The z-Transform
(ch.10)

- The z-transform
- The region of convergence for the z-transforms
- The inverse z-transform
- Geometric evaluation of the Fourier transform from the pole-zero plot
- Properties of the z-transform
- Some common z-transform pairs
- Analysis and characterization of LTI systems using z-transforms
- System function algebra and block diagram representations
- The unilateral z-transform

The z-transform

Recall

- The response of LTI systems to complex exponentials z^n

$$y[n] = H(z)z^n$$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}$$

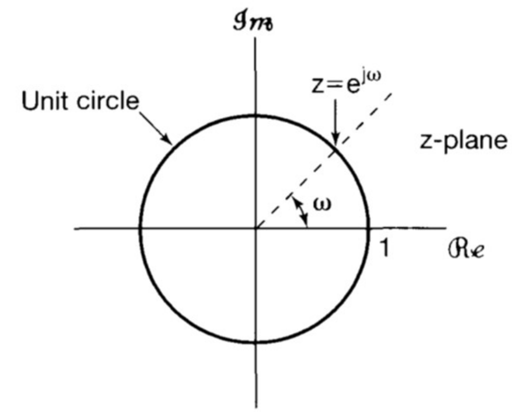
Definition

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

$$\boxed{X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n}}$$

The z-transform

Z-transform vs Fourier transform



$$x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)$$

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$z = e^{j\omega}$
$|z| = 1$ $\Downarrow$

$\Downarrow$ $z = re^{j\omega}$

$$X(z)\Big|_{z=e^{j\omega}} = X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$\boxed{X(z)\Big|_{z=e^{j\omega}} = \mathcal{F}\{x[n]\}}$$

$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} x[n] \left(re^{j\omega}\right)^{-n}$$

$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} \{x[n] r^{-n}\} e^{-j\omega n}$$

$$\boxed{X\left(re^{j\omega}\right) = \mathcal{F}\{x[n] r^{-n}\}}$$

The z-transform

Examples

$$x[n] = a^n u[n] \qquad X(z) = ?$$

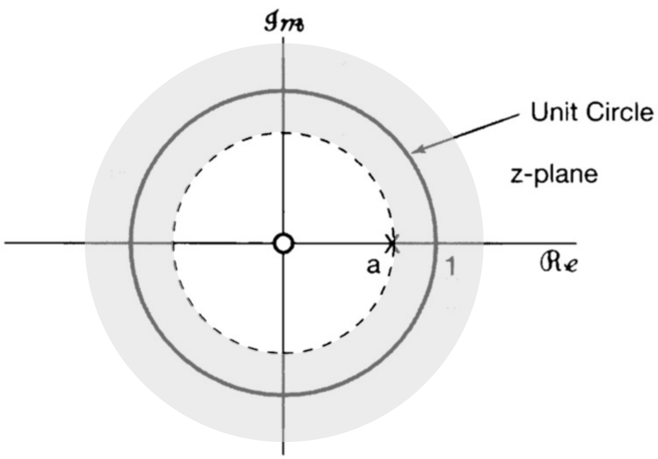
Solution

$$X(z) = \sum_{n=-\infty}^{+\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-a} \qquad |z| > |a|$$

$$\Downarrow \; a = 1$$

$$u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}} \qquad |z| > 1$$



The z-transform

Examples

$$x[n] = -a^n u[-n-1] \qquad X(z) = ?$$

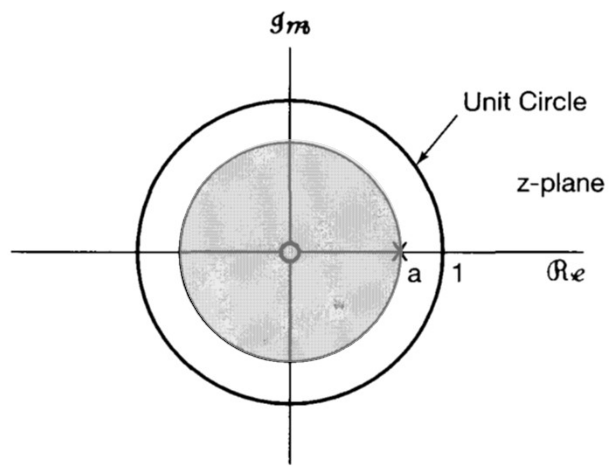
Solution

$$X(z) = -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$



$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|$$

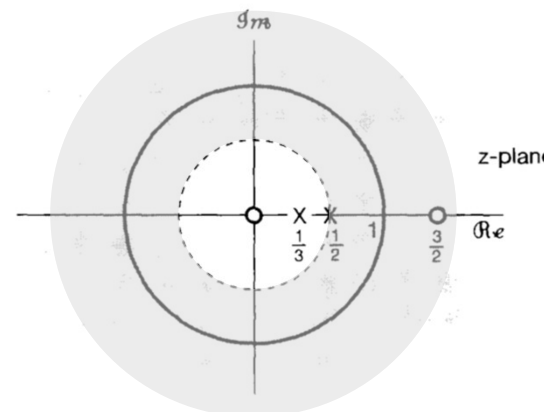
Examples

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \qquad X(z) = ?$$

Solution

$$\left(\frac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{3}z^{-1}} \qquad |z| > \frac{1}{3}$$

$$\left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$



$$7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

The z-transform

Examples

$$x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n] = \frac{1}{2j}\left(\frac{1}{3}e^{j\pi/4}\right)^n u[n] - \frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n u[n] \qquad X(z) =?$$

Solution

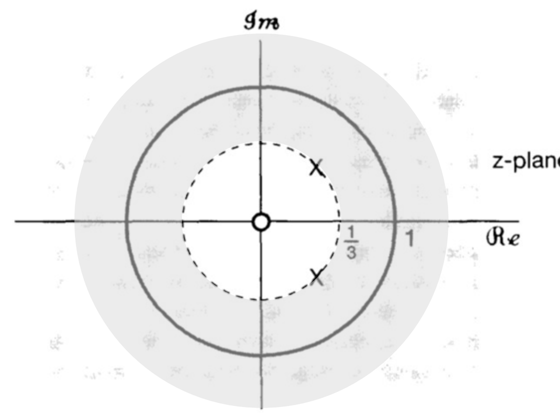
$$X(z) = \sum_{n=-\infty}^{+\infty}\left\{\frac{1}{2j}\left(\frac{1}{3}e^{j\pi/4}\right)^n u[n] - \frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n u[n]\right\} z^{-n}$$

$$= \frac{1}{2j}\sum_{n=0}^{+\infty}\left(\frac{1}{3}e^{j\pi/4}\right)^n z^{-n} - \frac{1}{2j}\sum_{n=0}^{+\infty}\left(\frac{1}{3}e^{-j\ \ /4}\right)^n z^{-n}$$

$$= \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{j\pi/4}z^{-1}} - \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{-j\pi/4}z^{-1}}$$

