HOJA 4ª

PROBLEMA 1:]

$$f(x) = e^{iax} = \cos ax + i \operatorname{sen} ax$$

EL PERIODO DE ESTA FUNCIÓN ES $T = \frac{2\pi}{a}$

YA QUE $f(x + \frac{2\pi}{a}) = e^{ia(x+\frac{2\pi}{a})} =$

$$= e^{iax + i2\pi} \underset{\text{VER EJERCICIO 1}}{=} e^{iax} e^{i2\pi} = e^{iax}$$

PROBLEMA 2] VER APUNTES

PROBLEMA 3:] SI $f, g \in L_2[a,b]$ SON ORTOGONALES

SE TIENE QUE $\langle f, g \rangle = 0$ Y $\|f\|_2 = \|g\|_2 = 1$

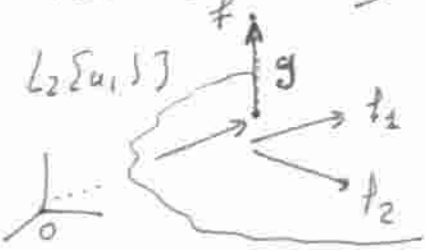
ASÍ $\|f-g\|_2^2 = \langle f-g, f-g \rangle = \int_a^b (f-g)(\overline{f-g})\, dt =$

$$= \int_a^b f\overline{f} - g\overline{f} - f\overline{g} + g\overline{g}\, dt =$$

$$= \|f\|_2^2 - \langle g, \overline{f} \rangle - \langle f, \overline{g} \rangle + \|g\|_2^2 = 2$$

POR TANTO $\|f-g\|_2 = \sqrt{2}$

PROBLEMA 4:]



SI $0 = \langle g, h \rangle \quad \forall h \in [f_1, f_2 \ldots f_n]$

TENDREMOS LO QUE NOS PIDEN

$h = \sum_{i=1}^n a_i f_i$ ES UNA COMBINACIÓN LINEAL DE LOS ELEMENTOS $\{f_1, f_2 \ldots f_n\}$

ASÍ $\langle f - \sum_{i=1}^n \langle f, f_i \rangle f_i, f_k \rangle = \int_a^b (f - \sum_{i=1}^n \langle f, f_i \rangle f_i) f_k\, dt =$

$$= \int_a^b f \cdot f_k\, dt - \sum_{i=1}^n \langle f, f_i \rangle \int_a^b f_i f_k\, dt \underset{\text{POR LA ORTONORMALIDAD}}{=} \int_a^b f f_k\, dt - \langle f, f_k \rangle = 0$$

POR TANTO $\langle f - \sum_{i=1}^n \langle f, f_i \rangle f_i, \sum_{k=1}^n a_k f_k \rangle = 0.$

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PROBLEMA 5º] SI $x_n \xrightarrow[n\to\infty]{} x$

ENTONCES $\left| x - \dfrac{x_1 + \cdots + x_n}{n} \right| = \left| \dfrac{nx}{n} - \dfrac{x_1 + \ldots + x_n}{n} \right| =$

$$= \frac{1}{n} \left| (x - x_1) + (x - x_2) + \cdots + (x - x_n) \right|$$

ASI DADO $\varepsilon > 0$

PARA $\varepsilon/2$ $\exists k_0$ TAL QUE SI $k \geq k_0 \Rightarrow |x - x_k| \leq \varepsilon/2$

POR OTRO LADO $\exists n_0$ TAL QUE SI $n \geq n_0$

$$\frac{\left| (x - x_1) + \cdots + (x - x_{k_0}) \right|}{n} < \varepsilon/2$$

ASI SI $n \geq n_0 \geq k_0$

$$\left| x - \frac{x_1 + \cdots + x_n}{n} \right| \leq \frac{1}{n} \left| (x - x_1) + \cdots + (x - x_{k_0}) \right| + \frac{1}{n} \sum_{k = k_0 + 1}^{n} |x - x_k| \leq$$

$$\leq \varepsilon/2 + \frac{1}{n} (n - k_0)\, \varepsilon/2 < \varepsilon$$

LUEGO $\sigma_n = \dfrac{x_1 + \cdots + x_n}{n} \xrightarrow[n\to\infty]{} x$

PROBLEMA 6º] SI f ES 2π-PERIÓDICA Y DERIVABLE ENTONCES SU SERIE DE FOURIER $\dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \operatorname{sen} nx$ CONVERGE PUNTUALMENTE A f $\forall x \in [-\pi, \pi]$

