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a. $T(n) = 3T(n/3) + \sqrt{n}$

Master theorem:

$a=3, b=3 \rightarrow \log_b^a = 1, f(n) = \sqrt{n}$

Case 1: $\sqrt{n} = O(n^{1-\varepsilon})$ for some $\varepsilon > 0$ ✓ $0 < \varepsilon < 1/2$

$\rightarrow T(n) = \Theta(n^{\log_b^a}) \rightarrow T(n) = \Theta(n)$.

b. $T(n) = 3T(n/4) + n\log n$

Master theorem

$a=3, b=4, f(n) = n\log n, \log_b^a = \log_4^3 = 0.793$

case 1: $n\log n = O(n^{\log_4^3 - \varepsilon})$ for some $\varepsilon > 0$ X

case 2: $n\log n = \Theta(n^{\log_4^3})$ X

case 3: $n\log n = \Omega(n^{\log_4^3 + \varepsilon})$ with $\underset{\varepsilon \simeq 0.2}{\varepsilon > 0}$ and $3(\frac{n}{4}\log(\frac{n}{4})) < c.n\log n$

for some $c < 1$ $\frac{3n}{4}(\log n - \overset{2}{\cancel{\log 4}}) < c.n\log n$

$\rightarrow \frac{3n}{4}\log n - 3n/2 < c.n\log n$ for $3/4 \leq c < 1$ ✓

$\rightarrow$ case 3: ✓

$\rightarrow T(n) = \Theta(f(n)) \rightarrow T(n) = \Theta(n\log n)$

c. $T(n) = 4T(n/2) + n^2$

$a=4, b=2, f(n)=n^2, \log_b^a = 2$

case 1: $n^2 = O(n^{2-\varepsilon})$ for some $\varepsilon > 0$ X

case 2: $n^2 = \Theta(n^2)$ ✓

$\rightarrow T(n) = \Theta(n^{\log_b^a}.\log n) \rightarrow T(n) = \Theta(n^2\log n)$

e. $T(n) = 2T(n/2) + n/\log n$

$T(n) = 2(2T(n/4) + \frac{n/2}{\log n/2}) + n/\log n = 2^2 T(\frac{n}{2^2}) + \frac{n}{\log n - 1} + n/\log n$

$= 2(2(2T(n/8) + \frac{n/4}{\log n/4}) + \frac{n/2}{\log n/2}) + \frac{n}{\log n} = 2^3 T(\frac{n}{2^3}) + \frac{n}{\log n - 2} + \frac{n}{\log n - 1}$

$+ \frac{n}{\log n} \rightarrow T(n) = 2^k T(n/2^k) + \sum_{i=0}^{k-1} \frac{n}{\log n - i}$

$\frac{n}{2^k} = 1 \rightarrow n = 2^k \rightarrow k = \log_2 n$

$$T(n) = 2^{\log n} T(1) + \sum_{i=0}^{\log n - 1} \frac{n}{\log n - i} = n + n \cdot \sum_{i=0}^{\log n - 1} \frac{1}{\log n - i}$$

$$= n + n \cdot \sum_{i=1}^{\log n} \frac{1}{i} = O(n H_{\log n}) = O(n \log \log n)$$

f. $T(n) = T(\sqrt{n}) + 1$

$$T(n) = T(n^{1/2}) + 1 = T(n^{1/2^2}) + 2 = \cdots = T(n^{1/2^k}) + k$$

$$n^{1/2^k} = 0 \times \quad n^{1/2^k} = 1 \to 1/2^k = 0 \times$$

$$n^{1/2^k} = 2 \to 2^{2^k} = n \to k = \log \log n \checkmark$$

$$\to T(n) = O(\log \log n).$$

d. $T(n) = 4T(n/2) + n^2 \log^2 n$

$$T(n) = 4\left(4T(n/4) + \frac{n^2}{4} \log^2 n/2\right) + n^2 \log^2 n$$

$$= 4\left(4(4T(n/8) + \frac{n^2}{16} \log^2 n/4) + \frac{n^2}{4} \log^2 n/2\right) + n^2 \log^2 n$$

$$\to T(n) = 4^3 T(n/2^3) + n^2 \log^2 n/2^2 + n^2 \log^2 n/2 + n^2 \log^2 n$$

$$\to k: T(n) = 2^{2k} T(n/2^k) + n^2 \log^2 \frac{n}{2^{k-1}} + n^2 \log^2 \frac{n}{2^{k-2}} + \cdots + n^2 \log^2 \frac{n}{2^{k-k}}$$

$$n/2^k = 1 \to 2^k = n \to k = \log n$$

$$\to T(n) = 2^{2\log n} T(1) + n^2 \sum_{i=1}^{\log n} \log^2 \frac{n}{2^{(\log n) - i}}$$

$$\log^2 \frac{n}{2^{(\log n) - i}} = (\log n - \log_2 2^{\log n - i})^2 = (\log n - \log n + i)^2 = i^2$$

$$\to T(n) = 2^{2\log n} \overset{1}{\cancel{T(1)}} + n^2 \underbrace{\sum_{i=1}^{\log n} i^2}_{\Theta(\log^3 n)} \to T(n) = \Theta(n^2 \log^2 n)$$