Semidefinite programming.

$$\left[\begin{array}{l} \min_{X \in \mathrm{Sym}^{n\times n}} \operatorname{Tr}(CX) \\ \text{such that } \operatorname{Tr}(A_i X) = b_i \\ X \succeq 0 \end{array}\right]$$

1. Reductions

A_i and C can be assumed symmetric, since

$$\operatorname{Tr}(AX) = \operatorname{Tr}((AX)^T) = \operatorname{Tr}(XA^T) = \operatorname{Tr}(A^T X)$$

so $$\operatorname{Tr}\left(\frac{A+A^T}{2} X\right) = \operatorname{Tr}(AX)$$

2. Is it a convex optimization problem?

a) Feasible set is convex : • PSD matrices form a convex set

spectrahedron

• Intersection with the other convex set (affine constraints)

(can have infinitely many vertices)

b) Does not satisfy the definition of convex opt. problem:

• $X \succeq 0 \leftrightarrow y^T X y \geq 0$ is a convex constraint for every $y \in \mathbb{R}^n$

infinitely many convex conditions

• finitely many polynomial constraints (equivalent to the original problem) — from Sylvester criterion.

3. Existence of optimal value $\not\Rightarrow$ existence of optimal solution

Example: $\circledast$ $\left[\begin{array}{l} \min y \\ \frac{1}{x} \le y \\ x, y \geq 0 \end{array}\right]$

This is an SDP problem

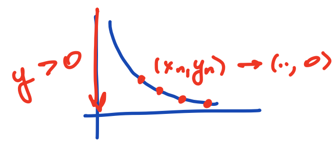
• $\left\{\begin{array}{l} xy \geq 1 \\ x,y \geq 0 \end{array}\right\} \Leftrightarrow \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \succeq 0$ (Sylvester)

• $\operatorname{Tr}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix}\right) = \operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 1 & y \end{pmatrix} = y$

• $\operatorname{Tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & z \\ z & y \end{pmatrix}\right) = 2z = 2$

$y_n \to 0$ as $x_n \to 0$; 0 - optimal value that is not attained at a feasible point

Convexity of the constraint set is not enough



there is no 'closeness' in $\mathrm{Tr}(CX) = y$ $(0, \infty)$

Compare with LP:
$$\begin{bmatrix} \min c^T x \\ Ax \le b \\ x \ge 0 \end{bmatrix}$$
if $c^T x_n \to p_*$

$\exists x: \quad p_* = c^T x$ The set $\{c^T x \mid x \text{ is feasible}\}$ is closed

4. Duality

Let p_* be an optimal value

$$L(X, \lambda, \mu) = \mathrm{Tr}(CX) + \sum \lambda_i (b_i - \mathrm{Tr}(A_i X)) - \mathrm{Tr}(X \cdot \mu)$$ - Lagrangian

$$D(\lambda, \mu) = \min_X L(X, \lambda, \mu)$$ - dual function

Lemma $\forall \lambda, \mu \succeq 0 \quad D(\lambda, \mu) \le p^*$

Note: we prove it for the case when optimal solution x^* exists

(Pf)
$$L(X^*, \lambda, \mu) = \mathrm{Tr}(CX^*) - \mathrm{Tr}(X^* \mu) = p^* - \mathrm{Tr}(X^* \mu) \le p$$
[since $B, A \succeq 0 \Rightarrow \mathrm{Tr}(AB) \ge 0$ (HW 1)]

So, $\min L \le p^*$ as well ∎

Dual problem should provide the best lower bound for p_*.

Idea: maximize $D(\lambda, \mu)$

$$\begin{bmatrix} \max D(\lambda, \mu) \\ \text{s.t. } \mu \succeq 0 \end{bmatrix}$$

$$D(\lambda, \mu) = \min L(X, \lambda, \mu) = \begin{cases} \lambda^T b & \text{if } C - \sum_i \lambda_i A_i - \mu = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Solving for $C - \sum \lambda_i A_i - \mu = 0$ for some $\mu \succeq 0$ we have

$$\boxed{C \succeq \sum_i \lambda_i A_i}$$

Dual problem:

$$\left[\begin{array}{ll} \max\limits_{\lambda \in \mathbb{R}^m} & b^T\lambda \\ \text{s.t.} & \sum_i \lambda_i A_i \preceq C \end{array}\right]$$

$$A_0 + \underset{\substack{\uparrow \\ \text{decision variables}}}{\lambda_1} A_1 + \dots + \lambda_n A_n \succeq 0$$

