9.1.5

证明 记 $\lim_{n\to\infty} M_n = M$, 则对 $\varepsilon = 1, \exists N \in \mathbb{N}^*$, 使得当 $n > N$ 时, 有

$$\rho(O, M_n) - \rho(O, M) \leqslant \rho(M_n, M) \leqslant 1 \implies \rho(O, M_n) \leqslant \rho(O, M) + 1,$$

取 $\rho_M = \max\{\rho(O, M_1), \rho(O, M_2), \cdots, \rho(O, M_N), \rho(O, M) + 1\}$, 则对 $\forall n \in \mathbb{N}^*$, 有 $\rho(O, M_n) \leqslant \rho_M$, 故点列 $\{M_n\}$ 有界. □

9.1.11 ~~略~~

9.1.12 $x+y=2,\ \dfrac{y}{x}=3 \quad x=\dfrac{1}{2}\ y=\dfrac{3}{2}$

$f(2,3)=\dfrac{1}{4}-\dfrac{9}{4}=-2$

$x+y=a,\ \dfrac{y}{x}=b$ 得 $x=\dfrac{a}{1+b} \quad y=\dfrac{ab}{1+b}$

$f(x,y)=\dfrac{a^2(1-b^2)}{(1+b)^2}$

9.1.13 设 $f(x, y) = x^y, \varphi(x, y) = x + y, \psi(x, y) = x - y$, 求

$$f[\varphi(x, y), \psi(x, y)], \quad \varphi[f(x, y), \psi(x, y)], \quad \psi[\varphi(x, y), f(x, y)].$$

解 计算得:

$$\begin{aligned} f[\varphi(x, y), \psi(x, y)] &= (x + y)^{x-y}, \\ \varphi[f(x, y), \psi(x, y)] &= x^y + x - y, \\ \psi[\varphi(x, y), f(x, y)] &= x + y - x^y. \end{aligned}$$

9.1.14

解 (1) 对 $\forall \varepsilon > 0$, 取 $\delta = \varepsilon$, 则当 $|x| < \delta, |y| < \delta$ 且 $(x, y) \neq (0, 0)$ 时, 有

$$\begin{cases} x^2 \leqslant \delta\,|x|\,, \\ y^2 \leqslant \delta\,|y| \end{cases} \implies \frac{x^2 + y^2}{|x| + |y|} \leqslant \frac{\delta(|x| + |y|)}{|x| + |y|} = \delta = \varepsilon,$$

故 $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2 + y^2}{|x| + |y|} = 0$.

另解 设 $\begin{cases} x = r\cos\theta, \\ y = r\sin\theta \end{cases}$ $(0 \leqslant \theta < 2\pi)$, 则有 $|\cos\theta| + |\sin\theta| \geqslant 1$.

对 $\forall \varepsilon > 0$, 取 $\delta = \varepsilon$, 则当 $r = \rho(O, (x, y)) < \delta$ 时, 有

$$\frac{x^2 + y^2}{|x| + |y|} = \frac{r}{|\cos\theta| + |\sin\theta|} \leqslant r < \delta = \varepsilon,$$

故 $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2 + y^2}{|x| + |y|} = 0$.

(5) 书P56例9.1.2，极限为0

(6) $x=r\cos\theta$，$y=r\sin\theta$

$$\frac{x^2+y^2}{x^4+y^4}=\frac{1}{r^2}\frac{1}{\sin^4\theta+\cos^4\theta}<\frac{1}{r^2}\times\frac{1}{2}$$ 因为 $\sin^4\theta+\cos^4\theta=\frac{1}{4}(\cos(4\theta)+3)$

易知 $\lim\limits_{\substack{x\to\infty\\y\to\infty}}\frac{x^2+y^2}{x^4+y^4}=0$

(10) 原式 $=\lim\limits_{\substack{x\to0\\y\to0}}\frac{xy}{(x+y)(\sqrt{xy+1}+1)}=\lim\limits_{\substack{x\to0\\y\to0}}\frac{xy}{2(x+y)}$

