EX2

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October 2021

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Definition 1.1 — Let

$$P_C = \{(X_1, X_2) \mid X_1 \cup X_2 = C \wedge X_1 \cap X_2 = \varnothing\}$$

$$M_C = \{(X_1, X_2) \mid (X_1, X_2) \in P_C \wedge V_1(X_1) \geq \frac{V_1(C)}{2} \wedge V_2(X_2) \geq \frac{V_2(C)}{2}\}$$

$$S_C = \{(X_1, X_2) \mid (X_1, X_2) \in M_C \wedge V_1(X_1) + V_2(X_2) = \underset{(X_i, X_j) \in M_C}{\arg\max}\ V_1(X_i) + V_2(X_j)\}$$

Theorem 1.2 — *Section 1, Consider $(X_1, X_2) \in S_C$ then (X_1, X_2) is Proportional*

Proof. In order to satisfy Proportional we need to show that $\forall_i V_i(X_i) \geq \frac{V_i(C)}{n} = \frac{V_i(C)}{2}$.
By the definition of $S_C \subseteq M_C$ we know that

$$(X_1, X_2) \in M_C \implies V_1(X_1) \geq \frac{V_1(C)}{2} \wedge V_2(X_2) \geq \frac{V_2(C)}{2}$$

and that proof the theorem. □

Theorem 1.3 — *Section 2, Consider $(X_1, X_2) \in S_C$ then (X_1, X_2) is Pareto Optimal*

Proof. Suppose in the negative that

$$\exists_{(X_i, X_j) \in P_C} (V_1(X_i) > V_1(X_1) \wedge V_2(X_j) \geq V_2(X_2)) \vee (V_1(X_i) \geq V_1(X_1) \wedge V_2(X_j) > V_2(X_2))$$

Let (X_i, X_j) be as such, then because

$$(V_1(X_i) > V_1(X_1) \wedge V_2(X_j) \geq V_2(X_2)) \vee (V_1(X_i) \geq V_1(X_1) \wedge V_2(X_j) > V_2(X_2))$$

It can be concluded that $V_1(X_i) \geq V_1(X_1) \geq \frac{V_1(C)}{2} \wedge V_2(X_j) \geq V_2(X_2) \geq \frac{V_2(C)}{2}$ therefore $(X_i, X_j) \in M_C$.

Furthermore because

$$(V_1(X_i) > V_1(X_1) \wedge V_2(X_j) \geq V_2(X_2)) \implies V_1(X_i) + V_2(X_j) > V_1(X_1) + V_2(X_2)$$

and also in other case

$$(V_1(X_i) \geq V_1(X_1) \wedge V_2(X_j) > V_2(X_2)) \implies V_1(X_i) + V_2(X_j) > V_1(X_1) + V_2(X_2)$$

therefore

$$V_1(X_i) + V_2(X_j) > V_1(X_1) + V_2(X_2)$$

and that a contradiction to definition of S_C because

$$V_1(X_1) + V_2(X_2) = \underset{(X_i, X_j) \in M_C}{\arg\max}\ V_1(X_i) + V_2(X_j)$$

but $(X_i, X_j) \in M_C$ and $V_1(X_i) + V_2(X_j) > V_1(X_1) + V_2(X_2)$ and that a contradiction. □

Theorem 1.4 — *Section 3, Consider* $(X_1, X_2) \in S_C$ *then* (X_1, X_2) *is Envy free*

Proof. In order to proof that (X_1, X_2) is envy free we need to show that

$$V_1(X_1) \geq V_1(X_2) \wedge V_2(X_2) \geq V_2(X_1)$$

(it is trivial that $V_1(X_1) \geq V_1(X_1)$,$V_2(X_2) \geq V_2(X_2)$).

From the fact that $(X_1, X_2) \in S_C \subseteq M_C$ we know for sure that

$$V_1(X_1) \geq \frac{V_1(C)}{2}, V_2(X_2) \geq \frac{V_2(C)}{2}$$

From the V_1, V_2 evaluation function definition we know that

$$V_1(X_1) + V_1(X_2) = V_1(C) \Longrightarrow V_1(X_1) = V_1(C) - V_1(X_2)$$

$$V_2(X_1) + V_2(X_2) = V_2(C) \Longrightarrow V_2(X_2) = V_2(C) - V_2(X_1)$$

therefore

$$\begin{aligned} V_1(C) - V_1(X_2) \geq \frac{V_1(C)}{2}, &\qquad V_2(C) - V_2(X_1) \geq \frac{V_2(C)}{2} \Longrightarrow \\ \frac{V_1(C)}{2} - V_1(X_2) \geq 0, &\qquad \frac{V_2(C)}{2} - V_2(X_1) \geq 0 \Longrightarrow \\ \frac{V_1(C)}{2} \geq V_1(X_2), &\qquad \frac{V_2(C)}{2} \geq V_2(X_1) \Longrightarrow \\ V_1(X_1) \geq \frac{V_1(C)}{2} \geq V_1(X_2), &\qquad V_2(X_2) \geq \frac{V_2(C)}{2} \geq V_2(X_1) \end{aligned}$$

And that is what is required to prove. ☐