Introduction to Robotics
Chapter III Position and Orientation Representation

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Position and Orientation Representation

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  - ➢ Coordinates Transformation
  - ➢ Homogeneous Coordinates Transformation
  - ➢ Dynamic Robot Posture and Position Representation
  - ➢ General Rotation Transformation
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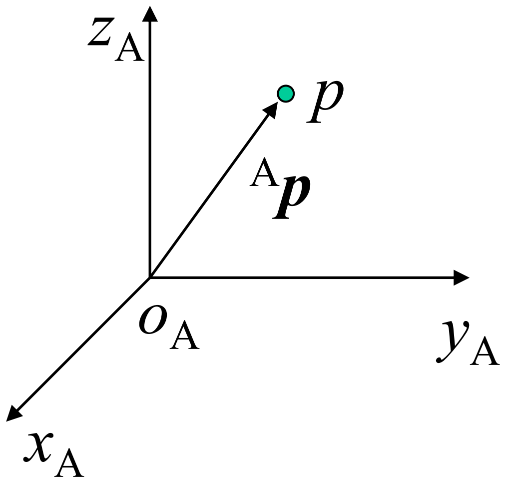
Position and Orientation Representation

- Position is represented by Position Vector

In Cartesian Coordinates {**A**}, Position Vector of Point A is : ${}^{A}\boldsymbol{p}$

Position Vector $${}^{A}\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$



Vector Addition $${}^{A}\boldsymbol{p} = p_x\vec{i} + p_y\vec{j} + p_z\vec{k}$$

l_2-norm $$|\boldsymbol{p}| = \sqrt{{p_x}^2 + {p_y}^2 + {p_z}^2}$$

Concepts: Unit Vector and Direction Vector

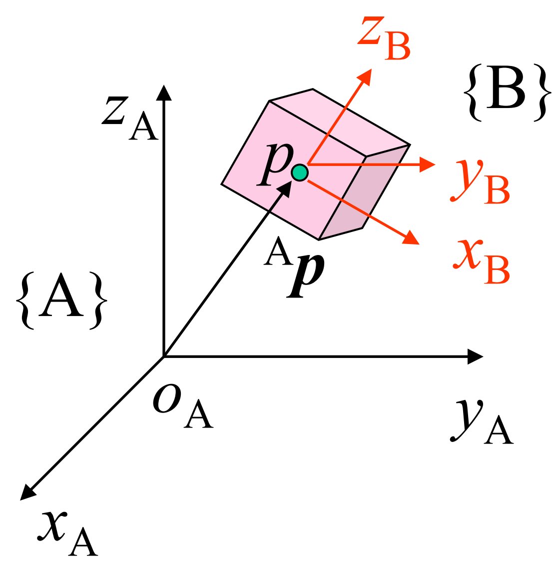
Position and Orientation Representation

- Orientation is represented by referring to a coordinates established on the object. Orientation is also named as posture.
- We establish a coordinates on object B, denoted as {**B**}, we use the three direction vectors of x_B, y_B, z_B-axis in {**A**} to represent B's orientation.

$$ {}_B^A\boldsymbol{R} = \begin{bmatrix} {}^A x_B & {}^A y_B & {}^A z_B \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

${}_B^A\boldsymbol{R}$ represents the orientation of B in {**A**}.

$$ {}^A x_B = r_{11}\vec{i} + r_{21}\vec{j} + r_{31}\vec{k} $$
$$ {}^A y_B = r_{12}\vec{i} + r_{22}\vec{j} + r_{32}\vec{k} $$
$$ {}^A z_B = r_{13}\vec{i} + r_{23}\vec{j} + r_{33}\vec{k} $$

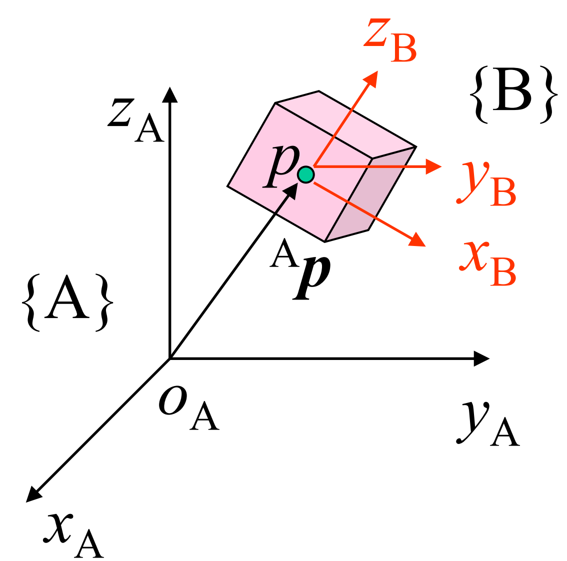


Rotation Matrix

$$^{A}x_B = r_{11}\vec{i} + r_{21}\vec{j} + r_{31}\vec{k}$$
$$^{A}y_B = r_{12}\vec{i} + r_{22}\vec{j} + r_{32}\vec{k}$$
$$^{A}z_B = r_{13}\vec{i} + r_{23}\vec{j} + r_{33}\vec{k}$$

$$\Rightarrow \begin{bmatrix} ^{A}x_B \\ ^{A}y_B \\ ^{A}z_B \end{bmatrix} = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = {}^{A}_{B}\boldsymbol{R}^T \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$$

Supposing we have $^{B}\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$



We will further have

$$\begin{bmatrix} p_x & p_y & p_z \end{bmatrix} \begin{bmatrix} ^{A}x_B \\ ^{A}y_B \\ ^{A}z_B \end{bmatrix} = \begin{bmatrix} p_x & p_y & p_z \end{bmatrix} {}^{A}_{B}\boldsymbol{R}^T \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = \left\{ {}^{A}_{B}\boldsymbol{R} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \right\}^T \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = {}^{A}\boldsymbol{p} \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$$

Conclusion: $^{A}\boldsymbol{p} = {}^{A}_{B}\boldsymbol{R}\, {}^{B}\boldsymbol{p}$

Another Interpretation: Vector rotation by rotation matrix

Rotation Matrix

- Rotation matrix ${}^{A}_{B}\boldsymbol{R}$ is orthogonal matrix

$$ {}^{A}x_B \cdot {}^{A}y_B = {}^{A}y_B \cdot {}^{A}z_B = {}^{A}z_B \cdot {}^{A}x_B = 0 $$

$$ {}^{A}x_B \cdot {}^{A}x_B = {}^{A}y_B \cdot {}^{A}y_B = {}^{A}z_B \cdot {}^{A}z_B = 1 $$

$$ {}^{B}_{A}R = {}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T}, \quad \left| {}^{A}_{B}R \right| = 1 $$

$Defination:$

$if\ A^T A = I$

$A\ is\ an\ othogonal\ matrix$

There are three independent parameters in ${}^{A}_{B}\boldsymbol{R}$, which means we need three parameters to represent an orientation in 3D space. (think about how many independent parameters in a unit vector?)

- Ref: Cross Product vs Dot Product

$$ {}^{A}x_B \cdot {}^{A}y_B = r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} $$

$$ {}^{A}x_B \times {}^{A}y_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \end{vmatrix} = (r_{21}r_{32} - r_{22}r_{31})\vec{i} + (r_{12}r_{31} - r_{11}r_{32})\vec{j} + (r_{11}r_{22} - r_{12}r_{21})\vec{k} $$

Position and Orientation Representation

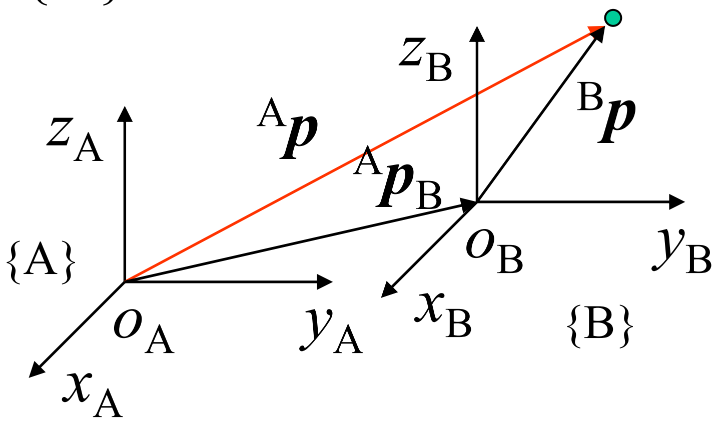
- Pose Representation： With respect to $\{\mathbf{A}\}$, the origin of $\{\mathbf{B}\}$ can be represented by a position vector, while the orientation of the axes of $\{\mathbf{B}\}$ can be represented by the rotation matrix. Thus we have

$$\{\mathbf{B}\} = \left\{ {}^{A}_{B}R \quad {}^{A}p_{B} \right\}$$

Coordinate Transformation

- Translation Coordinate Transformation： for a position vector in {**B**}, e.g., $^{B}\boldsymbol{p}$, its representation in {**A**} is