For convergence,

$$\left|\frac{1}{3}e^{j\pi/4}z^{-1}\right| < 1 \;\;\&\;\; \left|\frac{1}{3}e^{-j\pi/4}z^{-1}\right| < 1 \quad \Longrightarrow \quad |z| > 1/3$$



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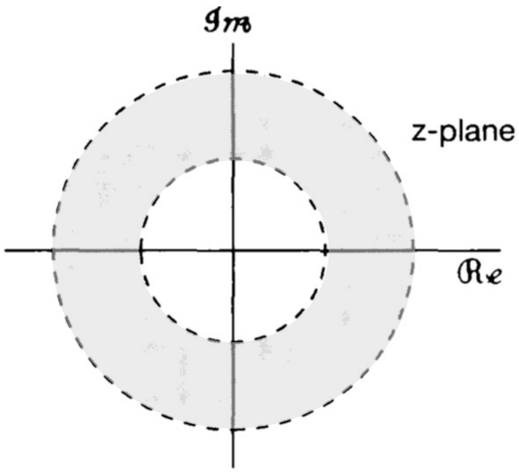
The region of convergence for z-transforms

Properties



- The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.

  ROC of $X(z)$: $x[n]r^{-n}$ converges (absolutely summable)

$$\sum_{n=-\infty}^{+\infty} |x[n]|r^{-n} < \infty$$

- The ROC does not contain any poles.

  $X(z)$ is infinite at a pole

The region of convergence for z-transforms

Properties

- If $x[n]$ is of finite duration ($x[n] \neq 0$ for $N_1 < n < N_2$), then the ROC is the entire z-plane, except possibly $z = 0$ and/or $z = \infty$

If $N_1 < 0$ and $N_2 > 0$

ROC does not include $z = 0$ or $z = \infty$

If $N_1 \geq 0$,

ROC includes $z = \infty$, not $z = 0$

If $N_2 \leq 0$,

ROC includes $z = 0$, not $z = \infty$

The region of convergence for z-transforms

Examples

$$\delta[n] \xleftrightarrow{\mathcal{Z}} \sum_{n=-\infty}^{+\infty} \delta[n] z^{-n} = 1$$

ROC = the entire z-plane

$$\delta[n-1] \xleftrightarrow{\mathcal{Z}} \sum_{n=-\infty}^{+\infty} \delta[n-1] z^{-n} = z^{-1}$$

ROC = the entire z-plane except $z = 0$

$$\delta[n+1] \xleftrightarrow{\mathcal{Z}} \sum_{n=-\infty}^{+\infty} \delta[n+1] z^{-n} = z$$

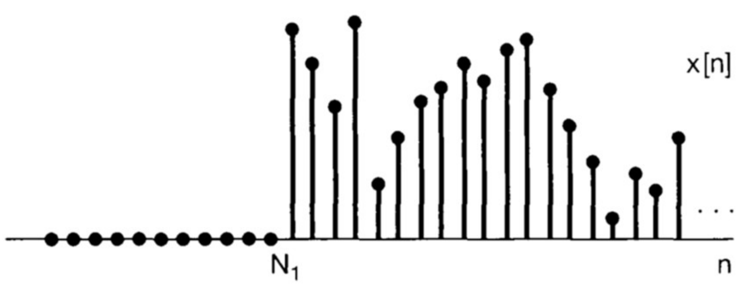
ROC = the entire finite z-plane (except $z = \infty$)

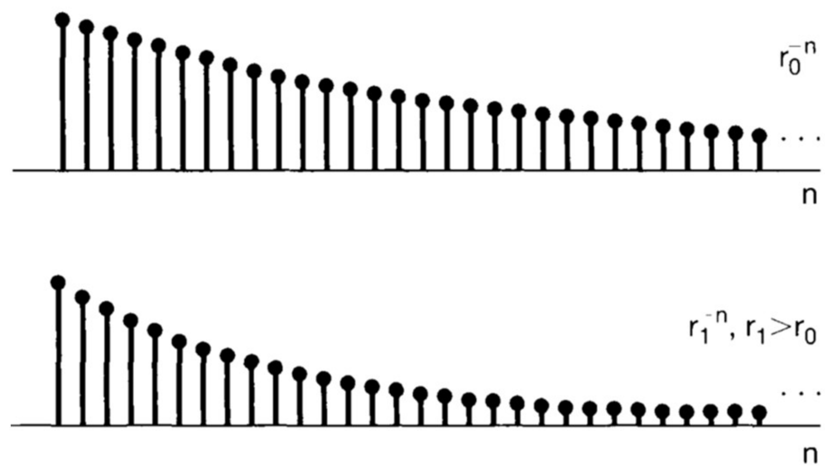
The region of convergence for z-transforms

Properties

- If $x[n]$ is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC.

Right-sided signal





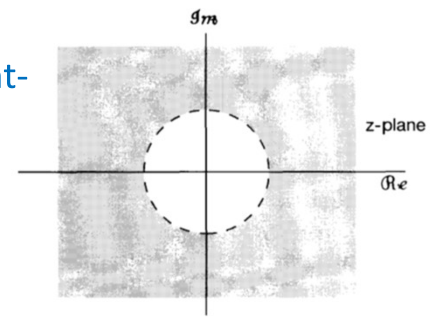
- If $x[n]$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $0 < |z| < r_0$ will also be in the ROC.

The region of convergence for z-transforms

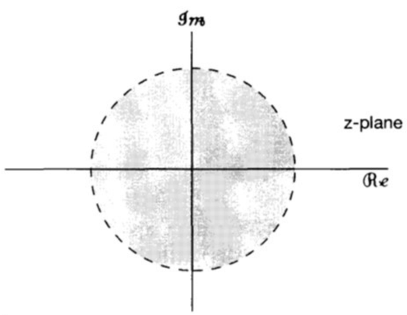
Properties

- If $x[n]$ is a two-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$.

