Approximate optimization

Recall an example from the 1st lecture (see slides 17-18)

<u>Max cut problem</u> (NP-hard)

Approximate solution: a cut that contains $\geq \frac{1}{2} \cdot |\text{size of max cut}|$ edges
(using probabilistic argument)

Another example: <u>integer optimization problem</u> (also NP-hard)

$$\boxed{\text{maximize} \quad \sum A_{ij} x_i x_j \qquad x_i = \pm 1 \qquad A \in Sym(n)} \quad \text{(INT)}$$
$i = 1 \dots n$

$\downarrow$ Approximate relaxation

$$\boxed{\max \sum_{i,j=1}^{n} A_{ij} \langle X_i, X_j \rangle \text{ s.t. } \|X_i\| = 1} \quad \text{(SDP-INT)}$$
$(X_i^T X_i = 1)$ $\qquad X_i \in \mathbb{R}^n \quad i = 1 \dots n$

Why is this an SDP program?

$W \rightarrow$ $w_{ij} = \langle X_i, X_j \rangle$
How to find X_i, X_j?
$W = VV^T$ (Cholesky)

$$\boxed{\begin{array}{l} \max \ tr(A^T W) \\ W \succeq 0 \\ W_{ii} = 1 \end{array}}$$

$W := (w_{ij}) \in \mathbb{R}^{n \times n}$
$w_{ij} = \langle X_i, X_j \rangle$
Gram matrix

Exercise: check that $X \succeq 0$.

Thm $INT(A)$ - maximum in integer optimization, $A \succeq 0$

$$INT(A) \overset{①}{\leq} SDP\text{-}INT(A) \overset{②}{\leq} 2K \cdot INT(A)$$

(Pf) ① Let $x_1, \dots x_n$ give max to $INT(A)$

Define $X_i := (x_i, 0, \dots 0)$

then $\sum a_{ij} \langle X_i, X_j \rangle = \sum a_{ij} x_i x_j$

② Follows from symmetric Grothendieck's inequality

$$\boxed{\begin{array}{l} A \in Sym(n), \ |\sum a_{ij} x_i x_j| \leq 1 \text{ for } \forall x_i \in \{\pm 1\} \\ A \succeq 0 \\ \text{Then for any } u_i, v_i \in \mathbb{R}^n, \ \|u_i\| = \|v_i\| = 1: \quad |\sum a_{ij} \langle u_i, v_j \rangle| \leq 2K \end{array}}$$

$\blacksquare$

TODOs: prove ② and learn how to translate solutions rather than optimal values

We will discuss a more general statement:

Grothendieck's inequality

$A \in \mathbb{R}^{m \times n}$, for any $x_i, y_j \in \{\pm 1\}$ and $\left|\sum_{ij} a_{ij} x_i y_j\right| \le 1$

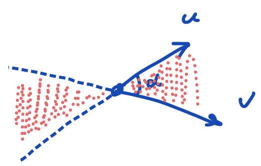
Then $\forall\, u_i, v_j \in \mathbb{R}^n, \mathbb{R}^m$ $\|u_i\| = \|v_j\| = 1$: $\left|\sum_{ij} a_{ij} \langle u_i, v_j \rangle\right| \le K$

Pf idea:

Grothendieck's identity $g \sim N(0, I_n)$, then for any fixed vectors $u, v \in S^{n-1}$

$$\mathbb{E}\, \mathrm{sgn}\langle g, u\rangle \cdot \mathrm{sgn}\langle g, v\rangle = \frac{2}{\pi} \arcsin \langle u, v\rangle$$

Why? Rotation invariance of g: let $g \in \mathrm{span}\{u, v\}$



$$\mathbb{P}\{\mathrm{sgn}\langle g, u\rangle \cdot \mathrm{sgn}\langle g, v\rangle = -1\} = \frac{\alpha}{\pi}$$

$$\mathbb{E}\ldots = (-1)\cdot\frac{\alpha}{\pi} + (1)\,\frac{\pi - \alpha}{\pi} = \frac{2}{\pi}\left(\frac{\pi}{2} - \alpha\right) = \frac{2}{\pi}\arcsin\langle u, v\rangle$$

since

$$\begin{cases} \cos\alpha = \langle u, v\rangle & (\|u\|\cdot\|v\| \cos\alpha = \langle u, v\rangle) \\ \arcsin z = \frac{\pi}{2} - \arccos z \end{cases}$$

