"DIVIDE THE LAND EQUALLY" (Ezekiel 47:14)

Fairly Dividing a Cake after Some Parts were Burnt in the Oven

Erel Segal-Halevi

ARIEL UNIVERSITY

Inspired by:

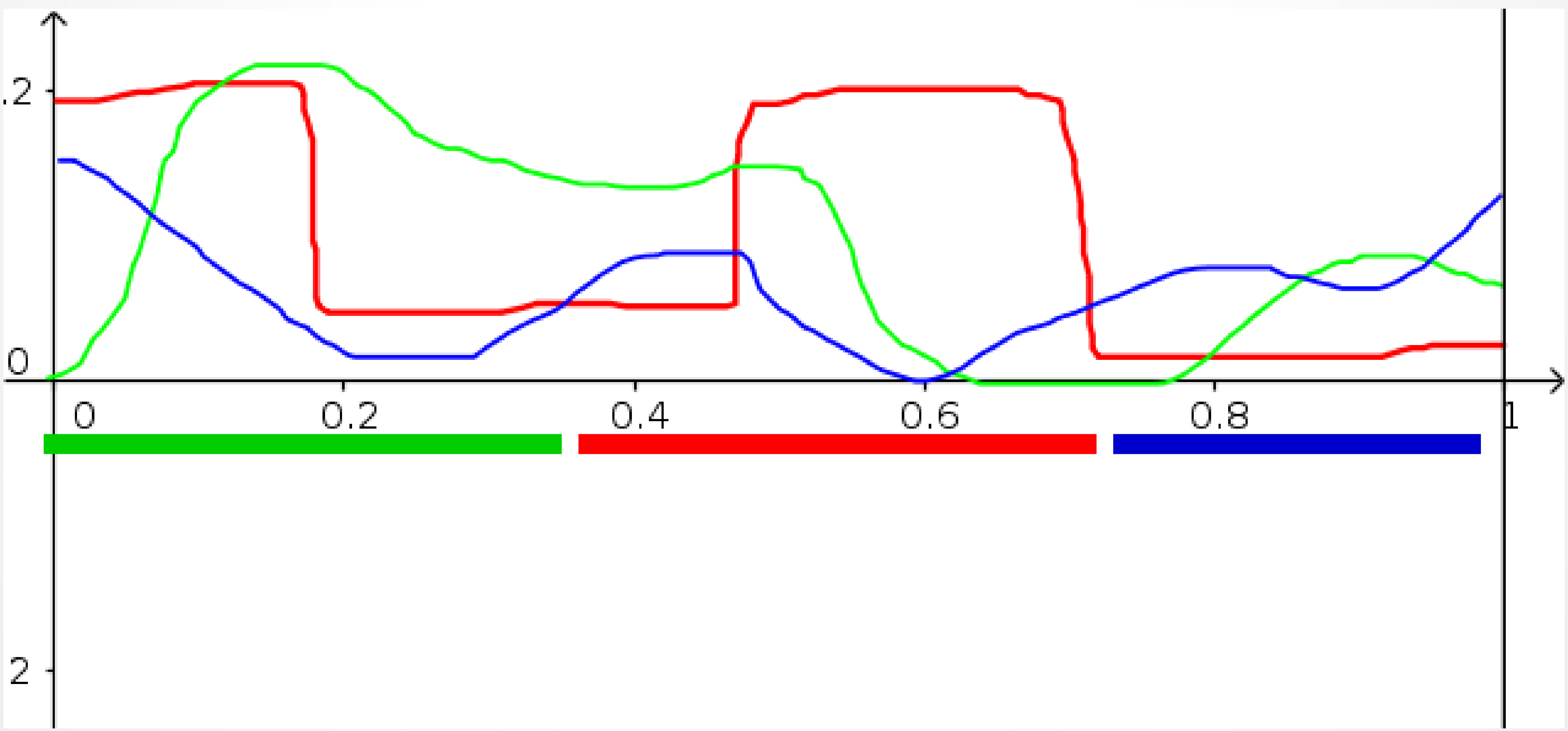
- Francis Su (1999): ”Sperner’s Lemma in Fair Division”.
- Oleg Musin (2014): ”Around Sperner’s lemma”.
- Anna Bogomolnaia, Herve Moulin, Fedor Sandomirskiy & Elena Yanovskaya (2017): ”Competitive Division of a Mixed Manna”.

Connected Envy-Free Cake-Cutting

Input: **Cake** C (interval [0,1]).

n agents with different **value-densities.**

v_i



Connected Envy-Free Cake-Cutting

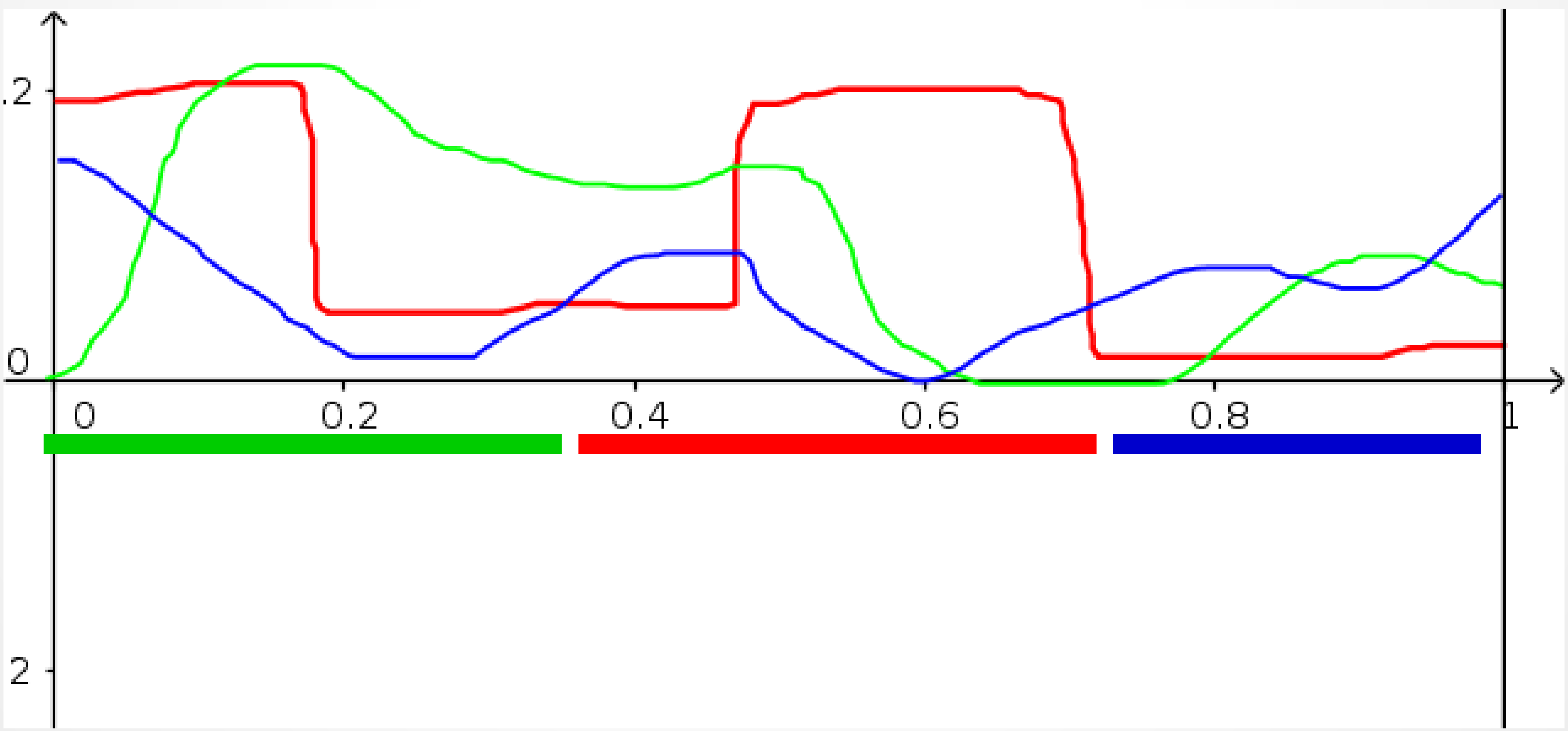
Input: **Cake** C (interval [0,1]).

n agents with different **value-densities.**

Goal: Partition C to n connected intervals

v_i

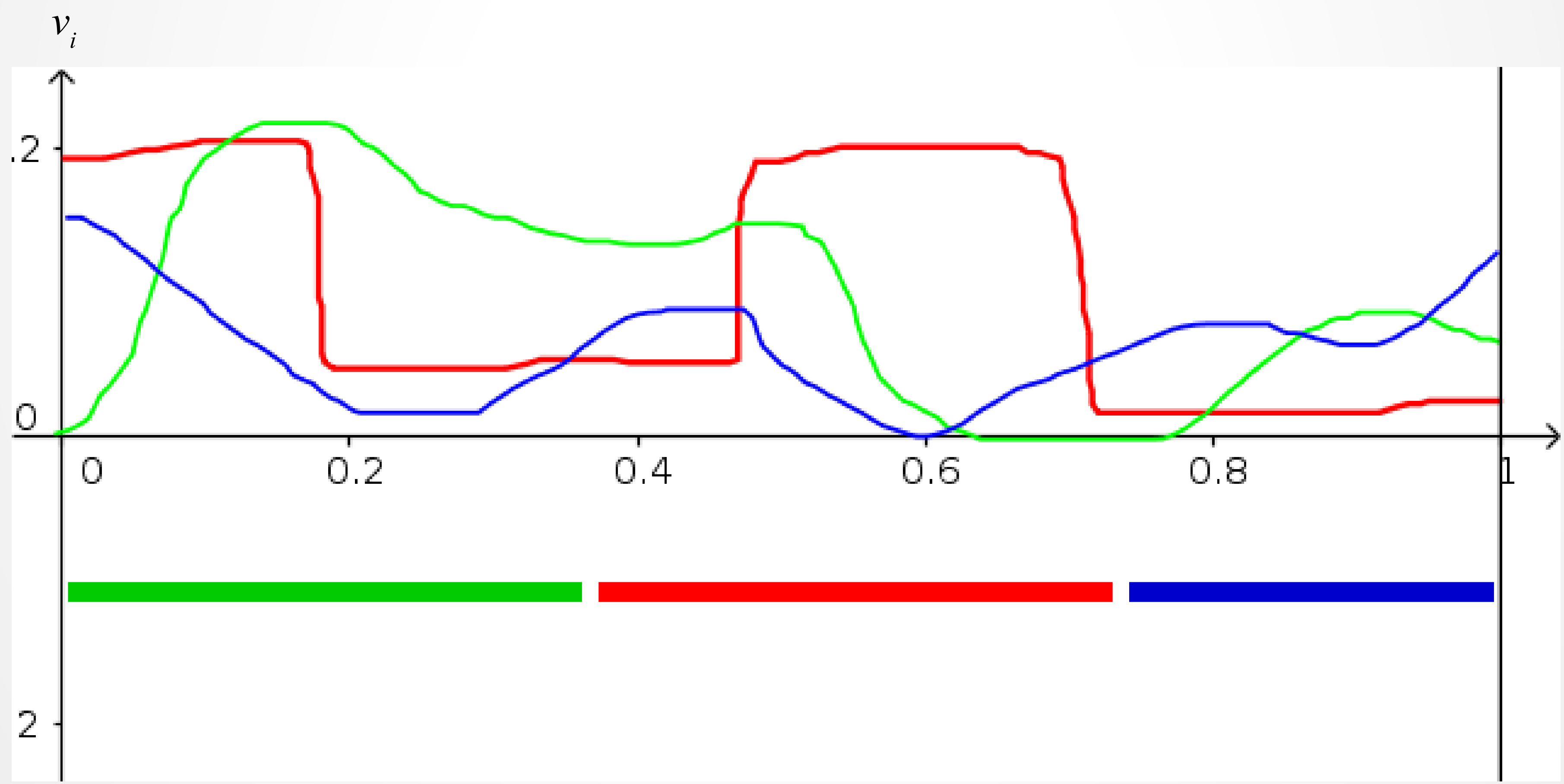
such that **no agent envies another agent.**