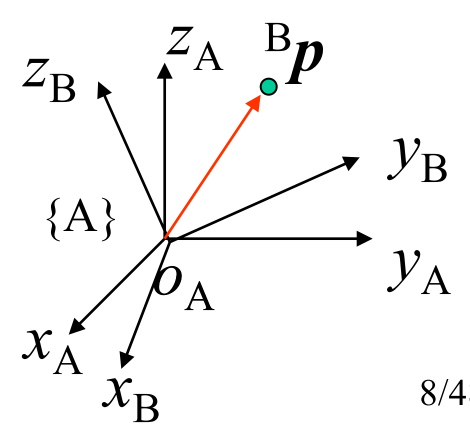
$$^{A}p = {}^{B}p + {}^{A}p_{B}$$



- Rotation Coordinate Transformation：

Supposing {**A**} and {**B**} are located on the same origin, representation of a point P in {**A**} and {**B**} is transformed by：

$$^{A}p = {}^{A}_{B}R\ {}^{B}p$$



Inversely $\quad {}^{B}p = {}^{B}_{A}R\ {}^{A}p = {}^{A}_{B}R^{T}\ {}^{A}p$

Coordinate Transformation

- Fundamental Rotation Transformation: rotation about *x*, *y* , *z*-axis respectively: An arbitration rotation transformation can be synthesized by fundamental rotation transformations.

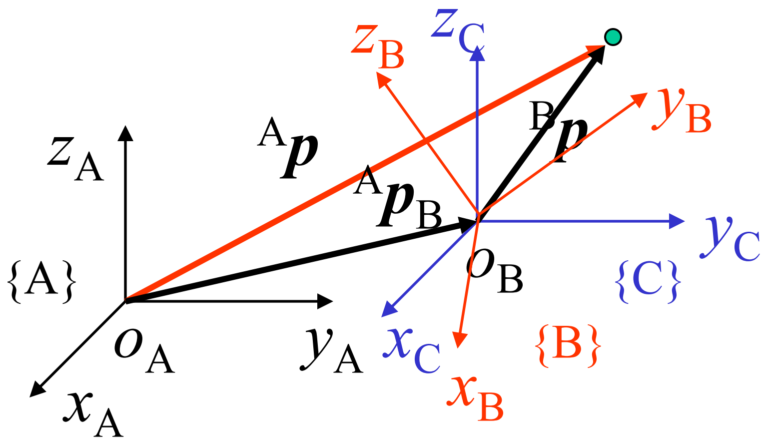
$$R(x,\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix},\qquad R(y,\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix},\qquad R(z,\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

- Hybrid Transformation： Transformation consisting of both translation transformation and rotation transformation

since we have $^{A}p = {}^{C}p + {}^{A}p_{C}$

and $^{C}p = {}^{C}_{B}R\,{}^{B}p = {}^{A}_{B}R\,{}^{B}p$, $^{A}p_{C} = {}^{A}p_{B}$

So $^{A}p = {}^{A}_{B}R\,{}^{B}p + {}^{A}p_{B}$



- Question: Transformation order?

Coordinate Transformation

- Case Study 1： Coordinates {**B**} initially overlaps {**A**}. Supposing {**B**} is rotated 30° about z_A, then translated along x_A by 12 and y_A by 6. What is the position vector ${}^A\boldsymbol{p}_B$ and rotation matrix ${}_B^AR$ from {B} to {A}? Supposing we have ${}^B\boldsymbol{p}=[5 \quad 9 \quad 0]^T$, what is ${}^A\boldsymbol{p}$?
- Answer：

$$ {}_B^AR = R(z,30^\circ) = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad {}^Ap_B = \begin{bmatrix} 12 \\ 6 \\ 0 \end{bmatrix} $$

$$ {}^Ap = {}_B^AR\,{}^Bp + {}^Ap_B = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \\ 0 \end{bmatrix} + \begin{bmatrix} 12 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 11.83 \\ 16.294 \\ 0 \end{bmatrix} $$

Sequential Transformation I

- Case 1-Intrinsic Rotation: For coordinates {**A**}, {**B**} and {**C**}, supposing orientation of {**B**} is represented in {**A**} by T_1, orientation of {**C**} is represented in {**B**} by T_2, and orientation of a rigid body is represented in {**C**} by T_3, then the rigid body can be represented in {**A**} by:

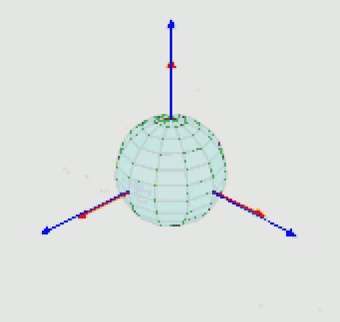
$$T = T_1 T_2 T_3$$

- Case 2-Extrinsic Rotation: For coordinates {**A**}, supposing a rigid body is transformed with respect to {**A**} by T_1, then the rigid body is transformed with respect to {**A**} by T_2, finally the rigid body is transformed with respect to {**A**} by T_3, then the rigid body can be represented in {**A**} by:

$$T = T_3 T_2 T_1$$

Question: How to interpret $R(y,\varphi)R(z,\theta)R(x,\alpha)$?

Sequential Transformation II



- Another Interpretation of Case 1:

For coordinates {**A**}, {**B**} and {**C**}, supposing orientation of {**B**} is represented in {**A**} by T_1, orientation of {**C**} is represented in {**B**} by T_2, and orientation of a rigid body is represented in {**C**} by T_3, then the rigid body can be represented in {**A**} by：

Can be interpreted as

For coordinates {**A**}, supposing a rigid body is transformed with respect to {**A**} by T_1, (we now get {A`}) then the rigid body is transformed with respect to {A`} by T_2, (we now get {A``}) finally the rigid body is transformed with respect to {A``} by T_3, (we now get {A```}) then the rigid body can be represented in {A} by：

Sequential Transformation

- Regarding the transformation order
  - Sequential translation rotation is independent of transformation order.
  - Rotation transformation and hybrid translation depends on transformation order. (**the order matters!**)

$$R(y,\varphi)R(z,\theta)R(x,\alpha)=\begin{bmatrix}\cos\varphi & 0 & \sin\varphi\\ 0 & 1 & 0\\ -\sin\varphi & 0 & \cos\varphi\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix}$$

$$=\begin{bmatrix}\cos\varphi\cos\theta & \sin\varphi\sin\alpha-\cos\varphi\sin\theta\cos\alpha & \sin\varphi\cos\alpha+\cos\varphi\sin\theta\sin\alpha\\ \sin\theta & \cos\theta\cos\alpha & -\cos\theta\sin\alpha\\ -\sin\varphi\cos\theta & \cos\varphi\sin\alpha+\sin\varphi\sin\theta\cos\alpha & \cos\varphi\cos\alpha-\sin\varphi\sin\theta\sin\alpha\end{bmatrix}$$

$$R(x,\alpha)R(z,\theta)R(y,\varphi)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\varphi & 0 & \sin\varphi\\ 0 & 1 & 0\\ -\sin\varphi & 0 & \cos\varphi\end{bmatrix}$$

$$=\begin{bmatrix}\cos\varphi\cos\theta & -\sin\theta & \sin\varphi\cos\theta\\ \sin\varphi\sin\alpha+\cos\varphi\sin\theta\cos\alpha & \cos\theta\cos\alpha & -\cos\varphi\sin\alpha+\sin\varphi s\theta\cos\alpha\\ -\sin\varphi\cos\alpha+\cos\varphi\sin\theta\sin\alpha & \cos\theta\sin\alpha & \cos\varphi\cos\alpha+\sin\varphi\sin\theta\sin\alpha\end{bmatrix}$$

Homogeneous Coordinate Transformation

- Definition

$$ {}^{A}p = {}^{A}_{B}R\ {}^{B}p + {}^{A}p_{B} \Rightarrow \begin{bmatrix} {}^{A}p \\ 1 \end{bmatrix} = \left[\begin{array}{c:c} {}^{A}_{B}R & {}^{A}p_{B} \\ \hdashline 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}p \\ 1 \end{bmatrix} $$

$$ {}^{A}p' = \begin{bmatrix} {}^{A}p \\ 1 \end{bmatrix},\quad {}^{A}_{B}T = \left[\begin{array}{c:c} {}^{A}_{B}R & {}^{A}p_{B} \\ \hdashline 0 & 1 \end{array}\right],\quad {}^{B}p' = \begin{bmatrix} {}^{B}p \\ 1 \end{bmatrix} \Rightarrow {}^{A}p' = {}^{A}_{B}T\ {}^{B}p' $$