POR SER f PAR

$$\boxed{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \operatorname{sen} nx \, dx} \underset{\substack{y = -x \\ dy = -dx}}{=} \frac{1}{\pi} \int_{\pi}^{-\pi} f(-y) \operatorname{sen}(n(-y)) (-1)\, dy \underset{\substack{f \text{ PAR} \\ \operatorname{sen}(n(-y)) = -\operatorname{sen} ny}}{=}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) (-\operatorname{sen} ny)\, dy = \boxed{-b_n.}$$

LUEGO $b_n = -b_n \Rightarrow b_n = 0 \quad \forall n \in \mathbb{N}.$

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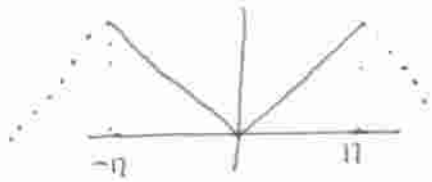
PROBLEMA 7º]

i) $f(x) = |x|$



f ES PAR, POR EL EJERCICIO 6º] $b_n = 0$

Y ASÍ $f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

(COMO LA EXTENSIÓN DE f 2π-PERIÓDICA ES CONTINUA Y EN $(-\pi,\pi)-\{0\}$ ES DERIVABLE, Y COMO EN $x=0$ Y $x=\pm\pi$ EXISTEN LAS DERIVADAS LATERALES, LA SERIE DE FOURIER DE f CONVERGE PUNTUALMENTE EN TODO $[-\pi,\pi]$ A f)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right] =$$

$$\underset{\substack{\downarrow \\ |x|\cos nx \text{ PAR}}}{=} \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx \underset{\substack{\downarrow \\ \text{POR PARTES}}}{=} \frac{2}{\pi}\left[x\frac{\operatorname{sen} nx}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\operatorname{sen} nx}{n}\,dx\right] =$$

$$= \frac{2}{\pi}\left[\frac{\cos nx}{n^2}\Big|_0^{\pi}\right] = \frac{2}{\pi}\left[\frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right] =$$

$$= \begin{cases} 0 & n \text{ PAR} \\ \frac{4}{\pi n^2} & n \text{ IMPAR} \end{cases}$$

$$|x| = \frac{\pi^2}{2} + \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)^2}\cos(2k+1)x.$$

ii) $\boxed{\cos^3 x} = \cos x \cos^2 x = \cos x\left(\frac{1+\cos 2x}{2}\right) = \frac{1}{2}\cos x + \frac{1}{2}\cos x\cos 2x =$

$$\underset{\substack{\downarrow \\ \cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]}}{=} \frac{1}{2}\cos x + \frac{1}{4}[\cos 3x + \cos x] = \boxed{\frac{3}{4}\cos x + \frac{1}{4}\cos 3x}$$

SERIE DE FOURIER.

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PROBLEMA 8:]

$$\int_{\alpha}^{\alpha+T} f(t)\,dt = \int_{\alpha}^{0} f(t)\,dt + \int_{0}^{\alpha+T} f(t)\,dt =$$

CAMBIO DE VARIABLE EN LA 1ª INTEGRAL

$t = x - T$

$dt = dx$

$$= \int_{\alpha+T}^{T} f(x-T)\,dx + \int_{0}^{\alpha+T} f(t)\,dt \underset{f\ T\text{-PERIÓDICA}}{=}$$

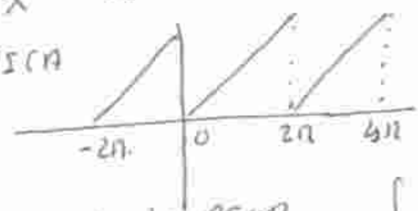
$$= \int_{\alpha+T}^{T} f(x)\,dx + \int_{0}^{\alpha+T} f(t)\,dt = \int_{0}^{T} f(t)\,dt$$