Now, $\sin\alpha \approx \alpha$ for small α

If we would have $\mathbb{E}\, \mathrm{sgn}\langle g, u\rangle\, \mathrm{sgn}\langle g, v\rangle = \frac{2}{\pi}\langle u, v\rangle$, then

$$\frac{2}{\pi}\sum a_{ij}\langle u_i, v_j\rangle = \sum a_{ij}\, \mathbb{E}\, \underbrace{\mathrm{sgn}\langle g, u\rangle}_{\substack{\parallel \\ x_i}}\, \underbrace{\mathrm{sgn}\langle g, v\rangle}_{\substack{\parallel \\ y_j}} \le 1 \quad \Rightarrow \quad K \approx 1.57 \left(\frac{\pi}{2}\right)$$

is wrong:)

Kernel trick: represent $\frac{2}{\pi}\arcsin\langle u, v\rangle = \langle u', v'\rangle$ in a $\underbrace{\text{higher-dimensional space}}$ ↳ tensor spaces

Namely, we define $u' = \Phi(u)$, $v' = \Phi(v)$

How to rectify non-linearities in scalar products?

Idea 1:

Rank-one tensor: $u \in \mathbb{R}^n$

$$\underbrace{u \otimes u \otimes \ldots \otimes u = u^{\otimes k}}_{\substack{\parallel \\ \mathcal{U}}} \in \mathbb{R}^{n \times \ldots \times n}$$

$\mathcal{U}$: $\mathcal{U}_{i_1 \ldots i_k} = u_{i_1} \cdot u_{i_2} \cdots u_{i_k}$

$\langle \mathcal{U}, \mathcal{V}\rangle := \sum \mathcal{U}_i \mathcal{V}_i$

Claim: $\langle u^{\otimes k}, v^{\otimes k}\rangle = \langle u, v\rangle^k$ (check!)

linear in $\mathbb{R}^{k \times \ldots \times k}$ — nonlinear

Idea 2: $\langle u, v\rangle + \langle u\otimes u, v\otimes v\rangle = \langle (u, u\otimes u), (v, v\otimes v)\rangle$ ⓥ

Indeed, $\langle (x_1 \dots x_n), (y_1 \dots y_n)\rangle = \sum x_i y_i$

Overall, we have

Lemma $f(x) = \sum_{k=0}^{\infty} a_k x^k$ real analytic function with $a_k > 0$:

Then $\exists\ \Phi : \mathbb{R}^n \to \mathcal{H}$: $\langle \Phi(u), \Phi(v)\rangle = f(\langle u, v\rangle)$

$\mathcal{H}$ — some higher-dimensional space.

Remark: if a_k can be negative, we have $\langle \Phi(u), \Psi(v)\rangle$:

$\Phi(u) := (|a_0|^{\frac{1}{2}}, |a_1|^{\frac{1}{2}} u, |a_2|^{\frac{1}{2}} \cdot u\otimes u, |a_3| \cdot u\otimes u\otimes u, \dots)$

$\Psi(v) := (|a_0|^{\frac{1}{2}}, \mathrm{sgn}(a_1)|a_1|^{\frac{1}{2}} v, \mathrm{sgn}(a_2)\cdot |a_2|^{\frac{1}{2}} v\otimes v, \dots)$

$\langle \Phi(u), \Psi(v)\rangle = \sum_{i=0}^{\infty} \mathrm{sgn}(a_i) \cdot |a_i| \cdot \underbrace{\langle u^{\otimes i}, v^{\otimes i}\rangle}_{= (\langle u, v\rangle)^i}$

Example $\langle \Phi(u), \Phi(v)\rangle = \underbrace{2\langle u, v\rangle^2 + 5\langle u, v\rangle^3}$

$\Phi : \mathbb{R}^n \to \mathbb{R}^{n\times n} \oplus \mathbb{R}^{n\times n\times n}$

$\oplus$ — cartesian product

ⓥ $= 2\langle u^{\otimes 2}, v^{\otimes 2}\rangle + 5\langle u^{\otimes 3}, v^{\otimes 3}\rangle = \langle (2u^{\otimes 2}, 5u^{\otimes 3}), (2v^{\otimes 2}, 5v^{\otimes 3})\rangle$

Note: if we need to subtract a term, we have $\Psi \neq \Phi$.

