A non-classical approach to finding mathematical truth: Constructive mathematics

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Although it is conceived by many students as torture and a pile of unnecessary problems, mathematics was seen as the only source of certainty and absolute universal truth by many philosophers for over two thousand years after Ancient Greece (Ernest, 1999, p.xi). Most of modern mathematicians have more or less this stance. This phrase from a modern mathematics book indicates that stance: "There are only two sciences that should count as exact sciences: mathematics and hindsight" (Leary, 2015, p.287). This conventional view of mathematics has been left by some mathematicians mainly in the last century, and different approaches to both mathematics and its philosophy occurred. Constructive mathematics is one of these non-classical approaches interpreting the existence of mathematical objects in a certainly different manner from classical mathematics. This raises the question of whether constructive mathematics is reliable for finding mathematical truth or not. In this paper, I will show that the methodology of constructive mathematics is not suitable from the perspective of the coherence theory of truth, and the consequences of constructive mathematics lead one not to believe it is true from the viewpoint of the pragmatic theory of truth. Although the proposition $x \neq 0$ or $x = 0$ is not fully applicable in today's, or even future's, computational power, the constructive argument defending the proposition is not valid.

Historical and philosophical background

Euclid is the one who started the axiomatic system of mathematics with his famous book Elements. Axioms are a set of statements that are accepted as true. Euclid used axioms and definitions to get other true statements called theorems in mathematics. A mathematical proof is a finite sequence of either axioms or proven theorems that end up with a given claim. Mathematical proofs can also be seen as "truth transmission" from axioms to theorems (Ernest, 1999, p.6). Logic, the method of inference, has to be used, by definition of logic, throughout the transmission of truth. Euclid first used this system of mathematics, and then many philosophers admired and used it. Newton and Spinoza used the mathematical system of Euclid in their books Principia and Ethics to strengthen their claims (p.1). “Let no one ignorant of geometry enter.” is also a well-known phrase of Plato (Suzanne, 2004), showing the importance of mathematics as the only valid way of reasoning.

Truth has many definitions in the philosophy of mathematics, depending on one’s position. The “realist”s, the position of most mathematicians, argue there is a mathematical universe in which every mathematical object exists, like the realm of ideas of Plato. A mathematical statement is true for a realist if it matches the facts of the mathematical universe. Hence, a statement’s truth value is determined by something external (Putnam, 1975, p.531), not affected by any “human activities, beliefs and capacities (Blanchette, 2022). This perception also implies that a mathematical statement is true or false independent of its proof (Andrews, 1996). Reversely, antirealism argues that mathematical objects do not exist, and in particular, constructivism claims that mathematics is a creation of the human mind (Bridges, 2018). Of course, these antirealist philosophies did not occur out of the blue.

The rise of antirealism happened after the two big crises of classical mathematics, i.e., the mathematics done with the classical logic: non-euclidian geometries and Cantor’s works on sets. Euclidian geometry was seen as the most certain truth. Hobbes said that “Geometry … is the only science … bestow on mankind.” (as cited in Ernest, 1999, p.1). Kant thought Euclidian geometry is a “necessary logical outcome of the reason.” (p.2). The discovery of non-euclidian geometries, such as hyperbolic and elliptic geometries, had a huge impact on nearly every philosopher. The universal truth of mathematics was attacked seriously. People started to think that there must be a problem with axioms because logic and definitions were still as sound as before. Then, axioms are considered in terms of consistency, completeness, and independence, not in terms of being true or false (Andrews, 1996) which is beyond the scope of this paper. This was the first crisis of classical mathematics.

The second crisis of classical mathematics is Cantor's theory of infinite sets. Cantor showed that, within the boundaries of classical mathematics, the infinity is not unique; there are different degrees of infinities (Flood, 2022). This was a big shock for mathematicians. How come the sacred mathematics allows this kind of craziness? People started to think more about the foundations of mathematics; as a result, different philosophies occurred. In this paper, the focus will be on constructive mathematics.

