Applications of SDP - 1

SDP: optimize a linear function of the entries of a matrix, with (a) affine constraints on the entries, and (b) a matrix with the entries that are affine expressions of the decision variables is PSD.

Standard form:

$$\begin{bmatrix} \min \ \mathrm{Tr}(C^T X) \\ \text{s.t.} \quad \mathrm{Tr}(A^T X) = b \\ X \succeq 0 \end{bmatrix}$$

Example

$$\begin{bmatrix} \min \ x + y \\ \text{s.t.} \ 3x - 2y \geq 3 \\ \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \succeq 0 \end{bmatrix}$$

variables must appear affinely in the matrix (can have $3x+5y$, cannot have x^2)

Then it can be transformed to a standard form.

Lemma: $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \succeq 0 \iff A \succeq 0$ and $B \succeq 0$

This is a way to combine all inequalities into one PSD constraint:

$$\left(\begin{array}{c|cc} 3x-2y-3 & 0 & 0 \\ \hline 0 & x & 1 \\ 0 & 1 & y \end{array}\right) \succeq 0$$

Application-1: Eigenvalue optimization

$$A(x) := A_0 + \sum_{i=1}^{n} x_i A_i$$

Toy example: completion problem

$$\begin{bmatrix} ? & 1 & ? \\ 2 & ? & ? \\ & \cdots & \end{bmatrix}$$

Part 1: $A_i \in \mathrm{Sym}(n)$

$$\max_{x \in \mathbb{R}^n} \lambda_{min}[A(x)]$$

$$\lambda_i(B + \alpha I) = \lambda_i(B) + \alpha$$

$$\lambda_{min} = \begin{bmatrix} \max_{x \in \mathbb{R}^n, t} t \\ \text{s.t.} \ tI \preceq A_0 + \sum_{i=1}^{n} x_i A_i \end{bmatrix}$$

$\leftarrow$ eigs of $(A_0 + \sum x_i A_i) \geq t$

recall: dual form of SDP

Part 2 $\min_{x\in\mathbb{R}^n} \lambda_{max}(A(x)) \longleftrightarrow \left[\begin{array}{l} \min_{x,t} t \\ A_0 + \sum x_i A_i \preceq t I \end{array}\right]$

Lemma: $\lambda_{max}(A)$ is a convex function of entries of A

(Proof) A, B

$\lambda_{max}(A)\ \lambda_{max}(B)$

$C := \alpha A + (1-\alpha) B$

$$\lambda_{max}(C) \overset{?}{\leq} \alpha \underbrace{\lambda_{max}(A)}_{\min_{A \preceq tI} t} + (1-\alpha) \underbrace{\lambda_{max}(B)}_{\min_{B\preceq tI} t}$$

$A \preceq \lambda_{max}(A)\cdot I \qquad B \preceq \lambda_{max}(B)\cdot I$

$$\alpha A + (1-\alpha) B \preceq \left(\alpha \lambda_{max}(A) + (1-\alpha)\lambda_{max}(B)\right)\cdot I$$

$$\lambda_{max}(\alpha A + (1-\alpha) B) = \min_{\alpha A + (1-\alpha) B \preceq tI} t$$

this is one feasible t

So, $\lambda_{max}(\alpha A + (1-\alpha) B) \leq \alpha \lambda_{max}(A) + (1-\alpha)\lambda_{max}(B)$ ∎

Part 3:

Given $A_0, A_1, \dots, A_m \in \mathbb{R}^{n\times p}$, let $A(x) := A_0 + \sum_{i=1}^{n} x_i A_i$

$\min_{x\in\mathbb{R}^m} \|A(x)\|$ – ? $\qquad \|A\| = \sup_{x:\|x\|_2=1} \|Ax\|_2 = \sqrt{\lambda_{max}(A^T A)}$

$$\left[\begin{array}{l} \min_{t,x} t \\ \|A(x)\|^2 \leq t \end{array}\right]$$

$\Updownarrow$

$$\left[\begin{array}{l} \min_{t,x} t \\ A^T(x)\cdot A(x) \preceq t I_p \end{array}\right]$$

$\Updownarrow$ Schur complement

$$\left[\begin{array}{l} \min_{t,x} t \\ \left(\begin{array}{c|c} I_n & A(x) \\ \hline A^T(x) & t I_p \end{array}\right) \succeq 0 \end{array}\right]$$

This is an SDP.