PHYS414 FINAL PROJECT

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1 Newton

1.1 Part A

Combining the hydrostatic equilibrium equations, we obtain:

$$\frac{1}{r^2}\frac{d}{dr}(\frac{r^2}{\rho}\frac{dP}{dr}) = -4\pi G\rho \tag{1}$$

Polytropic relation suggests:

$$P = K\rho^{1+\frac{1}{n}} \tag{2}$$

where K and n are real, positive constants, and n is called a polytropic index. Let us define the dimensionless variables:

$$\rho = \rho_c\theta^n, P = P_c\theta^{n+1}, r = \alpha\xi \tag{3}$$

where

$$\alpha^2 = \frac{K(n+1)\rho_c^{\frac{1-n}{n}}}{4\pi G} \tag{4}$$

By changing the variables to new varibles in eq.3, and inserting into eq.1 we obtain the Lane-Emden Equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\theta^n \tag{5}$$

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DSolve[{y''[x] + (2/x)*y'[x] - y[x]== 0, y[0.0001] == 1, y'[0.0001] == 0}, y, {x, 0.0001, 10}]
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$$\{\{y \to \text{Function}[\{x\}, \frac{e^{-x}(-0.5+0.5\,e^{2x})}{x}]\}\}$$

Figure 1: Mathematica Lane-Emden solution.

As seen from the Fig.1 and Fig.2, The solution goes as 1 - $\frac{1}{6}\xi^2 + \frac{n}{120}\xi^4$ Total radius of the star is notated by R which is equal to

$$R = \alpha\xi_n \tag{6}$$

$$R = (\frac{K}{G}\frac{n+1}{4\pi})^{\frac{1}{2}}\rho_c^{\frac{1-n}{2n}}\xi_n \tag{7}$$

$$\text{Series}\left[\frac{e^{-x}(-0.4999999974998369` + 0.4999999975001702` e^{2x})}{x}, \{x, 0, 4\}\right]$$

$$\frac{3.33289\times10^{-13}}{x} + 1. + 1.66644\times10^{-13}\,x + 0.166667\,x^2 + 1.3887\times10^{-14}\,x^3 + 0.00833333\,x^4 + O[x]^5$$

Figure 2: Series expansion of the solution

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi\alpha^3\rho_c \int_0^{\xi_n} \xi^2\theta^n d\xi = 4\pi\alpha^3\rho_c \int_0^{\xi_n} [-\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi})]d\xi \tag{8}$$

$$M = 4\pi(\frac{K}{G}\frac{n+1}{4\pi})^{\frac{3}{2}}\rho_c^{\frac{3-n}{2n}}(-\xi^2\frac{d\theta}{d\xi})_{\xi=\xi_n} \tag{9}$$

By combining eq.7 and eq.9, we obtain

$$M^{\frac{n-1}{n}}R^{\frac{3-n}{n}} = \frac{K}{GN_n} \tag{10}$$

where,

$$N_n = \frac{(4\pi)^{\frac{1}{n}}}{n+1}([-\xi^2\frac{d\theta}{d\xi}]_{\xi=\xi_n})^{\frac{1-n}{n}}\xi_n^{\frac{n-3}{n}} \tag{11}$$

1.2 Part B

The code to read the csv data can be found in the GitHub newton.py. Mass vs. Radius graph is found to be:

1.3 Part C

Using Mathematica, we obtain the series expansion for the eq.8 of the Final Project.

From Fig.4, we can see that

$$P = \frac{8C\rho^{\frac{5}{q}}}{5D^{\frac{5}{q}}} = K_*\rho^{1+\frac{1}{n_*}} \tag{12}$$

From eq.12 we can observe that

$$K_* = \frac{8C}{5D^{\frac{5}{q}}} \tag{13}$$

and,

$$n_* = \frac{q}{5-q} \tag{14}$$

To find these values, we take the logarithm of both sides in eq.10 and get the result:

$$lnM = \frac{n}{n-1}ln(\frac{K}{GN_n}) + \frac{3-n}{1-n}lnR \tag{15}$$

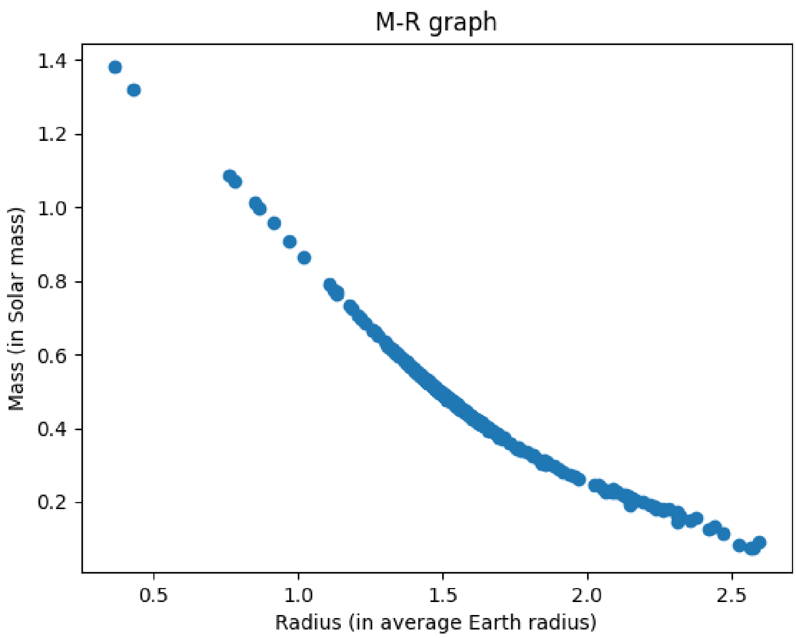

Figure 3: Mass vs. Radius graph for White Dwarfs

series $x\,(2\,x^2-3)\,\sqrt{x^2+1}\;+3\log\left(x+\sqrt{x^2+1}\right)$

Series expansion at x=0

$$\frac{8\,x^5}{5}-\frac{4\,x^7}{7}+\frac{x^9}{3}+O(x^{11})$$

Figure 4: Series expansion of the EOS for cold WDs

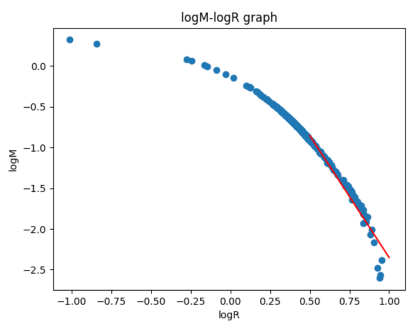

Figure 5: logM vs. logR fitted

So, by fitting the graph lnM vs. lnR, and calculating the slope, we can find n and hence q. Also, y-intercept of the fit should give us the factor containing K

Fig. 5 shows the logM vs. logR graph. We determine the cutoff as logR = 0.50, so R = 1.64, and apply a linear fit to the data.

From the fit, we find that n=1.58, and q is very close to 3. To find K, we need to find N. To find N, we apply shooting method to Lane-Emden equation with n=1.5. N is found to be 0.35010044899818915, and K is calculated as 0.115120399093098

2 Einstein

2.1 Part A

We solve Tolman-Oppenheimer-Volkoff (TOV) equations by integrating them with the shooting method. Initial p(0) is taken as 1e-4, and we increment it by 0.0001 in each step. We take 50 different points with different p(0), and find the Mass and Radius dependence curve given in Fig. 6.

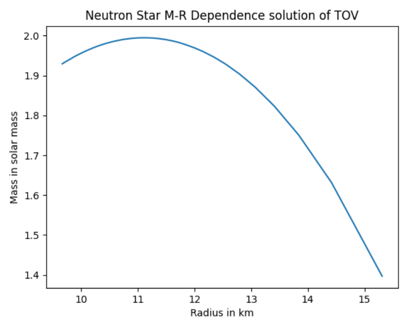

Figure 6: Neutran Star Mass vs. Radius

2.2 Part B

By introducing another variable to the shooting method, which is baryonic mass.

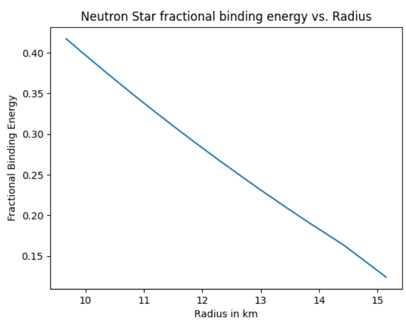

Figure 7: Neutran Star Fractional Binding Energy vs. Radius

2.3 Part C

The plot of M vs ρc curve can be found in Fig. 8. For the stability analysis, we check where the derivative of the graph is 0, and it is found where mass is around 1.9947 in solar units.

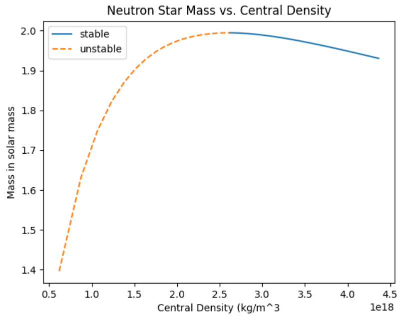

Figure 8: Neutran Star Mass vs. Central Density. Unstable (dashed), stable (Curve)