ROC for right-sided signal

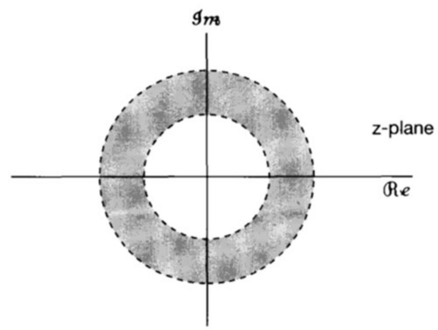


ROC for left-sided signal



$\Longrightarrow$

ROC for two-sided signal



The region of convergence for z-transforms

Examples

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1, a > 0 \\ 0 & otherwise \end{cases} \qquad X(z) = ?$$

Solution

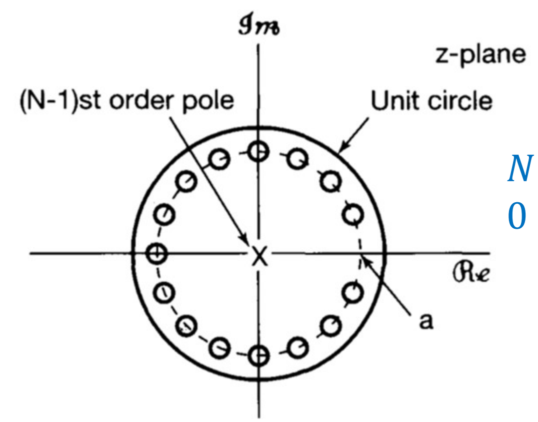
$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} \quad = \sum_{n=0}^{N-1} (az^{-1})^n \quad = \frac{1-(az^{-1})^N}{1-az^{-1}} \quad = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

The N roots of the numerator polynomial:

$$z_k = ae^{j\left(\frac{2\pi k}{N}\right)}, \qquad k = 0, 1, \cdots, N-1$$

When $k = 0$, the zero cancels the pole at $z = a$

$$z_k = ae^{j\left(\frac{2\pi k}{N}\right)}, \qquad k = 1, \cdots, N-1$$

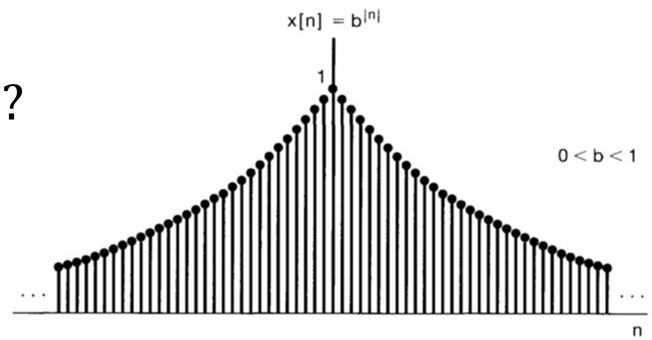


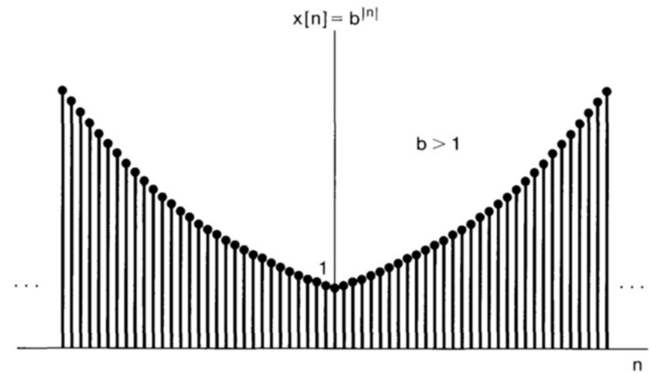
$N = 16$
$0 < a < 1$

The region of convergence for z-transforms

Examples

$$x[n] = b^{|n|}, b > 0 \qquad X(z) = ?$$



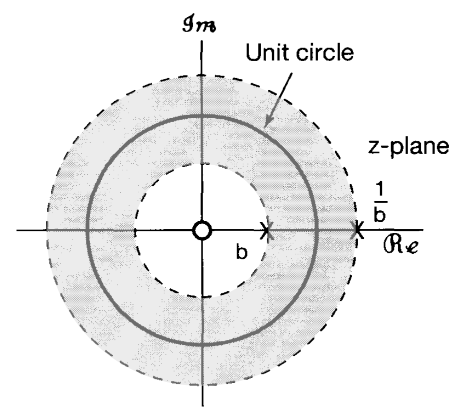


Solution

$$x[n] = b^n u[n] + b^{-n} u[-n-1]$$

$$b^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - bz^{-1}} \qquad |z| > b$$

$$b^{-n} u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{-1}{1 - b^{-1}z^{-1}} \qquad |z| < \frac{1}{b}$$



For convergence, $b < 1$

$$X(z) = \frac{1}{1 - bz^{-1}} - \frac{1}{1 - b^{-1}z^{-1}} \qquad b < |z| < \frac{1}{b}$$

The region of convergence for z-transforms

Properties

- If the z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity.

- If the z-transform $X(z)$ of $x[n]$ is rational, then if $x[n]$ is right-sided, the ROC is the region in the z-plane outside the outer-most pole.
  If $x[n]$ is causal, the ROC also includes $z = \infty$.

- If the z-transform $X(z)$ of $x[n]$ is rational, then if $x[n]$ is left-sided, the ROC is the region in the z-plane inside the inner-most nonzero pole.
  If $x[n]$ is anti-causal, the ROC also includes $z = 0$.

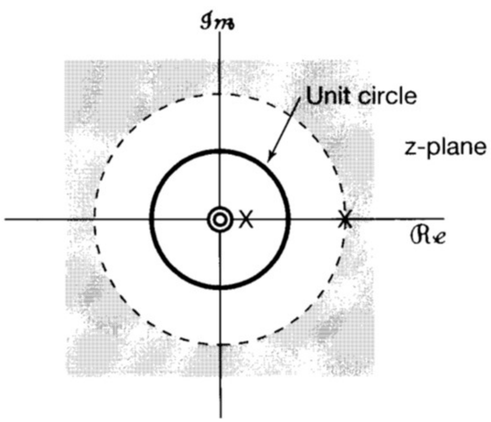
The region of convergence for z-transforms

Examples

$$X(z) = \frac{1}{\left(1 - \frac{1}{3}z^{-1}\right)(1 - 2z^{-1})}$$

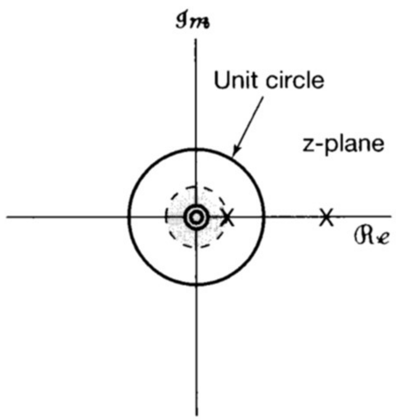
ROC ?