Constructive mathematics and its logic

Constructive mathematics is a mathematics based on intuitionistic logic (Bridges, 2018). To understand this definition, classical and intuitionistic logic must be defined first.

Classical logic basically has these three properties: The identity law, everything is equal to itself; the law of contradiction, a proposition cannot be both true and false; and the law of excluded middle, a proposition is either true or false (Andrews, 1996). It was first established by Aristo, and there has been no significant change from those times to today. Intuitionistic logic can roughly be described as "classical logic without the law of excluded middle" (Moschovakis, 2018). As it is evident, the main difference between classical and intuitionistic logic is the law of excluded middle.

The law of excluded middle (hereon LEM) is the logical law stating that either a statement or its negation is true (Andrews, 1996). For instance, any real number is either zero or non-zero. In classical logic, the identity law, LEM, and double negation law —if a statement is not false, then it is true— are equivalent (Andrews, 1996). Hence, rejection of one implies the rejection of others.

The mathematics using intuitionistic logic is called constructive mathematics and it was founded by L.E.J. Brouwer. According to his philosophy intuitionism, the statement "there exists" is interpreted as "it can be constructed"; moreover, he thinks that mathematics is a "free creation of human mind and an [mathematical] object exists if and only if it can be (mentally) constructed" (Bridges, 2018). The word construction might be ambiguous.

Here it can be conceived as showing something exists directly. Think of the set of humans, for example. To assert that this set is not empty, at least one human must be shown. It might seem trivial at first glance, but this philosophical stance, intuitionism, implies the rejection of reductio ad absurdum, i.e., a proof technique that proves a statement by showing that its denial leads to a contradiction with what is known before (Encyclopedia Britannica, 2022). For instance, the set of humans being empty implies that “I” do not exist, but “I” know “I” exist; therefore, the set of humans cannot be empty. Reductio ad absurdum, also known as indirect proof, is one of the main differences between constructive and classical mathematics and is very useful in proving classical theorems, generally harder than “the set of humans is not empty.”

Brouwer, obviously, runs counter to classical mathematics. He thinks that “classical logic is abstracted from finite sets," and mathematicians forgot that fact and used classical logic for all mathematics, including infinite sets (Bridges, 2018). According to Weyl, Brouwer says, “It is not that such contradictions [Russel’s paradox, explained in next paragraph] showed up that is surprising, but that they showed up at such a late stage of the game” (as cited in Bridges, 2018). That point of view leads him to set intuitionistic logic and therefore, rejection of LEM.

Russel’s paradox was such a massive depression for classical mathematics such that it can be counted as third crisis of classical mathematics. British philosopher and mathematician Bertrand Russel came up with this paradox, a proposition which is true when false and false when true: Think of a set **S**, containing only the sets that are not an element of themselves. The paradox is here: If **S** does not contain itself, it should contain itself by its definition; if **S** contains itself, then it should not because of the definition of **S** (Irvine, 2020). Many solutions are offered, but some mathematicians, like Brouwer, chose to establish a new mathematics that would not have these paradoxes.

A standard example given to demonstrate the difference between classical and constructive mathematics in practice is proof of this theorem (Çevik, 2021): There exist two irrational numbers, meaning that they cannot be written as ratios of two integers, *a* and *b*, such that a^b is rational. A non-constructive proof is as follows. Consider $\sqrt{2}^{\sqrt{2}}$. If this number is rational, the proof is done since $\sqrt{2}$ is irrational. If it is irrational, consider $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$ which is equal to 2, a rational number. Constructivists do not accept this proof because here, LEM is used in the implicit argument "$\sqrt{2}^{\sqrt{2}}$ is either rational or irrational." A constructive proof of this is as follows. Take a = $\log_2 9$ and b = $\sqrt{2}$. Hence $a^b = 3$ which is rational. Here, a and b are given explicitly; there is no division into cases, and therefore, no use of LEM such as "if rational …, if not …" As seen, proofs in constructive mathematics can be more difficult than classical mathematics because finding examples may take a plenty of time, but it is more convincing to see direct examples.

