Modewise methods for tensor dimension reduction

Liza Rebrova

Department of Mathematics
UCLA

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Joint work with Mark Iwen, Deanna Needell, and Ali Zare

Tensors and Kronecker/outer products

$$\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \ldots \times n_d} - d\text{-way tensor}$$

(for simplicity, in this talk, let's assume all $n_i = n$)

Rank 1 matrix can be defined as $\mathbf{x} \otimes \mathbf{y}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

$$\mathbf{x} \otimes \mathbf{y} = \begin{bmatrix} \mathbf{x}(1)\mathbf{y}(1) & \ldots & \mathbf{x}(1)\mathbf{y}(n) \\ \mathbf{x}(2)\mathbf{y}(1) & \ldots & \mathbf{x}(2)\mathbf{y}(n) \\ \ldots & \ldots & \ldots \\ \mathbf{x}(n)\mathbf{y}(1) & \ldots & \mathbf{x}(n)\mathbf{y}(n) \end{bmatrix} = \begin{bmatrix} \mathbf{x}(1) \\ \mathbf{x}(2) \\ \ldots \\ \mathbf{x}(n) \end{bmatrix} \circ \begin{bmatrix} \mathbf{y}(1) & \ldots & \mathbf{y}(n) \end{bmatrix}$$

By analogy, we define rank 1 tensor as $\mathcal{X} := \mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_d$,

$$\mathcal{X}(i_1, \ldots, i_d) = \mathbf{x}_1(i_1)\mathbf{x}_2(i_2)\ldots\mathbf{x}_d(i_d).$$

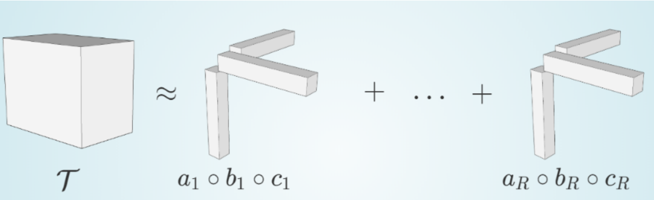
Tensor (CP) rank

(Candecomp-Parafac) rank r tensor as

$$\mathcal{X} = \sum_{i=1}^{r} \alpha_i \mathbf{x}_1^i \otimes \ldots \otimes \mathbf{x}_d^i$$

Normalization: we always assume $\|\mathbf{x}_j^i\|_2 = 1$. Clearly, $r \leq n^d$.

For example, for a 3-way (3 modes) tensor,



Picture is taken from Medium post "Tensor Factorization for Graph Analysis in Python" by Claudio Stamile

Fitting problem

For an arbitrary tensor $\mathcal{Y}$, find the closest rank r tensor $\mathcal{X}$:

$$\arg\min_{\mathcal{X}} \|\mathcal{X} - \mathcal{Y}\|^2$$

Tensor norm here is a generalization of the Frobenius matrix norm (sum of squares of all entries of the tensor)

This problem includes finding the best set of vectors $\{\mathbf{x}_j^i\}$ (basis) and the best set of coefficients $\{\alpha_i\}_{i=1}^r$:

$$\arg\min_{\mathcal{X}} \|\mathcal{X} - \mathcal{Y}\|^2 = \arg\min_{\mathbf{x}_j^i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}} \| \sum_{i=1}^r \alpha_i \bigotimes_{j=1}^d \mathbf{x}_j^i - \mathcal{Y}\|^2$$

Solving the fitting problem

Idea:

- Start with random basis for $\mathcal{X}$: take random unit vectors $\mathbf{x}_j^i \in \mathbb{R}^n$ for $j = 1, \ldots, d$, $i = 1, \ldots, r$
- Fix all but one mode $j \in [d]$, namely, $\mathbf{x}_j^1, \ldots, \mathbf{x}_j^r$
- Optimize over j-th mode
- Repeat for the other modes until some error threshold

This turns out to be equivalent to solving n_j separate problems of the form:

Find

$$\operatorname*{arg\,min}_{\alpha_1,\ldots,\alpha_r \in \mathbb{R}} \| \sum_{i=1}^{r} \alpha_i \bigotimes_{j=1 \neq j'}^{d} \mathbf{x}_j^i - \mathcal{Y}' \|^2$$

That is, looking for the best fit in some fixed basis

Dimension reduction for the fitting problem

Goal: reduce the size of this problem.