${}^{A}p'$ and ${}^{B}p'$ is homogeneous coordinate, ${}^{A}_{B}T$ is homogeneous coordinate transformation matrix

Case Study 2: Revise Case Study 1 by homogeneous coordinate transformation.

$$ {}^{A}_{B}T = \begin{bmatrix} {}^{A}_{B}R & {}^{A}p_{B} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 & 12 \\ 0.5 & 0.866 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\qquad {}^{A}p' = {}^{A}_{B}T\,{}^{B}p' = \begin{bmatrix} 0.866 & -0.5 & 0 & 12 \\ 0.5 & 0.866 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 11.83 \\ 16.294 \\ 0 \\ 1 \end{bmatrix} $$

Homogeneous Coordinate Transformation

- Translation Homogeneous Coordinate Transformation

$$\mathrm{Trans}(a,b,c)=\begin{bmatrix}1&0&0&a\\0&1&0&b\\0&0&1&c\\0&0&0&1\end{bmatrix}$$

- Rotation Homogenous Coordinate Transformation

$$\mathrm{Rot}(x,\theta)=\begin{bmatrix}1&0&0&0\\0&\cos\theta&-\sin\theta&0\\0&\sin\theta&\cos\theta&0\\0&0&0&1\end{bmatrix},\quad \mathrm{Rot}(y,\theta)=\begin{bmatrix}\cos\theta&0&\sin\theta&0\\0&1&0&0\\-\sin\theta&0&\cos\theta&0\\0&0&0&1\end{bmatrix},\quad \mathrm{Rot}(z,\theta)=\begin{bmatrix}\cos\theta&-\sin\theta&0&0\\\sin\theta&\cos\theta&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

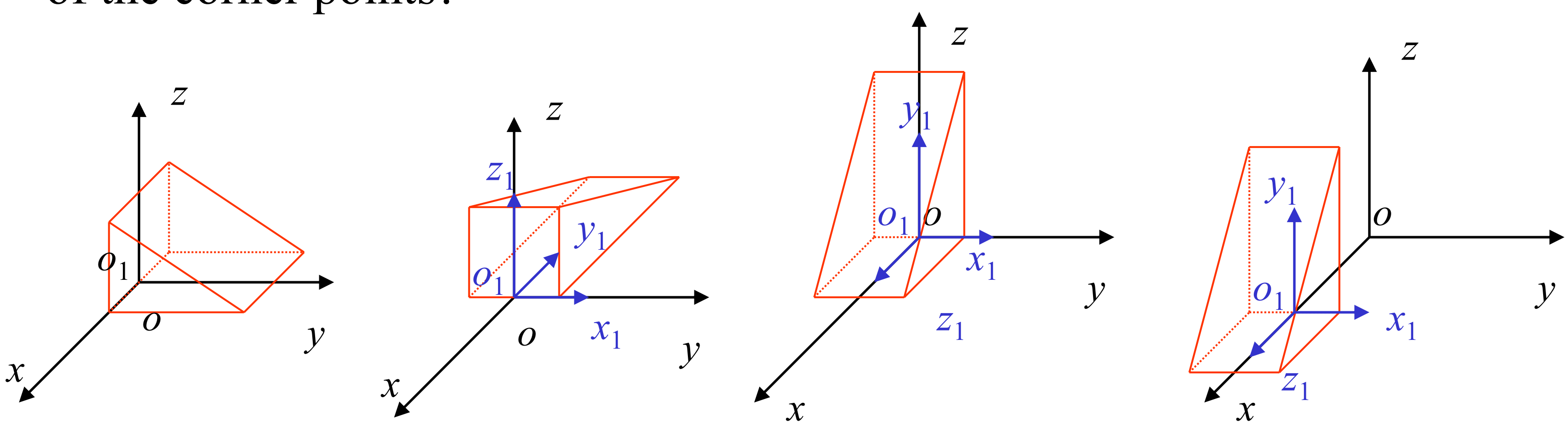
- Hybrid Homogeneous Coordinate Transformation

Rotation after translation $${}^{A}_{B}T=\begin{bmatrix}I&{}^{A}p_B\\0&1\end{bmatrix}\begin{bmatrix}{}^{A}_{B}R&0\\0&1\end{bmatrix}=\begin{bmatrix}{}^{A}_{B}R&{}^{A}p_B\\0&1\end{bmatrix}$$

Translation after rotation $${}^{A}_{B}T=\begin{bmatrix}{}^{A}_{B}R&0\\0&1\end{bmatrix}\begin{bmatrix}I&{}^{A}p_B\\0&1\end{bmatrix}=\begin{bmatrix}{}^{A}_{B}R&{}^{A}_{B}R\,{}^{A}p_B\\0&1\end{bmatrix}$$

Homogeneous Coordinate Transformation

- Case Study 3： Supposing a rigid body is defined in base coordinates by its corner points, e.g., {(1,0,0), (-1,0,0), (-1,0,2), (1,0,2), (1,4,0), (-1,4,0)}. If the rigid body is rotated about z-axis by 90°, then rotated by 90° about y-axis, and finally translated by 4 along x-axis. What are now the coordinates of the corner points?



We first establish an object coordinates on the rigid body which overlaps the base coordinates initially. Then in object coordinates, the corner points' coordinates keep constant regardless the transformation. The object coordinates can be represented in base coordinates as:

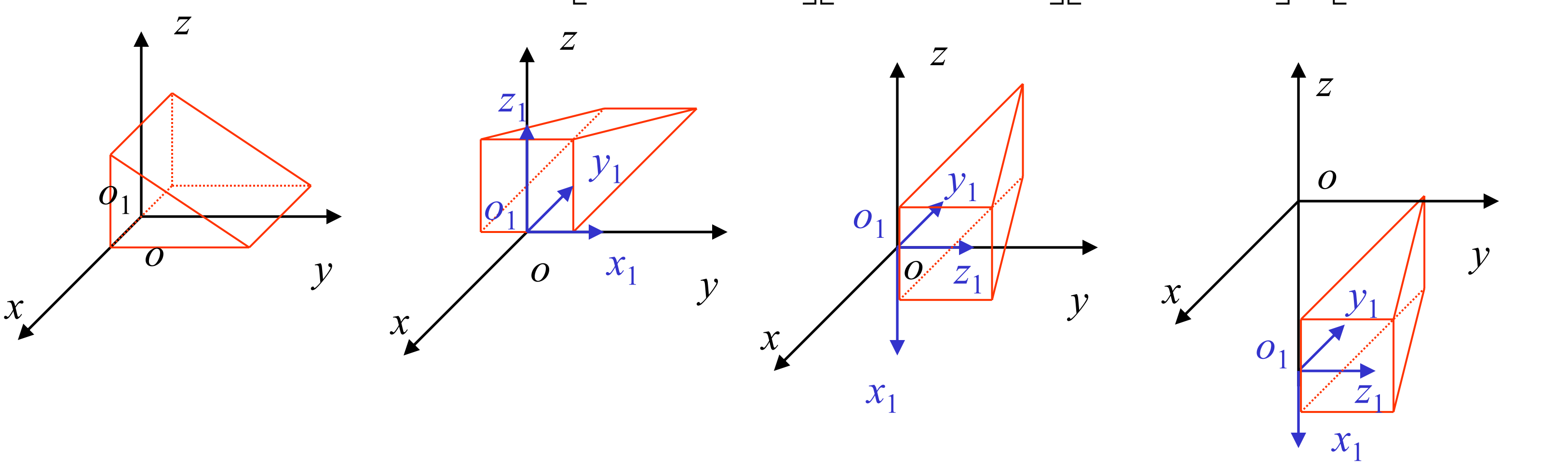
Homogeneous Coordinate Transformation

$$T = \mathrm{Trans}(4,0,0)\mathrm{Rot}(y,90)\mathrm{Rot}(z,90) = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

✓

$$T = \mathrm{Rot}(z,90)\mathrm{Rot}(y,90)\mathrm{Trans}(4,0,0) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

✗



Homogeneous Coordinate Transformation

Corner points' coordinates in base coordinates are: ⇒ ${}^{A}p' = {}^{A}_{B}T\ {}^{B}p'$

$$\begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 6 & 6 & 4 & 4 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

