Introduction to Robotics

Chapter V Inverse Kinematics and Differential Kinematics

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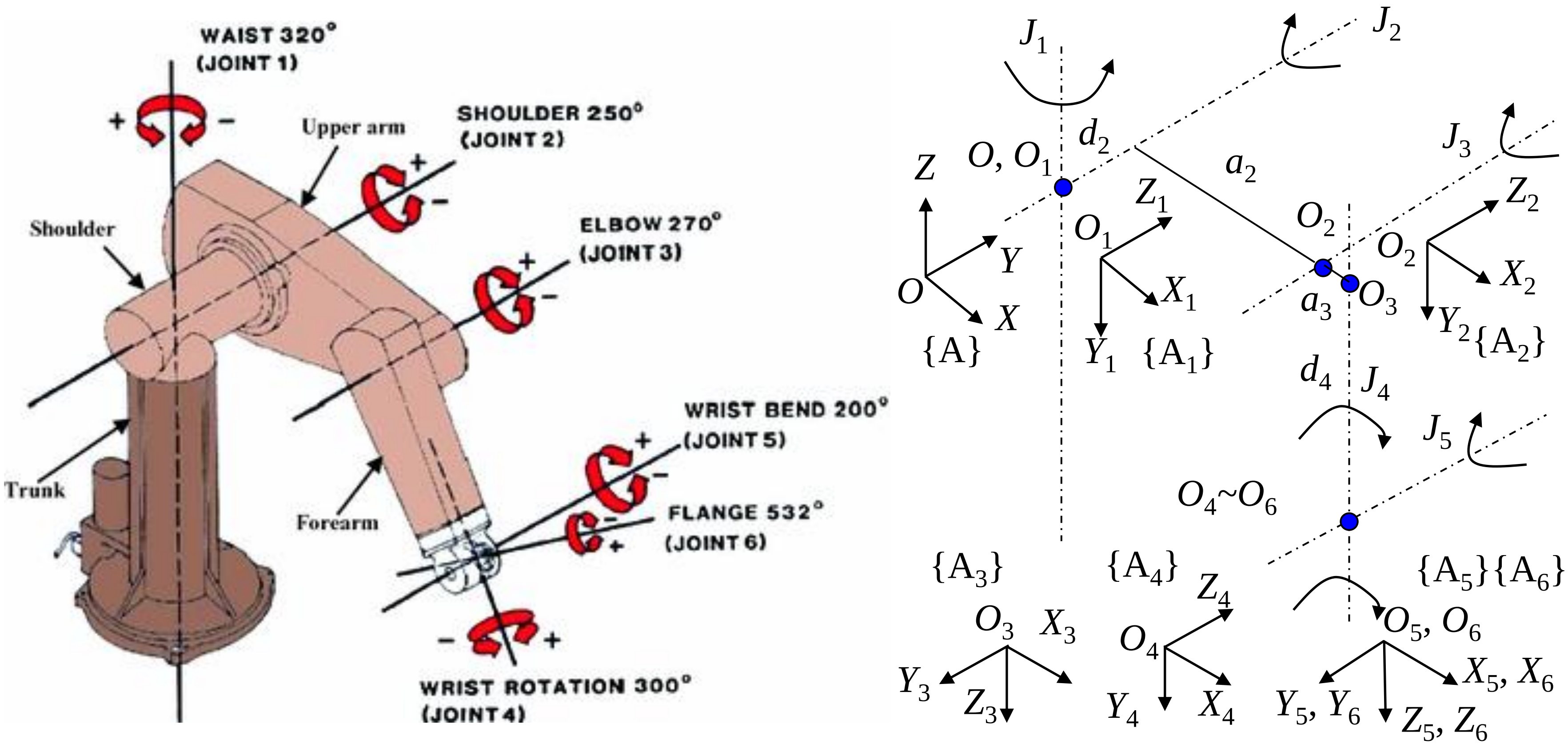
Inverse Kinematics

- Content
  - Inverse Kinematics of PUMA560
  - Differential Kinematics

4.3 Forward Kinematics for Spherical Coordinates Articulated Robot

4.3.1 PUMA560



Unimation PUMA560 Coordinates Establishment

Note no end-effector is attached to robot end; O overlaps O_1

3.1 Forward Kinematics for Spherical Coordinates Articulated Robot

- D-H Parameters：

Link	θ_i	α_i	a_i	d_i
1	θ_1	-90°	0	0
2	θ_2	0°	a_2	d_2
3	θ_3	-90°	a_3	0
4	θ_4	90°	0	d_4
5	θ_5	-90°	0	0
6	θ_6	0°	0	0

3.1 Forward Kinematics for Spherical Coordinates Articulated Robot

- D-H Matrix：

$$T_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_3 = \begin{bmatrix} \cos\theta_3 & 0 & -\sin\theta_3 & a_3\cos\theta_3 \\ \sin\theta_3 & 0 & \cos\theta_3 & a_3\sin\theta_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4 = \begin{bmatrix} \cos\theta_4 & 0 & \sin\theta_4 & 0 \\ \sin\theta_4 & 0 & -\cos\theta_4 & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_5 = \begin{bmatrix} \cos\theta_5 & 0 & -\sin\theta_5 & 0 \\ \sin\theta_5 & 0 & \cos\theta_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_6 = \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Robot End Pose/Position： $T = T_1 T_2 T_3 T_4 T_5 T_6$

Inverse Kinematics for Spherical Coordinates Articulated Robots

Mathematical Method for Inverse Kinematics of PUMA560

Supposing we have Link Transformation Matrices $T_1 \sim T_6$ of PUMA560, we can calculate T_2T_3, T_5T_6, $T_4T_5T_6$, $T_2T_3T_4T_5T_6$ as follows.

