HOJA 5ª

PROBLEMA 1:] SI f ES PAR Y REAL

ENTONCES $f(x)\cos\lambda x$ ES PAR $\forall\lambda\in\mathbb{R}$

$f(x)\,\text{sen}\,\lambda x$ ES IMPAR $\forall\lambda\in\mathbb{R}$ [COMPROBAR)

ASÍ $\int_{-\infty}^{\infty} f(x)\,\text{sen}\,\lambda x = 0$ Y POR TANTO

$$\mathcal{F}\{f\}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx = \int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx \in\mathbb{R}$$

(YA QUE $\left|\int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx\right| \leq \int_{-\infty}^{\infty}|f(x)|\,dx = \|f\|_1 < \infty$.

SI f ES IMPAR, AHORA $f(x)\cos\lambda x$ ES IMPAR

Y $\mathcal{F}\{f\}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-ix\lambda}dx = -i\int_{-\infty}^{\infty} f(x)\,\text{sen}\,\lambda x\,dx \in i\mathbb{R}$.

PROBLEMA 2ª] a) $\widehat{\chi_{[-\delta,\delta]}}(\lambda) = \int_{-\infty}^{\infty}\chi_{[-\delta,\delta]}(x)e^{-i\lambda x}dx =$

$$\underset{\substack{\Downarrow\\ \chi_{[-\delta,\delta]}\text{ PAR}\\ \text{(VER EJERCICIO 1)}}}{=} \int_{-\delta}^{\delta}\cos\lambda x\,dx = \left.\frac{\text{sen}\,\lambda x}{\lambda}\right|_{-\delta}^{\delta} = 2\frac{\text{sen}\,\lambda\delta}{\lambda}$$

- $\lim_{\lambda\to 0}\frac{2\,\text{sen}\,\lambda\delta}{\lambda} = 2\delta$
- Y $f(x) = \frac{2\,\text{sen}\,\lambda\delta}{\lambda}$ ES PAR

c) $\hat{f}(\lambda) = \int_{-\infty}^{\infty}\cos(2\pi\alpha x)\,\chi_{[-\delta,\delta]}(x)\,e^{-i\lambda x}dx \underset{\substack{\downarrow\\ f\text{ PAR}}}{=} \int_{-\delta}^{\delta}\cos 2\pi\alpha x\cdot\cos\lambda x\,dx =$

$$\underset{\substack{\uparrow\\ \cos a\cos b = \frac{1}{2}[\cos(a+b)+\cos(a-b)]}}{=} \frac{1}{2}\int_{-\delta}^{\delta}\cos(2\pi\alpha x+\lambda x) + \cos(2\pi\alpha x-\lambda x)\,dx =$$

$$= \frac{1}{2}\left[\left.\frac{\text{sen}\,(2\pi\alpha+\lambda)x}{2\pi\alpha+\lambda}\right|_{-\delta}^{\delta} + \left.\frac{\text{sen}\,((2\pi\alpha-\lambda)x)}{2\pi\alpha-\lambda}\right|_{-\delta}^{\delta}\right] =$$

$$= \frac{(2\pi\alpha-\lambda)\,\text{sen}(2\pi\alpha+\lambda)\delta + (2\pi\alpha+\lambda)\,\text{sen}(2\pi\alpha-\lambda)\delta}{4\pi^2\alpha^2-\lambda^2}$$

HOJA 5ª

PROBLEMA 3º) $f_n(x) = \cos(2\pi\alpha x)\,\chi_{[-\frac{n}{\alpha},\, \frac{n}{\alpha}]}(x)$

$$F[f_n](\lambda) = \hat{f}_n(\lambda) \underset{\text{EJERCICIO 2}}{=} \frac{(2\pi\alpha-\lambda)\,\mathrm{sen}(2\pi\alpha+\lambda)\frac{n}{\alpha} + (2\pi\alpha+\lambda)\,\mathrm{sen}(2\pi\alpha-\lambda)\frac{n}{\alpha}}{4\pi^2\alpha^2-\lambda^2} =$$

$$= \frac{(2\pi\alpha-\lambda)\,\mathrm{sen}(\lambda\frac{n}{\alpha}) - (2\pi\alpha+\lambda)\,\mathrm{sen}(\lambda\frac{n}{\alpha})}{4\pi^2\alpha^2-\lambda^2} =$$