Constructive mathematics and coherence theory

The coherence theory of truth claims that a proposition is true if and only if it is consistent with someone's beliefs (Walker, 2017. p.240). Consistency can also be understood as non-contradictory to what is known as true (Safitri et.al., 2022). Hence, if a system of information is known to be true, any information contradictory to that system must be false.

The Discovery of positron by Paul Dirac and of Neptune are good counter-examples against the method of constructive mathematics. In 1928, Dirac found an equation for electrons moving at relativistic speeds, but this equation had a problem: just as x^2=4 has two solutions, 2 and -2, the equation of Dirac fully allowed the existence of a positively charged electron, now called "positron" (CERN, 2022). Dirac saw that there would be no contradiction if that new particle existed. Although at that time, there was no experimental evidence, he concluded that particle should have existed. That is the exact procedure of the coherence theory of truth, a proposition being non-contradictory to former beliefs is true. After six years, the prediction of Dirac was confirmed by experiments. A constructivist should not accept that kind of indirect existence "proof" but should wait until the experiments are fully convincing. Nevertheless, it is seen that indirect proofs, such as Dirac's totally mathematical prediction of positron, work in the world of physics, as opposed to intuitionistic viewpoint of existence of an object.

Another counter-example to the methodology of constructivism is the discovery of Neptune. This eighth planet Neptune was discovered when it was realized that there were anomalies in the orbit of Uranus, which was against the Newtonian laws of gravitation (Uri, 2021). After the mathematical calculations, it is seen that there must be a planet in order for Uranus to orbit like that (id.). The situation is a good example of coherence theory since the non-existence of such a plane would be contradictory to what is known before, i.e., Newton's laws. Therefore, scientists concluded that a planet, now called Neptune, should have existed, and a few years later, it was seen by telescope too. Here, the process could be thought like reductio ad absurdum, a proof technique of classical mathematics saying that if the negation of a proposition leads to a contradiction, the proposition itself must be true. As mentioned before, constructivists do not accept reductio ad absurdum. However, scientists' prediction of the existence of Neptune is as follows: a planet being not existing is contradictory; therefore, it exists. As mentioned above, this method is not accepted by constructivists but works totally fine in the physical universe, meaning the methodology of constructive mathematics is not enough for getting physical information. While the methodology of classical mathematics enables one to make these discoveries, constructive mathematics fails to achieve that. Therefore, from the perspective of coherence theory of truth, methodology of constructive mathematics fails to be true.

Constructive mathematics and pragmatic theory

The pragmatic theories of truth mainly have two versions: C.S. Pearce's proposal that a claim will be accepted true “at the end of the inquiry” and W. James’s proposal that “truth is defined in terms of utility” (Capps, 2019). In this paper, Pearce’s pragmatic theory of truth will be meant when the term “pragmatic theory" is used.

According to pragmatic theory, a claim is true if its results are acceptable. The word “acceptable" makes the theory overly subjective. To overcome this subjectiveness, it can be interpreted as being acceptable by a rational mind. The arguments that will be given in this paper will base on this interpretation of the pragmatic theory.

For the sake of argument, assume constructive mathematics is the right way to find mathematical truth. This assumption leads one to so many abnormal results that it is not plausible to assume constructive mathematics is the right way to find mathematical truth. These abnormal consequences of constructive mathematics can be classified into two classes: theorems that are classically valid but not constructively and theorems that are constructively valid but not classically. Classically valid but constructively not valid theorems will be called as “lost” theorems meaning lost when switching to constructive mathematics.

The first lost theorem is the trichotomy principle; every real number is either negative, positive, or zero. Constructivists do not accept this principle because, in the proof of trichotomy, LEM is used when asserting that " if a number is not positive, then it is non-positive." An equivalent theorem which is again false in constructivism, this may surprise even ones not surprised in the first part of the paragraph, is that for every real number x, $x = 0$ or $x \neq 0$. The reason for rejection is the same: LEM being used in “zero or not zero”. An argument given by constructivists to defend this rejection will be investigated later on, but these are such natural results that rejecting them does not seem plausible.