$y=x\to0$

$y=x^2-x\to\frac{1}{2}$

极限不存在

(11) $x=r\cos\theta$ $y=r\sin\theta$

原式 $=\frac{1-\cos r^2}{r^4\sin\theta\cos\theta}$ 取 $\theta=\frac{\pi}{4}$ 可知其显然发散

(12) 令 $y=x$, 则 $y\to0\ (x\to0)$,

$$\lim_{\substack{x\to0\\y\to0}}(1+xy)^{\frac{1}{x+y}}=\lim_{x\to0}(1+x^2)^{\frac{1}{2x}}=\lim_{x\to0}(1+x^2)^{\frac{1}{x^2}\cdot\frac{x}{2}}=1,$$

又令 $y=x^2-x$, 同样满足 $y\to0\ (x\to0)$, 此时

$$\lim_{\substack{x\to0\\y\to0}}(1+xy)^{\frac{1}{x+y}}=\lim_{x\to0}[1+x(x^2-x)]^{\frac{1}{x^2}}=\frac{1}{\mathrm{e}},$$

由式 (9.1)(9.2) 知, $\lim\limits_{\substack{x\to0\\y\to0}}(1+xy)^{\frac{1}{x+y}}$ 不存在.

9.1.16 证明: 当极限 $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=A$ 存在时,

(1) 若 $y\neq y_0$ 时, $\lim\limits_{x\to x_0} f(x,y)$ 存在, 则 $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)=A$;

(2) 若 $x\neq x_0$ 时, $\lim\limits_{y\to y_0} f(x,y)$ 存在, 则 $\lim\limits_{x\to x_0}\lim\limits_{y\to y_0} f(x,y)=A$.

提示 对 $y\neq y_0$, 记 $\lim\limits_{x\to x_0} f(x,y)=l(y)$.

证明 (1) 记 $\lim\limits_{x\to x_0} f(x,y)=l(y)\ (y\neq y_0)$. 则对 $\forall\varepsilon>0$, 由 $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=A$ 知,

$\exists\delta_1>0$, 使得当 $\begin{cases}|x-x_0|<\delta_1,\\ |y-y_0|<\delta_1,\\ (x,y)\neq(x_0,y_0)\end{cases}$ 时, 有

$$|f(x,y)-A|<\frac{\varepsilon}{2},$$

由 $\lim\limits_{x\to x_0} f(x,y)=l(y)\ (y\neq y_0)$ 知, $\exists\delta_2>0$, 使得当 $0<|x-x_0|<\delta_2$ 时, 有

$$|f(x,y)-l(y)|<\frac{\varepsilon}{2},$$

令 $\delta=\min\{\delta_1,\delta_2\}$, 取 $x'=x_0+\dfrac{\delta}{2}$, 则有

$$\begin{cases}|f(x',y)-A|<\dfrac{\varepsilon}{2},\\ |f(x',y)-l(y)|<\dfrac{\varepsilon}{2}\end{cases}$$

对 $\forall 0<|y-y_0|<\delta_1$ 成立, 此时

$$|l(y)-A|\leqslant|f(x',y)-A|+|f(x',y)-l(y)|<\varepsilon,$$

这正是

$$\lim_{y\to y_0}\lim_{x\to x_0} f(x,y)=A.$$

(2) 同理可证. □

9.1.17

解 (2) 显然, 函数在 $y\neq 0$ 处连续, 当 $y=0$ 时,

1° $x_0\neq 0$, 则 $\lim\limits_{\substack{x\to x_0\\ y\to 0}} x\sin\dfrac{1}{y}$ 不存在, 函数 $f(x,y)$ 在 $(x_0,0)\ (x_0\neq 0)$ 处不连续;

2° $x_0=0$, 则 $\lim\limits_{(x,y)\to(0,0)} f(x,y)=\lim\limits_{(x,y)\to(0,0)} x\sin\dfrac{1}{y}=0=f(0,0)$, 故 $f(x,y)$ 在 $(0,0)$ 处连续.

