Time and frequency characterization of signals and systems (ch.6)

- The magnitude-phase representation of Fourier Transform
- The magnitude-phase representation of the frequency response of LTI systems
- Time-domain properties of ideal frequency-selective filters
- Time-domain and frequency-domain aspects of non-ideal filters
- First- and second-order system

The magnitude-phase representation of FT

Magnitude and phase spectrum

- Continuous FT $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$ $X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$

- Discrete FT $x[n] \longleftrightarrow X\left(e^{j\omega}\right)$ $X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$

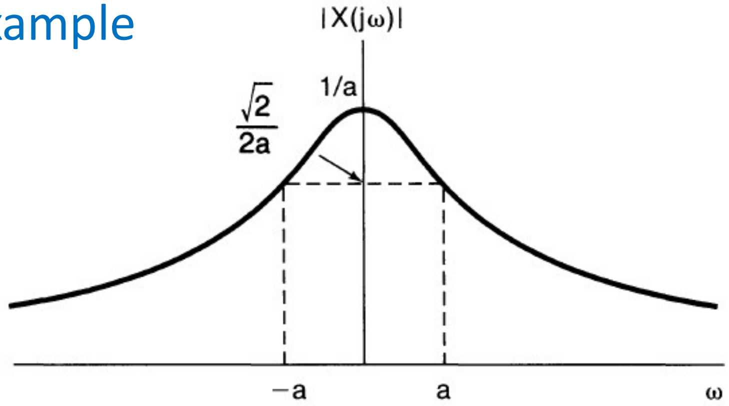
- Amplitude spectrum: $|X(j\omega)|$ and $|X(e^{j\omega})|$

- Phase spectrum (angle): $\angle X(j\omega)$and $\angle X(e^{j\omega})$

The magnitude-phase representation of FT

Magnitude spectrum

Continuous time as an example



IFT: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$

- IFT: decomposition of the signal $x(t)$ into a "sum" of complex exponentials at different frequencies
- $|X(e^{j\omega})|$: describes the basic frequency content of a signal, and the relative magnitude of the each frequency (complex exponential)
- $|X(j\omega)|^2$: energy-density spectrum of $x(t)$
- $|X(j\omega)|^2 d\omega/2\pi$: energy in the signal between ω and $\omega + d\omega$

The magnitude-phase representation of FT

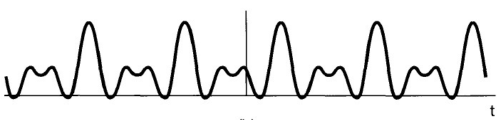
Phase spectrum

- ❑ $\angle X(j\omega)$
  - relative phase of the each complex exponential
  - significant effect on the nature of the signal
  - changes in $\angle X(j\omega)$ lead to phase distortion

❑ Example 1: $x(t) = 1 + \frac{1}{2}\cos(2\pi t + \varphi_1) + \frac{1}{2}\cos(4\pi t + \varphi_2) + \frac{1}{2}\cos(6\pi t + \varphi_3)$

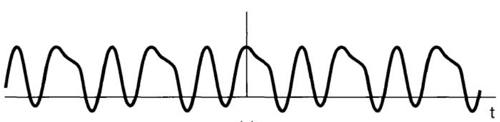
(a)

$$\varphi_1 = \varphi_2 = \varphi_3 = 0$$



(b)

$$\varphi_1 = 4rad, \varphi_2 = 8rad, \varphi_3 = 12rad$$



(c)

$$\varphi_1 = 6rad, \varphi_2 = -2.7rad, \varphi_3 = 0.93rad$$

(d)

$$\varphi_1 = 1.2rad, \varphi_2 = 4.1rad, \varphi_3 = -7.02rad$$

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The magnitude-phase representation of LTI

Gain and phase shift

- For LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

- The frequency response $H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$

- $|H(j\omega)|$: Gain of the LTI system; $\angle H(j\omega)$: phase shift of the LTI system

$$Y(j\omega) = H(j\omega)X(j\omega) = |H(j\omega)||X(j\omega)|e^{j(\angle H(j\omega)+\angle X(j\omega))}$$

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)| \qquad \angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

The magnitude-phase representation of LTI

Linear phase system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

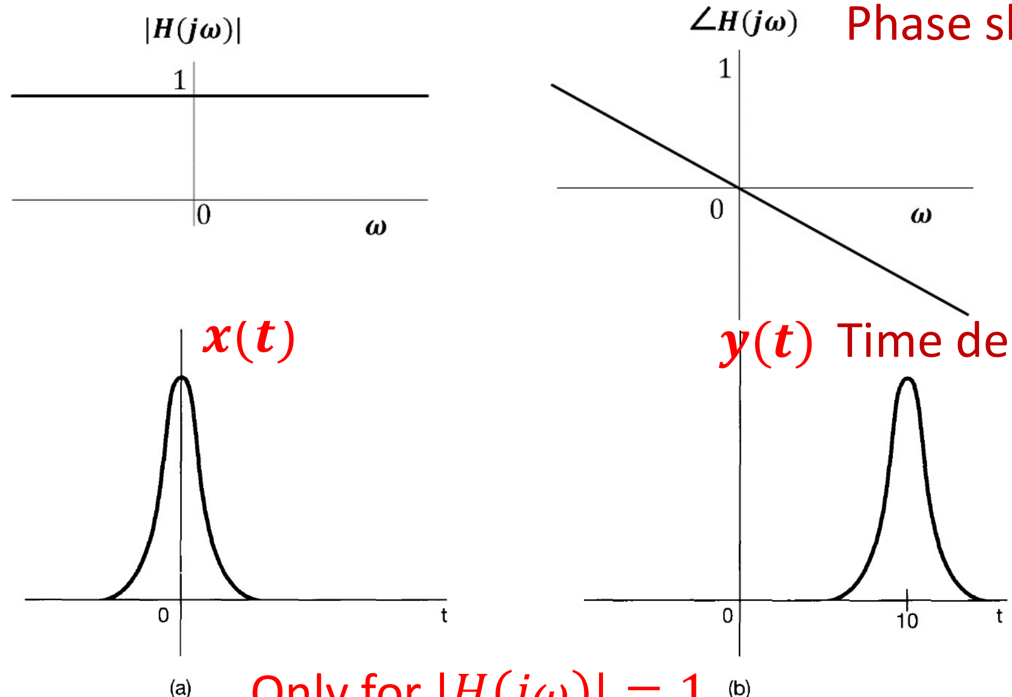
For $H(j\omega) = e^{-j\omega t_0}$

$|H(j\omega)| = 1$ $\boxed{\angle H(j\omega) = -\omega t_0}$

$\angle H(j\omega)$ is a linear function of ω

All-pass system

Phase shit



Time delay

Only for $|H(j\omega)| = 1$

Output of system:

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= X(j\omega)e^{-j\omega t_0}$$

$$\mathrm{y(t)} = x(t - t_0)$$

The magnitude-phase representation of LTI

Non-linear phase system

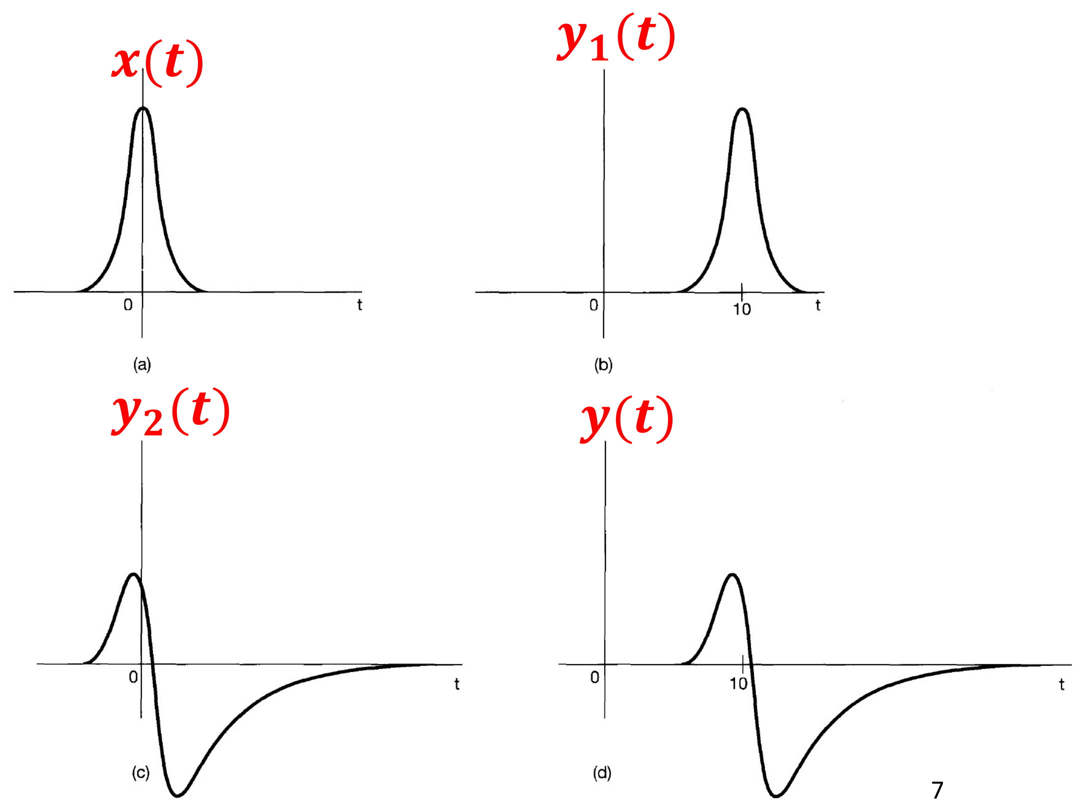
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

For $\boldsymbol{H(j\omega) = H_1(j\omega)H_2(j\omega)}$

$$H_1(j\omega) = e^{-j\omega t_0}$$

$$H_2(j\omega) = e^{\angle H_2(j\omega)}$$

$\angle H_2(j\omega)$ is a nonlinear function of ω



$$|H(j\omega)| = 1$$

$$\angle H(j\omega) = -\omega t_0 + \angle H_2\,(j\omega)$$

The magnitude-phase representation of LTI

Group delay

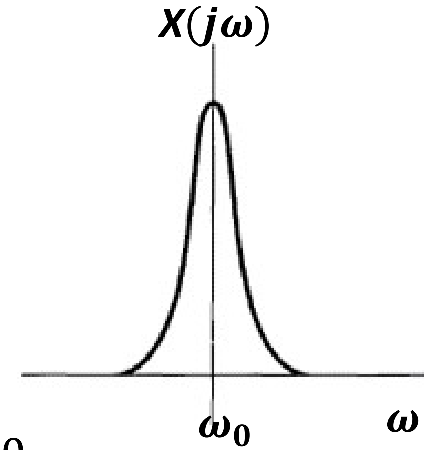
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- Consider a system with $\angle H(j\omega)$ a nonlinear function of ω

- For a narrow band input $x(t)$, $\angle H\,(j\omega) \simeq -\phi - \alpha\omega$

$$Y(j\omega) \simeq X(j\omega)|H(j\omega)|e^{-j\phi}e^{-j\alpha\omega}$$

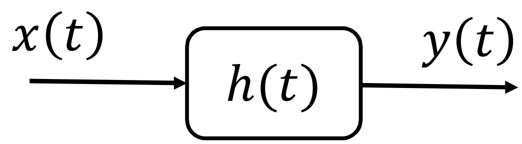
- The time delay α is referred to as the group delay at $\omega = \omega_0$



$$\tau(\omega) = -\frac{d}{d\omega}\{\angle H\,(j\omega)\}$$

The magnitude-phase representation of LTI

Group delay: example



- Consider

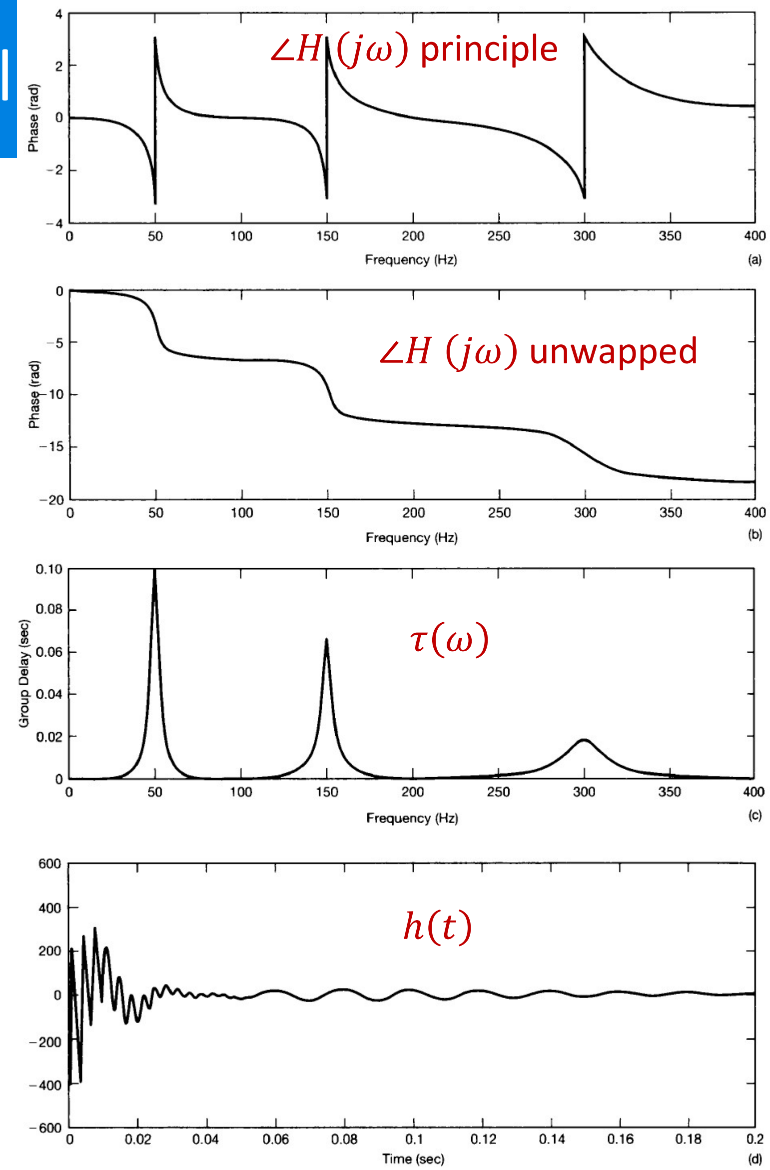
$$H(j\omega) = \prod_{i=1}^{3} H_i(j\omega) \qquad H_i(j\omega) = \frac{1 + (j\omega/\omega_i)^2 - 2j\zeta_i(\omega/\omega_i)}{1 + (j\omega/\omega_i)^2 + 2j\zeta_i(\omega/\omega_i)}$$

$$\begin{cases} \omega_1 = 315 \text{ rad/sec and } \zeta_1 = 0.066, \\ \omega_2 = 943 \text{ rad/sec and } \zeta_2 = 0.033, \\ \omega_3 = 1888 \text{ rad/sec and } \zeta_3 = 0.058. \end{cases}$$

$$|H_i(j\omega)| = 1 \Rightarrow |H(j\omega)| = 1$$

$$\angle H_i\,(j\omega) = -2\arctan\left[\frac{2\zeta_i(\omega/\omega_i)}{1 - (\omega/\omega_i)^2}\right]$$

$$\angle H(j\omega) = \sum_{i=1}^{3} \angle H_i\,(j\omega) \qquad \tau(\omega) = -\frac{d}{d\omega}\{\angle H\,(j\omega)\}$$



The magnitude-phase representation of LTI

Log-Magnitude and Bode Plots

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Time domain: $y(t) = x(t) * h(t)$ Convolution

Frequency domain: $Y(j\omega) = H(j\omega)X(j\omega)$ Multiplication

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$

$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Logarithmic amplitude: $\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$ Summation

Logarithmic amplitude scale: $20\log_{10}$, referred to as *decibels* (dB).