$$T_2T_3 = \begin{bmatrix} \cos(\theta_2+\theta_3) & 0 & -\sin(\theta_2+\theta_3) & a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3) \\ \sin(\theta_2+\theta_3) & 0 & \cos(\theta_2+\theta_3) & a_2\sin\theta_2 + a_3\sin(\theta_2+\theta_3) \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_5T_6 = \begin{bmatrix} \cos\theta_5\cos\theta_6 & -\cos\theta_5\sin\theta_6 & -\sin\theta_5 & 0 \\ \sin\theta_5\cos\theta_6 & -\sin\theta_5\sin\theta_6 & \cos\theta_5 & 0 \\ -\sin\theta_6 & -\cos\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4T_5T_6 = \begin{bmatrix} \cos\theta_4 & 0 & \sin\theta_4 & 0 \\ \sin\theta_4 & 0 & -\cos\theta_4 & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_5\cos\theta_6 & -\cos\theta_5\sin\theta_6 & -\sin\theta_5 & 0 \\ \sin\theta_5\cos\theta_6 & -\sin\theta_5\sin\theta_6 & \cos\theta_5 & 0 \\ -\sin\theta_6 & -\cos\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_4\cos\theta_5\cos\theta_6 - \sin\theta_4\sin\theta_6 & -\cos\theta_4\cos\theta_5\sin\theta_6 - \sin\theta_4\cos\theta_6 & -\cos\theta_4\sin\theta_5 & 0 \\ \sin\theta_4\cos\theta_5\cos\theta_6 + \cos\theta_4\sin\theta_6 & -\sin\theta_4\cos\theta_5\sin\theta_6 + \cos\theta_4\cos\theta_6 & -\sin\theta_4\sin\theta_5 & 0 \\ \sin\theta_5\cos\theta_6 & -\sin\theta_5\sin\theta_6 & \cos\theta_5 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics for Spherical Coordinates Articulated Robots

Mathematical Method for Inverse Kinematics of PUMA560

Supposing we have Link Transformation Matrices $T_1 \sim T_6$ of PUMA560, we can calculate T_2T_3, T_5T_6, $T_4T_5T_6$, $T_2T_3T_4T_5T_6$ as follows.

$$T_2T_3T_4T_5T_6 = \begin{bmatrix} b_{111} & b_{112} & b_{113} & -d_4\sin(\theta_2+\theta_3)+a_2\cos\theta_2+a_3\cos(\theta_2+\theta_3) \\ b_{121} & b_{122} & b_{123} & d_4\cos(\theta_2+\theta_3)+a_2\sin\theta_2+a_3\sin(\theta_2+\theta_3) \\ b_{131} & b_{132} & b_{133} & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$b_{111}=\cos(\theta_2+\theta_3)(\cos\theta_4\cos\theta_5\cos\theta_6-\sin\theta_4\sin\theta_6)-\sin(\theta_2+\theta_3)\sin\theta_5\cos\theta_6$$

$$b_{121}=\sin(\theta_2+\theta_3)(\cos\theta_4\cos\theta_5\cos\theta_6-\sin\theta_4\sin\theta_6)+\cos(\theta_2+\theta_3)\sin\theta_5\cos\theta_6$$

$$b_{131}=-\sin\theta_4\cos\theta_5\cos\theta_6-\cos\theta_4\sin\theta_6$$

$$b_{112}=-\cos(\theta_2+\theta_3)(\cos\theta_4\cos\theta_5\sin\theta_6+\sin\theta_4\cos\theta_6)+\sin(\theta_2+\theta_3)\sin\theta_5\sin\theta_6$$

$$b_{122}=-\sin(\theta_2+\theta_3)(\cos\theta_4\cos\theta_5\sin\theta_6+\sin\theta_4\cos\theta_6)-\cos(\theta_2+\theta_3)\sin\theta_5\sin\theta_6$$

$$b_{132}=\sin\theta_4\cos\theta_5\sin\theta_6-\cos\theta_4\cos\theta_6$$

$$b_{113}=-\cos(\theta_2+\theta_3)\cos\theta_4\sin\theta_5-\sin(\theta_2+\theta_3)\cos\theta_5$$

$$b_{123}=-\sin(\theta_2+\theta_3)\cos\theta_4\sin\theta_5+\cos(\theta_2+\theta_3)\cos\theta_5$$

$$b_{133}=\sin\theta_4\sin\theta_5$$

Inverse Kinematics of PUMA560

(1) Calculation of θ_1

$$T_1^{-1}T = T_2T_3T_4T_5T_6$$

$$T_1^{-1}T = \begin{bmatrix} n_x\cos\theta_1 + n_y\sin\theta_1 & o_x\cos\theta_1 + o_y\sin\theta_1 & a_x\cos\theta_1 + a_y\sin\theta_1 & p_x\cos\theta_1 + p_y\sin\theta_1 \\ -n_z & -o_z & -a_z & -p_z \\ -n_x\sin\theta_1 + n_y\cos\theta_1 & -o_x\sin\theta_1 + o_y\cos\theta_1 & -a_x\sin\theta_1 + a_y\cos\theta_1 & \boxed{-p_x\sin\theta_1 + p_y\cos\theta_1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2T_3T_4T_5T_6 = \begin{bmatrix} b_{111} & b_{112} & b_{113} & -d_4\sin(\theta_2+\theta_3) + a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3) \\ b_{121} & b_{122} & b_{123} & d_4\cos(\theta_2+\theta_3) + a_2\sin\theta_2 + a_3\sin(\theta_2+\theta_3) \\ b_{131} & b_{132} & b_{133} & \boxed{d_2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From:

$$-p_x\sin\theta_1 + p_y\cos\theta_1 = d_2$$

And Define:

$$\alpha = \text{atan}(p_y, p_x)$$

$$\begin{cases} p_x = \sqrt{p_x^2 + p_y^2}\cos\alpha \\ p_y = \sqrt{p_x^2 + p_y^2}\sin\alpha \end{cases}$$

We have:

$$\begin{cases} \sin(\alpha - \theta_1) = \dfrac{d_2}{\sqrt{p_x^2 + p_y^2}} \\ \cos(\alpha - \theta_1) = \pm\dfrac{\sqrt{p_x^2 + p_y^2 - d_2^2}}{\sqrt{p_x^2 + p_y^2}} \end{cases}$$

