HW I

Note: In some places, some primitive recursive functions from previous exercises are recalled.

Zero function : $\theta_n : \mathbb{N}^n \to \mathbb{N} \qquad \theta(x_1, \ldots, x_n) = 0$

Successor function : $s : \mathbb{N} \to \mathbb{N} \qquad s(x) = x + 1$

Projection : $I_k^n : \mathbb{N}^n \to \mathbb{N} \qquad I_k^n(x_1, \ldots, x_n) = x_k$

Zadanie 3. Niech f, g oraz p będą pierwotnie rekurencyjnymi predykatami. Wykaż, że następujące predykaty są pierwotnie rekurencyjne:

📌 Let $\vec{x} = (x_1, \ldots, x_n)$.

(a) $\wedge(x_1, ..., x_n) = f(x_1, ..., x_n) \wedge g(x_1, ..., x_n)$

Since f and g are predicate primitive recursive functions, we can write them as a characteristics such that $F(\vec{x}) = X_f(\vec{x})$ and $G(\vec{x}) = X_g(\vec{x})$. So we have,

$$F(\vec{x}) = X_f(\vec{x}) = \begin{cases} 1, & if\ f(\vec{x})\ is\ true \\ 0, & if\ f(\vec{x})\ is\ false \end{cases}$$

$$G(\vec{x}) = X_g(\vec{x}) = \begin{cases} 1, & if\ g(\vec{x})\ is\ true \\ 0, & if\ g(\vec{x})\ is\ false \end{cases}$$

As we know from conjunction logic operator, we have:

$$\wedge(\vec{x}) = f(x_1, ..., x_n) \wedge g(x_1, ..., x_n) = F(\vec{x}).G(\vec{x}) = mul(F(\vec{x}), G(\vec{x}))$$

Hence it is also predicate primitive recursive function.

(b) $\sim (x_1, \ldots, x_n) = \sim f(x_1, \ldots, x_n)$

Let $F(\vec{x}) = X_f(\vec{x})$. Moreover,

$$F(\vec{x}) = X_f(\vec{x}) = \begin{cases} 1, & if\ f(\vec{x})\ is\ true \\ 0, & if\ f(\vec{x})\ is\ false \end{cases}$$

Observe that:

$$sg(F(\vec{x})) = \begin{cases} 0, & X_f(\vec{x}) = 0 \\ 1, & wff. \end{cases}$$

Therefore:

$$\overline{sg}(F(\vec{x})) = \begin{cases} 1, & X_f(\vec{x}) = 0 \\ 0, & wff. \end{cases}$$

We can conclude that:

$$\overline{sg}(F(\vec{x})) = \begin{cases} 1, & if\ f(\vec{x})\ is\ false \\ 0, & wff. \end{cases} = \sim F = \sim X_f(x_1, \ldots, x_n)$$

Since we know negation of signum function is primitive recursive, this predicate function is also primitive recursive.

(c) $if - then - else_p(x_1, ..., x_n, t, f) =$
$\begin{cases} t, & je\dot{z}eli\ p(x_1, ..., x_n) \\ f, & je\dot{z}eli\ \sim p(x_1, ..., x_n) \end{cases}$

Let $P(\vec{x}) = X_p(\vec{x})$.
Also let $T(\vec{x}) = t$, $F(\vec{x}) = f$ be the constant functions.

We can write:

$$if - then - else_p(\vec{x}, t, f) = \begin{cases} T(\vec{x}), & P(\vec{x}) = 1 \\ F(\vec{x}), & P(\vec{x}) = 0 \end{cases}$$

Define a function M such that

$M(h_1, h_2, h_3)(\vec{x}) = M(h_1(\vec{x}), h_2(\vec{x}), h_3(\vec{x}))$
$= \begin{cases} I_2^3(h_1(\vec{x}), h_2(\vec{x}), h_3(\vec{x})), & h_1(\vec{x}) = 1 \\ I_3^3(h_1(\vec{x}), h_2(\vec{x}), h_3(\vec{x})), & h_1(\vec{x}) = 0 \end{cases} = \begin{cases} h_2(\vec{x}), & h_1(\vec{x}) = 1 \\ h_3(\vec{x}), & h_1(\vec{x}) = 0 \end{cases}$

which is primitive recursive function.

Since we can write:

$$if - then - else_p(\vec{x}, t, f) = M(P, T, F)(\vec{x}) = \begin{cases} T(\vec{x}), & P(\vec{x}) = 1 \\ F(\vec{x}), & P(\vec{x}) = 0 \end{cases}$$

$if - then - else$ function is also primitive recursive function.