综上, 函数 $f(x,y)$ 在 $(x,y)=(x_0,0)\ (x_0\neq 0)$ 处间断, 在其余点处连续.

(4) 显然, 函数在 $(x_0,y_0)\in\{(x,y):x+y\neq 0\}$ 处连续;

当 $x_0+y_0=0$ 时, 取 $(x_n,y_n)=\left(x_0+\dfrac{2}{n},y_0+\dfrac{1}{n}\right)$, 则有 $(x_n,y_n)\to(x_0,y_0)\ (n\to\infty)$,

1° $x_0\neq y_0$, 则 $\lim\limits_{n\to\infty} f(x_n,y_n)=\lim\limits_{n\to\infty}\dfrac{x_0-y_0+\frac{1}{n}}{\frac{3}{n}}$ 不存在;

2° $x_0=y_0=0$, 则 $\lim\limits_{n\to\infty} f(x_n,y_n)=\lim\limits_{n\to\infty}\dfrac{\frac{1}{n}}{\frac{3}{n}}=\dfrac{1}{3}\neq f(0,0)=0$,

故 $f(x,y)$ 在 $(x_0,y_0)\ (x_0+y_0=0)$ 处不连续.

综上, 函数 $f(x,y)$ 在点 $(x,y)\in\{(x,y)\in\mathbb{R}^2:x+y\neq 0\}$ 处连续, 在其余点处间断. □

(1) 在 $\{(x,y)\mid x\neq y\}$ 上显然连续

对于 $\{(x,y)\mid x=y\}$ 在 $x\neq 0$ 时易知其不连续

对于 $(0,0)$，$x=y+y^3$ 则 $\lim\limits_{(x,y)\to(0,0)}\frac{xy}{x-y}=\lim\limits_{(x,y)\to(0,0)}\frac{(y+y^3)y}{y^3}$ 极限不存在

$\therefore$ 在 $\{(x,y)\mid x=y\}$ 上不连续

(3) $x=r\cos\theta$ $y=r\sin\theta$ $f(x,y)=\frac{r^2\cos^2\theta\sin\theta}{r^2}$ $r<r$ 知其极限为0

$\therefore f(x,y)$ 在 R^2 上连续

9.1.18

$\lim\limits_{t\to 0}f(t\cos\alpha,t\sin\alpha)=0$ 易知

但由例 9.1.3 (P57) 知其极限不存在

证明 对 $\forall(x_0,y_0)\in D$, 往证 $f(x,y)$ 在 (x_0,y_0) 处连续.

对 $\forall\varepsilon>0$, 由于 $f(x_0,y)$ 在 y_0 处连续, 从而 $\exists\delta_1>0$, 使得

$$|f(x_0,y_0+\delta_1)-f(x_0,y_0-\delta_1)|<\frac{\varepsilon}{4}, \tag{9.3}$$

又函数 $f(x,y_0+\delta_1)$ 在 x_0 处连续, 故 $\exists\delta_2>0$, 使得当 $|x-x_0|<\delta_2$ 时, 有