Connected Envy-Free Cake-Cutting

Cake is **good** - all value-densities are **positive:**

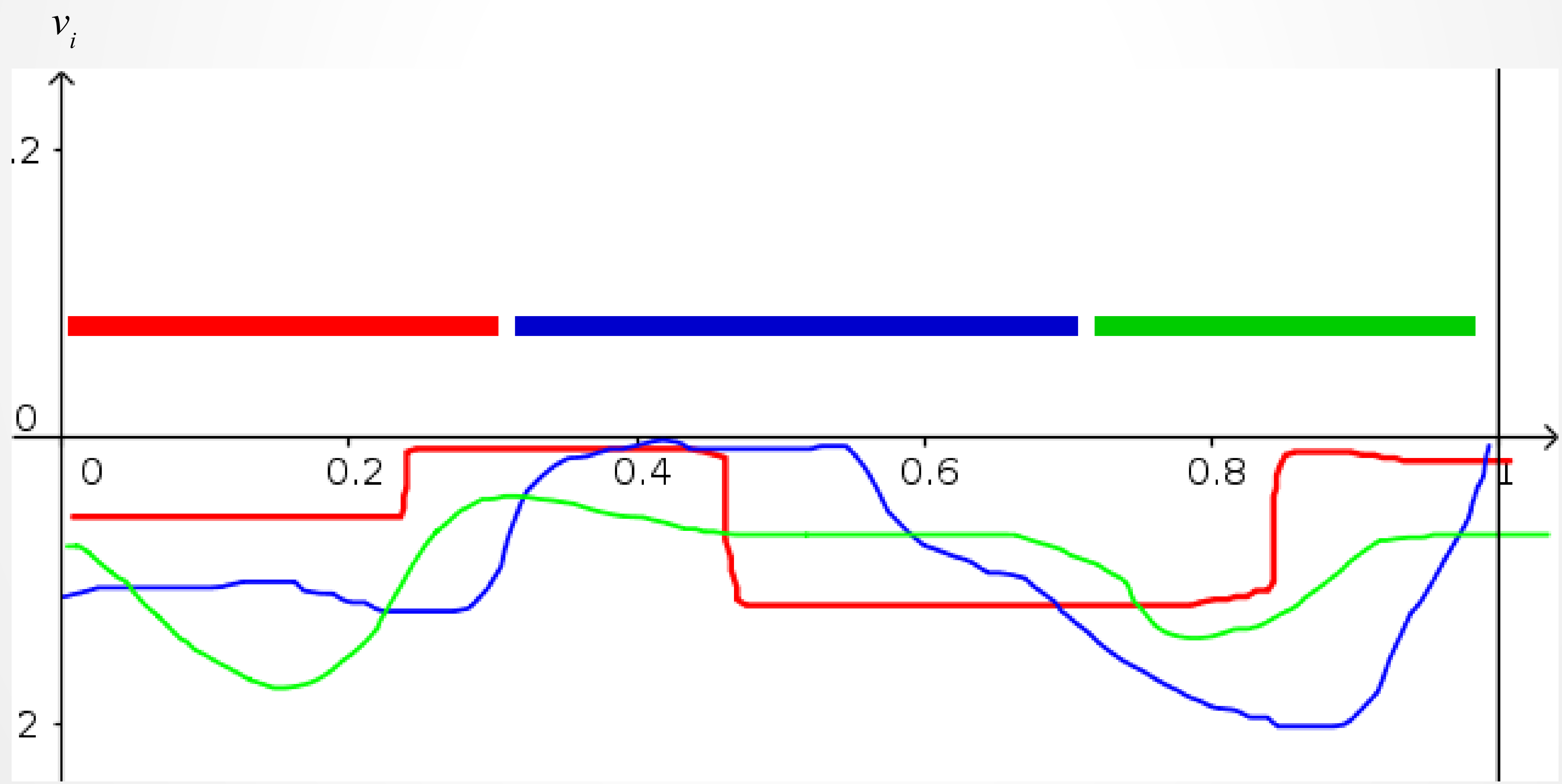
– solution always exists (Stromquist 1980, Simmons 1980)



Connected Envy-Free Cake-Cutting

Cake is **bad** - all value-densities are **negative**:

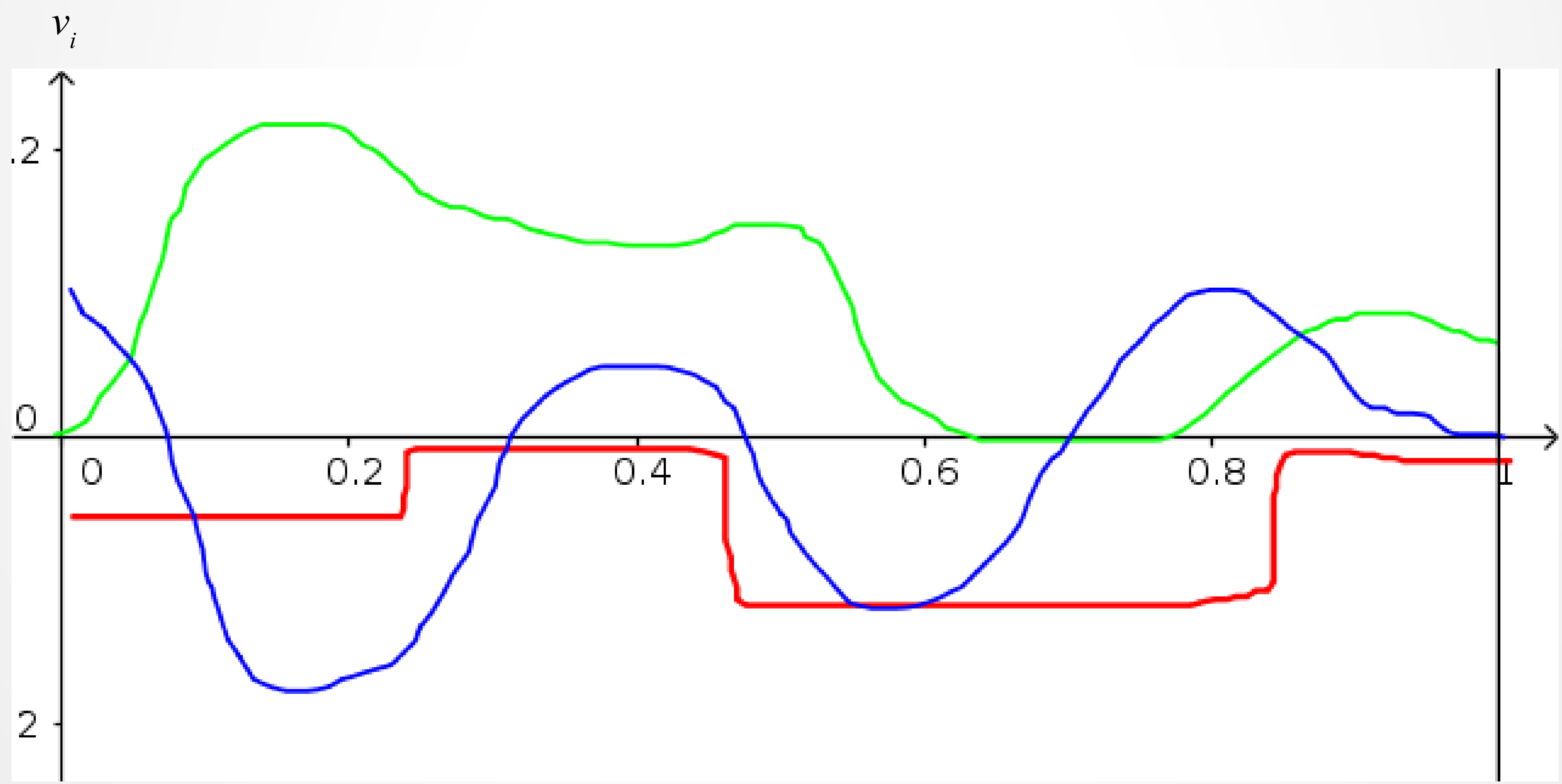
– solution always exists (Su 1999)



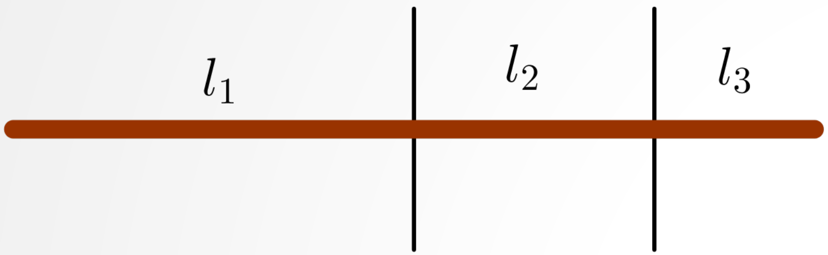
Connected Envy-Free Cake-Cutting

Cake is **mixed** – value densities are **general**:

- **does a solution always exist?**



Simplex of Partitions
(based on Stromquist 1980)



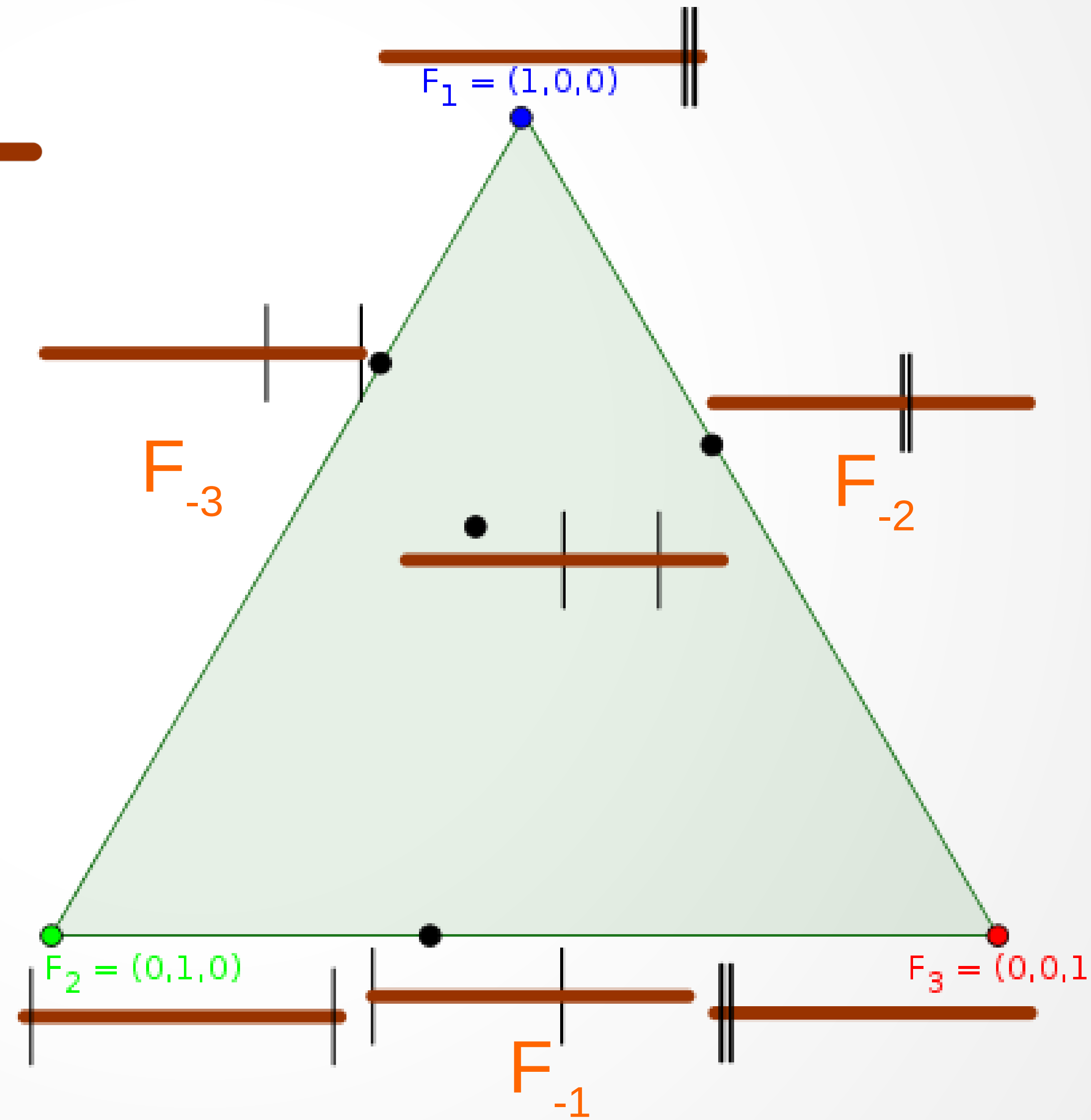
Partition for 3 agents:

(l_1, l_2, l_3)

$l_1+l_2+l_3 = 1$

Envy-free division:

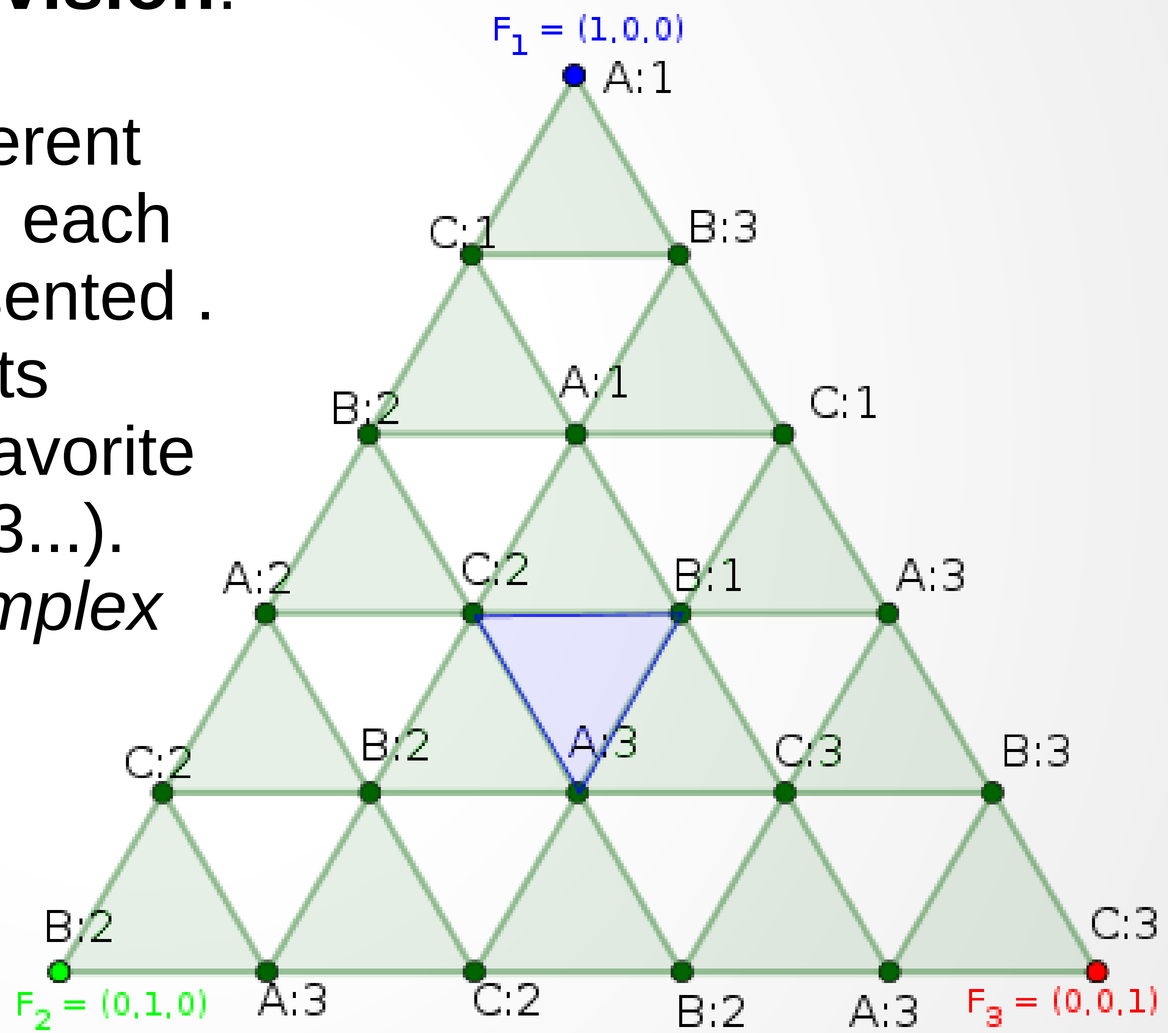
a point in which each agent prefers a different piece.



Triangulating the Simplex of Partitions
(Su 1999)

Approximate envy-free division:

- Triangulate the simplex.
- Assign each vertex to a different agent (A,B,C...) such that in each baby-simplex, all are represented .
- Ask each agent to label all its vertices by the index of its favorite piece in that partition (1, 2, 3...).
- Find a *fully-labeled baby-simplex* (with all labels $1,..,n$).



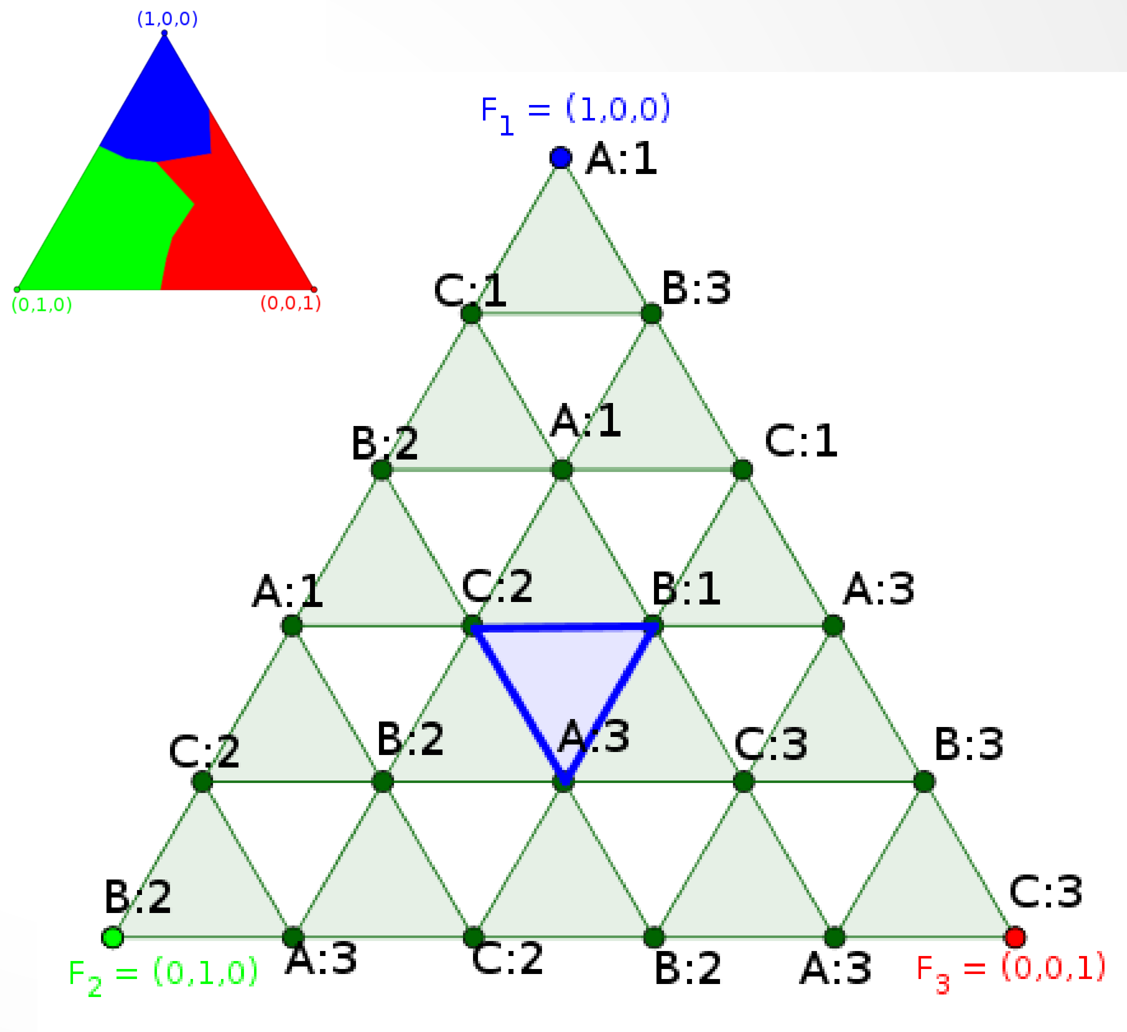
Old Key: Spener's lemma

A **positive** agent prefers a non-empty piece.

→ A face is labeled only with its endpoints' labels.

→ Sperner's lemma: fully-labeled baby-simplex exists.

→ Approximately-envy-free division exists.



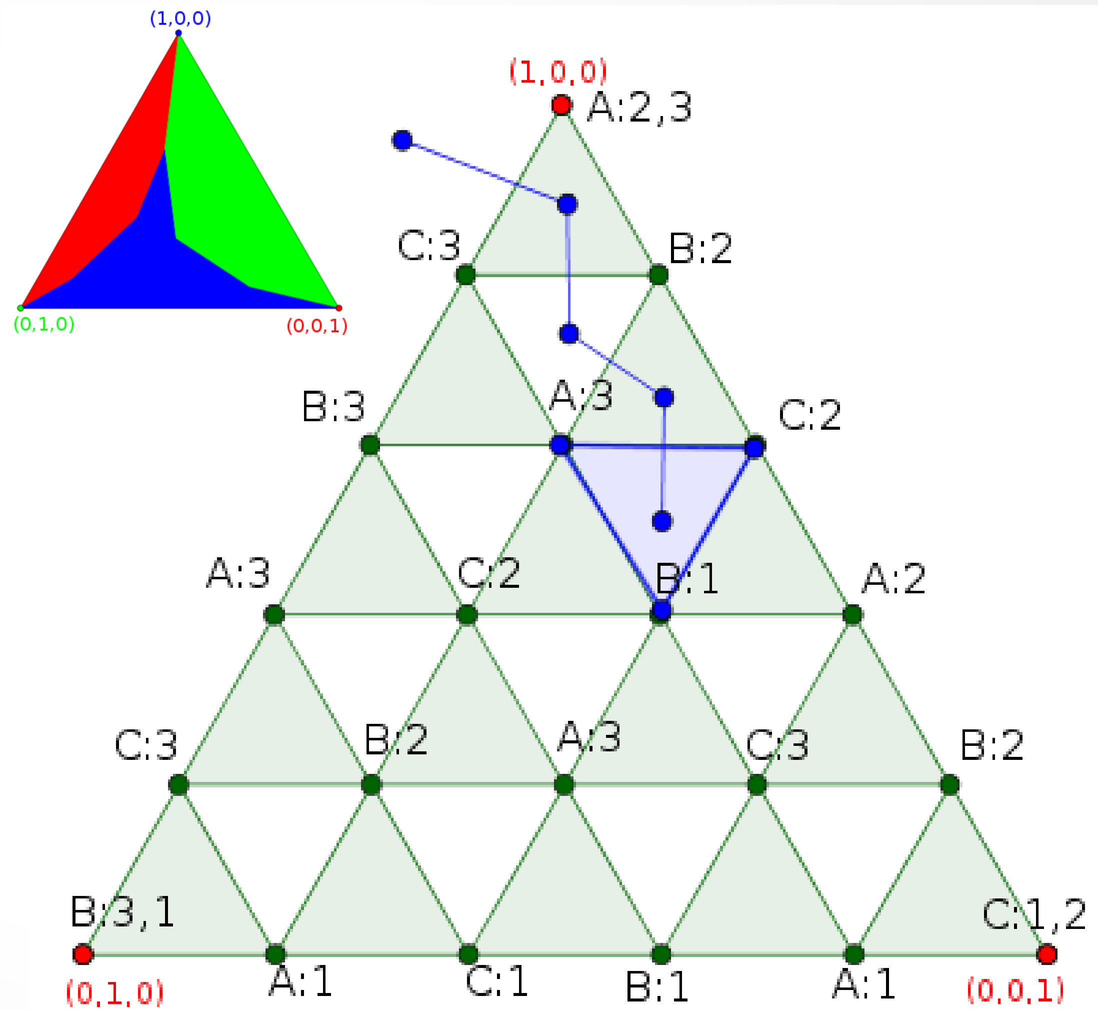
Old Key: Spener's lemma

A **negative** agent always prefers an empty piece.