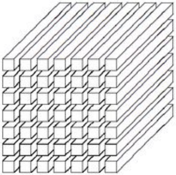
Preferably,

- in a subspace oblivious way (to have the same simple operation for the multiple applications in various bases)
  For example, classical dimension reduction lemma

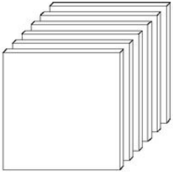
Lemma (Johnson-Lindenstrauss)

Take small $\eta > 0$. Random projection from $\mathbb{R}^n \to \mathbb{R}^m$ ε-preserves distances between $e^{c(\eta)\varepsilon^2 m}$ points with probability $1 - \eta$.

- without vectorization of the tensors



(c) Mode-3 (tube) fibers: $\mathbf{x}_{ij:}$



(c) Frontal slices: $\mathbf{X}_{::k}$ (or $\mathbf{X}_k$)

Picture is taken from Kolda&Bader paper

Modewise products: tensor $\times_j$ matrix

Definition (j-mode product)

A tensor $\mathcal{X} \in \mathbb{R}^{n^d}$ can be multiplied by a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ to get a tensor $(\mathcal{X} \times_j \mathbf{A}) \in \mathbb{R}^{n \times \ldots \times m \times \ldots \times n}$ with the coordinates

$$(\mathcal{X} \times_j \mathbf{A})(\ldots, i_{j-1}, \ell, i_{j+1}, \ldots) = \sum_{i_j=1}^{n} \mathbf{A}(\ell, i_j) \mathcal{X}(\ldots, i_j, \ldots).$$

for any $j = 1, \ldots, n$.

For example, for a 2 way tensor (matrix)

$$\mathcal{X} \times_1 \mathbf{A}_1 \times_2 \mathbf{A}_2 = \mathbf{A}_1 \mathcal{X} \mathbf{A}_2^T$$

For the CP representation, it is equivalent to

$$\mathcal{X} \times_1 \mathbf{A}_1 \times_2 \mathbf{A}_2 \ldots \times_d \mathbf{A}_d = \sum_{i=1}^{r} \alpha_i (\mathbf{A}_1 \mathbf{x}_1^i) \otimes \ldots \otimes (\mathbf{A}_d \mathbf{x}_d^i)$$

So, instead of

Fitting problem: $\|\mathcal{X} - \mathcal{Y}\|^2 \to \min$

$$\operatorname*{arg\,min}_{\alpha_1,\dots,\alpha_r \in \mathbb{R}} \| \sum_{i=1}^{r} \alpha_i \bigotimes_{j=1 \neq j'}^{d} \mathbf{x}_j^i - \mathcal{Y} \|^2$$

let us find

Reduced fitting problem: $\|\mathcal{X} \times_{j=1}^{d} \mathbf{A}_j - \mathcal{Y} \times_{j=1}^{d} \mathbf{A}_j\|^2 \to \min$

$$\operatorname*{arg\,min}_{\alpha_1,\dots,\alpha_r \in \mathbb{R}} \| \sum_{i=1}^{r} \alpha_i \bigotimes_{j=1}^{d} \mathbf{A}_j \mathbf{x}_j^i - \mathcal{Y} \bigtimes_{j=1}^{d} \mathbf{A}_j \|^2$$

Will it find us a good solution for the original (non-reduced) problem?

Subspace oblivious dimension reduction for tensors

For now: let $\mathcal{Y} = 0$.

We want

$$\left| \|\mathcal{X}\|^2 - \|\mathcal{X} \bigtimes_{j=1 \neq j}^{d} \mathbf{A}_j\|^2 \right| \leq \varepsilon \|\mathcal{X}\|^2$$

for any low r-rank tensor $\mathcal{X}$ from a fixed CP subspace, and for $m \times n$ matrices $\mathbf{A}_j$'s taken from some general (subspace oblivious!) model.

Johnson-Lindenstrauss embeddings

We are going to consider matrices $\mathbf{A}_j$ such that

Definition (η-optimal family of JL embeddins)

A $m \times n$ matrix $\mathbf{A}$ is an η-optimal JL embedding if for any $\varepsilon \in (0,1)$ and $\mathcal{S} \subset \mathbb{R}^n$ of cardinality $|\mathcal{S}| \leq \eta e^{\varepsilon^2 m/C}$,

$$\left| \|\mathbf{A}\mathbf{x}\|_2^2 - \|\mathbf{x}\|_2^2 \right| \leq \varepsilon \|\mathbf{x}\|_2^2 \text{ for any } \mathbf{x} \in \mathcal{S}$$

with probability at least $1-\eta$.

Gaussian, Fourier matrices, random projection matrices (to a subspace uniformly selected from the Grassmanian) ...

Definition is inspired by Johnson-Lindenstrauss Lemma:
for any small $\eta > 0$, random projection from $\mathbb{R}^n \to \mathbb{R}^m$ ε-preserves distances between $e^{c(\eta)\varepsilon^2 m}$ points with probability $1-\eta$.

