From linear to semidefinite programming:

classes of convex optimization problems.

1. Linear programs (LP) [ORF 522]

$$\begin{bmatrix} \min c^T x \\ \text{s.t. } Ax \leq b \end{bmatrix}$$
$(x \geq 0)$

$A \in \mathbb{R}^{m \times n}$
$b \in \mathbb{R}^m$

2. Quadratic programs (QP)

$$\begin{bmatrix} \min x^T A_0 x + b_0^T x + c_0 \\ \text{s.t. } A_q x \leq b_q \end{bmatrix}$$

$A_0 \in \mathbb{R}^{n \times n}$ $\quad A_q \in \mathbb{R}^{m \times n}$
$b_0 \in \mathbb{R}^n$ $\quad b_q \in \mathbb{R}^m$
$c_0 \in \mathbb{R}$

Convex quadratic program, if $A_1 \succeq 0$. (Example: SVM)

3. Quadratically constrained QP (QCQP)

$$\begin{bmatrix} \min x^T A_0 x + b_0^T x + c_0 \\ \text{s.t. } x^T A_i x + b_i^T x + c_i \leq 0 \end{bmatrix}$$

$A_i \in \mathbb{R}^{n \times n}$
$b_i \in \mathbb{R}^n$
$c_i \in \mathbb{R}$

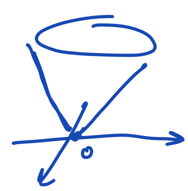
Convex if all the matrices are PSD symmetric

4. SOCP: Second order cone programs

$$\begin{bmatrix} \min_x f^T x \\ \|A_i x + b_i\|_2 \leq c_i^T x + d_i \end{bmatrix} \quad i = 1 \ldots m$$

Lorentz cone

$$L^{n+1} = \{(x, t) : \|x\|_2 \leq t\}$$



Example: LASSO with block sparsity

$$\min_x \|Ax - y\|_2$$

$x = (x_1, \ldots x_p)^T$
$x_i \in \mathbb{R}^{n_i}$
$\sum n_i = n$

We want as many blocks x_i's to be zero as possible.

$$\left[\min_{x} \|Ax-y\|_2 + \gamma \sum_{i=1}^{p} \|x_i\|_2\right] \longleftarrow$$

ℓ_1-norm of the vector $\begin{pmatrix} \|x_1\|_2 \\ \vdots \\ \|x_p\|_2 \end{pmatrix}$

$$\Updownarrow$$

$$\left[\begin{array}{l} \min_{x,z,t} \; z + \gamma \sum_{i=1}^{p} t_i \\ \|Ax-y\|_2 \le z \\ \|x_i\|_2 \le t_i \end{array}\right]$$

Block-sparsified lasso appears, e.g. in model selection problem

5. SDP : Semidefinite programming

$$\left[\begin{array}{l} \min_{X \in Sym^{n \times n}} \; \mathrm{Tr}(CX) \\ \text{s.t. } \mathrm{Tr}(A_i X) = b_i \\ X \succeq 0 \end{array}\right] \longleftrightarrow \left[\begin{array}{l} \min_{X \in S^{n \times n}} \; \sum_{k\ell} C_{k\ell} X_{k\ell} \\ \sum_{k,\ell} (A_i)_{k\ell} X_{k\ell} = b_i \\ X \succeq 0 \end{array}\right]$$

So Traces mean there are affine constraints on the individual entries of X

The only "matrix condition" is $X \succeq 0$

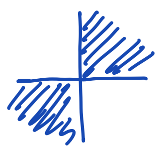
6. CP: Conic program

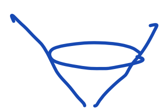
$$\left[\begin{array}{l} \min \; f(x) \\ \text{s.t. } Ax \le b \\ x \in C \end{array}\right]$$

f is convex

C is a convex cone

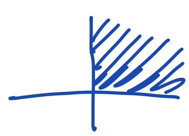
convex set — $\forall x \;\; \alpha x \in X, \; \alpha > 0$

Examples : • Non-convex cone 

• Not a conv hull of a finite set 

• Convex cones: • PSD matrices

• 2nd order cone $\{\|x\|_2 < t\}$

• $\mathbb{R}^n_+$ 

$$\boxed{LP \subset SOCP \subset SDP \subset CP}$$