习题 11.7

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1. (1)

$\int_{L_1} F\cdot dr=\int_0^1 dy=1$

$\int_{L_2} F\cdot dr=\int_0^1[(-x^2)+(x)(2x)]dx=\int_0^1 x^2dx=\frac{1}{3}$

$\int_{L_3} F\cdot dr=\int_0^1[(-x)+(x)]dx=0$

$\int_{L_4} F\cdot dr=0+\int_0^1(-1)dx=-1$

不相等，因为 $\mathrm{rot}\vec{F}\neq\vec{0}$，即其不是保守场.

(2) $\int_{L_1} F\cdot dr=\int_0^1 dy=1$

$\int_{L_2} F\cdot dr=\int_0^1[(2x)(x^2)+(x^2)(2x)]dx=4\int_0^1 x^3dx=1$

$\int_{L_3} F\cdot dr=\int_0^1(2x^2+x^2)dx=3\int_0^1 x^2dx=1$

$\int_{L_4} F\cdot dr=\int_0^1 2x\,dx=1$

相等，因为 $\mathrm{rot}\vec{F}=\vec{0}$，即其为保守场.

2. (1) 记 $\vec{v}=(2x+y,x+4y+2z,2y-6z)$

$\nabla\times\vec{v}=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x+y & x+4y+2z & 2y-6z\end{vmatrix}=\vec{0}$，故原积分与路径无关

构造 $\overrightarrow{P_1O}$ 平行于x轴，$\overrightarrow{OP_3}$ 平行于z轴

$\int_L\vec{v}\cdot dr=\int_{\overrightarrow{P_1O}}\vec{v}\cdot dr+\int_{\overrightarrow{OP_3}}\vec{v}\cdot dr=\int_a^0 2x\,dx+\int_0^a-6z\,dz=-4a^2$

(2) 记 $\vec{v}=(x^2-yz,y^2-zx,z^2-xy)$

$\nabla\times\vec{v}=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2-yz & y^2-zx & z^2-xy\end{vmatrix}=\vec{0}$，故原积分与路径无关

构造 $\overrightarrow{AB}$ 平行于z轴

$\int_L\vec{v}\cdot dr=\int_{\overrightarrow{AB}}\vec{v}\cdot dr=\int_0^h z^2dz=\frac{1}{3}h^3$

3. (1) 记 $\vec{v}=P\vec{i}+Q\vec{j}\Rightarrow\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$，故 $\vec{v}$ 为无旋场，从而是有势场

其势函数

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$$\varphi(x,y)=\int_{(0,0)}^{(x,y)} v\cdot dr+C=\left(\int_{(0,0)}^{(x,0)}+\int_{(x,0)}^{(x,y)}\right)v\cdot dr+C$$

$$=\int_0^x 2x\,dx+\int_0^y(2y\cos x-x^2\sin y)dy+C$$

$$=x^2+y^2\cos x+x^2\sin x-x^2+C$$

$$=y^2\cos x+x^2\cos y+C$$

(2) $\operatorname{rot}\vec{v}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz(2x+y+z) & xz(2y+z+x) & xy(2z+x+y) \end{vmatrix}=\vec{0}$

故 $\vec{v}$ 为无旋场，从而是有势场，

$$\varphi(x,y)=\int_{(0,0,0)}^{(x,y,z)} v\cdot dr=\left(\int_{(0,0,0)}^{(x,0,0)}+\int_{(x,0,0)}^{(x,y,0)}+\int_{(x,y,0)}^{(x,y,z)}\right)v\cdot dr+C$$

$$=0+0+\int_0^z xy(2z+x+y)dz+C$$

$$=xyz^2+x^2yz+xy^2z$$

(3) $\vec{v}=r^2\vec{r}=(x^2+y^2+z^2)(x\vec{i}+y\vec{j}+z\vec{k})$

$\operatorname{rot}\vec{v}=\vec{0}$，故 $\vec{v}$ 为无旋场，从而是有势场.

$$\varphi(x,y,x)=\left(\int_{(0,0,0)}^{(x,0,0)}+\int_{(x,0,0)}^{(x,y,z)}+\int_{(x,y,0)}^{(x,y,z)}\right)v\cdot dr+C$$

$$=\int_0^x x^3dx+\int_0^y(x^2+y^2)y\,dy+\int_0^z(x^2+y^2+z^2)z\,dz+C$$

$$=\frac{1}{4}x^4+\frac{1}{2}x^2y^2+\frac{1}{4}y^4+\frac{1}{2}x^2z^2+\frac{1}{2}y^2z^2+\frac{1}{4}z^4+C$$

$$=\frac{1}{4}r^4+C$$

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4. $\operatorname{rot}\vec{F} = \nabla\times\vec{F} = \left((2-2a)x, (1-a)y, (3a-3)z+5-5a\right) = \vec{0}$

