LA homework Dec.01 name: 王柯皓 id: 202153302

§ 8.3 ( Page 817 )

**6.** Suppose that the linear transformations $T_1: P_2 \to P_2$ and $T_2: P_2 \to P_3$ are given by the formulas $T_1(p(x)) = p(x+1)$ and $T_2(p(x)) = xp(x)$. Find $(T_2 \circ T_1)(a_0 + a_1 x + a_2 x^2)$.

$(T_2 \circ T_1)(p(x)) = T_2(T_1(p(x))) = T_2(p(x+1)) = (x+1)p(x+1)$

$(T_2 \circ T_1)(a_0 + a_1 x + a_2 x^2) = a_0(x+1) + a_1(x+1)^2 + a_2(x+1)^3$

**7.** Let $q_0(x)$ be a fixed polynomial of degree $m$, and define a function $T$ with domain $P_n$ by the formula $T(p(x)) = p(q_0(x))$. Show that $T$ is a linear transformation.

let $T_1: q_0(x) \to t$ $T_2: t \to p(t)$.

which are all linear transformations,

so $(T_2 \circ T_1)$ is also a linear transformation

that is $T(p(x)) = p(q_0(x))$

**10.** In each part, let $T: R^2 \to R^2$ be multiplication by $A$. Determine whether $T$ has an inverse; if so, find

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

**(a)** $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

**(b)** $A = \begin{bmatrix} 6 & -3 \\ 4 & -2 \end{bmatrix}$

(a) $A^{-1} = \frac{1}{5\times1 - 2\times2}\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ -2x_1 + 5x_2 \end{bmatrix}$

(b) $\det(A) = 6\times(-2) - 4\times(-3) = 0$

so $T$ has no inverse

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&\\0&1\end{bmatrix}$$

**12.** In each part, determine whether the linear operator $T: R^n \rightarrow R^n$ is one-to-one; if so, find $T^{-1}(x_1, x_2, \ldots, x_n)$.

(a) $T(x_1, x_2, \ldots, x_n) = (0, x_1, x_2, \ldots, x_{n-1})$

(b) $T(x_1, x_2, \ldots, x_n) = (x_n, x_{n-1}, \ldots, x_2, x_1)$

(c) $T(x_1, x_2, \ldots, x_n) = (x_2, x_3, \ldots, x_n, x_1)$

(b) $[T] = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & 0 \\ \vdots & \vdots & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} = [T^{-1}]$ So $T^{-1}(x_1, x_2, \cdots, x_n)$
$= (x_n, x_{n-1}, \cdots, x_2, x_1)$

(c). $[T] = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$ $[T^{-1}] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & & \vdots \\ \vdots & \vdots & \vdots & \cdots & & 1 \\ 1 & 0 & 0 & \cdots & & 0 \end{bmatrix}$

$T^{-1}(x_1, x_2, \cdots, x_n) = (x_n, x_1, x_2, \cdots, x_{n-2}, x_{n-1})$

**17.** Let $T: P_1 \rightarrow R^2$ be the function defined by the formula

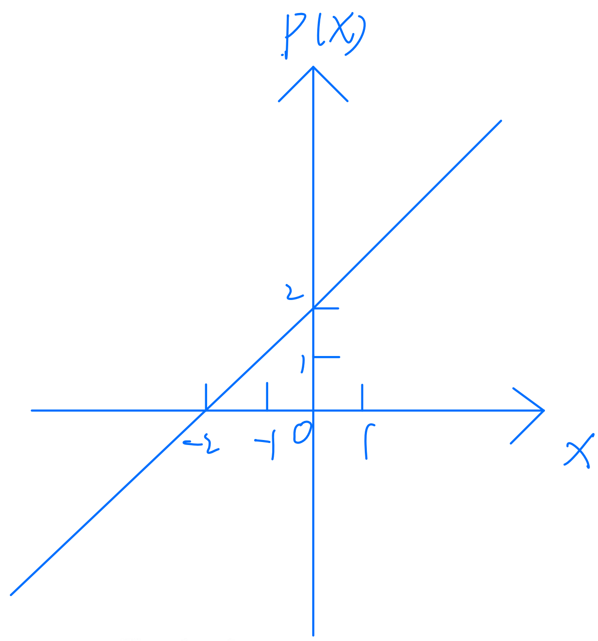
$$T(p(x)) = (p(0), p(1))$$

(a) Find $T(1 - 2x)$.

(b) Show that $T$ is a linear transformation.

(c) Show that $T$ is one-to-one.

(d) Find $T^{-1}(2, 3)$, and sketch its graph.




(a) $T(1-2x) = (1, -1)$

(b).

(c) let $p(x) = a_0 + a_1 x$. then $T(p(x)) = (a_0, a_0 + a_1)$ so if $T(p(x)) = (0, 0)$ then $a_0 = a_1 = 0$ and $p$ is the zero polynomial so $\ker(T) = \{0\}$

(d) $T(p(x)) = (a_0, a_0 + a_1)$ $a_0 = 2$ $a_1 = 1$

$T^{-1}(2, 3) = 2 + x$

**18.** Let $T: R^2 \rightarrow R^2$ be the linear operator given by the formula $T(x, y) = (x + ky, -y)$. Show that $T$ is one-to-one and that $T^{-1} = T$ for every real value of $k$.

$[T] = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$ $\ker([T]) = \{0\}$ so $T$ is one-to-one.

