FAIRNESS IN REAL ESTATE DIVISION: REBUILD & DIVIDE

- WORK IN PROGRESS -

Noga Klein-Elmalem,
Open University

Rica Gonen,
Open University

Erel Segal-Halevi
Ariel University

REBUILD & DIVIDE: PROCESS

1. An old apartment house is evacuated and demolished.
2. A new, taller house is built in its place.
3. Each old home-owner receives exactly one apartment in the new house for free.
4. The remaining new apartments are given to the construction company, who sells them for profit.

REBUILD & DIVIDE: RATIONALE

A common urban renewal process in Israel.

Main goals:

- Increase supply of homes in city centers.
- Improve security & earthquake resistance.

Data:

- First project: 2001.
- Since then, about 6000-7000 projects per year.
- About 1/7 of all new construction projects.

RULES

1. At least 2/3 of home owners must agree.

2. Owners are entitled to apartments that are at least as large as their old ones (usually larger).

3. The details of the new assignment are not written in law; they are determined in the contract between the owners and the company.

REALITY

- Only ~20% of houses with potential to Rebuild&Divide manage to sign a contract.

- Main obstacle: disagreement about assignment of new apartments.

EXAMPLE: COURT PROCEEDINGS

Claims:

1. "My new apartment has 19 sqm more than my old apartment; other owners, with larger old apartments, got 23 sqm more".
2. "I got an apartment 1.5 floors higher (3.5 to 5); other owners got 2 floors higher".
3. "My old apartment is on its own half-floor, but the new apartment is not; I am entitled to unique compensation".
4. "My old apartment is square-shaped; my new apartment is not. This makes it harder to place furniture. Other owners did get a square-shaped apartment."

Court replies:

1. Home-owners with larger old apartments belong to a different "class" – incomparable;
2. Rounding the floor number down is as fair as rounding up;
3. The added value of a half-floor is negligible;
4. The new square-shaped apartments have longer corridors; hard to place furniture too.

EXAMPLE: COURT PROCEEDINGS

One word summary:

Envy

OUR GOAL:
ELIMINATE ENVY
USING MONEY

BASELINE: RENTAL HARMONY

Input:

- *n* rooms;
- *n* tenants with subjective valuations of rooms;
- Fixed total rent *R*.

Output:

- Assign exactly one room to each tenant;
- Charge a price from each tenant.

Constraints:

- Sum of all prices = *R*;
- No tenant envies the bundle (room+price) of another tenant.

RENTAL HARMONY: EXAMPLE INPUT

	Room 1	Room 2
	₪ 1200	₪ 800
	₪ 1500	₪ 700

RENTAL HARMONY: EXAMPLE OUTPUT

An envy-free solution exists for any *n*. Example for *n*=2:

	Room 1	Room 2	Payment
	₪ 1200	₪ 800	₪ 700
	₪ 1500	₪ 700	₪ 1350

Online implementations: NYT, Spliddit, PrefTools.

RENTAL HARMONY VS. REBUILD AND DIVIDE

Rental Harmony	Rebuild&Divide
One room per person.	One unit per person.
Equal entitlements.	Different entitlements – depending on endowments.
Envy depends only on room values.	Envy depends on **difference** between old values and new values.
Agents **might** gain from manipulating their values.	Agents **obviously** gain from manipulating value of old unit.

REBUILD AND REDIVIDE

Input:

- *n* old units; *n* agents with subjective valuations.
- Each agent owns exactly one old unit.
- *n* new units; agents have subjective valuations.

Output:

- Assign exactly one new unit to each agent;
- Charge a (positive or negative) price from each agent.

Constraints:

- Sum of all prices = 0 (balanced budget)
- No agent envies the **improvement** (new minus old) of another agent.

ELICITING TRUE OLD VALUES

Idea:

- Instead of asking about unit values, we ask about values of *characteristics*, determined based on common practice in house appraisal.
- Still manipulable, but now agents might lose from manipulating – there is an incentive to tell true values.

