1. (20 points) Let A and B be any sets. Show that if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$.

(**Remark:** $\mathcal{P}(A)$ is the power set of A, i.e., the set of all subsets of A)

Since we know $A \subseteq A$, we know $A \in P(A)$

Since $P(A) = P(B)$. we know that $A \in P(B)$.

Therefore. $A \subseteq B$. (1)

Reason in exactly the same way we can deduce that $B \subseteq A$ (2)

Now combining (1) and (2) yields the result

$A = B$

2. (20 points) Construct a bijection from $A = (0, 1) \cup [2, 3) \cup (4, 5]$ to $B = (6, 7) \cup [8, +\infty)$.

$A \to B: x \mapsto x+6$

Reasons: (1) for $x \in (0,1)$ in A

$x+6 \in (6,7)$ in B

(2) for $x \in [2,3) \cup (4,5]$

$x+6 \in [8, +\infty)$ in B

So $x \mapsto x+6$ is a bijection from A to B

3. (20 points) Prove or disprove $|\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}| = |\mathbb{R}|$.

Considering the functions $g(x,y) = f(x,y) - f(-a,-b)$

and $h(\theta) = (\cos\theta, \sin\theta)$.

the latter maps any angle to a point on unit circle

Consider $g \circ h$. This is a continuous function

from $[0, 2\pi]$ to R.

By definition of g, it follows that $g \circ h(0) = -g \circ h(2\pi)$

By the Intermediate Value Theorem, there is a point

$\theta \in [0, 2\pi]$ where $g \circ h(\theta) = 0$

At that point, by definition of g and h,

we have $f(h(\theta)) = f(-h(\theta))$. Since h is nonzero

on $[0, 2\pi]$. this proves that $\{(x,y) \in R^2 : x^2+y^2=1\} \to R$

is not injective. so it is not bijective as well

So $|\{(x,y) \in R^2 : x^2+y^2=1\}| \neq |R|$

4. (20 points) Prove or disprove $|\{(a_1, a_2, a_3, \ldots) : a_i \in \{1, 2, 3\} \text{ for all } i = 1, 2, 3, \ldots\}| = |\mathbb{Z}^+|$.

Suppose $A = \{(a_1, a_2, a_3, \cdots) : a_i \in \{1, 2, 3\} \text{ for all } i = 1, 2, 3, \cdots\}$

if $i = 1 \in Z^+$ then $A = \{1\}$ or $\{2\}$ or $\{3\}$

if $i = 2 \in Z^+$ then $A = \{1\}$ or $\{2\}$ or $\{3\}$ or $\{1, 2\}$ or $\{1, 3\}$ or $\{2, 3\}$

if $i \geq 3 \in Z^+$ since $a_i \in \{1, 2, 3\}$ for all $i = 1, 2, 3, \cdots$

we know its elements can be arranged as a sequence

which is composed by $\{1, 2, 3\}$

So A is countably infinite

thus $|A| = |Z^+|$

namely $|\{(a_1, a_2, a_3, \cdots) : a_i \in \{1, 2, 3\} \text{ for all } i = 1, 2, 3, \cdots\}| = |Z^+|$

5. (20 points) Find a countably infinite number of subsets of $\mathbb{Z}^+$, say $A_1, A_2, \ldots \subseteq \mathbb{Z}^+$ such that the following requirements are simultaneously satisfied:

- $|A_i| = |\mathbb{Z}^+|$ for all $i = 1, 2, \ldots$;
- $A_i \cap A_j = \emptyset$ for all $i \neq j$;
- $\cup_{i=1}^{\infty} A_i = \mathbb{Z}^+$.

$A_1 = \{x_{11}, x_{12}, x_{13}, \cdots, x_{1n}, \cdots\}$

$A_2 = \{x_{21}, x_{22}, x_{23}, \cdots, x_{2n}, \cdots\}$

$\vdots$

$A_m = \{x_{m1}, x_{m2}, x_{m3}, \cdots, x_{mn}, \cdots\}$

$\vdots$

$S = \{2^k \cdot 3^n, n, k \in N\}$

let $f: S \to \bigcup_{i=1}^{\infty} A_i$ defined by $f(2^k \cdot 3^n) = x_{kn}$