12.1.1 作出下列周期 2π 的函数的图形, 并把它们展开成 Fourier 级数, 并说明收敛情况.

(1) 在 $[-\pi, \pi)$ 中, $f(x) = \begin{cases} -\pi, & -\pi \leqslant x \leqslant 0, \\ x, & 0 < x < \pi; \end{cases}$

(2) 在 $[-\pi, \pi)$ 中, $f(x) = \cos \dfrac{x}{2}$;

(3) 在 $[-\pi, \pi)$ 中, $f(x) = \begin{cases} e^x, & -\pi \leqslant x \leqslant 0, \\ 1, & 0 \leqslant x < \pi. \end{cases}$

解 (1)

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\pi\,dx + \int_0^{\pi} x\,dx\right) = -\frac{1}{2}\pi, \\
a_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right) \\
&= -\left(\frac{1}{n}\sin nx\Big|_{-\pi}^{0}\right) + \frac{1}{n\pi}\left(x\sin nx + \frac{1}{n}\cos nx\right)\Big|_0^{\pi} \\
&= \frac{1}{n^2\pi}(\cos n\pi - 1) = \frac{1}{n^2\pi}((-1)^n - 1), \\
b_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right) \\
&= \left(\frac{1}{n}\cos nx\Big|_{-\pi}^{0}\right) - \frac{1}{n\pi}\left(x\cos nx - \frac{1}{n}\sin nx\right)\Big|_0^{\pi} \\
&= \frac{1}{n}(1 - 2\cos n\pi) = \frac{1}{n}(1 - 2(-1)^n),
\end{aligned}
$$

$$
\Longrightarrow f(x) \sim -\frac{1}{4}\pi + \sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi}((-1)^n - 1)\cos nx + \frac{1}{n}(1 - 2(-1)^n)\sin nx\right)
$$

收敛

$$
= \begin{cases} f(x), & x \neq k\pi, \\ -\dfrac{\pi}{2}, & x = 2k\pi, \\ 0, & x = (2k-1)\pi, \end{cases} \qquad k \in \mathbb{Z}.
$$

(2) $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{1}{\pi}$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \dfrac{4(-1)^n}{(1-4n^2)\pi}$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$

收敛

(3)

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\left(\int_{-\pi}^{0} \mathrm{e}^x\,\mathrm{d}x + \int_0^{\pi} 1\cdot\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\mathrm{e}^x\Big|_{-\pi}^{0} + \pi\right) = \frac{1}{\pi}(1-\mathrm{e}^{-\pi}+\pi), \\
a_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0} \mathrm{e}^x\cos nx\,\mathrm{d}x + \int_0^{\pi}\cos nx\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\frac{\mathrm{e}^x\cos nx + n\mathrm{e}^x\sin nx}{1+n^2}\Big|_{-\pi}^{0} + \frac{1}{n}\sin nx\Big|_0^{\pi}\right) \\
&= \frac{1-\mathrm{e}^{-\pi}\cos n\pi}{\pi(1+n^2)} = \frac{1-\mathrm{e}^{-\pi}(-1)^n}{\pi(1+n^2)}, \\
b_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0} \mathrm{e}^x\sin nx\,\mathrm{d}x + \int_0^{\pi}\sin nx\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\frac{\mathrm{e}^x\sin nx - n\mathrm{e}^x\cos nx}{1+n^2}\Big|_{-\pi}^{0} - \frac{1}{n}\cos nx\Big|_0^{\pi}\right) \\
&= \frac{1}{\pi}\left(\frac{-n-(-n\mathrm{e}^{-\pi}\cos n\pi)}{1+n^2} + \frac{1}{n}(1-\cos n\pi)\right) \\
&= \frac{n(-1+\mathrm{e}^{-\pi}(-1)^n)}{\pi(1+n^2)} + \frac{1}{\pi n}(1-(-1)^n),
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow f(x) &\sim \frac{1}{2\pi}(1-\mathrm{e}^{-\pi}+\pi) \\
&\quad + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\mathrm{e}^{-\pi}(-1)^n}{1+n^2}(\cos nx - \sin nx) + \frac{1}{n}(1-(-1)^n)\sin nx\right) \\
&= \begin{cases} f(x), & x \neq (2k-1)\pi, \\ \dfrac{\mathrm{e}^{-\pi}+1}{2}, & x = (2k-1)\pi, \end{cases} \quad k\in\mathbb{Z}.
\end{aligned}
$$

收敛

12.1.2 将下列函数展开成以指定区间长度为周期的 Fourier 级数, 并说明收敛情况.

