# Applications of SDP: combinatorial optimization.

## Independent set problem

$G = (V, E)$, $|V| = n$

Independent set – no inner edges. Stability number – largest stable set.

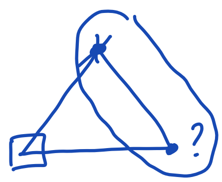
It is NP-hard to test if $\alpha(G)$ is NP-hard; SDP relaxation–?

$$\alpha(G) = \begin{bmatrix} \max\limits_x \sum x_i \\ x_i + x_j \le 1 \quad \leftarrow \{i,j\} \in E \\ x_i \in \{0,1\} \end{bmatrix} \xrightarrow{\text{LP relaxation}} \alpha_{LP}(G) = \begin{bmatrix} \max\limits_x \sum x_i \\ x_i + x_j \le 1 \\ 0 \le x_i \le 1 \end{bmatrix}$$

$$\alpha \le \alpha_{LP}$$

Better relaxation?

We can add more valide inequalities to LP.



$(C_2)$: For a 2-clique, $x_i + x_j \le 1$ (2-clique – edge between 2 vertices)

$(C_3)$ For a 3-clique ($\triangle$), $x_i + x_j + x_k \le 1$

$(C_4)$ For a 4-clique ($\boxtimes$), $x_i + x_j + x_k + x_e \le 1$ – Clique inequalities

⋮

Number of cliques is exponential in the size of the graph...

$$\alpha_{LP}^{(k)} := \begin{bmatrix} \max \sum x_i \\ 0 \le x_i \le 1 \\ C_2, \dots C_k \end{bmatrix}$$

SDP relaxation (Lovász, 1979)

$$\alpha_{SDP}(G) = \begin{bmatrix} \max\limits_{X \in Sym(n)} \sum\limits_{i,j} X_{ij} \\ Tr(X) = 1 \\ X_{ij} = 0 \quad \{i,j\} \in E \\ X \succeq 0 \end{bmatrix} \quad (*)$$

$\leftarrow Tr(JX)$, where $J$ is a matrix of all 1s

$\boxed{\text{Theorem}}$ $\forall G$: $\alpha(G) \overset{①}{\le} \alpha_{SDP}(G) \overset{②}{\le} \alpha_{LP}^{(k)}(G)$ $\forall k \geq 2$

Proof of ①: Let $S$ be a maximum stable set of $G$, $|S| = k$

Let $X = x \cdot x^T$, $x \in \mathbb{R}^n$ $x_i = \begin{cases} \frac{1}{\sqrt{k}} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$

$X_{ij} = x_i x_j = 0$ if $\{i,j\} \in E$ (at least one of $x_i$ and $x_j = 0$)

$y^T X y = \sum_{i,j} X_{ij} y_i y_j = \sum_{i,j} x_i x_j y_i y_j = \sum_{i,j} (x_i y_i) \cdot (x_j y_j) = (\sum x_i y_i)^2 \geq 0$

$\mathrm{Tr}(X) = \sum X_{ii} = \sum_i x_i^2 = \sum_{i \in S} \frac{1}{k} = 1$

$X$ is feasible for an SDP.

$\sum X_{ij} = \sum_{i,j} x_i x_j = (\sum x_i)^2 = \left(\frac{|S|}{\sqrt{k}}\right)^2 = \frac{|S|^2}{k} = |S|$

So, the value of the objective $\alpha_{SDP} \geq \alpha$.

Let's further study $\alpha_{SDP}(G)$, what is its dual?

$$(P) \left[ \begin{array}{ll} \min & \mathrm{Tr}(CX) \\ & \mathrm{Tr}(A_i X) = b_i \\ & X \succeq 0 \end{array} \right] \qquad (D) \left[ \begin{array}{ll} \max_y & b^T y \\ & \sum_{i=1}^m y_i A_i \preceq C \end{array} \right]$$

$$(P^*) \left[ \begin{array}{ll} -\min & \mathrm{Tr}(-JX) \\ & \mathrm{Tr}\, X = 1 \\ & X_{ij} = 0 \quad (i,j) \in E \\ & X \succeq 0 \end{array} \right] \longleftrightarrow \left[ \begin{array}{ll} -\max_{t, y_{ij}} & t \\ & tI + \sum y_{ij} E_{ij} \preceq -J \end{array} \right]$$

$$E_{ij} = \begin{array}{c} \\ (i,j) \to \end{array} \begin{array}{c} (i,j) \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

$\Updownarrow$

$$\left[ \begin{array}{ll} \min_{t,y} & -t \\ & -tI - \sum y_{ij} E_{ij} \succeq J \end{array} \right]$$

$\Updownarrow$ $-t \leftarrow t$

$-y_{ij} \leftarrow y_{ij}$

$$\left[ \begin{array}{ll} \min_{t, y_{ij}} & t \\ & tI + \sum y_{ij} E_{ij} \succeq J \end{array} \right]$$

$$(D^\star)\quad \left[\begin{array}{l} \min\ t \\ t\in\mathbb{R},\ Z\in \mathrm{Sym}(n) \\ tI+Z-J\succeq 0 \\ Z_{ij}=0 \quad \text{if } i=j \text{ or } \{i,j\}\notin E \end{array}\right]$$

$Z=0$
$t>n$
$\mathrm{eig}(J)=(n,0,0,\dots,0)$
(rank one matrix)

Both $(P^\star)$ and $(D^\star)$ are strictly feasible, so there is no duality gap! And they share the same optimal value.