→ In each main vertex, we can select one of the possible n-1 labels s.t. each face is labeled only with its endpoints' labels.

→ Sperner's lemma: fully-labeled simplex exists.

→ Approximately-envy-free division exists.



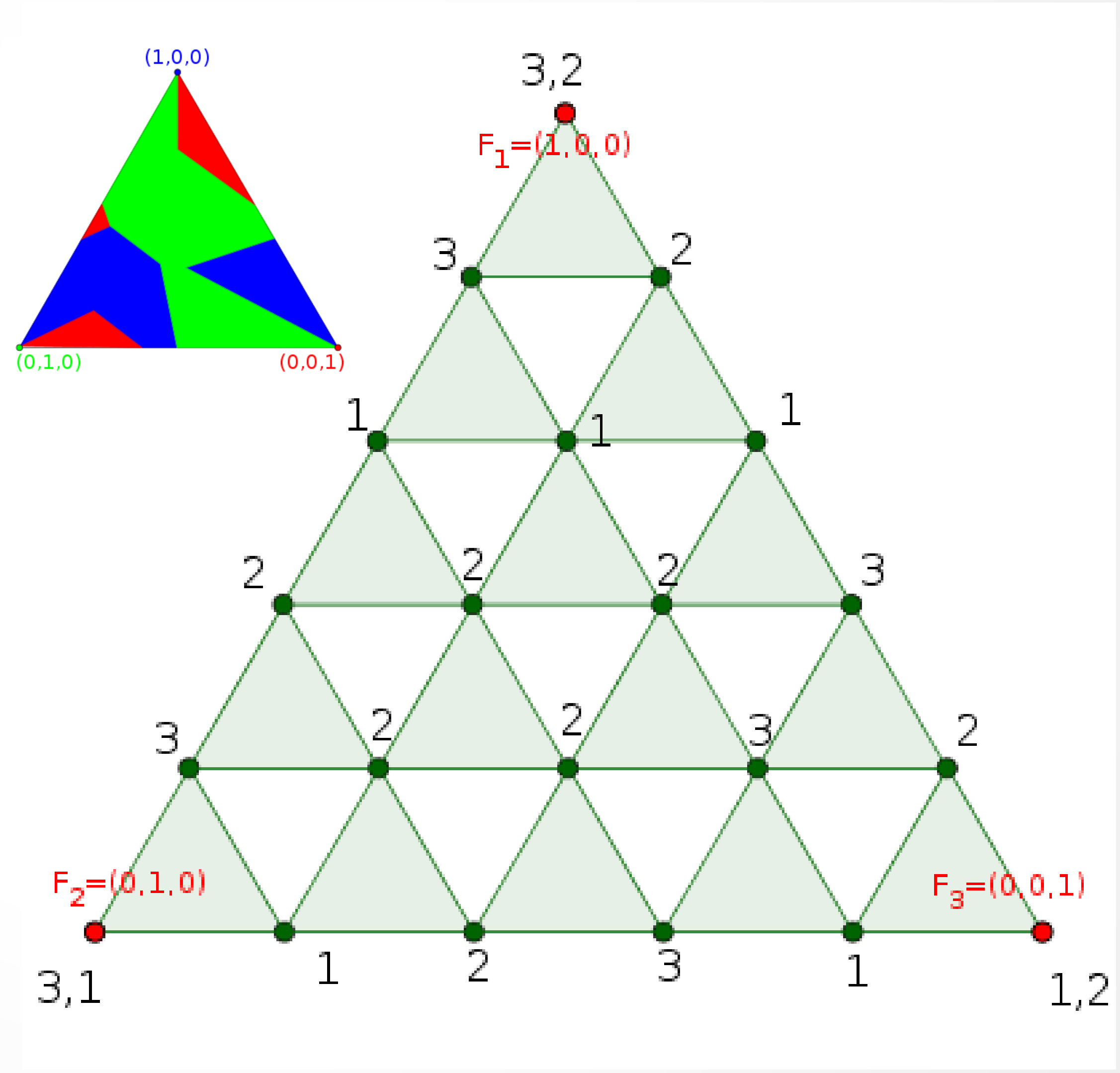
Old Key: Spener‘s lemma

A **general** agent sometimes prefers an empty piece and sometimes prefers a non-empty piece.

→ All labels may appear on a face.

→ **Sperner‘s lemma is inapplicable!**

- *Do you see any other pattern in the coloring or labeling?*

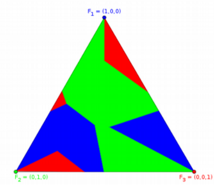


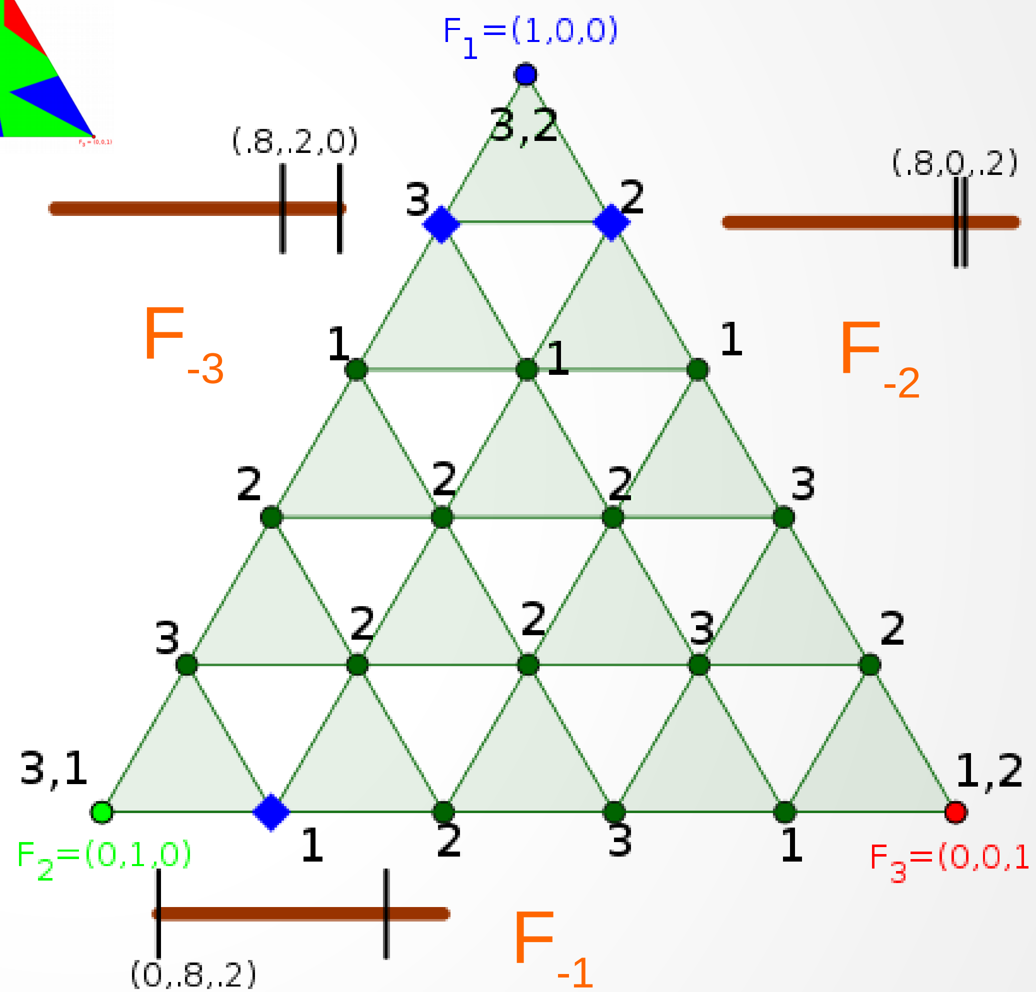
New Key #1: Consistency

Friends := points with the same ordered list of non-zero coordinates.

Consistent labeling :=
For every two friends x,y, x in F_{-3} and y in F_{-k}:
$\text{Lbl}(y) = \pi_{-k}(\text{Lbl}(x))$:

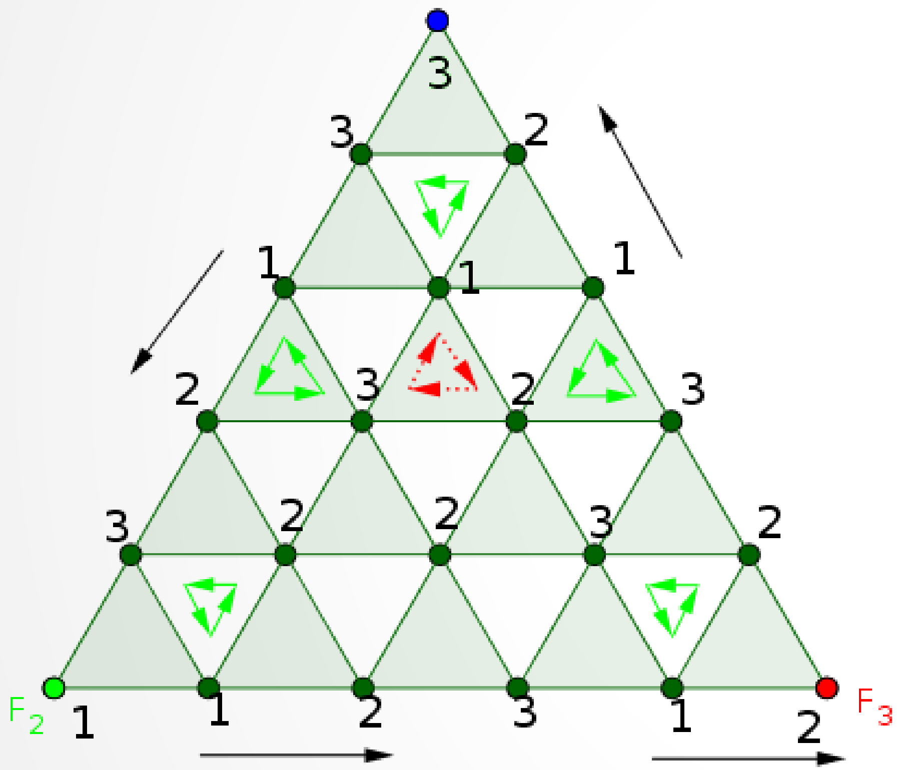
Lbl(x)	1 (Left)	2 (Right)	3 (Empty)
π_{-3}	1	2	3
π_{-2}	1	3	2
π_{-1}	2	3	1

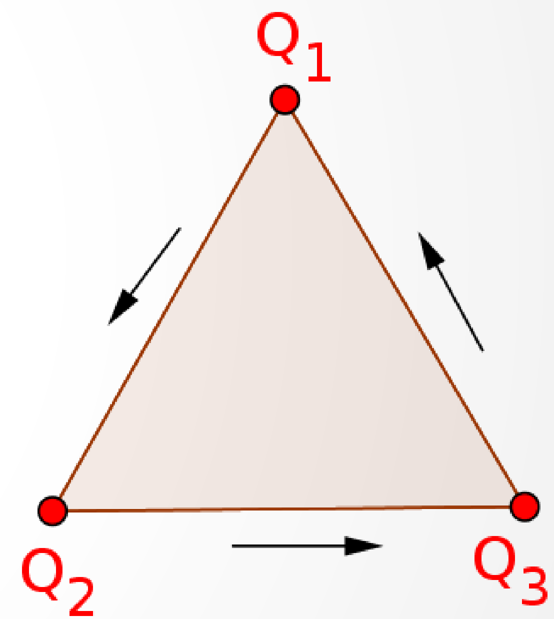




New Key #2: Degree of Labeling

Labeling ≡ linear mapping from triangulation vertices to vertices of some reference simplex Q:





Degree of mapping = net number of rounds *(CCW=positive)*.
Interior degree = sum of degrees of baby-simplices (5-1=4).
Non-zero interior degree → fully-labeled baby-simplex.
Boundary degree = degree along boundary (1+1+1+1=4).