(1) $f(x) = 1-\sin\dfrac{x}{2}$ $(0 \leqslant x \leqslant \pi)$;

(2) $f(x) = \dfrac{x}{3}$ $(0 \leqslant x \leqslant T)$;

(3) $f(x) = \mathrm{e}^{ax}$ $(-l \leqslant x \leqslant l)$;

(4) $f(x) = \begin{cases} 1, & |x|<1, \\ -1, & 1 \leqslant |x| \leqslant 2. \end{cases}$

(3) 记

$$A=a,\quad B=\frac{n\pi}{l}.$$

$$\begin{aligned}
a_0&=\frac{1}{l}\int_{-l}^{l}\mathrm{e}^{Ax}\,\mathrm{d}x=\frac{1}{l}\left(\frac{1}{A}\mathrm{e}^{Ax}\Big|_{-l}^{l}\right)=\frac{1}{Al}(\mathrm{e}^{Al}-\mathrm{e}^{-Al}),\\
a_n&=\frac{1}{l}\int_{-l}^{l}\mathrm{e}^{Ax}\cos Bx\,\mathrm{d}x=\frac{1}{l}\,\frac{A\mathrm{e}^{Ax}\cos Bx+B\mathrm{e}^{Ax}\sin Bx}{A^2+B^2}\Big|_{-l}^{l}=\frac{1}{l}\frac{A(\mathrm{e}^{Al}-\mathrm{e}^{-Al})(-1)^n}{A^2+B^2},\\
b_n&=\frac{1}{l}\int_{-l}^{l}\mathrm{e}^{Ax}\sin Bx\,\mathrm{d}x=\frac{1}{l}\,\frac{A\mathrm{e}^{Ax}\sin Bx-B\mathrm{e}^{Ax}\cos Bx}{A^2+B^2}\Big|_{-l}^{l}=\frac{1}{l}\frac{-B(\mathrm{e}^{Al}-\mathrm{e}^{-Al})(-1)^n}{A^2+B^2},
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow f(x)&\sim\frac{1}{2Al}(\mathrm{e}^{Al}-\mathrm{e}^{-Al})+\frac{1}{l}\sum_{n=1}^{\infty}\frac{(-1)^n(\mathrm{e}^{Al}-\mathrm{e}^{-Al})}{A^2+B^2}(A\cos Bx-B\sin Bx)\\
&=\begin{cases}\mathrm{e}^{Ax}, & x\neq(2k-1)l,\\ \dfrac{\mathrm{e}^{-Al}+\mathrm{e}^{Al}}{2}, & x=(2k-1)l,\end{cases}\quad k\in\mathbb{Z}.
\end{aligned}$$

收敛

(4) $a_0=\frac{1}{2}\int_{-2}^{2}f(x)\,dx=0$

$a_n=\frac{1}{2}\int_{-2}^{2}f(x)\cos\frac{n\pi}{2}x\,dx=\int_0^1\cos\frac{n\pi}{2}x\,dx-\int_1^2\cos\frac{n\pi}{2}x\,dx$

$=\frac{4\sin\frac{n}{2}\pi}{n\pi}$

$b_n=\frac{1}{2}\int_{-2}^{2}f(x)\sin\frac{n\pi}{2}x\,dx=0$

12.1.3 把下列函数展开成正弦级数和余弦级数:

(1) $f(x) = 2x^2\ (0 \leqslant x \leqslant \pi)$;

(2) $f(x) = \begin{cases} A, & 0 \leqslant x < \dfrac{1}{2}, \\ 0, \dfrac{1}{2} \leqslant x \leqslant l; \end{cases}$

