Laplace Transform and Inverse Laplace Transform

Fourier transform:

The Fourier transform of f(t): $F(j\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$

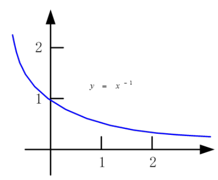
The inverse Fourier transform: $f(t) = \dfrac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)e^{i\omega t}d\omega$

Laplace transform:

Bilateral Laplace transform pair:

$$F(s) = \int_{-\infty}^{+\infty} f(t)e^{-\mathrm{st}}dt, \quad s = \sigma + j\omega \quad \Rightarrow \quad F(s) = \int_{-\infty}^{+\infty} f(t)e^{-\mathrm{st}}dt = \int_{-\infty}^{+\infty} [f(t)e^{-\sigma t}]e^{-j\omega t}dt = F(f(t)e^{-\sigma t})$$

$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{\mathrm{st}}ds, \quad s = \sigma + j\omega$$



Unilateral Laplace transform pair:

$$F(s) = \int_{0}^{+\infty} f(t)e^{-\mathrm{st}}dt, \quad s = \sigma + j\omega$$

$$f(t) = \left[\frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{\mathrm{st}}ds\right]u(t), \quad s = \sigma + j\omega$$

Laplace Transform with Function *laplace*

Find the Laplace transform of $f(t) = e^{-t}sin(at)u(t)$

```
clear
syms t a
f = exp(-t)*sin(a*t)*heaviside(t);
L = laplace(f)
```

L =

$$\frac{a}{(s+1)^2 + a^2}$$

Inverse Laplace Transform with Function *ilaplace*

Find the inverse Laplace transform of $F(s) = \dfrac{s^2}{s^2 + 1}$.

```
clear
syms s
F = s^2/(s^2+1);
```

```
ft = ilaplace(F)
```

ft = $\delta(t) - \sin(t)$

Poles and Zeros

Relationship between Poles/Zeros and the Impulse Response

$$H(s) = \frac{(s-b)}{s \cdot (s-a)}$$

```
syms s a b
H = (s-b)/(s*(s-a))
```

H =

$$\frac{b-s}{s\ (a-s)}$$

```
ilaplace(H)
```

ans =

$$\frac{b}{a} + \frac{e^{a\,t}\ (a-b)}{a}$$

实轴上的一阶极点：$\mathrm{H1}(s) = \frac{1}{(s+0.1)}, \mathrm{H2}(s) = \frac{1}{s}, \mathrm{H3}(s) = \frac{1}{(s-0.1)}$

s 平面上的一阶极点：$\mathrm{H4}(s) = \frac{1}{(s+0.1)^2+1}; \mathrm{H5}(s) = \frac{1}{s^2+1}; \mathrm{H6}(s) = \frac{1}{(s-0.1)^2+1}$

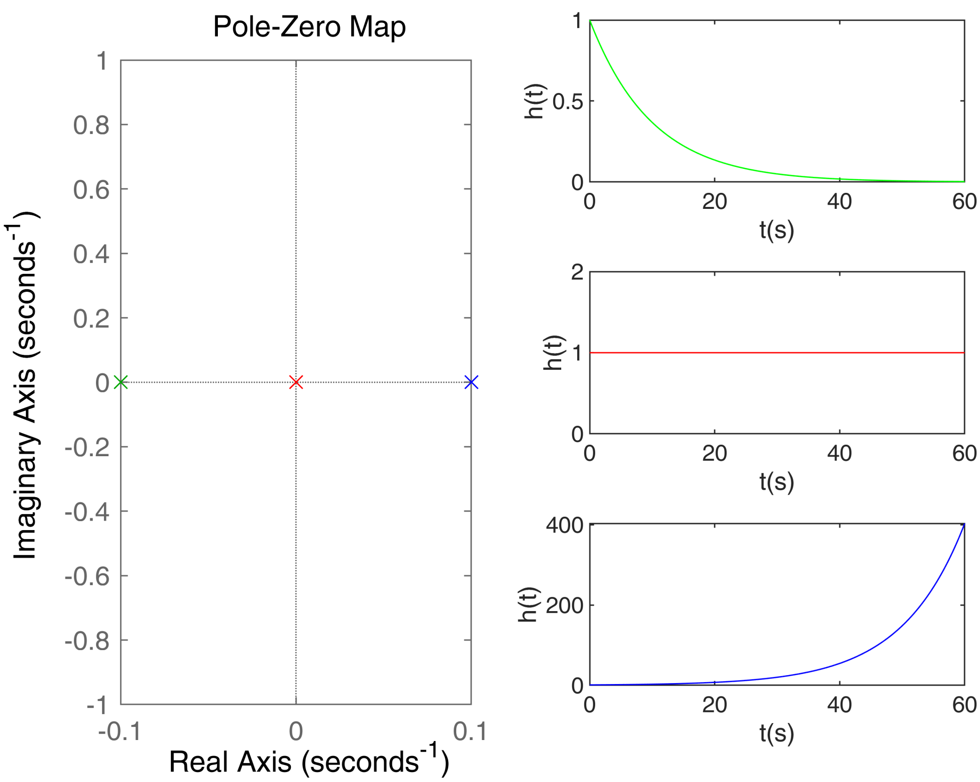
实轴上的二阶极点：$\mathrm{H7} = \frac{1}{(s+0.1)^2}, \mathrm{H8} = \frac{1}{s^2}, \mathrm{H9} = \frac{1}{(s-0.1)^2}$

