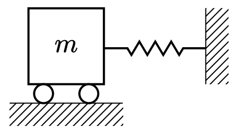
Mass spring system



force F exerted by spring (acceleration of the block)

$$
\begin{aligned}
F &= m * a \\
&= -k * x \quad \text{(Hooke's law)} \\
&= \frac{dv}{dt} \qquad \text{(velocity)} \\
&= \frac{d^2x}{xt^2} \qquad \text{(acceleration)}
\end{aligned}
$$

solving the differential equation

$$\frac{d^2x}{dt^2} = \frac{-k * x}{m}$$

yields

$$x(t) = x_0 * \cos(\omega * t) + \frac{v_0}{\sqrt{\frac{k}{m}}} * \sin(\omega * t)$$

which somehow translates to

$$x(t) = A * \cos\left(\omega * t - \phi\right)$$

with

$$\omega = \sqrt{\frac{k}{m}}$$

since ω is the angular rate in radians, $\frac{\omega}{2\pi} = T$ is the period, the time it takes to complete one full circle.

$$
\begin{aligned}
T &= \frac{\omega}{2\pi} \\
&= \frac{\sqrt{\frac{k}{m}}}{2\pi} \\
&= \sqrt{\frac{m}{k}} * 2\pi
\end{aligned}
$$

Energy

the spring holds potential energy PE, the mass holds kinetic energy KE.

$$\mathrm{PE} = \frac{kx^2}{2}$$

$$\mathrm{KE} = \frac{mv^2}{2}$$

then, the total energy E is

$$E = \frac{kx^2 + mv^2}{2}$$

The total energy in the system stays constant over time.

$$\frac{dE}{dt} = 0$$

Total energy

It is possible to calculate the total amount of energy present, when the mass stops and changes its direction of movement about the extreme points of the vibration $(x = -A, x = A)$.

$$E(v = 0, x = A) = \frac{kx^2 + \cancel{mv^2}}{2} = \frac{kA^2}{2}$$

Maximum velocity

Similarly about the equilibrium point $(x = 0)$, the spring has no energy and the mass has all of the energy.

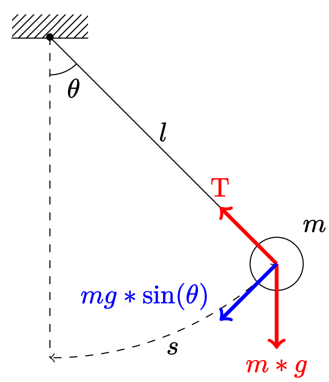
$$\begin{aligned}
E(x = 0) &= \frac{\cancel{kx^2} + mv^2}{2} = \frac{mv_{\max}^2}{2} \\
\frac{kA^2}{2} &= \frac{mv_{\max}^2}{2} \\
\frac{kA^2}{m} &= v_{\max}^2 \\
\sqrt{\frac{kA^2}{m}} &= v_{\max} \\
\sqrt{\frac{k}{m}} &=
\end{aligned}$$

Velocity in function of position

(The total energy in the system stays constant over time.)

$$
\begin{aligned}
\frac{kA^2}{2} &= \frac{kx^2 + mv^2}{2} \\
kA^2 &= kx^2 + mv^2 \\
k * (A^2 - x^2) &= mv^2 \\
\frac{k}{m} * (A^2 - x^2) &= v^2 \\
\sqrt{\frac{k}{m} * (A^2 - x^2)} &= v(x)
\end{aligned}
$$

Pendulum



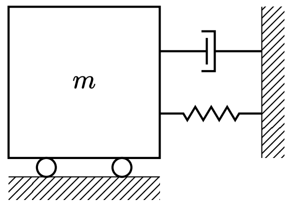
Two forces acting on the commuter, the gravity force $|F_g| = m * g$ and tension force T are pictured. The tangent force will be the projection of mg onto the tangent, as T is perpendicular to the arc of movement and thus does not contribute to the tangential movement of the commuter.

Since the force and thus acceleration of the commuter is not proportional to θ but $\sin(\theta)$, this is not simple harmonic motion. However for small angles θ, $\sin(\theta) \simeq \theta$, thus for large distances l, assume $F_{\text{total}} = mg * \theta$. And since $\theta = \frac{s}{l}$, we can directly fix acceleration in function of position:

$$F = mg * \frac{s}{l}$$

notice that this is directly analogous to $F = k * x$.

Mass-spring-damper system



$$F_{\text{damper}} = \underbrace{-b}_{\text{damping coefficient}} * v$$

While the force of the spring is dependent on position, the force of the damper is dependent on velocity.

$$\begin{aligned} m * a = F_{\text{total}} &= F_{\text{damper}} + F_{\text{spring}} \\ &= -ks + -bv \end{aligned}$$

Our differential equation now becomes

$$m * \frac{d^2x}{dt^2} = -b * \frac{dx}{dt} + -k * s$$

When solved, yields

$$x = A * e^{-\gamma * t} * \cos(\omega' * t)$$

with

$$\gamma = \frac{b}{2m} \quad \wedge \quad \omega' = \sqrt{\frac{k}{m} - \gamma^2}$$

Forced oscillation

mass-spring-damper system, however now an external force, which is also a sin wave, is present.

$$m * a = F_{\text{total}} = -kx - bv + F_{\text{ext,initial}} * cos(\omega * t)$$

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F_{\text{ext,initial}} * cos(\omega * t)$$

solution of which is

$$x = A_0 * \sin(\omega * t + \phi_0)$$

with

$$A_0 = \frac{F_0}{m * \sqrt{(\omega^2 - \omega_0^2) + b^2 * \omega^2 * m^{-2}}} \quad \wedge \quad \phi_0 = \text{atan}\left(\frac{\omega_0^2 - \omega}{\omega * (\frac{b}{m})}\right)$$

Amplitude A_0 of the resultant motion is dependent on the frequency of the applied force $\frac{\omega}{2\pi}$. As the applied force frequency reaches the natural frequency of the oscillator, a spike in A_0 is observed.