Linear Algebra Concepts

KTT Data Science Group

1 Group

1.1 Theory

A group is a set G, combined with a binary operation $\cdot$, that satisfies the following four axioms:

- **Closure:** For all $a, b \in G$, the result of the operation, $a \cdot b$, is also in G.
- **Associativity:** For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **Identity Element:** There exists an element $e \in G$ such that for every element $a \in G$, the equation $a \cdot e = e \cdot a = a$ holds.
- **Inverse Element:** For each element $a \in G$, there exists an inverse element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

This structure ensures that elements can be combined consistently, with a neutral element and the ability to reverse operations.

1.2 Practical Examples

1. The set of integers $\mathbb{Z}$ with the addition operation, denoted as $(\mathbb{Z}, +)$.
   - **Closure:** The sum of any two integers is an integer.
   - **Associativity:** $(a + b) + c = a + (b + c)$.
   - **Identity:** The identity element is 0, since $a + 0 = a$.
   - **Inverse:** The inverse of an integer a is $-a$, since $a + (-a) = 0$. For instance, the inverse of 3 is -3.
2. The set of non-zero real numbers $\mathbb{R}^*$ with the multiplication operation, denoted as $(\mathbb{R}^*, \cdot)$.
   - **Closure:** The product of any two non-zero real numbers is a non-zero real number.
   - **Associativity:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
   - **Identity:** The identity element is 1, since $a \cdot 1 = a$.
   - **Inverse:** The inverse of a non-zero real number a is $\frac{1}{a}$, since $a \cdot \frac{1}{a} = 1$. For instance, the inverse of 2 is $\frac{1}{2}$.

2 Abelian (Commutative) Group

2.1 Theory

An Abelian group is a group where the binary operation is also commutative. This means that for all elements a, b in the group G, the following condition holds:

$$a \cdot b = b \cdot a$$

This property allows the order of operations to be changed without affecting the result.

2.2 Practical Examples

1. $(\mathbb{Z}, +)$: The group of integers under addition is Abelian because for any two integers a and b, $a + b = b + a$.
2. $(\mathbb{R}^*, \cdot)$: The group of non-zero real numbers under multiplication is Abelian because for any two non-zero real numbers a and b, $a \cdot b = b \cdot a$.

3 General Linear Group

3.1 Theory

The General Linear Group of degree n over the real numbers, denoted $GL(n, \mathbb{R})$, is the set of all $n \times n$ invertible matrices with real entries. A matrix is invertible if and only if its determinant is non-zero. This group represents all reversible linear transformations of an n-dimensional vector space.

$$GL(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det(A) \neq 0\}$$

3.2 Practical Examples

1. Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. To check if $A \in GL(2, \mathbb{R})$, we compute its determinant:

   $$\det(A) = (1)(4) - (3)(2) = 4 - 6 = -2$$

   Since $\det(A) = -2 \neq 0$, the matrix A is in $GL(2, \mathbb{R})$.
2. Let $B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. To check if $B \in GL(2, \mathbb{R})$, we compute its determinant:

   $$\det(B) = (2)(1) - (0)(1) = 2 - 0 = 2$$

   Since $\det(B) = 2 \neq 0$, the matrix B is in $GL(2, \mathbb{R})$.

4 Span

4.1 Theory

The span of a set of vectors $S = \{v_1, v_2, \ldots, v_k\}$, denoted as $span(S)$, is the set of all possible linear combinations of these vectors. It represents the entire subspace that can be reached using these vectors.

$$span(S) = \{\sum_{i=1}^{k} \alpha_i v_i : \alpha_i \in \mathbb{R}, v_i \in S\}$$

4.2 Practical Examples

1. Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. Any vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ can be written as a linear combination:

   $$\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   Thus, the span of S is all of $\mathbb{R}^2$.
2. Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$. To show this spans $\mathbb{R}^2$, we must find scalars α_1, α_2 such that for any vector $\begin{pmatrix} x \\ y \end{pmatrix}$:

   $$\alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

   This gives the system of equations: $\alpha_1 + \alpha_2 = x$ and $\alpha_1 - \alpha_2 = y$. Solving gives $\alpha_1 = \frac{x+y}{2}$ and $\alpha_2 = \frac{x-y}{2}$. Since we can find such scalars for any x, y, the set spans $\mathbb{R}^2$.

5 Generating Set

5.1 Theory

A set of vectors S is a generating set for a vector space V if the span of S is equal to V. This means that every vector in V can be expressed as a linear combination of the vectors in S.

$$span(S) = V$$

5.2 Practical Examples

1. The set $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a generating set for $\mathbb{R}^2$ because, as shown in the span example, any vector in $\mathbb{R}^2$ can be formed from a linear combination of these two vectors.
2. The set $S = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ is also a generating set for $\mathbb{R}^2$. As demonstrated previously, we can find the specific linear combination for any target vector in $\mathbb{R}^2$.