- Supposing {**B**} is represented in {**A**} by：

$$ {}^{A}_{B}T = \begin{bmatrix} {}^{A}_{B}R & {}^{A}p_{B} \\ 0 & 1 \end{bmatrix} $$

- Then we have {**A**} represented in {**B**} as：

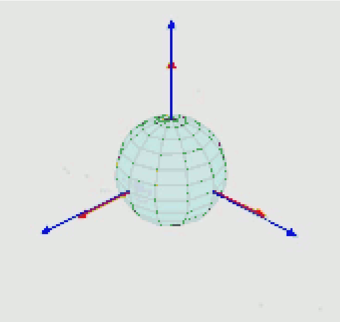
$$ {}^{B}_{A}T = {}^{A}_{B}T^{-1} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}p_{B} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\,{}^{A}p_{B} \\ 0 & 1 \end{bmatrix} $$

- And

$$ T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix},\ T^{-1} = \begin{bmatrix} n_x & n_y & n_z & -p\cdot n \\ o_x & o_y & o_z & -p\cdot o \\ a_x & a_y & a_z & -p\cdot a \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

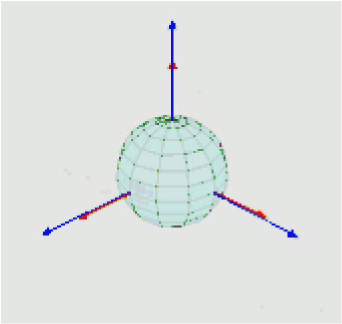
Mobile Robot Orientation and Position Representation

- Robot Motion Orientation Representation
  - Euler Angles： are three angles introduced by Leonhard Euler to describe the orientation of a rigid body with respect to a fixed coordinate system.
  - Euler Angles： typically used to represent orientation of a mobile frame of reference in physics.
  - Euler Angles can be defined by composition of rotations, which means that three composed elemental rotations are always sufficient to reach any target frame.
  - The three elemental rotations may be extrinsic (rotations about the axes xyz of the original coordinate system, which is assumed to remain motionless), or intrinsic (rotations about the axes of the rotating coordinate system XYZ, solidary with the moving body, which changes its orientation after each elemental rotation).
  - Euler angles are typically denoted as α, β, γ, or ψ, θ, φ.
  - For either extrinsic or intrinsic rotation, we have 12 possible rotation sequences.



Mobile Robot Orientation and Position Representation

- For Better Clarity, we have the following appointment
  - We have two coordinates, the base coordinates xyz and the object coordinates XYZ. We also call the object coordinates as mobile coordinates. In math and robotics, coordinates is synonym of frame.
  - Base coordinates xyz is fixed, while object coordinates XYZ is rotating along with the rotation of the object.
  - Extrinsic rotation happens about x-, y-, z-axis, while intrinsic rotation happens about X-, Y-, Z-axis.
  - Now one rotation sequence can be possibly described as x-y-z or z-Y-Z.



- Robot Motion Orientation Representation
  - Euler Angles - Proper Euler Angles： a mobile frame first rotate about z axis by φ, then rotate about Y axis by θ, finally rotate about Z axis by ϕ. the z-Y-Z order (or Z-Y-Z order) transformation is calculated as

$$\mathrm{Euler}(\varphi,\theta,\phi)=\mathrm{Rot}(z,\varphi)\mathrm{Rot}(y,\theta)\mathrm{Rot}(z,\phi)$$

$$=\begin{bmatrix}\cos\varphi & -\sin\varphi & 0 & 0\\ \sin\varphi & \cos\varphi & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\phi & -\sin\phi & 0 & 0\\ \sin\phi & \cos\phi & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\cos\varphi\cos\theta\cos\phi-\sin\varphi\sin\phi & -\cos\varphi\cos\theta\sin\phi-\sin\varphi\cos\phi & \cos\varphi\sin\theta & 0\\ \sin\varphi\cos\theta\cos\phi+\cos\varphi\sin\phi & -\sin\varphi\cos\theta\sin\phi+\cos\varphi\cos\phi & \sin\varphi\sin\theta & 0\\ -\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Robot Motion Orientation and Position Representation

- Robot Motion Orientation Representation
  - Euler Angles - Tait-Bryan Angles： a mobile frame first rotate about z axis by φ, then rotate about Y axis by θ, finally rotate about X axis by ϕ. the z-Y-X order (or Z-Y-X order) transformation is calculated as

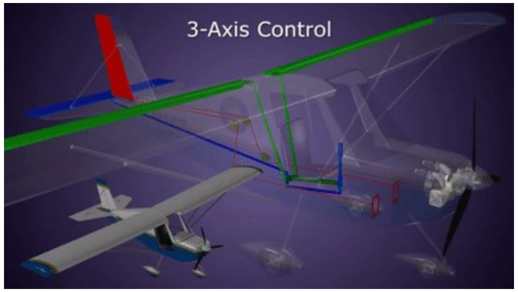
$$\mathrm{RPY}(\varphi,\theta,\phi)=\mathrm{Rot}(z,\varphi)\mathrm{Rot}(y,\theta)\mathrm{Rot}(x,\phi)$$

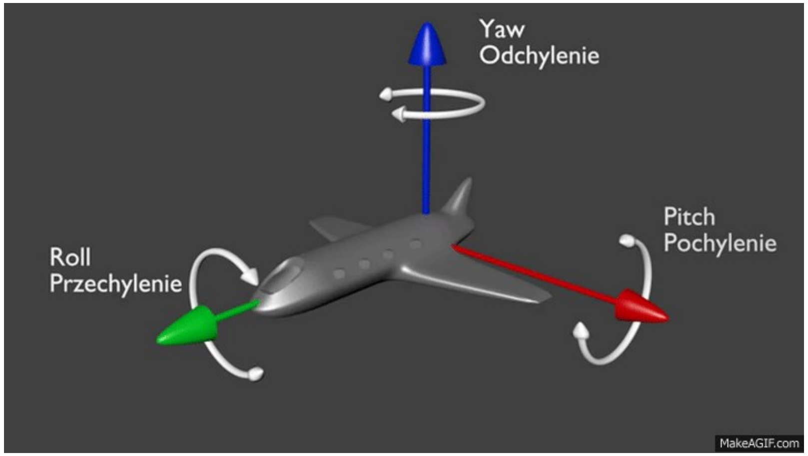
$$=\begin{bmatrix}\cos\varphi & -\sin\varphi & 0 & 0\\ \sin\varphi & \cos\varphi & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\phi & -\sin\phi & 0\\ 0 & \sin\phi & \cos\phi & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\cos\varphi\cos\theta & \cos\varphi\sin\theta\sin\phi-\sin\varphi\cos\phi & \cos\varphi\sin\theta\cos\phi+\sin\varphi\sin\phi & 0\\ \sin\varphi\cos\theta & \sin\varphi\sin\theta\sin\phi+\cos\varphi\cos\phi & \sin\varphi\sin\theta\cos\phi-\cos\varphi\sin\phi & 0\\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Robot Motion Orientation and Position Representation

- Robot Motion Orientation Representation
  - Both Tait-Bryan Angles and Proper Euler Angles describe rotations in robotics by intrinsic rotation (rotations happens on mobile frame).
  - Regarding Tait-Bryan Angles： Also call Roll-Pitch-Yaw
  - Popularly used in aircraft control
  - Rotation about X – Pitch (俯仰)
  - Rotation about Y – Roll (横滚)
  - Rotation about Z – Yaw (偏转)





Symbols do not matter.

Robot Motion Orientation and Position Representation

➢ Conversion from rotation matrix to Proper Euler Angles

$$ {}^{A}R_{B}=\begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}=\begin{bmatrix} \cos\varphi\cos\theta\cos\phi-\sin\varphi\sin\phi & -\cos\varphi\cos\theta\sin\phi-\sin\varphi\cos\phi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta\cos\phi+\cos\varphi\sin\phi & -\sin\varphi\cos\theta\sin\phi+\cos\varphi\cos\phi & \sin\varphi\sin\theta \\ -\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix} $$

(1) If a_z=1， we have θ=0. In addition, since ${}_{B}^{A}\boldsymbol{R}$ is orthogonal matrix, we further have n_z=0， o_z=0， a_x=0， a_y=0, which finally yields

$$ \begin{bmatrix} n_x & o_x & 0 \\ n_y & o_y & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} \cos(\varphi+\phi) & -\sin(\varphi+\phi) & 0 \\ \sin(\varphi+\phi) & \cos(\varphi+\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ \phi+\varphi=\operatorname{atan2}(n_y,o_y) $$

Physical Meaning： from the definition of Proper Euler Angles, if θ=0, z-axis keeps unchanged, and rotation about z-axis by ϕ and φ twice equals rotation about z-axis by $\phi+\varphi$ once.