Now this can be applied to $f(x) = \sin\left(\frac{\beta\pi}{2}\langle u, v\rangle\right)$

Proof $u_i' := \Phi(u_i)$, $v_i' := \Psi(v_i)$: $\frac{2}{\pi}\arcsin\langle u_i', v_j'\rangle = \beta\langle u_i, v_i\rangle$

$$\beta\sum A_{ij}\langle u_i, v_j\rangle = \sum A_{ij}\frac{2}{\pi}\arcsin\langle u_i', v_j'\rangle = \sum A_{ij}\cdot \mathbb{E}\,\underbrace{\mathrm{sgn}\langle g, u_i'\rangle \cdot \mathrm{sgn}\langle g, v'\rangle}_{\le 1}$$

if u_i', v_j' have unit norms

defines selection of β:

$\|u_i'\|^2 = \|\Phi(u_i)\|^2 = \sum_{k=0}^{\infty} |a_i|\langle u_i, u_i\rangle^k$; $\|v_i'\|^2 = \|\Psi_i\|^2 = \sum_{k=0}^{\infty} \underbrace{\mathrm{sgn}^2(a_i)}_{=1} |a_i|\langle v_i, v_i\rangle^k$

$\Phi = \sin\left(\frac{\beta\pi}{2}x\right)$ $\qquad \sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots$

$\|\Phi(u)\|^2 = \underbrace{\sinh\left(\frac{\beta\pi}{2}\right) = 1}$ $\qquad \sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots$

$\Downarrow$ $\qquad \|x\| = 1$

$\frac{\beta\pi}{2} = \sinh^{-1}(1) = \ln(1+\sqrt{2})$

$\beta \approx \frac{1}{1.783}$

Notes: ① Grothendieck's inequality can be proved without kernel tricks with worse constant K (≤ 288)

Recall: $\sum A_{ij} \langle u_i, v_j \rangle \leq K$

General plan: $U_i = \langle g, u_i \rangle$
$V_j = \langle g, v_j \rangle$ $\quad g \sim N(0, I_N)$

Then $U_i, V_j \sim N(0,1)$ rotation invariance

$\mathbb{E}\, U_i \cdot V_j = \langle u_i, v_j \rangle$ define correlations

$K \geqslant \sum A_{ij} \langle u_i, v_j \rangle = \mathbb{E}\left(\sum A_{ij} U_i V_j\right)$

Truncation $U_i = \underbrace{U_i \cdot \mathbb{1}_{\{|U_i| \leq R\}}}_{U_i^-} + \underbrace{U_i \cdot \mathbb{1}_{\{|U_i| > R\}}}_{U_i^+}$

Tails are small and

$\mathbb{E}\left|\sum_{ij} A_{ij} U_i^- V_i^-\right| \leq R^2$ (so K can be taken as R^2)

② Kernel trick is useful beyond Grothendieck's inequality:

$K: \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ let it be some function. When is there a transformation $\Phi: \mathcal{X} \to \mathbb{R}^M$: $\langle \Phi(u), \Phi(v) \rangle = K(u,v)$

(Generalization of a statement about $K(u,v) = f(\langle u,v \rangle)$)

__Mercer's thm:__ It happens if and only if K is a positive-semidefinite kernel: for any finite collection of points $u_1, \dots, u_N \in \mathcal{X}$, the matrix

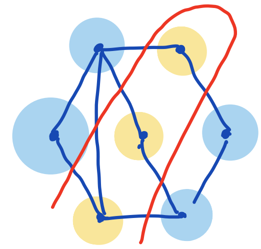
$$\begin{pmatrix} K(u_1,u_1) & K(u_1,u_2) & \dots & K(u_1,u_n) \\ K(u_2,u_1) & & & \\ \vdots & & & \\ K(u_n,u_1) & & \dots & K(u_n,u_n) \end{pmatrix} \succcurlyeq 0$$

