# EX6

Ido Kahana

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**Definition 0.1 —**

- Let $D = \{1, \dots, n\}$ be the set of ads
- Let $P = \{1, \dots, m\}$ be the set of places for ads
- we are discussing a case when $n \geq m$
- Let $V_j \in \mathbb{R}$ be the click value for ads $j \in D$
- Let $\vec{V}$ be the vectors of click values for ads
- Let $r_{j,k}$ be the probability of ad $j \in D$ to be clicked when it is shown on place $k \in P$
- Let $A$ be the set of all legal assignments

  $$A = \{\vec{u} \in D^m \mid \forall_{i \in D, j \in D, i \neq j} \vec{u}_i \neq \vec{u}_j\}$$

  (no ads happened on two places)
- Let $O$ be the set of optimal assignments

  $$O = \{\vec{u} \in A \mid \vec{u} \in \arg\max_{u \in A} \sum_{i=1}^{n} V_i \cdot r_{i,\vec{u}_i}\}$$

- Let $AdAuction : \{\mathbb{R}^n, \mathbb{R}^m\} \longrightarrow A$ be the greedy algorithm from the class

# 1 Q4 section 1

**Theorem 1.1** — *Consider a case when* $r_{j,k} = q_j \cdot r_k$ *for some series of* $r_j, q_k$ *then the following algorithm finding optimal assignments*($\vec{u} \in O$)

**Algorithm 1** Find optimal assignments

**Input** $\vec{V}, r$
**Output** $\vec{u} \in O$

1: **for** $j \in D$ **do**
2: $\quad \overline{V}_j \longleftarrow V_j \cdot \sum_{k \in P} r_{j,k}$.
3: **end for**
4: **for** $k \in P$ **do**
5: $\quad t_k \longleftarrow \frac{\sum_{j \in D} r_{i,k}}{n}$
6: **return** $AdAuction(t, \overline{V})$

*Proof.* Note that the input for $AdAuction(t, \overline{V})$ is valid since ($t$ represent probabilities ,$n = |D|$)

$$\forall_{k \in P, j \in D} \quad 0 \leq r_{j,k} \leq 1 \Longrightarrow \forall_{k \in P} \quad 0 \leq \frac{\sum_{j \in D} r_{j,k}}{n} \leq 1$$

Consider

$$c_r = \sum_{k \in P} r_k, \quad c_q = \sum_{i \in D} q_i$$

Then

$$\forall_{i \in D} \sum_{k \in P} r_{i,k} = \sum_{k \in P} q_i \cdot r_k = q_i \cdot \sum_{k \in P} r_k = q_i \cdot c_r$$

$$\forall_{k \in P} \sum_{i \in D} r_{i,k} = \sum_{i \in D} r_k \cdot q_i = r_k \cdot \sum_{i \in D} q_i = r_k \cdot c_q$$

From the algorithm line 2 and 5 we know that

$$\overline{V}_i = V_i \cdot \sum_{k \in P} r_{i,k} = V_i \cdot q_i \cdot c_r, \quad t_{\vec{u}_i} = \frac{\sum_{j \in D} r_{j,\vec{u}_i}}{n} = \frac{r_{\vec{u}_i} \cdot c_q}{n}$$

Now let check the result of the algorithm

$$AdAuction(t, \overline{V}) \in \underset{\vec{u} \in A}{\arg\max} \sum_{j=1}^{n} \overline{V}_i \cdot t_{\vec{u}_i} = \underset{\vec{u} \in A}{\arg\max} \frac{\sum_{i=1}^{n} V_i \cdot q_i \cdot c_r \cdot r_{\vec{u}_i} \cdot c_q}{n} =$$

$$\underset{\vec{u} \in A}{\arg\max} \frac{c_r \cdot c_q}{n} \cdot \sum_{i=1}^{n} V_i \cdot q_i \cdot r_{\vec{u}_i} = \underset{\vec{u} \in A}{\arg\max} \sum_{i=1}^{n} V_i \cdot r_{i,\vec{u}_i} \Longrightarrow AdAuction(t, \overline{V}) \in O$$

and that proof the theorem. □

## 2 Q4 section 2

Ads 1: Don't click me if i am in 3 place(You will get a virus).
Ads 2: Don't click me if i am in 2 place(You will get a virus).
Ads 3: Don't click me if i am in 1 place(You will get a virus).

$$r_{j,k} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}_{j,k}, V_i = i, n = m = 3$$

The output will be $(3, 2, 1)$ and this will provide utility of

$$3 \cdot 0 + 2 \cdot 0 + 1 \cdot 0 = 0$$

however there is a better solution $(1, 2, 3)$ this will provide

$$3 \cdot 1 + 2 \cdot 1 + 1 \cdot 1 = 5$$

**Theorem 2.1** — *There is no $q, r$ series such that $r_{j,k} = q_j \cdot r_k$ for this matrix*

$$r_{j,k} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}_{j,k}$$

*Proof.* Assume in the negative that there is such $q, r$ then because

$$\exists_{j,k} \quad r_{j,k} = 0 \Longrightarrow \exists_j q_j = 0 \quad \vee \quad \exists_k r_k = 0$$

So one of the rows or one of the columns in $r_{j,k}$ must be Stuffed with zeros and that a contradiction. □