Robot Motion Orientation and Position Representation

➢ Conversion from rotation matrix to Proper Euler Angles

(2) If a_z=-1, then θ=π. We also have, n_z=0, o_z=0, a_x=0, a_y=0. We get similar results from scenario (1).

(3) If $a_z \neq \pm 1$, then $s\theta \neq 0$. (*$s\theta$ is short for sin(θ)*)

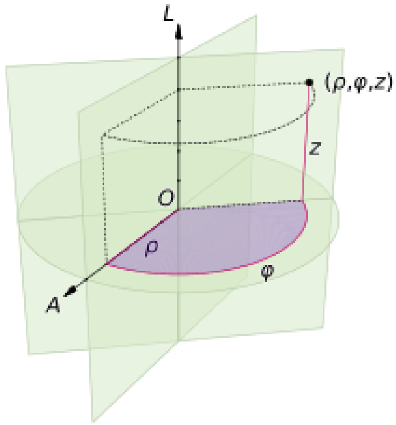
$$\begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix} = \begin{bmatrix} \cos\varphi\cos\theta\cos\phi - \sin\varphi\sin\phi & -\cos\varphi\cos\theta\sin\phi - \sin\varphi\cos\phi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta\cos\phi + \cos\varphi\sin\phi & -\sin\varphi\cos\theta\sin\phi + \cos\varphi\cos\phi & \sin\varphi\sin\theta \\ -\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}$$

$$\text{Solution} \left\{ \begin{array}{l} \theta = \operatorname{atan2}(a_x\cos\varphi + a_y\sin\varphi,\ a_z) \\ \text{if } s\theta > 0\text{:} \quad \left\{ \begin{array}{l} \phi = \operatorname{atan2}(o_z, -n_z) \\ \varphi = \operatorname{atan2}(a_y, a_x) \end{array} \right. \\ \text{if } s\theta < 0\text{:} \quad \left\{ \begin{array}{l} \phi = \operatorname{atan2}(-o_z, n_z) \\ \varphi = \operatorname{atan2}(-a_y, -a_x) \end{array} \right. \end{array} \right\}$$

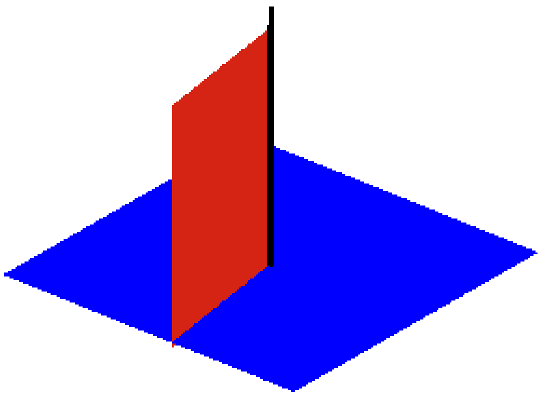
Robot Motion Orientation and Position Representation

- Pose Representation in Cylindrical and Spherical Coordinates
  - ➢ Cylindrical Coordinates Definition：

- The three coordinates (ρ, φ, z) of a point P are defined as:
- The *axial distance* or *radial distance* ρ is the Euclidean distance from the z-axis to the point P.
- The *azimuth* φ is the angle between the reference direction on the chosen plane and the line from the origin to the projection of P on the plane.
- The *axial coordinate* or *height* z is the signed distance from the chosen plane to the point P.



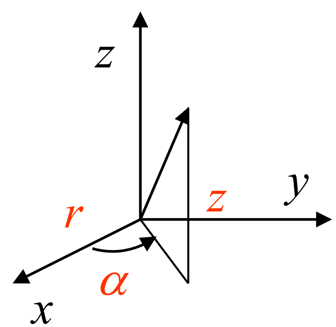
A cylindrical coordinate system with origin O, polar axis A, and longitudinal axis L. The dot is the point with radial distance ρ = 4, angular coordinate φ = 130°, and height z = 4.



Cylindrical coordinate surfaces. The three orthogonal components, ρ (green), φ (red), and z (blue), each increasing at a constant rate. The point is at the intersection between the three colored surfaces.

Robot Motion Orientation and Position Representation

- Pose Representation in Cylindrical and Spherical Coordinates
  - In Cylindrical Coordinates： a Cartesian coordinates is first translated along x-axis by r, then rotated z-axis by α, finally translated along z-axis by z. Now the cylindrical coordinates is represented in by



$$\mathrm{Cyl}(z,\alpha,r) = \mathrm{Trans}(0,0,z)\mathrm{Rot}(z,\alpha)\mathrm{Trans}(r,0,0)$$

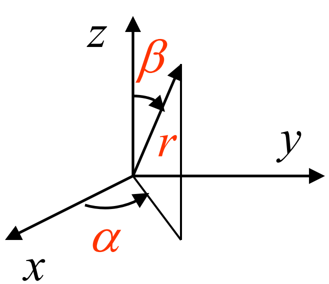
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & r\cos\alpha \\ \sin\alpha & \cos\alpha & 0 & r\sin\alpha \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Robot Motion Orientation and Position Representation

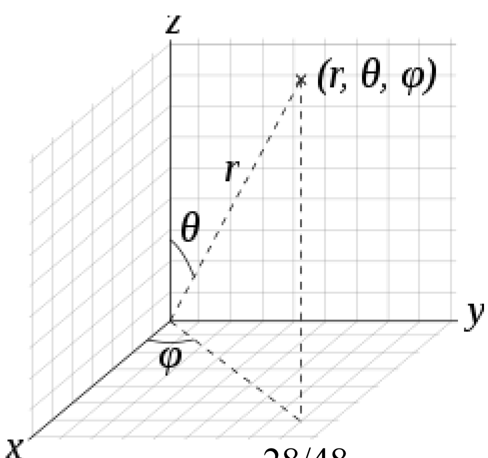
- In Spherical Coordinates： A Cartesian Coordinates is first translated along z-axis by r, then rotated about y-axis by β, and finally rotated by z-axis by α. Now the spherical coordinates is represented by

$$\mathrm{Sph}(\alpha,\beta,r) = \mathrm{Rot}(z,\alpha)\mathrm{Rot}(y,\beta)\mathrm{Trans}(0,0,r)$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta & -\sin\alpha & \cos\alpha\sin\beta & r\cos\alpha\sin\beta \\ \sin\alpha\cos\beta & \cos\alpha & \sin\alpha\sin\beta & r\sin\alpha\sin\beta \\ -\sin\beta & 0 & \cos\beta & r\cos\beta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



In mathematics, a **spherical coordinate system** is a coordinate system for 3D space where the position of a point is specified by three numbers: the *radial distance* of that point from a fixed origin, its *polar angle* measured from a fixed zenith direction, and the *azimuthal direction* of its orthogonal direction on a reference plane that passes through the origin and is orthogonal to the zenith, measured from a fixed reference direction on that plane. It can be seen as the three-dimensional version of the polar coordinate system.



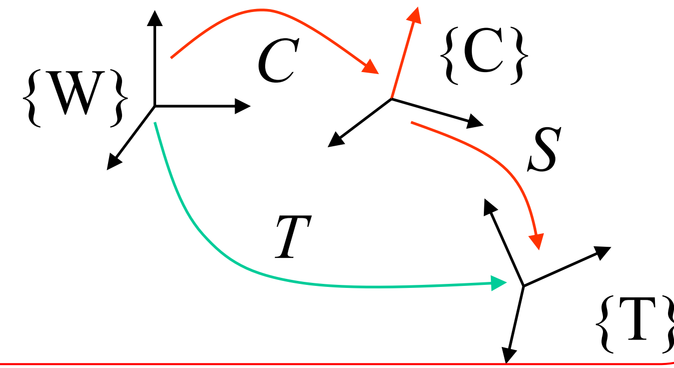
General Rotation Transformation

- General Rotation Transformation: Rotation about an arbitrary axis through the origin

