Lemma Finite set convex cone is a closed set

$C = \text{cone}\{a_1, \dots a_m\}$

Proof: [Assuming that $a_1 \dots a_n$ are linearly independent]

$x^k \in C \to x$ Then x must be in C - checking this

$$x^k = \sum_{i=1}^m \lambda_i^k a_i \to x \qquad \lambda_i^k \geq 0$$

Since finite subspaces of $\mathbb{R}^n$ are closed $x = \sum_{i=1}^m \lambda_i a_i$

To show that $\lambda_i \geq 0$, we can show that

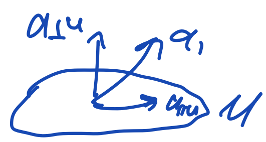
$\lambda_i^k \to \lambda_i$ for any i. Wlog, take $i=1$

$$\|x^k - x\|^2 = \|\sum_{i=1}^m \lambda_i^k a_i - \sum_{i=1}^m \lambda_i a_i\|^2 =$$

$$= \|(\lambda_1^k - \lambda_1) a_1 + \underbrace{\sum_{i=2}^m (\lambda_i^k - \lambda_i) a_i}_{\in \mathcal{U}}\|^2$$

$\mathcal{U} = \text{span}\{a_2 \dots a_m\}$

$a_1 = a_{\|\mathcal{U}} + a_{\perp \mathcal{U}}$



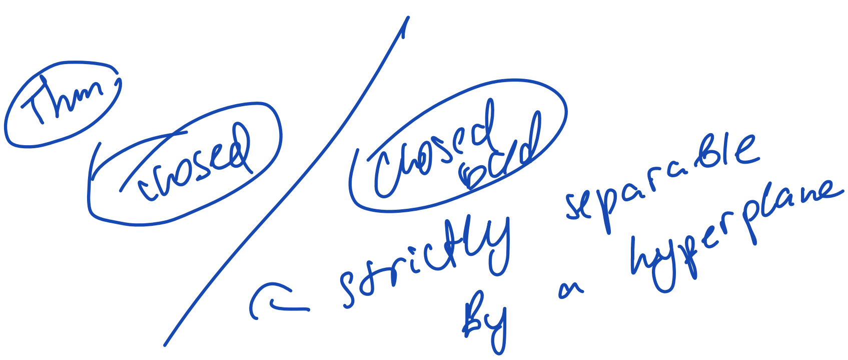
$$= \|(\lambda_1^k - \lambda_1) a_{\perp \mathcal{U}} + \tilde{a}\|^2 = (\lambda_1^k - \lambda_1)^2 \|a_{\perp \mathcal{U}}\|^2 + \|\tilde{a}\|^2$$

$\tilde{a} \in \mathcal{U}$

orthogonal!

$$\geq (\lambda_1^k - \lambda_1)^2 \underbrace{\|a_{\perp \mathcal{U}}\|^2}_{>0}$$

So, if $\lambda_1^k \not\to \lambda_1$ then $x^k \not\to x$ Contradiction ◻



Farkas lemma and its convex analysis proof

$\exists x$:
$$\begin{cases} Ax = b \\ x \geq 0 \end{cases}$$

(strict) or

$\exists y$
$$\begin{cases} y^T A \leq 0 & ① \\ y^T b > 0 & ② \end{cases}$$

Pf $\Rightarrow$

$$Ax = b$$
$$y^T A x = y^T b \overset{②}{>} 0$$
$$\underbrace{\sum_i \underbrace{(y^T A)_i}_{\leq 0} \underbrace{x_i}_{\geq 0}}_{\leq 0} \leq 0 \quad ①$$

contradiction!

$\Leftarrow$ If there is no x: $Ax = b$, $x \geq 0$, then

$\{b\} \cap \text{cone}\{a_1, \dots a_n\} = \emptyset$

By separation thm (checked convexity closedeness of both and $\{b\}$ is bounded)

So any $z \in \text{cone}\{a_1, \dots a_n\}$

$$\exists y, r: \begin{cases} y^T z - r < 0 \\ y^T b - r > 0 \end{cases} \longrightarrow y^T A - \begin{pmatrix} r \\ \vdots \\ r \end{pmatrix} < 0$$

Why r can be taken 0?