Solution



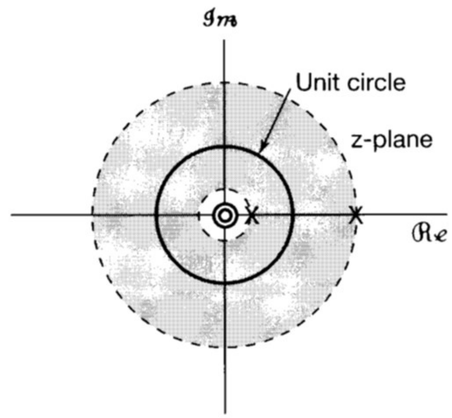
Right-sided sequence

Has no FT



Left-sided sequence

Has no FT



Two-sided sequence

FT converges

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The inverse z-transform

$$X\left(re^{j\omega}\right) = \mathcal{F}\{x[n]r^{-n}\}$$

$$x[n]r^{-n} = \mathcal{F}^{-1}\left\{X\left(re^{j\omega}\right)\right\} = \frac{1}{2\pi}\int_{2\pi} X\left(re^{j\omega}\right)e^{j\omega n}d\omega$$

$$x[n] = r^n \frac{1}{2\pi}\int_{2\pi} X\left(re^{j\omega}\right)e^{j\omega}\ \ d\omega = \frac{1}{2\pi}\int_{2\pi} X\left(re^{j\omega}\right)\left(re^{j\omega}\right)^n d\omega$$

$z = re^{j\omega}$

$dz = jre^{j\omega}d\omega = jzd\omega$

$$\boxed{x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz}$$

The inverse z-transform

Examples

$$X(z)=\frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)},\qquad |z|>\frac{1}{3}\qquad\qquad x[n]=?$$

Solution

$$X(z)=\frac{1}{1-\frac{1}{4}z^{-1}}+\frac{2}{1-\frac{1}{3}z^{-1}}$$

$$x[n]=x_1[n]+x_2[n]$$

$$\left.\begin{array}{lll} x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} \dfrac{1}{1-\frac{1}{4}z^{-1}} & & |z|>\dfrac{1}{4} \\ x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} \dfrac{2}{1-\frac{1}{3}z^{-1}} & & |z|>\dfrac{1}{3} \end{array}\right\} \Longrightarrow \quad x[n]=\left(\frac{1}{4}\right)^n u[n]+2\left(\frac{1}{3}\right)^n u[n]$$

The inverse z-transform

Examples

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}, \qquad \frac{1}{4} < |z| < \frac{1}{3} \qquad x[n] = ?$$

Solution

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}$$

$$x[n] = x_1[n] + x_2[n]$$

$$\left.\begin{aligned} x_1[n] &\xleftrightarrow{\mathcal{Z}} \frac{1}{1 - \frac{1}{4}z^{-1}} \quad |z| > \frac{1}{4} \\ x_2[n] &\xleftrightarrow{\mathcal{Z}} \frac{2}{1 - \frac{1}{3}z^{-1}} \quad |z| < \frac{1}{3} \end{aligned}\right\} \Longrightarrow x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$

The inverse z-transform

Examples

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}, \qquad |z| < \frac{1}{4} \qquad x[n] = ?$$

Solution

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}$$

$$x[n] = x_1[n] + x_2[n]$$

$$\left.\begin{aligned} x_1[n] &\overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{4}z^{-1}} \quad |z| < \frac{1}{4} \\ x_2[n] &\overset{\mathcal{Z}}{\longleftrightarrow} \frac{2}{1 - \frac{1}{3}z^{-1}} \quad |z| < \frac{1}{3} \end{aligned}\right\} \Longrightarrow x[n] = -\left(\frac{1}{4}\right)^n u[-n-1] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$

The inverse z-transform

Examples

$$X(z) = 4z^2 + 2 + 3z^{-1}, \qquad 0 < |z| < \infty \qquad x[n] = ?$$

Solution 1

$$x[n] = \begin{cases} 4, & n = -2 \\ 2, & n = 0 \\ 3, & n = 1 \\ 0, & otherwise \end{cases}$$

$$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$$

Solution 2

$$\delta[n + n_0] \xleftrightarrow{\mathcal{Z}} z^{n_0}$$

$$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$$

The inverse z-transform

Examples

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a| \qquad x[n] = ?$$

Solution

If $|z| > |a|$,

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots$$

$$x[n] = a^n u[n]$$

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \hline 1 - az^{-1} & 1 \\ & \underline{1 - az^{-1}} \\ & \quad az^{-1} \\ & \quad \underline{az^{-1} - a^2z^{-2}} \\ & \qquad\quad a^2z^{-2} \end{array}$$

If $|z| < |a|$,

$$\frac{1}{1 - az^{-1}} = -a^{-1}z - a^{-2}z^2 + \cdots$$

$$x[n] = -a^n u[-n-1]$$

$$\begin{array}{r|l} & -a^{-1}z - a^{-2}z^2 - \cdots \\ \hline -az^{-1} + 1 & 1 \\ & \underline{1 - a^{-1}z} \\ & \qquad a^{-1}z \end{array}$$

The inverse z-transform

Examples

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a| \qquad x[n] = ?$$

Solution

$$\log(1 + v) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} v^n}{n}, \qquad |v| < 1$$

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

$$x[n] = \begin{cases} (-1)^{n+1} a^n / n & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

$$= -\frac{(-a)^n}{n} u[n-1]$$

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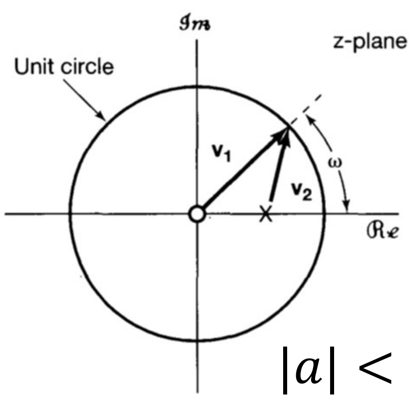
First-order systems

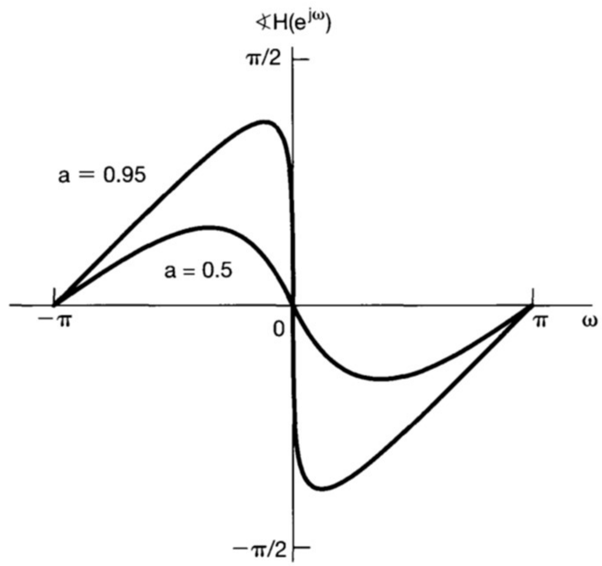
Consider $h[n] = a^n u[n]$

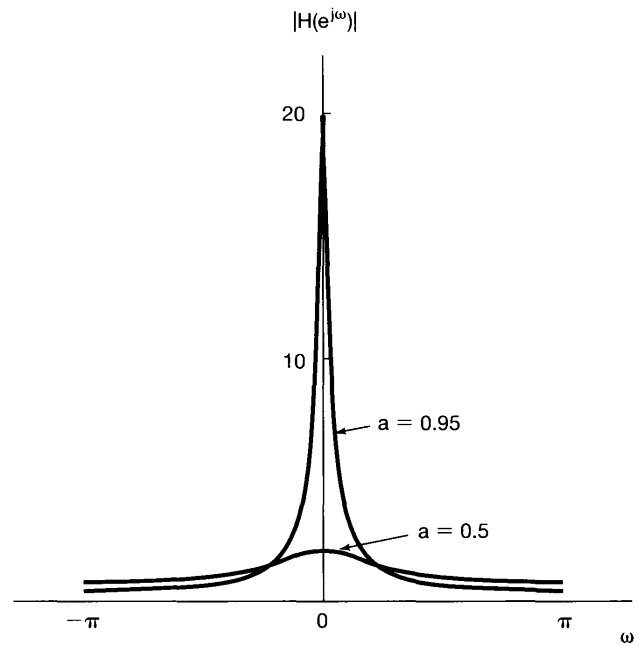
$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|$$