EXAMPLE: OLD UNITS

	Direction of airflow	Natural light	Parking availability	High Floor level
Agent 1	₪100,000	₪50,000	₪140,000	₪60,000
Agent 2	₪75,000	₪100,000	₪150,000	₪100,000

Valuations

	Direction of airflow	Natural light	Parking availability	High Floor level	Agent 1 valuation	Agent 2 valuation
Apartment of Agent 1	V	X	X	V	₪160,000	₪175,000
Apartment of Agent 2	V	V	X	V	₪210,000	₪275,000

Old apartments characteristics

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	Direction of airflow	Natural light	Parking availability	High Floor level
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Valuations

	Direction of airflow	Natural light	Parking availability	High Floor level	Agent 1 valuation	Agent 2 valuation
Apartment 1	X	V	V	V	₪250,000	₪350,000
Apartment 2	V	V	V	X	₪290,000	₪325,000

New apartments characteristics

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Apartment 1	X	V	V	V	₪250,000	₪350,000
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New apartments characteristics

EXAMPLE: INPUT

	Old unit of Agent 1	Old unit of Agent 2	New unit 1	New unit 2
Agent 1	₪160,000	₪210,000	₪250,000	₪290,000
Agent 2	₪175,000	₪275,000	₪350,000	₪325,000

Valuations of apartments

EXAMPLE: ATTEMPT 1

	Apartment of Agent 1	Apartment of Agent 2	New apartment 1	New apartment 2
Agent 1	₪160,000	₪210,000	₪250,000	₪290,000
Agent 2	₪175,000	₪275,000	₪350,000	₪325,000

Valuations of apartments

	Agent 1 improvement	Agent 2 improvement
Agent 1	₪90,000	₪80,000
Agent 2	₪175,000	₪50,000

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	Apartment of Agent 1	Apartment of Agent 2	New apartment 1	New apartment 2
Agent 1	₪160,000	₪210,000	₪250,000	₪290,000
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Valuations of apartments

	Agent 1 improvement	Agent 2 improvement
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Valuations of apartments

	Agent 1 improvement	Agent 2 improvement
Agent 1	₪90,000	₪80,000
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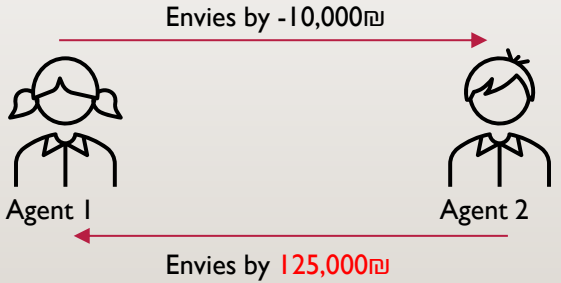

Cannot eliminate envy with payments

EXAMPLE: ATTEMPT 2

	Apartment of Agent 1	Apartment of Agent 2	New apartment 1	New apartment 2
Agent 1	₪160,000	₪210,000	₪250,000	₪290,000
Agent 2	₪175,000	₪275,000	₪350,000	₪325,000

Valuations of apartments

	Agent 1 improvement	Agent 2 improvement
Agent 1	₪130,000	₪40,000
Agent 2	₪150,000	₪75,000

EXAMPLE: ATTEMPT 2

	Apartment of Agent 1	Apartment of Agent 2	New apartment 1	New apartment 2
Agent 1	₪160,000	₪210,000	₪250,000	₪290,000
Agent 2	₪175,000	₪275,000	₪350,000	₪325,000

Valuations of apartments

	Agent 1 improvement	Agent 2 improvement
Agent 1	₪130,000	₪40,000
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Valuations of apartments

	Agent 1 improvement	Agent 2 improvement
Agent 1	₪130,000	₪40,000
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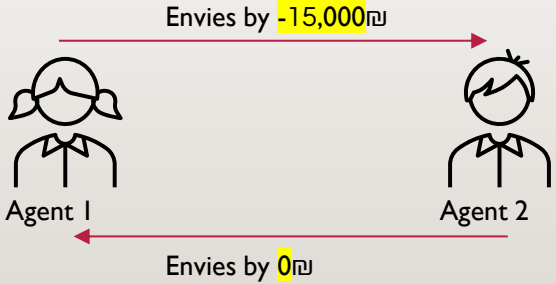
EXAMPLE: ATTEMPT 2 – SUCCESS!

	Apartment of Agent 1	Apartment of Agent 2	New apartment 1	New apartment 2
Agent 1	₪160,000	₪210,000	₪250,000	₪290,000
Agent 2	₪175,000	₪275,000	₪350,000	₪325,000

Valuations of apartments

	Agent 1 improvement	Agent 2 improvement	Payment
Agent 1	₪130,000	₪40,000	₪37,500
Agent 2	₪150,000	₪75,000	₪-37,500



ENVY-FREE ASSIGNMENT: EXISTENCE

Does there always exist an envy-free assignment of the new units?

No!

		Apartment of Agent 1	Apartment of Agent 2		New apartment 1	New apartment 2
Agent 1		200	100		300	300
Agent 2		100	200		300	300

Each agent envies the other agent's improvement.

