

Fairly Dividing a Cake after Some Parts were Burnt in the Oven

Erel Segal-Halevi

Fair Division — Definition

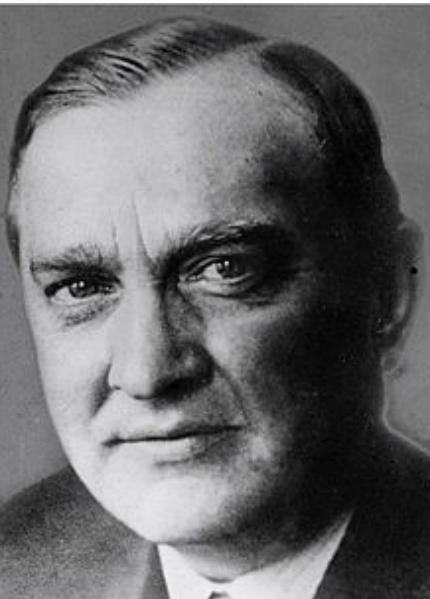
Dividing a heterogeneous resource to agents with different preferences such that everyone's share is "fair" by their preferences.

Fair Division — Then

Dividing a heterogeneous resource to agents with different preferences such that everyone's share is "fair" by their preferences.



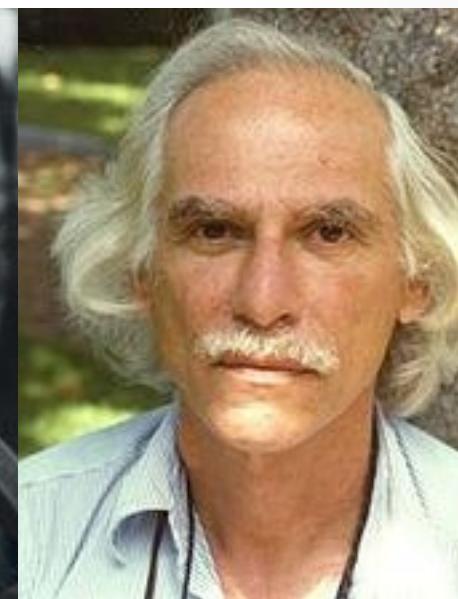
Banach



Knaster

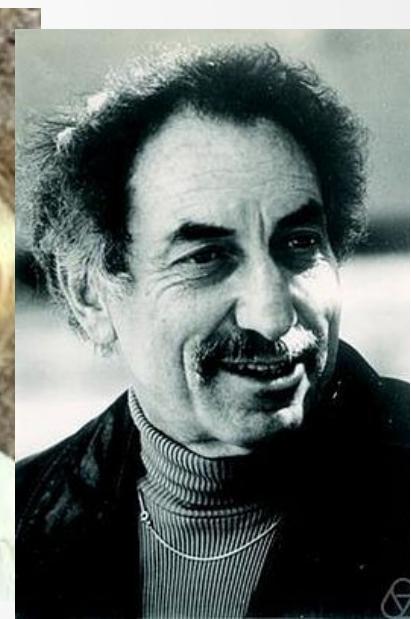


Dubins



Spanier

Steinhaus



Fair Division – Today

Dividing a heterogeneous resource to agents with different preferences such that everyone's share is "fair" by their preferences.

<http://fairoutcomes.com>

Fair Outcomes, Inc.

Game-Theoretic Solutions for Disputes and Negotiations
System Design - System Administration - Consultative and Online Services

[Home](#) [Fair Buy-Sell](#) [Fair Division](#) [Fair Proposals](#) [Fair Reputations](#)



Fair Outcomes, Inc.

Fair Outcomes, Inc. provides parties involved in disputes or difficult negotiations with access to newly developed proprietary systems that allow fair and equitable outcomes to be achieved with remarkable efficiency. Each of these systems is grounded in mathematical theories of fair division and of games.

Our founders and staff include game theorists, computer scientists, and practicing attorneys with extensive experience in designing, administering, utilizing, and providing consulting and online services with respect to such systems.

Further information about our company and our services may be obtained by using the contact information appearing on this page. Additional information about four of our systems, each of which can be accessed and used online (and examined and tested free of charge), can be obtained by clicking on the links appearing below:



<http://spliddit.org>

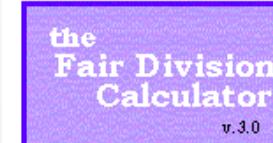
PROVABLY FAIR SOLUTIONS.

Spliddit offers quick, free solutions to everyday fair division problems, using methods that provide indisputable fairness guarantees and build on decades of research in economics, mathematics, and computer science.

<https://math.hmc.edu/~su/fairdivision>

Francis Su's Fair Division Page

Click on [The Fair Division Calculator](#) which has recently been updated! (version 3.01, 4/12/00)



A java applet for interactive decision making to find envy-free divisions of goods, burdens, or rent.



Share Rent

Moving into a new apartment with roommates? Create harmony by fairly assigning rooms and sharing the rent.

[START >](#)



Split Fare

Fairly split taxi fare, or the cost of an Uber or Lyft ride, when sharing a ride with friends.

[START >](#)



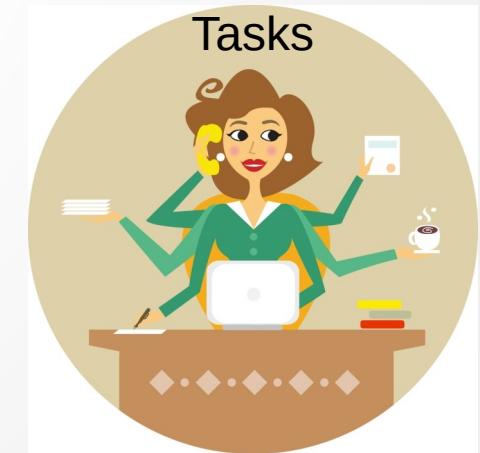
Assign Credit

Determine the contribution of each individual to a school project, academic paper, or business endeavor.

[START >](#)

Fair Division — Examples

Dividing a heterogeneous resource to agents with different preferences such that everyone's share is "fair" by their preferences.



Fair Division — Examples

Dividing a heterogeneous resource to agents with different preferences such that everyone's share is "fair" by their preferences.



Continuous Resource

Continuous Resource

Cake = Interval $[0,1]$.

n agents. Value-densities

Value = integral:

$$v_i : \text{Cake} \rightarrow \mathbf{R}$$

$$V_i(X_i) = \int_{X_i} v_i(x) dx$$

Continuous Resource

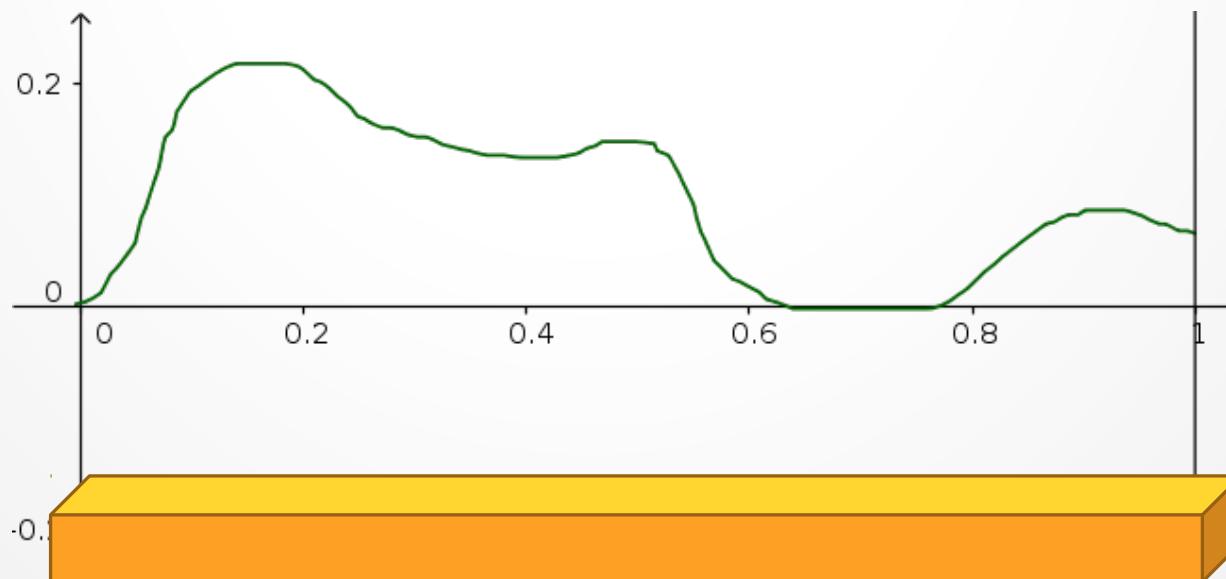
Cake = Interval $[0,1]$.

n agents. Value-densities

Value = integral:

$$v_i : \text{Cake} \rightarrow \mathbf{R}$$

$$V_i(X_i) = \int_{X_i} v_i(x) dx$$



Continuous Resource

Cake = Interval $[0,1]$.

n agents. Value-densities

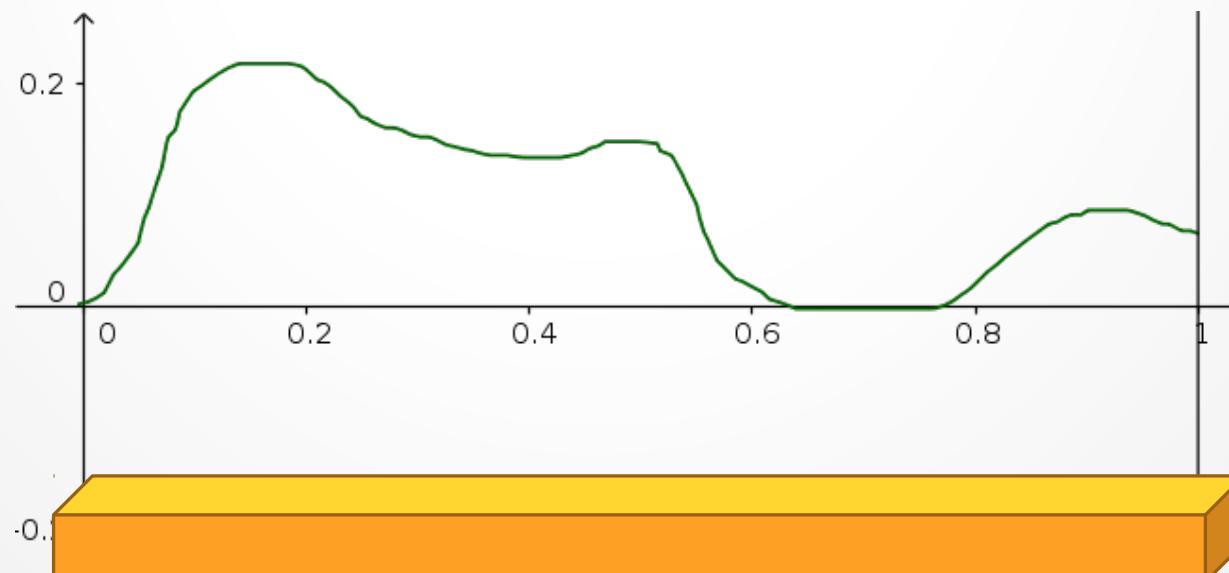
$$v_i : \text{Cake} \rightarrow \mathbf{R}$$

Value = integral:

$$V_i(X_i) = \int_{X_i} v_i(x) dx$$

Fairness (envy-freeness): For all i, j : $V_i(X_i) \geq V_i(X_j)$

For all i : X_i is **connected**.



Continuous Resource

Cake = Interval $[0,1]$.

n agents. Value-densities

$$v_i : \text{Cake} \rightarrow \mathbf{R}$$

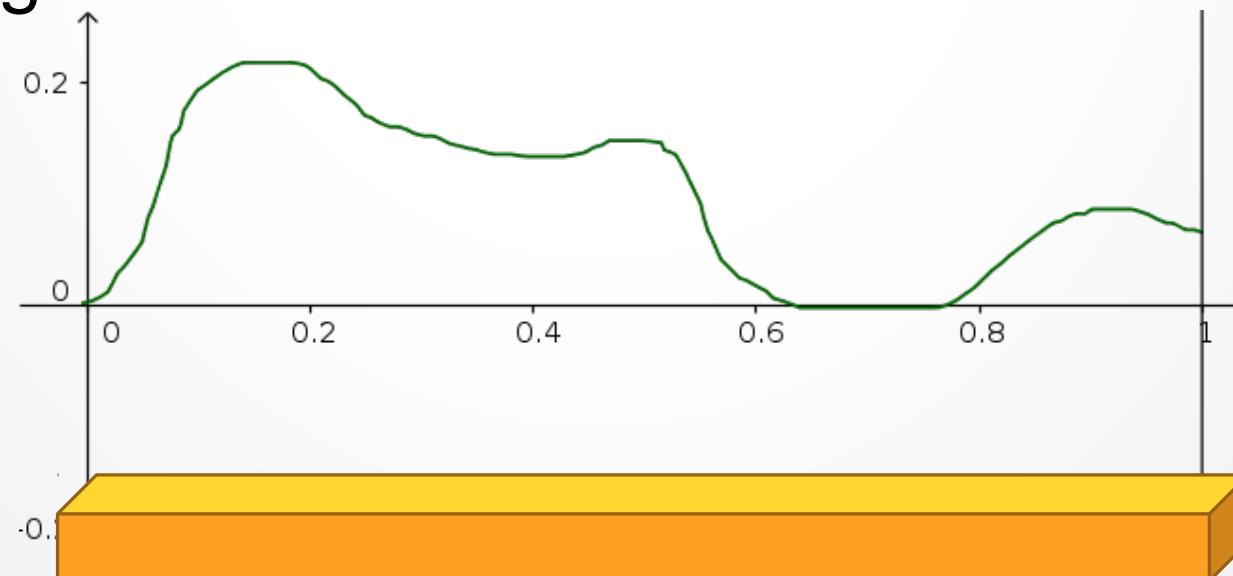
Value = integral:

$$V_i(X_i) = \int_{X_i} v_i(x) dx$$

Fairness (envy-freeness): For all i, j : $V_i(X_i) \geq V_i(X_j)$

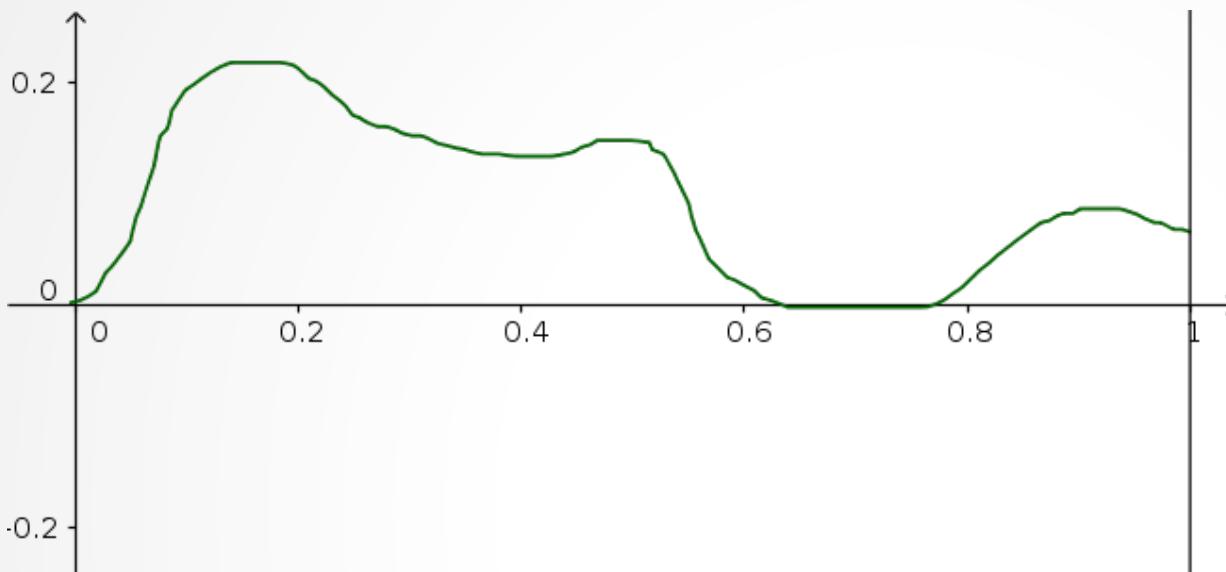
For all i : X_i is connected.

Easy for 2 agents. Difficult for 3 or more.



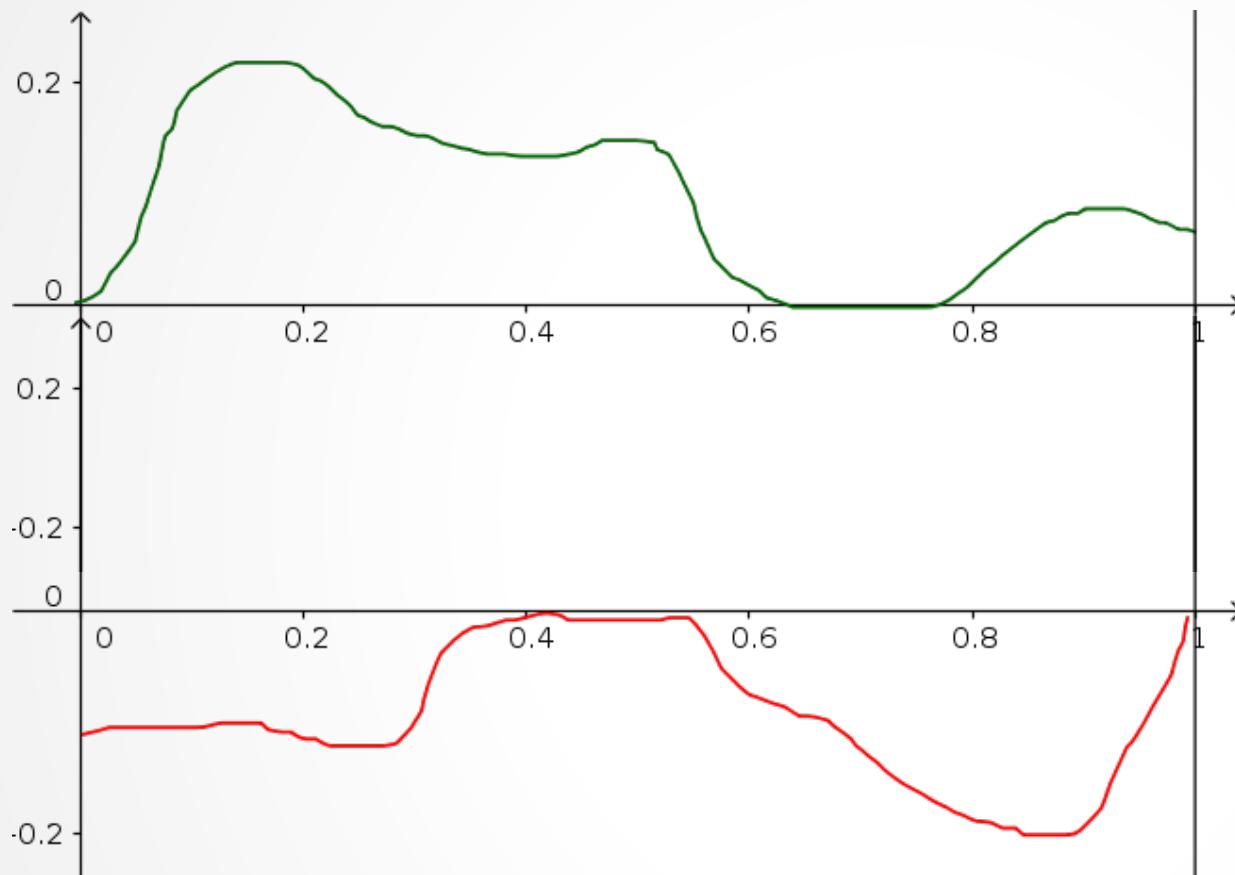
Valuation types

Valuation types



All positive -
solved by
Stromquist (1980),
Simmons (1980)

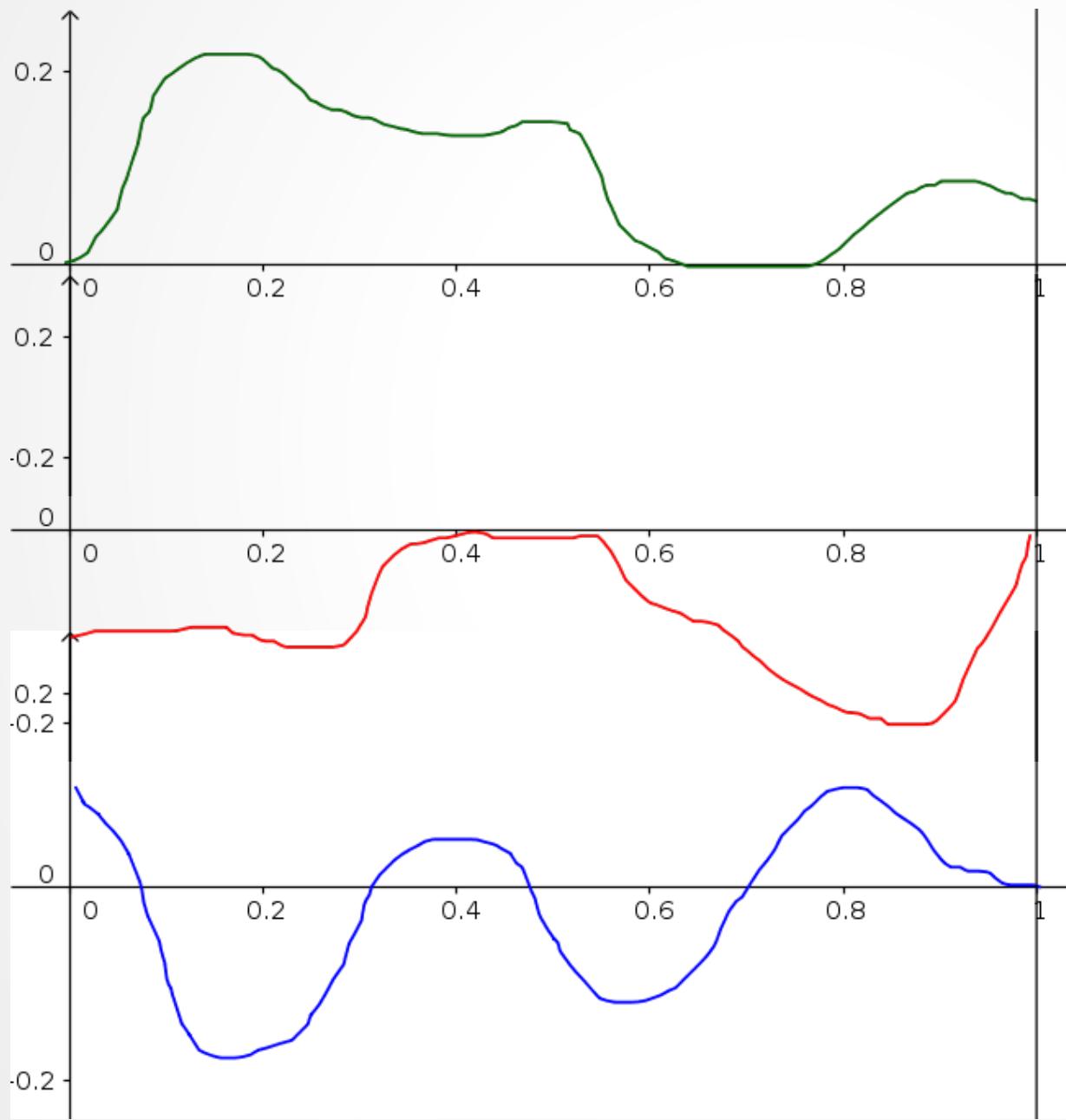
Valuation types



All positive -
solved by
**Stromquist (1980),
Simmons (1980)**

All negative –
solved by
Su (1999)

Valuation types



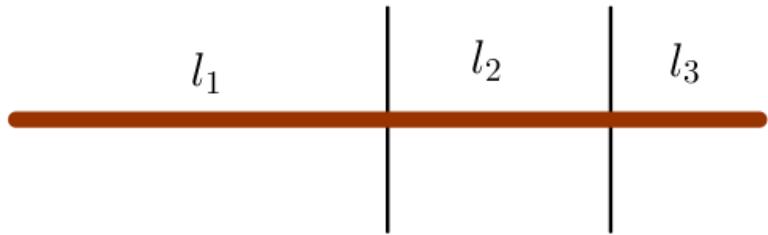
All positive -
solved by
**Stromquist (1980),
Simmons (1980)**

All negative –
solved by
Su (1999)

General –
this work.