6 Rank

6.1 Theory

The rank of a set of vectors is the number of linearly independent vectors in that set. It indicates the dimension of the subspace spanned by the vectors. A set of vectors $\{v_1, \ldots, v_k\}$ is linearly independent if the only solution to $\alpha_1 v_1 + \ldots + \alpha_k v_k = 0$ is $\alpha_1 = \ldots = \alpha_k = 0$.

6.2 Practical Examples

1. Consider the set $S = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$. The second vector is a multiple of the first: $\begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Since the vectors are linearly dependent, the set contains only one independent vector. Therefore, the rank is 1.
2. Consider the set $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. The first two vectors are the standard basis vectors and are linearly independent. The third vector can be written as a combination of the first two: $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The maximum number of linearly independent vectors in this set is 2. Therefore, the rank is 2.

7 Mapping

7.1 Theory

A mapping (or function) f from a set V to a set W, denoted $f : V \to W$, is a rule that assigns to each element in V exactly one element in W.

7.2 Practical Examples

1. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$. This mapping takes a real number, multiplies it by 2, and adds 1. For example, $f(3) = 2(3) + 1 = 7$.
2. $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x + y$. This mapping takes a 2D vector and assigns its sum of components. For example, $g\left(\begin{pmatrix} 5 \\ 2 \end{pmatrix}\right) = 5 + 2 = 7$.

8 Linear Mapping

8.1 Theory

A mapping $T : V \to W$ is a linear mapping if it preserves vector addition and scalar multiplication. For any vectors $u, v \in V$ and any scalar $\alpha \in \mathbb{R}$, the following two conditions must hold:

1. $T(u + v) = T(u) + T(v)$ (Additivity)
2. $T(\alpha v) = \alpha T(v)$ (Homogeneity)

8.2 Practical Examples

1. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$. This mapping is linear.
   - Additivity: $T\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}\right) = \begin{pmatrix} 2(x_1 + x_2) \\ 3(y_1 + y_2) \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 3y_1 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ 3y_2 \end{pmatrix} = T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + T\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right)$.
   - Homogeneity: $T\left(\alpha \begin{pmatrix} x \\ y \end{pmatrix}\right) = T\left(\begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}\right) = \begin{pmatrix} 2\alpha x \\ 3\alpha y \end{pmatrix} = \alpha \begin{pmatrix} 2x \\ 3y \end{pmatrix} = \alpha T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$.
2. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + y \\ 2y \end{pmatrix}$. This mapping is linear.
   - Additivity: $T\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}\right) = \begin{pmatrix} (x_1 + x_2) + (y_1 + y_2) \\ 2(y_1 + y_2) \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ 2y_1 \end{pmatrix} + \begin{pmatrix} x_2 + y_2 \\ 2y_2 \end{pmatrix} = T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + T\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right)$.
   - Homogeneity: $T\left(\alpha \begin{pmatrix} x \\ y \end{pmatrix}\right) = T\left(\begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}\right) = \begin{pmatrix} \alpha x + \alpha y \\ 2\alpha y \end{pmatrix} = \alpha \begin{pmatrix} x + y \\ 2y \end{pmatrix} = \alpha T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$.

9 Matrix Representation of Linear Mappings

9.1 Theory

Every linear mapping $T : V \to W$ can be represented by a matrix A. If v is a vector in V and $T(v)$ is its image in W, their coordinate representations with respect to bases B_V and B_W are related by the matrix equation:

$$[T(v)]_{B_W} = A[v]_{B_V}$$

This allows linear transformations to be performed using matrix multiplication.

9.2 Practical Examples

1. Consider the linear mapping $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x + y \\ x + 3y \end{pmatrix}$. We can rewrite this as a matrix-vector product:

   $$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

   The matrix representation is $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$.

2. Consider the linear mapping $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x - y \\ 3x + 2y \end{pmatrix}$. This can be expressed as:

   $$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

   The matrix representation is $A = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$.

10 Transformation Matrix

10.1 Theory

A **transformation matrix** is a matrix that represents a linear transformation in a coordinate system. Linear transformations are operations that map vectors to other vectors while preserving vector addition and scalar multiplication.

Formally, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $v \in \mathbb{R}^n$ is a vector, then there exists a matrix A such that:

$$T(v) = Av$$

Here, A encodes the transformation, meaning that multiplying a vector by A applies the transformation directly. Common linear transformations include rotation, scaling, reflection, and shear.

Key properties of transformation matrices:

- The columns of A are the images of the standard basis vectors under T.
- The composition of two linear transformations corresponds to the multiplication of their matrices.
- If a transformation is invertible, its matrix is also invertible.

10.2 Practical Examples

1. **Rotation in 2D:** To rotate a vector in $\mathbb{R}^2$ by an angle θ counterclockwise, the transformation matrix is:

   $$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   For example, rotating the vector $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by 90° ($\theta = \pi/2$) gives:

   $$R_{\pi/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   This rotates the vector from the x-axis to the y-axis.

2. **Scaling in 2D:** To stretch or shrink a vector along the x and y axes by factors s_x and s_y, the transformation matrix is:

   $$S = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}$$

   For example, scaling $v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ by $s_x = 3$ and $s_y = 2$ gives:

   $$S \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

   Both components are scaled independently according to the transformation.