Setup: world coordinates denoted as {**W**}, and coordinates {**C**} is represented by ***C*** in {**W**}, *f* is unit vector along Z-axis of {**C**} (C_z denotes Z-axis of {**C**}):

$$C = \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad f = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

$$or \quad f = a$$



Obviously, rotation about *f* equals rotation about C_z

$$\mathrm{Rot}(f,\theta) = \mathrm{Rot}(C_z,\theta)$$

For another coordinates {**T**}, its representation in {**W**} is *T*, and S in {**C**} , we have:

$$T = CS$$

General Rotation Transformation

Since rotation about f equals rotation about C_z, we have

$$\because \begin{cases} T = CS \\ \mathrm{Rot}(f,\theta)T = C\mathrm{Rot}(z,\theta)S \end{cases}$$ Both sides represent the same rotation

$$\therefore \mathrm{Rot}(f,\theta) = C\mathrm{Rot}(z,\theta)ST^{-1} = C\mathrm{Rot}(z,\theta)SS^{-1}C^{-1} = C\mathrm{Rot}(z,\theta)C^{-1}$$

$$\mathrm{Rot}(f,\theta) = \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & n_y & n_z & 0 \\ o_x & o_y & o_z & 0 \\ a_x & a_y & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n_x n_x \cos\theta - n_x o_x \sin\theta + n_x o_x \sin\theta + o_x o_x \cos\theta + a_x a_x & n_x n_y \cos\theta - n_x o_y \sin\theta + n_y o_x \sin\theta + o_y o_x \cos\theta + a_x a_y & n_x n_z \cos\theta - n_x o_z \sin\theta + n_z o_x \sin\theta + o_z o_x \cos\theta + a_x a_z & 0 \\ n_y n_x \cos\theta - n_y o_x \sin\theta + n_x o_y \sin\theta + o_y o_x \cos\theta + a_y a_x & n_y n_y \cos\theta - n_y o_y \sin\theta + n_y o_y \sin\theta + o_y o_y \cos\theta + a_y a_y & n_y n_z \cos\theta - n_y o_z \sin\theta + n_z o_y \sin\theta + o_z o_y \cos\theta + a_y a_z & 0 \\ n_z n_x \cos\theta - n_z o_x \sin\theta + n_x o_z \sin\theta + o_z o_x \cos\theta + a_z a_x & n_z n_y \cos\theta - n_z o_y \sin\theta + n_y o_z \sin\theta + o_y o_z \cos\theta + a_z a_y & n_z n_z \cos\theta - n_z o_z \sin\theta + n_z o_z \sin\theta + o_z o_z \cos\theta + a_z a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

General Rotation Transformation

$$C = \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$f = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

$${}^{A}x_B \cdot {}^{A}y_B = r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32}$$

$${}^{A}x_B \times {}^{A}y_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \end{vmatrix}$$

$$= (r_{21}r_{32} - r_{22}r_{31})\vec{i} + (r_{12}r_{31} - r_{11}r_{32})\vec{j} + (r_{11}r_{22} - r_{12}r_{21})\vec{k}$$

Define versθ=1-cosθ, and considering $a=f$ (both a and f are vectors)：

$$\begin{aligned} & n_x n_x \cos\theta - n_x o_x \sin\theta + n_x o_x \sin\theta + o_x o_x \cos\theta + a_x a_x \\ &= (n_x n_x + o_x o_x)\cos\theta + a_x a_x \\ &= (1 - a_x a_x)\cos\theta + a_x a_x \\ &= a_x a_x (1 - \cos\theta) + \cos\theta \\ &= f_x f_x \text{vers}\theta + \cos\theta \end{aligned}$$

Line 1 Column 1

$$\begin{aligned} & n_y n_x \cos\theta - n_y o_x \sin\theta + n_x o_y \sin\theta + o_y o_x \cos\theta + a_y a_x \\ &= (n_y n_x + o_y o_x)\cos\theta + (n_x o_y - n_y o_x)\sin\theta + a_y a_x \\ &= -a_y a_x \cos\theta + a_z \sin\theta + a_y a_x \\ &= a_y a_x (1 - \cos\theta) + a_z \sin\theta \\ &= f_y f_x \text{vers}\theta + f_z \sin\theta \end{aligned}$$

Line 2 Column 1

General Rotation Transformation

$$n_z n_x \cos\theta - n_z o_x \sin\theta + n_x o_z \sin\theta + o_z o_x \cos\theta + a_z a_x$$
$$= (n_z n_x + o_z o_x)\cos\theta + (n_x o_z - n_z o_x)\sin\theta + a_z a_x$$
$$= -a_z a_x \cos\theta - a_y \sin\theta + a_z a_x$$
$$= f_x f_z \operatorname{vers}\theta - f_y \sin\theta$$

Line 3 Column 1

$$n_x n_y \cos\theta - n_x o_y \sin\theta + n_y o_x \sin\theta + o_y o_x \cos\theta + a_x a_y$$
$$= (n_x n_y + o_y o_x)\cos\theta + (n_y o_x - n_x o_y)\sin\theta + a_x a_y$$
$$= -a_x a_y \cos\theta - a_z \sin\theta + a_x a_y$$
$$= f_x f_z \operatorname{vers}\theta - f_z \sin\theta$$

Line 1 Column 2

$$n_y n_y \cos\theta - n_y o_y \sin\theta + n_y o_y \sin\theta + o_y o_y \cos\theta + a_y a_y$$
$$= (n_y n_y + o_y o_y)\cos\theta + a_y a_y$$
$$= (1 - a_y a_y)\cos\theta + a_y a_y$$
$$= f_y f_y \operatorname{vers}\theta + \cos\theta$$

Line 2 Column 2

$$n_z n_y \cos\theta - n_z o_y \sin\theta + n_y o_z \sin\theta + o_y o_z \cos\theta + a_z a_y$$
$$= (n_z n_y + o_y o_z)\cos\theta + (n_y o_z - n_z o_y)\sin\theta + a_z a_y$$
$$= -a_z a_y \cos\theta + a_x \sin\theta + a_z a_y$$
$$= f_z f_y \operatorname{vers}\theta + f_x \sin\theta$$

Line 3 Column 2

$$n_x n_z \cos\theta - n_x o_z \sin\theta + n_z o_x \sin\theta + o_z o_x \cos\theta + a_x a_z$$
$$= (n_x n_z + o_z o_x)\cos\theta + (n_z o_x - n_x o_z)\sin\theta + a_x a_z$$
$$= -a_x a_z \cos\theta + a_y \sin\theta + a_x a_z$$
$$= f_x f_z \operatorname{vers}\theta + f_y \sin\theta$$

Line 1 Column 3

$$n_y n_z \cos\theta - n_y o_z \sin\theta + n_z o_y \sin\theta + o_z o_y \cos\theta + a_y a_z$$
$$= (n_y n_z + o_z o_y)\cos\theta + (n_z o_y - n_y o_z)\sin\theta + a_y a_z$$
$$= -a_y a_z \cos\theta - a_x \sin\theta + a_y a_z$$
$$= f_y f_z vers\theta - f_x \sin\theta$$

Line 2 Column 3

$$n_z n_z \cos\theta - n_z o_z \sin\theta + n_z o_z \sin\theta + o_z o_z \cos\theta + a_z a_z$$
$$= (n_z n_z + o_z o_z)\cos\theta + a_z a_z$$
$$= (1 - a_z a_z)\cos\theta + a_z a_z$$
$$= f_z f_z \operatorname{vers}\theta + \cos\theta$$

Line 3 Column 3

General Rotation Transformation

The general rotation transformation is now described as:

$$\mathrm{Rot}(f,\theta)=\begin{bmatrix} f_x f_x \mathrm{vers}\theta+\cos\theta & f_y f_x \mathrm{vers}\theta - f_z\sin\theta & f_z f_x \mathrm{vers}\theta + f_y\sin\theta & 0 \\ f_x f_y \mathrm{vers}\theta + f_z\sin\theta & f_y f_y \mathrm{vers}\theta+\cos\theta & f_z f_y \mathrm{vers}\theta - f_x\sin\theta & 0 \\ f_x f_z \mathrm{vers}\theta - f_y\sin\theta & f_y f_z \mathrm{vers}\theta + f_x\sin\theta & f_z f_z \mathrm{vers}\theta+\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Equivalent rotation axis and angle