Simplex of Partitions – Definition

(based on Stromquist 1980)



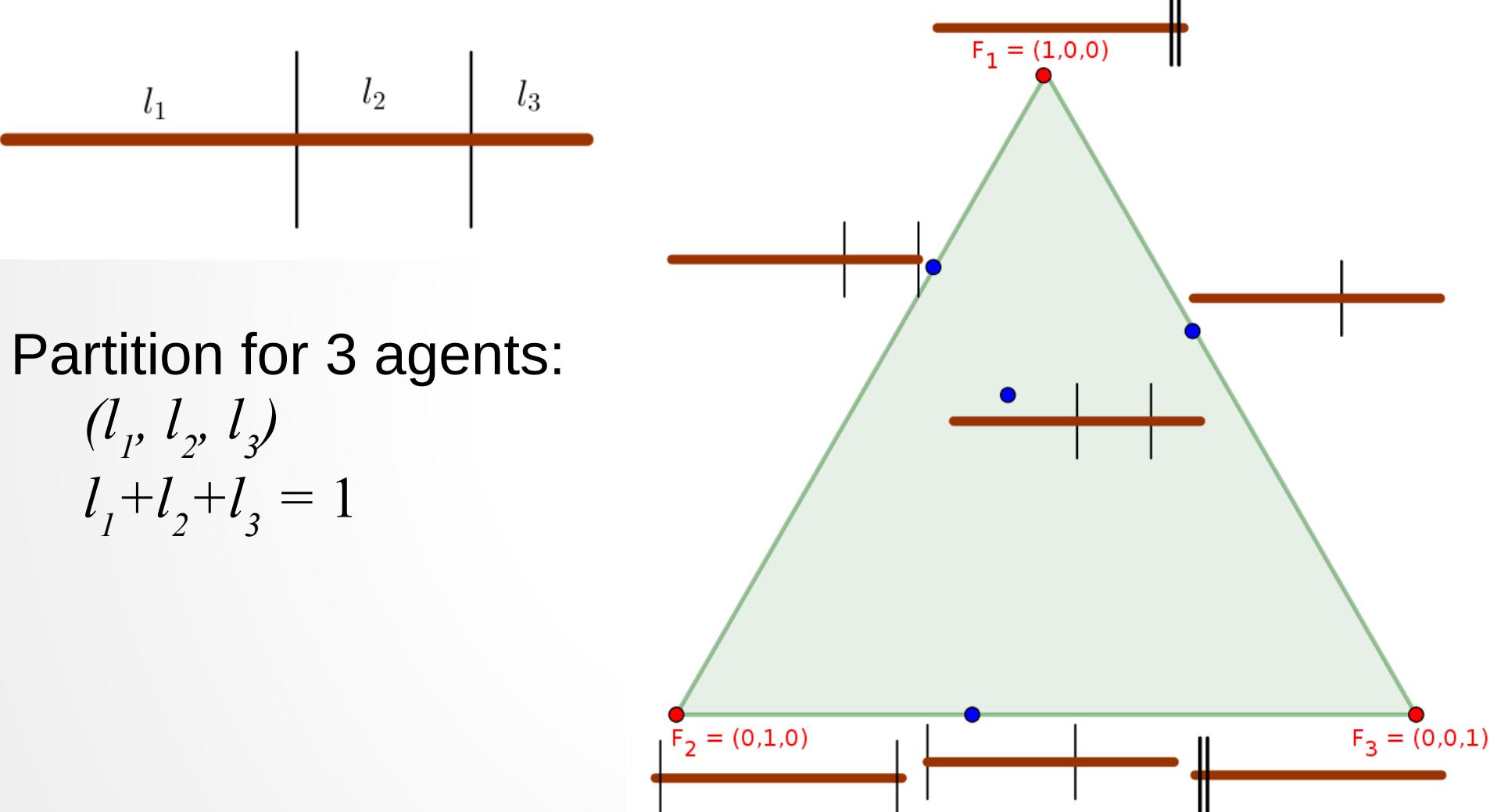
Partition for 3 agents:

$$(l_1, l_2, l_3)$$

$$l_1 + l_2 + l_3 = 1$$

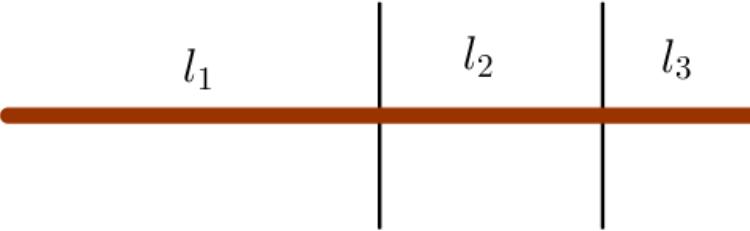
Simplex of Partitions – Definition

(based on Stromquist 1980)



Simplex of Partitions – Definition

(based on Stromquist 1980)

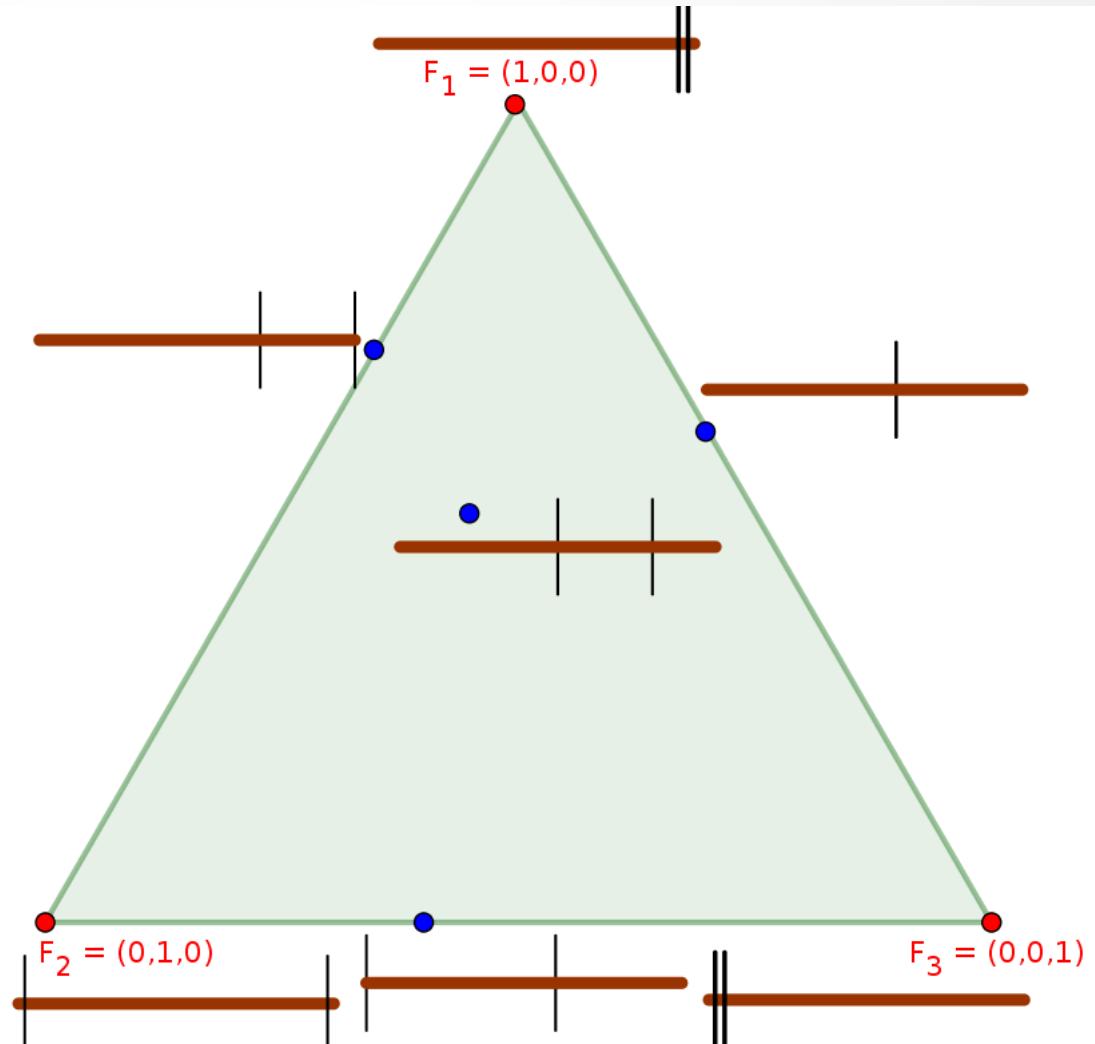


Partition for 3 agents:

$$(l_1, l_2, l_3)$$

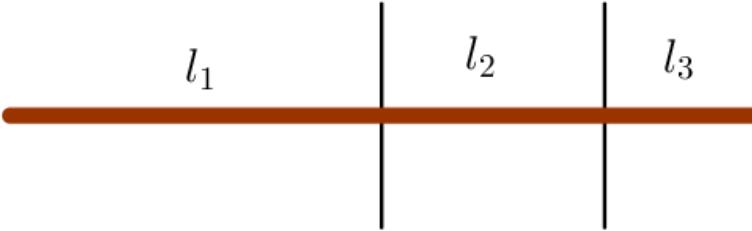
$$l_1 + l_2 + l_3 = 1$$

Envy-free division =



Simplex of Partitions – Definition

(based on Stromquist 1980)

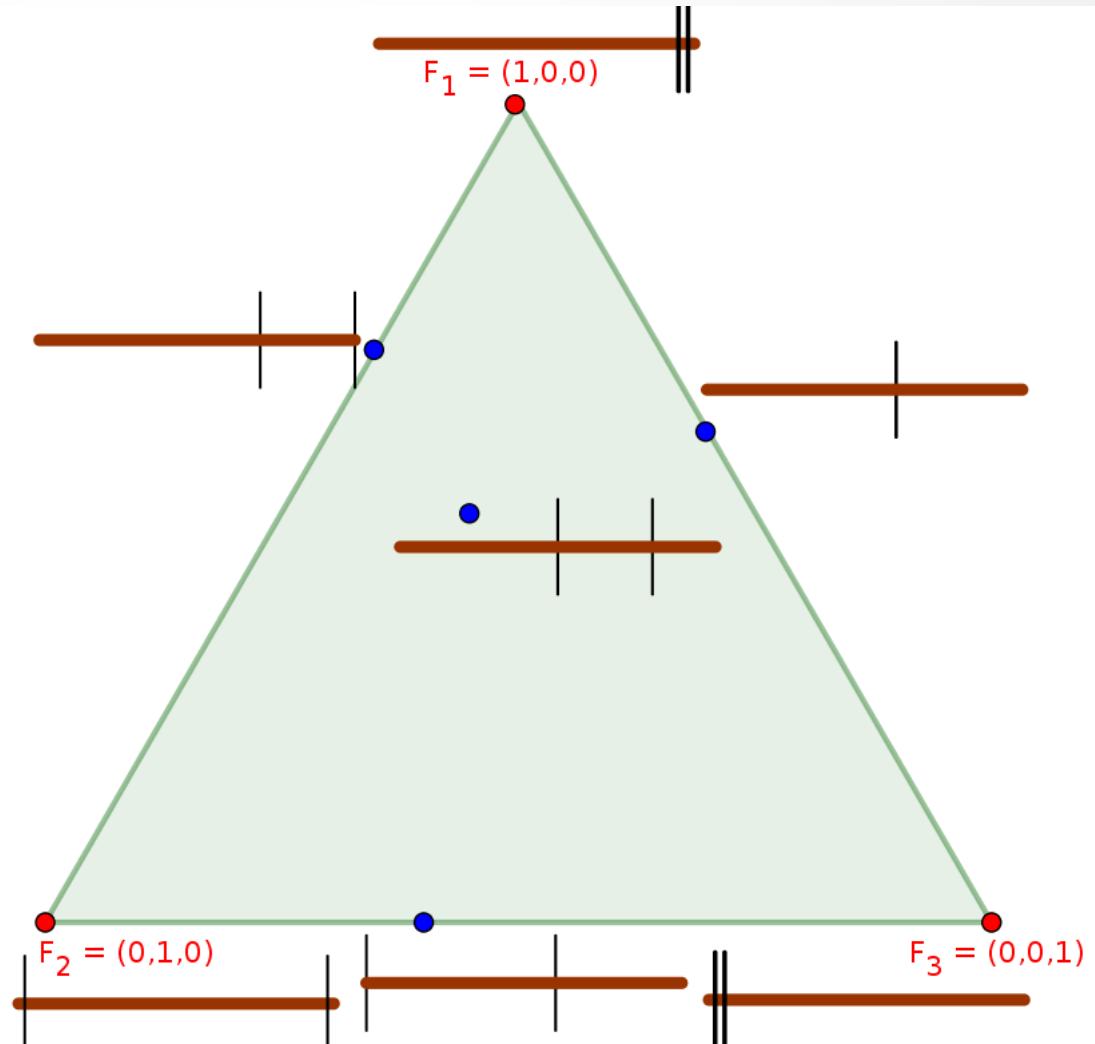


Partition for 3 agents:

$$(l_1, l_2, l_3)$$

$$l_1 + l_2 + l_3 = 1$$

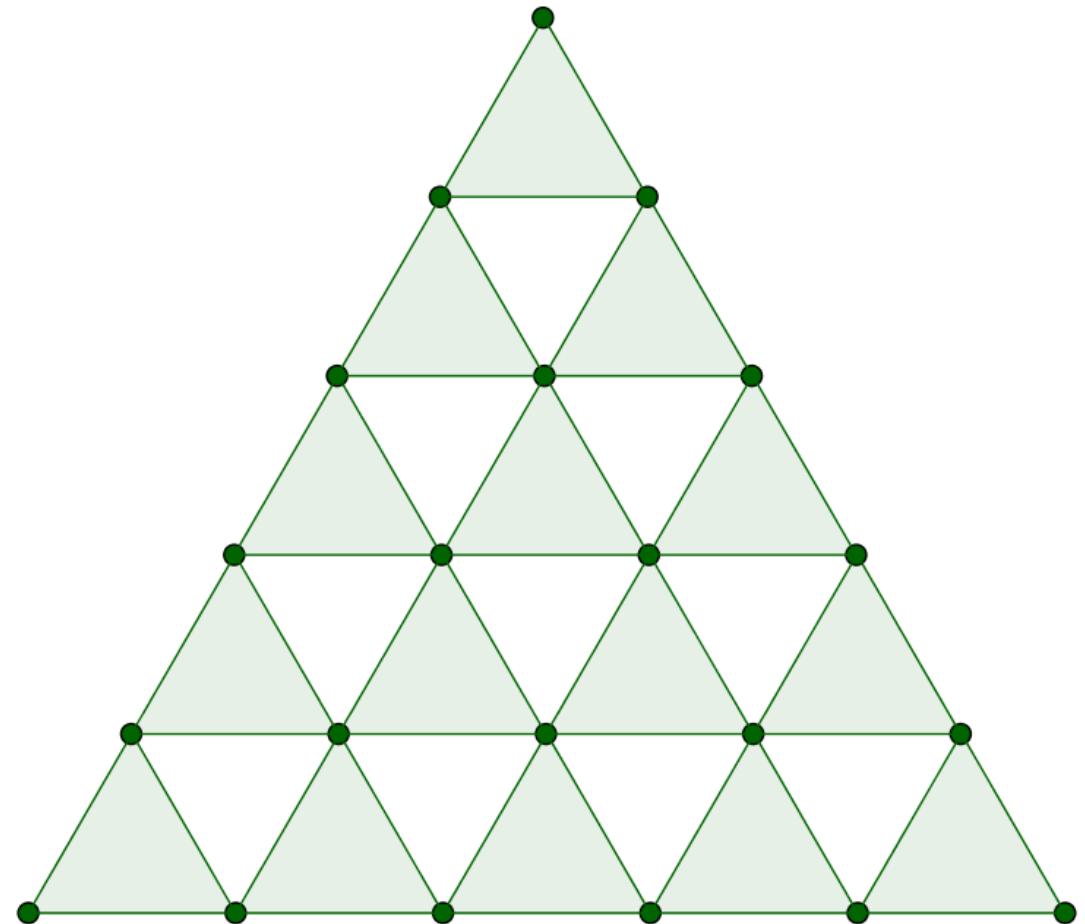
*Envy-free division =
point in which
each agent prefers
a different piece.*



Simplex of Partitions – Triangulation

(based on Simmons 1980, Su 1999)

a. Triangulate the simplex.

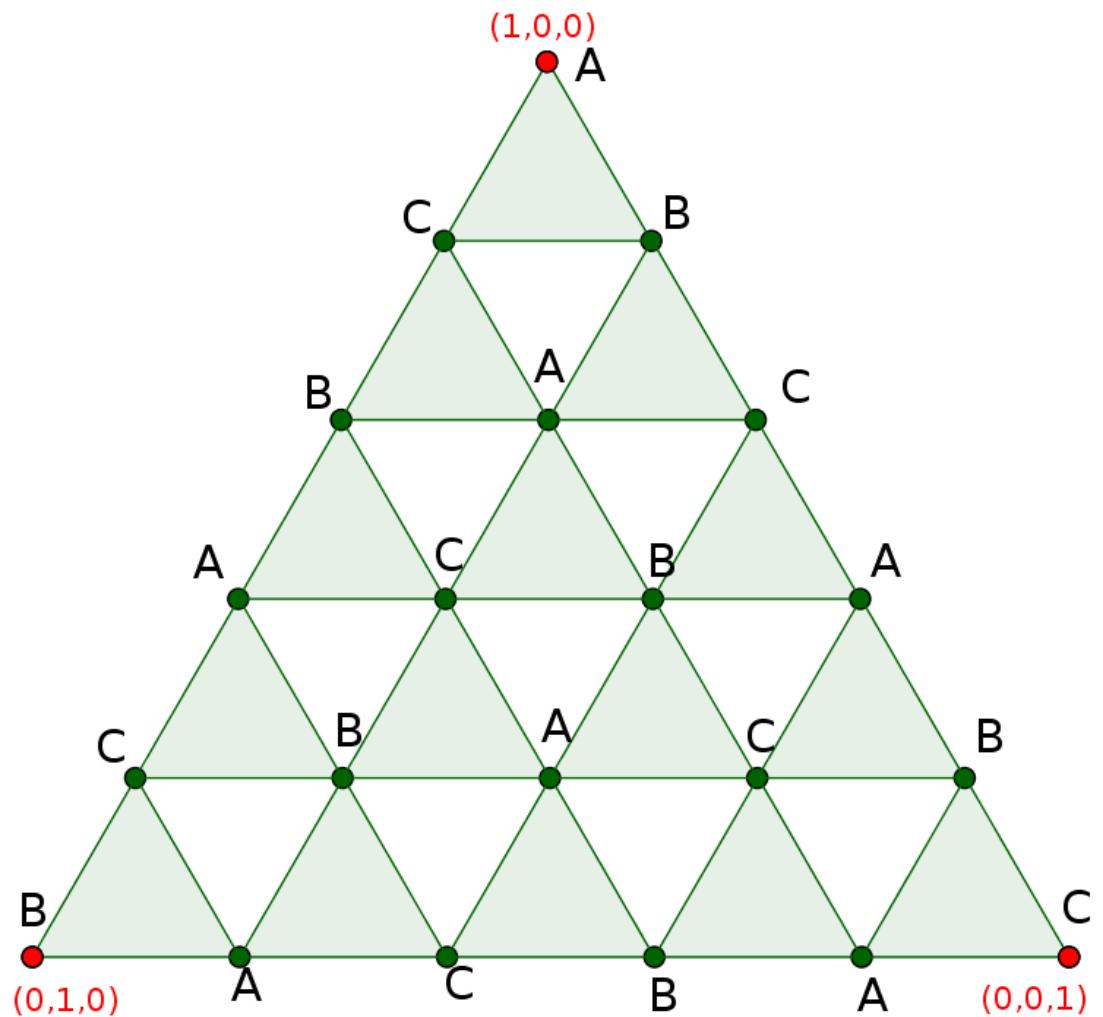


Simplex of Partitions – Triangulation

(based on Simmons 1980, Su 1999)

a. Triangulate the simplex.

b. Assign each vertex to a different agent such that in each sub-simplex, all agents are represented.



