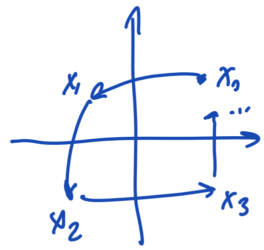
Applications of SDP - 2

Dynamical systems and stability (optimal control)

Discrete time

$x_{k+1} = f(x_k) \quad k = 0, 1, 2, \dots$

$x_k \in \mathbb{R}^n; \quad f: \mathbb{R}^n \to \mathbb{R}^n$

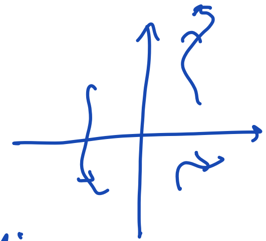


Continuous time

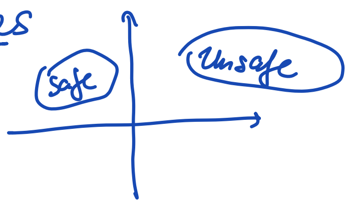
$\dot{x} = \frac{dx}{dt} = f(x)$

f is a vector field

$f: \mathbb{R}^n \to \mathbb{R}^n$



Examples



- Collision avoidance question:
  Is it true that $\forall x_0 \in S$
  $x(t, x_0) \notin \mathcal{U}$

- Stability question

DT: $f(\bar{x}) = \bar{x}$

Equilibrium point $\bar{x} \in \mathbb{R}^n$ CT: $\dot{x}(\bar{x}) = 0$ $\quad (f(\bar{x}) = 0)$

If we start close to $\bar{x}$, does the trajectory come to $\bar{x}$?

If we start anywhere, does the trajectory come to $\bar{x}$?

Turbulence of a plane, standing position of a robot, prices, complex systems...

2 questions: (1) is a current system stable? (2) Can we create a controller to make it stable?

$\boxed{\text{Def 1}}$ Locally asymptotically stable: (a) $\forall \varepsilon > 0 \ \exists \delta > 0 : x(0) \in B_\delta \Rightarrow x(t) \in B_\varepsilon \ \forall t$

(b) $\exists \alpha > 0 : x(0) \in B_\alpha \Rightarrow \lim_{t \to \infty} x(t) = 0$

Globally asymptotically stable: (a) same as above

(b) $\forall x_0 \in \mathbb{R}^n \ \lim_{t \to \infty} x(t) = 0$

Part 1:

STABILITY OF LINEAR SYSTEM $\boxed{y = Ax}$ linear system $A \in \mathbb{R}^{n \times n}$

For each $x_0 \in \mathbb{R}^n$, consider a sequence $\{x_n\}$: $x_n = Ax_{n-1}$

It is a dynamical system; it is GAS (Globally Asymptotically Stable) if $\{x_n\} \to 0$ for $\forall x_0 \in \mathbb{R}^n$

attractor of the system

a)
How can one characterize if the system is GAS?

① The system is GAS $\iff$ its spectral radius < 1 (all eigenvalues have norms < 1)

② $\boxed{\text{Thm 1} \quad A \text{ defines a GAS} \iff \exists P \in Sym(n): \ P \succ 0, \ P \succ A^T P A.}$

Note: search for such P is a semidefinite program

Proof ($\Leftarrow$) $V(x) := x^T P x$ Lyapunov function

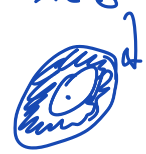
- $V(0) = 0$
- $V(x) > 0$ (P is a PD matrix) $x \neq 0$
- $V(Ax) < V(x)$ $\forall x \neq 0$ $\quad (P \succ A^T P A)$

$\Rightarrow$ for any $x_0 \in \mathbb{R}^n$ $\{V(x_n)\} \searrow, \ \geq 0 \Rightarrow \{V(x_n)\} \to c \geq 0$

If $c = 0$ then $V(x_n) \to 0$, then $x_n \in \{x \mid V(x_n) \leq t\}$ for some t and this is a compact set ($V(x)$ is radially unbounded), so $\{x_n\}$ has an accumulation point that must be 0 ($V(x) = 0$ at 0 only)

If $c > 0$, so all $x_n \in \{x \mid c \leq x^T P x \leq x_0^T P x_0\} =: S \Rightarrow x > c'' > 0$ for $x \in S$

We will show that $V(A^k(x))$ must decay below c



Indeed, $\delta(x) = V(x) - V(Ax)$

separated from 0

$$= x^T P x - x^T A^T P A x = x^T (P - A^T P A) x > c' > 0$$

positive definite

$\Rightarrow$

$$V(x) := \sum_{j=0}^{\infty} \|A^j x\|^2 = \sum_{j=0}^{\infty} x^T A^{j^T} A^j x = x^T \overbrace{\left(\sum_{j=0}^{\infty} A^{j^T} A^j \right)}^{=:P} x = x^T P x$$