EN EL CASO PARTICULAR DE QUE $\alpha = -T/2$, SE SIGUE QUE

$$\int_{-T/2}^{T/2} f(t)\,dt = \int_{0}^{T} f(t)\,dt$$

PROBLEMA 9:]

a) SEA $f(x) = x$ SI $x \in [0, 2\pi)$ Y SE REPLICA DE FORMA 2π-PERIÓDICA



POR EL PROBLEMA ANTERIOR, $\int_{-\pi}^{\pi} f(x)\cdot\cos nx\,dx = \int_{0}^{2\pi} f(x)\cos nx\,dx$

Y LO MISMO PARA EL SENO.

ASÍ $a_0 = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left.\frac{x^2}{2}\right|_0^{2\pi} = 2\pi$; $\frac{a_0}{2} = \pi$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx \underset{\text{POR PARTES}}{=} \frac{1}{\pi}\left[\left. x\frac{\operatorname{sen} nx}{n}\right|_0^{2\pi} - \int_0^{2\pi}\frac{\operatorname{sen} nx}{n}\,dx\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\operatorname{sen} nx\,dx = \frac{1}{\pi}\left[\left.-x\frac{\cos nx}{n}\right|_0^{2\pi} + \underbrace{\int_0^{2\pi}\frac{\cos nx}{n}\,dx}_{0}\right]$$

$$= -\frac{2\pi}{\pi n} = -\frac{2}{n}$$

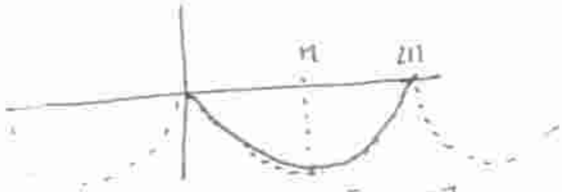
LUEGO $x = \pi - 2\sum\frac{\operatorname{sen} nx}{n}$ SI $0 < x < 2\pi$. OBSERVEMOS QUE $\forall x \in (0, 2\pi)$ f ES DERIVABLE, LUEGO LA SERIE DE FOURIER CONVERGE PUNTUALMENTE A $f(x)$, PARA $x = 0$ Ó $x = 2\pi$ LA SERIE CONVERGE A $\pi \neq f(0) = f(2\pi)$

HOJA 4^a

PROBLEMA 9:] b) SEA $g(x) = \frac{x^2}{2} - \pi x \quad x \in (0, 2\pi)$

$0 = g(0) = g(2\pi) = \frac{4\pi^2}{2} - 2\pi^2$, LUEGO PUEDO EXTENDER

g DE FORMA 2π-PERIÓDICA



$\left[\; g'(x) = x - \pi \begin{cases} <0 & \text{SI } x<\pi \\ >0 & \text{SI } x\in(\pi,2\pi) \end{cases} \right.$

POR EL PROBLEMA 8: PUEDO TRABAJAR EN $[0,2\pi]$

ADEMÁS COMO g ES PAR $b_n = 0$ USANDO (PROBLEMA 6:)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{x^2}{2} - \pi x \, dx = \frac{1}{\pi}\left[\frac{x^3}{6} - \frac{\pi x^2}{2} \right|_0^{2\pi} =$$

$$= \frac{1}{\pi}\left[\frac{8\pi^3}{6} - \frac{4\pi^3}{2} \right] = \pi^2 \left[4/3 - 2 \right] = -\frac{2\pi^2}{3}$$

ASÍ $\frac{a_0}{2} = -\frac{\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{x^2}{2} - \pi x \right) \cos nx \, dx \underset{\text{PARTES}}{=} \frac{1}{\pi}\left[\left(\frac{x^2}{2} - \pi x \right) \frac{\operatorname{sen} nx}{n} \right|_0^{2\pi} -$$

$$- \int_0^{2\pi} (x-\pi) \frac{\operatorname{sen} nx}{n} \, dx = -\frac{1}{\pi}\int_0^{2\pi} (x-\pi)\frac{\operatorname{sen} nx}{n} \, dx \underset{\text{PARTES}}{=}$$

$$= -\frac{1}{\pi}\left[-(x-\pi)\frac{\cos nx}{n^2} \right|_0^{2\pi} + \underbrace{\int_0^{2\pi} \frac{\cos nx}{n^2}\, dx}_{0} \Big] =$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n^2} + \frac{\pi}{n^2} \right] = \frac{2}{n^2}$$

