# vibrations summary

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# Part I

# Simple Harmonic Motion

## 1   Mass spring system
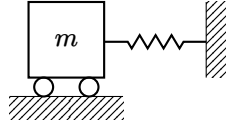


force $F$ exerted by spring (acceleration of the block)

$$F = m * a$$
$$= -k * x \quad \text{(Hooke's law)}$$
$$= \frac{d^2 s}{dt^2} \quad \text{(acceleration)}$$

solving the differential equation

$$\frac{d^2 s}{dt^2} = \frac{-k * s}{m}$$

yields

$$s(t) = s_0 * \cos(\omega * t) + \frac{v_0}{\sqrt{\frac{k}{m}}} * \sin(\omega * t)$$

which somehow translates to

$$s(t) = A * \cos\left(\omega * t - \phi\right) \mid \omega = \sqrt{\frac{k}{m}}$$

## 1.1   Displacement, velocity, acceleration

since

$$s(t) = A * \cos(\omega * t - \phi)$$

therefore,

$$v(t) = \frac{ds}{dt}$$
$$= -A * \omega * \sin(-\omega * t + \phi)$$

and

$$a(t) = \frac{d^2 s}{dt^2}$$
$$= -A * \omega^2 * \sin(-\omega * t + \phi)$$

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## 1.2 Frequency, period

since $\omega$ is the angular rate in radians, $\frac{\omega}{2\pi} = T$ is the period, the time it takes to complete one full circle.

$$T = \frac{\omega}{2\pi}$$

$$= \frac{\sqrt{\frac{k}{m}}}{2\pi}$$

$$= \sqrt{\frac{m}{k}} * 2\pi$$

## 1.3 Energy

the spring holds potential energy PE, the mass holds kinetic energy KE.

$$\text{PE} = \int ks\,ds = \frac{ks^2}{2}$$

$$\text{KE} = \frac{mv^2}{2}$$

then, the total energy $E$ is

$$E = \frac{ks^2 + mv^2}{2}$$

The total energy in the system stays constant over time.

$$\frac{dE}{dt} = 0$$

### 1.3.1 Total energy

At extreme points, all of the energy will be either on the spring, or on the mass.

When the mass is at the outer extremes $(x = +A, x = -A)$, where the mass changes direction, it briefly stops. At this point, all of the energy will be stored as potential energy in the spring.

It is possible to calculate the total amount of energy present, when the mass stops and changes its direction of movement about the extreme points of the vibration $(x = -A, x = A)$.

$$E = \frac{kx^2 + mv^2}{2}$$

$$E(v = 0, x = A) = \frac{k * (A)^2 + m * (0)^2}{2} = \frac{kA^2}{2}$$

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### 1.3.2 Maximum velocity

Similarly, when the mass is passing through the rest position ($x = 0$), the spring will have no energy and the mass will have all of the energy.

$$E(x = 0) = \frac{k * (0)^2 + mv^2}{2} = \frac{mv_{\text{max}}^2}{2}$$

since energy stays constant,

$$\frac{kA^2}{2} = \frac{mv_{\text{max}}^2}{2}$$
$$\frac{kA^2}{m} = v_{\text{max}}^2$$
$$\sqrt{\frac{kA^2}{m}} = v_{\text{max}}$$
$$\sqrt{\frac{k}{m}} =$$

### 1.3.3 Velocity in function of position

(The total energy in the system stays constant over time.)

$$\frac{kA^2}{2} = \frac{kx^2 + mv^2}{2}$$
$$kA^2 = kx^2 + mv^2$$
$$k * (A^2 - x^2) = mv^2$$
$$\frac{k}{m} * (A^2 - x^2) = v^2$$
$$\sqrt{\frac{k}{m} * (A^2 - x^2)} = v(s)$$

## 1.4 Pendulum

Gravity $|F_g| = m * g$ and tension $T$ act on the commuter.

$T$ is perpendicular to the arc of movement, thus does not contribute to the acceleration.

The acceleration is thus the projection of $mg$ onto the tangent.