Bode plots: Plots of $20\log_{10}|H(j\omega)|$ and $\angle H(j\omega)$ versus $\log_{10}(\omega)$

Log-Magnitude and Bode Plots

Magnitude: Plot of $20\log_{10}|H(j\omega)|$ vs. $\log_{10}(\omega)$

Phase: Plot of $\angle H(j\omega)$ vs. $\log_{10}(\omega)$

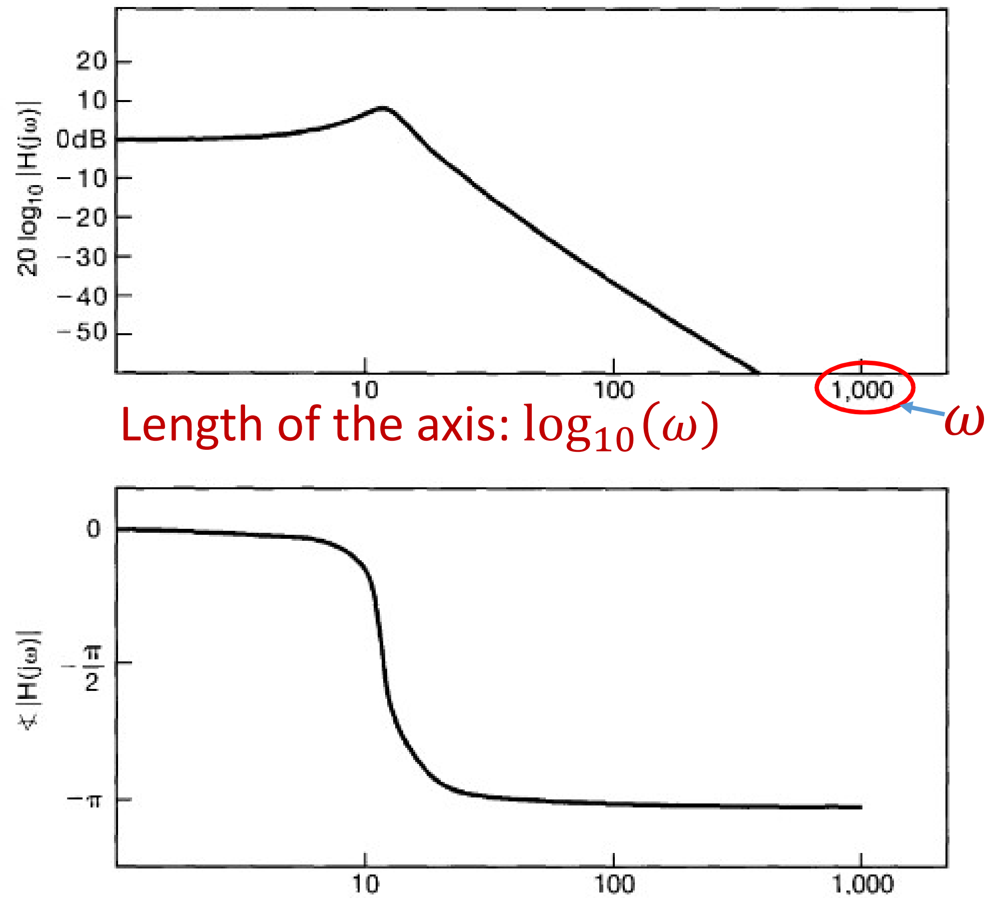


Figure 6.8 A typical Bode plot. (Note that ω is plotted using a logarithmic scale.)

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Time-domain properties of ideal frequency-selective filters

Frequency-selective filters

Low-pass filter

High-pass filter

Band-pass filter

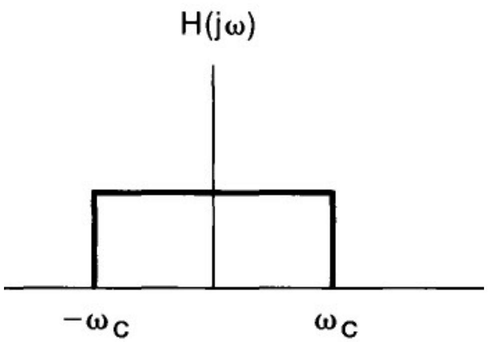
We focus on low-pass filter, similar concepts and results for high-pass and band-pass filters.

Time-domain properties of ideal frequency-selective filters

Ideal low-pass filters: zero phase

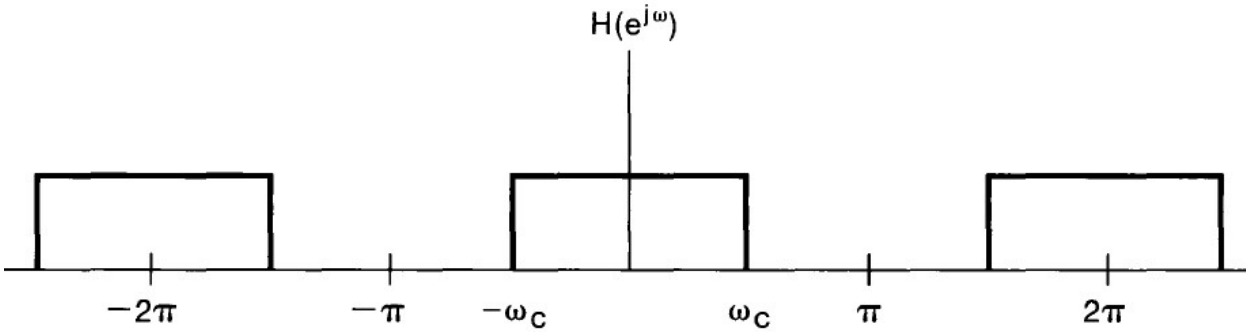
CT

$$\mathrm{H}(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



DT

$$\mathrm{H}\left(e^{j\omega}\right) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$



Time-domain properties of ideal frequency-selective filters

Ideal low-pass filters: zero phase

- Impulse response:

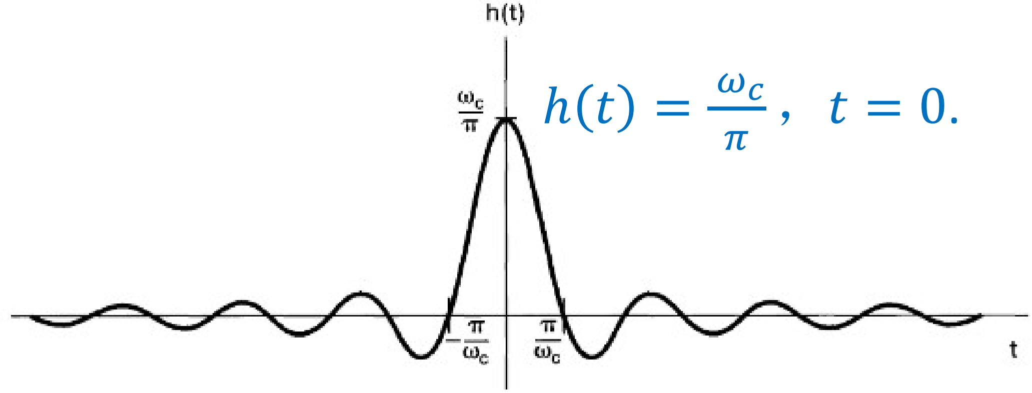
$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{H}(j\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\cdot\frac{1}{jt}e^{j\omega t}|_{-\omega_c}^{\omega_c}$$

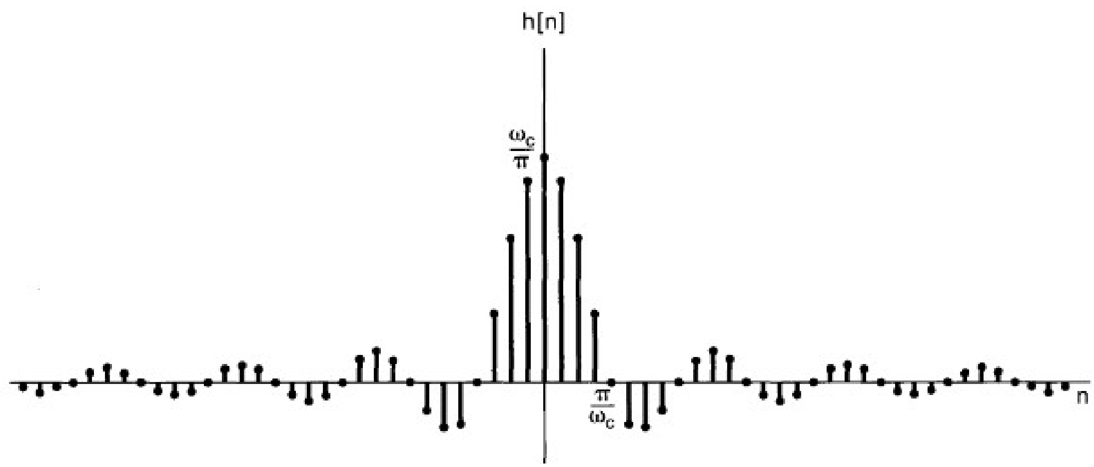
$$= \frac{1}{2\pi}\cdot\frac{1}{jt}\cdot 2j\sin(\omega_c t) = \frac{\sin\omega_c t}{\pi t}$$

$$h(n) = \frac{\sin\omega_c n}{\pi n}$$



$h(t) = \frac{\omega_c}{\pi}, \quad t = 0.$

$h(t) = 0, \ \omega_c t = k\pi, k \neq 0. \ \omega_c\uparrow$, width $of\ h(t) \downarrow$

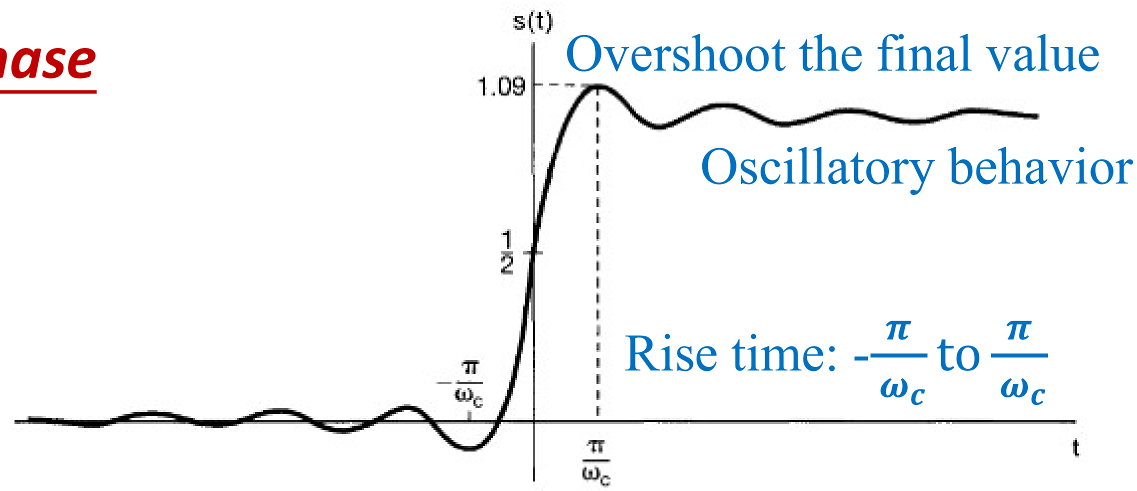


Time-domain properties of ideal frequency-selective filters

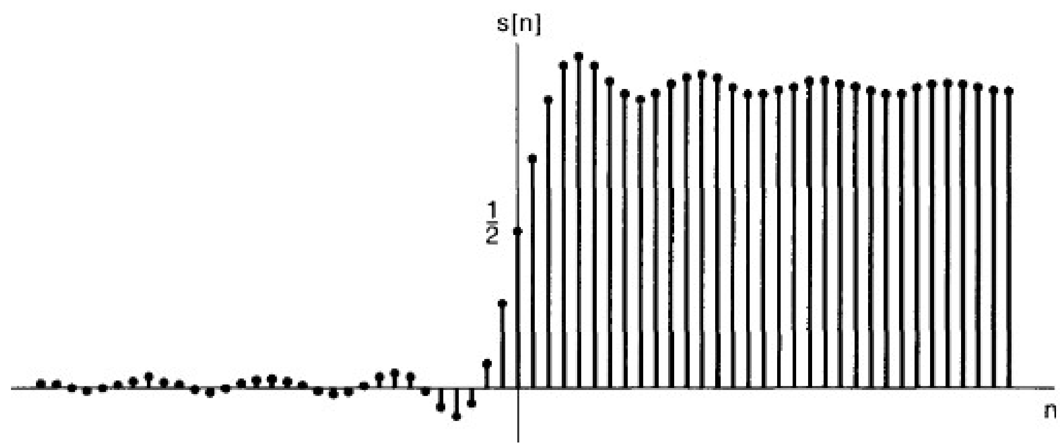
Ideal low-pass filters: zero phase

- Step response:

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$



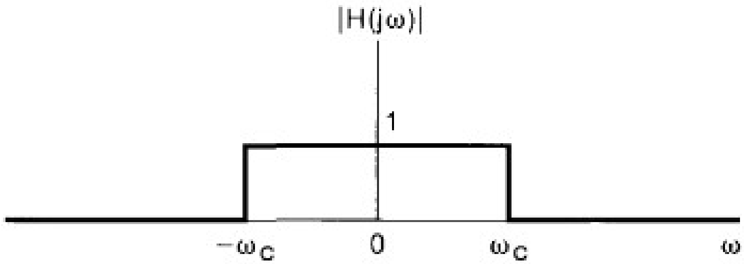
$$s(n) = \sum_{m=-\infty}^{n} h(m)$$



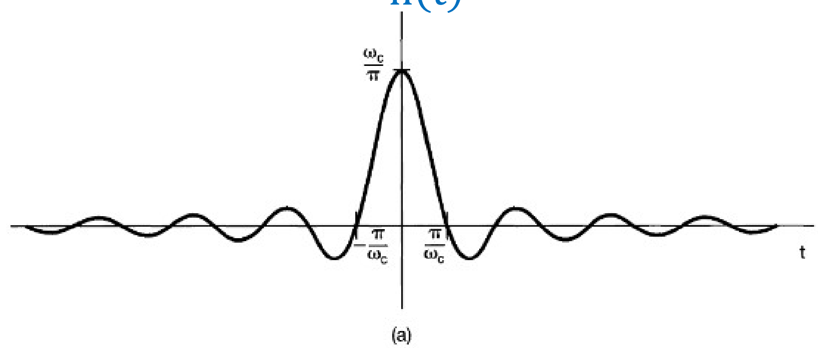
Time-domain properties of ideal frequency-selective filters