ASÍ $\frac{x^2}{2} - \pi x = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} \cos nx \quad 0 \le x \le 2\pi$

YA QUE g ES DERIVABLE EN $x \in (0, 2\pi)$ Y EN $x=0$ Y $x=2\pi$

EXISTEN LAS DERIVADAS LATERALES Y $g(0^+) = g(0^-) = 0$

Y $g(2\pi^+) = g(2\pi^-) = 0$.

PROBLEMA 10:] a) TOMANDO $x=0$ EN PROBLEMA ANTERIOR

PARTE b) $\quad 0 = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2}\cos 0 \iff \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$

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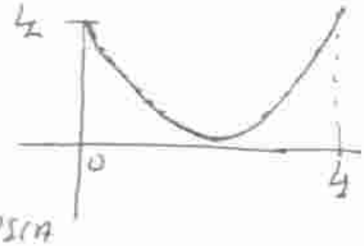
PROBLEMA 11:

SEA $f(x) = (x-2)^2$, $x \in [0,4]$



ESTA FUNCIÓN SE PUEDE EXTENDER DE FORMA 4-PERIÓDICA

CONSIDEREMOS $\cos n\frac{2\pi}{4}x$, $\operatorname{sen} \frac{n\pi}{2}x$ $n \in \mathbb{N}$

FAMILIA 4-PERIÓDICA Y ORTOGONAL EN $L_2[0,4]$

COMO LA EXTENSIÓN DE f ES PAR $b_n = 0$

$$a_0 = \frac{1}{2}\int_0^4 (x-2)^2\,dx = \frac{1}{2}\left.\frac{(x-2)^3}{3}\right|_0^4 = \frac{1}{6} + \frac{8}{2\cdot 3} = \frac{9}{6} = \frac{3}{2}$$

$$\frac{a_0}{2} = \frac{3}{4}$$

$$a_n = \frac{1}{2}\int_0^4 (x-2)^2 \cos\frac{n\pi}{2}x\,dx \underset{\text{PARTES}}{=}$$

$$= \frac{1}{2}\left[\underbrace{\left.\frac{(x-2)^2 \operatorname{sen} n\pi/2\, x}{n\pi/2}\right|_0^4}_{=0} - 2\int_0^4 (x-2)\frac{\operatorname{sen} n\pi/2\, x}{n\pi/2}\,dx\right] =$$

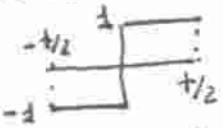
$$= -\int_0^4 (x-2)\frac{\operatorname{sen} n\pi/2\, x}{n\pi/2}\,dx \underset{\text{PARTES}}{=} \left.\frac{(x-2)\cos n\pi/2\, x}{n^2\pi^2/4}\right|_0^4 - \underbrace{\int_0^4 \frac{\cos n\pi/2\, x}{n^2\pi^2/4}}_{=0}$$

$$= \frac{8}{n^2\pi^2} + \frac{8}{n^2\pi^2} = \frac{16}{n^2\pi^2}$$

ASÍ $(x-2)^2 = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{16}{n^2\pi^2}\cos n\frac{\pi}{2}x$ $\forall x \in [0,4]$

POR SER f DERIVABLE EN $(0,4)$ Y POR EXISTENCIAS DERIVADAS LATERALES EN $x=0$ Y $x=4$

PROBLEMA 12: a)