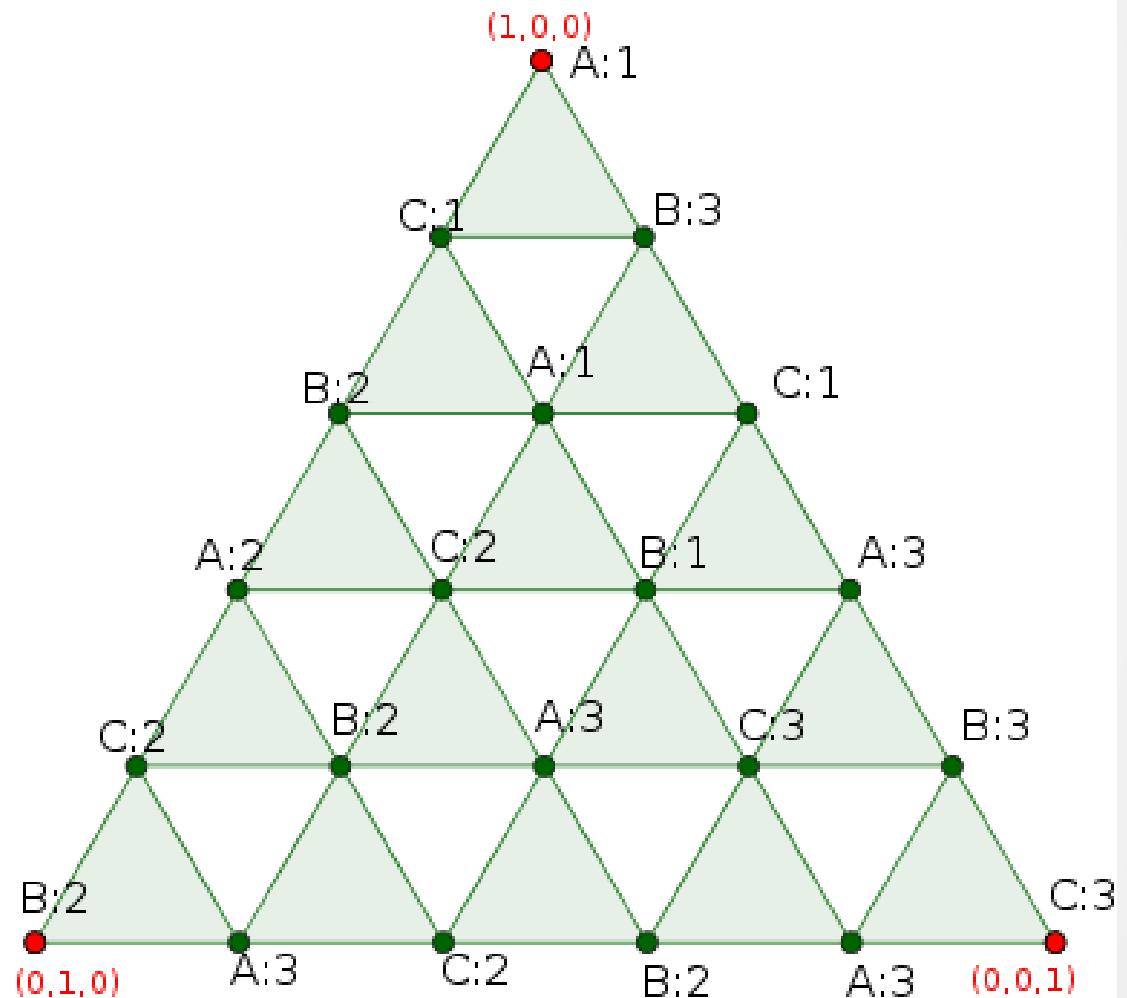
Simplex of Partitions – Triangulation

(based on Simmons 1980, Su 1999)

a. Triangulate the simplex.

b. Assign each vertex to a different agent such that in each sub-simplex, all agents are represented.

c. Ask each agent to label all its vertices by the index of his favorite piece.



Simplex of Partitions – Triangulation

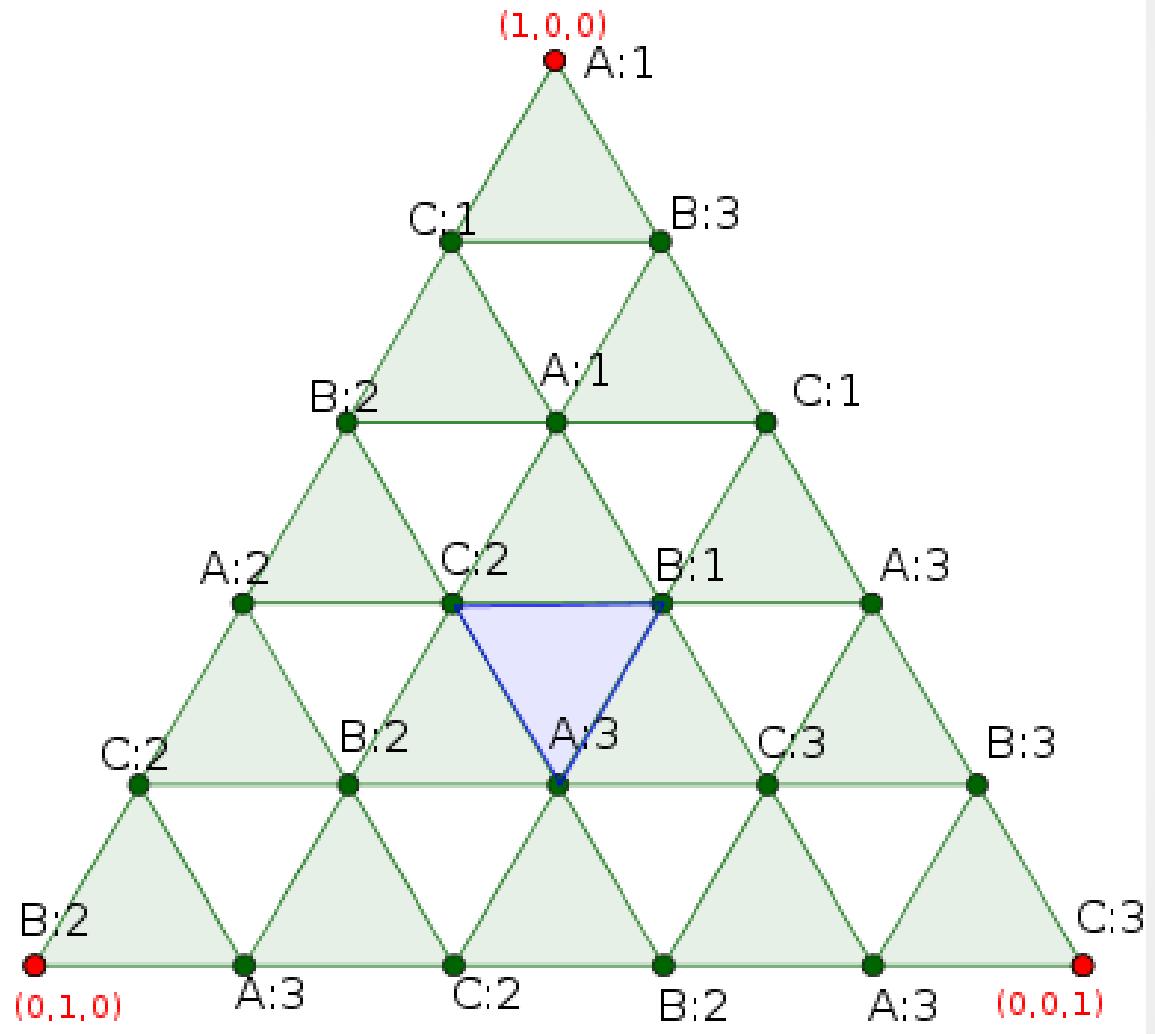
(based on Simmons 1980, Su 1999)

a. Triangulate the simplex.

b. Assign each vertex to a different agent such that in each sub-simplex, all agents are represented.

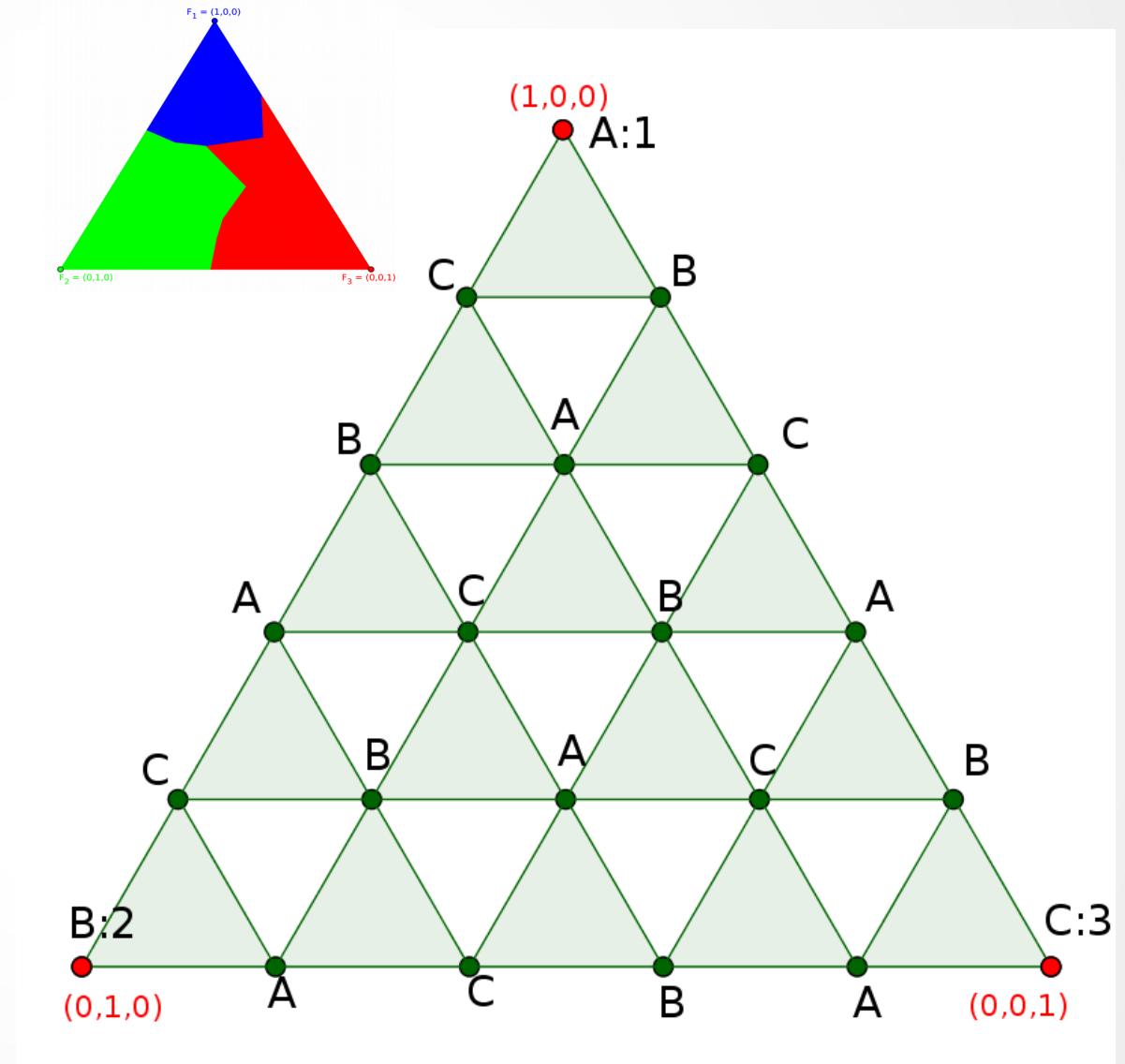
c. Ask each agent to label all its vertices by the index of his favorite piece.

d. A simplex labeled by all n labels = an approximately-envy-free division.



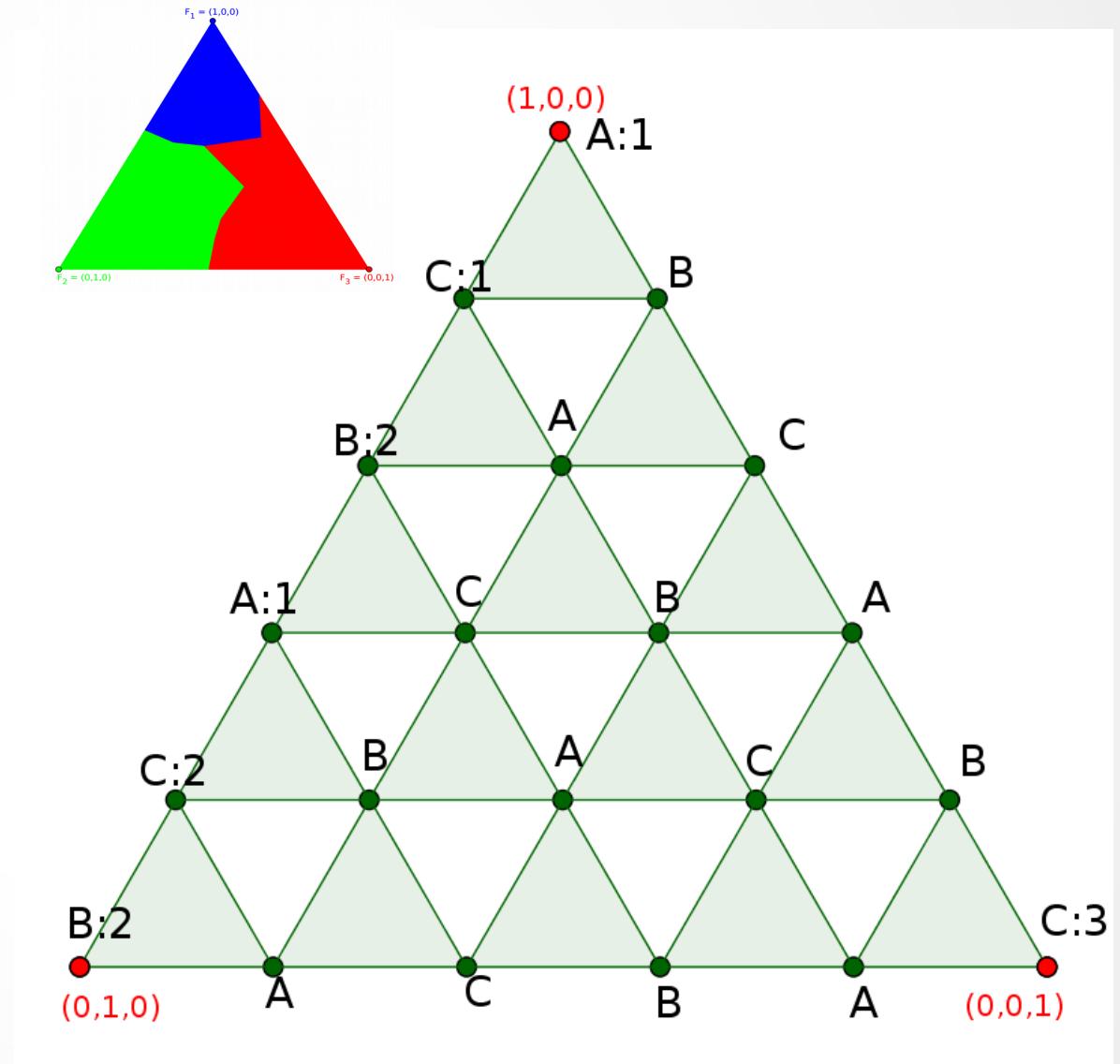
Triangulation – Positive Agents

Fact: When all agents have positive valuations, each face is labeled only with the labels of its endpoints.



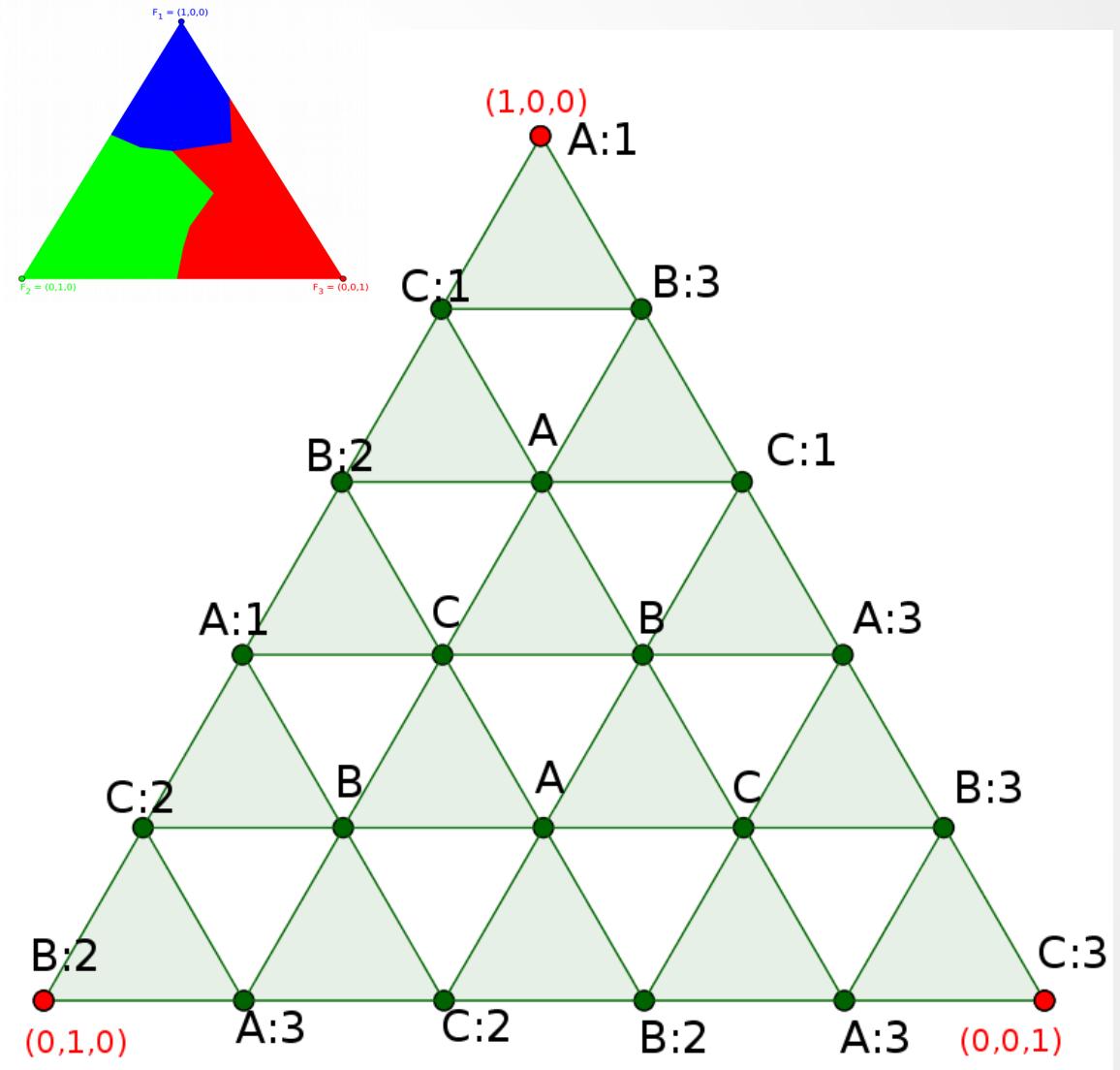
Triangulation – Positive Agents

Fact: When all agents have positive valuations, each face is labeled only with the labels of its endpoints.



Triangulation – Positive Agents

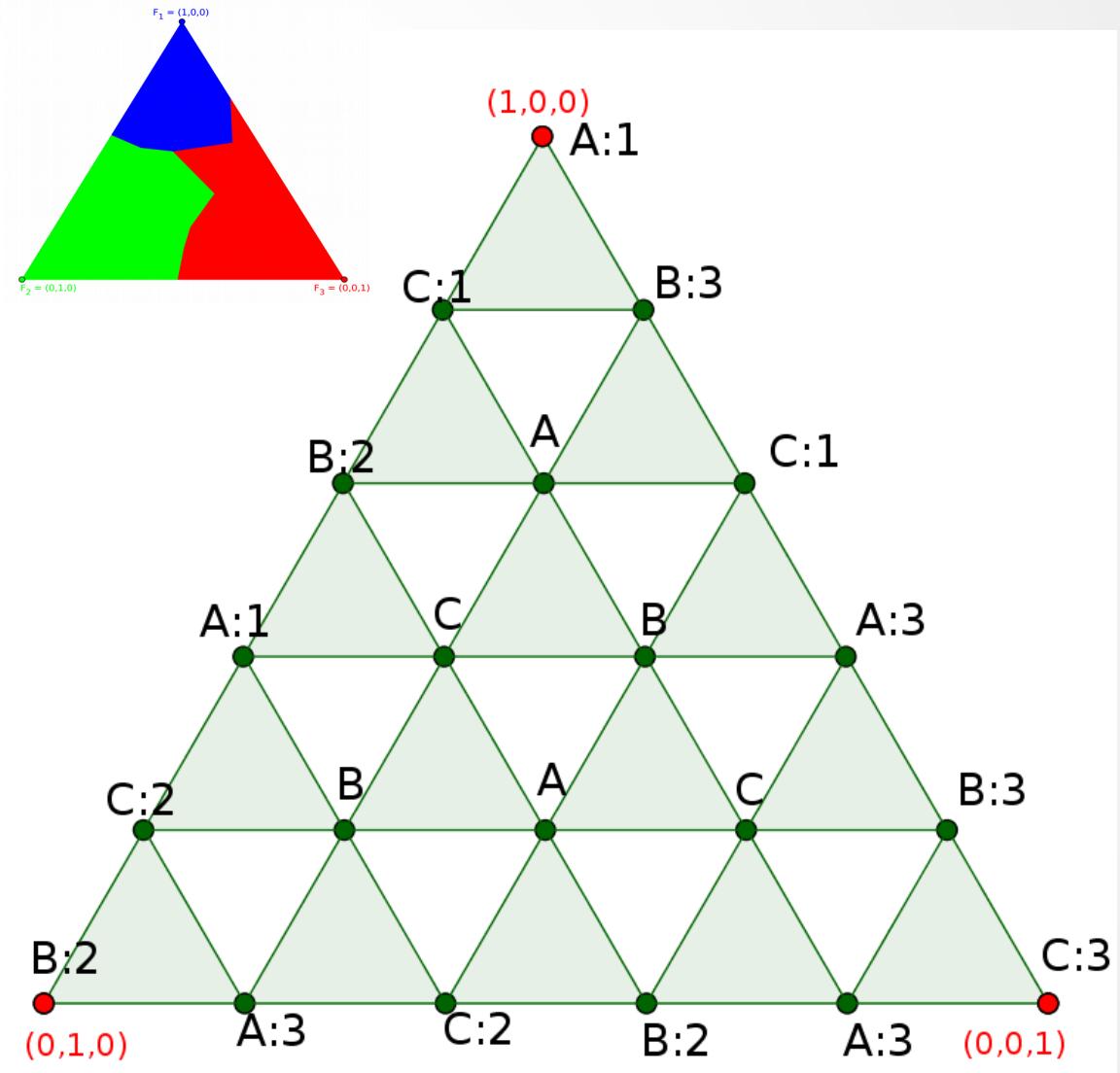
Fact: When all agents have positive valuations, each face is labeled only with the labels of its endpoints.