- ≥ 0
- all eigenvalues < 1, so convergent

$$V(Ax) - V(x) = \sum_{j=1}^{\infty} \|A^j x\|^2 - \sum_{j=0}^{\infty} \|A^j x\|^2 = -\|x\|^2 < 0$$

So, $A^T P A \prec P$ ■

General thm:
$f: \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$ $\quad x_{k+1} = f(x_k)$
If $\exists$ a continuous Lyapunov function $V: \mathbb{R}^n \to \mathbb{R}$:
1. $V(0) = 0$, $V(x) > 0$ $\forall x \neq 0$ — positivity
2. $V(f(x)) < V(x)$ $\forall x \neq 0$ — decrease
3. if $\|x\| \to \infty \Rightarrow V(x) \to \infty$ — radially unbounded

Remark: <u>continuous time</u>:

$\dot{x} = Ax$ is GAS $\iff$ $\exists P \in Sym(n)$: $P \succ 0$, $A^T P + PA \prec 0$
$\Downarrow$
$V(x) := x^T P x$: $\dot{V}(x) = \langle \nabla V(x), \dot{x} \rangle < 0$ if $x \neq 0$

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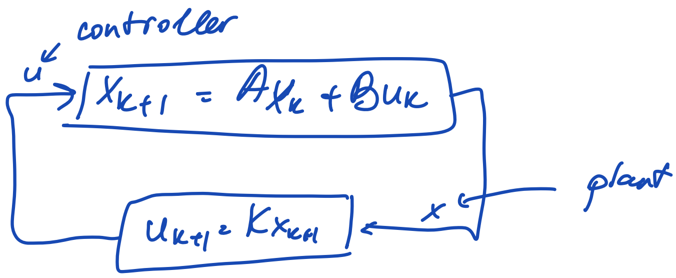
b) Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, does there exist $K \in \mathbb{R}^{k \times n}$: $A + BK$ is stable?
$(\rho(A+BK) < 1)$

We would like to design a linear controller $u = Kx$ to make a closed-loop system stable

$$x_{k+1} = Ax_k + Bu_k = Ax_k + B(Kx_k) = (A+BK)x_k$$



Control theory application

So, stability $\iff$ $\exists P \succ 0$ such that

$$\boxed{(A+BK)^T P (A+BK) \prec P}$$

This is <u>not</u> an SDP problem (not linear in K, P)

We will create a re-formulation of this problem that is SDP.

Lemma (Schur complement)

$X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ $\qquad S := C - B^T A^{-1} B$

- if $A \succ 0 \Rightarrow X \succeq 0 \iff S \succeq 0$. ← proved that
- $X \succ 0 \iff A \succ 0$ and $S \succ 0$. ← similar

Step1 $\quad A+BK$ is stable $\iff A^T+K^TB^T$ is stable

(E is stable $\iff E^T$ is stable, since they share the same eigenvalues)

Note:

$$\left[\begin{array}{c|c} P^{-1} & E \\ \hline E^T & P \end{array}\right] \succ 0 \iff P - E^TPE \succ 0 \qquad V(x) = x^TPx$$

$$\iff P^{-1} - E I E^T \succ 0 \qquad V(x) = x^TP^{-1}x$$

Step 2: $\quad P^{-1} - (A^T+K^TB^T)^T P^{-1} (A^T+K^TB^T) \succ 0$

$$\Updownarrow$$

$$\left[\begin{array}{c|c} P^{-1} & P^{-1}(A^T+K^TB^T) \\ \hline (A^T+K^TB^T)^T P^{-1} & P^{-1} \end{array}\right] \succ 0$$

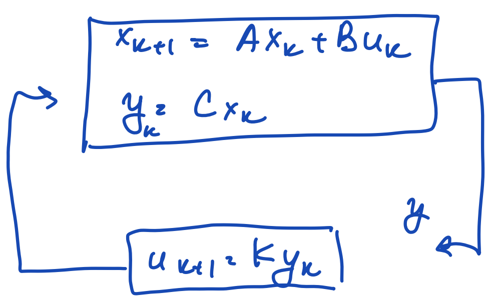
$$\Updownarrow \quad L = KP^{-1}$$

$$\left[\begin{array}{c|c} \tilde{P} & \tilde{P}A^T + L^TB^T \\ \hline A\tilde{P} + BL & \tilde{P} \end{array}\right] \succ 0$$

where $\tilde{P} = P^{-1}$

SDP in $\tilde{P}$ and L. We can find them and then solve for K and P.

c) Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{r\times n}$, is there a $K \in \mathbb{R}^{k\times r}$: $A+BKC$ is stable?



instead of x, the controller K "sees" it through $y=Cx$

stabilization with output feedback

- It is an open question in control theory whether this can be formulated as an SDP problem
- If the controller is bounded $k_{ij} \in [a_{ij}, b_{ij}]$, it is NP-hard

Without the condition, the complexity is unknown