Apart from lost theorems, there are also theorems that are constructively proven but not valid in classical mathematics. To see the absurdity of constructive mathematics, following constructive theorem can be given as example. Not every subset, a set that is a part of the whole set, of a finite set, meaning there is a finite number of elements in the set, needs to be finite (Bauer, 2016, p.486). This theorem holds in constructive mathematics because the statement “every subset of a finite set is finite” is equivalent to LEM (p.487). Rejecting LEM leads to such absurd results, and any rational human would agree that these results are not plausible. It can be concluded that from the viewpoint of pragmatic theory, constructive mathematics is not a suitable way of finding mathematical truth.

A rejection might come to mind at this point: In the argument given above, the reductio ad absurdum method which constructivists reject used when saying “assume constructivism is the right way” and then showing consequences are not acceptable; therefore, the theory is wrong. However, this is not a problem for the argument because the argument is not restricted by constructive method but based on the pragmatic theory of truth, and in this theory, the structure used in the argument is totally acceptable. Furthermore, the argument is not a contradiction argument, which means it is not restricted by acceptances of constructive mathematics. Hence, constructivists' prohibition of reductio ad absurdum does not cause a problem for the argument.

Defending $x = 0$ or $x \neq 0$

As mentioned above, constructivist does not accept that for every real number x, x is either zero or non-zero (hereon, * will refer to this claim). The main reason for this is LEM. However, there is also an argument from constructive mathematics saying that this result is actually practically reasonable. The following version of the argument is from Bridges (2018). In order to assert * constructively, a computer algorithm deciding whether $x \neq 0$ or $x = 0$ must be found. This is because constructivism wants to have an explicit construction as proof, which corresponds to an algorithm in the computer science terminology. Since computers have limited memory and computational power, they can “handle real numbers only by means of finite rational approximations." For instance, an irrational number $\sqrt{2}$ might be stored as 1.4142; but in fact, it has infinitely many non-repeating digits after the point, as being irrational implies. A computer, or any computational device such as the human mind, cannot know/ compute the exact irrational number. Say, the computer can only compute the first 100 digits after the point of a real number. Then this computer cannot distinguish 10^{-101} and 0. If one improves the power of that computer, say, to 10000 digits, then again, $10^{-100001}$ and zero would be identical for the computer. No matter how powerful it is, unless infinitely powerful, which is impossible, there exist infinitely many numbers that computers cannot distinguish from zero. That is the explanation of rejecting * by Bridges.

The argument in favor of constructivism above actually does not defend constructivism; on the contrary, it accepts the validness of *. The reason for this lies behind the word “limited." It is true that for computers * does not hold, but this does not mean * is not true at all. There are such small numbers that computers cannot differ from zero, but still, there are**.** If those numbers would not exist, then the argument would not work. Hence, although * is practically not applicable to the real world, as an abstract theorem, it is true.

Conclusion

To sum up, main difference between constructive mathematics and classical mathematics is, rejection of LEM. According to coherence theory, if non-existence of something is contradictory to what is known before, then this thing exists. A mathematical proof technique which is not accepted by constructivists called reductio ad absurdum, uses coherence theory of truth in this way: If the negation of a proposition leads to contradictions, then the proposition is true. Two physical discoveries, of which positron and Neptune, confirms the validity of this proof technique, and therefore the invalidity of constructive mathematics. The abnormal consequences of constructivism, such as lost theorems and theorems that are only valid in constructivism, indicate the voidness of constructive mathematics according to the pragmatic theory of truth. Lastly the division of real numbers into two, $x = 0$ or $x \neq 0$, is not accepted in constructivism, but the argument given to support this claim is not valid. These are the three parts of this paper showing the wrongness of constructive mathematics. In the mathematical sense, constructivism fails but from the perspective of philosophy of mathematics, the attempt was really remarkable, which is, unfortunately, beyond the scope of this paper.

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