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$



$|a| < 1$





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Properties of the z-transform

Linearity

$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) \quad \mathrm{ROC} = R1$$

$$\Longrightarrow \quad \boxed{ax_1[n]+bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z)}$$

with ROC containing $R_1 \cap R_2$

$$x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z) \quad \mathrm{ROC} = R2$$

Time shifting

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \quad \mathrm{ROC} = R$$

$$\Downarrow$$

$$x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z)$$

$\mathrm{ROC} = R$ except for the possible addition or deletion of the origin or infinity

Properties of the z-transform

Scaling in the z-domain

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z) \qquad \text{ROC} = R$$

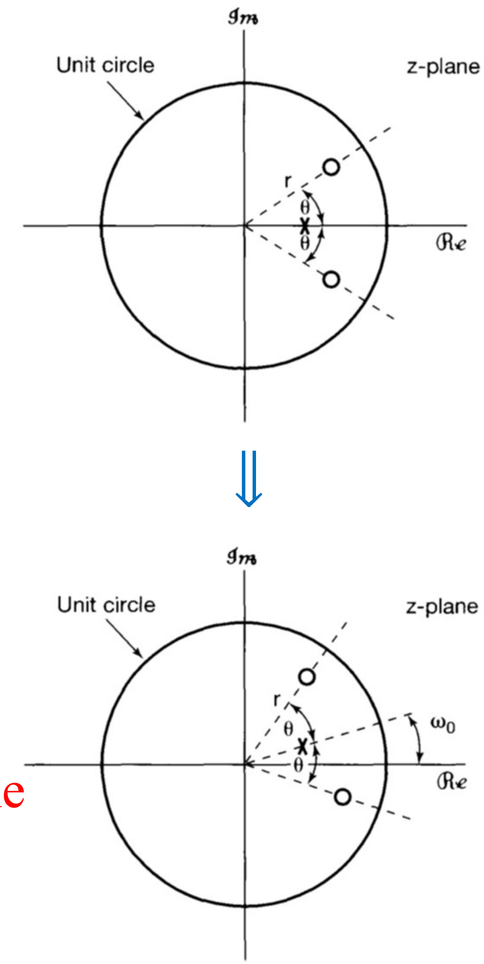
$$\Downarrow$$

$$z_0^n x[n] \xleftrightarrow{\mathcal{Z}} X(z/z_0) \qquad \text{ROC} = |z_0|R$$

$$\Downarrow \; z_0 = e^{j\omega_0}$$

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{Z}} X\left(e^{-j\omega_0} z\right) \qquad \text{ROC} = R$$

Multiplication by $e^{j\omega_0 n}$ $\Leftrightarrow$ Rotation by ω_0 in the Z-plane



Properties of the z-transform

Time reversal

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z) \qquad \text{ROC} = R$$

$$\Downarrow$$

$$x[-n] \xleftrightarrow{\mathcal{Z}} X\left(\frac{1}{z}\right) \qquad \text{ROC} = \frac{1}{R}$$

Time expansion

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k \\ 0 & \text{if } n \text{ is not a multiple of } k \end{cases}$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z) \qquad \text{ROC} = R$$

$$\Downarrow$$

$$x_{(k)}[n] \xleftrightarrow{\mathcal{Z}} X(z^k) \qquad \text{ROC} = R^{1/k}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$\Downarrow$$

$$X(z^k) = \sum_{n=-\infty}^{+\infty} x[n]z^{-kn}$$

Properties of the z-transform

Conjugation

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z) \qquad \mathrm{ROC} = R$$

$$\Downarrow$$

$$x^*[n] \xleftrightarrow{\mathcal{Z}} X^*(z^*) \qquad \mathrm{ROC} = R$$

Convolution

$$x_1[n] \xleftrightarrow{\mathcal{Z}} X_1(z) \qquad \mathrm{ROC} = R_1$$

$$x_2[n] \xleftrightarrow{\mathcal{Z}} X_2(z) \qquad \mathrm{ROC} = R_2$$

$$\Longrightarrow \quad x_1[n] * x_2[n] \xleftrightarrow{\mathcal{Z}} X_1(z)X_2(z)$$

with ROC contains $R_1 \cap R_2$

Properties of the z-transform

First-difference

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z) \qquad \mathrm{ROC} = R$$

$$x[n] - x[n-1] \xleftrightarrow{\mathcal{Z}} (1 - z^{-1})X(z)$$ ROC $= R$, possible deletion of $z = 1$ and/or addition of $z = 0$

Accumulation

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z) \qquad \mathrm{ROC} = R$$

$$w[n] = \sum_{k=-\infty}^{n} x[k] \xleftrightarrow{\mathcal{Z}} \frac{1}{(1 - z^{-1})} X(z)$$ ROC contains $R \cap \{|z| > 1\}$

Properties of the z-transform

Differentiation in the z-domain

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z) \qquad \mathrm{ROC} = R$$

$$\Downarrow$$

$$nx[n] \xleftrightarrow{\mathcal{Z}} -\mathrm{z}\frac{dX(z)}{dz} \qquad \mathrm{ROC} = R$$

Properties of the z-transform

Examples

$$X(z) = \log(1 + az^{-1}) \qquad |z| > |a| \qquad x[n] = ?$$

Solution

$$nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1 + az^{-1}} \qquad |z| > |a|$$

$$a(-a)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{a}{1 + az^{-1}} \qquad |z| > |a|$$

$$a(-a)^{n-1} u[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{1 + az^{-1}} \qquad |z| > |a|$$

$$x[n] = -\frac{(-a)^n}{n} u[n-1]$$

Properties of the z-transform

Examples

$$X(z) = \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z| > |a| \qquad x[n] = ?$$

Solution

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$na^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z| > |a|$$

Properties of the z-transform

The initial-value theorem

If

$$x[n] = 0 \text{ for } n < 0,$$

Then,

$$x[0] = \lim_{z\to\infty} X(z)$$

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

For $n > 0$, $z \to \infty \implies z^{-n} \to 0$

For $n = 0$, $z^{-n} = 1$

❑ Examples

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

$$X(z) = \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}}$$

$\implies$

$$x(0) = 1$$

$$\lim_{z\to\infty} X(z) = 1$$

Properties of the z-transform

Summary

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
10.5.9	Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z\to\infty} X(z)$			

