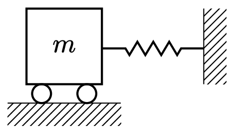
# Table of contents

# Part I

# Simple Harmonic Motion

## 1 Mass spring system




force $F$ exerted by spring (acceleration of the block)

$$\begin{aligned} F &= m * a \\ &= -k * x \quad \text{(Hooke's law)} \\ &= \frac{d^2 s}{dt^2} \quad \text{(acceleration)} \end{aligned}$$

solving the differential equation

$$\frac{d^2 s}{dt^2} = \frac{-k * s}{m}$$

yields

$$s(t) = s_0 * \cos(\omega * t) + \frac{v_0}{\sqrt{\frac{k}{m}}} * \sin(\omega * t)$$

which somehow translates to

$$s(t) = A * \cos(\omega * t - \phi) \mid \omega = \sqrt{\frac{k}{m}}$$

### 1.1 Displacement, velocity, acceleration

since

$$s(t) = A * \cos(\omega * t - \phi)$$

therefore,

$$\begin{aligned} v(t) &= \frac{ds}{dt} \\ &= -A * \omega * \sin(-\omega * t + \phi) \end{aligned}$$

and

$$\begin{aligned} a(t) &= \frac{d^2 s}{dt^2} \\ &= -A * \omega^2 * \sin(-\omega * t + \phi) \end{aligned}$$

## 1.2 Frequency, period

since $\omega$ is the angular rate in radians, $\frac{\omega}{2\pi} = T$ is the period, the time it takes to complete one full circle.

$$\begin{aligned} T &= \frac{\omega}{2\pi} \\ &= \frac{\sqrt{\frac{k}{m}}}{2\pi} \\ &= \sqrt{\frac{m}{k}} * 2\pi \end{aligned}$$

## 1.3 Energy

the spring holds potential energy PE, the mass holds kinetic energy KE.

$$\text{PE} = \int ks\, ds = \frac{ks^2}{2}$$

$$\text{KE} = \frac{mv^2}{2}$$

then, the total energy $E$ is

$$E = \frac{ks^2 + mv^2}{2}$$

The total energy in the system stays constant over time.

$$\frac{dE}{dt} = 0$$

### 1.3.1 Total energy

At extreme points, all of the energy will be either on the spring, or on the mass.

When the mass is at the outer extremes ($x = +A, x = -A$), where the mass changes direction, it briefly stops. At this point, all of the energy will be stored as potential energy in the spring.

It is possible to calculate the total amount of energy present, when the mass stops and changes its direction of movement about the extreme points of the vibration ($x = -A, x = A$).

$$\begin{aligned} E &= \frac{kx^2 + mv^2}{2} \\ E(v = 0, x = A) &= \frac{k * (A)^2 + m * (0)^2}{2} = \frac{kA^2}{2} \end{aligned}$$

### 1.3.2 Maximum velocity

Similarly, when the mass is passing through the rest position ($x = 0$), the spring will have no energy and the mass will have all of the energy.

$$E(x=0) = \frac{k*(0)^2 + mv^2}{2} = \frac{mv_{\text{max}}^2}{2}$$

since energy stays constant,

$$\begin{aligned}
\frac{kA^2}{2} &= \frac{mv_{\text{max}}^2}{2} \\
\frac{kA^2}{m} &= v_{\text{max}}^2 \\
\sqrt{\frac{kA^2}{m}} &= v_{\text{max}} \\
\sqrt{\frac{k}{m}} &=
\end{aligned}$$

### 1.3.3 Velocity in function of position

(The total energy in the system stays constant over time.)

$$\begin{aligned}
\frac{kA^2}{2} &= \frac{kx^2 + mv^2}{2} \\
kA^2 &= kx^2 + mv^2 \\
k*(A^2 - x^2) &= mv^2 \\
\frac{k}{m}*(A^2 - x^2) &= v^2 \\
\sqrt{\frac{k}{m}*(A^2 - x^2)} &= v(s)
\end{aligned}$$

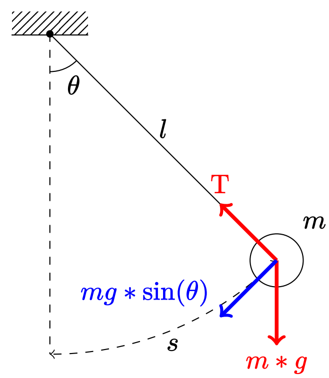
## 1.4 Pendulum

Gravity $|F_g| = m * g$ and tension $T$ act on the commuter.