Proof Outline

Step	Proved for
1. n consistent agent-labelings → Combined consistent labeling.	Any n
2. Consistent labeling (boundary condition) → Nonzero boundary degree.	$n = 3$ (this paper) $n = 4$, prime (Meunier and Zerbib, `18)
3. For *any* labeling, interior degree = boundary degree.	Any n
4. Nonzero interior degree → At least 1 fully-labeled simplex → Approximate envy-free division.	Any n

Conclusion

A connected envy-free division exists for agents with **arbitrary** valuations

v_i

(positive, negative or mixed), whenever their number is 4 or **prime!**

Open question:

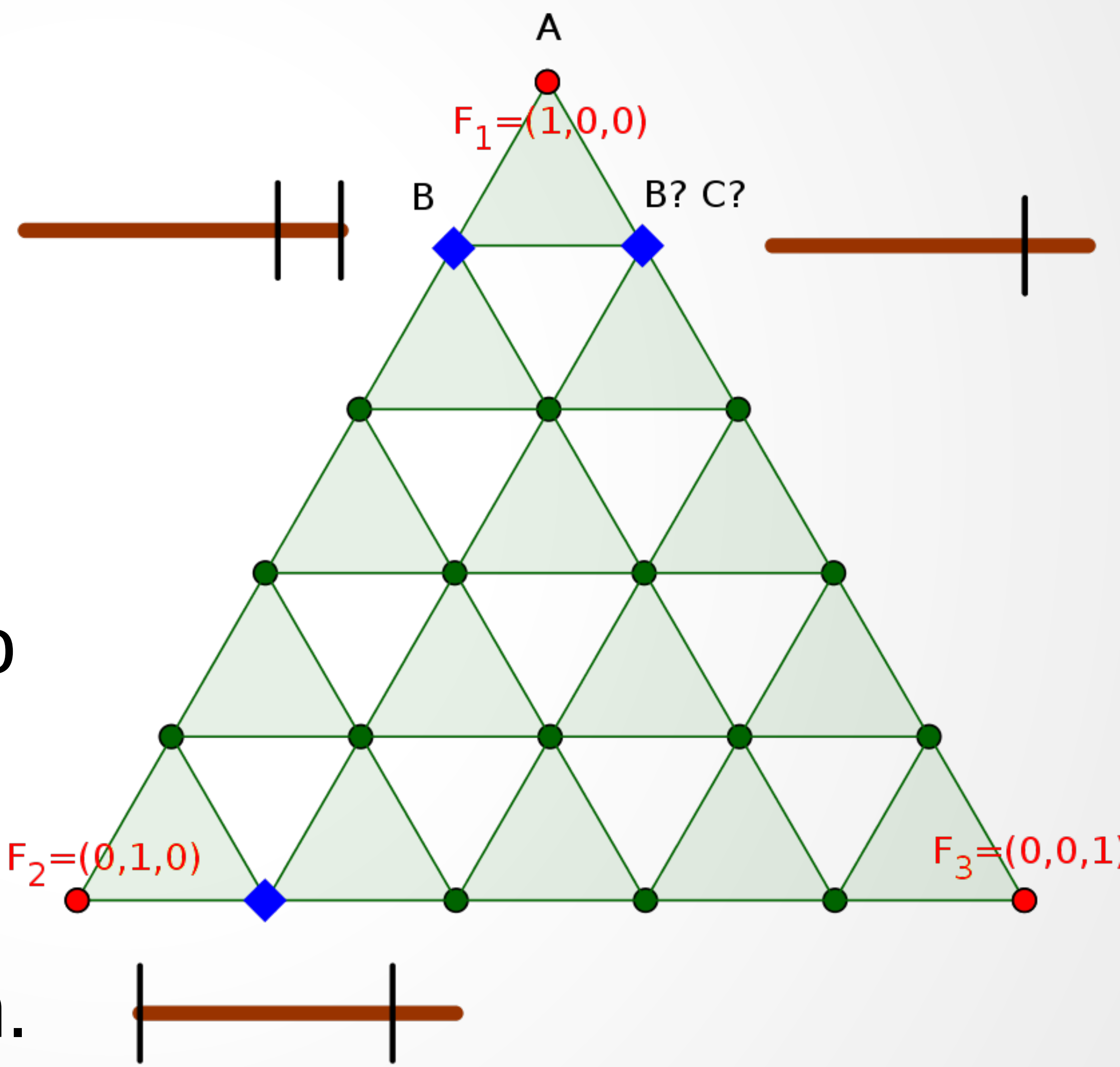
does a connected envy-free division exist for any number of agents **?**

Step 1: n labelings → 1 labeling

We need to assign owners to vertices s.t.:

- In each baby-simplex, each vertex belongs to a **different** owner.
- Friends are assigned to **the same** owner.

Does not work with the equilateral triangulation.



Step 1: n labelings → 1 labeling

We need to assign owners to vertices s.t.:

- In each baby-simplex, each vertex belongs to a **different** owner.
- Friends are assigned to **the same** owner.

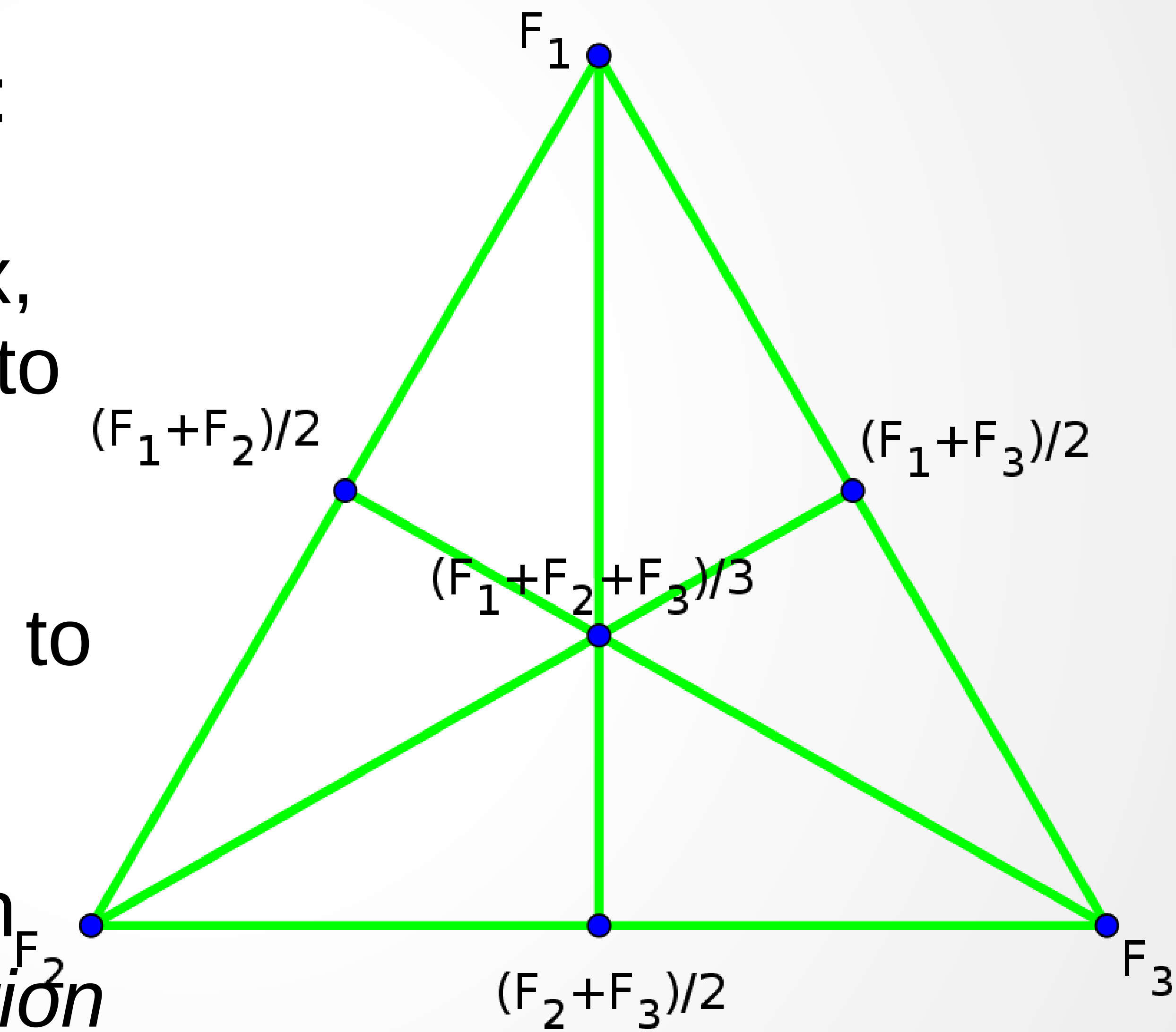
Lemma: it works with *barycentric triangulation*

Step 1: n labelings → 1 labeling

We need to assign owners to vertices s.t.:

- In each baby-simplex, each vertex belongs to a **different** owner.
- Friends are assigned to **the same** owner.

