LA homework Dec.22 name: 王柯皓 id: 2021533025

§ 6.4 (Page 672)

In Exercises 2–4, find the least squares solution of the linear equation $A\mathbf{x}=\mathbf{b}$.

2. (a) $A=\begin{bmatrix}1 & -1\\ 2 & 3\\ 4 & 5\end{bmatrix}$; $\mathbf{b}=\begin{bmatrix}2\\ -1\\ 5\end{bmatrix}$

$$A^T=\begin{bmatrix}1 & 2 & 4\\ -1 & 3 & 5\end{bmatrix}$$

$$A^TAx=A^Tb$$

$$\begin{bmatrix}1 & 2 & 4\\ -1 & 3 & 5\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 3\\ 4 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}1 & 2 & 4\\ -1 & 3 & 5\end{bmatrix}\begin{bmatrix}2\\ -1\\ 5\end{bmatrix}$$

$$\begin{bmatrix}21 & 25\\ 25 & 35\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}20\\ 20\end{bmatrix}\qquad \begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}\frac{20}{11}\\ -\frac{8}{11}\end{bmatrix}$$

8. Find the orthogonal projection of $\mathbf{u}$ on the subspace of R^3 spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(a) $\mathbf{u}=(2,1,3)$; $\mathbf{v}_1=(1,1,0)$, $\mathbf{v}_2=(1,2,1)$

(b) $\mathbf{u}=(1,-6,1)$; $\mathbf{v}_1=(-1,2,1)$, $\mathbf{v}_2=(2,2,4)$

Take $A=\begin{bmatrix}-1 & 2\\ 2 & 2\\ 1 & 4\end{bmatrix}$ $A^T=\begin{bmatrix}-1 & 2 & 1\\ 2 & 2 & 4\end{bmatrix}$

$$\begin{bmatrix}-1 & 2 & 1\\ 2 & 2 & 4\end{bmatrix}\begin{bmatrix}-1 & 2\\ 2 & 2\\ 1 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}-1 & 2 & 1\\ 2 & 2 & 4\end{bmatrix}\begin{bmatrix}1\\ -6\\ 1\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}-\frac{7}{3}\\ \frac{1}{3}\end{bmatrix}$$

$$proj_W(u)=A\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 2\\ 2 & 2\\ 1 & 4\end{bmatrix}\begin{bmatrix}-\frac{7}{3}\\ -\frac{1}{3}\end{bmatrix}=\begin{bmatrix}\frac{5}{3}\\ -\frac{16}{3}\\ -\frac{11}{3}\end{bmatrix}$$

18. Prove: If A has linearly independent column vectors, and if $A\mathbf{x}=\mathbf{b}$ is consistent, then the least squares solution of $A\mathbf{x}=\mathbf{b}$ and the exact solution of $A\mathbf{x}=\mathbf{b}$ are the same.

since A has linearly independent column vectors, so A^TA is invertible

$Ax=b \Rightarrow A^{-1}Ax=A^{-1}b \Rightarrow x=A^{-1}b$ which is the exact solution of $Ax=b$

$A^TAx=A^Tb \Rightarrow (A^TA)^{-1}A^TAx=(A^TA)^{-1}A^Tb \Rightarrow x=A^{-1}(A^T)^{-1}A^Tb=A^{-1}b$ which is the least squares solution of $Ax=b$

so the solutions are the same.

20. Let $P: R^m \to W$ be the orthogonal projection of R^m onto a subspace W.

(a) Prove that $[P]^2=[P]$.

(b) What does the result in part (a) imply about the composition $P \circ P$?

(c) Show that $[P]$ is symmetric.

(a) $[P]=A(A^TA)^{-1}A^T=(AA^{-1})((A^T)^{-1}A^T)=I$ $\quad [P]^2=I^2=I$

so $[P]^2=[P]$

(b)

(c) since $[P]=I$ so $[P]$ is symmetric

§ 7.1 (Page 708)

1. (a) Show that the matrix

$$A=\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

is orthogonal in three ways: by calculating A^TA, by using part (*b*) of Theorem 7.1.1, and by using part (*c*) of Theorem 7.1.1.

(b) Find the inverse of the matrix A in part (a).

1(a) $A^TA=\begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $A^{-1}=A^T=\begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}$

2. (a) Show that the matrix

$$A=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

is orthogonal.

(b) Let $T: R^3 \to R^3$ be multiplication by the matrix A in part (a). Find $T(\mathbf{x})$ for the vector $\mathbf{x}=(-2, 3, 5)$. Using the Euclidean inner product on R^3, verify that $\|T(\mathbf{x})\|=\|\mathbf{x}\|$.

(a) $A^TA=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $T(x)=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}=\begin{bmatrix} \frac{14}{3} \\ -\frac{5}{3} \\ \frac{11}{3} \end{bmatrix}$

$\|T(x)\|=\sqrt{(\frac{14}{3})^2+(-\frac{5}{3})^2+(\frac{11}{3})^2}=\sqrt{38}$

$\|x\|=\sqrt{(-2)^2+3^2+5^2}=\sqrt{38}$

So $\|T(x)\|=\|x\|$

13. What conditions must a and b satisfy for the matrix

$$\begin{bmatrix} a+b & b-a \\ a-b & b+a \end{bmatrix}$$

to be orthogonal?

$$\begin{bmatrix} a+b & a-b \\ b-a & a+b \end{bmatrix}\begin{bmatrix} a+b & b-a \\ a-b & a+b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(a+b)^2+(a-b)^2=1$$

$$a^2+b^2=\frac{1}{2}$$

14. Prove that a 2×2 orthogonal matrix A has only one of two possible forms:

$$A=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad A=\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

where $0\le\theta<2\pi$. [*Hint:* Start with a general 2×2 matrix $A=(a_{ij})$, and use the fact that the column vectors form an orthonormal set in R^2.]

the column vectors form an orthonormal set.

$$\|c_1\|^2=\cos^2\theta+\sin^2\theta=1$$

$$\|c_2\|^2=(-\sin\theta)^2+\cos^2\theta=1$$

$$\|c_3\|^2=\cos^2\theta+\sin^2\theta=1$$

$$\|c_4\|^2=\sin^2\theta+(-\cos\theta)^2=1$$

17. Find a, b, and c for which the matrix

$$\begin{bmatrix} a & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ b & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ c & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

is orthogonal. Are the values of a, b, and c unique? Explain.

the row vectors form an orthonormal set

$\|r_1\|^2=a^2+\frac{1}{2}+\frac{1}{2}=1$ $\qquad a=0$

$\|r_2\|^2=b^2+\frac{1}{6}+\frac{1}{6}=1$ $\qquad b=\pm\frac{2}{\sqrt{6}}$

$\|r_3\|^2=c^2+\frac{1}{3}+\frac{1}{3}=1$ $\qquad c=\pm\frac{1}{\sqrt{3}}$

$a=0 \quad b=\frac{2}{\sqrt{6}} \quad c=-\frac{1}{\sqrt{3}}$

or $a=0 \quad b=-\frac{2}{\sqrt{6}} \quad c=\frac{1}{\sqrt{3}}$