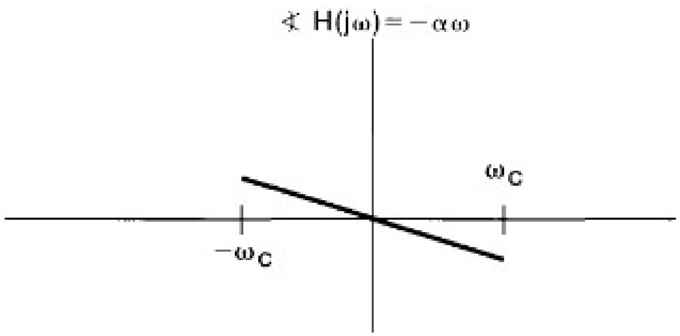
Ideal low-pass filters: linear phase

- Impulse response:

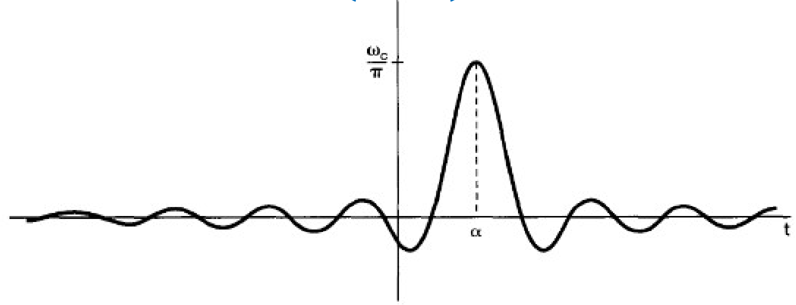


$h(t)$





$h(t-\alpha)$



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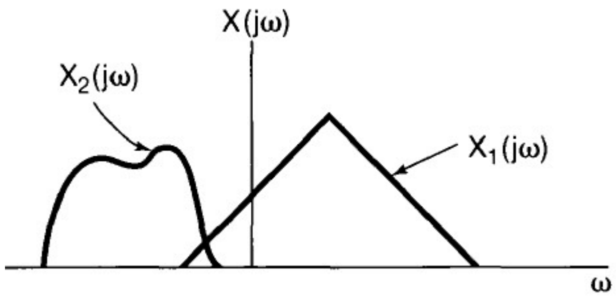
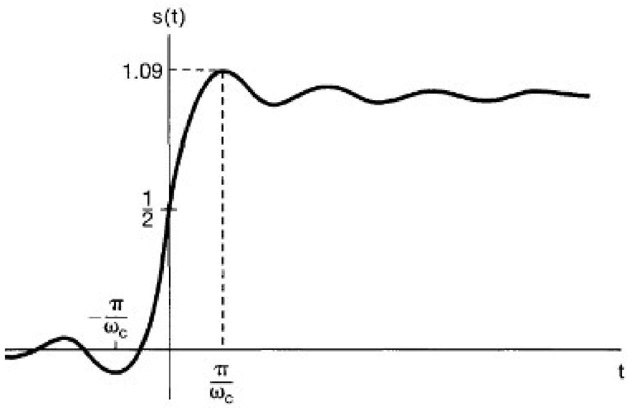
Non-ideal filters

Why non-ideal filters

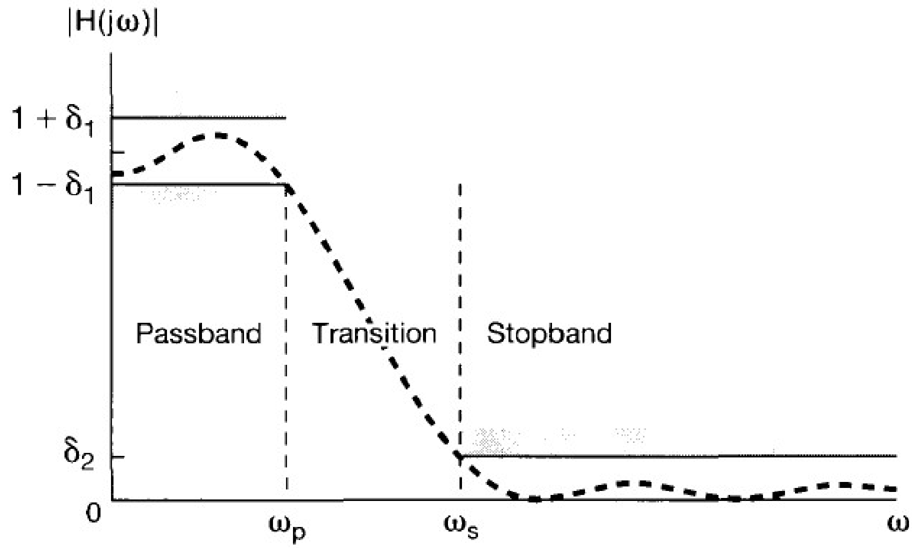
- Gradual transition band is sometimes preferable
- Idea Low-pass filter is not attainable (not causal)
- The more precisely frequency characteristics, the more complicated or costly the implementation
  - resistors, capacitors, and operational amplifiers in continuous time
  - memory registers, multipliers, and adders in discrete time