Lemma: it works with *barycentric triangulation*



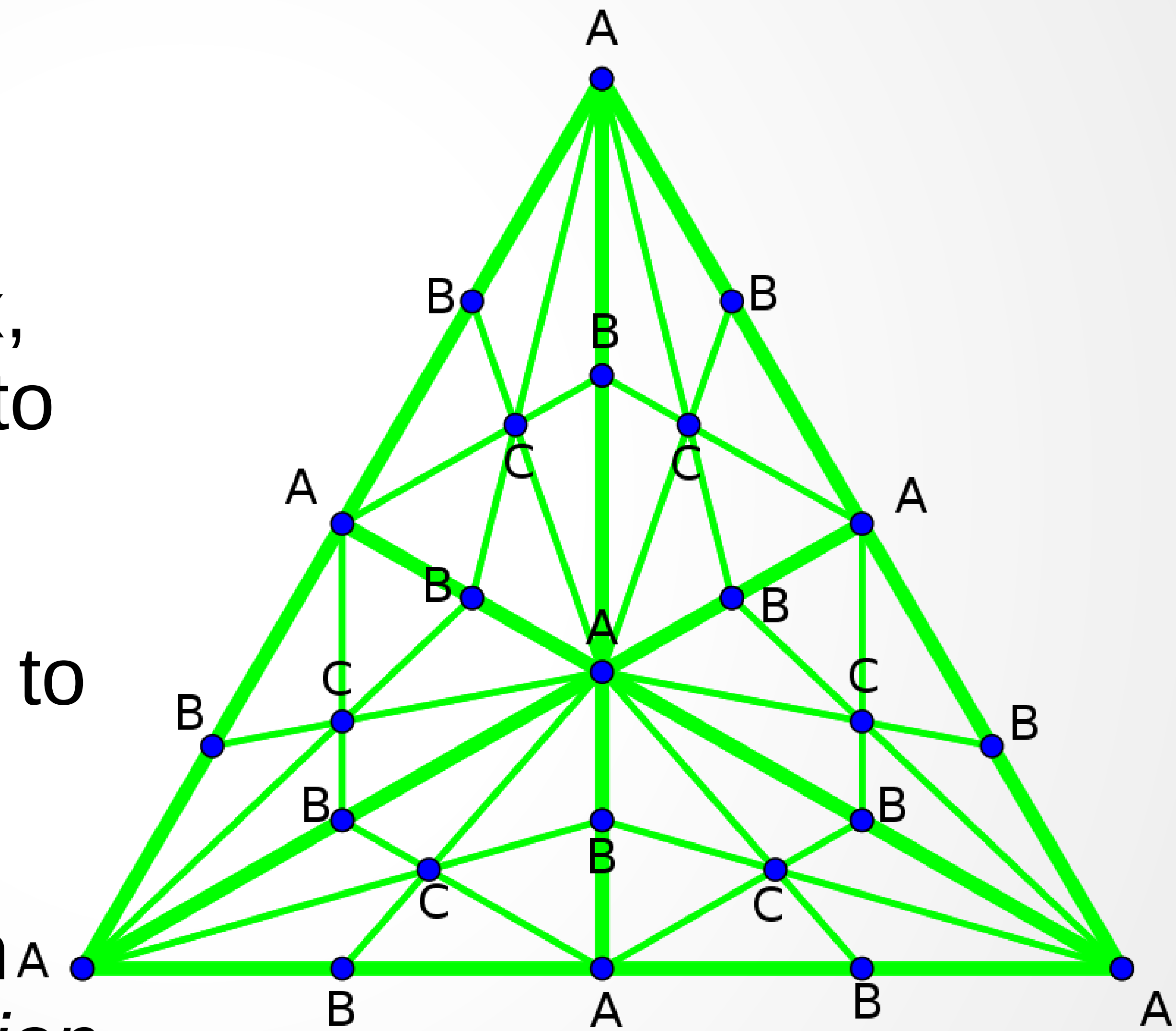
Step 1: n labelings → 1 labeling

We need to assign owners to vertices s.t.:

- In each baby-simplex, each vertex belongs to a **different** owner.

- Friends are assigned to **the same** owner.

Lemma: it works with *barycentric triangulation*

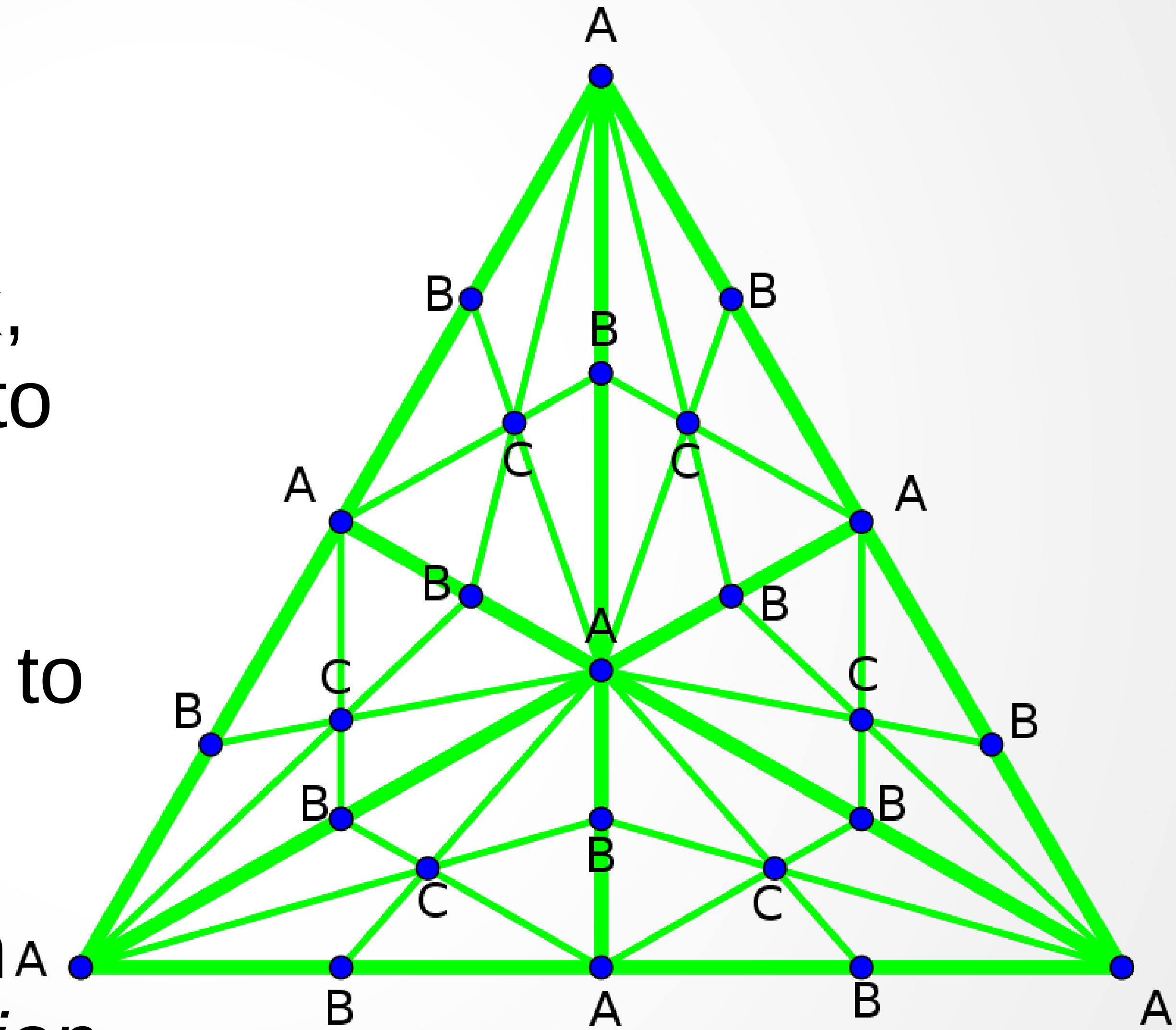


Step 1: n labelings → 1 labeling

We need to assign owners to vertices s.t.:

- In each baby-simplex, each vertex belongs to a **different** owner.
- Friends are assigned to **the same** owner.

Lemma: it works with *barycentric triangulation*



Open issue: *computational complexity.*

Step 2: Consistency → Boundary degree

Consistency:
If x,y are friends,
x in F_{-3} and y in F_{-k},
then $Lbl(y)=\pi_{-k}(Lbl(x))$.

π_{-3}	1	2	3	*Even*
π_{-2}	1	3	2	*Odd*
π_{-1}	2	3	1	*Even*

Lemma: *When n=3*, if a labeling is consistent,
then labels on main vertices can be chosen such that:
boundary-degree mod 3 $\neq$ 0.

Step 2: Consistency → Boundary degree

Consistency:
If x,y are friends,
x in F_{-3} and y in F_{-k},
then $\text{Lbl}(y)=\pi_{-k}(\text{Lbl}(x))$.

π_{-3}	1	2	3	*Even*
π_{-2}	1	3	2	*Odd*
π_{-1}	2	3	1	*Even*

Lemma: *When n=3*, if a labeling is consistent,
then labels on main vertices can be chosen such that:
boundary-degree mod 3 ≠ 0.

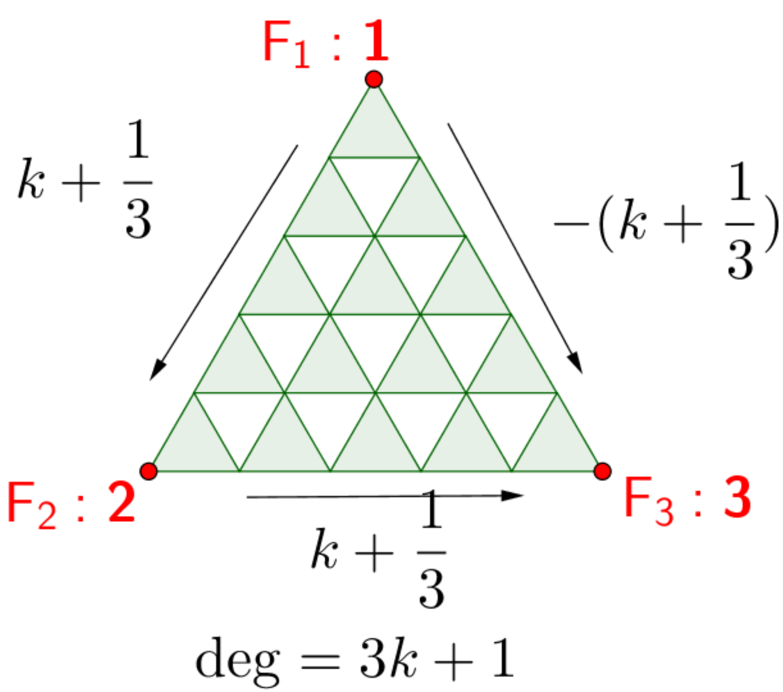
Proof: By consistency, there are two cases:
(+) In each main-vertex i, the label-set contains i, or:
(-) In each main-vertex i, the label-set contains all except i.

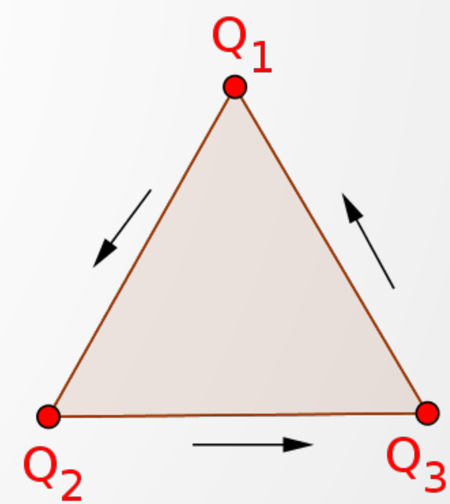
Step 2: Consistency → Boundary degree

Consistency:
If x,y are friends,
x in F_{-3} and y in F_{-k},
then Lbl(y)=π_{-k}(Lbl(x)).

π_{-3}	1	2	3	*Even*
π_{-2}	1	3	2	*Odd*
π_{-1}	2	3	1	*Even*

Case (+): In each main-vertex i, the label-set contains i.



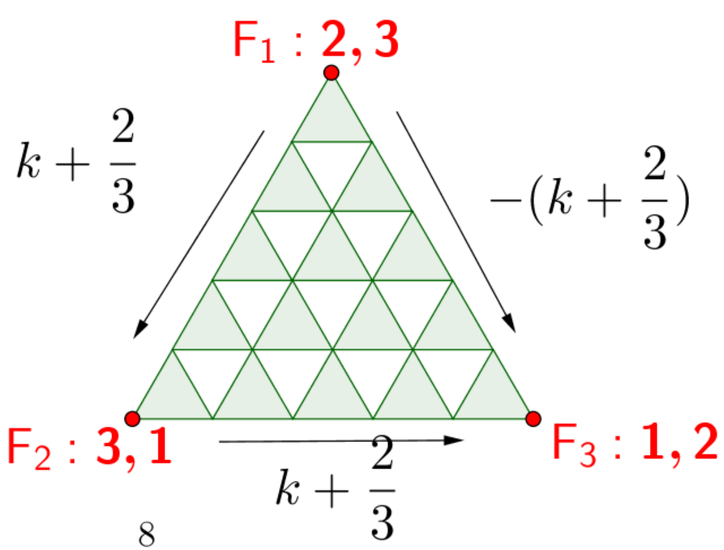


Step 2: Consistency → Boundary degree

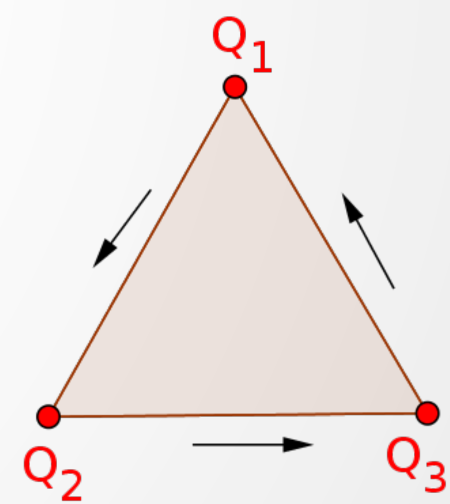
Consistency:
If x,y are friends,
x in F_{-3} and y in F_{-k},
then $Lbl(y)=\pi_{-k}(Lbl(x))$.

