Discrete Mathematics: Homework 2

(Deadline: 8:00am, March 4, 2022)

1. (20 points) Let $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Show that $\left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor$.

   (Hint: division algorithm)

2. (20 points) Let $a, b \in \mathbb{Z}, n \in \mathbb{Z}^+$ and $a \equiv b \pmod{n}$. Let $c_0, c_1, \ldots, c_k \in \mathbb{Z}$, where $k \in \mathbb{Z}^+$. Show that $c_0 + c_1 a + \cdots + c_k a^k \equiv c_0 + c_1 b + \cdots + c_k b^k \pmod{n}$.

   (Hint: show that $a^i - b^i$ is a multiple of n)

3. (20 points) Let x, y, z be integers such that $x^2 + y^2 = 3z^2$. Show that x, y, z must be all even. Based on this result, show that the equation $x^2 + y^2 = 3z^2$ has no other integer solutions except $(x, y, z) = (0, 0, 0)$.

4. (20 points) Let p be an odd prime and let $\mathbb{Z}_p^* = \{[1]_p, [2]_p, \ldots, [p-1]_p\}$.

   (1) Show that $([a]_p)^2 = [1]_p$ if and only if $[a]_p \in \{[1]_p, [p-1]_p\}$.

   (2) Show that $[1]_p \cdot [2]_p \cdots [p-1]_p = [-1]_p$ and thus conclude that $(p-1)! \equiv -1 (\bmod\ p)$. (This is called **Wilson's Theorem**.)

   (Hint: partition the elements of $\mathbb{Z}_p^*$ as $(p+1)/2$ subsets of the form $\{\alpha, \alpha^{-1}\}$)

5. (20 points) Let p be a prime and $p \notin \{2, 5\}$. Show that p divides infinitely many elements of the set $\{9, 99, 999, 9999, 99999, \ldots\}$.

   (Hint: consider $([10]_p)^{p-1}$)