Main theorem -1

Theorem (Iwen-Needell-R.-Zare)

Let $\mathcal{L}$ be an r-dimensional subspace of $\mathbb{R}^{n^d}$ spanned by a basis $\mathcal{B} := \left\{ \bigcirc_{\ell=1}^{d} \mathbf{x}_k^{(\ell)} \mid k \in [r] \right\}$. If all $\mathbf{A}_j \in \mathbb{R}^{m \times n}$ from an (η/d)-optimal family of JL embeddings, $m \gtrsim \varepsilon^{-2} r^{2/d} d^2$, then with probability at least $1-\eta$

$$\left| \|\mathcal{X}\|^2 - \|\mathcal{X} \bigtimes_{j=1}^{d} \mathbf{A}_j\|^2 \right| \le \varepsilon \, \|\mathbf{a}\|_2^2 \, ,$$

for all $\mathcal{X} = \sum_{i=1}^{r} \alpha_i \mathbf{x}_1^i \otimes \ldots \otimes \mathbf{x}_d^i \in \mathcal{L}$.

Total number of entries $N = n^d \to M \sim \varepsilon^{-2d} r^2 d^{2d}$.

Main theorem-1

Theorem (Iwen-Needell-R.-Zare)

Let $\mathcal{L}$ be an r-dimensional subspace of $\mathbb{R}^{n^d}$ spanned by a basis $\mathcal{B} := \left\{ \bigcirc_{\ell=1}^{d} \mathbf{x}_k^{(\ell)} \right\}_{k\in[r]}$ with modewise coherence $\mu_{\mathcal{B}}^{d-1} < 1/2r$.
If all $\mathbf{A}_j \in \mathbb{R}^{m\times n}$ from an (η/d)-optimal family of JL embeddings with $m \gtrsim \varepsilon^{-2} r^{2/d} d^2$, then with probability at least $1-\eta$

$$\left| \|\mathcal{X}\|^2 - \|\mathcal{X} \bigtimes_{j=1}^{d} \mathbf{A}_j\|^2 \right| \le \varepsilon \|\mathcal{X}\|^2,$$

for all $\mathcal{X} \in \mathcal{L}$.

Total number of entries $N = n^d \to M \sim \varepsilon^{-2d} r^2 d^{2d}$.

Modewise (in)coherence

$$\mu_{\mathcal{B}} := \max_{\ell \in [d]} \max_{\substack{k,h \in [r] \\ k \neq h}} \left| \left\langle \mathbf{x}_k^{(\ell)},\ \mathbf{x}_h^{(\ell)} \right\rangle \right|,$$

- measures angles between all basis vectors (from the same subspaces)
- orthogonal bases have coherence zero
- random (sub)gaussian tensors are incoherent enough with exponentially high probability:

Lemma

If all components of all vectors $\mathbf{x}_k^{(j)}$ are normalized independent mean zero K-subgaussian random variables, with probability at least $1 - 2r^2 d \exp\left(-c\mu^2 n\right)$ maximum modewise coherence parameter of the tensor $\mathcal{X}$ is at most μ.

Theorem 2: Fitting an arbitrary $\mathcal{X}$

Theorem (Iwen-Needell-R.-Zare)

Let $\mathcal{L}$ be an r-dimensional subspace of $\mathbb{R}^{n^d}$ spanned by a basis $\mathcal{B} := \left\{ \bigcirc_{\ell=1}^{d} \mathbf{x}_k^{(\ell)} \right\}_{k \in [r]}$ with $\mu_{\mathcal{B}}^{d-1} < 1/2r$ and $\mathcal{Y} \notin \mathcal{L}$.
If all $\mathbf{A}_j \in \mathbb{R}^{m \times n}$ are from an (η/d)-optimal family of JL embeddings with $m \gtrsim \varepsilon^{-2} r d^3$, then with probability at least $1 - \eta$

$$\left| \|\mathcal{Y} - \mathcal{X}\|^2 - \|(\mathcal{Y} - \mathcal{X}) \bigtimes_{j=1}^{d} \mathbf{A}_j\|^2 \right| \leq \varepsilon \|\mathcal{Y}\|^2 ,$$

for all $\mathcal{X} \in \mathcal{L}$.

Total number of entries $N = n^d \to M \sim \varepsilon^{-2d} r^d d^{3d}$.
Reason: we need to additionally compress a subspace spanned by $\{P_{\mathcal{L}^\perp}(\mathcal{Y}) \pm \mathcal{B}\}$, this basis is NOT low rank.

Can we do better?