$$|f(x,y_0+\delta_1)-f(x_0,y_0+\delta_1)|<\frac{\varepsilon}{4}, \tag{9.4}$$

同理, $\exists\delta_3>0$, 使得当 $|x-x_0|<\delta_3$ 时, 有

$$|f(x,y_0-\delta_1)-f(x_0,y_0-\delta_1)|<\frac{\varepsilon}{4}, \tag{9.5}$$

又由于 $f(x,y)$ 关于 y 单调, 从而当 $|y-y_0|<\delta_1$ 时, 有

$$|f(x,y)-f(x,y_0)|\leqslant|f(x,y_0+\delta_1)-f(x,y_0-\delta_1)|, \tag{9.6}$$

最后, 函数 $f(x,y_0)$ 在 x_0 处连续, 从而 $\exists\delta_4>0$, 使得当 $|x-x_0|<\delta_4$ 时, 有

$$|f(x,y_0)-f(x_0,y_0)|<\frac{\varepsilon}{4}. \tag{9.7}$$

由式 (9.3)~(9.7) 知, 对 $\forall\varepsilon>0$, 取 $\delta=\min\{\delta_1,\delta_2,\delta_3,\delta_4\}>0$, 则当 $\begin{cases}|x-x_0|<\delta,\\|y-y_0|<\delta,\\(x,y)\in D\end{cases}$ 时, 有

$$\begin{aligned}|f(x,y)-f(x_0,y_0)|&\leqslant|f(x,y)-f(x,y_0)|+|f(x,y_0)-f(x_0,y_0)|\\&\leqslant|f(x,y_0+\delta_1)-f(x,y_0-\delta_1)|+|f(x,y_0)-f(x_0,y_0)|\\&\leqslant|f(x,y_0+\delta_1)-f(x_0,y_0+\delta_1)|+|f(x_0,y_0+\delta_1)-f(x_0,y_0-\delta_1)|\\&\quad+|f(x_0,y_0-\delta_1)-f(x,y_0-\delta_1)|+|f(x,y_0)-f(x_0,y_0)|<\varepsilon,\end{aligned}$$

这正是 $f(x,y)$ 在 (x_0,y_0) 处连续. □

9.1.21

$\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)$ 的充分必要条件是：$\forall\varepsilon$, $\exists\delta$, 使对于 $\forall x_1, x_2$, $0<|x_1-x_0|<\delta$, $0<|y_1-y_0|<\delta$

$0<|x_2-x_0|<\delta$, $0<|y_2-y_0|<\delta$ 时, 有 $|f(x_1,y_1)-f(x_2,y_2)|<\varepsilon$

必要性：若 $\lim\limits_{x\to x_0} f(x,y)=a$ 则 $\forall\varepsilon$, $\exists\delta$, 使 $0<|x_1-x_0|<\delta$, $0<|y_1-y_0|<\delta$

$\Rightarrow$

$0<|x_2-x_0|<\delta$, $0<|y_2-y_0|<\delta$ 时 $|f(x_1,y_1)-a|<\frac{\varepsilon}{2}$, $|f(x_2,y_2)-a|<\frac{\varepsilon}{2}$ 三角不等式

$\therefore$ 有 $|f(x_1,y_1)-f(x_2,y_2)|<\varepsilon$

$\Leftarrow$

充分性：(核心是利用定理 9.5) 对于任意收敛到 (x_0,y_0) 的点列 (x_n,y_n)

由其收敛知, $\forall\varepsilon$, $\exists\delta$, $\exists N$, 对于 $\forall n,m>N$

使 $0<|x_n-x_0|<\delta$, $0<|y_n-y_0|<\delta$, $0<|x_m-x_0|<\delta$, $0<|y_m-y_0|<\delta$ 则有 $|f(x_n,y_n)-f(x_m,y_m)|<\varepsilon$

$\therefore$ 有 $\lim\limits_{n\to\infty} f(x_n,y_n)$ 存在, 由定理 9.5 知, $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)$ 存在

9.1.23 设 $f(x,y)=\dfrac{1}{1-xy}, (x,y)\in[0,1]\times[0,1], (x,y)\neq(1,1)$, 证明函数连续但不一致连续.

证明 先证 $f(x,y)$ 在 $D=[0,1]\times[0,1]\setminus\{(1,1)\}$ 上连续.