s 平面上的二阶极点：$\mathrm{H10} = \frac{1}{((s+0.1)^2+1)^2}, \mathrm{H11} = \frac{1}{(s^2+1)^2}, \mathrm{H12} = \frac{1}{((s-0.1)^2+1)^2}$

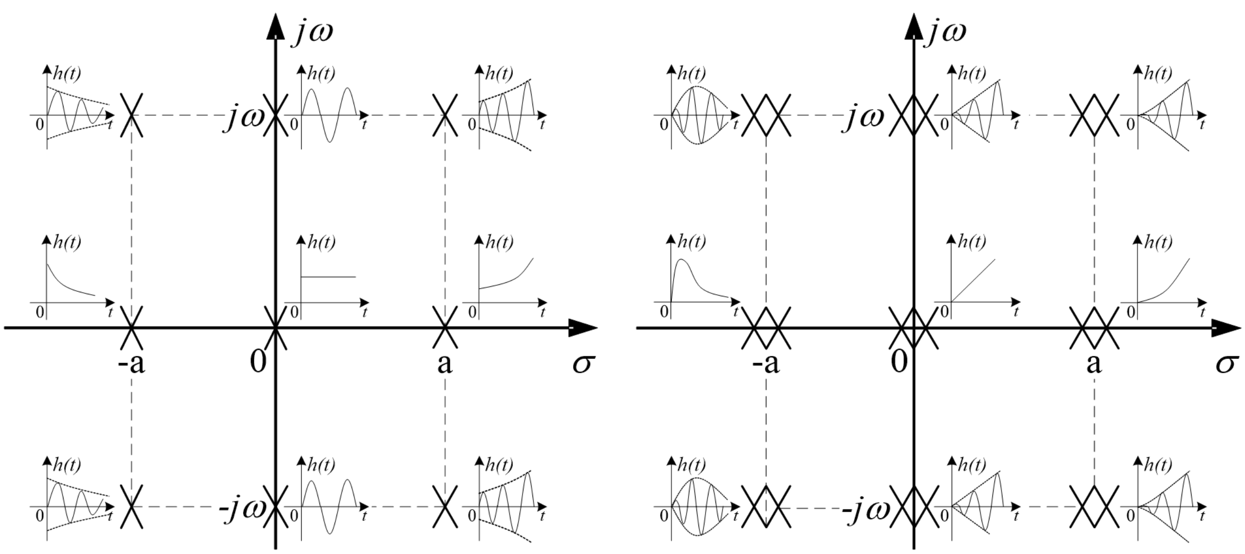
```
clear;clf;
t = 0:0.001:60;
b = 1;
a = [1 0.1;1 0; 1 -0.1];

color = ["g","r","b"];
subplot(3,2,[1 3 5]); hold on;
for i=1:3
    sys = tf(b,a(i,:));
    pzplot(sys,color(i));ylim([-1 1]);
end
for i =1:3
    sys = tf(b,a(i,:));
    subplot(3,2,i*2); plot(t,impulse(sys,t),color(i));
    xlabel('t(s)');ylabel('h(t)');
```

```
end
```



Relationship between Poles and System Stability (the shape of the impulse response).