Triangulation – Positive Agents

Fact: When all agents have positive valuations, each face is labeled only with the labels of its endpoints.

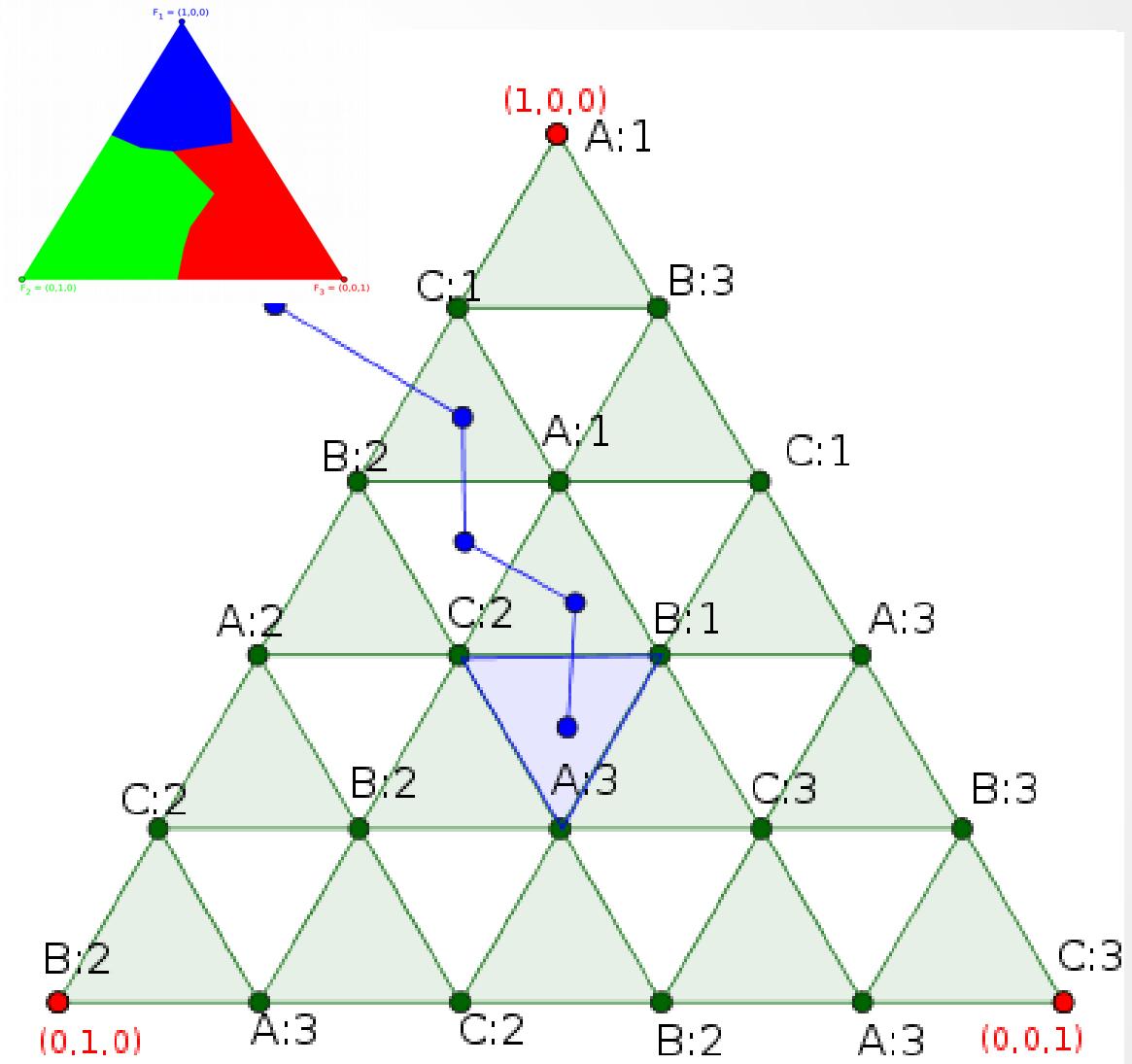
Lemma (Sperner 1929):
When each face is labeled only with the labels of its endpoints, a fully-labeled sub-simplex exists.



Triangulation – Positive Agents

Fact: When all agents have positive valuations, each face is labeled only with the labels of its endpoints.

Lemma (Sperner 1929):
When each face is labeled only with the labels of its endpoints, a fully-labeled subsimplex exists.

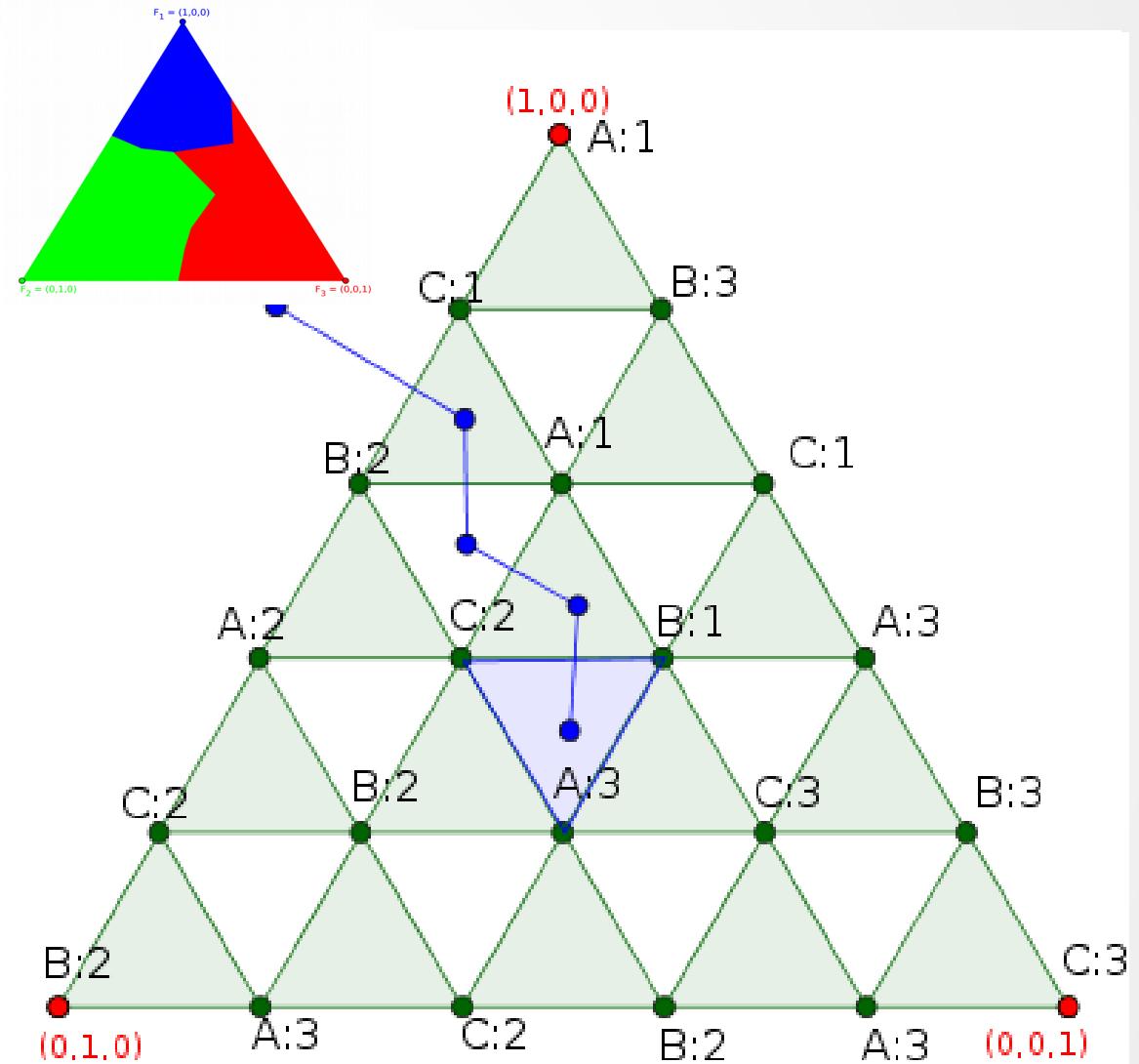


Triangulation – Positive Agents

Fact: When all agents have positive valuations, each face is labeled only with the labels of its endpoints.

Lemma (Sperner 1929):
When each face is labeled only with the labels of its endpoints, a fully-labeled subsimplex exists.

Corollary: when all valuations are positive, an approximately-envy-free division exists.



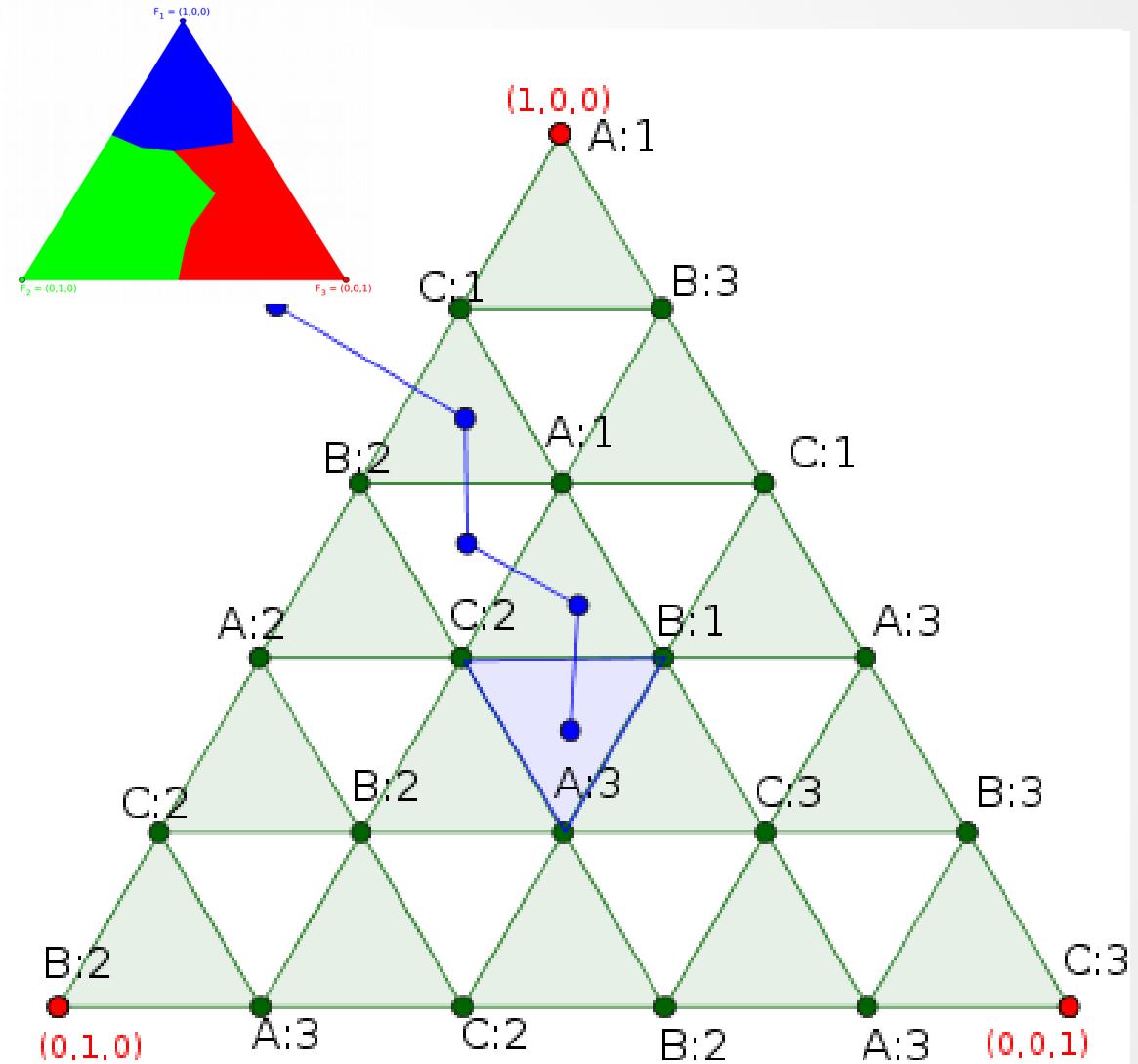
Triangulation – Positive Agents

Fact: When all agents have positive valuations, each face is labeled only with the labels of its endpoints.

Lemma (Sperner 1929):
When each face is labeled only with the labels of its endpoints, a fully-labeled subsimplex exists.

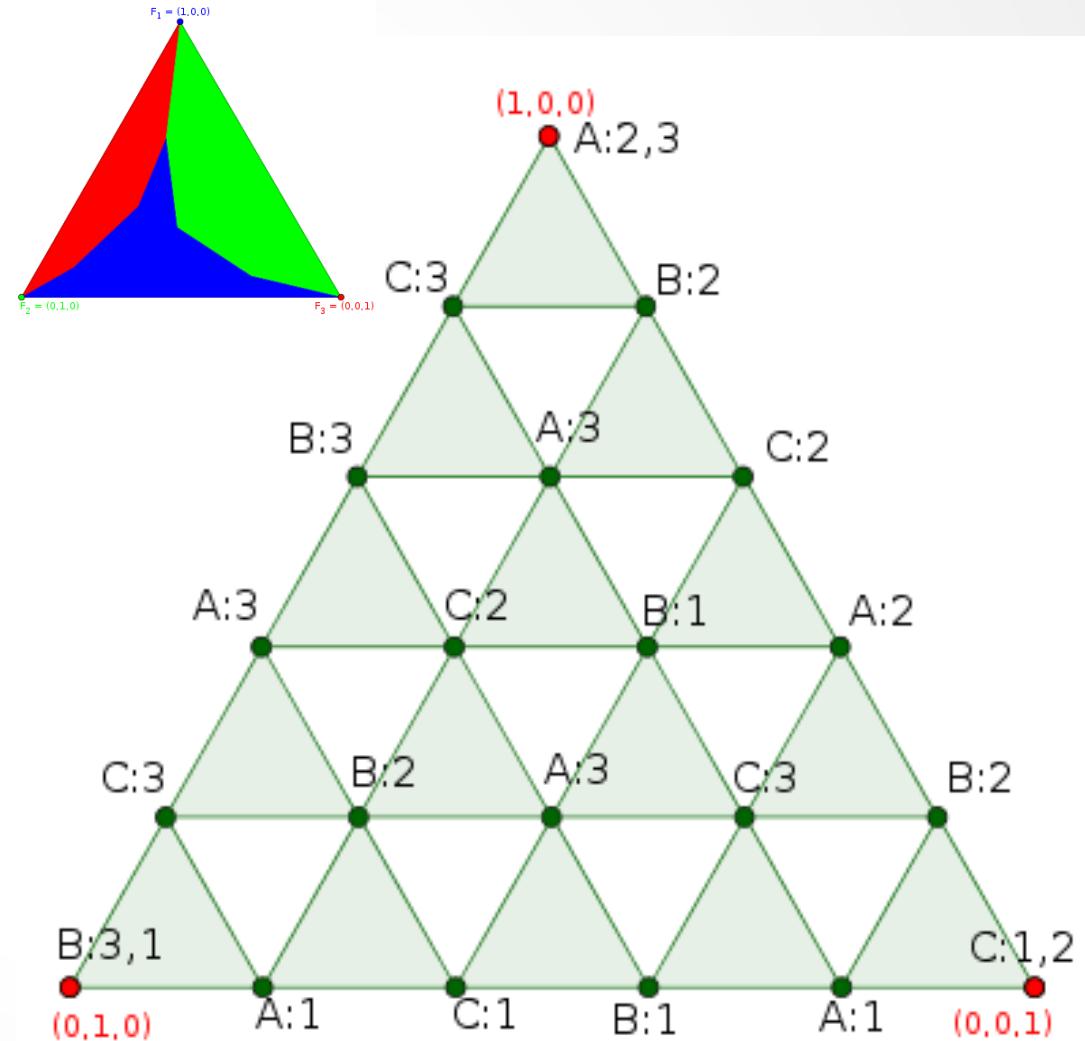
Corollary: when all valuations are positive, an approximately-envy-free division exists.

Corollary (Stromquist 1980, Simmons 1980, Su 1999):
when valuations are also continuous, an envy-free division exists.



Triangulation – Negative Agents

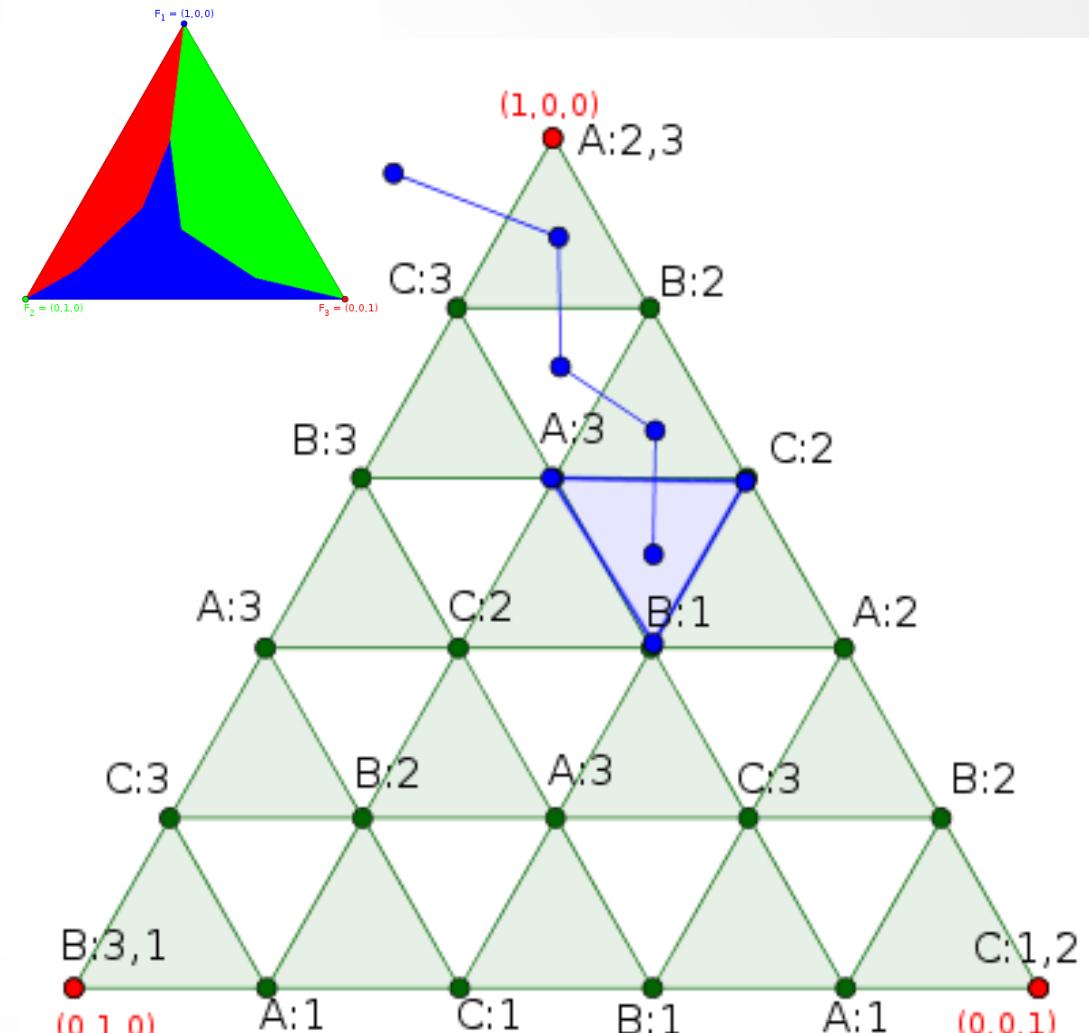
Fact: When all agents have negative valuations, it is possible to label the n main vertices such that each face is labeled only with the labels of its endpoints.



Triangulation – Negative Agents

Fact: When all agents have negative valuations, it is possible to label the n main vertices such that each face is labeled only with the labels of its endpoints.

Corollary: when all valuations are negative, an approximately-envy-free division exists.