The z-Transform
(ch.10)

- The z-transform
- The region of convergence for the z-transforms
- The inverse z-transform
- Geometric evaluation of the Fourier transform from the pole-zero plot
- Properties of the z-transform
- **Some common z-transform pairs**
- Analysis and characterization of LTI systems using z-transforms
- System function algebra and block diagram representations
- The unilateral z-transform

Some z-transform pairs

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

The z-Transform
(ch.10)

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- ❑ Properties of the z-transform
- ❑ Some common z-transform pairs
- ❑ **Analysis and characterization of LTI systems using z-transforms**
- ❑ System function algebra and block diagram representations
- ❑ The unilateral z-transform

Causality

Causal $\Longleftrightarrow$ ROC of $H(z)$ is the exterior of a circle, including infinity

A system with rational $H(z)$ is causal $\Longleftrightarrow$

- ROC is the exterior of a circle outside the outermost pole;
- With $H(z)$ expressed as a ratio of polynomials in z, the order of the numerator cannot be greater than the order of the denominator.

Examples

$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}$$ Noncausal

Examples

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}} \qquad |z| > 2$$

Solution 1

$|z| > 2$: ROC is the exterior of a circle outside the outermost pole.

$$H(z) = \frac{2 - \frac{5}{2}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})} = \frac{2z^2 - \frac{5}{2}z}{z^2 - \frac{5}{2}z + 1}$$

$\Longrightarrow$ Causal

Solution 2

$h[n] = [(1/2)^n + 2^n]u[n] \quad \Longrightarrow \quad h[n] = 0$ for $n < 0 \quad \Longrightarrow$ Causal

Stability

For an LTI system,

Stable $\Leftrightarrow$ The ROC of $H(z)$ includes the unit circle, $|z| = 1$

❑ Examples

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}$$

ROC	Causal	Stable
$\lvert z\rvert > 2$	Yes	No
$1/2 < \lvert z\rvert < 2$	No	Yes
$\lvert z\rvert < 1/2$	No	No

Stability

For a causal LTI system with rational system function $H(z)$,

Stable $\Leftrightarrow$ All of the poles of $H(z)$ lie inside the unit circle. (magnitude smaller than 1)

❑ Examples

$$H(z) = \frac{1}{1 - az^{-1}} \text{ is stable} \quad \Longrightarrow \quad |a| < 1$$

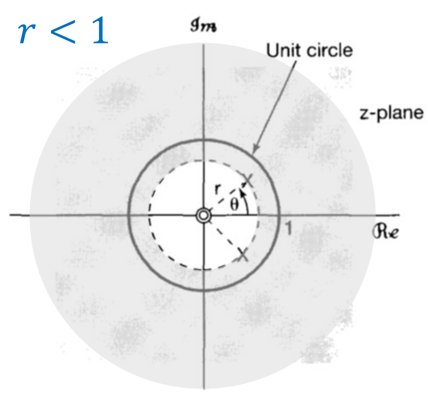
❑ Examples

$$H(z) = \frac{1}{1 - (2r\cos\theta)z^{-1} + r^2z^{-2}}$$

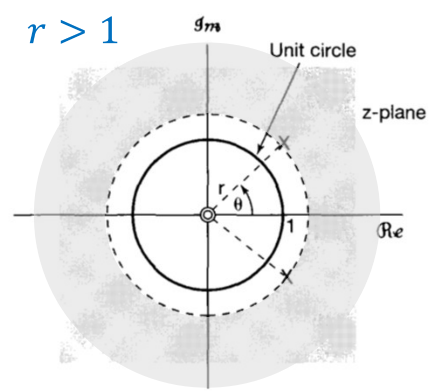
Poles: $z_1 = re^{j\theta}$ $z_2 = re^{-j\theta}$

Stable $\Longrightarrow$ $r < 1$

$r < 1$



$r > 1$



LTI systems characterized by linear constant-coefficient difference equations

❑ Examples

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

$$Y(z) = X(z)\left[\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \left[\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}\right] \Longrightarrow \begin{cases} |z| > \frac{1}{2} & h[n] = \left(\frac{1}{2}\right)^{n} u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1] \\ |z| < \frac{1}{2} & h[n] = -\left(\frac{1}{2}\right)^{n} u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n] \end{cases}$$

LTI systems characterized by linear constant-coefficient difference equations

❑ In general

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(s)\sum_{k=0}^{M} b_k z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad \Longrightarrow \qquad \begin{cases} \text{Poles at the solution of } \displaystyle\sum_{k=0}^{N} a_k z^{-k} = 0 \\ \text{Zeros at the solution of } \displaystyle\sum_{k=0}^{M} b_k z^{-k} = 0 \end{cases}$$

Examples relating system behavior to the system function

Given the following information about an LTI system, $H(z) =?\ \ h[n] =?$

- If $x_1[n] = (1/6)^n u[n]$, then $y_1[n] = \left[a\left(\frac{1}{2}\right)^n + 10\left(\frac{1}{3}\right)^n\right]u[n]$
- If $x_2[n] = (-1)^n$, then $y_2[n] = \frac{7}{4}(-1)^n$

Solution

$$X_1(z) = \frac{1}{1-\frac{1}{6}z^{-1}}, |z| > \frac{1}{6}$$

$$Y_1(z) = \frac{a}{1-\frac{1}{2}z^{-1}} + \frac{10}{1-\frac{1}{3}z^{-1}} = \frac{(a+10) - \left(5+\frac{a}{3}\right)z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}, |z| > \frac{1}{2}$$

$$H(z) = \frac{Y_1(z)}{X_1(z)} = \frac{\left[(a+10) - \left(5+\frac{a}{3}\right)z^{-1}\right]\left(1-\frac{1}{6}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)},$$

Examples relating system behavior to the system function

Solution continue

$$\frac{7}{4} = H(-1) = \frac{\left[(a+10) + \left(5 + \frac{a}{3}\right)\right]\left(\frac{7}{6}\right)}{\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)} \quad \Longrightarrow \quad a = -9$$

$$H(z) = \frac{(1-2z^{-1})\left(1-\frac{1}{6}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

ROC of $X_1(z)$: $|z| > \frac{1}{6}$ $\quad \Longrightarrow \quad$ ROC of $H(z)$: $|z| > \frac{1}{2}$

Examples relating system behavior to the system function

Consider a stable and causal system with impulse response $h[n]$ and rational system function $H(z)$, which contains a pole at $z = 1/2$ and a zero somewhere on the unit circle.