$\mathrm{sen}(a+b) = \cos a\,\mathrm{sen}\, b + \mathrm{sen}\, a \cos b$

$\mathrm{sen}\, 2n\pi = 0$

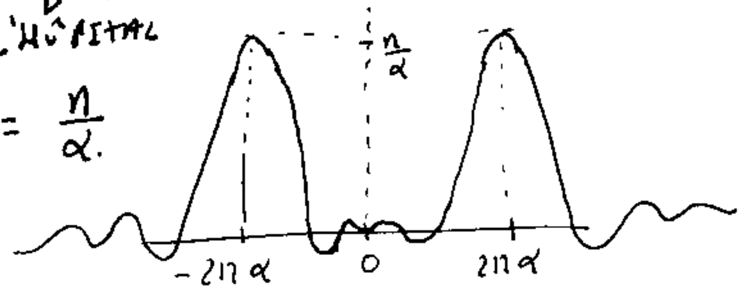
$\cos 2n\pi = 1$

$$= \frac{-2\lambda\,\mathrm{sen}(\lambda\frac{n}{\alpha})}{4\pi^2\alpha^2-\lambda^2} = \hat{f}_n(\lambda)$$ ESTA FUNCIÓN ES PAR

y $$\lim_{\lambda\to\pm2\pi\alpha} \frac{-2\lambda\,\mathrm{sen}(\lambda\frac{n}{\alpha})}{4\pi\alpha^2-\lambda^2} \underset{\text{L'HÔPITAL}}{=} \lim_{\lambda\to\pm2\pi\alpha} \frac{-2\,\mathrm{sen}(\lambda\frac{n}{\alpha}) - 2\lambda\frac{n}{\alpha}\cos(\lambda\frac{n}{\alpha})}{-2\lambda} =$$

$$= \frac{2\pi\alpha\frac{n}{\alpha}}{2\pi\alpha} = \frac{n}{\alpha}.$$

$$\lim_{\lambda\to\pm\infty} \hat{f}_n(\lambda) = 0$$



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Por otro lado $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \cos 2\pi\alpha x\, \chi_{[-\frac{n}{\alpha},\frac{n}{\alpha}]} = \cos 2\pi\alpha x$

Si $f(x) = \cos 2\pi\alpha x$, $\hat{f}(x) = \lim_{n\to\infty} \int_{-n/\alpha}^{\frac{n}{\alpha}} \cos 2\pi\alpha x\, e^{-ix\lambda}\,dx =$

$$= \lim_{n\to\infty} \frac{-2\lambda\,\mathrm{sen}(\lambda\frac{n}{\alpha})}{4\pi\alpha^2-\lambda^2} = \begin{cases} \infty & \text{si } \lambda = \pm 2\pi\alpha. \\ \text{y no existe si} & \lambda \neq \pm 2\pi\alpha. \end{cases}$$

HOJA 5ª

PROBLEMA 4:] SI $h(t) = A e^{-\alpha t} \chi_{[0,\infty)}(t)$

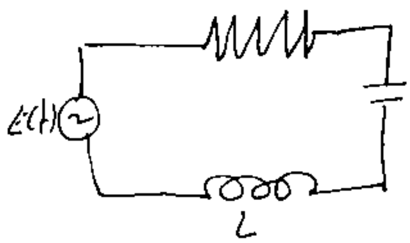
ENTONCES $\hat{h}(s) = \int_{-\infty}^{\infty} h(t)\, e^{-ist}\, dt = \int_0^{\infty} A e^{-\alpha t} e^{-ist}\, dt =$

$$= \int_0^{\infty} A\, e^{-t[\alpha + is]}\, dt = A \left. \frac{e^{-t[\alpha+is]}}{-[\alpha+is]} \right|_0^{\infty} =$$

$$= A\, \frac{1}{\alpha + is}.$$ (FILTRO DE BUTTERWORTH)

$\lim_{x\to\infty} e^{-x} = 0$

LA FUNCIÓN DE TRANSFERENCIA DE UN CIRCUITO RCL



ES $H(s) = \frac{1}{1 + RCis + (is)^2 LC}$

SI $L=0$ (NO HAY INDUCTANCIA; ES DECIR EN UN CIRCUITO RC)