Results:

$$\theta_1 = \text{atan2}(p_y, p_x) - \text{atan2}(d_2, \pm\sqrt{p_x^2 + p_y^2 - d_2^2})$$

Inverse Kinematics of PUMA560

(2) Calculation of θ_3

$$T_1^{-1}T = T_2T_3T_4T_5T_6$$

$$T_1^{-1}T = \begin{bmatrix} n_x\cos\theta_1 + n_y\sin\theta_1 & o_x\cos\theta_1 + o_y\sin\theta_1 & a_x\cos\theta_1 + a_y\sin\theta_1 & \boxed{p_x\cos\theta_1 + p_y\sin\theta_1} \\ -n_z & -o_z & -a_z & \boxed{-p_z} \\ -n_x\sin\theta_1 + n_y\cos\theta_1 & -o_x\sin\theta_1 + o_y\cos\theta_1 & -a_x\sin\theta_1 + a_y\cos\theta_1 & \boxed{-p_x\sin\theta_1 + p_y\cos\theta_1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2T_3T_4T_5T_6 = \begin{bmatrix} b_{111} & b_{112} & b_{113} & \boxed{-d_4\sin(\theta_2+\theta_3) + a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3)} \\ b_{121} & b_{122} & b_{123} & \boxed{d_4\cos(\theta_2+\theta_3) + a_2\sin\theta_2 + a_3\sin(\theta_2+\theta_3)} \\ b_{131} & b_{132} & b_{133} & \boxed{d_2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{cases} -d_4\sin(\theta_2+\theta_3) + a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3) = p_x\cos\theta_1 + p_y\sin\theta_1 \\ d_4\cos(\theta_2+\theta_3) + a_2\sin\theta_2 + a_3\sin(\theta_2+\theta_3) = -p_z \\ d_2 = -p_x\sin\theta_1 + p_y\cos\theta_1 \end{cases}$$

Square Sum

$$-2d_4a_2\sin\theta_3 + 2a_2a_3\cos\theta_3 = p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_2^2 - d_4^2$$

$$\theta_3 = \operatorname{atan2}(a_3, d_4) - \operatorname{atan2}(k, \pm\sqrt{a_3^2 + d_4^2 - k^2})$$

$$k = \frac{p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_2^2 - d_4^2}{2a_2}$$

Inverse Kinematics of PUMA560

(3) Calculation θ_2

$$T_3^{-1}T_2^{-1}T_1^{-1}T = T_4T_5T_6$$

$$T_3^{-1}T_2^{-1}T_1^{-1}T = \begin{bmatrix} b_{211} & b_{212} & b_{213} & \boxed{(p_x\cos\theta_1 + p_y\sin\theta_1)\cos(\theta_2+\theta_3) - p_z\sin(\theta_2+\theta_3) - a_2\cos\theta_3 - a_3} \\ b_{221} & b_{222} & b_{223} & p_x\sin\theta_1 - p_y\cos\theta_1 + d_2 \\ b_{231} & b_{232} & b_{233} & \boxed{-(p_x\cos\theta_1 + p_y\sin\theta_1)\sin(\theta_2+\theta_3) - p_z\cos(\theta_2+\theta_3) + a_2\sin\theta_3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4T_5T_6 = \begin{bmatrix} \cos\theta_4\cos\theta_5\cos\theta_6 - \sin\theta_4\sin\theta_6 & -\cos\theta_4\cos\theta_5\sin\theta_6 - \sin\theta_4\cos\theta_6 & -\cos\theta_4\sin\theta_5 & \boxed{0} \\ \sin\theta_4\cos\theta_5\cos\theta_6 + \cos\theta_4\sin\theta_6 & -\sin\theta_4\cos\theta_5\sin\theta_6 + \cos\theta_4\cos\theta_6 & -\sin\theta_4\sin\theta_5 & 0 \\ \sin\theta_5\cos\theta_6 & -\sin\theta_5\sin\theta_6 & \cos\theta_5 & \boxed{d_4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{cases} (p_x\cos\theta_1 + p_y\sin\theta_1)\cos(\theta_2+\theta_3) - p_z\sin(\theta_2+\theta_3) = a_2\cos\theta_3 + a_3 \\ (p_x\cos\theta_1 + p_y\sin\theta_1)\sin(\theta_2+\theta_3) + p_z\cos(\theta_2+\theta_3) = a_2\sin\theta_3 - d_4 \end{cases}$$

$$\begin{cases} \sin(\theta_2+\theta_3) = \dfrac{(p_x\cos\theta_1 + p_y\sin\theta_1)(a_2\sin\theta_3 - d_4) - (a_2\cos\theta_3 + a_3)p_z}{p_z^{\,2} + (p_x\cos\theta_1 + p_y\sin\theta_1)^2} \\ \cos(\theta_2+\theta_3) = \dfrac{(p_x\cos\theta_1 + p_y\sin\theta_1)(a_2\cos\theta_3 + a_3) + (a_2\sin\theta_3 - d_4)p_z}{p_z^{\,2} + (p_x\cos\theta_1 + p_y\sin\theta_1)^2} \end{cases}$$

Calculate $\sin(\theta_2+\theta_3)$ and $\cos(\theta_2+\theta_3)$

Inverse Kinematics of PUMA560

(3) Calculation of θ_2

$$\begin{cases} \sin(\theta_2+\theta_3) = \dfrac{(p_x\cos\theta_1 + p_y\sin\theta_1)(a_2\sin\theta_3 - d_4) - (a_2\cos\theta_3 + a_3)p_z}{{p_z}^2 + (p_x\cos\theta_1 + p_y\sin\theta_1)^2} \\ \cos(\theta_2+\theta_3) = \dfrac{(p_x\cos\theta_1 + p_y\sin\theta_1)(a_2\cos\theta_3 + a_3) + (a_2\sin\theta_3 - d_4)p_z}{{p_z}^2 + (p_x\cos\theta_1 + p_y\sin\theta_1)^2} \end{cases}$$

$$\begin{aligned}\theta_2 = \operatorname{atan2}[&(p_x\cos\theta_1 + p_y\sin\theta_1)(a_2\sin\theta_3 - d_4) - (a_2\cos\theta_3 + a_3)p_z, \\ &(p_x\cos\theta_1 + p_y\sin\theta_1)(a_2\cos\theta_3 + a_3) + (a_2\sin\theta_3 - d_4)p_z] - \theta_3\end{aligned}$$

Since θ_1 and θ_3 both has two solutions, θ_2 has 4 solutions.