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Another equivalent form of the dual is:

$$\left[\begin{array}{ll} \min\ Z_{n+1,n+1} & \\ Z\in \mathrm{Sym}(n+1) & \\ Z_{n+1,i}=Z_{ii}=1 & i=1,\dots,n \\ Z_{ij}=0 & \{i,j\}\in E \\ Z\succeq 0 & \end{array}\right] \quad (E^\star)$$

If $(t,A)$ is feasible for $(D^\star)$,

$(tI+A-J)_{11}=t-1\geq 0 \Rightarrow t>0$

$$Z:=\left(\begin{array}{c|c} I+\frac{1}{t}A & \vdots \\ \hline 1\ 1\ \cdots\ 1 & t \end{array}\right)$$

$Z\succeq 0 \iff tI+A-J\succeq 0$ (A version of Schur complement)

$\Rightarrow Z$ is feasible and obj value is the same

If $Z$ is feasible for $(E^\star)$, $Z=\left(\begin{array}{c|c} Z_n & \vdots \\ \hline 1\ 1\cdots 1 & z\end{array}\right)$

$\begin{pmatrix} 1 & 1 \\ 1 & z\end{pmatrix}\succeq 0$ (Sylvester)
$\Rightarrow z>0$

$(t,A):=(z,(Z_n-I_n)z)$

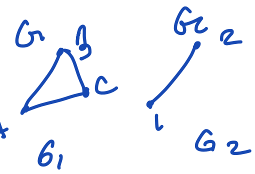
Part 2 How many $k$-letter words from the alphabet $v_1, \dots v_m$ can be transmitted without confusion?

It is $\alpha(G^k) \leftarrow$ maximal size of the independent set of $k$-th graph power.

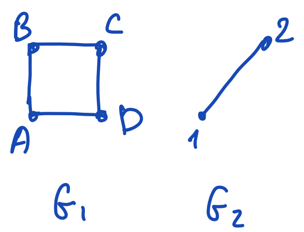
$\Sigma$. has nodes $(v_i, v_j) \in V \times V$

• $(v_i, v_j) \underset{\text{edge}}{\leftrightarrow} (v_k, v_\ell) \iff (k = i$ or $v_k \underset{\text{edge}}{\leftrightarrow} v_i)$ and $(\ell = j$ or $v_\ell \underset{\text{edge}}{\leftrightarrow} v_j)$.

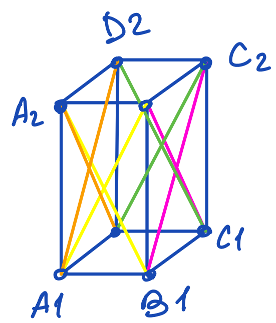
Example: 1)



$\longrightarrow$ $G_1 \otimes G_2$ is complete graph on 6 vertices



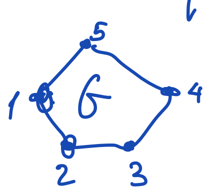
$\longrightarrow G_1 \otimes G_2$



Claim: $\alpha(G_1) \cdot \alpha(G_2) \leq \alpha(G_1 \otimes G_2)$

Pf: exercise

The inequality can be tight: $\alpha(G) = 2 \quad \alpha^2(G) = 4$



$\alpha(G \otimes G) = 5$ – construct a set to check this!

To compute the number of distinct words of any lenght can be computed via

Def (Shannon capacity) $\Theta(G) = \lim_k \alpha^{\frac{1}{k}}(G^k)$

$= \sup_k \alpha^{\frac{1}{k}}(G^k) \leftarrow$ can be shown using Claim above ($\log \alpha(G_1) + \log \alpha(G_2) \leq \log \alpha(G_1 \otimes G_2)$) gives "enough growth"

$\alpha(G_k) \leq n^k \Rightarrow \sup_k \leq n$ (finite) [and $\Theta \leq n$]

Note: $\Theta(G) \geq \alpha^{\frac{1}{k}}(G^k) \quad \forall k$ by definition

Claim: $\Theta(G) \leq \alpha_{SDP}(G)$ ?

