Review

1. Logical Equivalence

$A \equiv B$ means A and B has the same truth value for every truth assignment.

Prove: $A \equiv B$

- By definition, $A^{-1}(\mathrm{T}) = B^{-1}(\mathrm{T})$. i.e. $A^{-1}(\mathrm{T}) \subseteq B^{-1}(\mathrm{T})$ and $B^{-1}(\mathrm{T}) \subseteq A^{-1}(\mathrm{T})$
- $A \leftrightarrow B \equiv \mathrm{T},\ A \to B \equiv B \to A \equiv \mathrm{T}$, $A \Rightarrow B$ and $B \Rightarrow A$
- Laws. e.g. $A \to B \equiv \neg A \vee B$

2. Tautological Implications

$A \Rightarrow B$ means every truth assignment that causes A to be true causes B to be true.

Prove: $A \Rightarrow B$

- By definition, $A^{-1}(\mathrm{T}) \subseteq B^{-1}(\mathrm{T})$ or $B^{-1}(\mathrm{F}) \subseteq A^{-1}(\mathrm{F})$
- $A \to B \equiv \mathrm{T}$ or $\neg B \to \neg A \equiv \mathrm{T}$
- $A \wedge \neg B \equiv \mathrm{F}$
- Laws. e.g. $P \wedge Q \Rightarrow P$

3. Propositional function, Predicate logic

(i) Translation

$\forall$ uses $\to$, $\exists$ uses $\wedge$. Why?

- On empty domain, $\forall x P(x) \equiv \mathrm{T},\ \exists x P(x) \equiv \mathrm{F}$
- If $\forall x\ (P(x) \wedge \cdots)$, if x not in domain, making $P(x)$ false, the whole formula is directly F.
- If $\exists x\ (P(x) \to \cdots)$, if x not in domain, making $P(x)$ false, the whole formula is directly T.

(ii) Proof of $\equiv$ and $\Rightarrow$

4. Graph definitions

Homework 10

1.

(a) $\forall x(P(x) \rightarrow \exists y(P(y) \wedge \neg E(x,y) \wedge L(x,y)))$

(b) $\exists x(P(x) \wedge \forall y(P(y) \wedge \neg E(x,y) \rightarrow L(y,x)))$

5. Proof: $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$**.**

(i) Show that $\exists x(P(x) \vee Q(x)) \Rightarrow \exists x P(x) \vee \exists x Q(x)$.

Suppose that $\exists x(P(x) \vee Q(x))$ is T in an interpretation I.

- There is an x_0 s.t. $P(x_0) \vee Q(x_0)$ is T in I.
- At least one of $P(x_0)$ and $Q(x_0)$ is T in I. WLOG, suppose it's $P(x_0)$.
- $\exists x P(x)$ is T in I.
- $\exists x P(x) \vee \exists x Q(x)$ is T in I.

(ii) Show that $\exists x(P(x) \vee Q(x)) \Leftarrow \exists x P(x) \vee \exists x Q(x)$.

Suppose that $\exists x P(x) \vee \exists x Q(x)$ is T in an interpretation I.

- At least one of $\exists x P(x)$ and $\exists x Q(x)$ is T in I. WLOG, suppose it's $\exists x P(x)$.
- There is an x_0 s.t. $P(x_0)$ is T in I
- $P(x_0) \vee Q(x_0)$ is T in I.
- $\exists x(P(x) \vee Q(x))$ is T in I.

6. Proof: $\forall x(P(x) \rightarrow Q(x)) \Rightarrow \forall x P(x) \rightarrow \forall x Q(x)$

Suppose that $\forall x P(x) \rightarrow \forall x Q(x)$ is F in an interpretation I.

- $\forall x P(x)$ is T and $\forall x Q(x)$ is F in I .
- $P(x)$ is T for every x in I and there is an x_0 s.t. $Q(x_0)$ is F in I.
- $P(x_0) \rightarrow Q(x_0)$ is F in I.
- $\forall x(P(x) \rightarrow Q(x))$ is F in I.