Inverse Kinematics of PUMA560

(4) Calculation of θ_4

$$T_3^{-1}T_2^{-1}T_1^{-1}T = T_4T_5T_6$$

$$T_3^{-1}T_2^{-1}T_1^{-1}T = \begin{bmatrix} b_{211} & b_{212} & \boxed{b_{213}} & (p_x\cos\theta_1 + p_y\sin\theta_1)\cos(\theta_2+\theta_3) - p_z\sin(\theta_2+\theta_3) - a_2\cos\theta_3 - a_3 \\ b_{221} & b_{222} & \boxed{b_{223}} & p_x\sin\theta_1 - p_y\cos\theta_1 + d_2 \\ b_{231} & b_{232} & b_{233} & -(p_x\cos\theta_1 + p_y\sin\theta_1)\sin(\theta_2+\theta_3) - p_z\cos(\theta_2+\theta_3) + a_2\sin\theta_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4T_5T_6 = \begin{bmatrix} \cos\theta_4\cos\theta_5\cos\theta_6 - \sin\theta_4\sin\theta_6 & -\cos\theta_4\cos\theta_5\sin\theta_6 - \sin\theta_4\cos\theta_6 & \boxed{-\cos\theta_4\sin\theta_5} & 0 \\ \sin\theta_4\cos\theta_5\cos\theta_6 + \cos\theta_4\sin\theta_6 & -\sin\theta_4\cos\theta_5\sin\theta_6 + \cos\theta_4\cos\theta_6 & \boxed{-\sin\theta_4\sin\theta_5} & 0 \\ \sin\theta_5\cos\theta_6 & -\sin\theta_5\sin\theta_6 & \cos\theta_5 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{cases} \cos(\theta_2+\theta_3)(a_x\cos\theta_1 + a_y\sin\theta_1) - a_z\sin(\theta_2+\theta_3) = -\cos\theta_4\sin\theta_5 \\ a_x\sin\theta_1 - a_y\cos\theta_1 = -\sin\theta_4\sin\theta_5 \end{cases}$$

If $\sin\theta_5 > 0$:

$$\theta_4 = \mathrm{atan2}[-a_x\sin\theta_1 + a_y\cos\theta_1, -\cos(\theta_2+\theta_3)(a_x\cos\theta_1 + a_y\sin\theta_1) + a_z\sin(\theta_2+\theta_3)]$$

If $\sin\theta_5 < 0$:

$$\theta_4 = \mathrm{atan2}[a_x\sin\theta_1 - a_y\cos\theta_1, \cos(\theta_2+\theta_3)(a_x\cos\theta_1 + a_y\sin\theta_1) - a_z\sin(\theta_2+\theta_3)]$$

Inverse Kinematics of PUMA560

(4) Calculation of θ_4

If $\sin\theta_5=0$:

$$T_4T_5T_6=\begin{bmatrix} \cos(\theta_4+\theta_6) & -\sin(\theta_4+\theta_6) & 0 & 0 \\ \sin(\theta_4+\theta_6) & \cos(\theta_4+\theta_6) & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If $\sin\theta_5=0$, Joint 4 rotate about J_4 by θ_4 plus Joint 6 rotate about J_6 by θ_6 equals to Joint 4 rotate about J_4 by $\theta_4+\theta_6$.

$$\theta_4+\theta_6=\operatorname{atan2}[n_x\sin\theta_1-n_y\cos\theta_1,\cos(\theta_2+\theta_3)(n_x\cos\theta_1+n_y\sin\theta_1)-n_z\sin(\theta_2+\theta_3)]$$

Once $\theta_5=0$, only $\theta_4+\theta_6$ can be calculated, which means we have infinite number of solutions for θ_4 and θ_6 to reach the same pose. In this scenario, the inverse kinematics has infinite solutions.

Inverse Kinematics of PUMA560

(5) Calculation of θ_5 and θ_6

$$T_4^{-1}T_3^{-1}T_2^{-1}T_1^{-1}T = T_5T_6$$

$$T_4^{-1}T_3^{-1}T_2^{-1}T_1^{-1}T = \begin{bmatrix} b_{311} & b_{312} & \boxed{b_{313}} & b_{314} \\ b_{321} & b_{322} & \boxed{b_{323}} & b_{324} \\ \boxed{b_{331}} & \boxed{b_{332}} & b_{333} & b_{334} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_5T_6 = \begin{bmatrix} \cos\theta_5\cos\theta_6 & -\cos\theta_5\sin\theta_6 & \boxed{-\sin\theta_5} & 0 \\ \sin\theta_5\cos\theta_6 & -\sin\theta_5\sin\theta_6 & \boxed{\cos\theta_5} & 0 \\ \boxed{-\sin\theta_6} & \boxed{-\cos\theta_6} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$b_{313} = [(p_x\cos\theta_1 + p_y\sin\theta_1)\cos(\theta_2+\theta_3) - p_z\sin(\theta_2+\theta_3) - a_2\cos\theta_3 - a_3]\cos\theta_4 + (p_x\sin\theta_1 - p_y\cos\theta_1 + d_2)\sin\theta_4$$

$$b_{323} = -(p_x\cos\theta_1 + p_y\sin\theta_1)\sin(\theta_2+\theta_3) - p_z\cos(\theta_2+\theta_3) + a_2\sin\theta_3 - d_4$$

$$b_{331} = [\cos(\theta_2+\theta_3)(n_x\cos\theta_1 + n_y\sin\theta_1) - n_z\sin(\theta_2+\theta_3)]\sin\theta_4 - (n_x\sin\theta_1 - n_y\cos\theta_1)\cos\theta_4$$

$$b_{332} = [\cos(\theta_2+\theta_3)(o_x\cos\theta_1 + o_y\sin\theta_1) - o_z\sin(\theta_2+\theta_3)]\sin\theta_4 - (o_x\sin\theta_1 - o_y\cos\theta_1)\cos\theta_4$$