$[T^{-1}] = [T]^{-1}$ $\left[\begin{array}{cc|cc} 1 & k & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & k & 1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & k \\ 0 & 1 & 0 & -1 \end{array}\right]$

So $[T^{-1}] = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} = [T]$

In Exercises 20–21, determine whether $T_1 \circ T_2 = T_2 \circ T_1$.

**20.** (a) $T_1 : R^2 \to R^2$ is the orthogonal projection on the $x$-axis, and $T_2 : R^2 \to R^2$ is the orthogonal projection on the $y$-axis.

(b) $T_1 : R^2 \to R^2$ is the rotation about the origin through an angle $\theta_1$, and $T_2 : R^2 \to R^2$ is the rotation about the origin through an angle $\theta_2$.

(c) $T_1 : R^3 \to R^3$ is the rotation about the $x$-axis through an angle $\theta_1$, and $T_2 : R^3 \to R^3$ is the rotation about the $z$-axis through an angle $\theta_2$.

(a) $T_1 \circ T_2 = T_2 \circ T_1$

(b) $T_1 \circ T_2 \neq T_2 \circ T_1$

(c) $T_1 \circ T_2 = T_2 \circ T_1$

§ 8.5 ( Page 845 )

**2.** $T : R^2 \to R^2$ is defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 7x_2 \\ 3x_1 - 4x_2 \end{bmatrix}$$

and $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}; \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$[T]_B = \begin{bmatrix} 1 & 7 \\ 3 & -4 \end{bmatrix}$ $\quad [T(V_1)]_{B'} = \begin{bmatrix} 1 & 7 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 22 \\ -9 \end{bmatrix} = -\frac{31}{2}\vec{V_1} - \frac{75}{2}\vec{V_2}$

$[T(V_2)]_{B'} = \begin{bmatrix} 1 & 7 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \end{bmatrix} = \frac{9}{2}\vec{V_1} + \frac{25}{2}\vec{V_2}$

$[T]_{B'} = \begin{bmatrix} -\frac{31}{2} & -\frac{75}{2} \\ \frac{9}{2} & \frac{25}{2} \end{bmatrix}$

**6.** $T : R^2 \to R^2$ is defined by $T(\mathbf{x}) = 5\mathbf{x}$, and $B$ and $B'$ are the bases in Exercise 2.

$T(U_1) = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ $\quad T(U_2) = \begin{bmatrix} 20 \\ -5 \end{bmatrix}$ $\quad [T]_B = \begin{bmatrix} 10 & 20 \\ 10 & -5 \end{bmatrix}$

$U_1 = -2V_2 \quad U_2 = -\frac{5}{2}V_1 - \frac{13}{2}V_2$

$\begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix} -\frac{13}{2} & \frac{5}{2} \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{13}{10} & \frac{1}{2} \\ \frac{2}{5} & 0 \end{bmatrix}$ $\quad = \begin{bmatrix} \frac{75}{2} & -\frac{25}{2} \\ \frac{215}{2} & -\frac{85}{2} \end{bmatrix}$

$[T]_{B'} = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}\begin{bmatrix} 10 & 20 \\ 10 & -5 \end{bmatrix}\begin{bmatrix} -\frac{13}{10} & \frac{1}{2} \\ \frac{2}{5} & 0 \end{bmatrix} = \begin{bmatrix} -25 & \frac{25}{2} \\ -85 & -\frac{15}{2} \end{bmatrix}\begin{bmatrix} -\frac{13}{10} & \frac{1}{2} \\ \frac{2}{5} & 0 \end{bmatrix}$

**10.** Let $T: P_4 \to P_4$ be the linear operator given by the formula $T(p(x)) = p(2x+1)$.

(a) Find a matrix for $T$ relative to some convenient basis, and then use it to find the rank and nullity of $T$.

(b) Use the result in part (a) to determine whether $T$ is one-to-one.

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 - x_3 = 0$

$x_1 = -s + t$

$x_2 = s$

$x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

so $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is a basis corresponding to $\lambda = 1$

**12.** In each part, find a basis for $R^3$ relative to which the matrix for $T$ is diagonal.

(b) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_2 + x_3 \\ -x_1 + x_3 \\ x_1 + x_2 \end{bmatrix}$

$$[T] = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \det(\lambda I - T) = \begin{bmatrix} \lambda & 1 & -1 \\ 1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} = \lambda(\lambda^2 - 1) - (\lambda - 1) - (\lambda - 1)$$

$$= (\lambda - 1)(\lambda^2 - \lambda - 4) = (\lambda - 1)(\lambda + 1)(\lambda - 4) = 0$$

$\lambda = 1$ 或 $-1$ 或 $2$

**16.** (a) Prove that if $A$ and $B$ are similar matrices, then $A^2$ and $B^2$ are also similar. More generally, prove that $A^k$ and $B^k$ are similar if $k$ is any positive integer.

(b) If $A^2$ and $B^2$ are similar, must $A$ and $B$ be similar? Explain.

(a) since $A$ and $B$ are similar matrices

$A = P^{-1} B P \quad A^2 = P^{-2} B^2 P^2 = (P^{-1} P)^2 B^2 = B^2$

$A^k = P^{-k} B^k P^k = (P^{-1} P)^k B^k = B^k$

(b) $A^2 = P^{-1} B^2 P$

A and B are not always similar