π_{-3}	1	2	3	*Even*
π_{-2}	1	3	2	*Odd*
π_{-1}	2	3	1	*Even*

Case (-): In each main-vertex i, the label-set contains all except i.
Calculate the *sum* of degrees over all 8 choices:



$$\sum_{\text{choice}=1}^{8} \deg_{\text{choice}} = 3k+2$$



Step 2: Open question

Consistency for n agents:

If x,y are friends, x in F_{-n} and y in F_{-k},

then $\mathrm{Lbl}(y) = \pi_{-k}(\mathrm{Lbl}(x))$ where:

$$\pi_{-k}(j) = j \qquad j < k$$

$$\pi_{-k}(j) = j + 1 \qquad k <= j < n$$

$$\pi_{-k}(j) = k \qquad j = n$$

Conjecture: If a labeling is consistent,

then boundary-degree mod $n \neq 0$.

We proved the conjecture for n=3.

Meunier and Zerbib (2018) proved it for any prime n.

When conjecture is true, connected envy-free division exists!

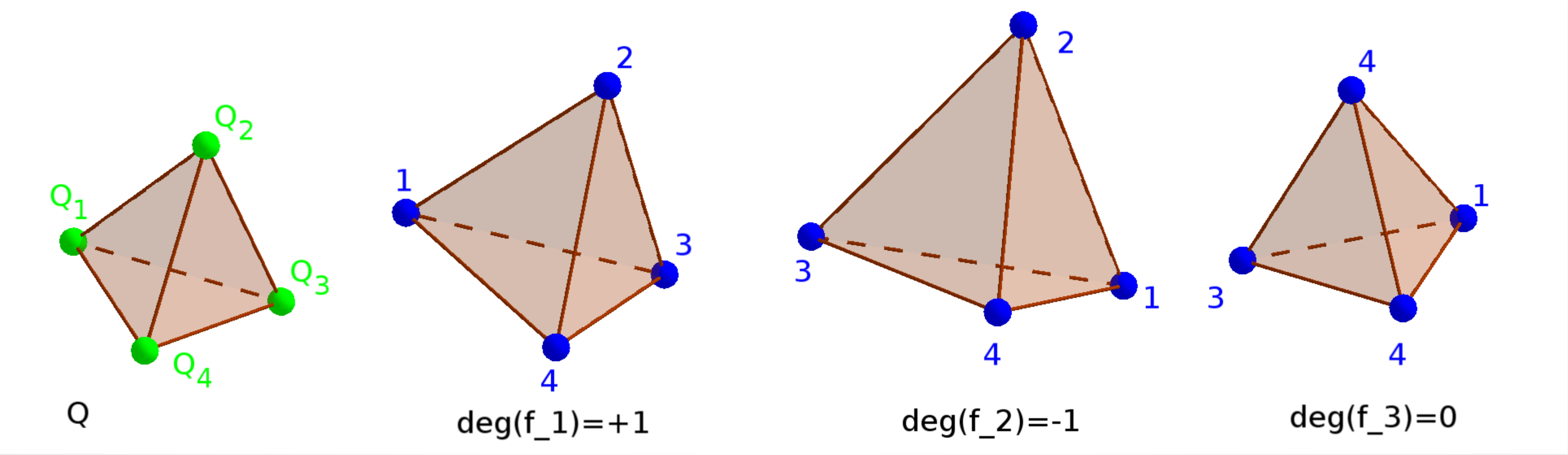
Step 3: Boundary degree = Interior degree

Definition:

Degree of labeling of an n-simplex in $R^{n\text{-}1}$

= sign of determinant of affine transformation to Q

= +1 if onto&no reflection, -1 if onto&one reflection, 0 if not onto.

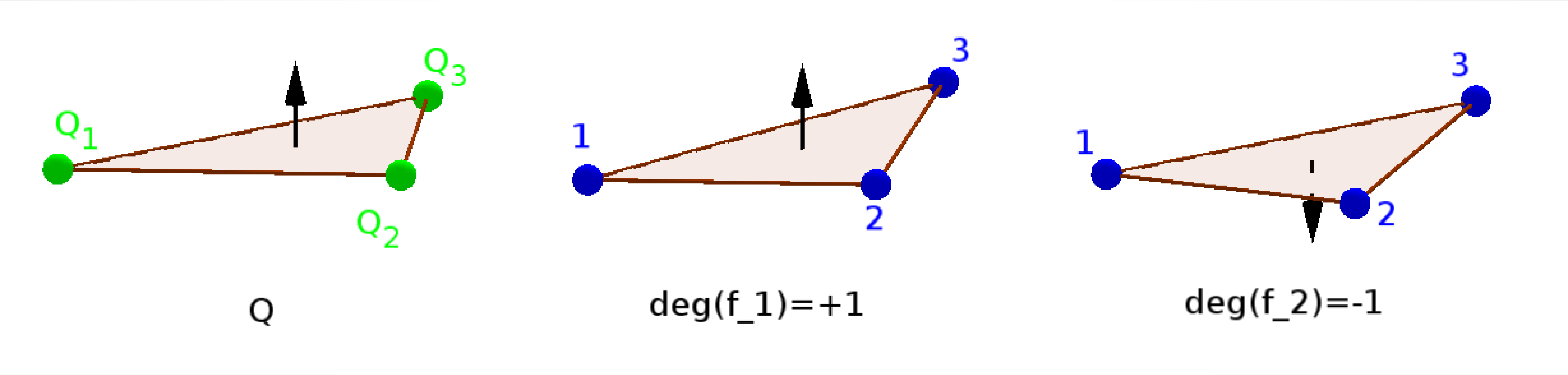


Step 3: Boundary degree = Interior degree

Definition:

Orientation of an $(n\text{-}1)$-simplex in $R^{n\text{-}1}$
= one of its two adjacent half-spaces.

Degree of labeling of an $(n\text{-}1)$-simplex in $R^{n\text{-}1}$
= sign of determinant of *any* affine transformation to Q that perserves the orientation.

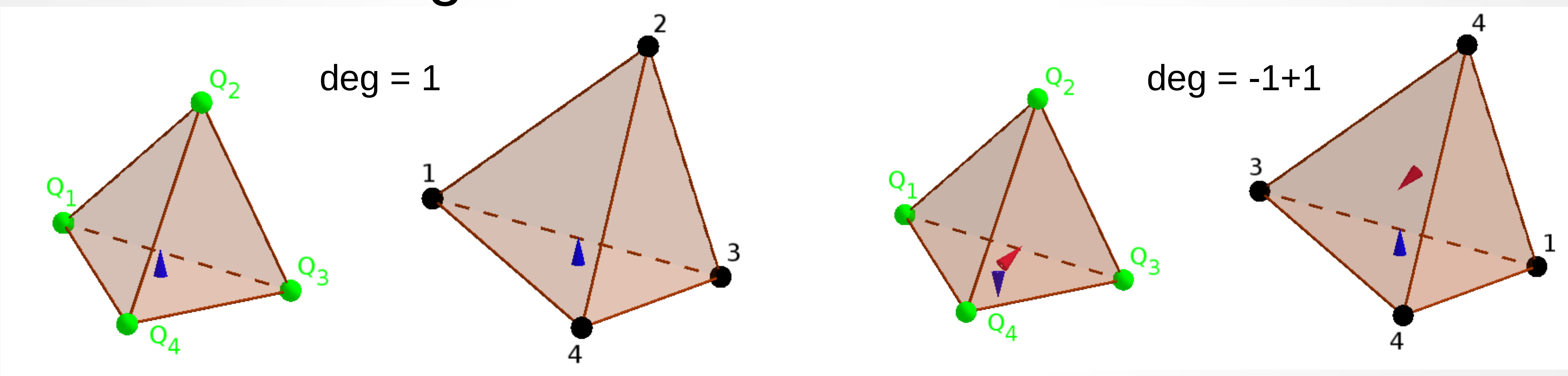


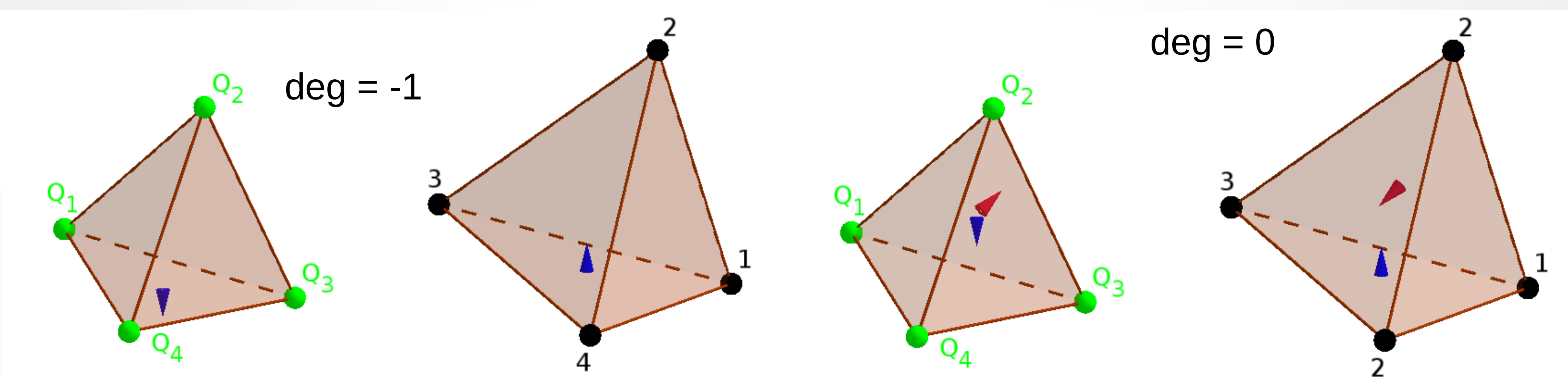
Step 3: Boundary degree = Interior degree

Lemma:

Degree of a labeling of an n-simplex in $R^{n\text{-}1}$,

= sum of degrees on each face oriented *inwards:*





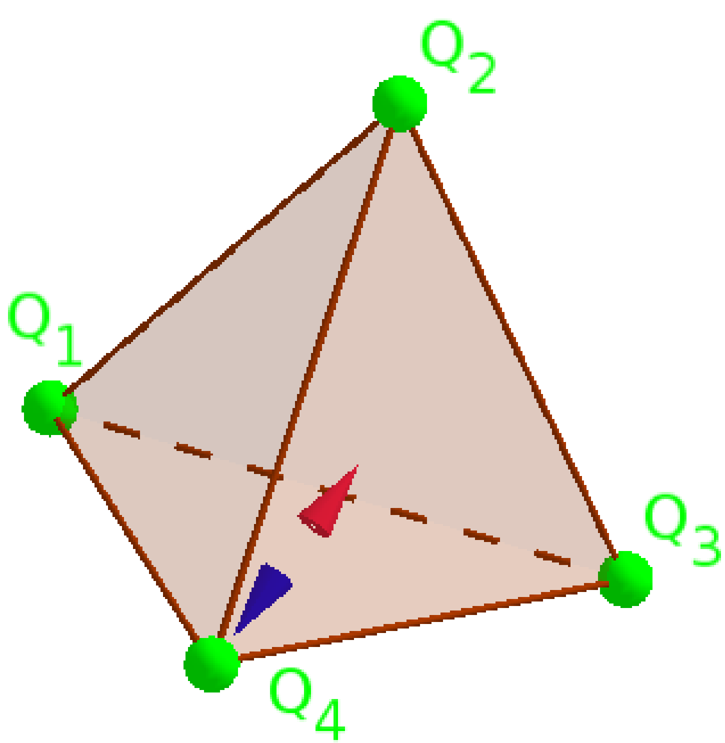
Step 3: Boundary degree = Interior degree