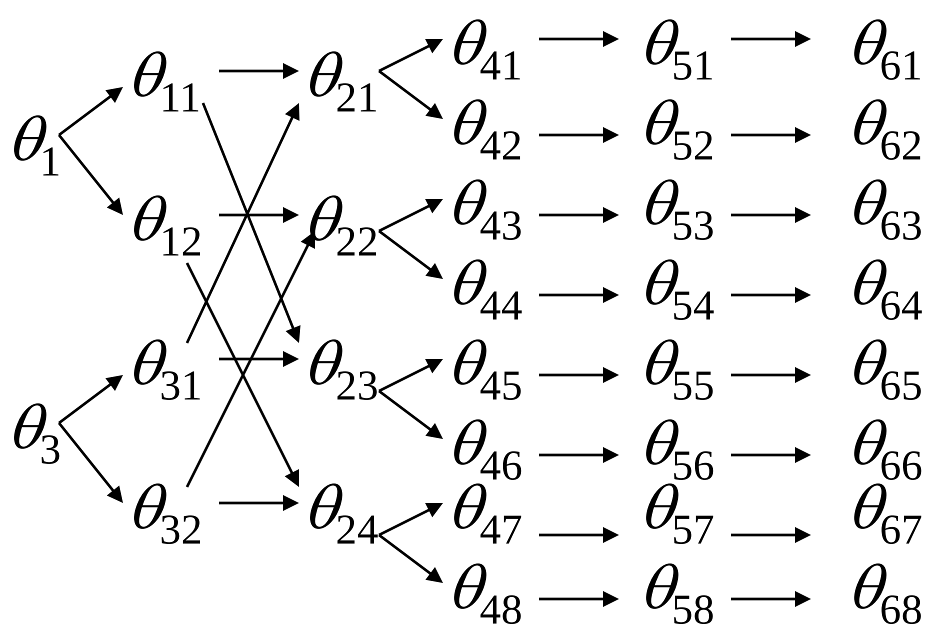
$$\begin{cases} \theta_5 = \text{atan2}(-b_{313}, b_{323}) \\ \theta_6 = \text{atan2}(-b_{331}, -b_{332}) \end{cases}$$

Inverse Kinematics of PUMA560

- Procedure of Inverse Kinematics Calculation

$\theta_1 \rightarrow \theta_3 \rightarrow \theta_2 \rightarrow \theta_4 \rightarrow \theta_5$ and θ_6

- Solutions:



Recommended Reading

- Dinesh Manocha and John F. Canny, Efficient inverse kinematics for general 6R manipulators, IEEE Transactions on Robotics and Automation, Vol. 10, No. 5, pp. 648-657, 1994

Differential Kinematics

- Differential Translation and Differential Rotation
  - {T} represented in world coordinates by T
  - Differential kinematics can be applied to both intrinsic transformation and extrinsic transformation
  - Differential kinematics is also known as differential transformation

➢ Differential Kinematics in Extrinsic Transformation

In Base Coordinates: $T + dT = \text{Trans}(d_x, d_y, d_z)\text{Rot}(f, d\theta)T$

We further have:

$$dT = \text{Trans}(d_x, d_y, d_z)\text{Rot}(f, d\theta)T - T$$

$$= [\text{Trans}(d_x, d_y, d_z)\text{Rot}(f, d\theta) - I]T = \Delta T$$

By default for differential translation:

$$\begin{cases} \lim\limits_{\theta \to 0} \sin\theta = \theta \\ \lim\limits_{\theta \to 0} \cos\theta = 1 \\ \lim\limits_{\theta \to 0} \text{vers}\,\theta = 0 \end{cases} \quad \text{Trans}(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Differential Kinematics

- Differential Translation and Differential Rotation
  - {T} represented in world coordinates by *T*
  - Differential kinematics can be applied to both intrinsic transformation and extrinsic transformation
  - Differential kinematics is also known as differential transformation

➢ Differential Kinematics in Extrinsic Transformation

$$T + dT = \mathrm{Trans}(d_x, d_y, d_z)\mathrm{Rot}(f, d\theta)T$$

By default for differential rotation:

$$\mathrm{Rot}(f, d\theta) = \begin{bmatrix} f_x f_x \mathrm{vers}(d\theta) + \cos(d\theta) & f_y f_x \mathrm{vers}(d\theta) - f_z \sin(d\theta) & f_z f_x \mathrm{vers}(d\theta) + f_y \sin(d\theta) & 0 \\ f_x f_y \mathrm{vers}(d\theta) + f_z \sin(d\theta) & f_y f_y \mathrm{vers}(d\theta) + \cos(d\theta) & f_z f_y \mathrm{vers}(d\theta) - f_x \sin(d\theta) & 0 \\ f_x f_z \mathrm{vers}(d\theta) - f_y \sin(d\theta) & f_y f_z \mathrm{vers}(d\theta) + f_x \sin(d\theta) & f_z f_z \mathrm{vers}(d\theta) + \cos(d\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -f_z d\theta & f_y d\theta & 0 \\ f_z d\theta & 1 & -f_x d\theta & 0 \\ -f_y d\theta & f_x d\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Differential Kinematics

- Differential Kinematics in Extrinsic Transformation

$$dT = \mathrm{Trans}(d_x, d_y, d_z)\mathrm{Rot}(f, d\theta)T - T$$

$$= [\mathrm{Trans}(d_x, d_y, d_z)\mathrm{Rot}(f, d\theta) - I]T = \Delta T$$

The differential factor Δ is:

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -f_z d\theta & f_y d\theta & 0 \\ f_z d\theta & 1 & -f_x d\theta & 0 \\ -f_y d\theta & f_x d\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -f_z d\theta & f_y d\theta & d_x \\ f_z d\theta & 0 & -f_x d\theta & d_y \\ -f_y d\theta & f_x d\theta & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \begin{cases} \delta_x = f_x d\theta \\ \delta_y = f_y d\theta \\ \delta_z = f_z d\theta \end{cases}$$

$\vec{d} = d_x\vec{i} + d_y\vec{j} + d_z\vec{k}$ is named as differential translation vector.