Simple harmonic motion is when acceleration is in function of position.
Since acceleration here is in function of $\sin(\theta)$ rather than $\theta$, this is not simple harmonic motion.
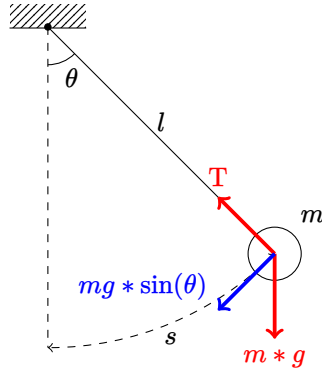
However, for very long pendulums (very large $l$), the angle $\theta$ will be very small. Approximating $\sin(\text{small } \theta) \simeq \theta$, we get
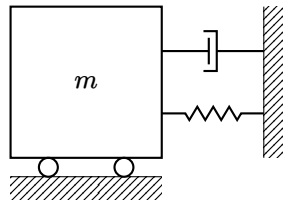
$$F = mg * \sin(\theta) \simeq mg * \theta$$

Note that conveniently, $\theta = \dfrac{s}{l}$

$$F = mg * \frac{s}{l}$$
$$= \frac{mg}{l} * s$$

notice that this is directly analogous to $F = k * s$.

## 2 Mass-spring-damper system



Just like a spring applies force in function of distance,
A damper applies force in function of velocity.

$$F_{\text{damper}} = \overbrace{-b}^{\text{damping coefficient}} * v$$

$$ma = F_{\text{total}} = F_{\text{spring}} + F_{\text{damper}}$$
$$= \underbrace{-ks}_{\text{spring}} + \underbrace{-bv}_{\text{damper}}$$

Our differential equation now becomes

$$m\frac{d^2 s}{dt^2} = -ks + -b\frac{ds}{dt}$$

When solved, yields

$$s = A * e^{-\gamma * t} * \cos(\omega' * t)$$

with

$$\gamma = \frac{b}{2m} \quad \wedge \quad \omega' = \sqrt{\frac{k}{m} - \gamma^2}$$

## 2.1 Damping

is underdamped when $b^2 < 4 * m * k$, it will overshoot and oscillate for a while.

is critically damped when $b^2 = 4mk$, it reaches equilirium point in the shortest time

is overdamped when $b^2 > 4mk$, it takes longer time to stabilise

| | |
|---|---|
| underdamped | $b^2 < 4 * m * k$ |
| critical | $b^2 = 4 * m * k$ |
| overdamped | $b^2 > 4 * m * k$ |

💡 Tip

there is a link between this $b^2 = 4mk$ and $\omega'$. find it.

# 3 Forced oscillation

mass-spring-damper system, with an external periodic force present

$$F_{\text{total}} = m * a = \underbrace{-ks}_{\text{spring}} + \underbrace{-bv}_{\text{damper}} + \underbrace{F_{\text{ext,initial}} * \cos(\omega * t)}_{\text{periodic forcing}}$$

$$m\frac{d^2 x}{dt^2} = -b\frac{dx}{dt} - kx + F_{\text{ext,initial}} * \cos(\omega * t)$$

solution of which is

$$x = A_0 * \sin(\omega * t + \phi_0)$$

with

$$\text{amplitude } A_0 = \frac{F_0}{m * \sqrt{(\omega^2 - \omega_0^2) + b^2 * \omega^2 * m^{-2}}} \quad \wedge \quad \phi_0 = \text{atan}\left(\frac{\omega_0^2 - \omega}{\omega * \left(\frac{b}{m}\right)}\right)$$

where

- $\omega$ is the frequency of the applied force
- $\omega_0$ is the resonant frequency

The natural (resonant) frequency $\omega_0$ of an mass-spring system is

$$\omega_0 = 2\pi * f = \sqrt{\frac{k}{m}}$$

The amplitude of oscillation of a forced mass-spring-damper system is greatest when forcing occurs at the natural frequency.