Is our dependence on r and on ε (and on d) good?

Lemma (Larsen, Nelson, 2016)

For any $n, d \geq 2$, there exists a set of n vectors in $\mathbb{R}^d$ so that any linear map $\mathbb{R}^d \to \mathbb{R}^m$, ε-preserving distances between them, must have

$$m \gtrsim \varepsilon^{-2} \ln n.$$

Modewise Fourier JL for a finite set

Consider a special modewise operator:

$$L_{\mathrm{FJL}}(\mathcal{X}) \;:=\; \mathbf{R}\left(\mathrm{vect}\left(\mathcal{X} \times_1 \mathbf{F}_1\mathbf{D}_1 \cdots \times_d \mathbf{F}_d\mathbf{D}_d\right)\right),$$

$\mathrm{vect} : \mathbb{R}^{n\times\cdots\times n} \to \mathbb{R}^{n^d}$ is the vectorization operator,
$\mathbf{R}$ is a matrix containing m random rows from $Id_{n^d \times n^d}$,
$\mathbf{F}_i \in \mathbb{R}^{n\times n}$ is a unitary discrete Fourier transform matrix,
$\mathbf{D}_i \in \mathbb{R}^{n\times n}$ is a diagonal matrix with n random ± 1 entries.

Theorem (Yin, Kolda, Ward, 2019)

Let $\eta \gtrsim n^{-d}$. Consider $\mathcal{S} \subset \mathbb{R}^{n^d}$ of cardinality $|\mathcal{S}| = p$. Then with probability at least $1-\eta$ the linear operator L_{FJL} is an ε-JL embedding of $\mathcal{S}$ into $\mathbb{R}^m$, where

$$m \;\gtrsim\; \varepsilon^{-2} \cdot \log^{2d-1}\left(\frac{\max(p, n^d)}{\eta}\right) \cdot \log n^d.$$

Moreover, if $d = 1$, then we may replace $\max(p, n^1)$ with p.

Yin–Kolda–Ward Theorem/Theorem 2:

Let us compare these two modewise JL-type embedding results:

- For a fixed finite set S / for a fixed subset $\mathcal{L}$
- Special Fourier modewise transform /large class of JL-type modewise maps
- $m \gtrsim \varepsilon^{-2}$ / $m \gtrsim \varepsilon^{-2d}$
- for any subset of tensors / only for incoherent bases

Idea: using Yin–Kolda–Ward Theorem to improve ε-dependence and to get rid of the incoherence assumption

How can this help with subspace embeddings?

Two ways to apply JL-type results to a low r-dimensional subspace (it is enough to approximate unit norm tensors only!):

- To an ε-net on $\mathcal{S}^{r-1}$:

Lemma (JL discretization)

Fix $\varepsilon \in (0,1)$. Let $\mathcal{L}$ be an r-dimensional subspace of $\mathbb{R}^n$, and let $\mathcal{N} \subset \mathcal{L}$ be an $(\epsilon/16)$-net of the unit sphere $\mathcal{S}^{r-1} \subset \mathcal{L}$. Then, if $\mathbf{A} \in \mathbb{R}^{m \times n}$ is an $(\varepsilon/2)$-JL embedding of $\mathcal{N}$ it will also satisfy

$$(1-\varepsilon)\|\mathbf{x}\|_2^2 \leq \|\mathbf{A}\mathbf{x}\|_2^2 \leq (1+\varepsilon)\|\mathbf{x}\|_2^2 \text{ for all } \mathbf{x} \in \mathcal{L}.$$

There exists an $(\varepsilon/16)$-net such that $|\mathcal{N}| \leq \left(\frac{47}{\varepsilon}\right)^r$.

- To a set of r basis vectors:
  Recall Theorem 1 above

Using Yin–Kolda–Ward theorem: wrong way

1. Apply Yin–Kolda–Ward Theorem to the approximation net $S = \mathcal{N}$ of cardinality $\left(\frac{47}{\varepsilon}\right)^r$
2. Use JL Discretization Lemma

Resulting dimension is at least

$$m \gtrsim \varepsilon^{-2} r^{2d-1} \cdot \log^{2d-1}\left(\frac{47}{\eta^{1/r}\varepsilon}\right) \cdot \log n^d.$$

So, ε dependence improves, but dependence on the rank even become worse: r^{2d-1} instead of r^d (Theorem 2)

Using Yin–Kolda–Ward theorem: right way

1. Apply Yin–Kolda–Ward Theorem to the set of r basis vectors
2. Proceed like we did for Theorem 2 to get the estimate for all others

Resulting dimension (since $r < n^d$):

$$m \gtrsim \varepsilon^{-2} r^2 \cdot \log^{2d-1}\left(\frac{n^d}{\eta}\right) \cdot \log n^d.$$

Much better! :)
Still quadratic dependence on rank...

