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11.3

1 (4) $\int_L y^2dx + xydy + xzdz$

$= \int_{L_1} y^2dx + \int_{L_2} xydy + \int_{L_3} xzdz$

$= 0 + \int_0^1 ydy + \int_0^1 zdz$

$= 1$

1 (5) $\int_L e^{x+y+z}dx + e^{x+y+z}dy + e^{x+y+z}dz$

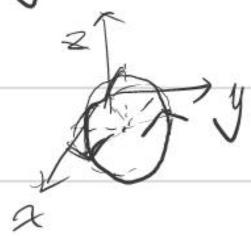
$= \int_L e^{x+y+z}d(x+y+z)$

$= \int_1^{\frac{3}{2}} e^t dt$

$= e^{\frac{3}{2}} - e$

1 (6) 先求交线：$\begin{cases} x+y=2 \\ x^2+y^2+z^2=2(x+y) \end{cases} \Rightarrow \begin{cases} (x-1)^2+\frac{z^2}{2}=1 \\ x+y=2 \end{cases}$

故设 $\begin{cases} x=1+\cos\theta \\ y=\sqrt{2}\sin\theta \\ z=1-\cos\theta \end{cases}$, $\theta\in[0,2\pi]$



$\int_L ydx+zdy+xdz$

$= \int_0^{2\pi}[\sqrt{2}\sin\theta\cdot(-\sin\theta)+(1-\cos\theta)\cdot(\sqrt{2}\cos\theta)+(1+\cos\theta)\sin\theta]d\theta$

$= \int_0^{2\pi}[-\sqrt{2}\sin^2\theta+\sqrt{2}\cos\theta-\sqrt{2}\cos^2\theta+\sin\theta+\sin\theta\cos\theta]d\theta$

$= \int_0^{2\pi}(-\sqrt{2}+\sin\theta+\sqrt{2}\cos\theta+\sin\theta\cos\theta)d\theta$

$= -2\sqrt{2}\pi$

3. $\vec{F} = -k\vec{r} = -k(x,y)$　注意方向

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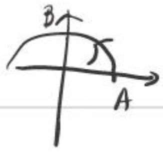

$$L: \begin{cases} x = a\cos\theta, \\ y = b\sin\theta \end{cases} \quad \theta \in [0, \frac{\pi}{2}]$$

$$W = (-k) \int_L x\,dx + y\,dy$$
$$= (-k) \int_0^{\frac{\pi}{2}} [(a\cos\theta)(-a\sin\theta) + (b\sin\theta)(b\cos\theta)]d\theta$$
$$= k(a^2-b^2) \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,d\theta$$
$$= \frac{1}{2}k(a^2-b^2)$$

4. (3) $\oint (yx^3+e^y)dx + (xy^3+xe^y-2y)dy$

$$= \iint_D \frac{\partial}{\partial x}(xy^3+xe^y-2y) - \frac{\partial}{\partial y}(yx^3+e^y)\ dxdy$$
$$= \iint_D (y^3+e^y-x^3-e^y)\,dxdy$$
$$= \iint_D (y^3-x^3)\,dxdy$$

因对称性,原式 $=0$.

4. (4) L: $y^2=x-1$ 与 $x=2$ 围成的曲线,逆时针.

$$\oint_L \sqrt{x^2+y^2}\,dx + y[xy+\ln(x+\sqrt{x^2+y^2})]dy$$
$$= \iint_D y^2 + \frac{y}{x+\sqrt{x^2+y^2}} \cdot \left(1+\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{y}{\sqrt{x^2+y^2}}\ dxdy$$
$$= \iint_D y^2\,dxdy$$
$$= \int_{-1}^{1} y^2 dy \int_{y^2+1}^{2} dx$$
$$= \int_{-1}^{1} (1-y^2)y^2\,dy$$
$$= \frac{4}{15}$$

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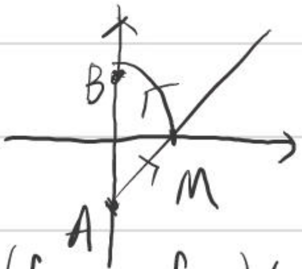
4. (5)



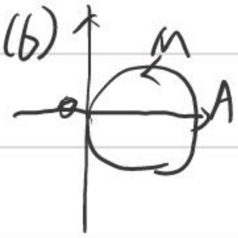
$$原式 = \left(\oint_{AMBA} + \int_{AB}\right)(x^2+2xy-y^2)dx + (x^2-2xy+y^2)dy$$

$$= \iint_D (2x-2y)-(2x-2y) \quad dxdy + \int_{AB}(x^2+2xy-y^2)dx+(x^2+2xy+y^2)dy$$

$$= 0 + \int_{-1}^{1} y^2 dy$$

$$= \frac{2}{3}$$

4 (6)



$$原式 = \left(\oint_{AMOA} + \int_{AO}\right)(e^x \sin y - my)dx + (e^x \cos y - m)dy$$