(3) $f(x) = \begin{cases} 1 - \dfrac{x}{2h}, & 0 \leqslant x \leqslant 2h, \\ 0, & 2h < x \leqslant \pi. \end{cases}$

(1) 奇延拓 $f(x) = \begin{cases} 2x^2 & 0 \leq x \leq \pi \\ -2x^2 & -\pi \leq x < 0 \end{cases}$

$$b_n = \frac{2}{\pi}\int_0^{\pi} 2x^2 \sin nx\, dx$$

$$= \frac{2}{\pi}\,\frac{-4 + (4 - 2n^2\pi^2)(-1)^n}{n^3} = \frac{-8 + (8 - 4n^2\pi^2)(-1)^n}{n^3\pi}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

偶延拓. $f(x) = 2x^2 \quad -\pi \leq x \leq \pi$

$$a_0 = \frac{4}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} 2x^2 \cos nx\, dx$$

$$= \frac{2}{\pi} \times \frac{4n\pi(-1)^n}{n^3} = \frac{8(-1)^n}{n^2}$$

$$f(x) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty} a_n \cos nx$$

(2) 奇延拓:

$$b_n = \frac{2}{l}\int_0^{l} f(x) \sin \frac{n\pi}{l} x\, dx$$

$$= \frac{2}{l}\int_0^{\frac{1}{2}} A \sin \frac{n\pi}{l} x\, dx = -\frac{2A}{n\pi}\left(\cos \frac{n\pi}{2l} - 1\right) = \frac{4A\sin^2\left(\frac{n\pi}{4l}\right)}{n\pi}$$

偶延拓:

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\, dx = \frac{A}{l}$$

$$a_n = \frac{2}{l}\int_0^{l} f(x) \cos \frac{n\pi}{l} x\, dx$$

$$= \frac{2}{l}\int_0^{\frac{1}{2}} A \cos \frac{n\pi}{l} x\, dx$$

$$= \frac{2A\sin\left(\frac{n\pi}{2l}\right)}{n\pi}$$

(3) 记 $f(x)$ 的奇延拓和偶延拓分别为 $f_{\mathrm{o}}(x), f_{\mathrm{e}}(x)$.

先考虑正弦级数.

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{2h}\left(1-\frac{x}{2h}\right)\sin nx\,\mathrm{d}x \\
&= \left(-\frac{2}{\pi n}\cos nx\Big|_0^{2h}\right) - \frac{1}{\pi h}\cdot\left(-\frac{1}{n}\left(x\cos nx-\frac{1}{n}\sin nx\right)\Big|_0^{2h}\right) \\
&= \frac{1}{\pi}\left(\frac{2}{n}-\frac{1}{n^2h}\sin 2nh\right), \\
\implies f_{\mathrm{o}}(x) &= \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{2}{n}-\frac{1}{n^2h}\sin 2nh\right)\sin nx, \quad x\in\mathbb{R}.
\end{aligned}$$

下面考虑余弦级数.

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{2h}\left(1-\frac{x}{2h}\right)\mathrm{d}x = \frac{2}{\pi}\left(x-\frac{1}{4h}x^2\right)\Big|_0^{2h} = \frac{2h}{\pi}, \\
a_n &= \frac{2}{\pi}\int_0^{2h}\left(1-\frac{x}{2h}\right)\cos nx\,\mathrm{d}x \\
&= \left(\frac{2}{n\pi}\sin nx\Big|_0^{2h}\right) - \frac{1}{h\pi}\cdot\frac{1}{n}\left(x\sin nx+\frac{1}{n}\cos nx\right)\Big|_0^{2h} \\
&= \frac{1}{\pi n^2 h}(1-\cos 2nh), \\
\implies f_{\mathrm{e}}(x) &= \frac{h}{\pi}+\frac{1}{\pi h}\sum_{n=1}^{\infty}\frac{1}{n^2}(1-\cos 2nh)\cos nx, \quad x\in\mathbb{R}.
\end{aligned}$$

12.1.4 已知函数的 Fourier 级数展开式, 求常数 a 的值.