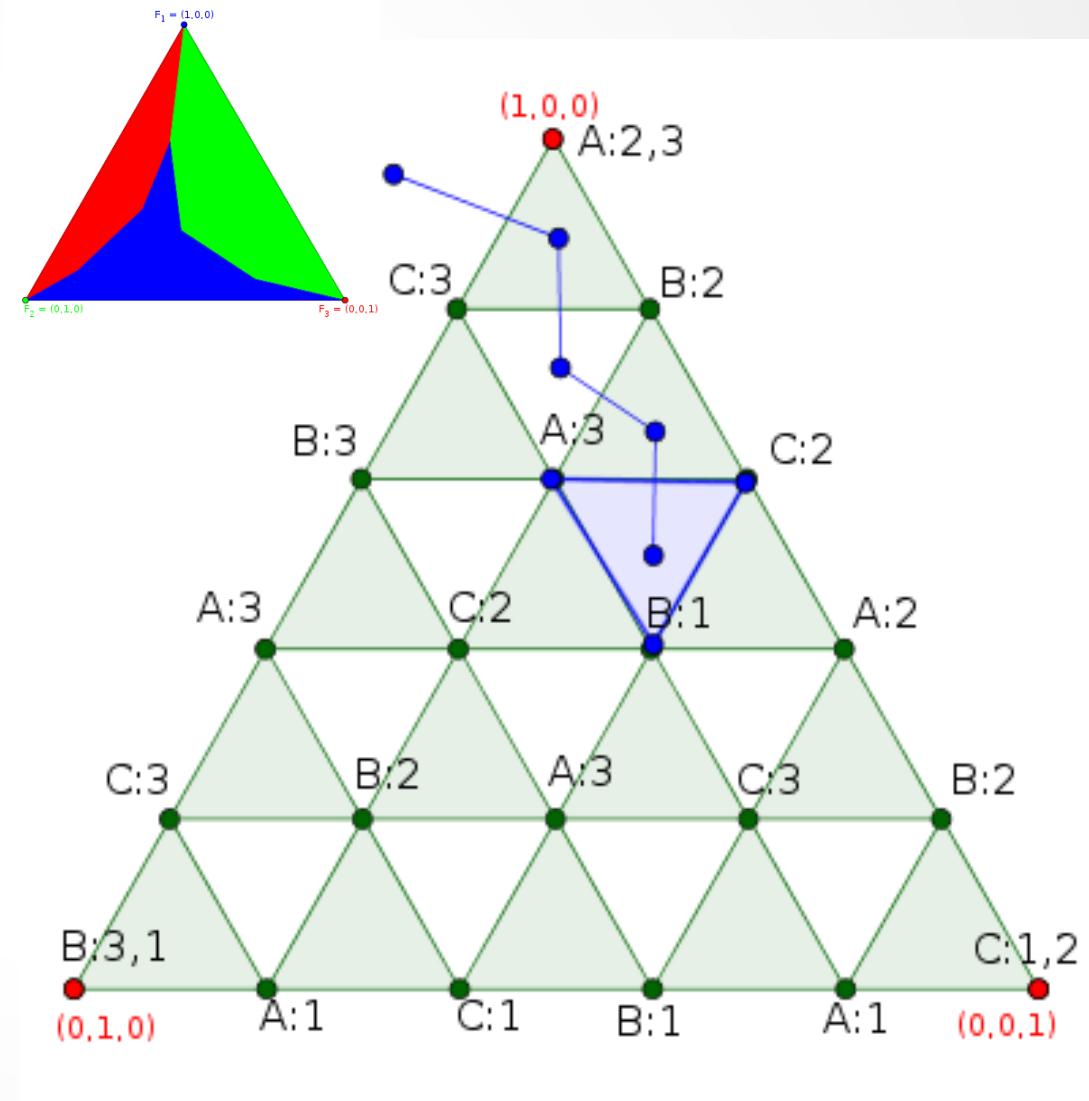


Triangulation – Negative Agents

Fact: When all agents have negative valuations, it is possible to label the n main vertices such that each face is labeled only with the labels of its endpoints.

Corollary: when all valuations are negative, an approximately-envy-free division exists.

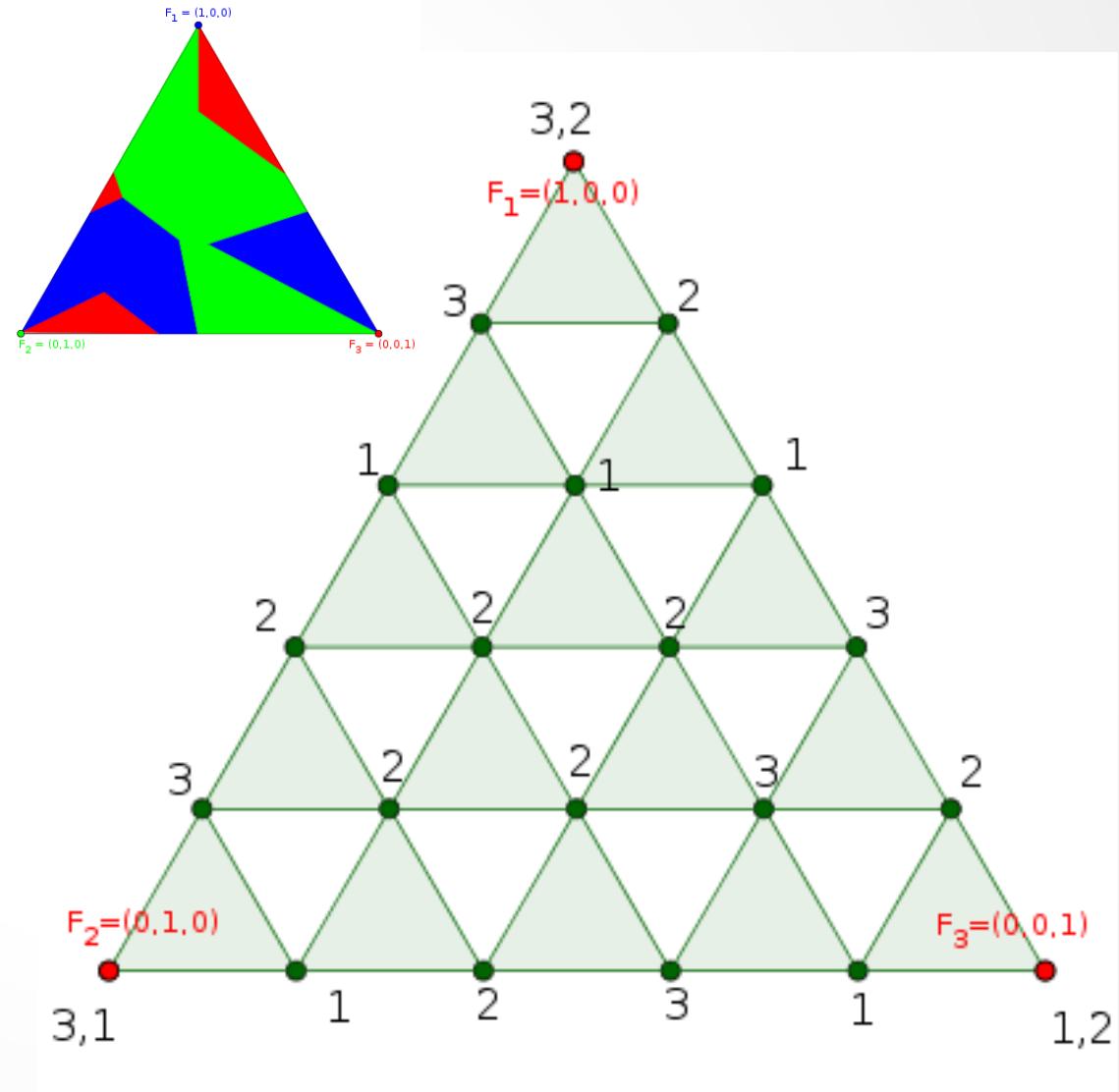
Corollary (Su 1999): when valuations are also continuous, an envy-free division exists.



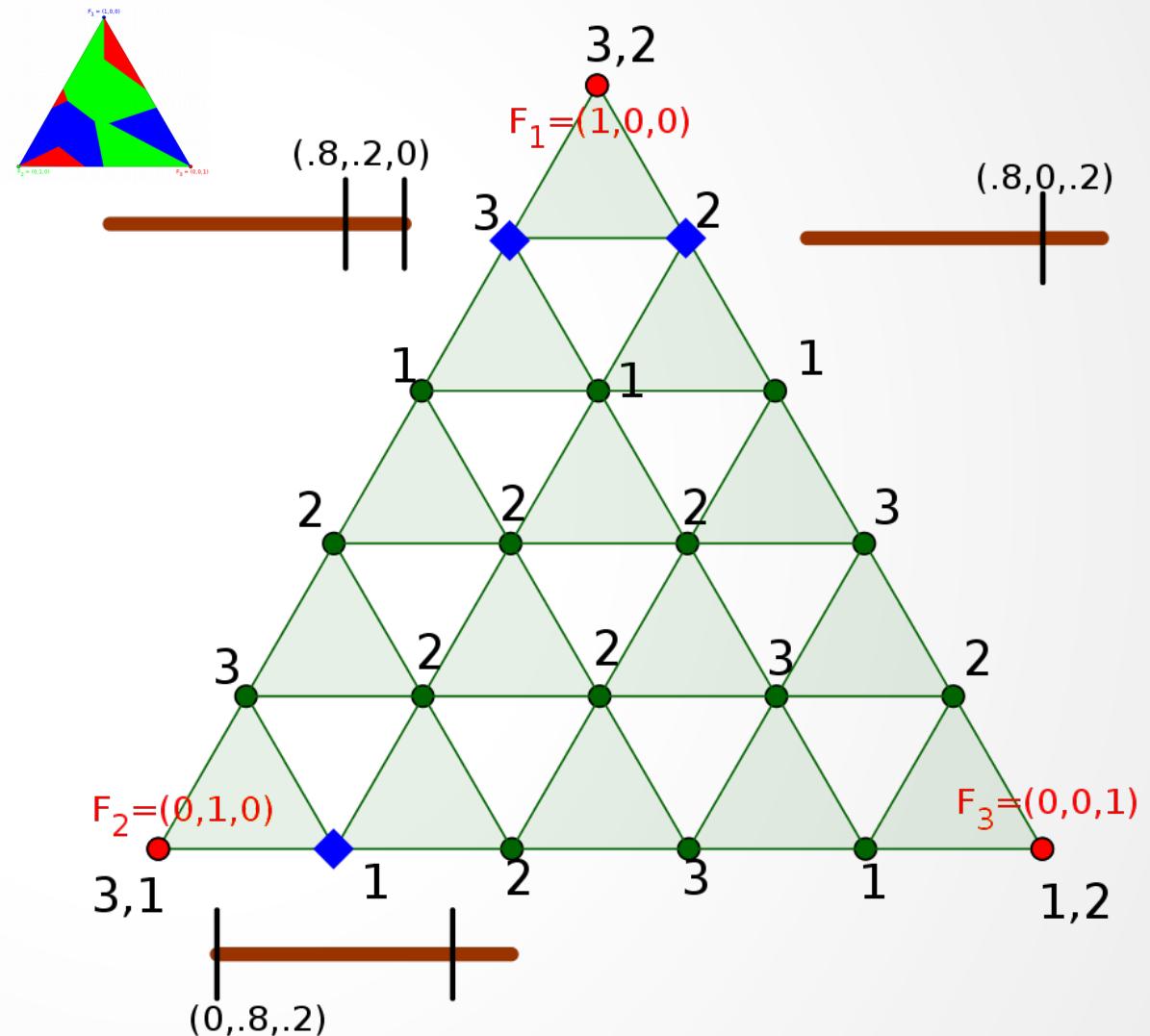
Triangulation – General Agents

In general, the conditions for Sperner's lemma are **not** satisfied.

What can we do?

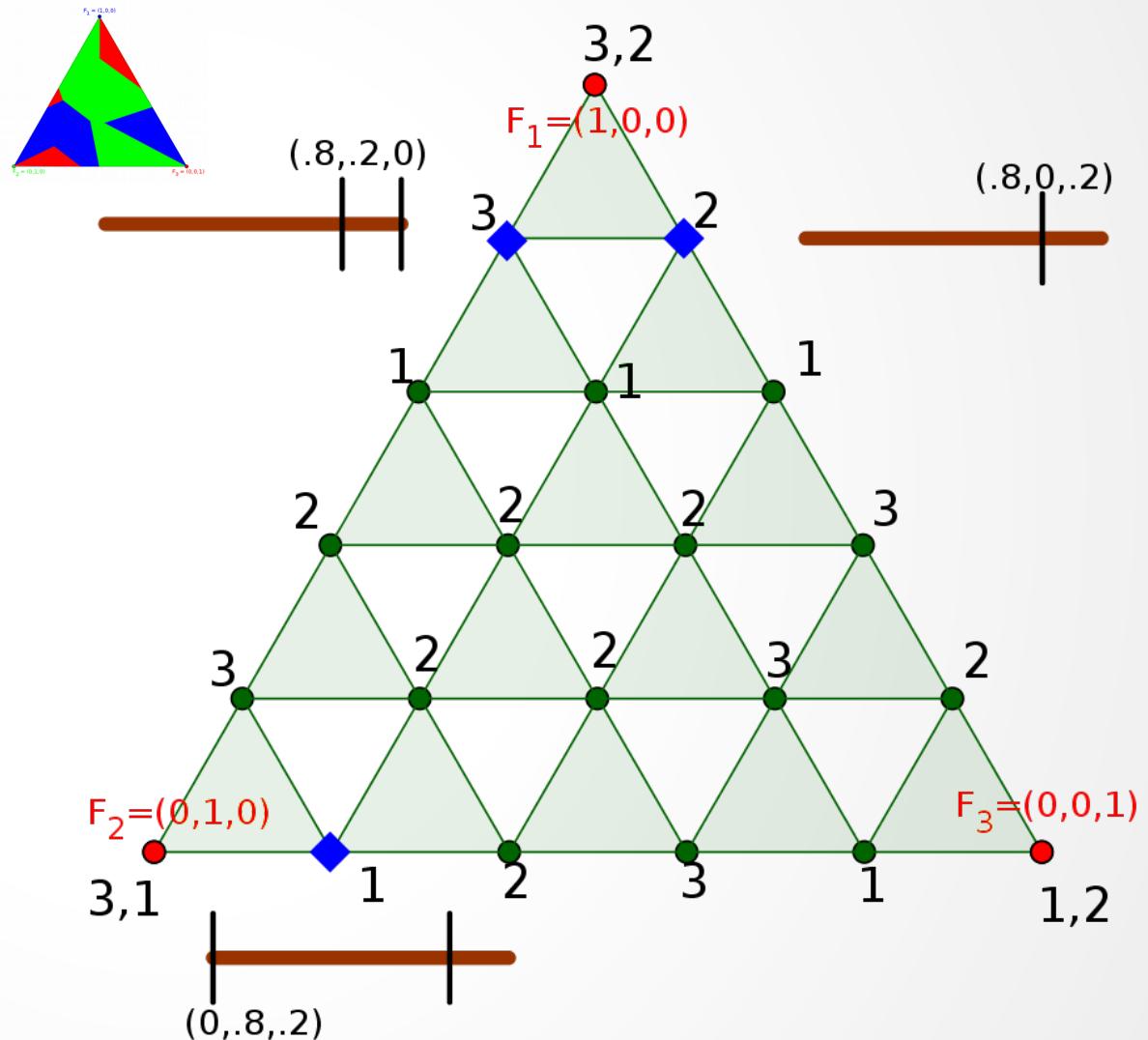


Boundary Permutation Condition



Boundary Permutation Condition

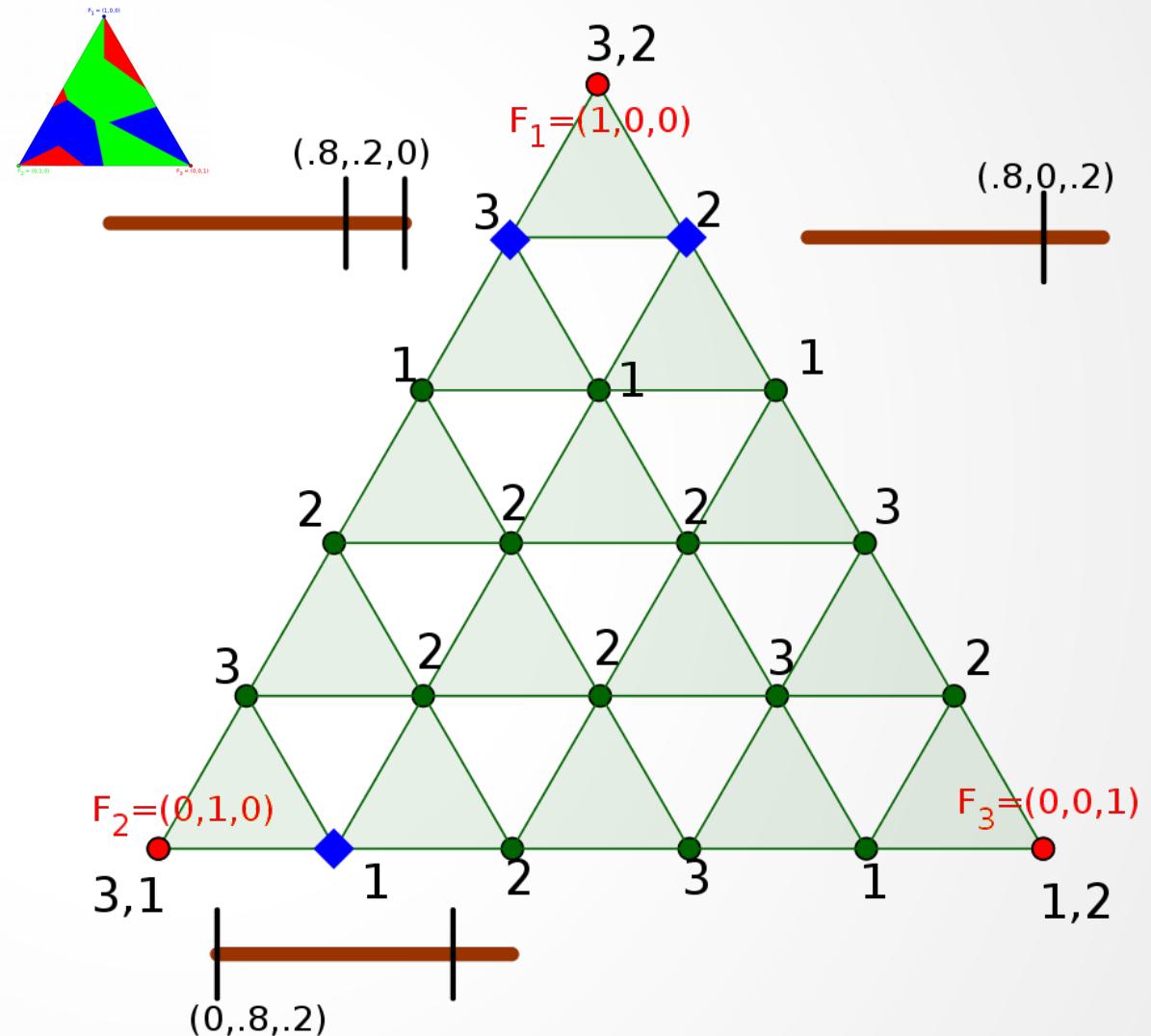
Definition: Two vertices in the simplex are called *friends* if they have the same ordered list of non-zero coordinates.



Boundary Permutation Condition

Definition: Two vertices in the simplex are called *friends* if they have the same ordered list of non-zero coordinates.

Fact: Each agent's labelings on friends are same up to permutation:

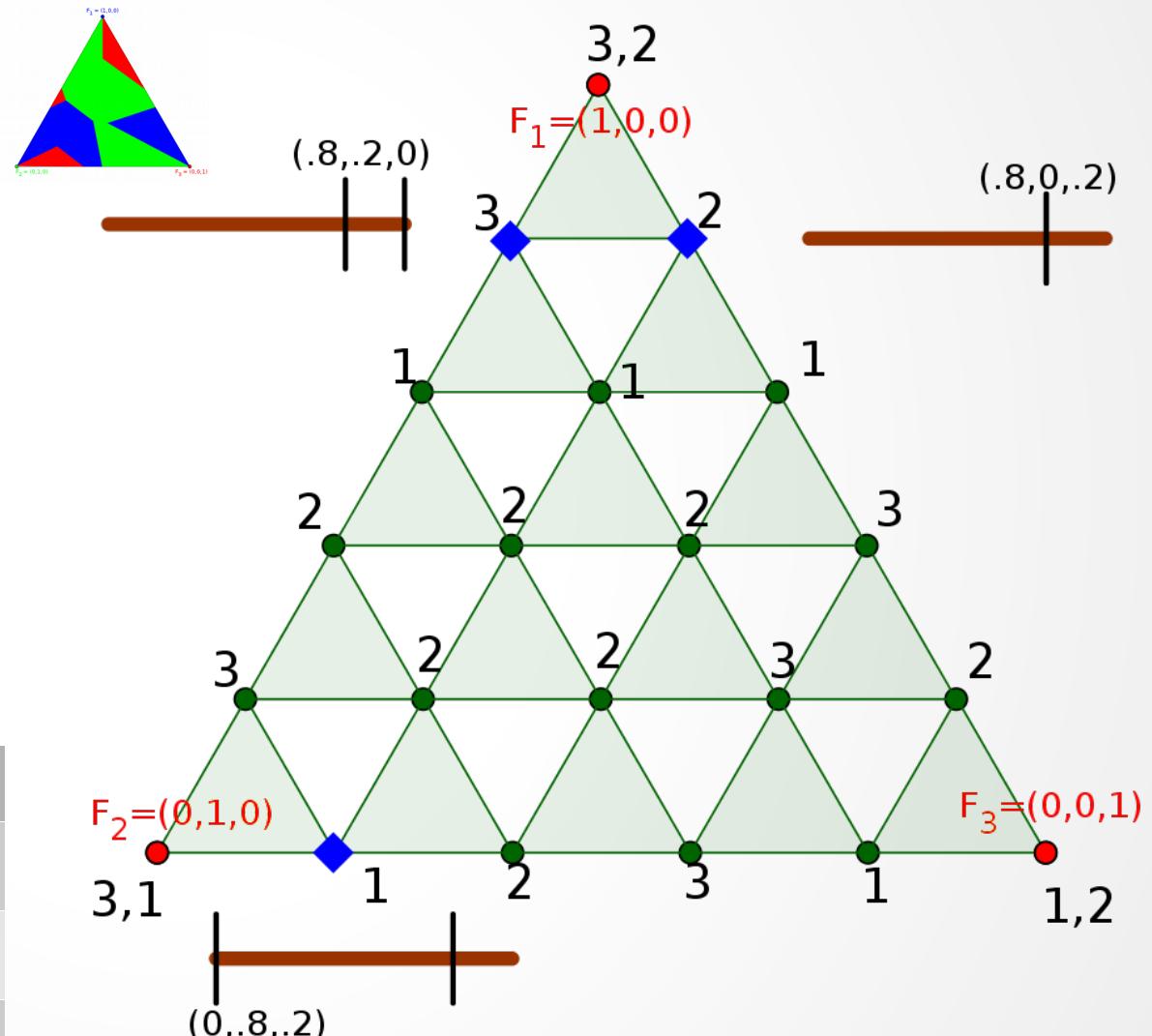


Boundary Permutation Condition

Definition: Two vertices in the simplex are called *friends* if they have the same ordered list of non-zero coordinates.