System Analysis

Solving Differential Equation

Differential Properties of Laplace Transform

$f(t) \to F(s)$

$y(t) \to Y(s)$

$y'(t) \to s \cdot Y(s) - y(0_-)$
$y''(t) \to s^2 \cdot Y(s) - s \cdot y(0_-) - y'(0_-)$

Apply Laplace Transform on Differential Equation

$$\begin{cases} y''(t) + 3y'(t) + 2y(t) = f(t), f(t) = e^{-t}u(t) \\ y(0_-) = 1, y'(0_-) = 2 \end{cases}$$

$$\left[s^2 \cdot Y(s) - s \cdot y(0_-) - y'(0_-)\right] + 3 \cdot [s \cdot Y(s) - y(0_-)] + 2 \cdot Y(s) = F(s)$$
$$Y(s) \cdot [s^2 + 3s + 2] + [-s \cdot y(0_-) - y'(0_-) - 3 \cdot y(0_-)] = F(s)$$
$$Y(s) \cdot [s^2 + 3s + 2] = (s - 2 - 3) + F(s)$$

$$Y(s) = \underbrace{\frac{s+5}{s^2+3s+2}}_{Y_{\mathrm{zi}}} + \underbrace{\overbrace{\frac{1}{s^2+3s+2}}^{H(s)} \cdot F(s)}_{Y_{\mathrm{zs}}} \qquad \begin{matrix} h(t) \\ f(t) \\ y_{\mathrm{zs}}(t) = h(t) * f(t) \end{matrix}$$

$$\begin{aligned} Y(s) &= \frac{s+5}{s^2+3s+2} + \frac{1}{s^2+3s+2} \cdot \frac{1}{s+1} \\ &= \left[\frac{-3}{s+2} + \frac{4}{s+3}\right] + \left[\frac{1}{(s+1)^2} + \frac{-1}{s+1} + \frac{1}{s+2}\right] \end{aligned}$$
$$y(t) = [-3e^{-2t} + 4e^{-3t}] + [te^{-t} - e^{-t} + e^{-2t}]$$

Find out **Zreo-state** Response $y''(0_-) = 0$

$$y''(t) + 3y'(t) + 2y(t) = f(t), f(t) = e^{-t}u(t)$$

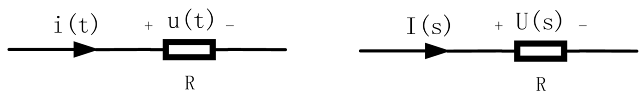
```
syms t s
Hs = 1/(s^2+3*s+2);
ft = exp(-t)*heaviside(t);
Fs = laplace(ft);
Ys = Fs*Hs;
yt = ilaplace(Ys)
```

yt = $\mathrm{e}^{-2t} - \mathrm{e}^{-t} + t\,\mathrm{e}^{-t}$

Circuits Analysis

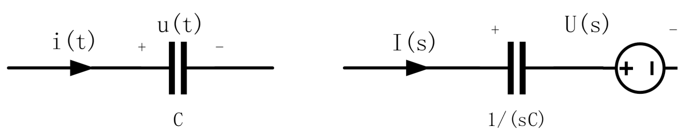
For resistance:

$$u(t) = R \cdot i(t)$$
$$\mathcal{L}[u(t)] = \mathcal{L}[R \cdot i(t)]$$
$$\frac{U(s)}{I(s)} = R$$



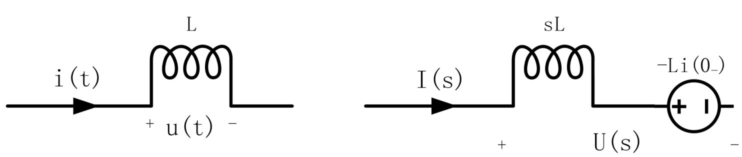
For capacitor:

$$i(t) = C\frac{du(t)}{dt}$$
$$\mathcal{L}[i(t)] = \mathcal{L}[C\frac{du(t)}{dt}]$$
$$I(s) = C[sU(s) - u(0_-)]$$
$$\frac{U(s)}{I(s)} = \frac{1}{sC}$$

