From linear to semidefinite programming: classes of convex optimization problems.

1. Linear programs (LP) [ORF 522]

$$\begin{bmatrix} \min & c^T x \\ \text{s.t.} & Ax \le b \\ & (x \ge 0) \end{bmatrix}$$

$A \in \mathbb{R}^{m \times n}$
$b \in \mathbb{R}^m$

2. Quadratic programs (QP)

$$\begin{bmatrix} \min & x^T A_0 x + b_0^T x + c_0 \\ \text{s.t.} & A_q x \le b_q \end{bmatrix}$$

$A_0 \in \mathbb{R}^{n \times n}$ $\quad A_q \in \mathbb{R}^{m \times n}$
$b_0 \in \mathbb{R}^n$ $\quad b_q \in \mathbb{R}^m$
$c_0 \in \mathbb{R}$

<u>Convex</u> quadratic program, if $A_1 \succeq 0$. (Exampe: SVM)

3. Quadratically constrained QP (QCQP)

$$\begin{bmatrix} \min & x^T A_0 x + b_0^T x + c_0 \\ \text{s.t.} & x^T A_i x + b_i^T x + c_i \le 0 \end{bmatrix}$$

$A_i \in \mathbb{R}^{n \times n}$
$b_i \in \mathbb{R}^n$
$c_i \in \mathbb{R}$

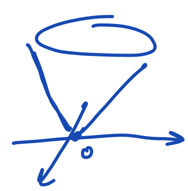
Convex if all the matrices are PSD symmetric

4. SOCP: Second order cone programs

$$\begin{bmatrix} \min_x & f^T x \\ & \|A_i x + b_i\|_2 \le c_i^T x + d_i \end{bmatrix} \quad i = 1, \dots m$$

Lorentz cone

$$L^{n+1} = \{(x, t) : \|x\|_2 \le t\}$$



<u>Example:</u> LASSO with block sparsity

$$\min_x \|Ax - y\|_2$$

$x = (x_1, \dots x_p)^T$
$x_i \in \mathbb{R}^{n_i}$
$\sum n_i = n$

We want as many blocks x_i's to be zero as possible.

$$\left[\min_{x} \|Ax-y\|_2 + \gamma \sum_{i=1}^{p} \|x_i\|_2\right]$$

ℓ_1-norm of the vector $\begin{pmatrix}\|x_1\|_2 \\ \vdots \\ \|x_p\|_2\end{pmatrix}$

$\Updownarrow$

$$\left[\begin{array}{l} \min\limits_{x,z,t} \; z + \gamma \sum_{i=1}^{p} t_i \\ \|Ax-y\|_2 \le z \\ \|x_i\|_2 \le t_i \end{array}\right]$$

Block-sparsified lasso appears, e.g. in model selection problem

5. SDP : Semidefinite programming

$$\left[\begin{array}{l} \min\limits_{X \in Sym^{n\times n}} \mathrm{Tr}(CX) \\ \text{s.t. } \mathrm{Tr}(A_i X) = b_i \\ X \succeq 0 \end{array}\right] \leftrightarrow \left[\begin{array}{l} \min\limits_{X \in S^{n\times n}} \sum_{kl} C_{kl} X_{kl} \\ \sum_{k,l} (A_i)_{kl} X_{kl} = b_i \\ X \succeq 0 \end{array}\right]$$

So Traces mean there are affine constraints on the individual entries of X

The only "matrix condition" is $X \succeq 0$

6. CP: Conic program

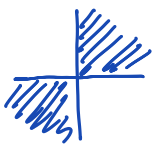
$$\left[\begin{array}{l} \min f(x) \\ \text{s.t. } Ax \le b \\ x \in C \end{array}\right]$$

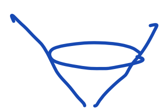
f is convex

C is a convex cone

convex set

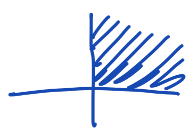
$\forall x \quad \alpha x \in X$, $\alpha > 0$

Examples: • Non-convex cone 

• Not a conv hull of a finite set 

• Convex cones: • PSD matrices

• 2nd order cone $\{\|x\|_2 \le t\}$

• $\mathbb{R}^n_+$ 

$$\boxed{LP \subset QP \subset QCQP \subset SOCP \subset SDP \subset CP}$$