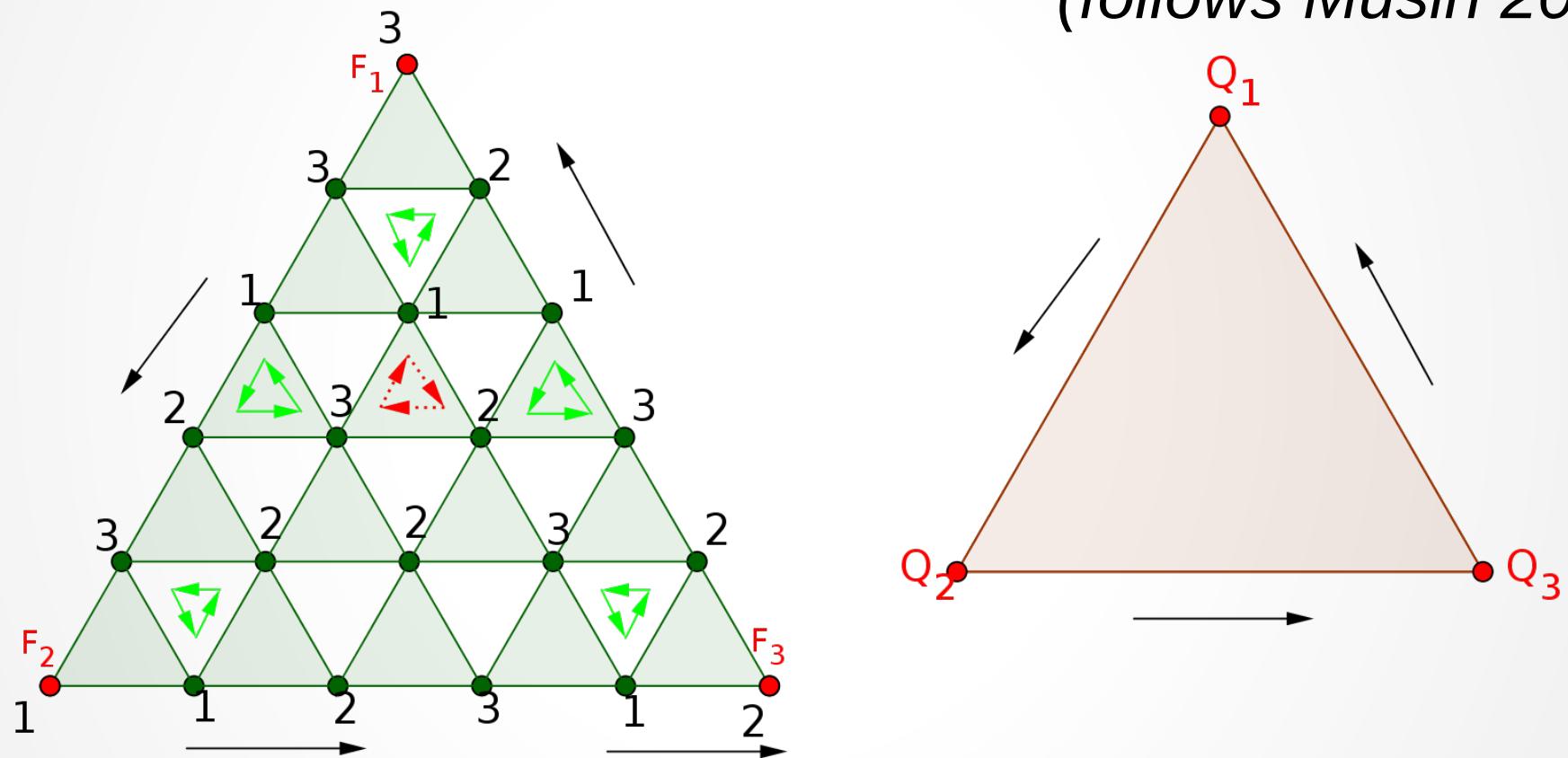
Fact: Each agent's labelings on friends are same up to permutation:

Pref:	Left	Right	Empty	
F_{12}	1	2	3	Even
F_{13}	1	3	2	Odd
F_{23}	2	3	1	Even



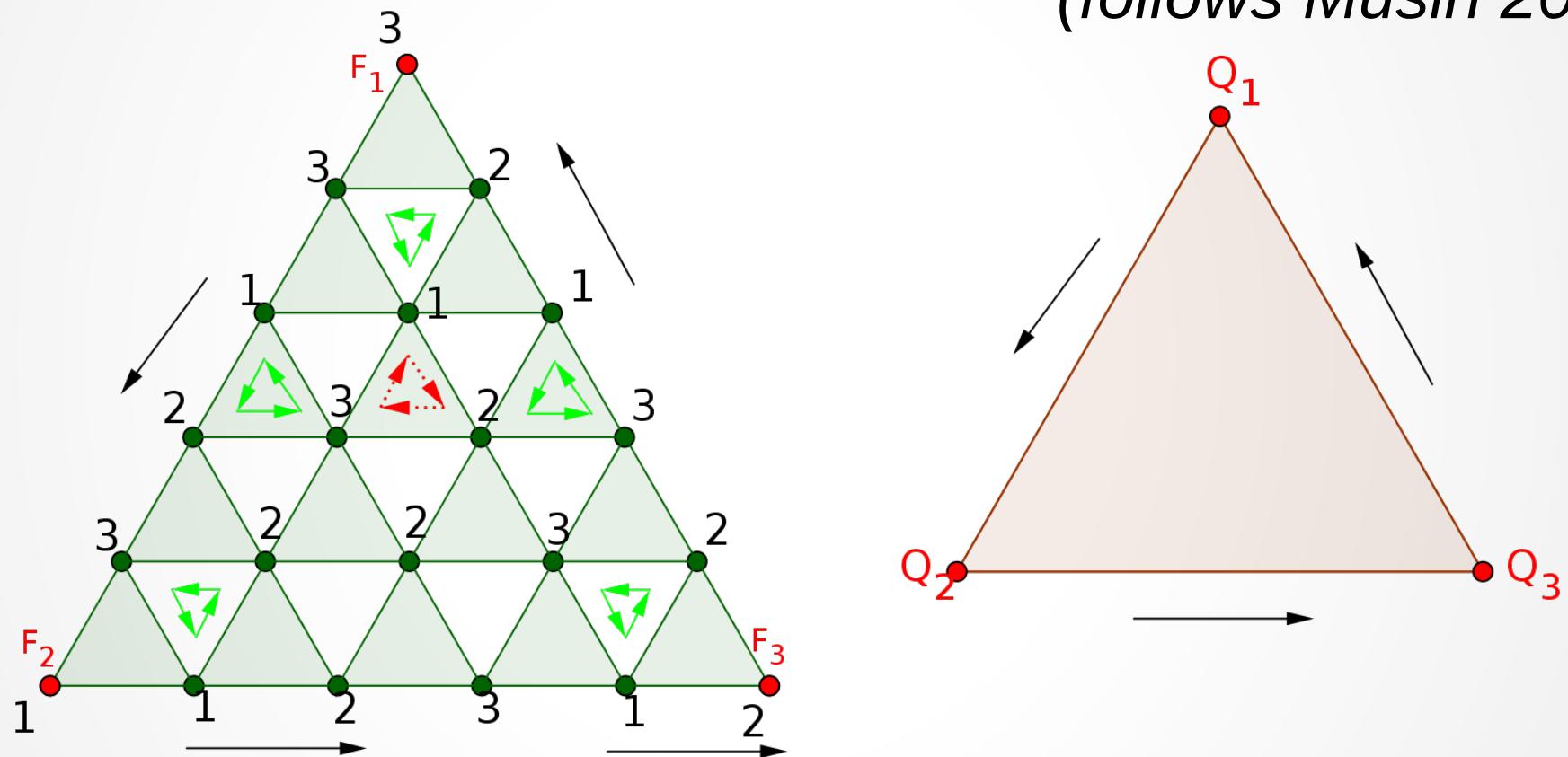
Degree of Labeling

Labeling \equiv mapping from triangulation vertices to vertices of Q
(follows Musin 2014)



Degree of Labeling

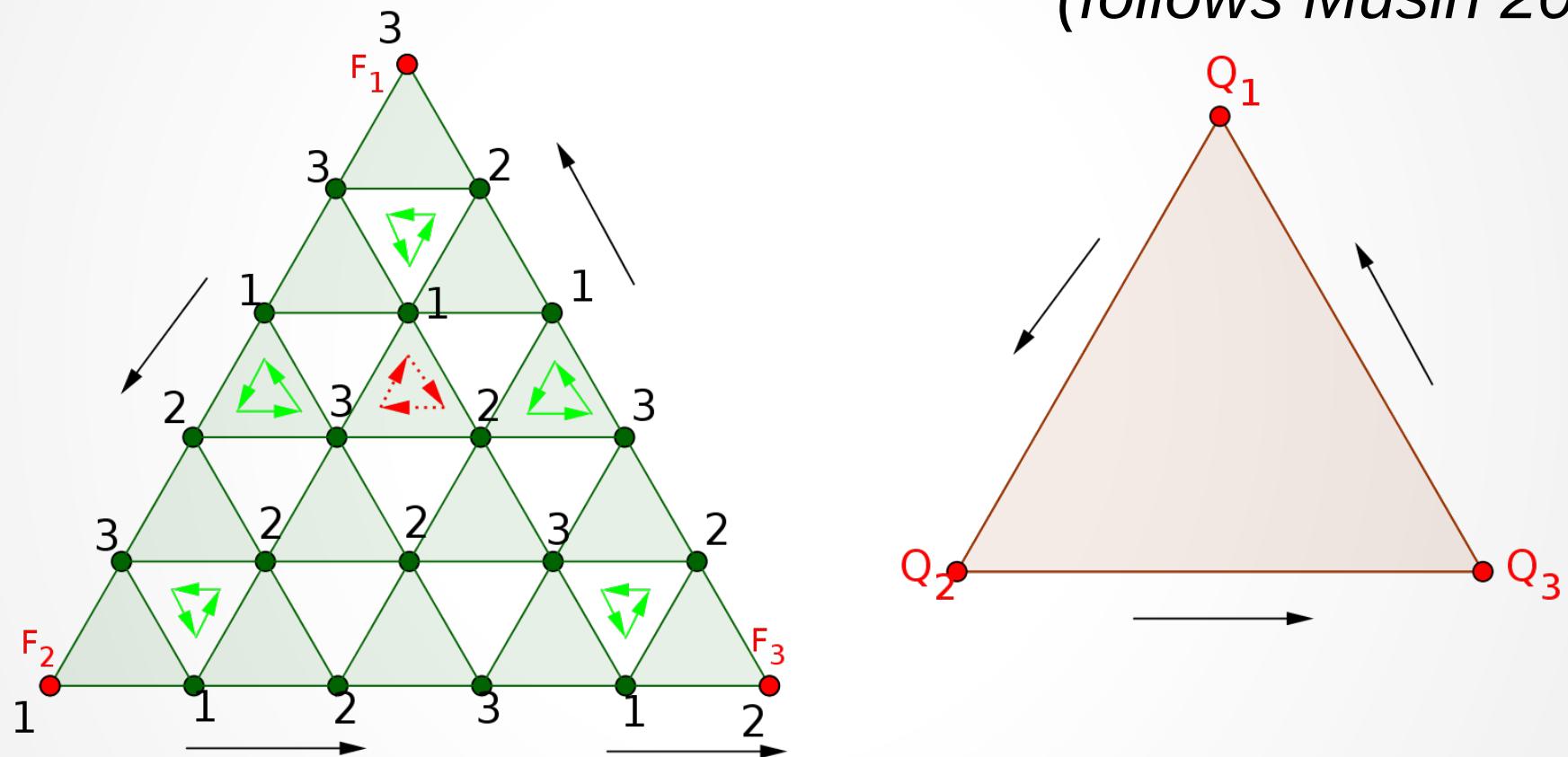
Labeling \equiv mapping from triangulation vertices to vertices of Q
(follows Musin 2014)



Degree of mapping = net number of rounds (CCW=positive).

Degree of Labeling

Labeling \equiv mapping from triangulation vertices to vertices of Q
(follows Musin 2014)



Degree of mapping = net number of rounds (CCW=positive).
Lemma: degree on boundary = degree in interior.

Steps in Existence Proof

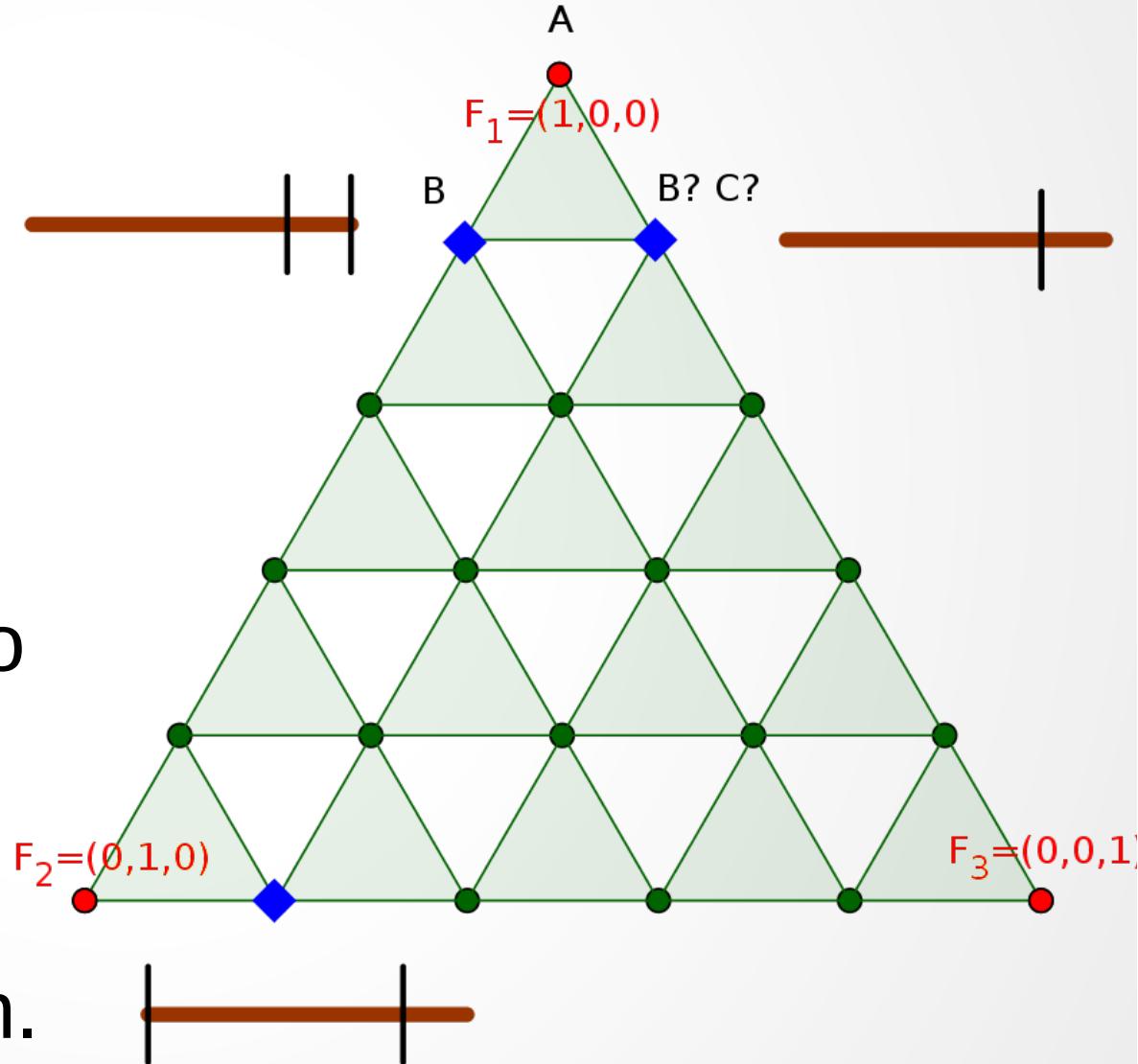
Step	Proved for
<p>1. n agent-labelings with permutation condition → Combined labeling with permutation condition</p>	Any n
<p>2. Permutation condition → Nonzero boundary degree</p>	$n = 3$
<p>3. Boundary degree = Interior degree</p>	Any n (?)

Step 1: n labelings \rightarrow 1 labeling

We need to assign owners to vertices s.t.:

- In each sub-simplex, each vertex belongs to a **different** owner.
- Friends are assigned to **the same** owner.

Does not work with the equilateral triangulation.



Step 1: n labelings \rightarrow 1 labeling

We need to assign owners to vertices s.t.:

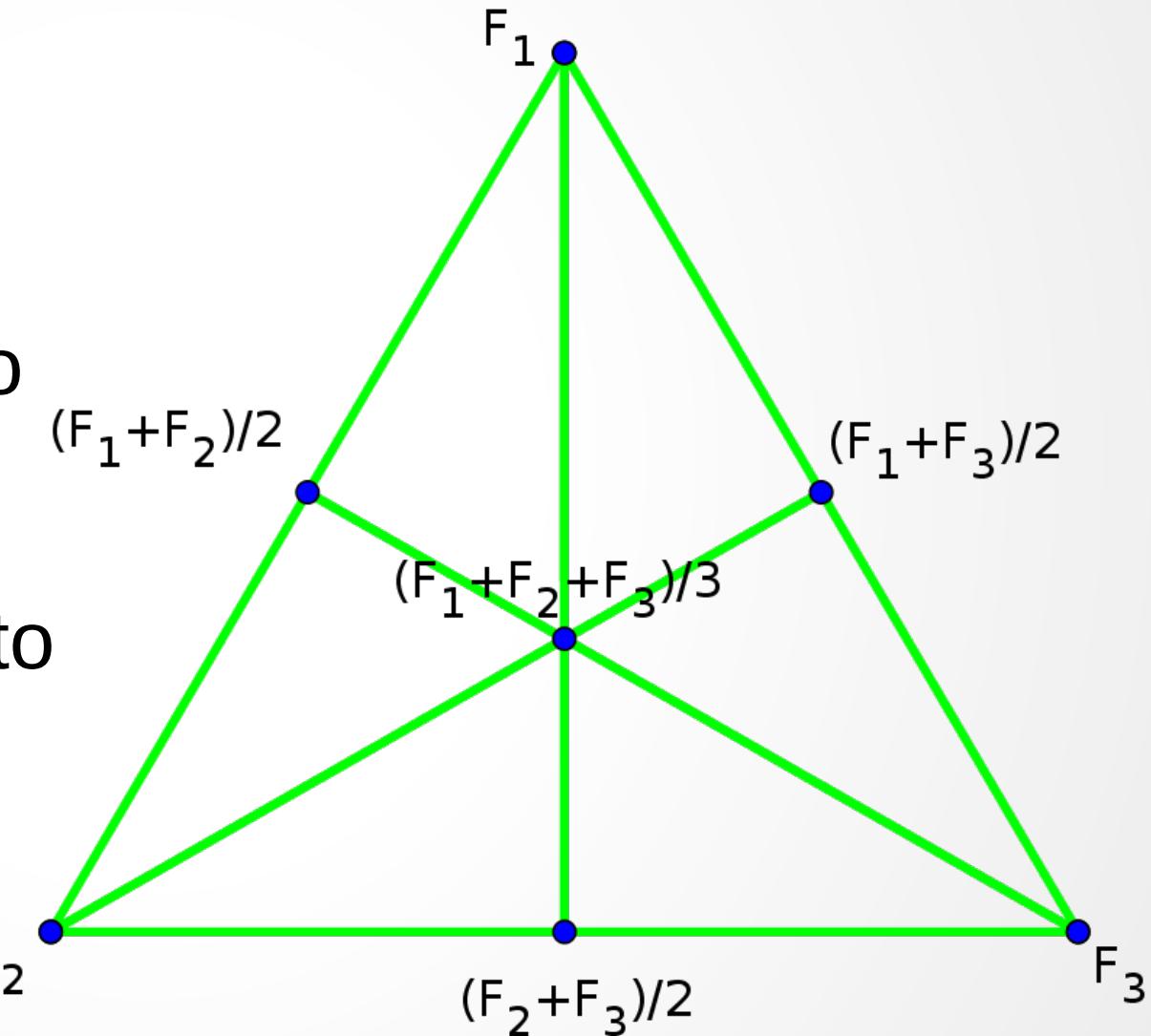
- In each sub-simplex, each vertex belongs to a **different** owner.
- Friends are assigned to **the same** owner.

Lemma: it works with
barycentric triangulation

Step 1: n labelings \rightarrow 1 labeling

We need to assign owners to vertices s.t.:

- In each sub-simplex, each vertex belongs to a **different** owner.
- Friends are assigned to **the same** owner.