- ❑ $\mathcal{F}\{(1/2)^n h[n]\}$ converges. True
- ❑ $H(e^{j\omega}) = 0$ for some ω True
- ❑ $h[n]$ has finite duration False
- ❑ $h[n]$ is real Insufficient information
- ❑ $g[n] = n\big[h[n] * h[n]\big]$ is the impulse response of a stable system True

The z-Transform
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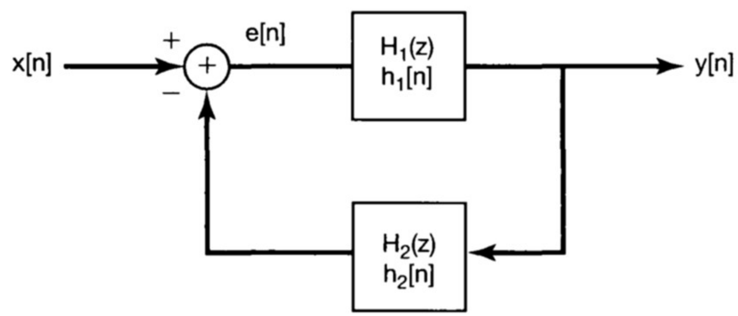
System functions for interconnections of LTI systems

$$\frac{Y(z)}{X(z)} = H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

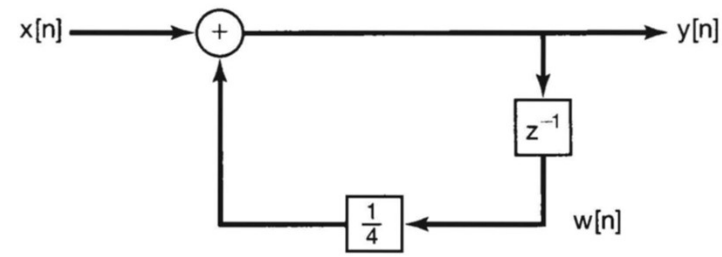


Block diagram representations for causal LTI systems

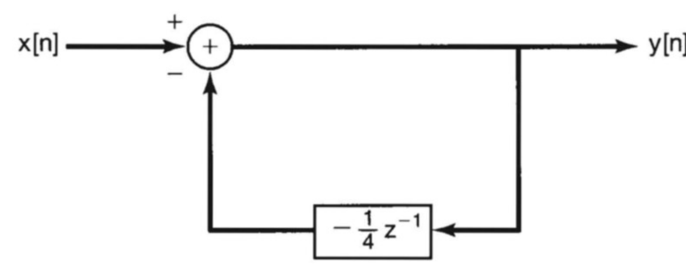
$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

$$y[n] - \frac{1}{4}y[n-1] = x[n]$$

$$w[n] = y[n-1]$$



Or equivalently

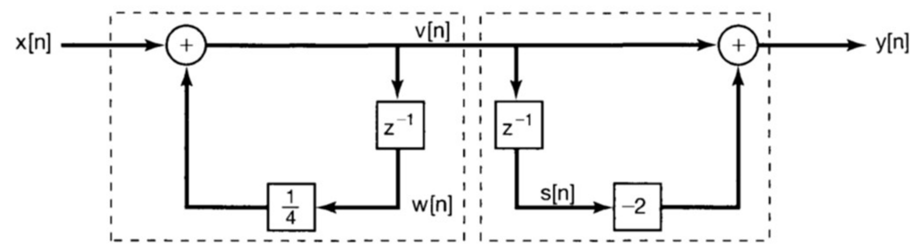


Examples: block diagram representations for causal LTI systems

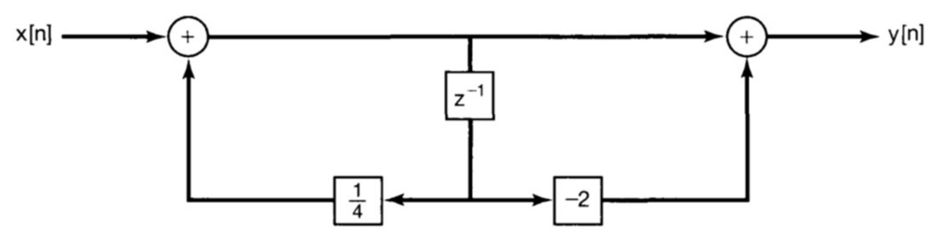
$$H(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}} = \left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right)(1 - 2z^{-1})$$

$$y[n] = v[n] - 2v[n-1]$$

$$w[n] = s[n] = v[n-1]$$



Or equivalently



System function algebra and block diagram representations

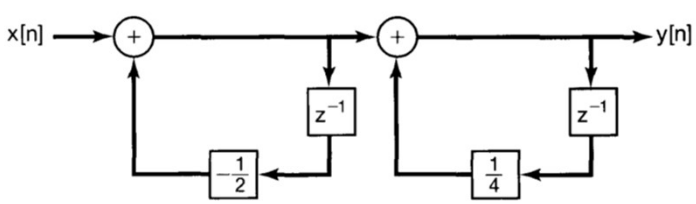
Examples: block diagram representations for causal LTI systems

$$H(z)=\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}=\frac{1}{\left(1+\frac{1}{2}z^{-1}\right)}\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)}=\frac{2/3}{\left(1+\frac{1}{2}z^{-1}\right)}+\frac{1/3}{\left(1-\frac{1}{4}z^{-1}\right)}$$

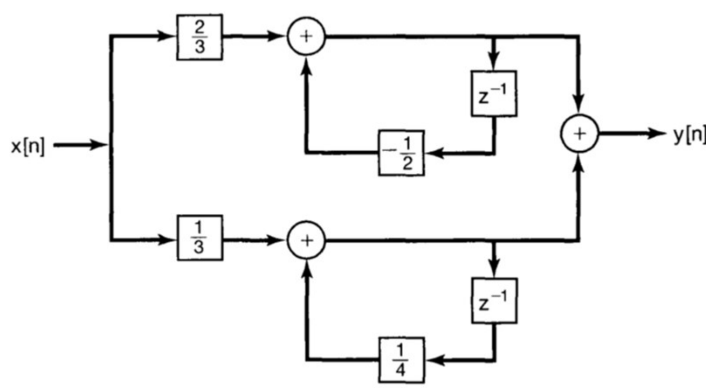
$$y[n]+\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2]=x[n]$$