$$= \frac{RC}{\frac{1}{RC} + is}$$

SI TOMAMOS $\frac{1}{RC}$ DE MODO QUE $\frac{1}{RC} = \alpha$

Y SI LA SEÑAL DE ENTRADA $E(t)$ LA AMPLIFICAMOS (ANTES DE PASAR POR EL FILTRO RC) αA (e.d. $A\alpha E(t)$),

ENTONCES $E(t) \longrightarrow A\alpha E(t)$ Y AL PASARLA POR EL FILTRO OBTENEMOS $\widehat{A\alpha E(t)} \cdot \frac{RC}{\frac{1}{RC} + is} = A\alpha\, \widehat{E(t)} \cdot \frac{\frac{1}{\alpha}}{\alpha + is} =$

$$= \widehat{E(t)}\, \frac{A}{\alpha + is}.$$

HOJA 5ª

PROBLEMA 5º]

a) $f(x) = \operatorname{sen} x \, \chi_{[-\pi,\pi]}(x)$ ES UNA FUNCIÓN IMPAR

ASÍ CON EL PROBLEMA 1º]

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} \operatorname{sen} x \, \chi_{[-\pi,\pi]} \, e^{-i\lambda x} dx = -i \int_{-\pi}^{\pi} \operatorname{sen} x \operatorname{sen} \lambda x \, dx =$$

$$= -i \int_{-\pi}^{\pi} \frac{1}{2} [\cos(x - \lambda x) - \cos(x + \lambda x)] =$$

$\operatorname{sen} a \operatorname{sen} b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$

$$= -i/2 \left[\frac{\operatorname{sen} x(1-\lambda)}{1-\lambda} \Big|_{-\pi}^{\pi} - \frac{\operatorname{sen} x(1+\lambda)}{1+\lambda} \Big|_{-\pi}^{\pi} \right] =$$

$$= -\cancel{i/2} \left[\frac{\cancel{2} \operatorname{sen} \pi(1-\lambda)}{1-\lambda} - \frac{\cancel{2} \operatorname{sen} \pi(1+\lambda)}{1+\lambda} \right] =$$

$\operatorname{sen}(a+b) = \cos a \operatorname{sen} b + \operatorname{sen} a \cos b$

$$= -i \left[\frac{(1+\lambda) \operatorname{sen} \lambda\pi + (1-\lambda) \operatorname{sen} \pi\lambda}{1-\lambda^2} \right] = -i \frac{2 \operatorname{sen} \lambda \pi}{1-\lambda^2}$$ (CONTINUA EN $\lambda = \pm 1$)

POR EL TEOREMA DE INVERSIÓN

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda =$$

$\hat{f}(\lambda)$ ES IMPAR

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} i \left(-i \frac{2 \operatorname{sen} \lambda \pi}{1-\lambda^2}\right) \operatorname{sen} \lambda x \, d\lambda = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{sen} \lambda \pi}{1-\lambda^2} \operatorname{sen} \lambda x \, d\lambda =$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\operatorname{sen} \lambda \pi}{1-\lambda^2} \operatorname{sen} \lambda x \, d\lambda.$$

$\frac{\operatorname{sen} \lambda \pi}{1-\lambda^2} \operatorname{sen} \lambda x$ PAR.

HOJA 5ª

PROBLEMA 6:

d) $\hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-is x} dx \underset{f(x) \in \mathbb{R}}{=} \int_{-\infty}^{\infty} \overline{f(x)} \, \overline{e^{-i(-s)x}} dx =$

$= \int_{-\infty}^{\infty} \overline{f(x) \, e^{-i(-s)x}} dx = \overline{\int_{-\infty}^{\infty} f(x) e^{-i(-s)x} dx} = \overline{\hat{f}(-s)}$

PROBLEMA 7:

d) $\int_{-\infty}^{\infty} f(t) \, \hat{g}(it) dt = \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} g(s) e^{-i(it)s} ds \, dt =$

$\underset{\text{T. FUBINNI}}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) g(s) \, e^{-i(is)t} dt \, ds =$

$= \int_{-\infty}^{\infty} g(s) \int_{-\infty}^{\infty} f(t) \, e^{-i(is)t} dt \, ds =$

$= \int_{-\infty}^{\infty} g(s) \, \hat{f}(is) \, ds$ c.q.d.