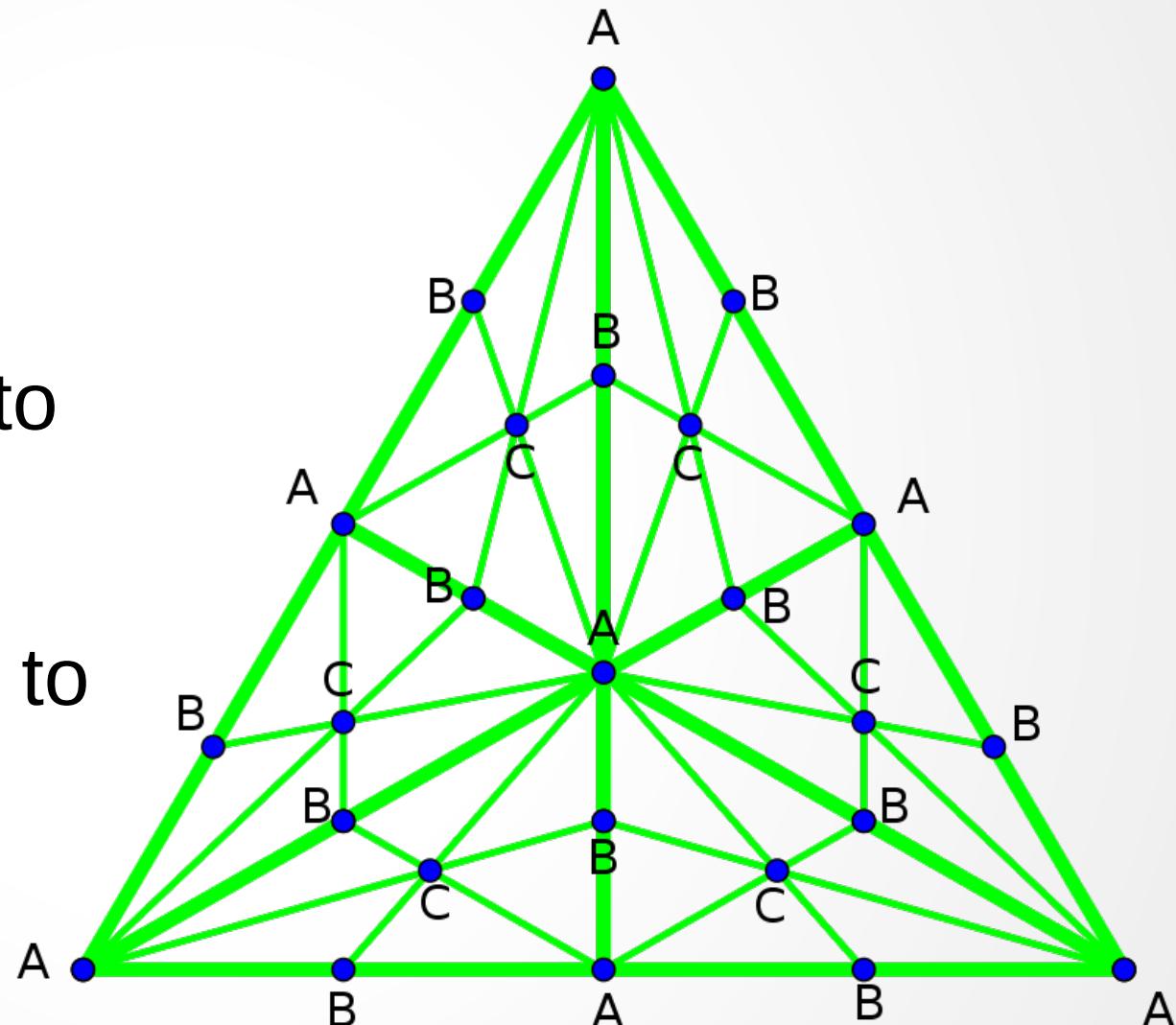


Lemma: it works with
barycentric triangulation

Step 1: n labelings \rightarrow 1 labeling

We need to assign owners to vertices s.t.:

- In each sub-simplex, each vertex belongs to a **different** owner.
- Friends are assigned to **the same** owner.



Lemma: it works with *barycentric triangulation*

Step 2: Permutation → Boundary degree

Permutation condition:

Pref:	Left	Right	Empty	
F_{12}	1	2	3	Even
F_{13}	1	3	2	Odd
F_{23}	2	3	1	Even

Agent condition: Either:
(+) In each main-vertex i , the label is i , or:
(-) In each main-vertex i , the label can be anything but i .

Lemma: When $n=3$, if labeling satisfies *permutation condition* and *agent condition*, then labels on main vertices can be chosen such that: boundary-degree mod 3 \leftrightarrow 0.

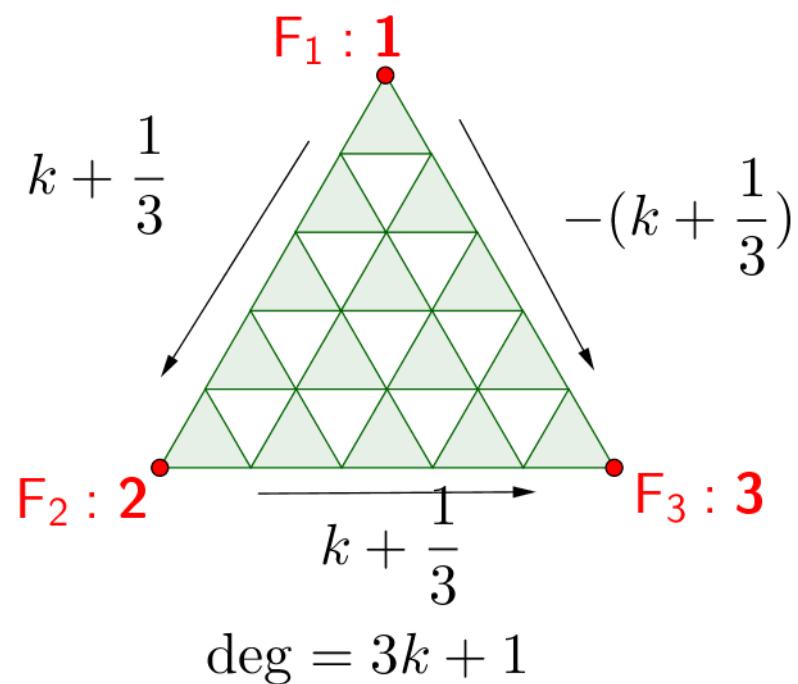
Step 2: Permutation → Boundary degree

Permutation condition:

Pref:	Left	Right	Empty	
F_{12}	1	2	3	Even
F_{13}	1	3	2	Odd
F_{23}	2	3	1	Even

Agent condition: Either:
 (+) In each main-vertex i , the label is i , or:
 (-) In each main-vertex i , the label can be anything but i .

Proof:



Step 2: Permutation → Boundary degree

Permutation condition:

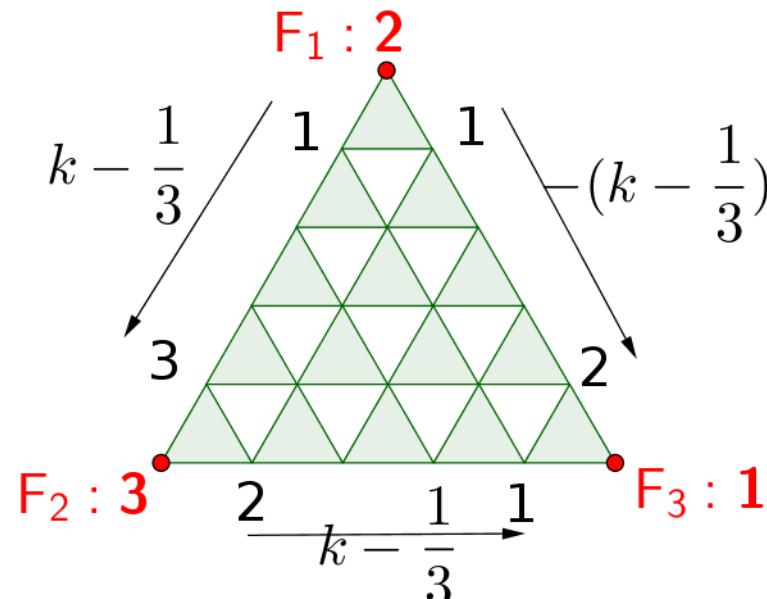
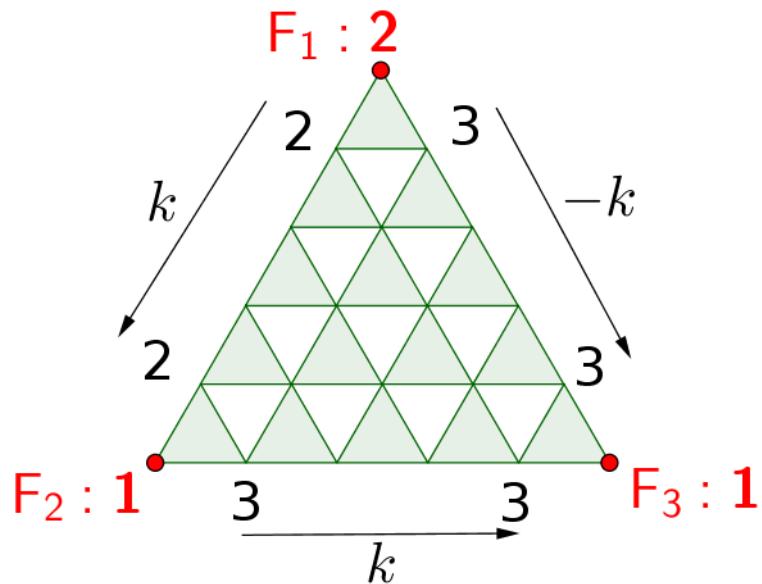
Pref:	Left	Right	Empty	
F_{12}	1	2	3	Even
F_{13}	1	3	2	Odd
F_{23}	2	3	1	Even

Agent condition: Either:

(+) In each main-vertex i , the label is i , or:

(-) In each main-vertex i , the label can be anything but i .

2 of 9 cases shown below:



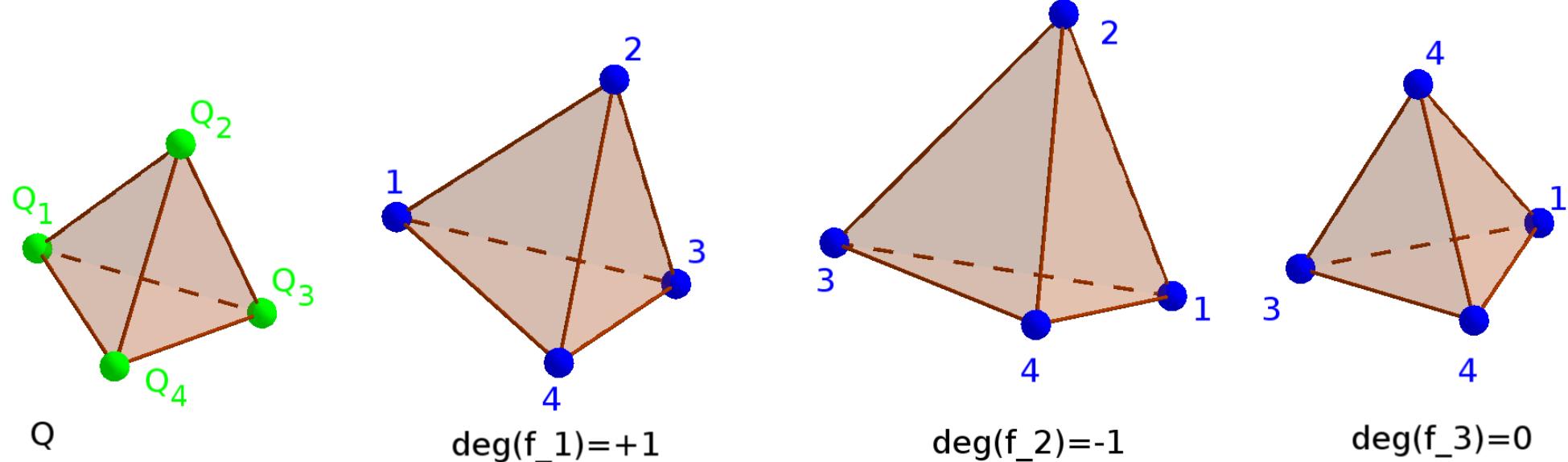
$$\deg = (3k) - 2/3 - 1/3 = 3k - 1 \quad \deg = (3k - 1) - 1/3 + 1/3 = 3k - 1$$

Step 3: Boundary degree = Interior degree

Definition:

Degree of labeling of an n -simplex in R^{n-1}

- = sign of determinant of affine transformation to Q
- = +1 if onto&no reflection, -1 if onto&one reflection,
- 0 if not onto.



Step 3: Boundary degree = Interior degree

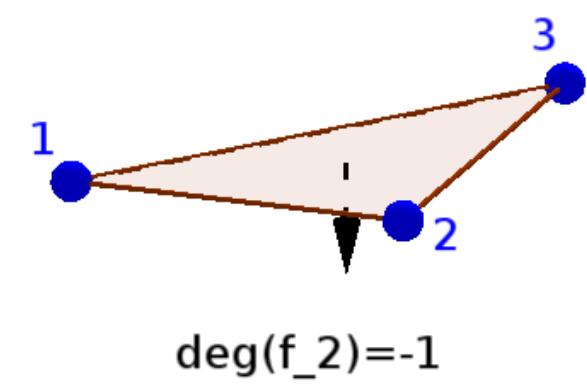
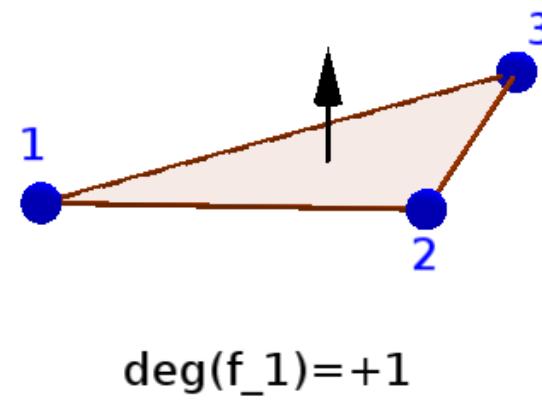
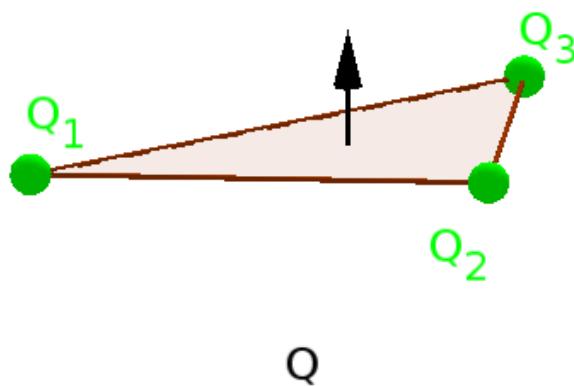
Definition:

Orientation of an $(n-1)$ -simplex in R^{n-1}

= one of its two adjacent half-spaces.

Degree of labeling of an $(n-1)$ -simplex in R^{n-1}

= sign of determinant of *any* affine transformation
to \mathbb{Q} that preserves the orientation.



Step 3: Boundary degree = Interior degree

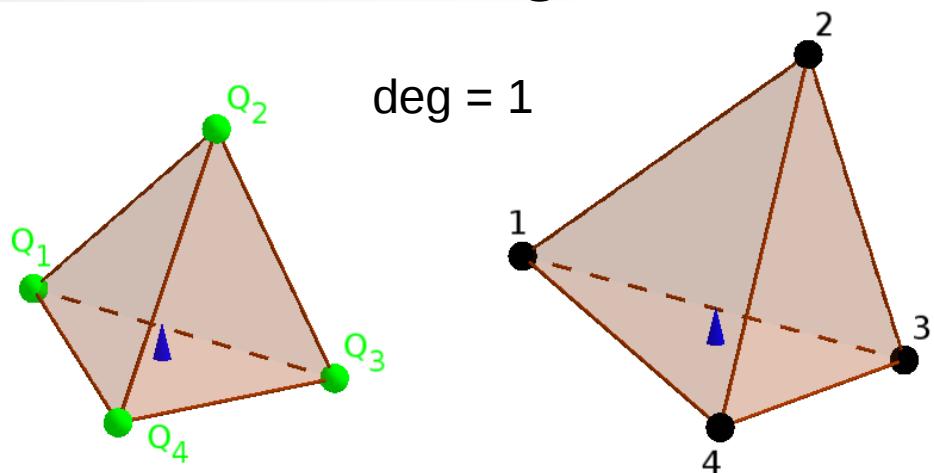
Lemma:

Degree of a labeling of an n -simplex in R^{n-1} ,
= sum of degrees on each face oriented *inwards*:

Step 3: Boundary degree = Interior degree

Lemma:

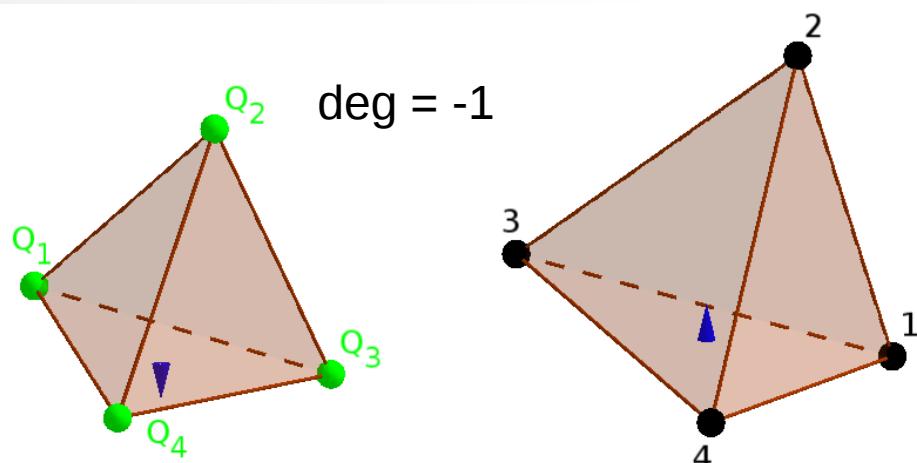
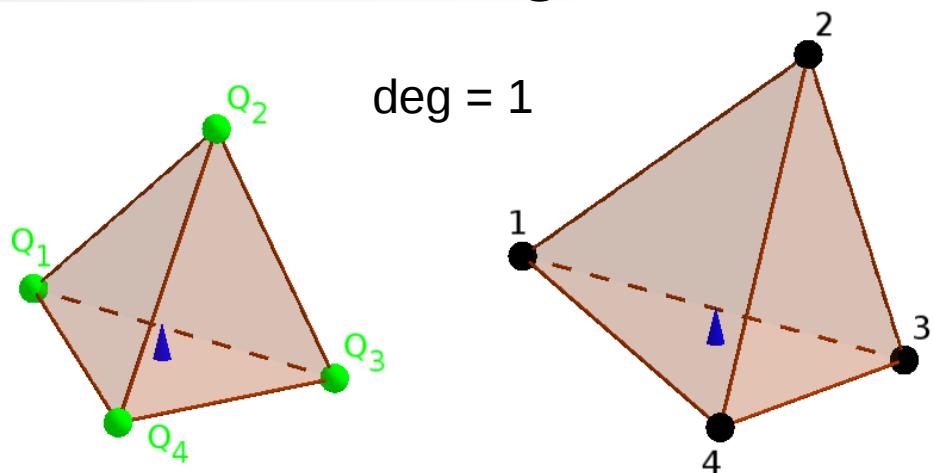
Degree of a labeling of an n -simplex in R^{n-1} ,
= sum of degrees on each face oriented *inwards*:



Step 3: Boundary degree = Interior degree

Lemma:

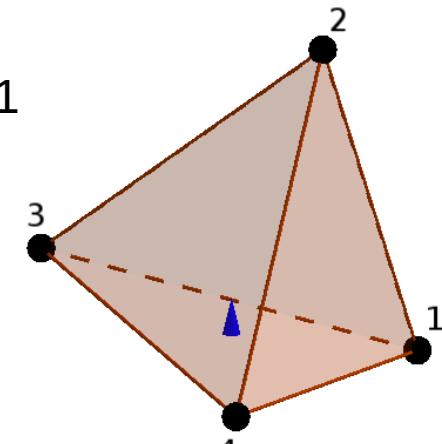
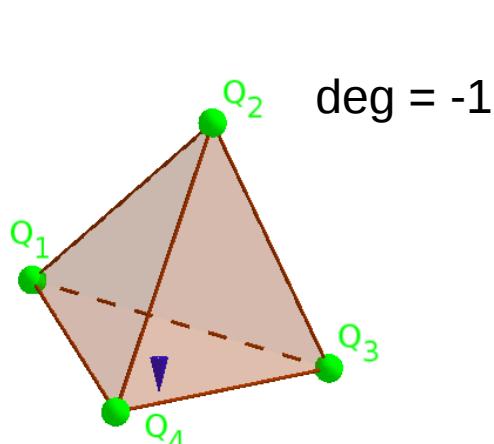
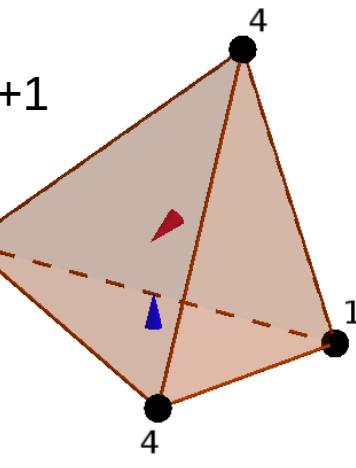
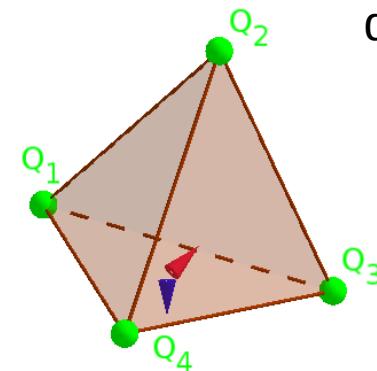
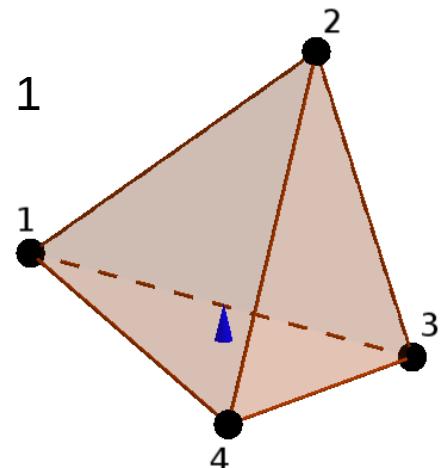
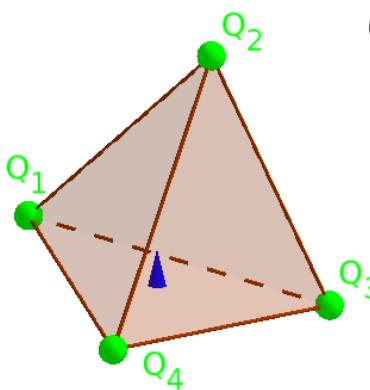
Degree of a labeling of an n -simplex in R^{n-1} ,
= sum of degrees on each face oriented *inwards*:



Step 3: Boundary degree = Interior degree

Lemma:

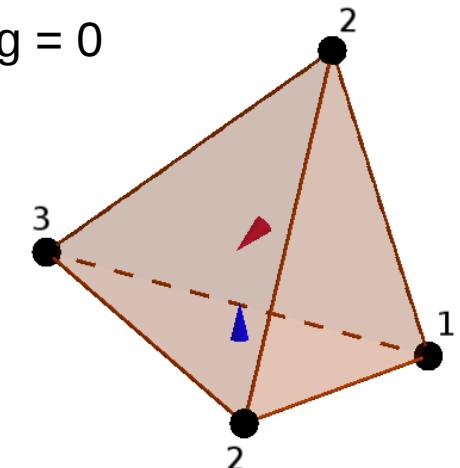
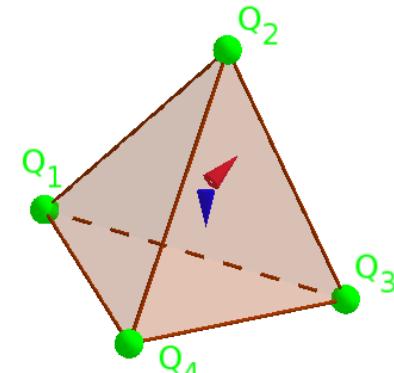
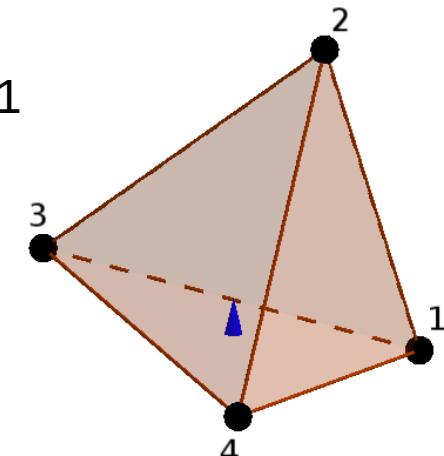
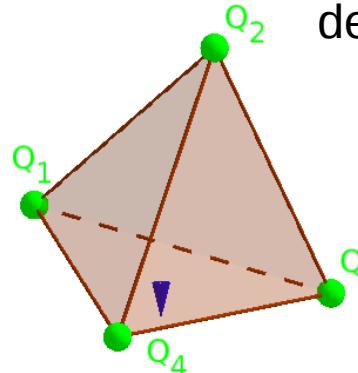
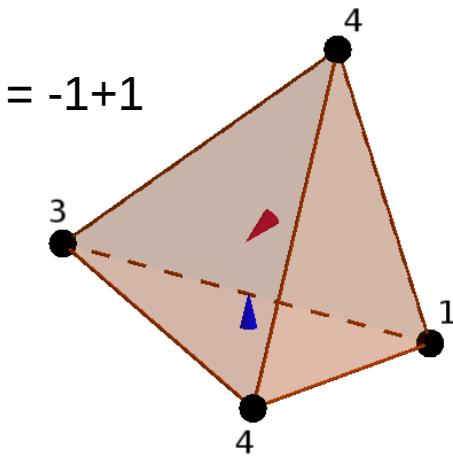
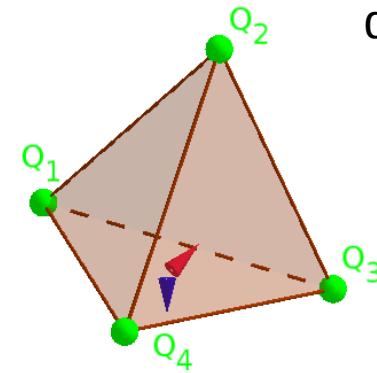
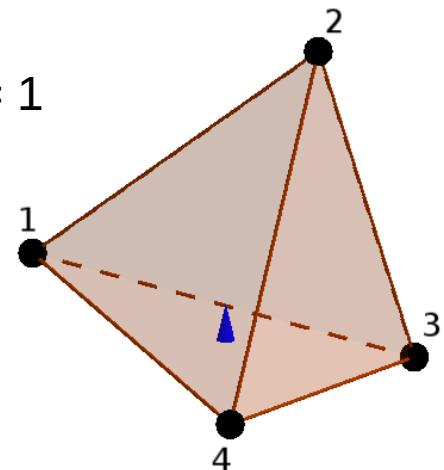
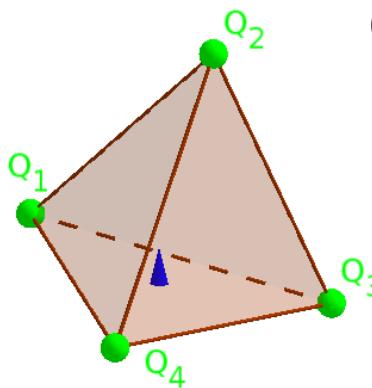
Degree of a labeling of an n -simplex in R^{n-1} ,
= sum of degrees on each face oriented *inwards*:



Step 3: Boundary degree = Interior degree

Lemma:

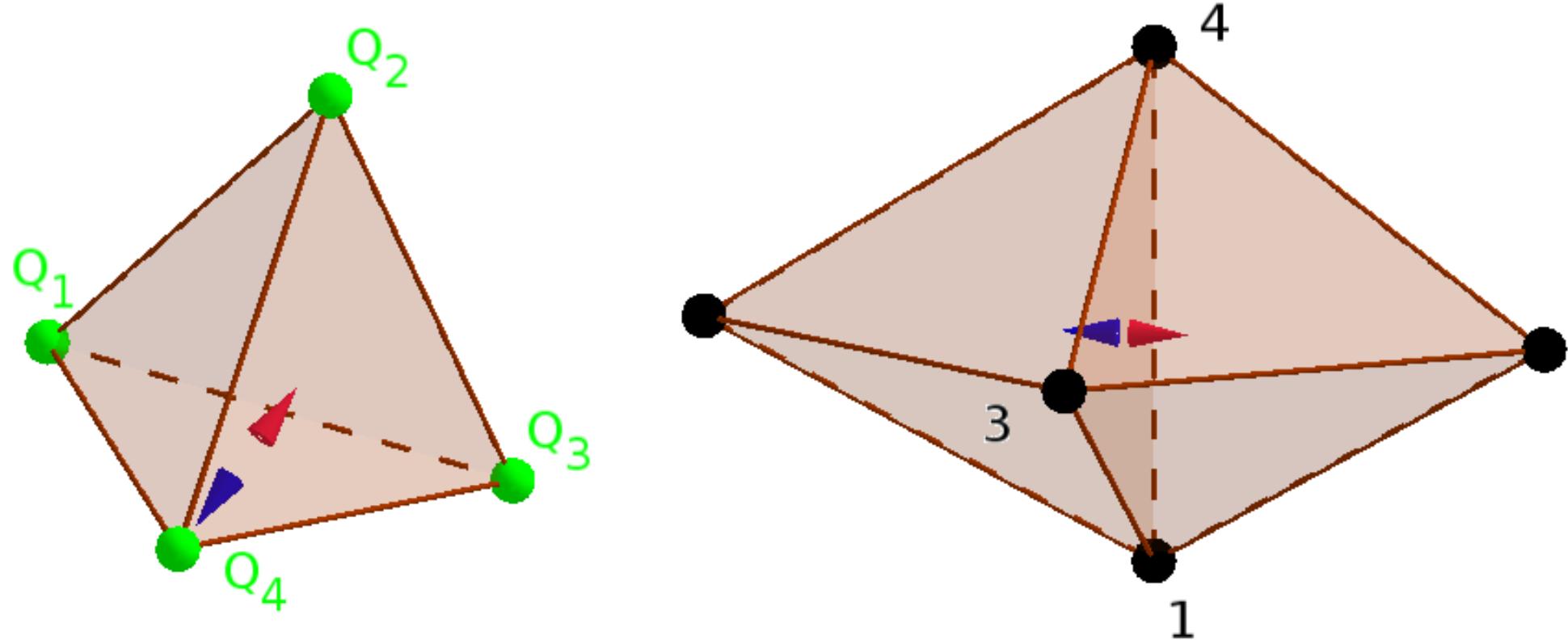
Degree of a labeling of an n -simplex in R^{n-1} ,
= sum of degrees on each face oriented *inwards*:



Step 3: Boundary degree = Interior degree

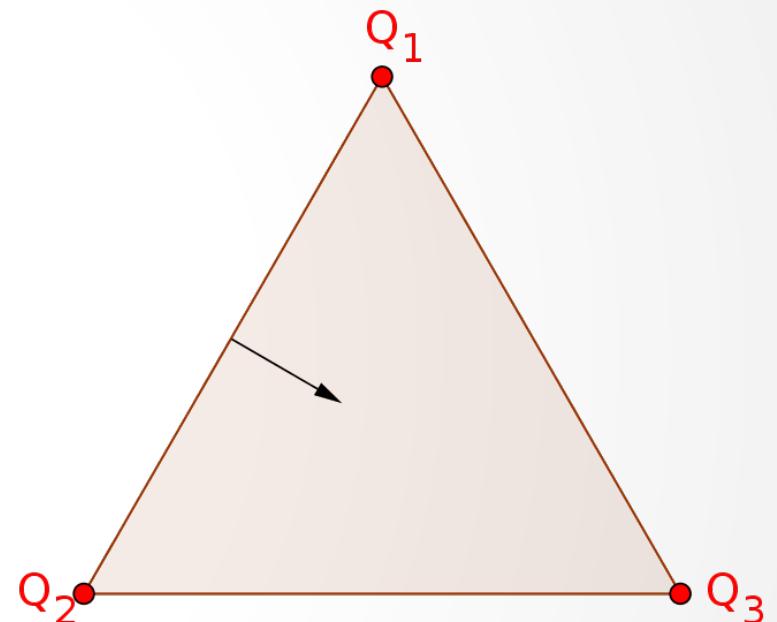
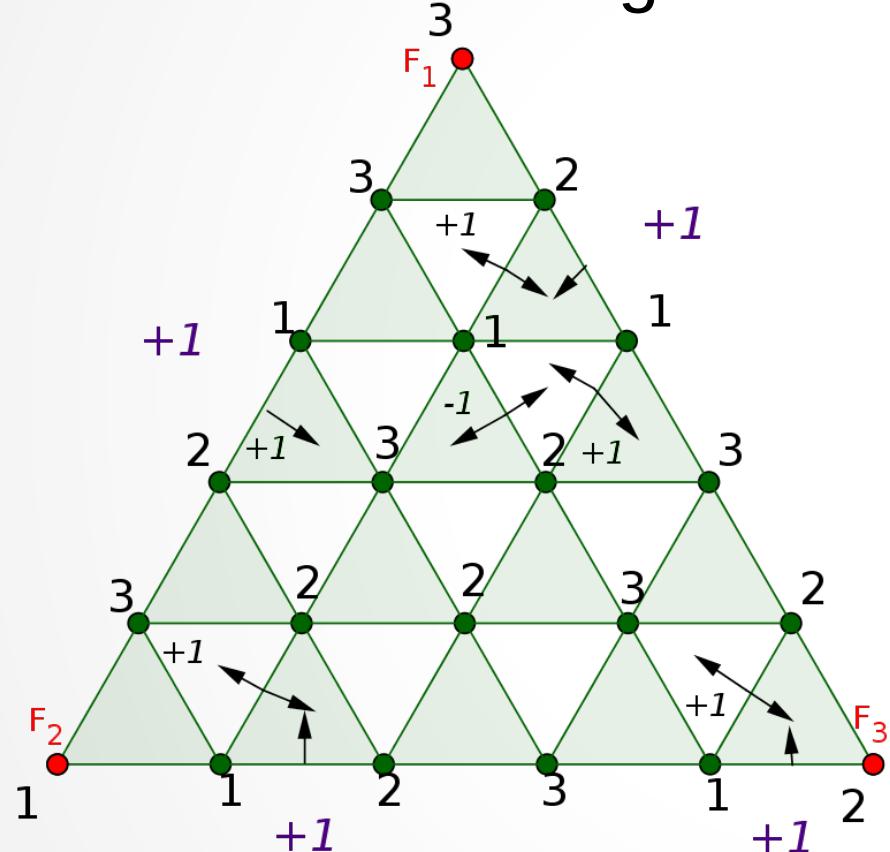
Lemma:

Sum of degrees of simplices in triangulation
= sum of degrees on each *boundary* face,
– since the internal faces cancel out:



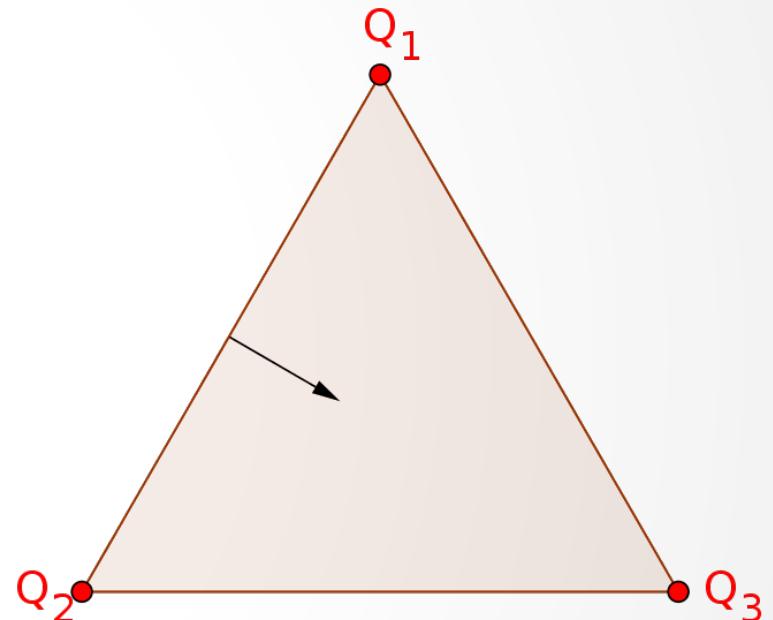
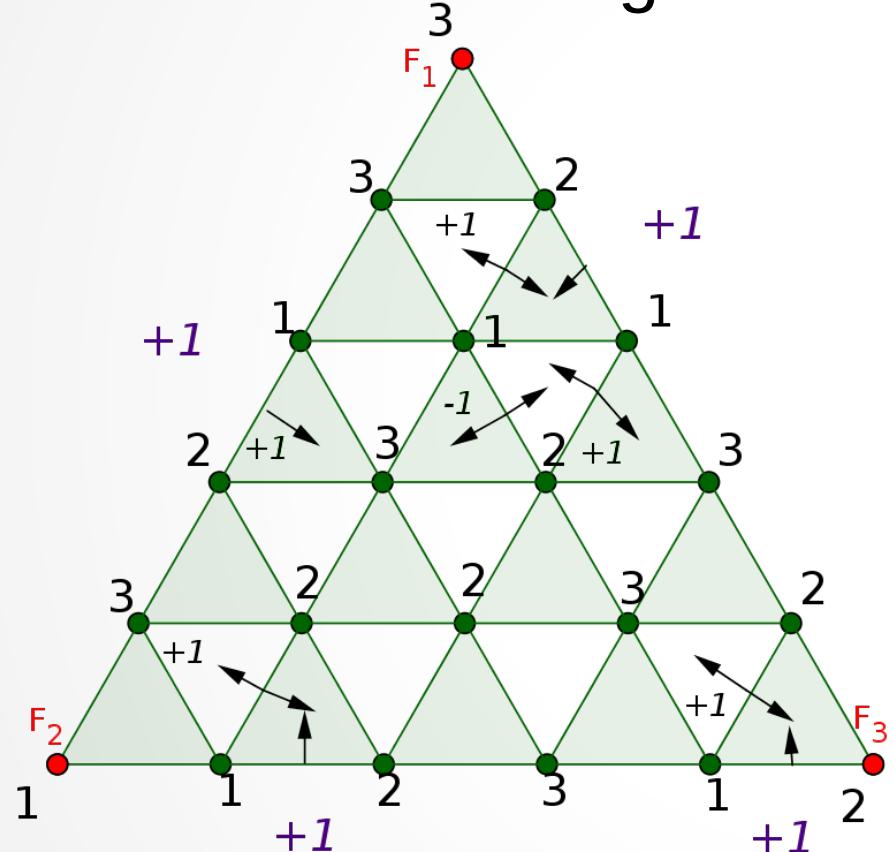
Step 3: Boundary degree = Interior degree

Definition: degree of triangulation labeling
= sum of degrees of each sub-simplex labeling.



Step 3: Boundary degree = Interior degree

Definition: degree of triangulation labeling
= sum of degrees of each sub-simplex labeling.



Lemma: interior degree = sum of degrees on faces
= sum of degrees on faces of boundary = boundary degree.

Conclusion

Step	Proved for
1. n agent-labelings with perm. condition → Combined labeling with perm. condition	Any n
2. Permutation condition → Nonzero boundary degree	$n = 3$
3. Boundary degree = Interior degree	Any n (?)

Theorem: for 3 agents with continuous valuations,
an envy-free connected division exists
for arbitrary mixed valuations.

Open question

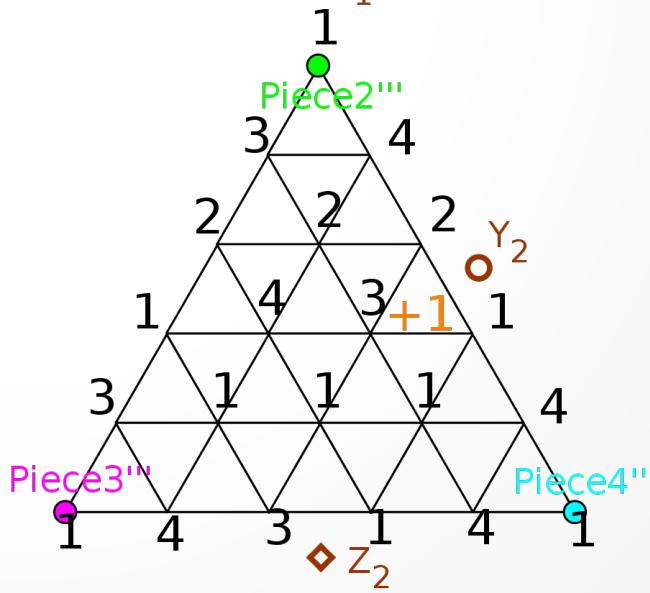
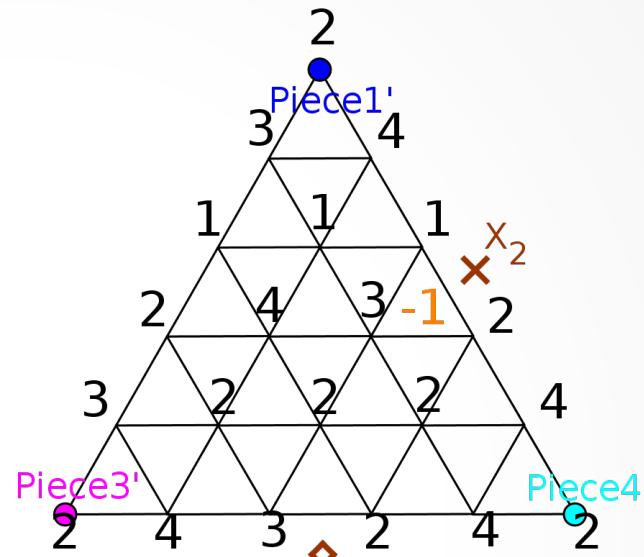
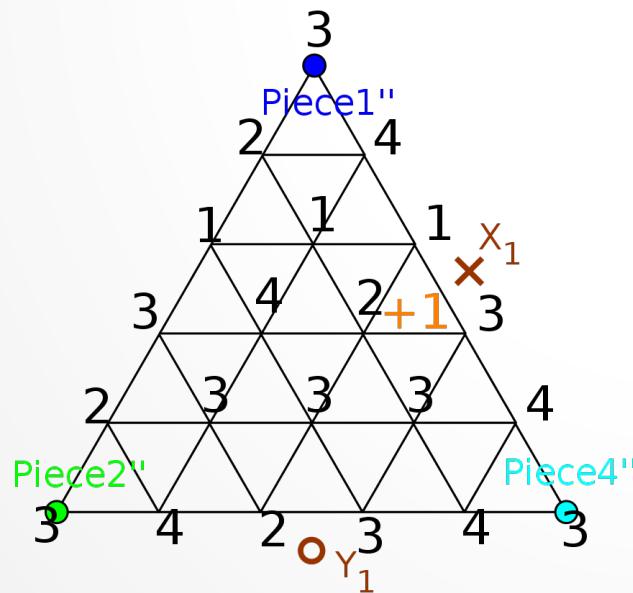
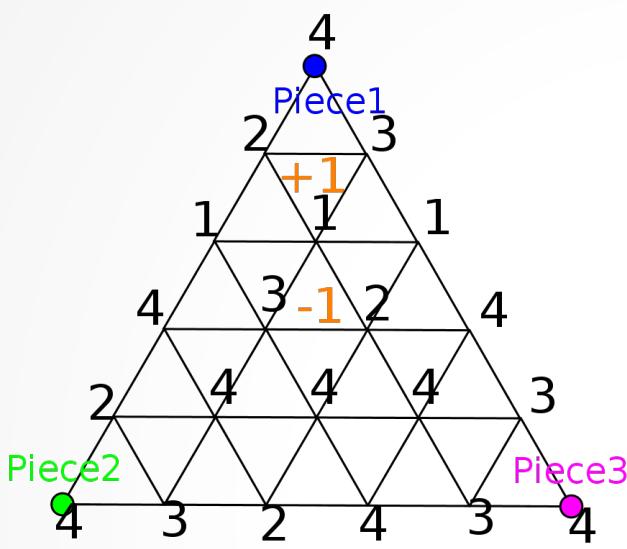
Permutation condition for 4 or more agents:

Pref:	Left	Middle	Right	Empty	
F_{123}	1	2	3	4	Even
F_{124}	1	2	4	3	Odd
F_{134}	1	3	4	2	Even
F_{234}	2	3	4	1	Odd

Conjecture: If labeling satisfies
permutation condition and *agent condition*,
then boundary-degree mod $n \neq 0$.

*If conjecture is true, then connected envy-free division exists
for arbitrary mixed valuations!*

Open question



Dividing Goods that are Bads

(Midrash Rabba, Genesis 33:1)



Dividing Goods that are Bads

(Midrash Rabba, Genesis 33:1)



Dividing Goods that are Bads

(Midrash Rabba, Genesis 33:1)



Dividing Goods that are Bads

(Midrash Rabba, Genesis 33:1)



Dividing Goods that are Bads

(Midrash Rabba, Genesis 33:1)