$$= \iint_D e^x \cos y - (e^x \cos y - m) \quad dxdy + 0$$

$$= m \cdot \iint_D dxdy$$

$$= m \cdot \frac{1}{2}\pi \left(\frac{1}{2}a\right)^2$$

$$= \frac{1}{8} m\pi a^2$$

5. (1)

$$S = \oint_L x dy$$

$$= 4 \int_0^{\frac{\pi}{2}} (a\cos^3 t)(3a\sin^2 t \cos t) dt$$

$$= 12a^2 \int_0^{\frac{\pi}{2}} \sin^2 t \cos^4 t dt$$

$$= 12a^2 \cdot \int_0^{\frac{\pi}{2}} (\cos^4 t - \cos^6 t) dt$$ Wallis公式·

$$= \frac{3}{8}\pi a^2$$

5.(2) $S = \oint_L x dy$

$$= -\int_0^{2\pi} a(t-\sin t)(a\sin t) dt$$

$$= -a^2 \int_0^{2\pi} (t\sin t - \sin^2 t) dt$$

$$= 3\pi a^2$$

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6. $\int_L \frac{-ydx+xdy}{x^2+y^2}$

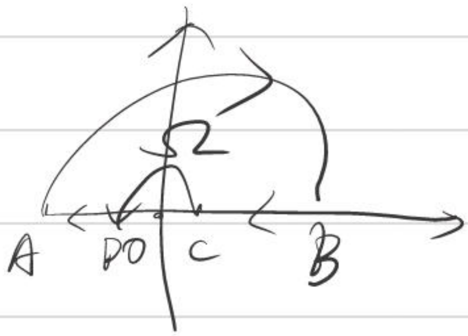
(1)

原式 $= \int_L \frac{-ydx+xdy}{a^2}$

$= \int_{\pi}^{0} \frac{(-a\sin\theta)(-a\sin\theta)+a\cos\theta)(a\cos\theta)}{a^2} d\theta$

$= \int_{\pi}^{0} d\theta = -\pi$

(2) 考虑简单闭合曲线：



$\overset{\frown}{ABCDA}$,

AB 为抛物线 $y=4-(x-1)^2$,

CD 为半径为 ε 的半圆, DA, BC 为直线.

则区域 Ω 内无奇点. 且有 $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$

故由 Green 公式：

$\oint \frac{-ydx+xdy}{x^2+y^2} = \iint_{\Omega} 0\, dxdy = 0$

$\int_{CB} \frac{-ydx+xdy}{x^2+y^2} = \int_{AD} \frac{-ydx+xdy}{x^2+y^2} = 0$

$\int_{DC} \frac{-ydx+xdy}{x^2+y^2} = -\pi$ (由(1)知)

故 原式 $= \left(\oint + \int_{CB} + \int_{DC} + \int_{AD}\right)\left(\frac{-ydx+xdy}{x^2+y^2}\right)$

$= -\pi$

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7. (1) 设 $\vec{n}=\cos\alpha\,\vec{i}+\cos\beta\,\vec{j}$

$\vec{\tau}=-\cos\beta\,\vec{i}+\cos\alpha\,\vec{j}$，有 $d\vec{r}=\vec{\tau}\,ds$

故 $\oint_L \frac{\partial f}{\partial n}ds=\oint_L\left(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta\right)ds$

$=\oint_L \frac{\partial f}{\partial x}\cos\alpha\,ds+\frac{\partial f}{\partial y}\cos\beta\,ds$

$=\oint \frac{\partial f}{\partial x}dy-\frac{\partial f}{\partial y}dx$

$=\iint_D \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)-\frac{\partial}{\partial y}\left(-\frac{\partial f}{\partial y}\right)\,dxdy$

$=\iint_D \Delta f\,dxdy$

得证.

(2) $\oint_L \cos\langle\vec{a},\vec{n}\rangle ds$

$=\oint_L \frac{\vec{a}\cdot\vec{n}}{|a|}ds$

$=\frac{1}{|a|}\oint_L(a_x\cos\alpha+a_y\cos\beta)ds$

$=\frac{1}{|a|}\oint_L(a_x dy-a_y dx)$

$=\frac{1}{|a|}\iint_D\left(\frac{\partial}{\partial x}(a_x)+\frac{\partial}{\partial y}(a_y)\right)dxdy$

$=0$　（a是常值向量）

(3)

对于 $\oint_L v\frac{\partial u}{\partial n}ds=\oint_L v\left(\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta\right)ds$

$=\oint_L v\left(\frac{\partial u}{\partial x}dy-\frac{\partial u}{\partial y}dx\right)$

$=\oint_L -v\frac{\partial u}{\partial y}dx+v\frac{\partial u}{\partial x}dy$

$=\iint_D \frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right)\,dxdy$

$=\iint_D\left(v\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+v\frac{\partial^2 u}{\partial y^2}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right)dxdy$

$=\iint_D(v\Delta u+\nabla u\cdot\nabla v)dxdy$

同理可得 $\oint_L u\frac{\partial v}{\partial n}ds=\iint\limits_D(u\Delta v+\nabla u\cdot\nabla v)dxdy$

故 $\oint_L(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n})ds$

$=\iint\limits_D(v\Delta u-u\Delta v)dxdy$