Fekete's lemma: $a_{n+m} \geq a_n + a_m$
$\Rightarrow \lim_{k \to \infty} \frac{a_k}{k} = \sup_k \frac{a_k}{k}$

First reductions:

$\Theta(G) = \sup \alpha^{\frac{1}{k}}(G^k) \leq \sup \alpha_{SDP}^{\frac{1}{k}}(G^k) \overset{?}{\leq} \alpha_{SDP}(G)$

$\alpha \leq \alpha_{SDP}$ (proved above)

Notation: $\vartheta := \alpha_{SDP}$

For ?, it is enough to show $\boxed{\vartheta(G_1 \otimes G_2) \leq \vartheta(G_1) \cdot \vartheta(G_2)}$

Discussion:
$$\left[\begin{array}{ll} \vartheta(G) = \max_X \operatorname{Tr}(JX) & \\ \text{s.t. } \operatorname{Tr}(X) = 1 & \\ X_{ij} = 0 & (i,j) \in E \\ X \succeq 0 & \end{array}\right] \quad (*)$$

Proposed strategy:

$\vartheta(G_1) \longrightarrow X_1$ optimal solution

$\vartheta(G_2) \longrightarrow X_2$

$X_1, X_2 \longrightarrow$ construct $X$ $n^2 \times n^2$ and show that $X$ is feasible for $\vartheta(G_1 \otimes G_2)$

But then optimal solution can be even bigger and it does not help prove (*)

Taking a dual gives a minimization problem! See previous page for dual construction.

**Thm** $\forall G_1, G_2$ $\vartheta(G_1 \otimes G_2) \leq \vartheta(G_1) \cdot \vartheta(G_2)$

($G_1$: $n$ nodes, $G_2$: $m$ nodes)

**Proof**

$(t_1, A)$ - optimal solution
$(t_2, B)$ - $\Rightarrow$ $(t_1 \cdot t_2, C)$ is feasible to $G_1 \otimes G_2$

$C := t_1 \cdot I_n \otimes B + t_2 A \otimes I_m + A \otimes B.$

**Def** Kronecker product $X \otimes Y = \begin{bmatrix} x_{11} Y & x_{12} Y & \cdots & x_{1n} Y \\ \vdots & & & \\ & & & x_{nn} Y \end{bmatrix}$

**Lemma** $X \succeq 0, Y \succeq 0 \Rightarrow X \otimes Y \succeq 0$

Why?

Spectrum: $X v_i = \lambda_i v_i$ $\qquad (X \otimes Y)(v_i \otimes w_j) = X v_i \otimes Y w_j = \lambda_i v_i \otimes \mu_j w_j$

$Y w_i = \mu_i w_i$ $\qquad (A \otimes B) \cdot (C \otimes D) = AC \otimes BD$

(mixed product property for Kronecker products)

$t_1 \cdot I_n + A - J_n \succeq 0$ $\quad\Rightarrow\quad$ $t_1 \cdot I_m + A + J_n \succeq 0$

$t_2 \cdot I_m + B - J_m \succeq 0$ $\quad\quad\;\;$ $t_2 \cdot I_n + B + J_m \succeq 0$

$$(t_1 I_n + A - J_n) \otimes (t_2 I_m + B + J_m) \succeq 0$$

$$(t_1 I_n + A + J_n) \otimes (t_2 I_m + B - J_n) \succeq 0$$

Expanding and averaging (see notes [AAA lec11])

$$t_1 t_2 I_{nm} + t_1 I_n \otimes B + t_2 A \otimes I_m + A \otimes B - J_{nm} \succeq 0$$

So, for $C$, $t_1 t_2 I_{nm} + C - J_{nm} \succeq 0$

Need to check that $C_{ij} = 0$ if $ij$ is not an edge in the product graph.

No edge means that (a) $i_1 \neq j_1$ and $i_1 - j_1$ is not an edge in $G_1$ (means $A_{i_1 j_1} = 0$)
or (b) $i_2 \neq j_2$ and $i_2 - j_2$ is not an edge in $G_2$ (means $B_{i_2 j_2} = 0$)

Without loss of generality, assume (a).

$C_{(i_1 i_2\, j_1 j_2)} = A_{i_1 j_1} B_{i_2 j_2} + t_1 I_{i_1 j_1} B_{i_2 j_2} + t_2 A_{i_1 j_1} I_{i_2 j_2}$

$A_{i_1 j_1} = 0$ and $I_{i_1 j_1} = 0$ → done

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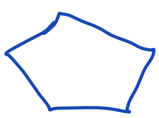
Discussion

For $C_5$, SDP can be solved in exact form: $\vartheta(C_5) = \sqrt{5}$ (done in original Lovasz paper)



$\alpha(C_5^2) = 5 \Rightarrow \alpha^{\frac{1}{2}}(C_5^2) \leq \Theta(G)$

$\Theta(C_7)$ – ? open problem

$3.2271 \leq \Theta(C_7) \leq 3.3177$ $(\vartheta(C_7))$

construction of a stable set in $\alpha^{\frac{1}{5}}(C_7^5)$

It is not known how to get upper bounds better in general.