$\Rightarrow \underline{a=1}$,

$\vec{F} = (x^2+5y+3yz, 5x+3xz-2, 3xy-4z)$

$$\varphi(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} \vec{F}\cdot d\vec{r} + C$$

$$= \left(\int_{(0,0,0)}^{(x,0,0)} + \int_{(x,0,0)}^{(x,y,0)} + \int_{(x,y,0)}^{(x,y,z)}\right)\vec{F}\cdot d\vec{r} + C$$

$$= \int_0^x x^2\,dx + \int_0^y (5x-2)\,dy + \int_0^z (3xy-4z)\,dz + C$$

$$= \frac{1}{3}x^3 + (5x-2)y + 3xyz - 2z^2 + C$$

5. (1) $u(x,y) = \int_{(0,0)}^{(x,y)} du + C = x^3 + 3x^2y^2 - y^4 + C$

(2) $u(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} du + C = \frac{1}{3}(x^3+y^3+z^3) - 2xyz + C$,

6. (2). $\vec{F} = \left(\frac{1}{y}\sin\frac{x}{y} - \frac{y}{x^2}\cos\frac{y}{x} + 1, \frac{1}{x}\cos\frac{y}{x} - \frac{x}{y^2}\sin\frac{x}{y} + \frac{1}{y^2}\right)$

$= P\vec{i} + Q\vec{j}$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$，故与路径无关.

构造 L：沿 $y=x$ 从 $(1,1)\to(2,2)$

原式 $= \int_L \vec{F}\cdot d\vec{r} = \int_1^2 \Big(\frac{1}{x}\sin 1 - \frac{1}{x}\cos 1 + 1 + \frac{1}{x}\cos 1$

$- \frac{1}{x}\sin 1 + \frac{1}{x^2}\Big)\,dx$

$= \int_1^2 \left(1+\frac{1}{x^2}\right)dx = 1 + \left(-\frac{1}{x}\right)\Big|_1^2$

$= \frac{3}{2}$

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(5) $\vec{F}=\left(1-\frac{1}{y}+\frac{y}{z},\ \frac{x}{z}+\frac{x}{y^2},\ -\frac{xy}{z^2}\right)$

$\nabla\times\vec{F}=\vec{0}$.

构造L: $x=y=z$, $(1,1,1)\to(2,2,2)$

原式 $=\int_L \vec{F}\cdot d\vec{r}=\int_1^2\left(2-\frac{1}{x}+1+\frac{1}{x}-1\right)dx$

$=\int_1^2 2dx=2$

(6): $\vec{F}=\frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z)$ 则

$\nabla\times\vec{F}=\vec{0}$, 构造 $L\in$ 球面 $(x_1,y_1,z_1)\to(x_2,y_2,z_2)$

$$\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)}\frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}}=\int_{(a,\theta_1,\varphi_1)}^{(a,\theta_2,\varphi_2)}\frac{1}{a}\,\vec{r}\cdot d\vec{r}=0$$

球面上 $\vec{r}\perp d\vec{r}$, 故 $\vec{r}\cdot d\vec{r}=0$
简化计算

7.(1) $\oint_L f(x^2+y^2)(xdx+ydy)$ 连续不一定可微 但一定可积 故 $\exists\varphi$

$=\frac{1}{2}\oint_L f(x^2+y^2)\,d(x^2+y^2)$

令 $\varphi(x,y)=\int_{(x_1,y_1)}^{(x,y)}\frac{f(x^2+y^2)}{2}d(x^2+y^2)\ \overset{t=x^2+y^2}{=\!=}\ \frac{1}{2}\int_{x_1^2+y_1^2}^{x^2+y^2}f(t)dt$

$\varphi(x,y)$ 满足 $\nabla\varphi=f(x^2+y^2)(x\vec{i}+y\vec{j})$

即 $\varphi(x,y)$ 为 $\vec{v}=f(x^2+y^2)(x\vec{i}+y\vec{j})$ 的势函数,

故其环量 $\equiv 0$，得证。

(2) $f(\sqrt{x^2+y^2+z^2})(x\,dx+y\,dy+z\,dz)$

$$= \frac{1}{2} f(\sqrt{x^2+y^2+z^2})\, d(x^2+y^2+z^2)$$

$$\overset{t=\sqrt{x^2+y^2+z^2}}{=\!=} \frac{1}{2} f(t)\, 2t\, dt$$

$$= t f(t)\,dt$$

令 $\varphi(x,y,z) = \int_{\sqrt{x_0^2+y_0^2+z_0^2}}^{\sqrt{x^2+y^2+z^2}} t f(t)\,dt$，满足 $\nabla\varphi = f(\sqrt{x^2+y^2+z^2})(x\vec{i}+y\vec{j}+z\vec{k})$

故 $\vec{F} = f(\sqrt{x^2+y^2+z^2})(x\vec{i}+y\vec{j}+z\vec{k})$ 为有势场

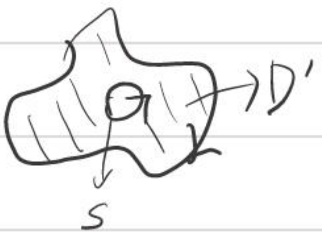
环量 $\equiv 0$. 得证.