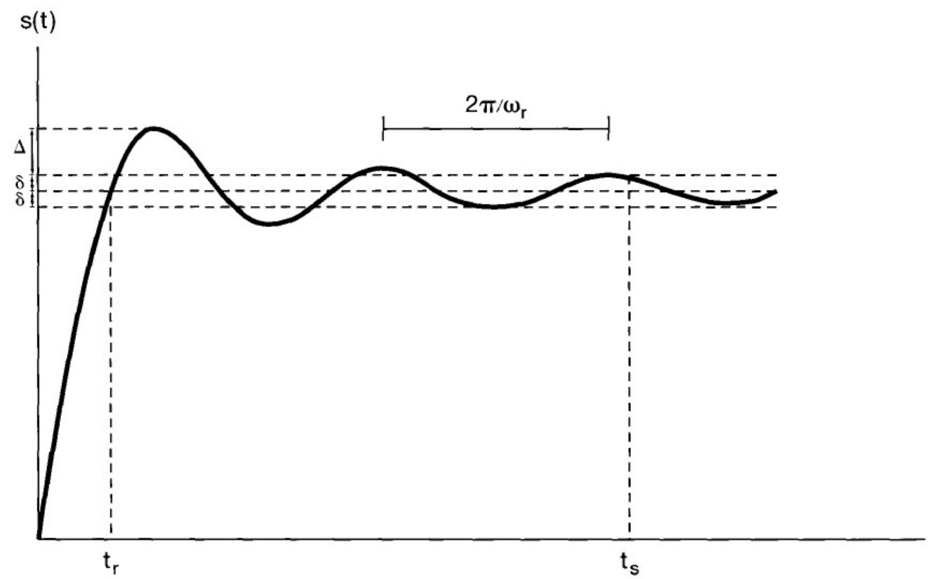


Non-ideal filters

Time and frequency domain



Step response of a CT low-pass filter

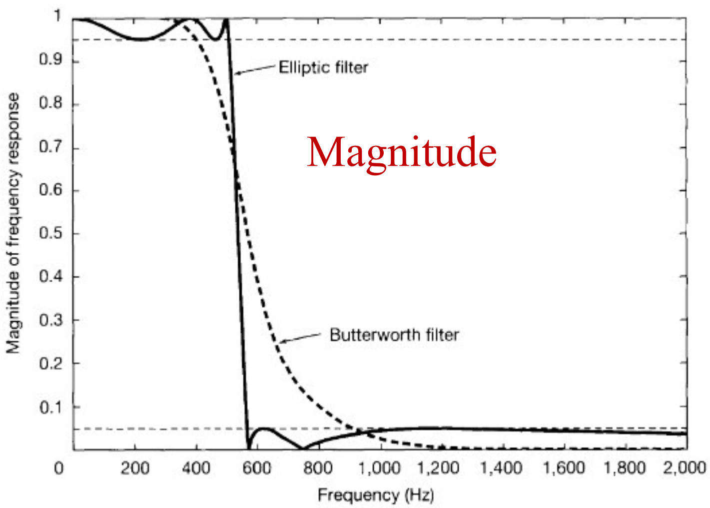


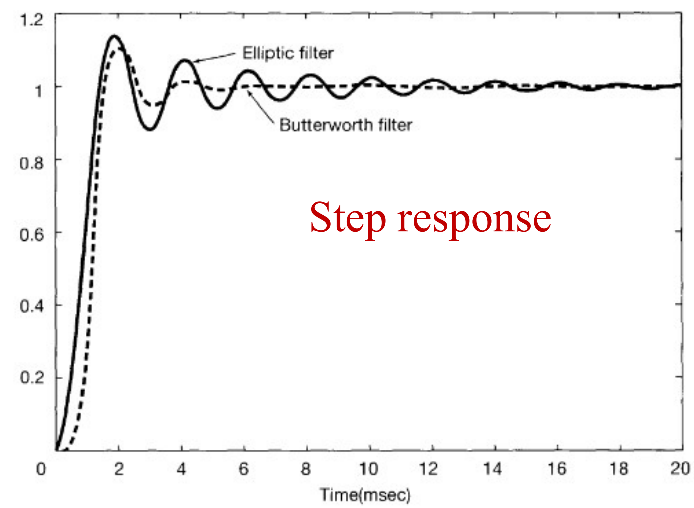
- Pass band 0 - ω_p, stop band ω>ω_s, transition ω_s - ω_p
- Pass-band ripple δ_1, stop-band ripple δ_2
- Linear (nearly) linear phase.

- Rise time: t_r
- Overshoot: Δ
- Ringing frequency: ω_r
- Settling time: t_s

Non-ideal filters

An example





- Fifth-order Butterworth filter and a fifth-order elliptic filter
- Same cutoff frequency
- Same passband and stopband ripple

Trade-off between time-domain (t_s) and frequency-domain (ω_s - ω_p).

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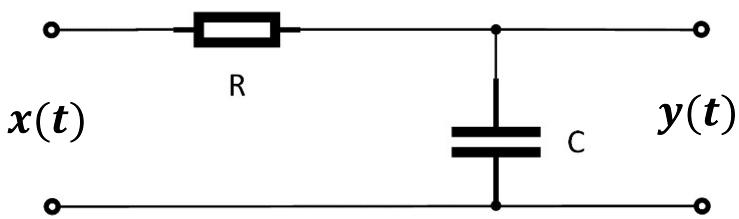
- ❑ First-and second-order system

First-order systems

First-order system (Continuous time)



- Differential equation:

$$C\frac{dy(t)}{dt} = \frac{x(t) - y(t)}{R}$$

$$\tau\frac{dy(t)}{dt} + y(t) = x(t), \tau = RC$$

- Frequency response:

$$\tau j\omega Y(j\omega) + Y(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{j\omega\tau + 1}$$

First-order systems

First-order system (Continuous time)

- ❑ Impulse response $H(j\omega) = \dfrac{1}{j\omega\tau + 1} = \dfrac{1/\tau}{j\omega + 1/\tau}$

$$e^{-at}u(t), a > 0 \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega + a}$$

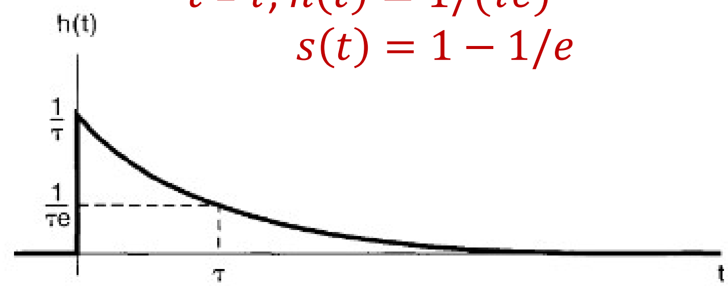
$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

- ❑ Step response

$$s(t) = \int_{-\infty}^{t} h(t')\,dt' = \frac{1}{\tau}\int_{0}^{t} e^{-t'/\tau}\,dt' = \begin{cases} 0, t < 0 \\ 1-e^{-t/\tau}), t \geq 0 \end{cases}$$

$$s(t) = (1-e^{-t/\tau})\,u(t)$$

- τ: time constant
- $t = \tau$, $h(t) = 1/(\tau e)$
  $s(t) = 1 - 1/e$



(a)

- $\tau \downarrow$, $h(t)$ decays more sharply
  $s(t)$ rises more sharply

(b)

First-order systems

Bold Plots (Continuous time)

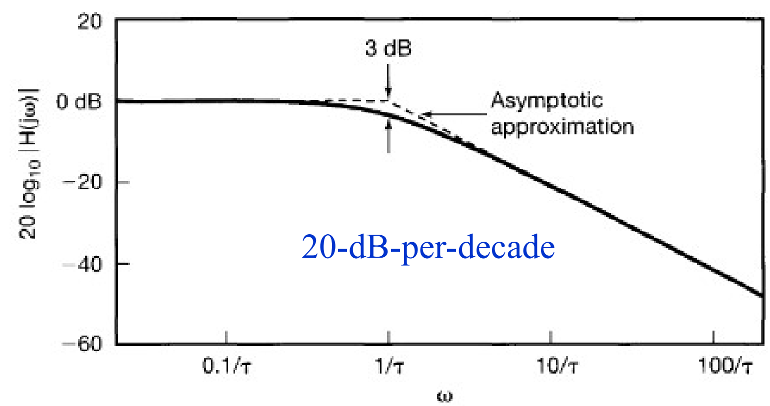
$$H(j\omega) = \frac{1}{j\omega\tau + 1}$$

❑ $\mathbf{20log_{10}|}\boldsymbol{H(j\omega)}\mathbf{|} = -10\log_{10}[(\omega\tau)^2 + 1]$

$$\simeq \begin{cases} 0, & \omega \ll 1/\tau \\ -20\log_{10}(\omega) - 20\log_{10}(\tau), & \omega \gg 1/\tau \end{cases}$$