> ⚠️ Warning
>
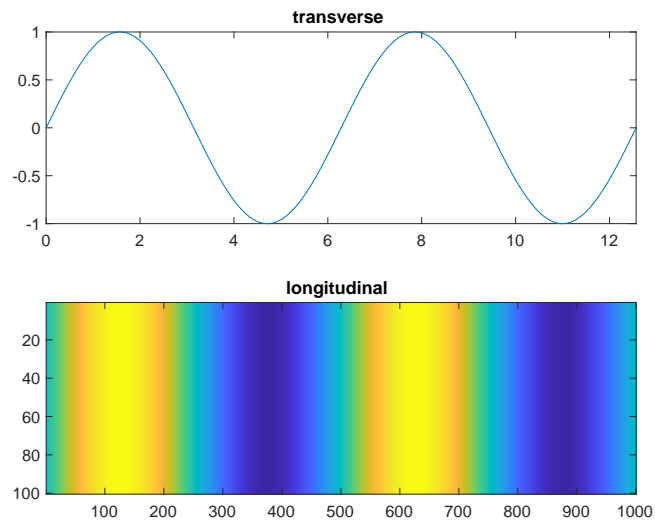> TODO this is an excellent simulation visualisation opportunity

# Part II

# Waves

wave is described by the properties amplitude, wavelength/wave speed, frequency/period.

converting between wavelength $\lambda$ and wave speed $\nu$:

$$\nu = \lambda * f$$

transverse and longitudinal waves:



example: tightrope is transverse, sound wave is longitudinal

wave speed can be generally described by

$$\nu = \sqrt{\frac{\text{stiffness}}{\text{inertia}}}$$

in the case of a jumping rope flung up and down,

$$\nu = \sqrt{\frac{\text{tension}}{\text{mass per length}}}$$

in the case of sound traveling through air,

$$\nu = \sqrt{\frac{\text{bulk modulus}}{\text{density}}}$$

# 1 Energy

## 1.1 Inverse square law

wave intensity $I$

$$I = \frac{\text{power}}{\text{area}}$$

remember that we established $E = \frac{k*A^2}{2}$ in the previous part. This tells us that energy is linearly proportional to amplitude square.

Power is energy per time, and thus power is linearly proportional to energy

And thus, $$I \propto A^2.$$

In the case of a wave propagating from a point source, the wave will diffuse in a spherical shape. The area bit is thus $4\pi r^2$.

If we fix power to be constant, then

$$I \propto \frac{1}{r^2}$$

this is the inverse square law.

from this, it also follows that $\quad A \propto \frac{1}{r}$

> **ℹ Note**
>
> the following equation is included, not sure if it will come handy
>
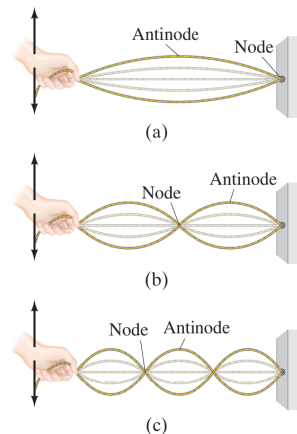> $$\frac{A_2}{A_1} = \frac{r_1}{r_2}$$

# 2 Standing wave

Reflection of a wave interferes with the transmission to create nodes and antinodes which appear not to move.

The distance over which this wave reflects should be a multiple of $\lambda/2$.

# 3 Refraction (Snell's law)

when a wave is transitioning between two mediums, in which the wave travels with two different speeds,
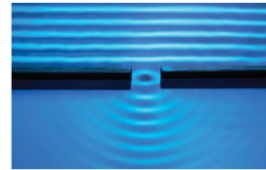


(a)

(b)

(c)

$$\frac{\sin(\theta_{\text{in}})}{\sin(\theta_{\text{out}})} = \frac{\nu_{\text{in}}}{\nu_{\text{out}}}$$
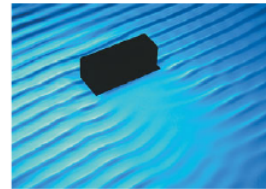
# 4  Diffraction

Waves bend around a blocking obstacle.

Waves will diffract more around obstacles which
are close to and smaller than the wavelength. A
rule of thumb says the turn angle diffraction can achieve around an obstacle is

$$\theta = \frac{\lambda}{l}$$

(a)

(b)

# Part III
# Sound