Φ - feature map

$K(u,v) = \exp\left(-\frac{\|u-v\|_2^2}{2\sigma^2}\right)$, $K(u,v) = (\langle u,v \rangle + r)^k \dots$

Φ gives separability conditions, and in many applications not need to be computed

Application to MAX-CUT



and solution approximation

0.878 - approximation algorithm (Goemans, Williamson)

$G = (V, E)$

Adjacency matrix A (binary)

$x \in \{\pm 1\}^n$ - partition onto two sets of edges.

$$\mathrm{CUT}(G, x) = \frac{1}{2} \sum_{x_i \cdot x_j = -1} A_{ij} = \frac{1}{4} \sum_{i,j=1}^{n} A_{ij} (1 - x_i x_j)$$

$$\text{MAX-CUT}(G) = \frac{1}{4} \max \{ A_{ij} (1 - x_i x_j) : x_i = \pm 1 \quad \forall i \}$$

$$\mathrm{SDP}(G) := \frac{1}{4} \max \{ \sum A_{ij} (1 - \langle X_i, X_j \rangle) : \|X\|_2 = 1 \quad \forall i \}$$

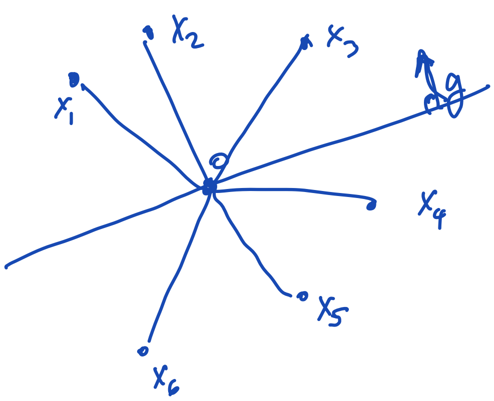
How to translate the solution (X_i) into labels $x_i = \pm 1$?

Randomized rounding step: choose a random hyperplane in $\mathbb{R}^n$

Assign $x_i = 1$, $x_i = -1$ depending on a subspace defined by a hyperplane.

Equivalently, we choose $g \sim N(0, I_n)$ and define $x_i = \operatorname{sgn} \langle X, g \rangle$

(random hyperplanes are defined by their random normals)

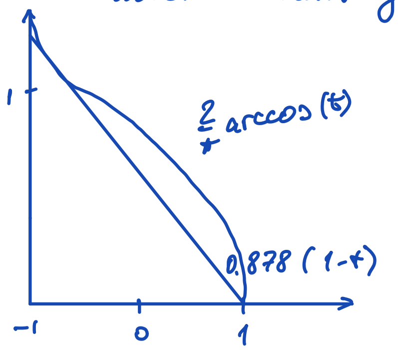


Theorem:

$$\mathbb{E}\, \mathrm{CUT}(G, x) \overset{(1)}{\geq} 0.878\, \mathrm{SDP}(G) \overset{(2)}{\geq} 0.878\, \text{MAX-CUT}(G)$$

where x is the result of randomized rounding of (X_i)

Recall Grothendieck's identity $\mathbb{E}\, \mathrm{sgn} \langle g,u\rangle\, \mathrm{sgn}\langle g,v\rangle = \frac{2}{\pi} \arcsin \langle u,v\rangle$, $g \sim N(0, I_n)$



$$1 - \frac{2}{\pi}\arcsin t = \frac{2}{\pi}\arccos t \geq 0.878\,(1-t)$$
$t \in [-1,1]$

Proof $\mathbb{E}\, \mathrm{CUT}(G,x) = \frac{1}{4}\sum_{ij} A_{ij}\,(1-\mathbb{E}\, x_i x_j)$

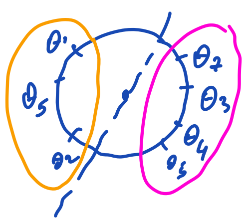
$$1-\mathbb{E}\, x_i x_j = 1-\mathbb{E}\,\mathrm{sgn}\langle X_i, g\rangle\, \mathrm{sgn}\langle X_j, g\rangle = 1-\frac{2}{\pi}\arcsin\langle X_i, X_j\rangle \geq 0.878\,(1-\langle X_i, X_j\rangle)$$