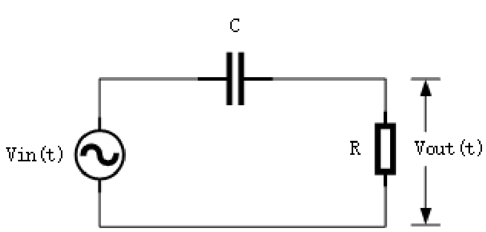


For inductor:

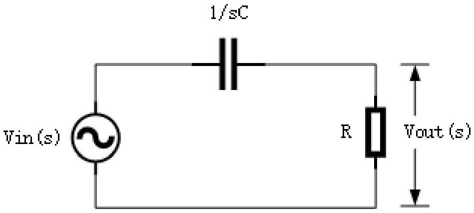
$$u(t) = L\frac{di(t)}{dt}$$
$$\mathcal{L}[u(t)] = \mathcal{L}[L\frac{di(t)}{dt}]$$
$$U(s) = L[sI(s) - i(0_-)]$$
$$\frac{U(s)}{I(s)} = sL$$



$C = 11.29\mathrm{nF}$, $R = 4.7k\Omega$. Observe [0,10kHz].



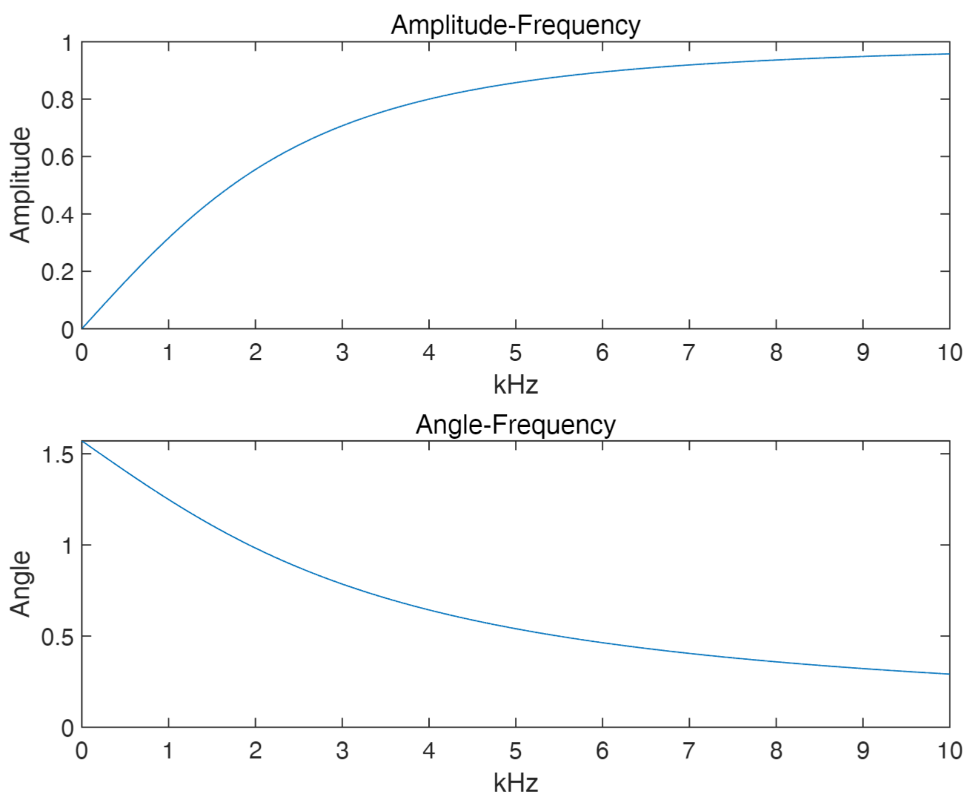
复频域等效电路



$$V_{\text{out}}(s) = V_{\text{in}}(s) \cdot \frac{R}{R + \frac{1}{\text{Cs}}}$$

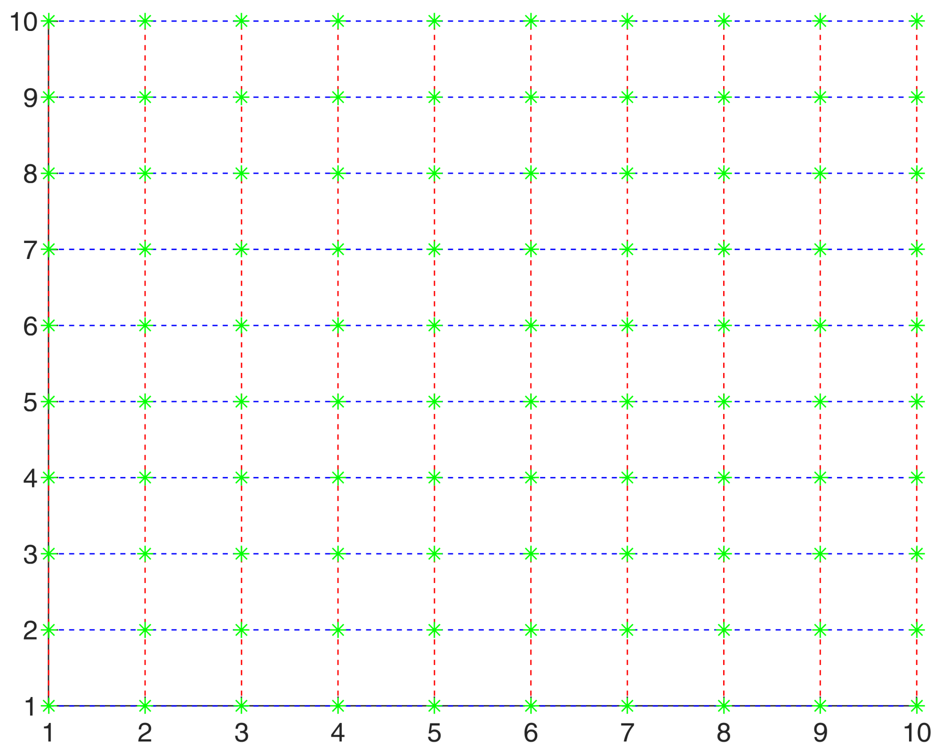
$$H(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{R}{R + \frac{1}{\text{Cs}}} = \frac{\text{RCs}}{1 + \text{RCs}} = \frac{s \cdot \frac{1}{18850}}{1 + s \cdot \frac{1}{18850}}$$

```
clear;clf;
b = [1/18850 0];
a= [1/18850 1];
w=linspace(1,10000*2*pi,1000);
H=freqs(b,a,w);
subplot(2,1,1);plot(w/(2*pi)/1000,abs(H));
title('Amplitude-Frequency');xlabel('kHz');ylabel('Amplitude');
subplot(2,1,2);plot(w/(2*pi)/1000,angle(H));