Given arbitrary rotation, we can get the equivalent rotation axis and angle, by

$$\begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} f_x f_x \mathrm{vers}\theta+\cos\theta & f_y f_x \mathrm{vers}\theta - f_z\sin\theta & f_z f_x \mathrm{vers}\theta + f_y\sin\theta & 0 \\ f_x f_y \mathrm{vers}\theta + f_z\sin\theta & f_y f_y \mathrm{vers}\theta+\cos\theta & f_z f_y \mathrm{vers}\theta - f_x\sin\theta & 0 \\ f_x f_z \mathrm{vers}\theta - f_y\sin\theta & f_y f_z \mathrm{vers}\theta + f_x\sin\theta & f_z f_z \mathrm{vers}\theta+\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summing up the diagonal elements yields:

$$n_x+o_y+a_z=(f_x^2+f_y^2+f_z^2)\mathrm{vers}\theta+3\cos\theta=1+2\cos\theta$$

$$\cos\theta=\frac{1}{2}(n_x+o_y+a_z-1)$$

General Rotation Transformation

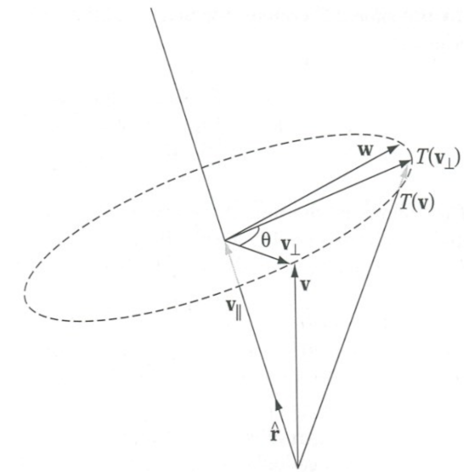
We further have:

$$o_z - a_y = 2 f_x \sin\theta$$

$$a_x - n_z = 2 f_y \sin\theta$$

$$n_y - o_x = 2 f_z \sin\theta$$

$$\therefore \sin\theta = \pm\frac{1}{2}\sqrt{(o_z - a_y)^2 + (a_x - n_z)^2 + (n_y - o_x)^2}$$



Rotation about a general axis through the origin, showing the axis of rotation and the plane of rotation

Supposing $s\theta > 0$ or $0 < \theta < 180°$ (正向旋转)

$$\theta = \mathrm{atan2}\left(\sqrt{(o_z - a_y)^2 + (a_x - n_z)^2 + (n_y - o_x)^2}, n_x + o_y + a_z - 1\right)$$

Supposing $s\theta < 0$ or $-180° < \theta < 0°$ (逆向旋转)

$$\theta = \mathrm{atan2}\left(-\sqrt{(o_z - a_y)^2 + (a_x - n_z)^2 + (n_y - o_x)^2}, 1 - n_x - o_y - a_z\right)$$

Equivalent Rotation Axis：

$$\begin{cases} f_x = (o_z - a_y)/(2\sin\theta) \\ f_y = (a_x - n_z)/(2\sin\theta) \\ f_z = (n_y - o_x)/(2\sin\theta) \end{cases}$$

Given a unit orthogonal matrix, we can get the equivalent rotation axis and rotation angle.

General Rotation Transformation

Case 3： Supposing a cylinder is located in a Cartesian space o-xyz. Its central axis goes through the origin but is not parallel to x- or y- or z- axis. The central axis unit direction vector is $f = (f_x, f_y, f_z)$. We want the cylinder to be rotated about f by θ. How can we do this by rotating the cylinder in o-xyz through rotating about Z-Y-X order?

General Rotation Transformation

Case 3： Supposing a cylinder is located in a Cartesian space o-xyz. Its central axis goes through the origin but is not parallel to x- or y- or z- axis. The central axis unit direction vector is $f = (f_x, f_y, f_z)$. We want the cylinder to be rotated about f by θ. How can we do this by rotating the cylinder in o-xyz through rotating about Z-Y-X order?

Solution: combine the following two Matrix.

$$\mathrm{Rot}(f,\theta)=\begin{bmatrix} f_x f_x \operatorname{vers}\theta+\cos\theta & f_y f_x \operatorname{vers}\theta-f_z\sin\theta & f_z f_x \operatorname{vers}\theta+f_y\sin\theta & 0 \\ f_x f_y \operatorname{vers}\theta+f_z\sin\theta & f_y f_y \operatorname{vers}\theta+\cos\theta & f_z f_y \operatorname{vers}\theta-f_x\sin\theta & 0 \\ f_x f_z \operatorname{vers}\theta-f_y\sin\theta & f_y f_z \operatorname{vers}\theta+f_x\sin\theta & f_z f_z \operatorname{vers}\theta+\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{RPY}(\varphi,\theta,\phi)=\mathrm{Rot}(z,\varphi)\mathrm{Rot}(y,\theta)\mathrm{Rot}(x,\phi)$$

$$=\begin{bmatrix} \cos\varphi\cos\theta & \cos\varphi\sin\theta\sin\phi-\sin\varphi\cos\phi & \cos\varphi\sin\theta\cos\phi+\sin\varphi\sin\phi & 0 \\ \sin\varphi\cos\theta & \sin\varphi\sin\theta\sin\phi+\cos\varphi\cos\phi & \sin\varphi\sin\theta\cos\phi-\cos\varphi\sin\phi & 0 \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Quaternion (四元数)

Definition: A quaternion q is defined as the sum of a scalar q_0 and a vector $\boldsymbol{q} = (q_1, q_2, q_3)$, namely

$$q = q_0 + q_1 i + q_2 j + q_3 k \ .$$

where：

$$ii = jj = kk = -1,$$
$$ij = k, \quad jk = i, \quad ki = j,$$
$$ji = -k, \ kj = -i, \ ik = -j.$$

So what's ijk?

The product of two quaternions $p=p_0+\boldsymbol{p}$ and $q=q_0+\boldsymbol{q}$ is

$$pq = p_0 q_0 - \boldsymbol{p}\cdot\boldsymbol{q} + p_0\boldsymbol{q} + q_0\boldsymbol{p} + \boldsymbol{p}\times\boldsymbol{q}.$$

which is another quaternion.

Question: Is pq equal to qp?

Quaternion (四元数)

Conjugate: Let $q=q_0+\boldsymbol{q}$, then the conjugate is

$$q^* = q_0 - \boldsymbol{q}$$

So we have:

$$q + q^* = 2q_0 \qquad\qquad q\, q^* = q^*\, q$$

And we can easily verify:

$$(pq)^* = q^* p^*$$

Norm: $|q| = \mathrm{sqrt}(q\, q^*)$

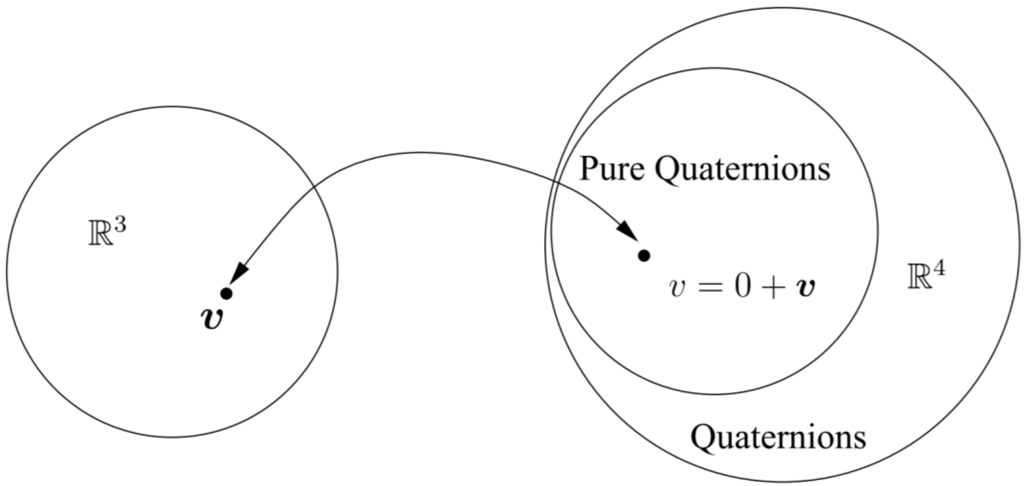
We can easily verify: $|pq|^2 = |q|^2\, |q|^2$

Inverse: the inverse of a quaternion q is defined as

$$q^{-1} = q^* / |q|^2$$

So, $qq^{-1} = q^{-1}q = 1$.

Quaternion Rotation Operator



A vector in $\mathbb{R}^3$ is a pure quaternion whose real part is zero.