(1) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} = a(2a-|x|)$, 其中 $-\pi\leqslant x\leqslant\pi$;

(2) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin nx = ax$, 其中 $-\pi<x<\pi$.

解 (1) 注意到,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,\mathrm{d}x = \frac{2a}{\pi}\int_0^{\pi}(2a-x)\,\mathrm{d}x = \frac{2a}{\pi}\left(2ax-\frac{1}{2}x^2\right)\Big|_0^{\pi} = \frac{2a}{\pi}\left(2a\pi-\frac{1}{2}\pi^2\right)=0$$

$$\implies a=\frac{\pi}{4}.$$

经验证, $a=\dfrac{\pi}{4}$ 时, $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$ 是 $\dfrac{\pi}{4}\left(\dfrac{\pi}{2}-|x|\right)$ $(-\pi\leqslant x\leqslant\pi)$ 的 Fourier 展开式.

(2) 注意到,

$$b_n = \frac{2}{\pi}\int_0^{\pi}ax\sin nx\,\mathrm{d}x = -\frac{2a}{n}\cos n\pi = \frac{(-1)^{n-1}}{n} \implies a=\frac{1}{2}.$$

□

12.1.5

(1) 设

$$f(x)=\begin{cases} x, & 0 \leqslant x \leqslant \dfrac{1}{2}, \\ 2-2x, & \dfrac{1}{2}<x<1, \end{cases}$$

$$S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\pi x, \quad -\infty<x<+\infty,$$

其中 $a_n=2\int_0^1 f(x)\cos n\pi x \,\mathrm{d}x \ (n=0,1,2,\cdots)$. 求 $S\left(\frac{9}{4}\right), S\left(-\frac{5}{2}\right)$;

(2) 设 $f(x)=\begin{cases} -1, & -\pi<x \leqslant 0, \\ 1+x^2, & 0<x \leqslant \pi, \end{cases}$ 则其以 2π 为周期的 Fourier 级数的和函数为 $S(x)\ (-\infty<x<+\infty)$. 求 $S(3\pi), S(-4\pi)$.

解 (1) 将 $f(x)$ 偶延拓, 周期 $T=2$, 则 $S(x)$ 为 $f(x)$ 的余弦级数, 从而

$$S\left(\frac{9}{4}\right)=S\left(\frac{1}{4}\right)=f\left(\frac{1}{4}\right)=\frac{1}{4},$$

$$S\left(-\frac{5}{2}\right)=S\left(\frac{5}{2}\right)=S\left(\frac{1}{2}\right)=\frac{1}{2}\left(f\left(\frac{1}{2}+\right)+f\left(\frac{1}{2}-\right)\right)=\frac{1}{2}\left(\frac{1}{2}+1\right)=\frac{3}{4}.$$

(2) $S(3\pi)=S(\pi)=\dfrac{f^+(-\pi)+f^-(\pi)}{2}=\dfrac{-1+1+\pi^2}{2}=\dfrac{\pi^2}{2}$

$S(-4\pi)=S(0)=\dfrac{f^+(0)+f^-(0)}{2}=\dfrac{-1+1}{2}=0$

12.1.6

(1) $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{\pi}\int_0^{\pi} f(x)\,dx-\int_{-\pi}^{0} f(x+\pi)=0$

$a_{2n}=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2nx\,dx=\frac{1}{\pi}\int_0^{\pi} f(x)\underline{\cos 2nx}\,dx-\int_{-\pi}^{0} f(x+\pi)\underline{\cos(2nx+2n\pi)}\,dx=0$

$b_{2n}=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 2nx\,dx=\frac{1}{\pi}\int_0^{\pi} f(x)\underline{\sin 2nx}\,dx-\int_{-\pi}^{0} f(x+\pi)\underline{\sin(2nx+2n\pi)}\,dx=0$

(2) $a_{2n-1}=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(2n-1)x\,dx=\frac{1}{\pi}\int_0^{\pi} f(x)\underline{\cos(2n-1)x}\,dx+\int_{-\pi}^{0} f(x+\pi)\underbrace{\cos((2n-1)x+(2n-1)\pi)}_{\text{相反数}}\,dx=0$