So, $\mathbb{E}\,\mathrm{CUT}(G,x) \geq 0.878 \cdot \frac{1}{4}\sum A_{ij}\,(1-\langle X_i, X_j\rangle) = 0.878\ \mathrm{SDP}(G)$

This proves ①. ② follows from the theorem on page ①.
(which also holds when A has zero diagonal)
see HW4 problem 5 ∎

It is believed that it is NP-hard to get better approximation (NP-hard, reduces to Unique Games Conjecture)

Another way to visualize randomized rounding



For the directions $\theta_1 \dots \theta_7$

How to avoid solving an SDP here?

in $\mathbb{R}^2$:
$\langle v_{\theta_i}, v_{\theta_j}\rangle = \cos(\theta_i - \theta_j)$ ← angle

$f(\theta_1 \dots \theta_n) = \sum_{ij \geq 1}^{n} A_{ij} \cos(\theta_i - \theta_j)$ - energy function

$f : [0, 2\pi]^n \to \mathbb{R}$, A - adjacency

f is minimized with gradient descent, ...

Then randomized rounding is used to separate θ_i's

"Rank 2 relaxation heuristics..." (Burer, Monteiro, Zhang) 2001

Steinerberger '21 — another ways to find small values of energy functions and related graph partitions

---

Other applications of Grothendieck's inequality for polynomial time approximation e.g. [Khot, Naor '11]

- Szemeredi partitions of graphs
- Friez-Kannan matrix decomposition
- Acyclic graphs
...

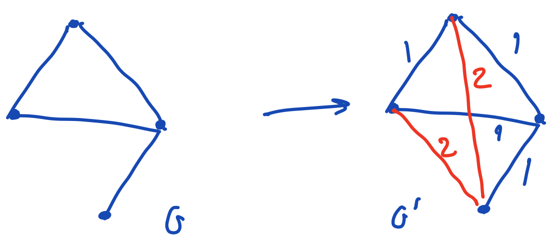
Other problems and proving inapproximability

① K-center problem: given a complete graph so that edges have "lengths" satisfying Δ inequality, find a K-element set minimizing max distance to it $\min_S \left(\max_{v \in V} d(v,S) \right)$

- For any $\varepsilon > 0$ it is not $(2-\varepsilon)$ approximable unless $NP = P$

Proof idea: reduction from Dominating Set (is there a set with K vertices so that each vertex is either in the set or is adjacent to 1 in the set?)

Exercise: STABLE SET reduces to it



Any $2-\varepsilon$ approximation algorithm on G' can decide whether there are K centers with distance 1 or only 2

This means deciding whether the same K-element set will work as K-dominating set in G.

(2) MAX-3-SAT - find an assignment that maximizes the number of satisfied clauses

$(\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \ldots$

NP-hard - reduces to satisfiability

T/F at random, for each clause $P(\text{satisfied}) = \frac{7}{8}$

$E(\#\text{ of satisfied clauses}) = \frac{7k}{8}$

$\exists$ assignment $\geq \frac{7}{8}k$ true clauses

$k \cdot P(\text{random assignment satisfies} \geq \frac{7k}{8} \text{ clauses}) =$

$$k \cdot P(\geq \tfrac{7}{8}k \text{ are satisfied}) \geq \sum_{j \geq \frac{7}{8}k} \underbrace{P(j \text{ clauses are satisfied})}_{p_j} = \frac{7}{8}k - \sum_{j < \frac{7k}{8}} j\, p_j \geq \frac{7k}{8} - \left(\frac{7k}{8} - \frac{1}{8}\right) \cdot 1$$

$$\geq \frac{1}{8}$$

(Thm) [Hastad, 97] No better approximation is possible unless P=NP

(3) MAX CUT $\rightarrow$ both rely on

PCP - probabilistically checkable proofs (accepts correct proofs and rejects incorrect proofs with high probability)

(Thm) $NP = PCP[O(\log n), O(1)]$

probabilistic verifiers in addition take a sequence of $O(\log n)$ random bits

can be simulated by deterministic algorithm at all $2^{\log n} \sim n$ possible random inputs R