<u>Totally unimodular matrices</u> $\leftarrow$ all determinants of minors are $\{\pm 1, 0\}$

Why? Create integral polytopes

Applications for many CS algorithms beyond König's theorem

<u>Example</u>: Binary matrix completion $\rightarrow$ integer linear program $\rightarrow$ LP

Binary Rank-1 approximation of a binary matrix $A$:

$$\sum_{ij} |(A - x_1 x_2^T)_{ij}| = \sum (A_{ij} - x_{1i} x_{2j})^2$$

$$= \sum (A_{ij}^2 - 2A_{ij} x_{1i} x_{2j} + x_{1i}^2 x_{2j}^2)$$

$$= \sum A_{ij}^2 - \sum x_{1i} (2A_{ij} - 1) x_{2j}$$

$$= \|A\|_F^2 - x_1^T W x_2$$

$$\left[\begin{array}{lll} \max \sum_{ij} W_{ij} z_{ij} & & \\ -x_{1i} - x_{2j} + 2z_{ij} \leq 0 & W_{ij} = 1 & \\ x_{1i} + x_{2j} - z_{ij} \leq 1 & W_{ij} = -1 & \\ x_{1i}, x_{2j}, z_{ij} \in \{0, 1\} & & \end{array}\right]$$

$A_{ij} = 1$

$A_{ij} = 0$

($z_{ij}$ is binary & $z_{ij} = 1 \iff x_{1i}, x_{2j} = 1$)

By total unimodularity, this can be relaxed to $x_{1i}, x_{2j}, z_{ij} \in [0, 1]$

[Shen, Ji, Ye, '09]

Can be extended for matrices with missing patterns.

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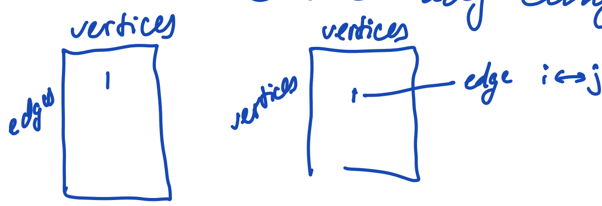
<u>How to test for TUM?</u>

- Incidence matrices of unoriented graphs are TUM only if graphs are bipartite
- In general, there are many graph-based criterions

[Schrijver "Theory of linear and integer programming"]
Chapters 19 - 21 "Recognizing total unimodularity"

e.g. for a 0-1 matrix, if there exists a permutation for every row, so that 1's appear consequitively, then it is TUM

- Note the distinction between incidence and adjacency matrices!




e.g. [Akbari, Kirkland, '07]
"On unimodular graphs"

- (Random?) Graph model $\longrightarrow$ Combinatorial graph properties $\longrightarrow$ TUM property

?
this is hard

**Singularity of random Bernoulli matrices**

$M_n \sim \pm 1$ independent entries with prob $p$.

$\mathbb{P}(M_n \text{ is singular}) = \left(\frac{1}{2} + \bar{o}_n(1)\right)^n$ ⊛

Linear independence of $n$ independent vectors sampled uniformly from a unit cube $\{\pm 1\}^n$ ?

- Komlós (67) have shown $\mathbb{P}(M_n \text{ is singular}) = o_n(1)$

⋮

- Tikhomirov (2019) got ⊛ which is exact (probability that two rows will coinside)