$\vec{\delta} = \delta_x\vec{i} + \delta_y\vec{j} + \delta_z\vec{k}$ is named as differential rotation vector.

Differential Kinematics

➢ Differential Kinematics in Intrinsic Transformation

In End Coordinates: $T + {}^T dT = T\ \mathrm{Trans}({}^T d_x, {}^T d_y, {}^T d_z)\mathrm{Rot}({}^T f, {}^T d\theta)$

We further have:

$$\begin{aligned} {}^T dT &= T\ \mathrm{Trans}({}^T d_x, {}^T d_y, {}^T d_z)\mathrm{Rot}({}^T f, {}^T d\theta) - T \\ &= T[\mathrm{Trans}({}^T d_x, {}^T d_y, {}^T d_z)\mathrm{Rot}({}^T f, {}^T d\theta) - I] = T\ {}^T\Delta \end{aligned}$$

Similarly, we can get

$${}^T\Delta = \begin{bmatrix} 0 & -{}^T f_z\, {}^T d\theta & {}^T f_y\, {}^T d\theta & {}^T d_x \\ {}^T f_z\, {}^T d\theta & 0 & -{}^T f_x\, {}^T d\theta & {}^T d_y \\ -{}^T f_y\, {}^T d\theta & {}^T f_x\, {}^T d\theta & 0 & {}^T d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -{}^T\delta_z & {}^T\delta_y & {}^T d_x \\ {}^T\delta_z & 0 & -{}^T\delta_x & {}^T d_y \\ -{}^T\delta_y & {}^T\delta_x & 0 & {}^T d_z \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\begin{cases} {}^T\delta_x = {}^T f_x\, {}^T d\theta \\ {}^T\delta_y = {}^T f_y\, {}^T d\theta \\ {}^T\delta_z = {}^T f_z\, {}^T d\theta \end{cases}$

Differential Kinematics

- Relationship between differential transformations in Intrinsic Transformation and Extrinsic Transformation

Supposing $T + \Delta T = T + T^T\Delta$, we can get relationship between Δ and ${}^T\Delta$:

$$\Delta T = T\,{}^T\Delta \rightarrow {}^T\Delta = T^{-1}\Delta T$$

Then for a given T:

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta T = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\delta_z n_y + \delta_y n_z & -\delta_z o_y + \delta_y o_z & -\delta_z a_y + \delta_y a_z & -\delta_z p_y + \delta_y p_z + d_x \\ \delta_z n_x - \delta_x n_z & \delta_z o_x - \delta_x o_z & \delta_z a_x - \delta_x a_z & \delta_z p_x - \delta_x p_z + d_y \\ -\delta_y n_x + \delta_x n_y & -\delta_y o_x + \delta_x o_y & -\delta_y a_x + \delta_x a_y & -\delta_y p_x + \delta_x p_y + d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Differential Kinematics

- Relationship between differential transformations in Intrinsic Transformation and Extrinsic Transformation

Supposing $T + \Delta T = T + T^T\Delta$, we can get relationship between Δ and ${}^T\Delta$:

$$\Delta T = T\,{}^T\Delta \rightarrow {}^T\Delta = T^{-1}\Delta T$$

Then for a given T:

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$\Delta T = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (\delta \times n)_x & (\delta \times o)_x & (\delta \times a)_x & (\delta \times p + d)_x \\ (\delta \times n)_y & (\delta \times o)_y & (\delta \times a)_y & (\delta \times p + d)_y \\ (\delta \times n)_z & (\delta \times o)_z & (\delta \times a)_z & (\delta \times p + d)_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Differential Kinematics

- Relationship between differential transformations in Intrinsic Transformation and Extrinsic Transformation

Supposing $T + \varDelta T = T + T^{T}\varDelta$, we can get relationship between $\varDelta$ and ${}^{T}\varDelta$:

$$\Delta T = T\,{}^{T}\Delta \rightarrow {}^{T}\Delta = T^{-1}\Delta T$$

Then for a given T:

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{T}\Delta = T^{-1}\Delta T = \begin{bmatrix} n_x & n_y & n_z & -p\cdot n \\ o_x & o_y & o_z & -p\cdot o \\ a_x & a_y & a_z & -p\cdot a \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} (\delta\times n)_x & (\delta\times o)_x & (\delta\times a)_x & (\delta\times p+d)_x \\ (\delta\times n)_y & (\delta\times o)_y & (\delta\times a)_y & (\delta\times p+d)_y \\ (\delta\times n)_z & (\delta\times n)_z & (\delta\times a)_z & (\delta\times p+d)_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} n\cdot(\delta\times n) & n\cdot(\delta\times o) & n\cdot(\delta\times a) & n\cdot(\delta\times p+d) \\ o\cdot(\delta\times n) & o\cdot(\delta\times o) & o\cdot(\delta\times a) & o\cdot(\delta\times p+d) \\ a\cdot(\delta\times n) & a\cdot(\delta\times o) & a\cdot(\delta\times a) & a\cdot(\delta\times p+d) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Differential Kinematics

$$ {}^{T}\Delta = T^{-1}\Delta T = \begin{bmatrix} n\cdot(\delta\times n) & n\cdot(\delta\times o) & n\cdot(\delta\times a) & n\cdot(\delta\times p+d) \\ o\cdot(\delta\times n) & o\cdot(\delta\times o) & o\cdot(\delta\times a) & o\cdot(\delta\times p+d) \\ a\cdot(\delta\times n) & a\cdot(\delta\times o) & a\cdot(\delta\times a) & a\cdot(\delta\times p+d) \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