对 $\forall(x_0,y_0)\in D, \forall\varepsilon>0$, 取 $\delta=\min\left\{\frac{\varepsilon}{4},1\right\}$, 记 $\Delta x=x-x_0, \Delta y=y-y_0$, 则当 $\begin{cases}|\Delta x|<\delta,\\ |\Delta y|<\delta\end{cases}$ 且 $(x,y)\in D$ 时, 有

$$\begin{aligned}|(1-xy)-(1-x_0y_0)|&=|x_0\Delta x+y_0\Delta y+\Delta x\Delta y|\leqslant|x_0\Delta x|+|y_0\Delta y|+|\Delta x\Delta y|\\&\leqslant 2(\Delta x+\Delta y)<4\delta\leqslant\varepsilon,\end{aligned}$$

故

$$\begin{aligned}&\lim_{(x,y)\to(x_0,y_0)}(1-xy)=1-x_0y_0\\ \Longrightarrow\ &\lim_{(x,y)\to(x_0,y_0)}f(x,y)=\frac{1}{\lim\limits_{(x,y)\to(x_0,y_0)}(1-xy)}=\frac{1}{1-x_0y_0}=f(x_0,y_0),\end{aligned}$$

从而 $f(x,y)$ 在 D 上连续.

下证其不一致连续.

取 $\varepsilon_0=\frac{1}{2}>0$, 对 $\forall\delta_n=\frac{1}{n}>0\ (n\in\mathbb{N}^*)$, 取点 $S_n(1-\delta_n,1), T_n\left(1-\frac{\delta_n}{2},1\right)$ 满足 $\rho(S_n,T_n)=\frac{\delta_n}{2}<\delta_n$, 但

$$\left|f(1-\delta_n,1)-f\left(1-\frac{\delta_n}{2},1\right)\right|=\frac{1}{\delta_n}=n\geqslant1>\varepsilon_0,$$

故 $f(x,y)$ 在 $[0,1]\times[0,1]\setminus\{(1,1)\}$ 上不一致连续. □

9.2

9.2.1

2) $f'_x = 2xy\cos(x^2y)$, $f'_x(1,\pi) = -2\pi$

3) $f'_x = \dfrac{y(2x+y)}{\sqrt{x^2y^2(x+y)^2+1}}$ $f'_x(1,y) = \dfrac{y(2+y)}{\sqrt{y^2(1+y)^2+1}}$

$f'_y = \dfrac{x(x+2y)}{\sqrt{x^2y^2(x+y)^2+1}}$ $f'_y(1,y) = \dfrac{2y+1}{\sqrt{y^2(1+y)^2+1}}$

9.2.2

4) $z = \ln(x+\sqrt{x^2+y^2})$ $z_x = \dfrac{1}{\sqrt{x^2+y^2}}$ $z_y = \dfrac{y}{x\sqrt{x^2+y^2}+x^2+y^2}$

(5)

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x+y}{x-y}\right)^2} \cdot \left(-\frac{2y}{(x-y)^2}\right) = -\frac{y}{x^2+y^2},$$
$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{x+y}{x-y}\right)^2} \cdot \left(\frac{2x}{(x-y)^2}\right) = \frac{x}{x^2+y^2}.$$

6) $u = e^{x(x^2+y^2+z^2)}$

$u_x = (3x^2+y^2+z^2)e^{x(x^2+y^2+z^2)}$ $u_y = 2xy\,e^{x(x^2+y^2+z^2)}$ $u_z = 2xz\,e^{x(x^2+y^2+z^2)}$

(7)

$$\frac{\partial u}{\partial x} = y^z x^{y^z-1}, \quad \frac{\partial u}{\partial y} = x^{y^z}\ln x \cdot z y^{z-1}, \quad \frac{\partial u}{\partial z} = x^{y^z}\ln x \cdot y^z \ln y = x^{y^z} y^z \ln x \ln y.$$

8) $u = xe^{-z} + \ln(x+\ln y) + z$

$u_x = e^{-z} + \dfrac{1}{x+\ln y}$. $u_y = \dfrac{1}{y}\dfrac{1}{x+\ln y}$ $u_z = -xe^{-z}+1$

9.2.3 设 $f(x,y)=\int_1^{x^2y}\frac{\sin t}{t}\,\mathrm{d}t$, 求 $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$.