$b_{2n-1}=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(2n-1)x\,dx=\frac{1}{\pi}\int_0^{\pi} f(x)\underline{\sin(2n-1)x}\,dx+\int_{-\pi}^{0} f(x+\pi)\underbrace{\sin((2n-1)x+(2n-1)\pi)}_{\text{相反数}}\,dx=0$

12.1.7 $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$f(x+h) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n(x+h) + \sum_{n=1}^{\infty} b_n \sin n(x+h)$

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \cos nh - a_n \sin nx \sin nh + \sum_{n=1}^{\infty} b_n \sin nx \cos nh + b_n \cos nx \sin nh$

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nh + b_n \sin nh) \cos nx + \sum_{n=1}^{\infty} (b_n \cos nh - a_n \sin nh) \sin nx$

$\therefore \overline{a_n} = a_n \cos nh + b_n \sin nh$, $\overline{b_n} = b_n \cos nh - a_n \sin nh$

12.1.8

(1) $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 - x^2 \, dx = 2 - \frac{2}{3}\pi^2$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1-x^2) \cos nx \, dx = -\frac{4(-1)^n}{n^2}$

$\therefore 1 - x^2 = 1 - \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} -\frac{4(-1)^n}{n^2} \cos nx$

取 $x = 0$. $1 = 1 - \frac{1}{3}\pi^2 + 4\sum \frac{(-1)^{n-1}}{n^2}$ $\therefore \sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

(2) $g(x) = x^4 \quad -\pi \le x \le \pi$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \cos nx \, dx = \left(\frac{8\pi^2}{n^2} - \frac{48}{n^4} \right) n^2$

$a_0 = \frac{2}{5}\pi^4$

$\therefore x^4 = \frac{1}{5}\pi^4 + \sum_{n=1}^{\infty} \left(\frac{8\pi^2}{n^2} - \frac{48}{n^4} \right) (-1)^n \cos nx$

由(1)中取 $x = \pi$ 得 $1 - \pi^2 = 1 - \frac{1}{3}\pi^2 - 4\sum \frac{1}{n^2}$

$\therefore \sum \frac{1}{n^2} = \frac{\pi^2}{6}$ (理应记住的结论)

得 $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{90}\pi^4$

12.1.9 将 $f(x)=1+x\ (0\leqslant x\leqslant\pi)$ 展开成周期为 2π 的余弦级数, 并求:

(1) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(2n-1)}{(2n-1)^2}$; (2) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos 4(2n-1)}{(2n-1)^2}$.

解

$$
\begin{aligned}
a_0&=\frac{2}{\pi}\int_0^{\pi}(1+x)\,\mathrm{d}x=\frac{2}{\pi}\left(x+\frac{1}{2}x^2\right)\bigg|_0^{\pi}=2+\pi,\\
a_n&=\frac{2}{\pi}\int_0^{\pi}(1+x)\cos nx\,\mathrm{d}x\\
&=\frac{2}{\pi}\cdot\frac{1}{n}\left(\sin nx+x\sin nx+\frac{1}{n}\cos nx\right)\bigg|_0^{\pi}\\
&=\frac{2}{\pi n^2}((-1)^n-1),\\
\Longrightarrow\ f(x)&=1+\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}\cos nx\\
&=1+\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2},\quad x\in\mathbb{R}.
\end{aligned}
$$

(1) 令 $x=1$ 得:

$$
\begin{aligned}
f(1)=1+\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)}{(2n-1)^2}&=2\\
\Longrightarrow\sum_{n=1}^{\infty}\frac{\cos(2n-1)}{(2n-1)^2}&=\frac{\pi}{4}\left(\frac{\pi}{2}-1\right),
\end{aligned}
$$

(2) 令 $x=4$ 得:

$$
f(4)=f(2\pi-4)=2\pi-3=1+\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 4(2n-1)}{(2n-1)^2}
$$

$$
\sum_{n=1}^{\infty}\frac{\cos 4(2n-1)}{(2n-1)^2}=\frac{\pi}{8}(8-3\pi).
$$