$y^T z < r$ for $z = 0$ (always in the cone)
$\Rightarrow r \geq 0$

if we have $z_0 \in \text{Cone}$: $y^T z_0 > 0$

then $\forall \alpha > 0$: $\alpha z_0 \in \text{Cone}$

$$y^T(\alpha z_0) > 0 \xrightarrow[\alpha \to \infty]{} r \to \infty$$

This is impossible
(r is upper bounded by $y^T b$)

So, we can write
$$\begin{cases} y^T z \leq 0 & \forall z \in \text{Cone} \\ y^T b > 0 \end{cases}$$

Next part: Farkas lemma $\Leftrightarrow$ LP strong duality

(P) $\begin{bmatrix} \min c^T x \\ Ax = b \\ x \geq 0 \end{bmatrix}$ (D) $\begin{bmatrix} \max b^T y \\ A^T y \leq c \end{bmatrix}$

Weak duality: if x is any feasible solution of (P)
y is any feasible solution of (D)

$$c^T x \geq b^T y$$

Strong duality if (P) has an finite optimal value $\Rightarrow$ (D) also has an optimal value and they are the same

$$c^T x_* = b^T y_* = p_*$$

Farkas from strong duality:

(P) $\begin{bmatrix} \min 0 \\ \text{s.t. } Ax = b \\ x \geq 0 \end{bmatrix}$ (D) $\begin{bmatrix} \max y^T b \\ \text{s.t. } A^T y \leq 0 \end{bmatrix}$

(D) is always feasible, but it only has finite optimal solution if $y^T b \leq 0$ $\forall y$

(otherwise we can consider (αy) $\alpha \to \infty$ and max of (D) will be ∞)

Strong duality of LP from Farkas lemma

(+ weak duality)

If p^* is finite optimal value for (P), then

(weak duality) $p_* \geq y^T b$ for any y: $A^T y \leq c$

So, it is enough to show

$$\exists y \left\{ \begin{array}{l} A^T y \le c \\ p_* \le y^T b \end{array} \right\}$$

(this means that $p_* = y^T b$, y, p_* → opt solution for (D), → opt value)

$$\Updownarrow$$

$$\boxed{? \; \exists y \left\{ \begin{pmatrix} A^T \\ \cdots \\ -b^T \end{pmatrix} \cdot y \le \begin{pmatrix} c \\ -p_* \end{pmatrix} \right\}}$$

(Lemma) (Variant of Farkas lemma)

$$\exists y : \{ \tilde{A} y \le \tilde{b} \} \iff \nexists \lambda \ge 0 : \left\{ \begin{array}{l} \lambda^T \tilde{A} = 0 \\ \lambda^T \tilde{b} < 0 \end{array} \right\}$$

Proof of lemma:

$$\exists y : \{ \tilde{A} y \le \tilde{b} \}$$

$$\Updownarrow$$

$$\exists \tilde{y} : \{ \tilde{y} = (y^+, y^-, s) \ge 0 \mid \tilde{A}(y^+ - y^-) + s = \tilde{b} \}$$

$$\Updownarrow$$

$$\exists \tilde{y} \left\{ \tilde{y} \ge 0 \;\middle|\; (\tilde{A} \mid -\tilde{A} \mid I) \begin{pmatrix} y^+ \\ y^- \\ s \end{pmatrix} = \tilde{b} \right\}$$

$$\exists x \left\{ \begin{array}{l} Ax = b \\ x \ge 0 \end{array} \right\}$$

$$\Updownarrow$$

$$\left\{ \begin{array}{l} \nexists y \\ y^T A \ge 0 \\ y^T b < 0 \end{array} \right\}$$

$$\Updownarrow \quad \text{Farkas}$$

$$\nexists \lambda \left\{ \begin{array}{l} \lambda^T (\tilde{A} \mid -\tilde{A} \mid I) \ge 0 \\ \lambda^T \tilde{b} < 0 \end{array} \right\}$$

$$\begin{pmatrix} \tilde{A}^T \\ -\tilde{A}^T \\ I \end{pmatrix} \lambda \ge 0$$

$$\left. \begin{array}{l} \tilde{A}^T \lambda \ge 0 \\ -\tilde{A}^T \lambda \ge 0 \end{array} \right\} \Longrightarrow \tilde{A}^T \lambda = 0$$

$$I \lambda \ge 0$$

Enough to show that

$$\boxed{\nexists\, \lambda \geq 0 : \begin{cases} \lambda^T \begin{pmatrix} A^T \\ -b^T \end{pmatrix} = 0 \\ \lambda^T \cdot \begin{pmatrix} c \\ -p_* \end{pmatrix} < 0 \end{cases}}$$

$\lambda^T = (\lambda_v^T \;\; \lambda_c^T)$ $\quad m \quad \begin{pmatrix} A^T \\ -b^T \end{pmatrix} \; n$

$$\begin{cases} \lambda_v^T \cdot A^T - \lambda_c^T b^T = 0 \\ \lambda_v^T \cdot c - \lambda_c^T \cdot p_* < 0 \end{cases}$$

$$\nexists\, \lambda \geq 0 \quad \begin{cases} A\lambda_v = \lambda_c \cdot b & (\star_1) \\ c^T \lambda_v < \lambda_c \cdot p_* & (\star_2) \end{cases} \quad (\star)$$