(d) $f(x_1, ..., x_n, k) = \exists_{i \leq k} p(x_1, ..., x_n, i)$

Let $F(\vec{x}, k) = X_f(\vec{x}, k)$ and $P(\vec{x}, i) = X_p(\vec{x}, i)$.

$$F(\vec{x}, k) = \begin{cases} 1, & \exists i \leq k \ P(\vec{x}, i) = 1 \\ 0, & wff. \end{cases}$$

The class of primitive recursive relations is closed under the logical operators and bounded quantifiers, hence this is also a primitive recursive functions.

Moreover we can write :

$$\begin{aligned} F(\vec{x}, k) &= P(\vec{x}, 0) \vee \cdots \vee P(\vec{x}, k) \\ &= sg(P(\vec{x}, 0) + \cdots + P(\vec{x}, k)) \\ &= sg(\sum_{i \leq k} P(\vec{x}, i)) \end{aligned}$$

Zadanie 6. Niech f : N → N będzie funkcją przyjmującą dla prawie wszystkich argumentów n stałą wartość f(n) = c. Wykaż, że f jest pierwotnie rekurencyjna.

Show that (Is $\forall n \in \mathbb{N} \ \exists c \in \mathbb{N} \ \ f(n) = c$ true?), is primitive recursive. (?)

First define a function $\sigma(c, x) = \begin{cases} 1, & f(x) = c \\ 0, & wff. \end{cases}$.

Let us define a function F such that:

$$F(c, x) = \begin{cases} 1, & \prod_{x \in \mathbb{N}} \sum_{c \in \mathbb{N}} (\sigma(c, x)) \\ 0, & wff. \end{cases}$$

Zadanie 7. Wykaż, że następujące funkcje są pierwotnie rekurencyjne:

Recall *prime* function.

$$prime(n) = \begin{cases} 1, & n > 1 \ \& \ \overline{sg}(\forall x < n(X_|(x,n))) \\ 0, & wff. \end{cases}$$

where $X_|(x,y) \iff x|y$ is another primitive recursive function.

(e) liczba liczb pierwszych mniejszych od n,

Define this function as NP (number of primes):

$$NP(n) = \sum_{i \le n} prime(i)$$

Hence primitive recursive function since it can be expressed with primitive recursive functions.

(f) n-ta liczba pierwsza,

Define this function as TP (nth prime):

$$NT(n) = \begin{cases} min\{i : i < n(\ln n + \ln\ln n) \ \& \ NP(i) = n\}, & n > 5 \\ 11, & n = 5 \\ \vdots & \vdots \\ 2, & n = 1 \\ 0, & n = 0 \end{cases}$$

Hence primitive recursive function since it can be expressed with primitive recursive functions.

By prime number theorem, n th number of prime bounded for n>5. So, this is bounded primitive recursive function since i is bounded.

(g) ⌊n√2⌋

Call this function FS.

From number theory, there is a fact that

$$\lfloor n\sqrt{x} \rfloor \leq n\sqrt{x} < \lfloor n\sqrt{x} \rfloor + 1$$

$$m = \lfloor n\sqrt{2} \rfloor \leq n\sqrt{2} < \lfloor n\sqrt{2} \rfloor + 1 = m + 1$$

$$m^2 \leq 2n^2 < m^2 + 2m + 1$$

Therefore $2n^2 = m^2 + k$ for some integer k. Here m is the answer where k is the remainder such that $k < 2m + 1$.

Observe that:
(iterate finite number of k until we find a integer m and m^2 which is perfect square)

$$FS(n) = \begin{cases} m, & if\ \exists k < 2m + 1 (2n^2 = m^2 + k) \\ 0 & wff. \end{cases}$$

This is bounded primitive recursive function since k is bounded by $2m + 1$.

(h) gcd(n, m) (gcd(0, 0) := 0)

Call this function GCD.
Just for minimalization assume that n>m.

$$GCD(n, m) = \prod_{i \leq NP(n)} C(n, m, NT(i))$$

where

$$C(n, m, i) = \begin{cases} i^k, & max\{k : i^k \leq m \,\&\, (i^k | n \,\&\, i^k | m) = 1\} \\ 1, & wff(\text{ if such k doesn't exist.}). \end{cases}$$

This is bounded primitive recursive function since k is bounded by $log_i m$.