$T$ is perpendicular to the arc of movement, thus does not contribute to the acceleration.

The acceleration is thus the projection of $mg$ onto the tangent.

Simple harmonic motion is when acceleration is in function of position.
Since acceleration here is in function of $\sin(\theta)$ rather than $\theta$, this is not simple harmonic motion.




However, for very long pendulums (very large $l$), the angle $\theta$ will be very small. Approximating sin(small $\theta$) $\simeq \theta$, we get

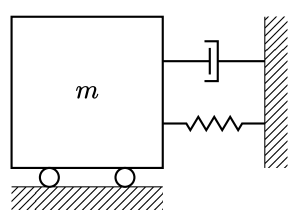
$$F = mg * \sin(\theta) \simeq mg * \theta$$

Note that conveniently, $\theta = \frac{s}{l}$

$$\begin{aligned} F &= mg * \frac{s}{l} \\ &= \frac{mg}{l} * s \end{aligned}$$

notice that this is directly analogous to $F = k * s$.

# 2 Mass-spring-damper system




Just like a spring applies force in function of distance,
A damper applies force in function of velocity.

$$F_{\text{damper}} = \overbrace{-b}^{\text{damping coefficient}} * v$$

$$
\begin{aligned}
ma = F_{\text{total}} &= F_{\text{spring}} + F_{\text{damper}} \\
&= \underbrace{-ks}_{\text{spring}} + \underbrace{-bv}_{\text{damper}}
\end{aligned}
$$

Our differential equation now becomes

$$m\frac{d^2s}{dt^2} = -ks + -b\frac{ds}{dt}$$

When solved, yields

$$s = A * e^{-\gamma * t} * \cos(\omega' * t)$$

with

$$\gamma = \frac{b}{2m} \quad \wedge \quad \omega' = \sqrt{\frac{k}{m} - \gamma^2}$$

## 2.1 Damping

is underdamped when $b^2 < 4 * m * k$, it will overshoot and oscillate for a while.

is critically damped when $b^2 = 4mk$, it reaches equilirium point in the shortest time

is overdamped when $b^2 > 4mk$, it takes longer time to stabilise

| | |
|---|---|
| underdamped | $b^2 < 4 * m * k$ |
| critical | $b^2 = 4 * m * k$ |
| overdamped | $b^2 > 4 * m * k$ |

> **Tip**
>
> there is a link between this $b^2 = 4mk$ and $\omega'$. find it.

# 3 Forced oscillation

mass-spring-damper system, with an external periodic force present

$$
\begin{aligned}
F_{\text{total}} = m * a = \underbrace{-ks}_{\text{spring}} + \underbrace{-bv}_{\text{damper}} + \underbrace{F_{\text{ext,initial}} * \cos(\omega * t)}_{\text{periodic forcing}} \\
m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F_{\text{ext,initial}} * \cos(\omega * t)
\end{aligned}
$$

solution of which is

$$x = A_0 * \sin(\omega * t + \phi_0)$$

with

$$\text{amplitude } A_0 = \frac{F_0}{m * \sqrt{(\omega^2 - \omega_0^2) + b^2 * \omega^2 * m^{-2}}} \quad \wedge \quad \phi_0 = \text{atan}\left(\frac{\omega_0^2 - \omega}{\omega * (\frac{b}{m})}\right)$$

where

- $\omega$ is the frequency of the applied force
- $\omega_0$ is the resonant frequency

The natural (resonant) frequency $\omega_0$ of an mass-spring system is

$$\omega_0 = 2\pi * f = \sqrt{\frac{k}{m}}$$

The amplitude of oscillation of a forced mass-spring-damper system is greatest when forcing occurs at the natural frequency.

> ⚠ Warning
>
> TODO this is an excellent simulation visualisation opportunity