REDEFINING THE PROBLEM

Input:

- *n* old units; *n* agents with subjective valuations.
- Each agent owns one old unit.
- *n* new units; agents has subjective valuations.

Output:

- Assign exactly one new unit to each agent;
- Charge a (positive or negative) price from each agent.

Constraints:

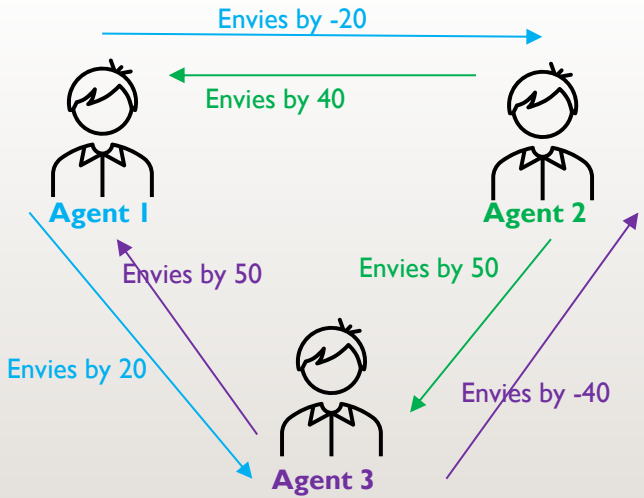
- Sum of all prices = 0 (balanced budget)
- **Subject to this, minimize the maximum envy per agent.**

TOOL: SIGNED ENVY GRAPH

Definition. Given old assignment *A* and new assignment *B*, the Signed Envy Graph $\mathbf{G_{AB}}$ is a complete directed weighted graph:

- The nodes of G_{AB} are the agents;
- $G_{AB}[i,j]$:= weight of edge i → j = the signed envy of *i* at *j*.

Similar tool used e.g. by Halpern and Shah (2019).



Definition. Given any directed graph *G*,
MACW(G) := *Maximum Average Cycle Weight* in G.

MIN MAX ENVY

Definition. Given old alloc. *A* and new alloc. B, MinMaxEnvy(A,B) := smallest maximum-envy that can be attained using payments.

Main Lemma.

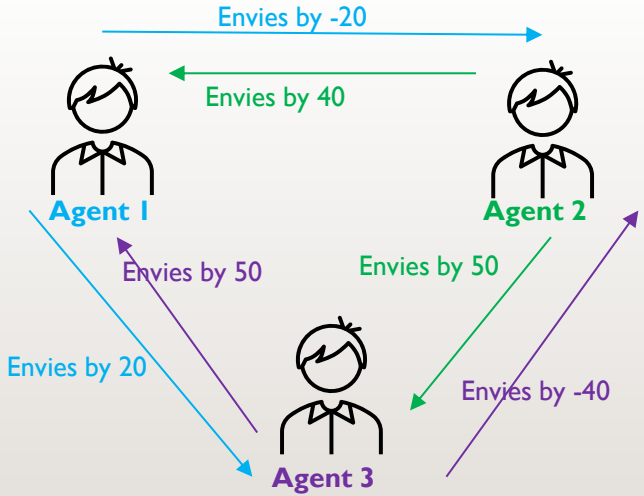
MinMaxEnvy(A,B) = MACW(G_{AB}).

Example. Five directed cycles.

Average weights: 10, 5, **35**; 6.66, 33.33.

Payment with envy MACW(G_{AB})=35:

- Agent 1 pays 15
- Agent 2 pays -15
- Agent 3 pays 0



MINIMIZING THE ENVY

Lemma. For any old assignment A and new assignment B,

$$\text{MinMaxEnvy}(A,B) = \text{MACW}(G_{AB}).$$

Proof. ≥ : Paying p_i to i subtracts p_i from any edge *outgoing i*, and adds p_i to any edge *incoming i*. Any cycle contains the same number (0 or 1) of such edges. So cycle weights do not change.

≤: Subtract $\text{MACW}(G_{AB})$ from all edges. Now there are no positive-weight cycles. Let L_i := maximum weight of directed path from i. Give L_i to each agent i to eliminate envy; then charge each agent $(\text{sum}_i L_i)/n$ for budget balance. Actual envy is ≤ $\text{MACW}(G_{AB})$.

MINIMIZING THE ENVY WITH GIVEN ASSIGNMENT

Lemma. For any old assignment A and new assignment B,

$$\text{MinMaxEnvy}(A,B) = \text{MACW}(G_{AB}).$$

There are strong-polytime algorithms for computing MACW(G) for any graph *G:* Karp (1978), Dasdan&Gupta (1998), Albrecht, Korte, Schietke&Vygen (2002). Applications to logic chip design.