According to $a\cdot(b\times c)=b\cdot(c\times a)=c\cdot(a\times b),\ \ a\cdot(a\times c)=0$, we have：

$$ {}^{T}\Delta = \begin{bmatrix} 0 & \delta\cdot(o\times n) & \delta\cdot(a\times n) & \delta\cdot(p\times n)+d\cdot n \\ \delta\cdot(n\times o) & 0 & \delta\cdot(a\times o) & \delta\cdot(p\times o)+d\cdot o \\ \delta\cdot(n\times a) & \delta\cdot(o\times a) & 0 & \delta\cdot(p\times a)+d\cdot a \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\delta\cdot a & \delta\cdot o & \delta\cdot(p\times n)+d\cdot n \\ \delta\cdot a & 0 & -\delta\cdot n & \delta\cdot(p\times o)+d\cdot o \\ -\delta\cdot o & \delta\cdot n & 0 & \delta\cdot(p\times a)+d\cdot a \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

Since：

$$ {}^{T}\Delta = \begin{bmatrix} 0 & -{}^{T}\delta_z & {}^{T}\delta_y & {}^{T}d_x \\ {}^{T}\delta_z & 0 & -{}^{T}\delta_x & {}^{T}d_y \\ -{}^{T}\delta_y & {}^{T}\delta_x & 0 & {}^{T}d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

Finally:

$$ \begin{cases} {}^{T}d_x = \delta\cdot(p\times n)+d\cdot n = n\cdot(\delta\times p+d) \\ {}^{T}d_y = \delta\cdot(p\times o)+d\cdot o = o\cdot(\delta\times p+d), \\ {}^{T}d_z = \delta\cdot(p\times a)+d\cdot a = a\cdot(\delta\times p+d) \end{cases} \quad \begin{cases} {}^{T}\delta_x = \delta\cdot n \\ {}^{T}\delta_y = \delta\cdot o \\ {}^{T}\delta_z = \delta\cdot a \end{cases} $$

Differential Kinematics

We use a compact structure to describe the relationship between $\{{}^T d, {}^T \delta\}$ and $\{d, \delta\}$:

$$\begin{bmatrix} {}^T d_x \\ {}^T d_y \\ {}^T d_z \\ {}^T \delta_x \\ {}^T \delta_y \\ {}^T \delta_z \end{bmatrix} = \begin{bmatrix} n_x & n_y & n_z & (p \times n)_x & (p \times n)_y & (p \times n)_z \\ o_x & o_y & o_z & (p \times o)_x & (p \times o)_y & (p \times o)_z \\ a_x & a_y & a_z & (p \times a)_x & (p \times a)_z & (p \times a)_z \\ 0 & 0 & 0 & n_x & n_y & n_z \\ 0 & 0 & 0 & o_x & o_y & o_z \\ 0 & 0 & 0 & a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}$$

This formula describes the relationship for a differential motion from the base coordinates to the end coordinates.

For instance, supposing we know T (the pose of end coordinates in base), and we tell a differential motion $\{d, \delta\}$ (with respect to base), then we can get the equivalent differential motion $\{{}^T d, {}^T \delta\}$ (with respect to end coordinates) from the above formular.

Content in this slide is Critically Important!

Why Differential Kinematics Important?

Reason 1:

For elementary rotations, differential transformation matrices are:

$$\mathrm{Rot}(x,\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad \mathrm{Rot}(y,\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad \mathrm{Rot}(z,\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\mathrm{Rot}(x,\delta_x)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -\delta_x & 0\\ 0 & \delta_x & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad \mathrm{Rot}(y,\delta_y)=\begin{bmatrix}1 & 0 & \delta_y & 0\\ 0 & 1 & 0 & 0\\ -\delta_y & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad \mathrm{Rot}(z,\delta_z)=\begin{bmatrix}1 & -\delta_z & 0 & 0\\ \delta_z & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Why Differential Kinematics Important?

Reason 1:

So supposing world coordinates is rotated about f by $d\theta$, then:

$$\Delta = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \begin{cases} \delta_x = f_x d\theta \\ \delta_y = f_y d\theta \\ \delta_z = f_z d\theta \end{cases}$$

Now world coordinates turns out to be:

$$T' = T + \Delta T = \begin{bmatrix} 1 & -\delta_z & \delta_y & 0 \\ \delta_z & 1 & -\delta_x & 0 \\ -\delta_y & \delta_x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{since } T = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Why Differential Kinematics Important?

Reason 1:

Now let's say the world coordinates rotates about x by δ_x, then rotates about y by δ_y, finally rotates about z by δ_z, we can validate:

$$\mathrm{Rot}(z,\delta_z)\mathrm{Rot}(y,\delta_y)\mathrm{Rot}(x,\delta_x) = \begin{bmatrix} 1 & -\delta_z & \delta_y & d_x \\ \delta_z & 1 & -\delta_x & d_y \\ -\delta_y & \delta_x & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which means

$$\mathrm{Rot}(z,\delta_z)\mathrm{Rot}(y,\delta_y)\mathrm{Rot}(x,\delta_x) = T'$$

So, we can see that:

Differential rotation about arbitrary axis about ***f*** by $d\theta$ is equal to Differential rotation about x- y- z- axis by $f_x d\theta$, $f_y d\theta$, *and* $f_z d\theta$.

A easy way for control!

And differential rotation is independent on rotation orders! See prove next slide.