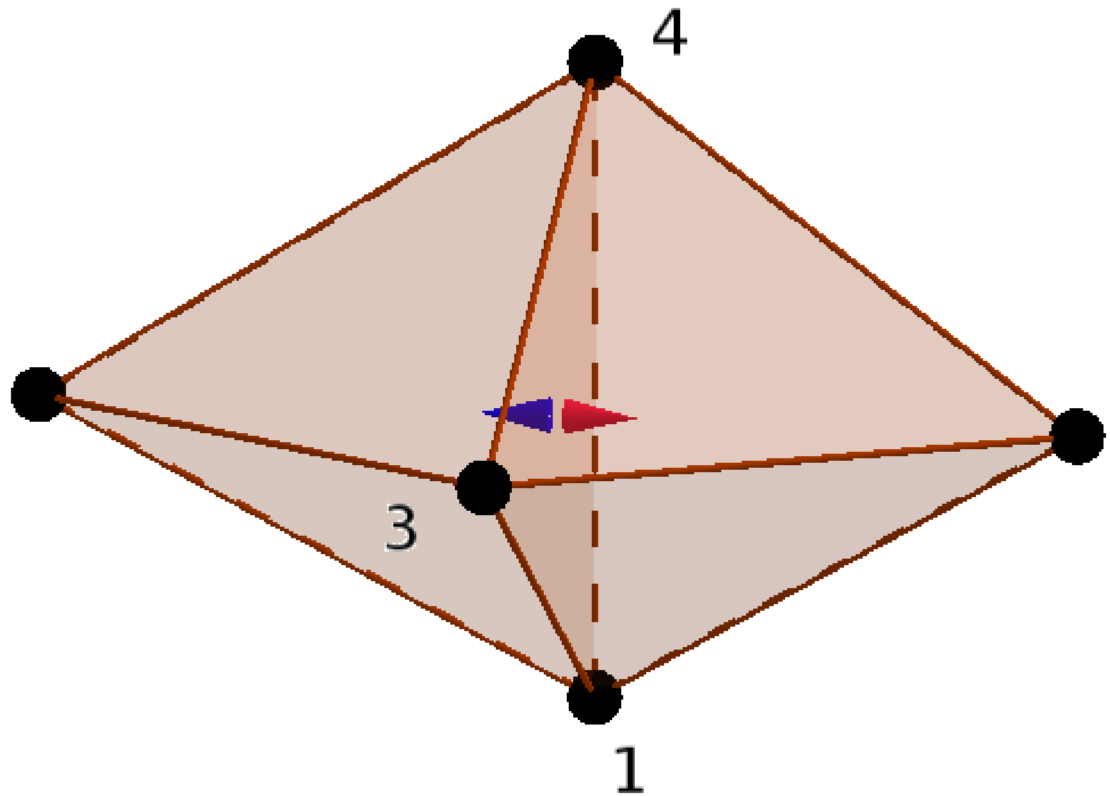
Lemma:

Sum of degrees of simplices in triangulation

= sum of degrees on each *boundary* face,

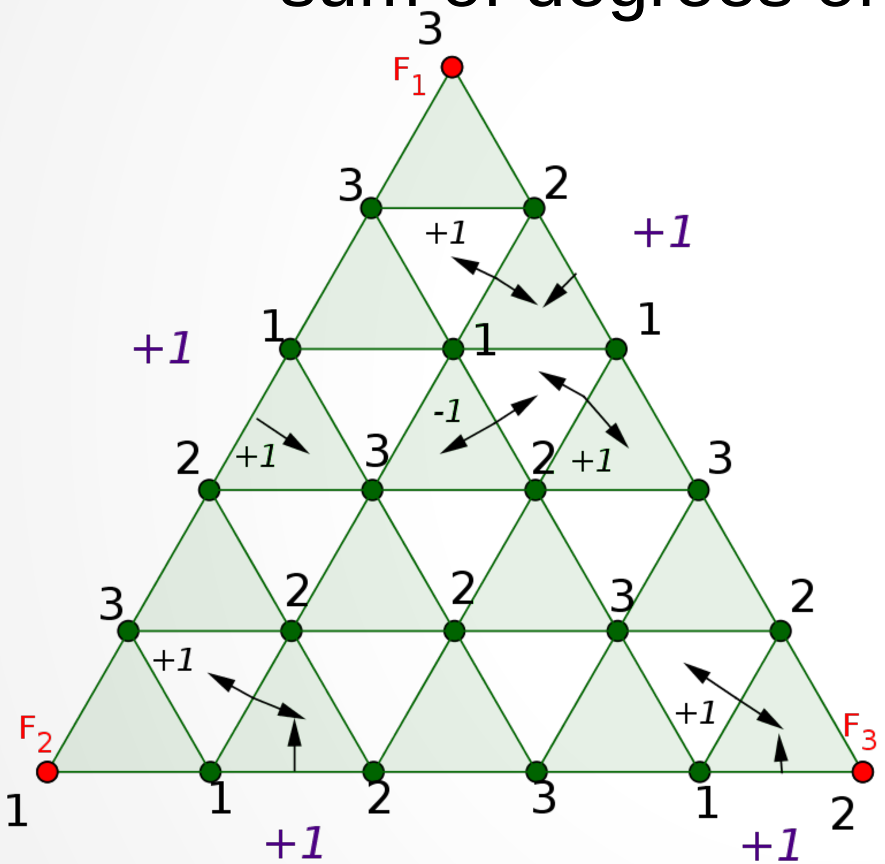
– since the internal faces cancel out:

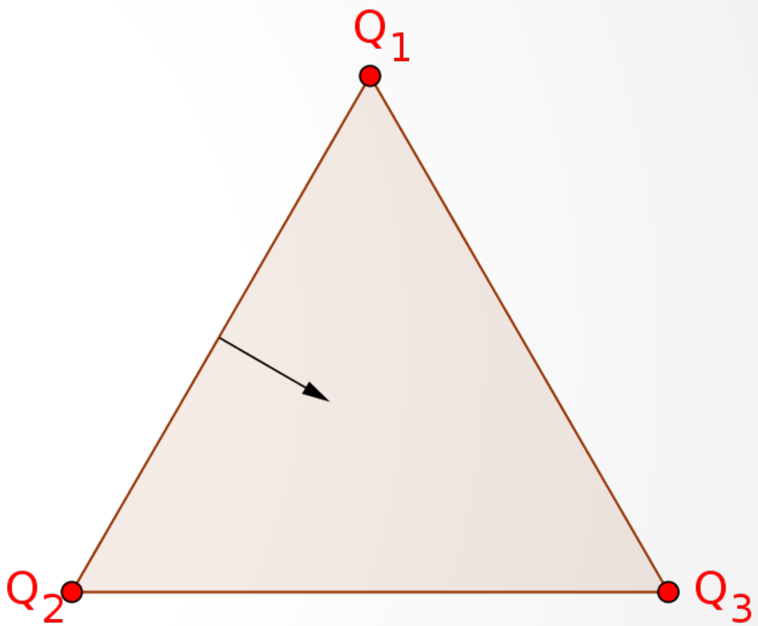




Step 3: Boundary degree = Interior degree

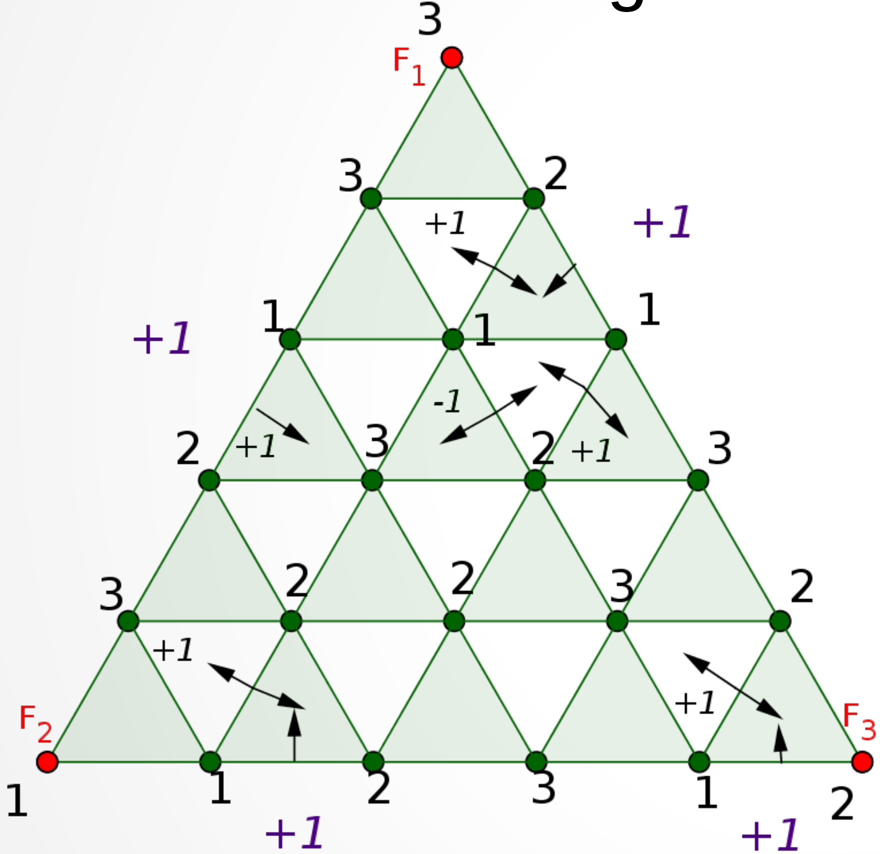
Definition: degree of triangulation labeling
= sum of degrees of each baby-simplex labeling.

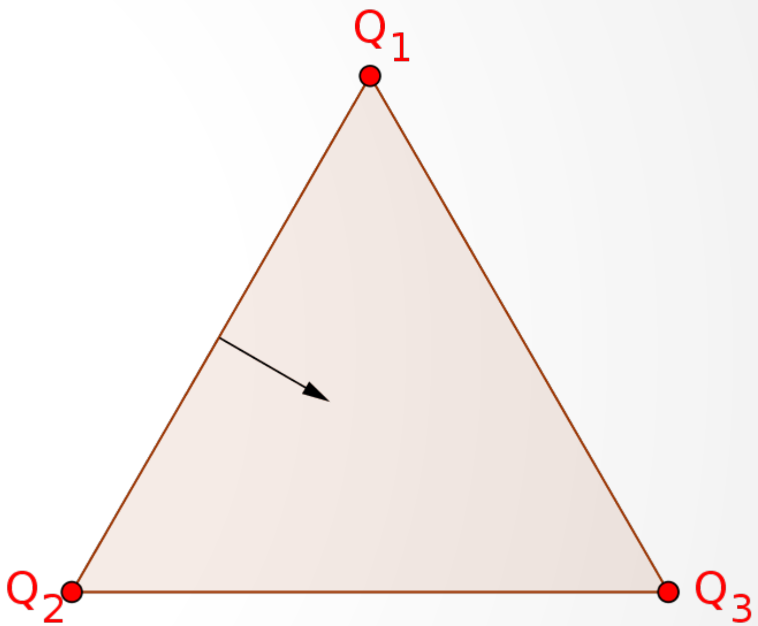




Step 3: Boundary degree = Interior degree

Definition: degree of triangulation labeling
= sum of degrees of each baby-simplex labeling.





Lemma: interior degree = sum of degrees on faces
= sum of degrees on faces of boundary = boundary degree.

Dividing Goods that are Bads

(Midrash Rabba, Genesis 33:1)

Dividing Goods that are Bads

(Midrash Rabba, Genesis 33:1)

Dividing Goods that are Bads

(Midrash Rabba, Genesis 33:1)

Dividing Goods that are Bads

(Midrash Rabba, Genesis 33:1)

Dividing Goods that are Bads
(Midrash Rabba, Genesis 33:1)