Theorem 1 *For any unit quaternion*

$$q = q_0 + \boldsymbol{q} = \cos\frac{\theta}{2} + \hat{\boldsymbol{u}}\sin\frac{\theta}{2},$$

and for any vector $\boldsymbol{v} \in \mathbb{R}^3$ *the action of the operator*

$$L_q(\boldsymbol{v}) = q\boldsymbol{v}q^* = \cos\theta \cdot \boldsymbol{v} + (1 - \cos\theta)(\hat{\boldsymbol{u}} \cdot \boldsymbol{v})\hat{\boldsymbol{u}} + \sin\theta \cdot (\hat{\boldsymbol{u}} \times \boldsymbol{v})$$

on $\boldsymbol{v}$ *is equivalent to a rotation of the vector through an angle* θ *about* $\hat{\boldsymbol{u}}$ *as the axis of rotation.*

Define: for unit quaternion q

$$\begin{aligned} L_q(\boldsymbol{v}) &= q\boldsymbol{v}q^* \\ &= (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{v} + 2(\boldsymbol{q} \cdot \boldsymbol{v})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{v}). \end{aligned}$$

Quaternion Operator Sequence

Let p and q be two unit quaternions. We first apply the operator L_p to the vector $\boldsymbol{u}$ and obtain the vector $\boldsymbol{v}$. To $\boldsymbol{v}$ we then apply the operator L_q and obtain the vector $\boldsymbol{w}$. Equivalently, we apply the composition $L_q \circ L_p$ of the two operators:

$$\begin{aligned}
\boldsymbol{w} &= L_q(\boldsymbol{v}) \\
&= q\boldsymbol{v}q^* \\
&= q(p\boldsymbol{u}p^*)q^* \\
&= (qp)\boldsymbol{u}(qp)^* \\
&= L_{qp}(\boldsymbol{u}).
\end{aligned}$$

- Because *p* and *q* are unit quaternions, so is the product *qp*. Hence the above equation describes a rotation operator whose defining quaternion is the product of the two quaternions *p* and *q*. The axis and angle of the composite rotation is given by the product *qp*

不同姿态表示之间的转换

姿态矩阵	$^{A}R_B=[n \quad o \quad a]=\begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}$
欧拉角	**欧拉角转换为姿态矩阵** $\mathbf{Euler}(\psi,\theta,\varphi)=$ $\begin{bmatrix} \cos\psi\cos\theta\cos\varphi-\sin\psi\sin\varphi & -\cos\psi\cos\theta\sin\varphi-\sin\psi\cos\varphi & \cos\psi\sin\theta \\ \sin\psi\cos\theta\cos\varphi+\cos\psi\sin\varphi & -\sin\psi\cos\theta\sin\varphi+\cos\psi\cos\varphi & \sin\psi\sin\theta \\ -\sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \end{bmatrix}$ **姿态矩阵转换为欧拉角** **姿态矩阵转换为欧拉角：** 若 a_z=1，则 θ=0。z 轴方向不变，两次分别绕 z 轴旋转角度 ψ和 φ 等价于一次绕 z 轴的旋转角度 $\psi+\varphi$。 $\psi+\varphi=\text{atan2}(n_y,o_y)$ 若 a_z=-1，则 $\theta=\pi$。绕 y 轴旋转之后，新的 z 轴方向与原来的 z 轴方向相反。两次分别绕 z 轴旋转角度 ψ和 φ 等价于一次绕 z 轴的旋转角度 $\varphi-\psi$。 $\varphi-\psi=\text{atan2}(n_y,o_y)$ 若 $a_z\neq\pm1$，则 $s\theta\neq0$。若 $s\theta>0$， $\begin{cases}\varphi=\text{atan2}(o_z,-n_z)\\ \psi=\text{atan2}(a_y,a_x)\end{cases}$。若 $s\theta<0$， $\begin{cases}\varphi=\text{atan2}(-o_z,n_z)\\ \psi=\text{atan2}(-a_y,-a_x)\end{cases}$。 $\theta=\text{atan2}(a_x c\psi+a_y s\psi, a_z)$。然后，利用式(2-10)和式(2-13)第一行的第一列和第二列、第二行的第一列和第二列的四个对应元素相等，判定出 ψ、 θ和 φ 的正确解。

不同姿态表示之间的转换

横俯偏角	横俯偏角转换为姿态矩阵

横俯偏角

横俯偏角转换为姿态矩阵

$$\mathrm{RPY}(\psi,\theta,\varphi)=\begin{bmatrix}\cos\psi\cos\theta & \cos\psi\sin\theta\sin\varphi-\sin\psi\cos\varphi & \cos\psi\sin\theta\cos\varphi+\sin\psi\sin\varphi \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\varphi+\cos\psi\cos\varphi & \sin\psi\sin\theta\cos\varphi-\cos\psi\sin\varphi \\ -\sin\theta & \cos\theta\sin\varphi & \cos\theta\cos\varphi\end{bmatrix}$$

姿态矩阵转换为横俯偏角

若 $n_z=1$，则 $\theta=-\pi/2$。绕 y 轴旋转后新的 x 轴与旋转前的 z 轴方向相同。分别绕 z 轴旋转角度 ψ 和绕新的 x 轴旋转角度 φ 等价于一次绕 z 轴的旋转角度 $\psi+\varphi$。

$$\psi+\varphi=\mathrm{atan2}(-a_y,o_y)$$

若 $n_z=-1$，则 $\theta=\pi/2$。绕 y 轴旋转后新的 x 轴与旋转前的 z 轴方向相反。分别绕 z 轴旋转角度 ψ 和绕新的 x 轴旋转角度 φ 等价于一次绕 z 轴的旋转角度 $\varphi-\psi$。

$$\varphi-\psi=\mathrm{atan2}(-a_y,o_y)$$

若 $n_z\neq\pm1$，则 $c\theta\neq0$。若 $c\theta>0$，$\begin{cases}\varphi=\mathrm{atan2}(o_z,a_z)\\ \psi=\mathrm{atan2}(n_y,n_x)\end{cases}$。若 $c\theta<0$，$\begin{cases}\varphi=\mathrm{atan2}(-o_z,-a_z)\\ \psi=\mathrm{atan2}(-n_y,-n_x)\end{cases}$。

$\theta=\mathrm{atan2}(-n_z,n_x c\psi+n_y s\psi)$。然后，利用式(2-10)和式(2-14)第一行的第二列和第三列、第二行的第二列和第三列的四个对应元素相等，判定出 ψ、θ 和 φ 的正确解。

不同姿态表示之间的转换

转轴转角

转轴转角转换为姿态矩阵

$$\mathbf{Rot}(f,\theta)=$$

$$\begin{bmatrix} f_x f_x(1-\cos\theta)+\cos\theta & f_y f_x(1-\cos\theta)-f_z\sin\theta & f_z f_x(1-\cos\theta)+f_y\sin\theta \\ f_x f_y(1-\cos\theta)+f_z\sin\theta & f_y f_y(1-\cos\theta)+\cos\theta & f_z f_y(1-\cos\theta)-f_x\sin\theta \\ f_x f_z(1-\cos\theta)-f_y\sin\theta & f_y f_z(1-\cos\theta)+f_x\sin\theta & f_z f_z(1-\cos\theta)+\cos\theta \end{bmatrix}$$

姿态矩阵转换为转轴转角：将旋转规定为绕矢量f的正向旋转，使得0≤θ ≤180°

$$\theta=\mathbf{atan2}\left(\sqrt{(o_z-a_y)^2+(a_x-n_z)^2+(n_y-o_x)^2},\ n_x+o_y+a_z-1\right)$$

$$\begin{cases} f_x=(o_z-a_y)/(2\sin\theta) \\ f_y=(a_x-n_z)/(2\sin\theta) \\ f_z=(n_y-o_x)/(2\sin\theta) \end{cases}$$

不同姿态表示之间的转换

四元数	四元数转换为转轴转角 $\theta = 2\,\mathrm{atan2}(\sqrt{q_1{}^2 + q_2{}^2 + q_3{}^2}, q_0)$ $\begin{cases} f_x = q_1 / \sin(\theta/2) \\ f_y = q_2 / \sin(\theta/2) \\ f_z = q_3 / \sin(\theta/2) \end{cases}$ 转轴转角转换为四元数： $q = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T = \begin{bmatrix} \cos(\theta/2) & f_x \sin(\theta/2) & f_y \sin(\theta/2) & f_z \sin(\theta/2) \end{bmatrix}^T$

习题

- 习题1： 已知矢量$\boldsymbol{u}$和坐标系F，$\boldsymbol{u}$为由F描述的一点。

$$u = 3\vec{i} + 2\vec{j} + 2\vec{k},\quad F = \begin{bmatrix} 0 & -1 & 0 & 10 \\ 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(1)确定表示同一点但由基坐标系描述的$\boldsymbol{v}$。

(2)首先让F绕基坐标系的y轴旋转90°，然后沿基坐标系的x轴平移20。求变换所得的新坐标系F′。

(3)确定表示同一点但由坐标系F′描述的矢量$\boldsymbol{u}'$。

(4)作图表示$\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{u}'$, F和F′之间的关系。

习题

- 习题2： 已知齐次变换矩阵

$$H=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

要求$Rot(f, \theta)=H$, 确定f和θ的值。

- 习题3： 矢量Q绕矢量f旋转θ角，产生新的矢量Q'，即$Q'= Rot(f, \theta)\ Q$。求证：

$$Q' = Qc\theta + s\theta(f \times Q) + (1-c\theta)[Q-(f \times Q)\times f]$$