Reason 2:
Differential rotation is independent on rotation orders

Proved by Enumeration:

$$\mathrm{Rot}(x,\delta_x)\mathrm{Rot}(y,\delta_y)\mathrm{Rot}(z,\delta_z)\mathrm{Trans}(d_x,d_y,d_z)=\begin{bmatrix}1 & -\delta_z & \delta_y & 0\\ \delta_z & 1 & -\delta_x & 0\\ -\delta_y & \delta_x & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & d_x\\ 0 & 1 & 0 & d_y\\ 0 & 0 & 1 & d_z\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & -\delta_z & \delta_y & d_x-\delta_z d_y+\delta_y d_z\\ \delta_z & 1 & -\delta_x & d_y+\delta_z d_x-\delta_x d_z\\ -\delta_y & \delta_x & 1 & d_z-\delta_y d_x+\delta_x d_y\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & -\delta_z & \delta_y & d_x\\ \delta_z & 1 & -\delta_x & d_y\\ -\delta_y & \delta_x & 1 & d_z\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\mathrm{Trans}(d_x,d_y,d_z)\mathrm{Rot}(x,\delta_x)\mathrm{Rot}(y,\delta_y)\mathrm{Rot}(z,\delta_z)=\begin{bmatrix}1 & 0 & 0 & d_x\\ 0 & 1 & 0 & d_y\\ 0 & 0 & 1 & d_z\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -\delta_z & \delta_y & 0\\ \delta_z & 1 & -\delta_x & 0\\ -\delta_y & \delta_x & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & -\delta_z & \delta_y & d_x\\ \delta_z & 1 & -\delta_x & d_y\\ -\delta_y & \delta_x & 1 & d_z\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Reason 1 & 2 have physical meanings in reality

Reason 1: Differential rotation about arbitrary axis about $\boldsymbol{f}$ by $d\theta$ is equal to Differential rotation about x- y- z- axis by $f_x d\theta$, $f_y d\theta$, *and* $f_z d\theta$.

Reason 2: Differential rotation is independent on rotation orders! See prove next slide.

The physical meaning of differential kinematics is:

In real world applications, we regularly need to modify the objects orientation by a robot manipulator **slightly**, which actually can be modelled as differential transformation.

In these cases, the task can be easily achieved by differential kinematics without considering the complicated general rotation transformation and then transform the transformation matrix into Euler angles for execution, nor will we consider the rotation orders.

We just need to **simply** get the rotation axis $\boldsymbol{f}$ and $d\theta$, and rotate the execution mechanism about x- y- z- axis by $f_x d\theta$, $f_y d\theta$, *and* $f_z d\theta$ respectively and simultaneously without considering rotation orders.

Differential Kinematics

Case study: Supposing the differential translation vector and differential rotation vector of {A} with respect to base coordinates are

$$\begin{array}{c} \begin{array}{cccc} n & o & a & p \end{array} \\ A=\begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}, \begin{cases} d = 1\vec{i} + 0\vec{j} + 0.5\vec{k} \\ \delta = 0\vec{i} + 0.1\vec{j} + 0\vec{k} \end{cases}$$

(1) what is dA ?

(2) what are the differential translation vector and differential rotation in {A}?

Answer：

$$\Delta = \begin{bmatrix} 0 & 0 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, dA = \Delta A = \begin{bmatrix} 0 & 0 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\delta \times p = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0.1 & 0 \\ 10 & 5 & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} - 1\vec{k}, \quad \delta \times p + d = 1\vec{i} + 0\vec{j} - 0.5\vec{k}$$

$$\begin{cases} {}^{T}d_x = n \cdot (\delta \times p + d) \\ {}^{T}d_y = o \cdot (\delta \times p + d), \\ {}^{T}d_z = a \cdot (\delta \times p + d) \end{cases} \quad \begin{cases} {}^{T}\delta_x = \delta \cdot n \\ {}^{T}\delta_y = \delta \cdot o \\ {}^{T}\delta_z = \delta \cdot a \end{cases}$$

$${}^{A}d = 0\vec{i} - 0.5\vec{j} + 1\vec{k}, \quad {}^{A}\delta = 0.1\vec{i} + 0\vec{j} + 0\vec{k}$$

Let's go deeper!

$\{{}^Td, {}^T\delta\}$ and $\{d, \delta\}$ can be viewed as differential motion of the end-effector and the differential motion regarding to the base.

$$\begin{bmatrix} {}^Td_x \\ {}^Td_y \\ {}^Td_z \\ {}^T\delta_x \\ {}^T\delta_y \\ {}^T\delta_z \end{bmatrix} = \begin{bmatrix} n_x & n_y & n_z & (p\times n)_x & (p\times n)_y & (p\times n)_z \\ o_x & o_y & o_z & (p\times o)_x & (p\times o)_y & (p\times o)_z \\ a_x & a_y & a_z & (p\times a)_x & (p\times a)_z & (p\times a)_z \\ 0 & 0 & 0 & n_x & n_y & n_z \\ 0 & 0 & 0 & o_x & o_y & o_z \\ 0 & 0 & 0 & a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}$$

Therefore, differential motion in end-effector coordinates can be easily converted into differential motion in base coordinates.

Let's go deeper!

$\{{}^{T}d,\ {}^{T}\delta\}$ and $\{d,\ \delta\}$ are actually the motion velocity of the end-effector and the motion velocity of the joint space.

$$\begin{bmatrix} {}^{T}d_x \\ {}^{T}d_y \\ {}^{T}d_z \\ {}^{T}\delta_x \\ {}^{T}\delta_y \\ {}^{T}\delta_z \end{bmatrix} = \begin{bmatrix} n_x & n_y & n_z & (p\times n)_x & (p\times n)_y & (p\times n)_z \\ o_x & o_y & o_z & (p\times o)_x & (p\times o)_y & (p\times o)_z \\ a_x & a_y & a_z & (p\times a)_x & (p\times a)_z & (p\times a)_z \\ 0 & 0 & 0 & n_x & n_y & n_z \\ 0 & 0 & 0 & o_x & o_y & o_z \\ 0 & 0 & 0 & a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}$$

Therefore, Differential kinematics actually concerns with the robot velocity control.

THANK YOU