$$y[n]=-\frac{1}{4}y[n-1]+\frac{1}{8}y[n-2]+x[n]$$

Cascade form

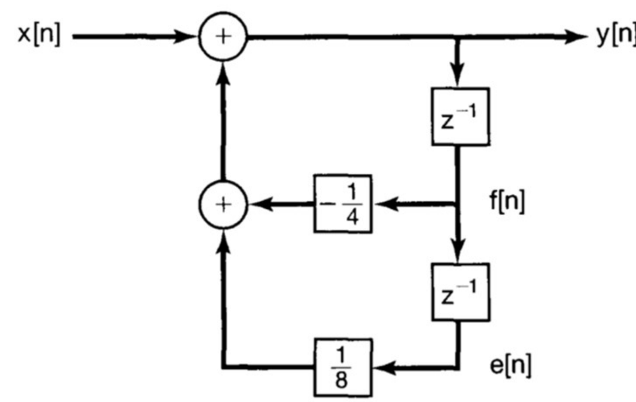


Parallel form



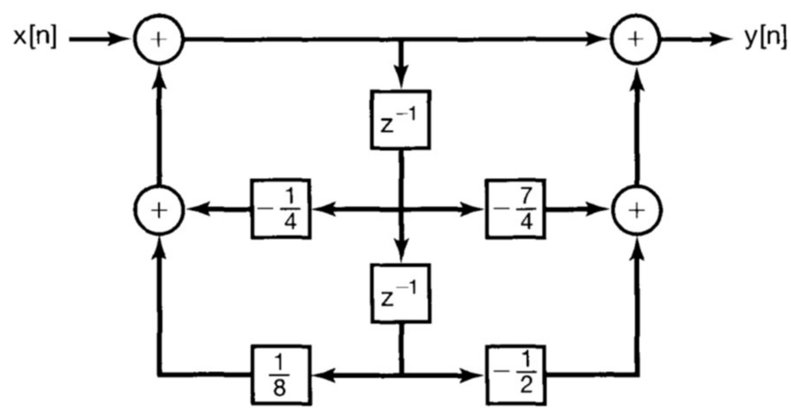
Direct form

$f[n]=y[n-1]$
$e[n]=f[n-1]=y[n]$



Examples: block diagram representations for causal LTI systems

$$H(z)=\frac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}=\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}\left(1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}\right)$$



The z-Transform
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The unilateral Laplace transform

$$x[n] \xleftrightarrow{\mathcal{UZ}} \mathcal{X}(z) = \mathcal{U}\mathfrak{L}\{x[n]\}$$

$$\mathcal{X}(z) \triangleq \sum_{n=0}^{\infty} x[n]z^{-n}$$

Examples

$$x[n] = a^n u[n]$$

$x[n] = 0$, for $n < 0$

$$\mathcal{X}(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

Examples

$$x[n] = a^{n+1}u[n+1]$$

$$X(z) = \frac{z}{1 - az^{-1}}, \qquad |z| > |a|$$

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} a^{n+1} z^{-n} = \frac{a}{1 - az^{-1}}, \qquad |z| > |a|$$

Not equal
$(x[-1] \neq 0)$

Examples

$$\mathcal{X}(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$$

Solution

$$\mathcal{X}(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}$$

$$x[n] = x_1[n] + x_2[n]$$

$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{4}z^{-1}} \qquad |z| > \frac{1}{4}$$

$$x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{2}{1 - \frac{1}{3}z^{-1}} \qquad |z| > \frac{1}{3}$$

$$\Longrightarrow \quad x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n], \qquad n \geq 0$$

Properties of the unilateral Laplace transform

Property	Signal	Unilateral z-Transform
—	$x[n]$	$\mathcal{X}(z)$
—	$x_1[n]$	$\mathcal{X}_1(z)$
—	$x_2[n]$	$\mathcal{X}_2(z)$
Linearity	$ax_1[n] + bx_2[n]$	$a\mathcal{X}_1(z) + b\mathcal{X}_2(z)$
Time delay	$x[n-1]$	$z^{-1}\mathcal{X}(z) + x[-1]$
Time advance	$x[n+1]$	$z\mathcal{X}(z) - zx[0]$
Scaling in the z-domain	$e^{j\omega_0 n}x[n]$ $z_0^n x[n]$ $a^n x[n]$	$\mathcal{X}(e^{-j\omega_0}z)$ $\mathcal{X}(z/z_0)$ $\mathcal{X}(a^{-1}z)$
Time expansion	$x_k[n] = \begin{cases} x[m], & n = mk \\ 0, & n \neq mk \end{cases}$ for any m	$\mathcal{X}(z^k)$
Conjugation	$x^*[n]$	$\mathcal{X}^*(z^*)$
Convolution (assuming that $x_1[n]$ and $x_2[n]$ are identically zero for $n < 0$)	$x_1[n] * x_2[n]$	$\mathcal{X}_1(z)\mathcal{X}_2(z)$
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})\mathcal{X}(z) - x[-1]$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\frac{1}{1-z^{-1}}\mathcal{X}(z)$
Differentiation in the z-domain	$nx[n]$	$-z\frac{d\mathcal{X}(z)}{dz}$

Initial Value Theorem

$$x[0] = \lim_{z\to\infty} \mathcal{X}(z)$$

Convolution Examples

A causal LTI system, initial rest condition

$$y[n] + 3y[n-1] = x[n] \qquad x[n] = \alpha u[n] \qquad y[n] = ?$$

Solution

$$\mathcal{H}(z) = \frac{1}{1+3z^{-1}}$$

$$\mathcal{Y}(z) = \mathcal{H}(z)\mathcal{X}(z) = \frac{\alpha}{(1+3z^{-1})(1-z^{-1})} = \frac{(3/4)\alpha}{1+3z^{-1}} + \frac{(1/4)\alpha}{1-z^{-1}}$$

$$y[n] = \alpha\left[\frac{1}{4} + \left(\frac{3}{4}\right)(-3)^n\right]u[n]$$

Shifting property

$$x[n+1] \xleftrightarrow{\mathcal{UZ}} z\mathcal{X}(z) - zx[0]$$

$$x[n-1] \xleftrightarrow{\mathcal{UZ}} z^{-1}\mathcal{X}(z) + x[-1]$$

$$x[n-2] \xleftrightarrow{\mathcal{UZ}} z^{-2}\mathcal{X}(z) + z^{-1}x[-1] + x[-2]$$

Consider $y[n] = x[n-1]$:

$$\begin{aligned}
\mathcal{Y}(z) &= \sum_{n=0}^{\infty} x[n-1]z^{-n} \\
&= x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n} \\
&= x[-1] + \sum_{n=0}^{\infty} x[n]z^{-(n+1)} \\
&= x[-1] + z^{-1}\mathcal{X}(z)
\end{aligned}$$

Solving differential equations using the unilateral z-transform

$y[n] + 3y[n-1] = x[n] \qquad x[n] = \alpha u[n] \qquad y[-1] = \beta$

$y[n] = ?$

Solution

$$\mathcal{Y}(z) + 3\beta + 3z^{-1}\mathcal{Y}(z) = \frac{\alpha}{1 - z^{-1}}$$

$$\mathcal{Y}(z) = \underbrace{\boxed{-\frac{3\beta}{1+3z^{-1}}}}_{\text{Zero-input response}} + \underbrace{\boxed{\frac{\alpha}{(1+3z^{-1})(1-z^{-1})}}}_{\text{Zero-state response}}$$

If $\alpha = 8, \beta = 1, y[n] = [3(-3)^n + 2]u[n]$, for $n \geq 0$