解

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial (x^2y)}\cdot\frac{\partial (x^2y)}{\partial x}=\frac{\sin x^2y}{x^2y}\cdot 2xy=\frac{2\sin x^2y}{x},$$
$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial (x^2y)}\cdot\frac{\partial (x^2y)}{\partial y}=\frac{\sin x^2y}{x^2y}\cdot x^2=\frac{\sin x^2y}{y}.$$

9.2.4 $\lim\limits_{\Delta x\to 0}\frac{f(0+\Delta x)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{0}{\Delta x}=0$ 存在

$\lim\limits_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=\lim\limits_{\Delta y\to 0}\frac{1}{(\Delta y)^2}$ 不存在

9.2.6 求曲面 $z=\frac{x^2+y^2}{4}$ 与平面 $y=4$ 的交线在点 $(2,4,5)$ 处的切线与 Ox 轴的正向所成的角度.

解

$$\frac{\partial z}{\partial x}(2,4)=\frac{1}{2}x\,|_{x=2}=1,$$

9.2.7 $\begin{cases} x=1 \\ y=t \\ z=\sqrt{t^2+2} \end{cases}$ $\frac{dx}{dt}=0$ $\frac{dy}{dt}=1$ $\frac{dz}{dt}=\frac{t}{\sqrt{2+t^2}}$ 代入有 $(0,1,\frac{\sqrt{3}}{3})$

方程为 $\begin{cases} x=1 \\ y=t+1 \\ z=\frac{\sqrt{3}}{3}t+\sqrt{3} \end{cases}$ 易知与x,y,z轴夹角为 $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{6}$,

9.2.10 式1 $=(1+3xyz+x^2y^2z^2)e^{xyz}$

式2 $=(2xz^2+x^2yz^3)e^{xyz}$

9.2.11 依求导计算. 求出相等即正确

略

9.2.13

(2)

$$\frac{\partial z}{\partial x}=\frac{y(y^2-x^2)}{(x^2+y^2)^2},\quad \frac{\partial z}{\partial y}=\frac{x(x^2-y^2)}{(x^2+y^2)^2}$$
$$\Longrightarrow\ \mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=\frac{y(y^2-x^2)}{(x^2+y^2)^2}\mathrm{d}x+\frac{x(x^2-y^2)}{(x^2+y^2)^2}\mathrm{d}y.$$

(4)
$$dz=-\frac{y}{x^2}\frac{1}{(\frac{y}{x})^2+1}dx+\frac{1}{x}\frac{1}{(\frac{y}{x})^2+1}dy$$
$$=-\frac{y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy$$

(6)

$$\frac{\partial z}{\partial x}(0,0)=\frac{\partial z}{\partial y}(0,0)=0,\quad \frac{\partial z}{\partial x}(1,1)=\frac{\partial z}{\partial y}(1,1)=-4$$
$$\Longrightarrow\ \mathrm{d}z(0,0)=0,\quad \mathrm{d}z(1,1)=-4(\mathrm{d}x+\mathrm{d}y).$$

9.2.15 根据可微的定义证明, 函数 $f(x,y)=\sqrt{|xy|}$ 在原点处不可微.