Recall: p_* – optimal value for (P)

$$\begin{cases} p_* = c^T x_* \\ A x_* = b & (1) \\ x_* \geq 0 & (2) \end{cases}$$

We will construct $\tilde{x}_*$ that gives smaller value for (P) (if $\lambda \geq 0$ satisfying $\star$ exists)

Case 1 $\quad \tilde{x}_* := \dfrac{\lambda_v}{\lambda_c}$

$\lambda_c \neq 0$

(1) $A\tilde{x}_* = A\dfrac{\lambda_v}{\lambda_c} \overset{(\star_1)}{=} \dfrac{1}{\lambda_c} \cdot A\lambda_v = \dfrac{\lambda_c b}{\lambda_c} = b$

(2) $\dfrac{\lambda_v}{\lambda_c} \geq 0 \quad (\lambda \geq 0)$

$c^T \tilde{x}_* = c^T \dfrac{\lambda_v}{\lambda_c} \overset{(\star_2)}{<} \dfrac{\lambda_c p_*}{\lambda_c} = p_*$

So p_* cannot be optimal value

Case $\lambda_c = 0$ $\quad (\star)$ becomes $\begin{cases} A\lambda_v = 0 & (\star_1) \\ c^T \lambda_v < 0 & (\star_2) \end{cases}$

$\tilde{x}_* := x_* + \lambda_v$

(1) $A\tilde{x}_* = A(x_* + \lambda_v) \overset{(\star_1)}{=} Ax_* = b$

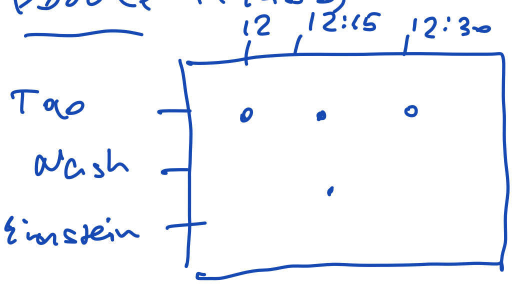
(2) $\tilde{x}_* \geq x_* \geq 0$

$c^T \tilde{x}_* = c^T(x_* + \lambda_v) = p_* + c^T \lambda_v \overset{(\star_2)}{<} p_*$

Next Application : strong duality of LP for combinatorial (graph) optimization

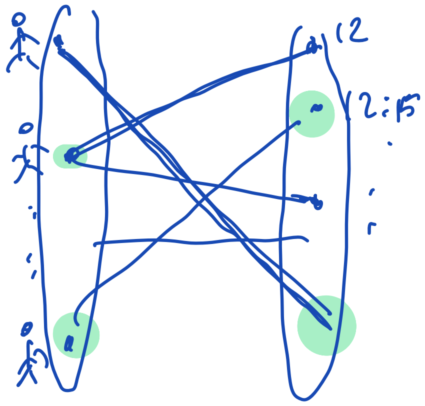
(and about relaxations)

Doodle (class 1)



Goal: to meet as many people as possible

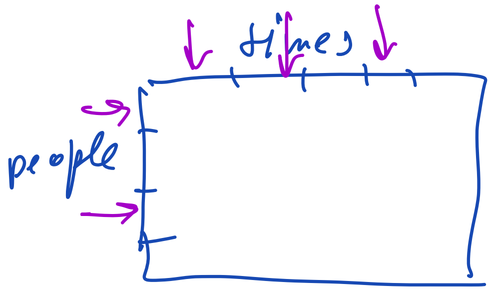
Can be represented as a bipartite graph where edges indicate availability.



people times

Goal: to select as many edges as possible so that each vertex used at most once

A matching



certificate: Selected rows and columns (arrows) so that each time slot touches one of them

Selecting a subset of vertices, so that every edge is connected to one of the selected vertices

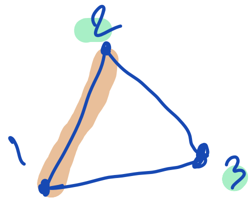
a vertex cover

Lemma $\boxed{\text{The cardinality of any matching} \leq \text{the cardinality of any vertex cover}}$

Thm (König) If G is bipartite then
$|\text{max matching}| = |\text{min vertex cover}|$

cardinality ($|X|$ = number of elements in the set X)

A counterexample for a not bipartite graph:



$|\text{min cover}| = 2$

$|\text{max matching}| = 1$

$$(M) = \left[\begin{array}{l} \max \sum_{i=1}^{E} e_i \\ \sum_{i \sim v} e_i \leq 1 \\ e_i = 0, 1 \quad \forall e_i \in E \leftarrow \text{set of edges} \end{array} \right]$$