EE150: Signals and Systems, Spring 2022
Comprehensive Problem Sets

(Due Monday, May.23 at 11:59am(noon) (CST))

1. [20 points] For each of the following statements, judge if it is true, and give a justification or counterexample.

   (a) If $x(t), t \in \mathbf{R}$ is a real-valued signal, then its Fourier transform $X(f)$, $f \in \mathbf{R}$, is also real-valued.

   (b) A linear causal continuous-time system is always time-invariant.

   (c) The inverse of a causal linear and time-invariant(LTI) system is always causal.

   (d) The system with real-valued input $x(t)$ and output

   $$y(t) = (1 + x^4(t))^{(\cos^2(5t) - \sin^2(5t))} \tag{1}$$

   is stable.

   (e) The discrete-time signal $x[n] = sin[\frac{3}{2}n]$ is a periodic signal.

   (f) The following two signals $x_1(t)$ and $x_2(t)$ are periodic with period $T = 1$, as shown in Figure 1.

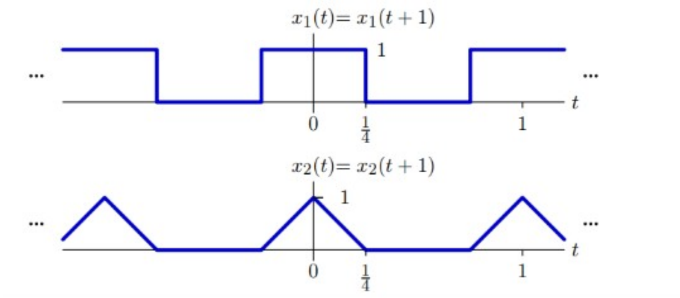

Figure 1: $x_1(t)$ and $x_2(t)$

   For the system shown in Figure 2, if $x(t) = x_1(t)$ and $y(t) = x_2(t)$, then this system cannot be a linear time-invariant system.

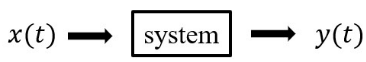

Figure 2: The system

   (g) If $f(t)$ and $h(t), t \in \mathbf{R}$ are real-valued signals, and the convolution satisfies $y(t) = f(t) * h(t)$, then $y(-t) = f(-t) * h(-t)$.

2. [20 points]

(a) Consider a linear, time-invariant system with impulse response

$$h[n] = (\frac{1}{2})^{|n|}$$

Find the Fourier series representation of the output $y[n]$ for each of the following inputs.

(i) $x[n] = sin(\frac{3\pi n}{4})$

(ii) $x[n] = j^n + (-1)^n$

(b) Repeat (a) for

$$h[n] = \begin{cases} 1, & 1 \leq n \leq 2 \\ -1, & -2 \leq n \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

3. [15 points] Consider a periodic signal $s(t)$ with period $\frac{1}{2}$ and Fourier coefficients $a_1 = a_{-1} = \frac{1}{2}, a_2 = a_{-2} = 1$, and $a_k = 0$ otherwise.

(a) Determine $s(t)$.

(b) Assume a system $y(t) = x(s(t))$. Is this system Memoryless, Time Invariant, Linear, Causal, Stable? Explain why.

(c) Consider an LTI system with impulse response

$$h(t) = \frac{sin(3(t-2))}{\pi(t-2)}$$

Determine the output $y_1(t)$ if the input is $s(t)$.

4. [20 points] When the input of a LTI system is $f(t)$, the corresponding output is

$$y(t) = \frac{1}{a}\int_{-\infty}^{\infty} g(\frac{x-t}{b})f(x-c)dx$$

where a, b are non-zero constants and we know that the Fourier Transform of $g(t)$ is $G(j\omega)$.

(a) Determine the frequency response $H(j\omega)$ of the system.

(b) Let the Fourier Transform of $f(t)$ be $F(j\omega) = 2\pi|d|\delta(\omega^2 - d^2)$, where d is a non-zero constant. By setting $G(j\omega) = \frac{a}{|b|}\frac{bd+j\omega}{bd-j\omega}e^{-j\frac{c}{b}\omega}$, determine the output of the LTI system, $y(t)$, by using the answer in part(a).

5. [25 points] In this problem, we will discuss two kinds of filters: RC filter and Gaussian filter.
Part 1. RC circuit
RC circuit is the most common low-pass filter.

(a) Determine the frequency response $H(j\omega)$ of the RC circuit below, which can be governed by

$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

(Hint: You can substitute $x(t) = e^{j\omega t}$ and $y(t) = H(j\omega)e^{j\omega t}$ in the differential equation and then you can obtain $H(j\omega)$)

(b) Explain why $H(j\omega)$ is a low-pass filter.

(c) Derive the continuous-time Fourier transform of the unit step function $u(t)$. And find the corresponding $Y(j\omega)$ when $x(t) = u(t)$.

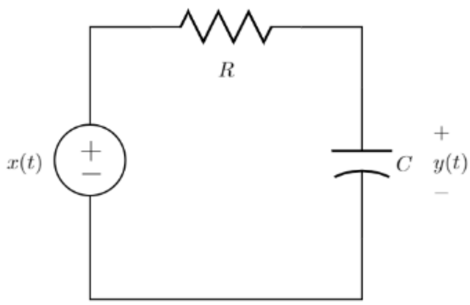

Figure 3: RC circuit

Part 2. Gaussian filter
Gaussian filter is widely used in computer vision. There are blurs under many natural situations and we can interpret them as Gaussian blur.

(a) Please find the continuous-time Fourier transform of $g(t) = e^{-t^2}$. (Hint: $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$)

(b) Now we define the one-dimensional Gaussian filter as $g(t) = \frac{1}{\sigma\sqrt{\pi}} e^{-\frac{t^2}{\sigma^2}}$. We also define the error function $erf(t) = \frac{1}{\sqrt{\pi}} \int_{-t}^{t} e^{-\tau^2} d\tau$. Error function is widely used in probability and statistics. $erf(t)$ can be seen in the graph below. It has the following property:

$$\int_{-\infty}^{t} e^{-\frac{\tau^2}{\sigma^2}} d\tau = \frac{\sigma\sqrt{\pi}}{2} + \frac{\sigma\sqrt{\pi}}{2} erf(\frac{t}{\sigma})$$

Please find and sketch $f(t) = u(t) * g(t)$ with $\sigma = 1$, where $u(t)$ is unit step function.

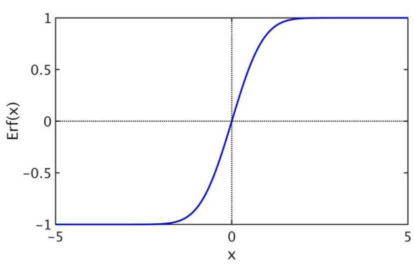

Figure 4: Error function