Linear Matrix Inequality (LMI)

Duality theorems:

Weak duality: Let X is feasible to SDP-primal, λ - feasible for SDP-dual. Then $\mathrm{Tr}(CX) \geq b^T\lambda$

Strong duality ✗

Examples: • only one of 2 optimal solutions are achieved

• there is a gap $\mathrm{Tr}(CX^*) < b^T\lambda^*$

Recall $b = 2$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\left[\begin{array}{l} \max 2\lambda \\ \lambda A \preceq C \end{array}\right]$$

$$\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \preceq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\lambda \\ -\lambda & 1 \end{pmatrix} \succeq 0$$

$$0 - (\lambda^2) \geq 0 \to \underline{\lambda = 0}$$

Theorem: If primal and dual SDP are strictly feasible (exists positive definite feasible X) then optimal values are achieved and $\mathrm{Tr}(CX^*) = b^T\lambda^*$.

Remark: dual form can be convenient to formulate problems:

Examples

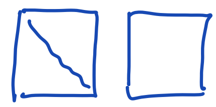
1) $\min y$: $\begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} y \succeq 0$

2) $\begin{pmatrix} 1+x & y \\ y & 1-x \end{pmatrix} \succeq 0$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \succeq 0 \iff$ $x \in [-1,1]$, $y \geq 0$, $x^2 + y^2 \leq 1$

6. LP and SOCP as special cases of SDP

LP

$$\left[\begin{array}{l} \min c^T x \\ a_i^T x = b_i \\ x \geq 0 \end{array}\right] \longleftrightarrow \left[\begin{array}{l} \min \operatorname{Tr}(\operatorname{diag}(c) X) \\ \operatorname{Tr}(\operatorname{diag}(a) X) = b_i \\ X \succeq 0 \end{array}\right]$$



non-diagonal entries do not participate in constraint and objective function

SOCP: we will show inclusion SOCP $\subset$ SDP

Lemma $X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ is a block matrix with $A \succ 0$

Then $X \succeq 0 \iff S := C - B^T A^{-1} B \succeq 0$

(Schur complement)

SOCP $\begin{cases} \min f^T x \\ \|A_i x + b_i\|_2 \leq c_i^T x + d_i \end{cases}$

Without loss of generality $c_i^T x + d_i > 0$

Then, $(c_i^T x + d_i)^2 \geq \|A_i x + b_i\|_2^2$

$\iff (c_i^T x + d_i) - (A_i x + b_i)^T \dfrac{1}{c_i^T x + d_i} (A_i x + b_i) \geq 0$

$\overset{\text{Lemma}}{\iff}$

$$\begin{pmatrix} (c_i^T x + d_i) I & (A_i x + b_i) \\ (A_i x + b_i)^T & (c_i^T x + d_i) \end{pmatrix} \succeq 0$$

(Proof of Lemma)

$f(u,v) = u^T A u + 2 v^T B^T u + v^T C v = \begin{pmatrix} u \\ v \end{pmatrix}^T X \begin{pmatrix} u \\ v \end{pmatrix}$

f is strictly convex in u.

$\frac{\partial f}{\partial u} = 2Au + 2Bv = 0 \quad u = -A^{-1}Bv \rightarrow$ gives global minimum in u.

$f_v^* = \min_u f = v^T (C - B^T A^{-1} B) v = v^T S v$

$\Rightarrow$ If $S \not\succeq 0$ $\quad v^T S v < 0$ implies $\min_u f < 0$ and $z^T X z < 0$ for $z = \begin{pmatrix} -A^{-1}Bv \\ v \end{pmatrix}$

$\Leftarrow \begin{pmatrix} u \\ v \end{pmatrix}^T X \begin{pmatrix} u \\ v \end{pmatrix} = f(u,v) \geq 0$ is S is PSD. ∎