PROBLEMA 8:

a) $f * (g+h)(x) = \int_{-\infty}^{\infty} f(y) \, (g+h)(x-y) dy = \int_{-\infty}^{\infty} f(y) \, g(x-y) + f(y) \, h(x-y) dy$

$= \int_{-\infty}^{\infty} f(y) \, g(x-y) \, dy + \int_{-\infty}^{\infty} f(y) \, h(x-y) dy = f*g(x) + f*h(x)$

c) $\| f*g(x) - f*g_n(x) \|_1 \underset{a)}{=} \| f*(g-g_n)(x) \|_1 =$

$= \int_{-\infty}^{\infty} | f*(g-g_n)(x) | dx = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(y) \, (g-g_n)(x-y) dy \right| dx \leq$

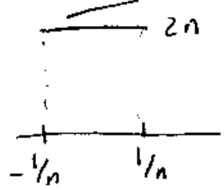
$\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y)| \, |g-g_n(x-y)| \, dy \, dx \underset{\text{T. FUBINNI}}{=}$

$= \int_{-\infty}^{\infty} |f(y)| \left(\int_{-\infty}^{\infty} |g-g_n(x-y)| dx \right) dy = \|f\|_1 \, \|g-g_n\|_1 \xrightarrow[n \to \infty]{} 0$

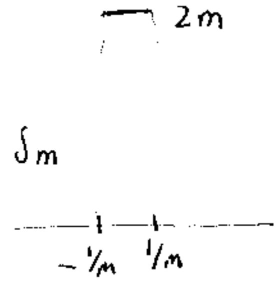
HOJA 5ª

PROBLEMA 9º)

δ_n



SI $m > n$ δ_m



$$\|\delta_n - \delta_m\|_1 = \int_{-\infty}^{\infty} |\delta_n(x) - \delta_m(x)|\,dx =$$

$$= 2\left|\frac{1}{n} - \frac{1}{m}\right|\cdot 2n + \frac{1}{2m}\left[2m - 2n\right] = 4\,\frac{m-n}{m} + 1 - \frac{n}{m}$$

SI $m = 2n$ $\quad \|\delta_n - \delta_{2n}\|_1 = 2n - 1/2 \xrightarrow[n\to\infty]{} \infty$; LUEGO $(\delta_n)_{n=1}^{\infty}$ NO ES UNA SUCESIÓN DE CAUCHY EN $L_1(\mathbb{R})$

$$\delta_n \longrightarrow \delta \quad \text{(DELTA DE DIRAC) PUNTUALMENTE}$$

$$y \quad h_n = f * \delta_n \xrightarrow[n\to\infty]{} f \quad \text{PUNTUALMENTE}$$

PROBLEMA 10º)

$$f_\delta(t) = \mathcal{F}^{-1}\left[\hat{f}(\lambda)\cdot \chi_{[-\delta,\delta]}(\lambda)\right] =$$

$$= \mathcal{F}^{-1}\left[\hat{f}(\lambda)\cdot \widehat{\mathcal{F}^{-1}\chi_{[-\delta,\delta]}}(\lambda)\right] = f * \mathcal{F}^{-1}\chi_{[-\delta,\delta]}(x)$$

COMO $\mathcal{F}^{-1}\chi_{[-\delta,\delta]} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi_{[-\delta,\delta]}(\lambda)\,e^{i\lambda x}\,d\lambda =$ $\quad \chi_{[-\delta,\delta]}$ PAR

$$= \frac{1}{2\pi}\int_{-\delta}^{\delta}\cos\lambda x\,d\lambda = \frac{1}{2\pi}\,\frac{\operatorname{sen}\lambda x}{x}\Big|_{-\delta}^{\delta} = \frac{1}{\pi}\,\frac{\operatorname{sen}\delta x}{x}$$

ASÍ $\quad f_\delta(t) = f(x) * \frac{1}{\pi}\,\frac{\operatorname{sen}\delta x}{x} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\,\frac{\operatorname{sen}\delta(t-x)}{t-x}\,dx.$

HOJA 5ª

PROBLEMA 11: Supongamos $0<a<b$

$y \quad h(t)=f(at) \quad y \quad g(t)=f(bt)$

SEA $y>0$ $\qquad h*g(y)=\int_{-\infty}^{\infty} h(t)\, g(y-t)\,dt=$

$$=\int_{-\infty}^{\infty} e^{-at}\chi_{[0,\infty)}(t)\; e^{-b(y-t)}\chi_{[0,\infty)}(y-t)\,dt$$

$$\chi_{[0,\infty)}(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$$

$$\chi_{[0,\infty)}(y-t)=\begin{cases}1 & t<y\\ 0 & t>y\end{cases}$$

$$=\int_0^y e^{-at}e^{bt}e^{-by}\,dt=e^{-by}\int_0^y e^{-t(a-b)}\,dt=$$

(*)
$$=e^{-by}\left[\frac{-e^{-t(a-b)}}{a-b}\right]_0^y=e^{-by}\left[\frac{e^{-y(a-b)}-1}{b-a}\right]=$$

$$=\frac{e^{-ya}-e^{-by}}{b-a}=\frac{f(ay)-f(by)}{b-a}$$

SI $a=b$ y $y>0$ en (*) tenemos

$$\int_0^y e^{-ay}\,dt=y\,e^{-ay}=y\,f(ay) \quad \text{c.q.d.}$$

HOJA 5ª

PROBLEMA 12:

LA SEÑAL f ES DE BANDA LIMITADA YA QUE

$$\operatorname{supp} \hat{f}(s) \subseteq [3\times10^2, 3\times10^4] \subseteq [-3\times10^4, 3\times10^4]$$

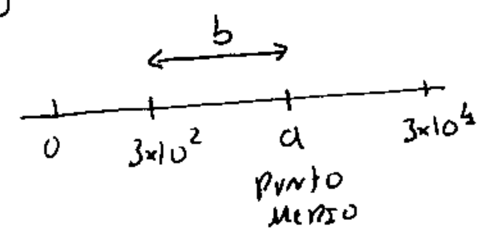
ASÍ LA FRECUENCIA DE NYQUIST ES $\nu = \frac{3\times10^4}{2\pi}$

Y LA TASA DE NYQUIST ES $\frac{3\times10^4}{\pi}$

SE PUEDE MUESTREAR A ESTA FRECUENCIA, PERO CLARAMENTE INCURRIREMOS EN SOBREMUESTREO, YA QUE EL INTERVALO $[3\times10^2, 3\times10^4]$ ES BASTANTE MAS PEQUEÑO QUE $[-3\times10^4, 3\times10^4]$

SEA $\hat{g}(s) = \hat{f}(s+a)$ ASÍ $\hat{g}(s)=0$ SI $s \in [-b, b]$

DONDE
$$\begin{cases} a = \dfrac{3\times10^2 + 3\times10^4}{2} \\ \text{y} \\ b = \dfrac{3\times10^4 - 3\times10^2}{2} \end{cases}$$



$$g(x) = \mathcal{F}^{-1}\{\hat{g}(s)\}(x) = \mathcal{F}^{-1}\{\hat{f}(s+a)\}(x) \underset{\text{EJERCICIO 6}}{=} e^{-ixa} f(x)$$

LUEGO CONOCER g ES COMO CONOCER f.

PARA g LA TASA DE NYQUIST (O DE MUESTREO)

ES PARA g $\quad \frac{b}{\pi} = \frac{3\times10^4 - 3\times10^2}{2\pi} \underset{\pi \geq 3}{\leq}$

$\leq \frac{10^2[10^2-1]}{2} = 50\times99$ FRECUENCIA DE MUESTREO

(ES DECIR, EL PERÍODO ENTRE MUESTRAS SERÁ $\sim \frac{1}{50\times99} \sim \frac{1}{5000}$.

POR OTRO LADO SI VAMOS A APLICAR FFT, TENEMOS QUE ENCONTRAR EL MENOR n_0 TAL QUE $50\times99 \leq 2^{n_0}$

COMO $2^{12} = 4096 < 4500 = 50\times99 < 2^{13}$

HOJA 5ª

PROBLEMA 13ª

$$k = 2m \quad y \quad w^k = 1$$

ASÍ $P(x) = \sum_{n=0}^{2m-1} a_n x^n = \sum_{j=0}^{m-1} a_{2j} x^{2j} + \sum_{j=0}^{m-1} a_{2j+1} x^{2j+1} =$

$$= \sum_{j=0}^{m-1} a_{2j} (x^2)^j + x \sum_{j=0}^{m-1} a_{2j+1} (x^2)^j$$

SEA $P_1(x) = \sum_{j=0}^{m-1} a_{2j} x^j$ y $P_2(x) = \sum_{j=0}^{m-1} a_{2j+1} x^j$

EL GRADO DE P_1 y P_2 ES $m - 1 = \frac{k}{2} - 1$

POR OTRO LADO $P(w) = \sum_{j=0}^{m-1} a_{2j} (w^2)^j + w \sum_{j=0}^{m-1} a_{2j+1} (w^2)^j =$

$$= P_1(w) + w P_2(w)$$

ADEMÁS w^2 ES UNA RAÍZ $k/2$-ÉSIMA DE LA UNIDAD YA QUE

$$(w^2)^{k/2} = w^k = 1.$$

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