title('Angle-Frequency');xlabel('kHz');ylabel('Angle');
```



Surface plot for Laplace transform

```
% meshgrid
clear;clf;
x = 1:10;
y = x;
[X Y] = meshgrid(x,y);
figure;hold on;axis([1 length(x) 1 length(y)])
for i = 1:length(y);
    plot(X(i,:),Y(i,:),'b--');
    pause(0.5)
end
for i=1:length(x)
    plot([X(1,i) X(1,i)],[Y(1,i) Y(end,i)],'r--');
    pause(0.5)
end
s = size(X);
for i = 1:s(1)
    for j = 1:s(2)
        plot([X(i,j) X(i,j)],Y(i,j),"g *")
    end
end
```



$F(s) = \dfrac{2(s-3)(s+3)}{(s-5)(s^2+10)}$, do the surface plot of F(s).

```
clf;clear;
x = -6:0.48:6;y=x;
```

```
[sigma,omega] = meshgrid(x,y);
s = sigma+1j*omega;
Fs = (2*(s-3).*(s+3))./((s-5).*(s.*s+10));
Fsabs = abs(Fs);
surf(sigma,omega,Fsabs);
axis([-6,6,-6,6,0,4.5]);
xlabel('sigma');ylabel('omega');zlabel('abs(F(s))')
title('Surface Plot of Laplace Transform');
colormap(hsv);
rotate3d on;
```