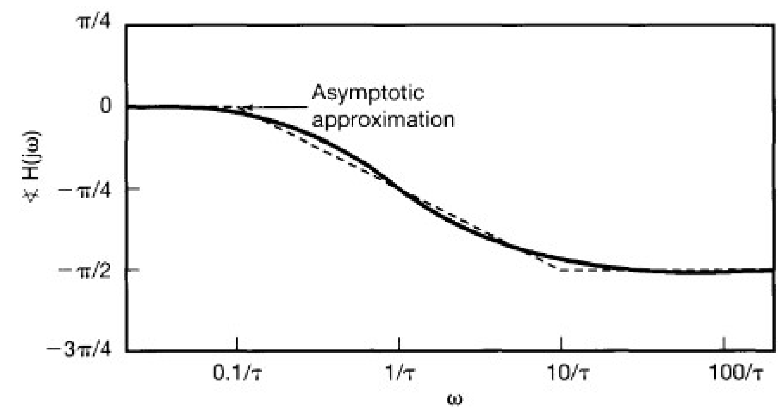
$\omega = 1/\tau, 20\log_{10}|H(j\omega)| = -10\log_{10}(2) \simeq -3dB$

$\omega = 1/\tau$： break frequency



❑ $\angle\boldsymbol{H(j\omega)} = -\tan^{-1}(\omega\tau)$

$$\simeq \begin{cases} 0, & \omega \leq 0.1/\tau \\ -\frac{\pi}{4}[\log_{10}(\omega\tau) + 1], & 0.1/\tau \leq \omega \leq 10/\tau \\ -\pi/2, & \omega \geq 10/\tau \end{cases}$$



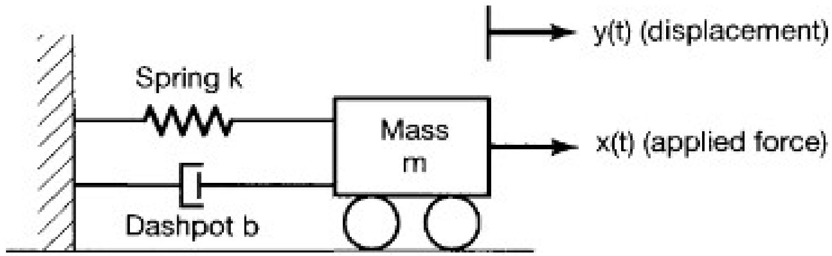
$\omega = 1/\tau, \angle H(j\omega) = -\pi/4$

$\tau \downarrow$, $h(t)$ and $s(t)$ more sharply, break frequency ↑.

Second-order systems

Differential equation

$$m\frac{d^2y(t)}{dt} = x(t) - ky(t) - b\frac{dy(t)}{dt}$$



$$\frac{d^2y(t)}{dt} + \left(\frac{b}{m}\right)\frac{dy(t)}{dt} + \left(\frac{k}{m}\right)y(t) = \frac{1}{m}x(t)$$

$${\omega_n}^2 = \frac{k}{m} \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{b}{2\sqrt{km}} \qquad 2\zeta\omega_n = \frac{b}{m}$$

$$\frac{d^2y(t)}{dt} + 2\zeta\omega_n\frac{dy(t)}{dt} + {\omega_n}^2y(t) = {\omega_n}^2x(t)$$

Second-order systems

- Frequency response: $$\frac{d^2y(t)}{dt}+2\zeta\omega_n\frac{dy(t)}{dt}+{\omega_n}^2y(t)={\omega_n}^2x(t)$$

$$(j\omega)^2Y(j\omega)+2\zeta\omega_n(j\omega)Y(j\omega)+{\omega_n}^2Y(j\omega)={\omega_n}^2X(j\omega)$$

$$H(j\omega)=\frac{{\omega_n}^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+{\omega_n}^2}$$

- Impulse response: $$H(j\omega)=\frac{{\omega_n}^2}{(j\omega-c_1)(j\omega-c_2)}=\frac{M_1}{(j\omega-c_1)}-\frac{M_2}{(j\omega-c_2)}$$

$\zeta\neq 1$

c_1, c_2: roots of $(j\omega)^2+2\zeta\omega_n(j\omega)+{\omega_n}^2=0$

$$c_1=-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1},\quad c_2=-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}$$

$$M_1=M_2=M=\frac{\omega_n}{2\sqrt{\zeta^2-1}}\longrightarrow h(t)=M[e^{c_1t}-e^{c_2t}]u(t)$$

Second-order systems

- Impulse response:

$\zeta = 1$ $\quad c_1 = c_1 = -\omega_n$ $\quad H(j\omega) = \dfrac{{\omega_n}^2}{(j\omega + \omega_n)^2}$

Critically damped

$$te^{-at}u(t) \xrightarrow{\mathcal{F}} H(j\omega) = \frac{1}{(j\omega + a)^2}$$

$$\therefore h(t) = {\omega_n}^2 te^{-\omega_n t}u(t)$$

- Recall $\zeta \neq 1$ $\quad h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$

$0 < \zeta < 1$

$$h(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}[e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} - e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}]\, u(t)$$

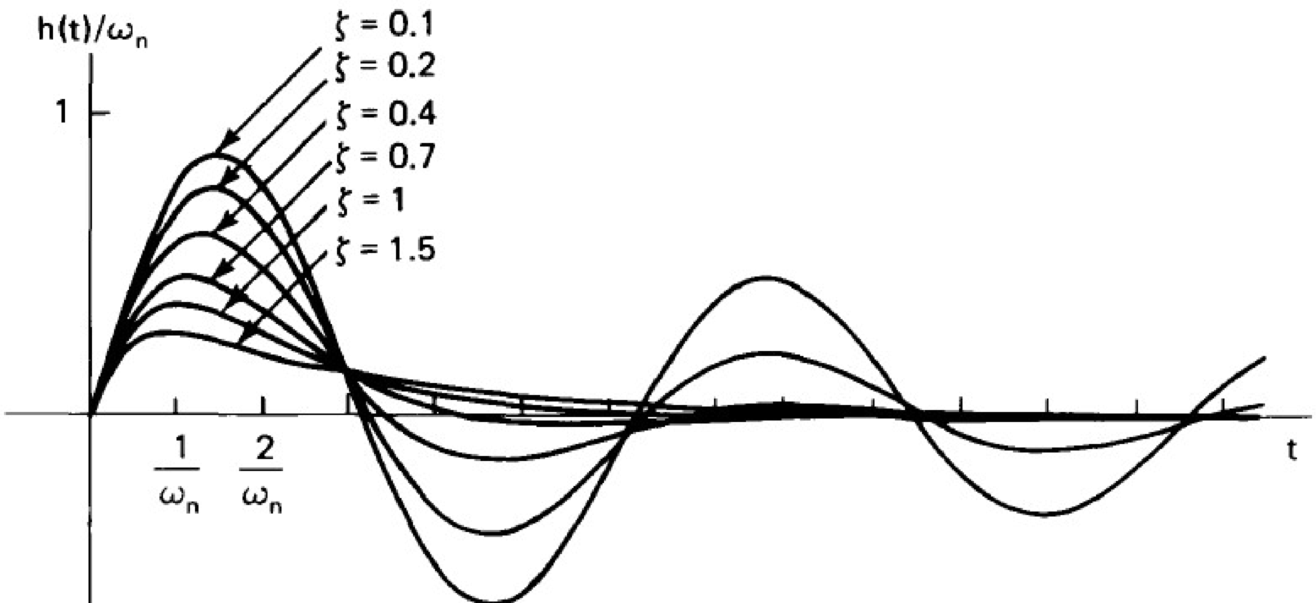
$$= \frac{\omega_n e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2 - 1}}\left[e^{j\omega_n\sqrt{1-\zeta^2}t} - e^{-j\omega_n\sqrt{1-\zeta^2}t}\right]u(t)$$

$\zeta > 1$: over damped

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2 - 1}}[2j\sin(\omega_n\sqrt{1-\zeta^2}\, t)]u(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}[\sin(\omega_n\sqrt{1-\zeta^2}\, t)]u(t)$$