证明 用反证法. 假设 $f(x,y)=\sqrt{|xy|}$ 在原点处可微, 根据定义, $\exists A,B\in\mathbb{R}$, 使得

$$\sqrt{|hk|}=Ah+Bk+o(\sqrt{h^2+k^2}),\quad \rho=\sqrt{h^2+k^2}\to 0,$$

上式中令 $k=0\ \Longrightarrow\ 0=Ah+o(|h|)\ (h\to 0)\ \Longrightarrow\ A=0$, 同理可得: $B=0$, 故

$$\sqrt{|hk|}=o(\sqrt{h^2+k^2}),\quad \rho=\sqrt{h^2+k^2}\to 0,$$

令 $h=k$, 得:

$$|h|=o(\sqrt{2}\,|h|),\quad \rho=\sqrt{2}\,|h|\to 0,$$

这显然是不成立的, 故矛盾, 从而假设不成立, 函数 $f(x,y)=\sqrt{|xy|}$ 在原点处不可微. □

说明 事实上, $f(x,y)$ 在 $(0,0)$ 处的偏导数 $\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=0$ 均存在, 故上述推出 $A=B=0$ 是自然的.

9.2.16 证明函数 $f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$ 在点 $(0,0)$ 连续且偏导数存在, 但在此点不可微.

证明 记 $O(0,0), M(x,y)$.

对 $\forall\varepsilon>0$, 取 $\delta=\varepsilon$, 则当 $\rho(M,O)<\delta$ 时, 有

$$|f(x,y)|\leqslant\left|\frac{x^2y}{x^2+y^2}\right|\leqslant|x|\cdot\frac{|xy|}{2|xy|}=\frac{1}{2}|x|\leqslant\rho(M,O)<\varepsilon,$$

故 $\lim\limits_{(x,y)\to(0,0)}f(x,y)=0=f(0,0)$, 函数 $f(x,y)$ 在 $(0,0)$ 处连续.

函数 $f(x,y)$ 在 $(0,0)$ 处的偏导数

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{0-0}{x-0}=0,\quad \frac{\partial f}{\partial y}(0,0)=0.$$

假设 $f(x,y)=\sqrt{|xy|}$ 在 $(0,0)$ 处可微, 则有

$$\frac{h^2k}{h^2+k^2}=\frac{\partial f}{\partial x}(0,0)h+\frac{\partial f}{\partial y}(0,0)k+o(\sqrt{h^2+k^2})=o(\sqrt{h^2+k^2}),\quad \rho=\sqrt{h^2+k^2}\to 0,$$

令 $h=k$ 得:

$$\frac{1}{2}h=o(\sqrt{2}\,|h|),\quad h\to 0,$$

这显然是不成立的, 故矛盾, 从而假设不成立, 函数 $f(x,y)$ 在 $(0,0)$ 处不可微. □

9.2.17 * 证明函数 $f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$ 在点 $(0,0)$ 连续且偏导数存在, 但偏导数在点 $(0,0)$ 处不连续, 而 f 在原点 $(0,0)$ 可微.

证明 注意到,

$$\lim_{(x,y)\to(0,0)}(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}=0=f(0,0),$$

故 $f(x,y)$ 在 $(0,0)$ 处连续.

(1) 当 $x^2+y^2\neq 0$ 时,

$$\begin{cases}\dfrac{\partial f}{\partial x}(x,y)=2x\sin\dfrac{1}{\sqrt{x^2+y^2}}+\dfrac{2x}{\sqrt{x^2+y^2}}\cos\dfrac{1}{\sqrt{x^2+y^2}},\\ \dfrac{\partial f}{\partial y}(x,y)=2y\sin\dfrac{1}{\sqrt{x^2+y^2}}+\dfrac{2y}{\sqrt{x^2+y^2}}\cos\dfrac{1}{\sqrt{x^2+y^2}}.\end{cases}$$

(2) 当 $(x,y)=(0,0)$ 时, 注意到,

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{x^2\sin\frac{1}{|x|}}{x}=\lim_{x\to 0}x\sin\frac{1}{|x|}=0,\quad \frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{y^2\sin\frac{1}{|y|}}{y}=\lim_{y\to 0}y\sin\frac{1}{|y|}=0,$$

而 $\displaystyle\lim_{(x,y)\to 0}\frac{\partial f}{\partial x}(x,y)$, $\displaystyle\lim_{(x,y)\to 0}\frac{\partial f}{\partial y}(x,y)$ 均不存在, 故偏导数在点 $(0,0)$ 处不连续.