Corollary. Given assignments A and B, there is a strong-polytime algorithm for computing a payment vector that minimizes the maximum envy.

MINIMIZING THE ENVY – FINDING AN ASSIGNMENT

Minimizing the maximum envy is equivalent to the problem:

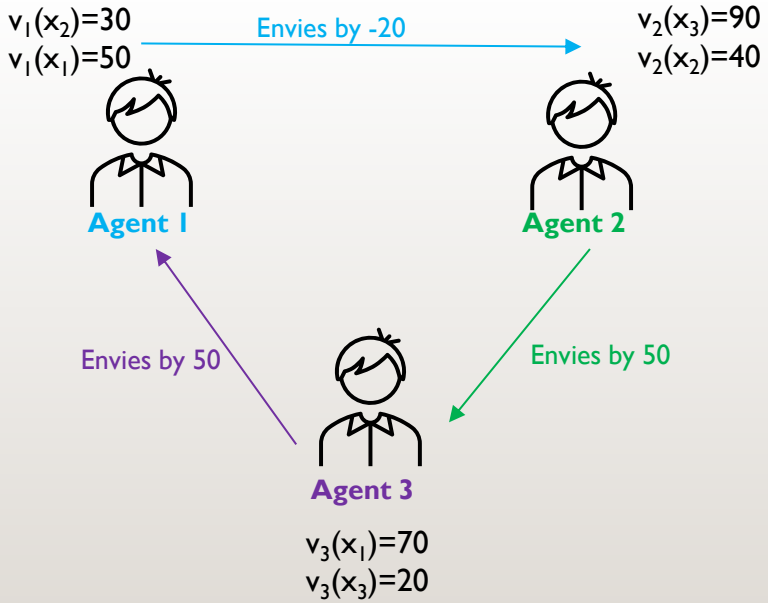
(*) Given the old assignment A, find a new assignment B for which MACW(G_{AB}) is smallest.

SPECIAL CASE: NO OLD UNITS

Special case where old units (in A) are worth 0 = Rental Harmony.

Weight of cycle in G_{AB} =
[sum of values in alternative assignment]
minus
[sum of values in assignment *B*].

Example. Weight of cycle =
(30+90+70) – (50+40+20) = 80.



SPECIAL CASE: NO OLD UNITS

Weight of cycle in G_{AB} =
[sum of values in alternative assignment]
minus [sum of values in assignment *B*].

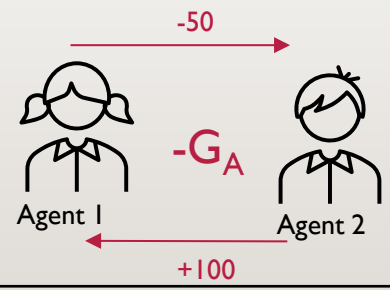
Corollary.

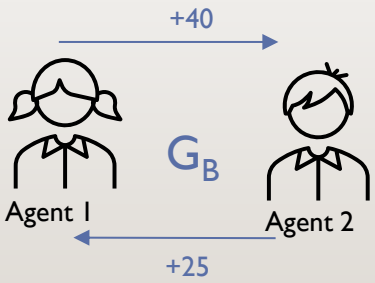
B is a maximum-weight matching ➔
All cycle weights are at most 0 ➔
$MACW(G_{AB}) \leq 0$ ➔
$MinMaxEnvy(A,B) \leq 0$ ➔
B is "envy-freeable" (Halpern and Shah, 2019).

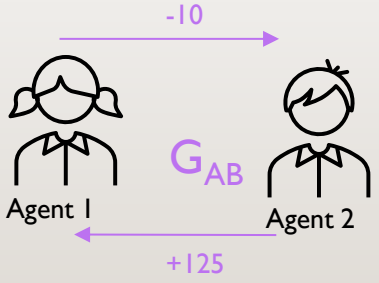
THE EFFECT OF OLD UNITS

Old units add to each edge a fixed amount, namely, minus the envy on that edge in the original assignment: $G_{AB} = -G_A + G_B$

	Old unit of 1	Old unit of 2	New unit of 1	New unit of 2	Improv. of 1	Improv. of 2
Agent 1	160	210	250	290	90	80
Agent 2	175	275	350	325	175	50







MINIMIZING THE ENVY – FINDING AN ASSIGNMENT

Minimizing the maximum envy is equivalent to the problem:

(*) Given a fixed matrix $-G_A$, find an assignment B for which $MACW(-G_A + G_B)$ is smallest.

When $-G_A=0$, problem (*) is solved by maximum-weight matching.

Open question. Can problem (*) be solved in polynomial time?