3. **Reflection across the x-axis (optional extra):**

   $$F_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F_x \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

   This flips any vector over the x-axis, changing the sign of the y-component.

11 Basis

11.1 Theory

A basis of a vector space is a set of vectors that is both linearly independent and spans the space. A basis provides a unique way to represent every vector in the space as a linear combination of its elements, serving as a coordinate system for the space.

11.2 Practical Examples

1. The standard basis for $\mathbb{R}^3$ is the set of vectors $B = \{[1,0,0]^T, [0,1,0]^T, [0,0,1]^T\}$.
   - **Spanning:** Any vector $[x,y,z]^T \in \mathbb{R}^3$ can be written as $x[1,0,0]^T + y[0,1,0]^T + z[0,0,1]^T$.
   - **Linearly Independent:** $x[1,0,0]^T + y[0,1,0]^T + z[0,0,1]^T = [0,0,0]^T$ only if $x = y = z = 0$.
2. A basis for the plane defined by $x + y + z = 0$ in $\mathbb{R}^3$ can be $B = \{[1,-1,0]^T, [0,1,-1]^T\}$.
   - **Spanning:** Any vector on the plane can be described as $[x, y, -x-y]^T$. We can show this is spanned by the basis vectors.
   - **Linearly Independent:** The vectors are not scalar multiples of each other, so they are linearly independent.

12 Basis Change

12.1 Theory

A vector v has different coordinate representations in different bases. A change-of-basis matrix $P_{B' \leftarrow B}$ transforms the coordinates of a vector from basis B to basis B'. The relationship is given by:

$$[v]_{B'} = P_{B' \leftarrow B}[v]_B$$

12.2 Practical Examples

1. Let $B = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ and $B' = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$. Let $v = \begin{pmatrix}2\\3\end{pmatrix}$. Find $[v]_{B'}$. We need to solve $c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$. This gives the system $c_1 + c_2 = 2$ and $c_1 - c_2 = 3$. Solving yields $c_1 = 2.5$ and $c_2 = -0.5$. So, $[v]_{B'} = \begin{pmatrix}2.5\\-0.5\end{pmatrix}$.
2. Let B be the standard basis for $\mathbb{R}^3$ and $B' = \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$. Given $v = \begin{pmatrix}3\\2\\1\end{pmatrix}$, find $[v]_{B'}$.

   We solve:
   $$c_1\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\end{pmatrix} + c_3\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}$$
   Solving the system gives $c_1 = 2, c_2 = 0, c_3 = 1$. Thus, $[v]_{B'} = \begin{pmatrix}2\\0\\1\end{pmatrix}$.

13 Image and Kernel

13.1 Theory

For a linear mapping $T : V \to W$:

- The **Image** of T, $Im(T)$, is the set of all possible output vectors in W.
  $$Im(T) = \{T(v) : v \in V\}$$
- The **Kernel** of T, $ker(T)$, is the set of all vectors in V that map to the zero vector in W.
  $$ker(T) = \{v \in V : T(v) = 0\}$$

13.2 Practical Examples

1. Let $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ x+y \end{pmatrix}$.

   - **Image:** The output is always a vector of the form $\begin{pmatrix} t \\ t \end{pmatrix}$. $Im(T) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.
   - **Kernel:** We solve $x + y = 0$, which gives $y = -x$. $ker(T) = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$.

2. Let $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ z \end{pmatrix}$.

   - **Image:** Any vector $\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbb{R}^2$ can be produced. Thus, $Im(T) = \mathbb{R}^2$.
   - **Kernel:** We solve $x + y = 0$ and $z = 0$. This gives $y = -x, z = 0$. $ker(T) = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\}$.

14 Affine Spaces

14.1 Theory

An affine space A is formed by shifting a vector space V by a fixed vector v_0.

$$A = v_0 + V = \{v_0 + v : v \in V\}$$

Unlike a vector space, an affine space does not necessarily have a fixed origin.

14.2 Practical Examples

1. A line in $\mathbb{R}^2$: $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $t \in \mathbb{R}$. Here, $v_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $V = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

2. A plane in $\mathbb{R}^3$: $A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$. This is the xy-plane.

15 Affine Subspaces

15.1 Theory

An affine subspace L is a subset of an affine space that is itself an affine space. It is formed by translating a vector subspace V by a vector v_0.

$$L = v_0 + V$$

15.2 Practical Examples

1. A plane in $\mathbb{R}^3$: $L = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$. This plane passes through $(1, 0, 0)$.

2. A line in $\mathbb{R}^3$: $L = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right\}$. This line passes through $(0, 1, 0)$.

16 Affine Mappings

16.1 Theory

An affine mapping $f(x)$ is a combination of a linear transformation (matrix A) and a translation (vector b).

$$f(x) = Ax + b$$

16.2 Practical Examples

1. Linear regression: $y = X\beta + \epsilon$. The term $X\beta$ is a linear map, and an intercept term within the model acts as the translation.

2. An affine transformation in 2D:

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   This transformation scales, shears, and then translates the input vector.