Second-order systems

- Impulse response:



$0 < \zeta < 1$: damped

$\zeta > 1$: overdamped

$\zeta = 1$: critically damped

ω_n: un-damped natural frequency

ζ: damping ratio

Second-order systems

- Step response

$\zeta \neq 1$

$$s(t)=\int_{-\infty}^{t} h(t')\,dt' = M\int_{0}^{t} e^{c_1 t'} - e^{c_2 t'}\,dt'$$

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

$$= \left\{ \begin{matrix} 0, t<0 \\ M\left(\frac{e^{c_1 t'}}{c_1} - \frac{e^{c_2 t'}}{c_2}\right)\Big|_0^t = 1 + M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right], t \geq 0 \end{matrix} \right\} = \left\{1 + M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right]\right\} u(t)$$

$\zeta = 1$ $\quad h(t) = {\omega_n}^2 t e^{-\omega_n t} u(t)$

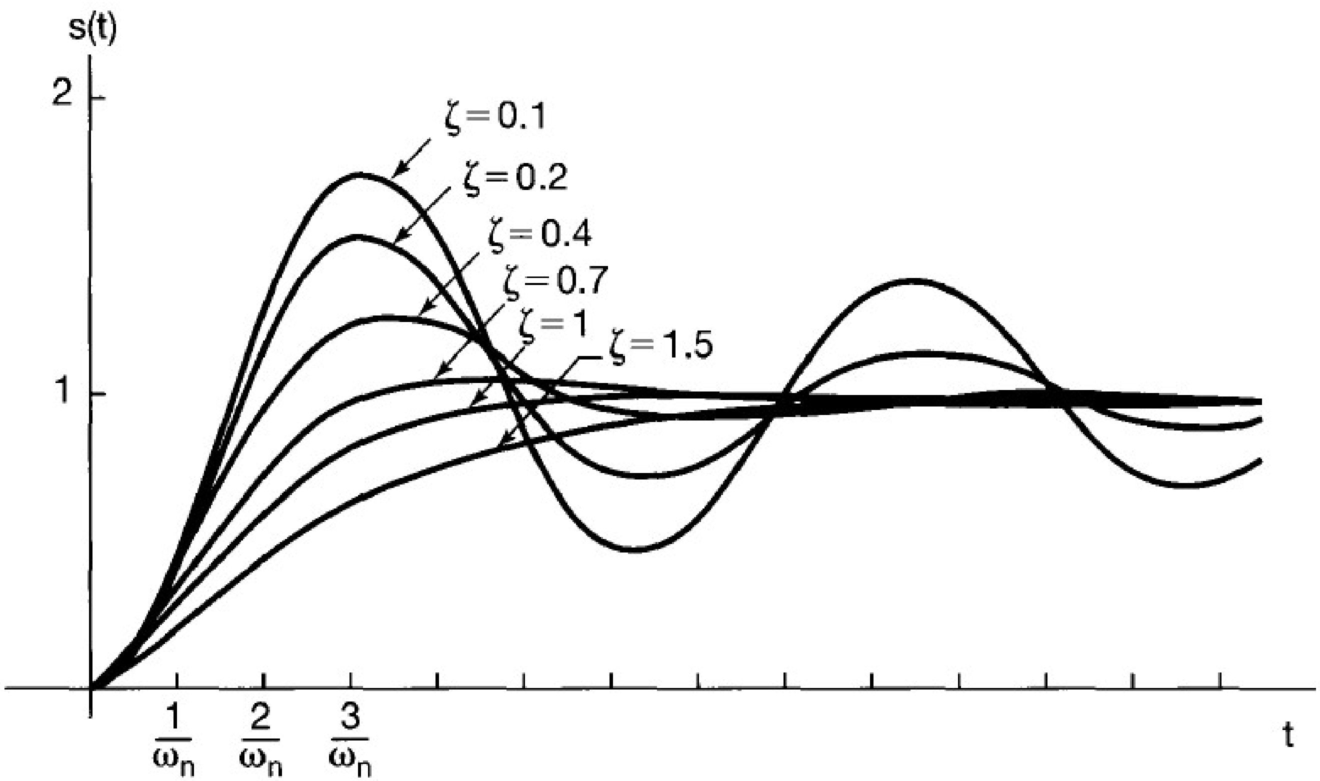
$$s(t)=\int_{0}^{t} {\omega_n}^2 t' e^{-\omega_n t'}\,dt' = -\omega_n \int_{0}^{t} t'\,de^{-\omega_n t'}$$

$$= \left\{ \begin{matrix} 0, t<0 \\ -\omega_n t' e^{-\omega_n t'}\Big|_0^t - \int_0^t e^{-\omega_n t'}\,d(-\omega_n t') = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}, t \geq 0 \end{matrix} \right\}$$

$$s(t) = [1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}]u(t)$$

Second-order systems

- Step response



Second-order systems

- Bold plots $$H(j\omega)=\frac{{\omega_n}^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+{\omega_n}^2}=\frac{1}{(j\omega/\omega_n)^2+2\zeta(j\omega/\omega_n)+1}$$

$$\mathbf{20log_{10}}|\boldsymbol{H(j\omega)}|=-20\log_{10}|(j\omega/\omega_n)^2+2\zeta(j\omega/\omega_n)+1|$$

$$=-10\log_{10}\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}$$

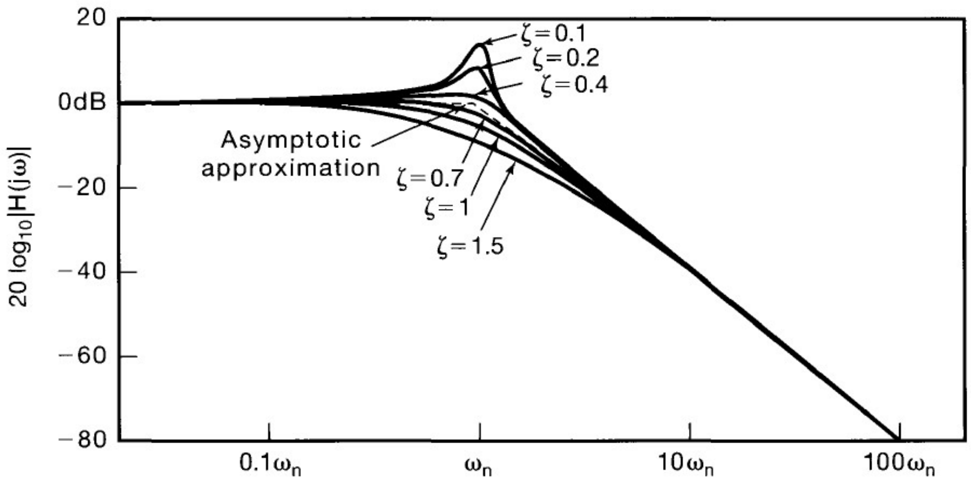
$$\simeq\begin{cases}0, & \omega\ll\omega_n\\ -40\log_{10}\omega+40\log_{10}\omega_n, & \omega\gg\omega_n\end{cases}$$

$$\angle\boldsymbol{H(j\omega)}=-\tan^{-1}\left[\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right]\simeq\begin{cases}0, & \omega\leq 0.1\omega_n\\ -\frac{\pi}{2}\left[\log_{10}\left(\frac{\omega}{\omega_n}\right)+1\right], & 0.1\omega_n\leq\omega\leq 10\omega_n\\ -\pi, & \omega\geq 10\omega_n\end{cases}$$

Second-order systems

❑ Bold plots

$20\log_{10}|H(j\omega)|$



$\angle H(j\omega)$