Polynomial optimization

(a) Quadratic feasibility is NP-hard

Reduction to Stable set problem

$$\alpha(G) \geq k \iff \exists x \in \mathbb{R}^n \left\{ \begin{array}{l} x_i \in \{0,1\} \\ x_i + x_j \leq 1 \quad \overset{\text{edge}}{i \leftrightarrow j} \\ \sum x_i \geq k \end{array} \right\} \iff \exists x \in \mathbb{R}^n, s \in \mathbb{R} \left\{ \begin{array}{l} x_i(x_i - 1) = 0 \\ x_i x_j = 0 \quad \overset{\text{edge}}{i \leftrightarrow j} \\ \sum x_i - k = s^2 \end{array} \right\}$$

Note: generic stable set problem is reduced to a particular case of quadratic feasibility problem

(b) Polynomial positivity (with rational coeff's) is NP-hard in deg 4

the first hard power (check that $p = 1, 2, 3 \to$ in P)

Generic quadratic feasibility reduces to specific polynomial positivity (deg 4)

$$q_i(x) = 0 \text{ is feasible} \iff p(x) = \sum q_i^2(x) \text{ is not always positive}$$

(c) Polynomial positivity reduction to
- 1-in-3 SAT (if deg 4)
- 3 SAT (if deg 6)

Any SAT $\to$

$$\varphi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

$$p = \sum_{i=1}^{3} (x_i(1-x_i))^2 + \left[(x_1 + (1-x_2) + x_3 - 1)(x_1 + (1-x_2) + x_3 - 2)(x_1 + (1-x_2) + x_3 - 2) \right]^2 + \dots$$

φ is satisfyiable $\iff$ p has a zero solution.

(d) Polynomial nonnegativity is also NP-hard

We can reduce it to matrix copositivity problem: $M \in \mathrm{Sym}(n)$? $x^T M x \geq 0 \quad \forall x \geq 0$

Indeed, consider $p(x) := v(x)^T M v(x)$ $\quad v(x) = \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix}$

Its non-negativity would imply copositivity for M.

→ Co-positivity of M is NP-hard: this is non-trivial.

In general, $\left[\begin{array}{l} \min \ x^T M x \\ \text{s.t. } x_i \geq 0 \end{array}\right]$ is a particular case of $\left(\begin{array}{l} \min \ x^T Q x + c^T x + d \\ \text{s.t. } Ax \leq b \end{array}\right)$ quadratic polytope constraints

And both can be reduced to max CLIQUE problem (by Motzkin – Strauss thm)

$f(x) = \sum_{i \overset{\text{edge}}{\leftrightarrow} j} x_i x_j$ for a graph on n vertices

$\min_{\Delta} f(x) = \frac{1}{2\omega} - \frac{1}{2}$ ← ω: largest clique size

$\Delta := \{ \bar{x} \in \mathbb{R}^n \mid \sum x_i = 1, \ x_i \geq 0 \}$

[AAA Lec 14
MS paper]

→ Fun fact: M is co-positive if $M = P + N$, $P \succeq 0$ and $N \geq 0$ (non-negative entries)

(e) Discussion: a set of copositive matrices (and nonnegative polynomials) is convex, but testing feasibility (and so, optimizing over it) is NP-hard

It is not known if it is in NP (as writing some polynomials with n unknowns is not polynomial in time)

So, convexity by itself is not super helpful

Is locality helpful?

$\boxed{\text{STRICT LOCAL-4: Given a polynomial } p \text{ of degree } 4, \ \bar{x} \in \mathbb{R}^n, \text{ is } \bar{x} \text{ an unconstrained \underline{local} min for } p?}$

(Proof) Reduction to polynomial positivity

$\bar{x}$ is a strict local min for some $p(x) \iff \tilde{p}(x): \tilde{p} > 0 \quad \forall x \in \mathbb{R}^n$

Idea: if $\bar{x} = 0$ (shift!) and p is homogeneous $(p(\alpha x) = \alpha p(x))$
this is true: $\forall x \in \mathbb{R}^n$ take $\alpha = \frac{\varepsilon}{\|x\|}$ and $p(\alpha x) \in \varepsilon$-neighborhood of $0 \Rightarrow p(\alpha x) > 0, \ \alpha > 0)$

Homogenization of $p(x)$: define $p_h(x,y) := y^d p(\frac{x}{y})$, d is degree.

For example,

$$p(x) = x^4 + 3x^3 + 2x + 5$$

$$p_h(x,y) = x^4 + 3x^3y + 2xy^3 + 5y^4$$

$$p_h(x,1) = p(x)$$

Does this operation preserve positivity?

In general, no: Example

$$(1 - x_1 x_2)^2 + x_1^2 \quad > 0$$

$$(y^2 - x_1 x_2)^2 + x_1^2 y^2 \not> 0 \text{ since}$$

one can take $y = 0 \quad x_1 = 1 \quad x_2 = 0$

extra condition

Lemma: $p(x) > 0 \ \forall x \iff p_h(x,y) > 0 \quad \forall (x,y) \neq 0, y \neq 0$

(Pf) $\Rightarrow$ $p_h(x,y) > 0 \ \forall (x,y) \neq 0 \Rightarrow p_h(x,1) \neq 0 = p_h(x)$

$\Leftarrow$ Suppose $\exists (x,y)$: $p_h(x,y) = 0$, $y \neq 0$, rescale $0 = \frac{1}{y^d} \cdot p_h(x,y) = p_h\left(\frac{x}{y}, 1\right)$

$= p(\tilde{x})$

$\tilde{x} = \frac{x}{y}$

Why is this enough for us?

We proved polynomial positivity from quadratic feasibility:

Polynomials we actually consider

$$p(x,s) = \sum (x_i (\overline{1 - x_i})^{0/1})^2 +$$

$$+ \underbrace{\sum x_i^2 x_j^2}_{\text{no edge}} + \sum (x_i^2 - k - s^2)^2$$

$\sum x_i - k = s^2$

If $y = 0$, the only terms that do not become 0 are those of maximal degree

$$p_h(x,s,0) =$$

$$= \sum x_i^4 + \sum x_i^2 x_j^2 + s^4$$

This is $= 0$ only if $x, s = 0$.

So, in this case positivity is preserved

(homogenized)
Hense, testing positivity of ~~homogenious~~ polynomials is also NP-hard!

Overall, local optimization can be also hard!

Relaxations...

Relaxation of nonnegativity of a polynomial

$p(x) = \sum_i q_i^2(x) \geq 0 \ \forall x$ - clear (SOS polynomials)

Existence of a SOS decomposition is a certificate of nonnegativity

Thm $p(x) \in \mathbb{R}^n$ of degree $2d$ is SOS $\Leftrightarrow$ $\exists\, Q \succeq 0 : p(x) = z^T Q z$

$z = [1, x_1, x_2 \dots x_n, x_1 x_2, \dots x_n^d]$
all monomials

$\boxed{\text{Finding such } Q \text{ is an SDP.}}$

So, not every nonnegative polynomial is SOS (but one can try to check for SOS instead of the nonnegativity since it is sufficient and easier)