LA EXTENSIÓN DE f π-PERIÓDICA ES IMPAR ASÍ $a_n = 0$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\operatorname{sen}\left(n\frac{2\pi}{\pi}x\right)dx \underset{\text{PROBLEMA 8}}{=} \frac{2}{\pi}\int_0^{\pi} f(x)\operatorname{sen}\left(n\frac{2\pi}{\pi}x\right)dx = \frac{2}{\pi}\int_0^{\pi/2}\operatorname{sen} n\frac{2\pi}{\pi}x -$$

$$-\frac{2}{\pi}\int_{\pi/2}^{\pi}\operatorname{sen} n\frac{2\pi}{\pi}x = \frac{2}{\pi}\left[-\cos\left(\frac{n2\pi x}{\pi}\right)\cdot\left.\frac{\pi}{2n\pi}\right|_0^{\pi/2} + \cos\left(\frac{n2\pi x}{\pi}\right)\left.\frac{\pi}{2n\pi}\right|_{\pi/2}^{\pi}\right] = \ldots\ldots$ de

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PROBLEMA 14:

5) $\int_{-\pi}^{\pi} e^{int}\,dt = \frac{e^{int}}{in}\Big|_{-\pi}^{\pi} = \frac{1}{in}\left[e^{in\pi} - e^{-in\pi}\right] =$

$= \frac{2i \operatorname{sen} n\pi}{ni} = \frac{2 \operatorname{sen} n\pi}{n} = 0$

$\downarrow$ b)

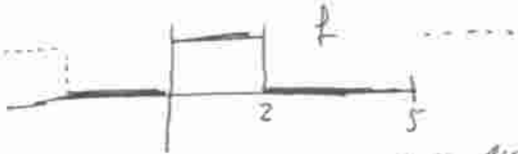
YA QUE $\int_{-\pi}^{\pi} e^{int} = \int_{-\pi}^{\pi} \cos nt + i \operatorname{sen} nt\,dt =$

$= \int_{-\pi}^{\pi} \cos nt + i \int_{-\pi}^{\pi} \operatorname{sen} nt\,dt =$

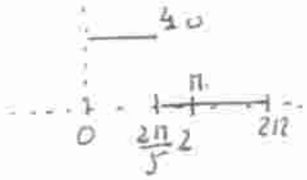
$= \frac{\operatorname{sen} nt}{n}\Big|_{-\pi}^{\pi} + i\left(-\frac{\cos nt}{n}\Big|_{-\pi}^{\pi}\right) = 0.$

PROBLEMA 16: $v(t) = \begin{cases} 40 & \text{si } 0<t<2 \\ 0 & \text{si } 2<t<5 \end{cases}$ y su extensión

5-periódica



SEA $g(t) = v\left(t\frac{5}{2\pi}\right)$ función 2π-periódica



$g(t) \sim \sum_{k=1}^{5} c_k e^{ikt}$

$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t) e^{-ikt}\,dt =$

$= \frac{1}{2\pi}\int_{0}^{2\pi} g(t) e^{-ikt}\,dt = \frac{1}{2\pi}\int_{0}^{\frac{4\pi}{5}} 40\, e^{ikt}\,dt = \frac{20}{\pi}\frac{e^{ikt}}{ik}\Big|_{0}^{\frac{4\pi}{5}} =$

$= \frac{20}{\pi}\left[\frac{1}{ik} - \frac{e^{ik\frac{4\pi}{5}}}{ik}\right] = \frac{20}{\pi i k}\left[1 - e^{ik\frac{4\pi}{5}}\right]$

ASÍ $v(t) = g\left(t\frac{2\pi}{5}\right) \simeq \sum_{k=1}^{5} \frac{20}{\pi i k}\left[1 - e^{ik\frac{4\pi}{5}}\right] e^{ikt\frac{2\pi}{5}}$

DONDE SI $k=1$, $c_1 = \frac{20}{\pi i}\left[1 - e^{i4\pi/5}\right]$, $c_2 = \frac{10}{\pi i}\left[1 - e^{i\frac{8\pi}{5}}\right]$, $c_3 = \frac{20}{3\pi i}\left[1 - e^{i\frac{12\pi}{5}}\right]$

$c_4 = \frac{5}{\pi i}\left[1 - e^{i\frac{16\pi}{5}}\right]$ $c_5 = \frac{4}{\pi i}\left[1 - e^{i4\pi}\right] = 0$