8. 记 $B = P\vec{i}+Q\vec{j}$，$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y} = \frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}+\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = 0$

注意有势场结论需满足区域单连通.

存在奇点 $O(0,0)$，区域为复连通区域. 不满足条件.

使用 Green 公式 + 挖个 ε-圆的方法:



$$\oint_{\partial D} B\cdot dr = \left(\oint_{\partial D'} - \oint_S\right) B\cdot dr$$

$$= \iint_{D'} \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dS + \oint_S B\,dr$$

$$= \frac{2I}{\varepsilon^2}\int_0^{2\pi} -y\,dx+x\,dy = 4\pi I$$

11. 由路径无关 $\Rightarrow Q'_x = 2x \Rightarrow Q(x,y) = x^2+f(y)$

$$\int_{(0,0)}^{(x,y)} 2xy\,dx + [x^2+f(y)]\,dy$$

$$= \left(\int_{(0,0)}^{(x,0)} + \int_{(x,0)}^{(x,y)}\right) 2xy\,dx + [x^2+f(y)]dy$$

$$= x^2 y + \int_0^y f(y)\,dy$$

代入 $(t,1)$, $(1,t)$

$\Rightarrow$ $t^2 + \int_0^1 f(y)\,dy = t + \int_0^t f(y)\,dy$

对t求导 $\Rightarrow$ $2t + 0 = 1 + f(t)$

$\Rightarrow f(t) = 2t-1$

$\Rightarrow Q(x,y) = x^2 + 2y - 1$

14. $\nabla \times \vec{v} = 0 \Rightarrow \dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

$$-2x(x^4+y^2)^{\lambda} - x^2(x^4+y^2)^{\lambda-1}\lambda\, 4x^3 = 2x(x^4+y^2)^{\lambda} + 2xy\,\lambda(x^4+y^2)^{\lambda-1}2y$$

$$\Rightarrow (x^4+y^2)^{\lambda}\cdot 4x + 4\lambda(x^4+y^2)^{\lambda-1}(x^5+xy^2) = 0$$

$$\Rightarrow 4[x(x^4+y^2) + \lambda(x^5+xy^2)](x^4+y^2)^{\lambda-1} = 0$$

$$\Rightarrow \lambda = -1$$

$$\vec{v} = \frac{2xy}{x^4+y^2}\vec{i} - \frac{x^2}{x^4+y^2}\vec{j}$$

$$\vec{u}(x,y) = \int_{(0,0)}^{(x,y)} \vec{v}\cdot ds + C = \left(\int_{(0,0)}^{(x,0)} + \int_{(x,0)}^{(x,y)}\right)\vec{v}\cdot ds + C$$

$$= \int_0^y -\frac{x^2}{x^4+y^2}\,dy + C = -\arctan\frac{y}{x^2} + C$$

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15. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

有 $\begin{cases} \vec{e_r} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} \\ \vec{e_\theta} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} \end{cases}$

$\Rightarrow \begin{cases} \vec{i} = \cos\theta\,\vec{e_r} - \sin\theta\,\vec{e_\theta} \\ \vec{j} = \sin\theta\,\vec{e_r} + \cos\theta\,\vec{e_\theta} \end{cases}$

又 $\begin{cases} \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\ \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta) \end{cases}$ $\Rightarrow$ $\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial f}{\partial \theta}\sin\theta \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial f}{\partial \theta}\cos\theta \end{cases}$

有 $\frac{\partial f}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e_\theta} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$

即 $\nabla = \frac{\partial}{\partial r}\vec{e_r} + \frac{\partial}{r\partial\theta}\vec{e_\theta}$

利用 $\begin{cases} \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)\cos\theta + \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)(-r\sin\theta) \\ \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)\sin\theta + \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)(r\cos\theta) \end{cases}$

$\Delta = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ 化简可得

$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

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16、极坐标 $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ ， $u = \ln r$

$\Delta u = (-\frac{1}{r^2}) + (\frac{1}{r})\cdot(\frac{1}{r}) + 0 = 0$ ✓

球面坐标： $\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$

$u = \frac{1}{r}$　　$\frac{\partial u}{\partial r} = -\frac{1}{r^2}$

$\Delta u = \frac{1}{r^2}\cdot 0 + \frac{1}{r^2\sin\theta}\cdot 0 + 0 = 0$

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