Improved two step dimension reduction

Let us vectorize the result of Step 2 to get a vector (tensor with $d = 1$) in $\mathbb{R}^m$,

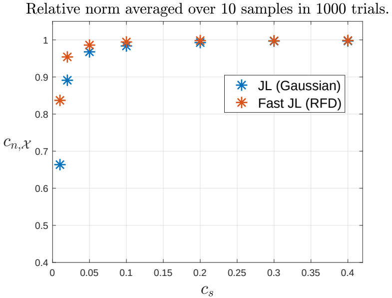
3. Now, apply Yin–Kolda–Ward Theorem to the approximation net $S = \mathcal{N}$ of cardinality $\left(\frac{47}{\varepsilon}\right)^r$ in $\mathbb{R}^m$

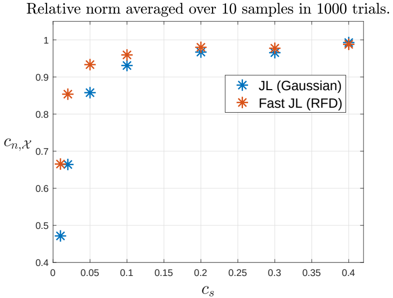
to get

$$\tilde{m} \gtrsim \varepsilon^{-2} r \cdot \log\left(\frac{47}{\varepsilon \eta^{1/r}}\right) \cdot \log m.$$

Optimal dependence on both ε and r! (and a bit of logarithmic multiples...)

Experiments-1





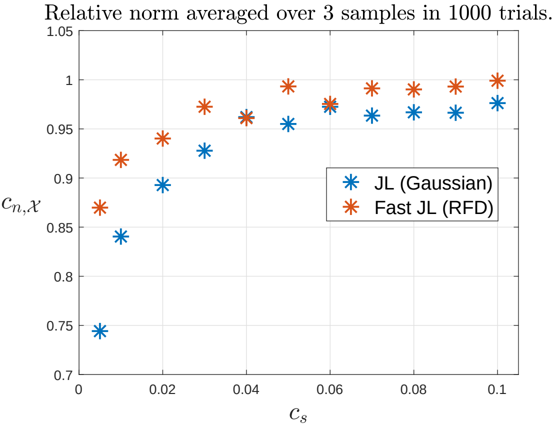
$c_s = m/n$ – compression ratio

$c_{n,\mathcal{X}} = \|\mathcal{X} \times_1 \mathbf{A_1} \dots \times_d \mathbf{A_d}\| / \|\mathcal{X}\|$ – relative norm

Both data sets contain 10 tensors with $d = 4$, $r = 10$, $n = 100$

Coherent tensors constructed as $1 + \sqrt{0.1} \cdot g$, $g \sim N(0, 1)$

Experiments-2

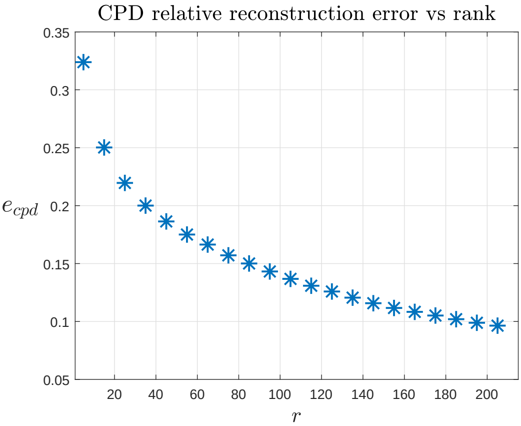


The same for MRI data: three 3-mode MRI images of size

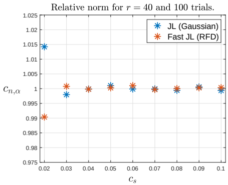
$240 \times 240 \times 155$

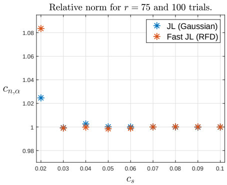
What was the rank r?

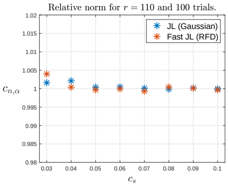
Experiments-3



Experiments-4







Some future directions

- Remove theoretical incoherence assumption in Theorem 2 (which is still the most general model for modewise compression!)
- Give JL-type guarantees for all CP-rank r tensors with high probability: get RIP (restricted isometry property) type results.

Thanks for your attention!

QUESTIONS?