下证函数 $f(x,y)$ 在 $(0,0)$ 处可微.

往证: $\Delta f=(h^2+k^2)\sin\dfrac{1}{\sqrt{h^2+k^2}}=o(\sqrt{h^2+k^2})$ $(\rho=\sqrt{h^2+k^2}\to 0)$.

事实上,

$$\lim_{\rho\to 0}\frac{(h^2+k^2)\sin\frac{1}{\sqrt{h^2+k^2}}}{\sqrt{h^2+k^2}}=\lim_{\rho\to 0}\rho\sin\frac{1}{\rho}=0,$$

故函数 $f(x,y)$ 在 $(0,0)$ 处可微. □

9.2.18

1) $Z_x = -2x\log(1+y)\ 2)$

$Z_y = -\dfrac{x^2}{1+y}$

$Z_{xx} = -2\log(1+y)$

$Z_{xy} = -\dfrac{2x}{1+y}$

$Z_{yy} = \dfrac{x^2}{(1+y)^2}$

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§9.2

18.(2) $z=u\arctan v$; $u=\dfrac{xy}{x-y}$. $v=x^2y+y-x$

$$\frac{\partial u}{\partial x}=\frac{1}{(x-y)^2}[y(x-y)-xy]=\frac{-y^2}{(x-y)^2}$$

$$\frac{\partial u}{\partial y}=\frac{x^2}{(x-y)^2}$$

$$\frac{\partial v}{\partial x}=2xy-1\ ;\quad \frac{\partial v}{\partial y}=x^2+1$$

$$\therefore\ \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$=\frac{-y^2}{(x-y)^2}\arctan v+\frac{u}{1+v^2}(2xy-1)$$

$$=\frac{-y^2}{(x-y)^2}\arctan(x^2y+y-x)+\frac{xy(2xy-1)}{(x-y)[1+(x^2y+y-x)^2]}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$=\frac{x^2}{(x-y)^2}\arctan(x^2y+y-x)+\frac{xy(x^2+1)}{(x-y)[1+(x^2y+y-x)^2]}$$

$$\therefore\ \frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\frac{\partial z}{\partial x}=\frac{2y^2}{(x-y)^3}\arctan(x^2y+y-x)$$

$$+\frac{-y^2}{(x-y)^2}\frac{2xy-1}{1+(x^2y+y-x)^2}+\frac{4xy^2-y}{(x-y)[1+(x^2y+y-x)^2]}$$

$$-\frac{[1+(x^2y+y-x)^2]+(x-y)2(x^2y+y-x)(2xy-1)}{\{(x-y)[1+(x^2y+y-x)^2]\}^2}xy(2xy-1)$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{2x(x-y)-2x^2}{(x-y)^3}\arctan(x^2y+y-x)$$

$$+\frac{x^2}{(x-y)^2}\frac{2xy-1}{1+(x^2y+y-x)^2}+\frac{3x^2y+y}{(x-y)[1+(x^2y+y-x)^2]}$$

$$-\frac{[1+(x^2y+y-x)^2]+(x-y)2(x^2y+y-x)(2xy-1)}{\{(x-y)[1+(x^2y+y-x)^2]\}^2}xy(x^2+1)$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{-2x^2}{(x-y)^3}\arctan(x^2y+y-x)$$

$$+\frac{x^2}{(x-y)^2}\frac{x^2+1}{1+(x^2y+y-x)^2}+\frac{x^3+x}{(x-y)[1+(x^2y+y-x)^2]}$$

$$-\frac{-[1+(x^2y+y-x)^2]+(x-y)2(x^2y+y-x)(x^2+1)}{\{(x-y)[1+(x^2y+y-x)^2]\}^2}xy(x^2+1)$$