff tI + A - J \succeq 0$ (A version of Schur complement)

$\Rightarrow Z$ is feasible and obj value is the same

If Z is feasible for (E^*), $Z = \left(\begin{array}{c|c} Z_n & \vdots \\ \hline 1\,1\cdots 1 & z \end{array}\right)$

$$\begin{pmatrix} 1 & 1 \\ 1 & z \end{pmatrix} \succeq 0 \quad \text{(Sylvester)}$$

$$\Rightarrow z > 0$$

$$(t,A) := (z, (Z_n - I_n)z)$$

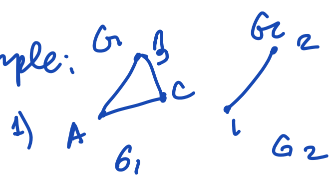
Part 2 How many k-letter words from the alphabet $v_1, \dots v_m$ can be transmitted without confusion?

It is $\alpha(G^k)$

Σ. has nodes $(v_i, v_j) \in V \times V$

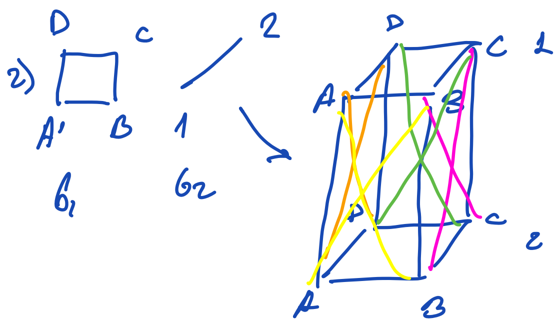
• $(v_i, v_j) \underset{\text{edge}}{\longleftrightarrow} (v_k, v_l) \iff (k = i$ or $v_k \underset{\text{edge}}{\longleftrightarrow} v_i)$ and $(l = j$ or $v_l \underset{\text{edge}}{\longleftrightarrow} v_j)$.

Example: 1)



$\longrightarrow$ $G_1 \otimes G_2$ is a complete graph on 6 vertices

2)



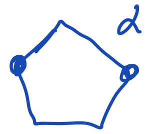
$G_1 \otimes G_2$

Claim: $\alpha(G_A \otimes G_B) \geq \alpha(G_A) \cdot \alpha(G_B)$

Pf: exercise

Example: $\alpha^2(G) = 4$ $\qquad \alpha(G \otimes G)$

$\alpha(G) = 2$



Def (Shannon capacity) $\theta(G) = \lim\limits_{k \to \infty} \alpha^{\frac{1}{k}}(G^k) = \sup\limits_k \alpha^{\frac{1}{k}}(G^k)$

can be shown using Claim above (also see Fekete's lemma)

The goal is to estimate Shannon capacity of a graph (alphabet).

$\theta(G) \geq \alpha^{\frac{1}{k}}(G^k) \quad \forall k$ by definition

Claim: $\theta(G) \leq \alpha_{SDP}(G)$?

Note: trivial bound $\theta(G) \leq n$

Note: $\theta(G) = \sup \alpha^{\frac{1}{k}}(G^k) \overset{\text{above}}{\leq} \sup \alpha_{SDP}^{\frac{1}{k}}(G^k) \overset{?}{\leq} \alpha_{SDP}(G)$

if $\alpha_{SDP}(G^k) \le \alpha_{SDP}^k(G)$

$\Uparrow$

? $\boxed{\alpha_{SDP}(G_1 \otimes G_2) \le \alpha_{SDP}(G_1) \cdot \alpha_{SDP}(G_2)}$

Theorem $\forall G_1, G_2$: $\alpha_{SDP}(G_1 \otimes G_2) \le \alpha_{SDP}(G_1) \cdot \alpha_{SDP}(G_2)$

(Proof) via solutions to the dual