Introduction To Robotics
Chapter VII Dynamics

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Dynamics

- Content
  - Rigid body dynamics
  - Manipulator Dynamics
  - Approximate Jacobian Control

Rigid Body Dynamics

- Robot dynamics is concerned with the relationship between the forces acting on a robot mechanism and the accelerations they produce.
- Rigid-body dynamics studies the movement of systems of interconnected bodies under the action of external forces. The assumption that the bodies are rigid (i.e. they do not deform under the action of applied forces) simplifies analysis, by reducing the parameters that describe the configuration of the system to the translation and rotation of reference frames attached to each body. This excludes bodies that display fluidic, highly elastic, and plastic behavior.
- The dynamics of a rigid body system is described by the laws of kinematics and by the application of Newton‘s second law (kinetics) or their derivative form, Lagrangian mechanics. The solution of these equations of motion provides a description of the position, the motion and the acceleration of the individual components of the system, and overall the system itself, as a function of time. The formulation and solution of rigid body dynamics is an important tool in the computer simulation of mechanical systems.
- Forward dynamics is used for simulation and robot system design;
- Inverse dynamics is used for robot control.

Rigid Body Dynamics

- Fundamentals

➢ Lagrangian: L=K-P, K: Kinetic Energy, P: Potential Energy

➢ System Dynamics Equation:

- Lagrange Approach

$$F_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}, \quad i = 1, 2, \ldots, n$$

Typically a system has i elements.

F_i : Force or torque

$\dot{q}_i$: Velocity

q_i : Position or Angular Position

- Newton-Euler Approach

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) + \frac{\partial D}{\partial \dot{q}_i} - \frac{\partial K}{\partial q_i} + \frac{\partial P}{\partial q_i} = \frac{\partial W}{\partial q_i}$$

D: Dissipated Energy related to damping

Rigid Body Dynamics

➢ System Dynamics Equation (Translation Motion)

- Newton-Euler Approach

Kinetic Energy $K = \frac{1}{2} M_1 \dot{x}_1^2 + \frac{1}{2} M_0 \dot{x}_0^2$

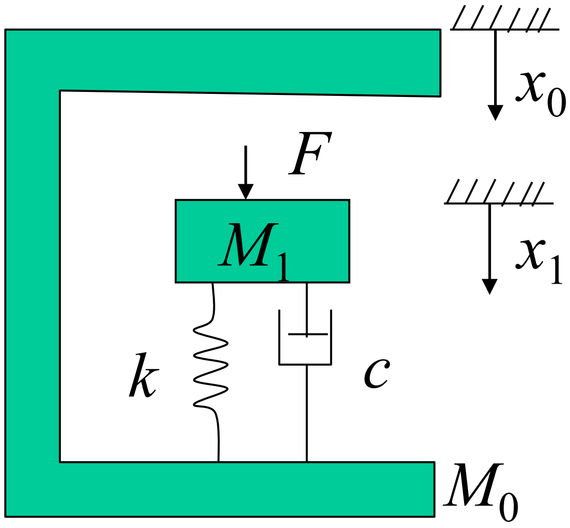
Dissipated Energy $D = \frac{1}{2} c(\dot{x}_1 - \dot{x}_0)^2$

Potential Energy $P = \frac{1}{2} k(x_1 - x_0)^2 - M_1 g x_1 - M_0 g x_0$

Work $W = F x_1$

x_0 and x_1 are initially 0.

x_0 and x_1 are generalized coordinates.



Rigid Body Dynamics

- System Dynamics Equation (Translation Motion)
  - Newton-Euler Approach

✓ If $x_0=0$:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}_1}\right)-\frac{\partial K}{\partial x_1}+\frac{\partial D}{\partial \dot{x}_1}+\frac{\partial P}{\partial x_1}=\frac{\partial W}{\partial x_1}$$

$$\frac{d}{dt}(M_1\dot{x}_1)-0+c\dot{x}_1+kx_1-M_1g=F \Longrightarrow M_1\ddot{x}_1+c\dot{x}_1+kx_1=F+M_1g$$

✓ If $x_0\neq0$:

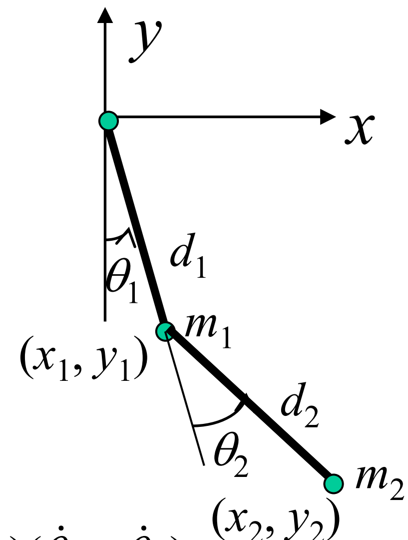
$$\begin{cases} M_1\ddot{x}_1+c(\dot{x}_1-\dot{x}_0)+k(x_1-x_0)-M_1g=F \\ M_0\ddot{x}_0+c(\dot{x}_1-\dot{x}_0)-k(x_1-x_0)-M_0g=0 \end{cases} \Longrightarrow$$

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_0 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_0 \end{bmatrix}+\begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_0 \end{bmatrix}+\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}-\begin{bmatrix} M_1g \\ M_0g \end{bmatrix}=\begin{bmatrix} F \\ 0 \end{bmatrix}$$

Rigid Body Dynamics

➢ System Dynamics Equation (Rotation Motion) Lagrange Approach

Assumption: Link mass is modeled as point mass on link's terminal



Link 1: $\begin{cases} K_1 = \dfrac{1}{2} m_1 d_1^2 \dot{\theta}_1^2 \\ P_1 = -m_1 g d_1 \cos\theta_1 \end{cases}$

Link 2: $\begin{cases} K_2 = \dfrac{1}{2} m_2 v_2^2 \\ P_2 = m_2 g y_2 \end{cases}$

$$\begin{cases} x_2 = d_1 \sin\theta_1 + d_2 \sin(\theta_1+\theta_2) \\ y_2 = -d_1\cos\theta_1 - d_2\cos(\theta_1+\theta_2) \end{cases} \Rightarrow \begin{cases} \dot{x}_2 = d_1\cos\theta_1\dot{\theta}_1 + d_2\cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \\ \dot{y}_2 = d_1\sin\theta_1\dot{\theta}_1 + d_2\sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \end{cases}$$

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 \Rightarrow v_2^2 = d_1^2\dot{\theta}_1^2 + d_2^2(\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) + 2d_1 d_2 \cos\theta_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)$$

$$\begin{cases} K_2 = \dfrac{1}{2} m_2 d_1^2\dot{\theta}_1^2 + \dfrac{1}{2} m_2 d_2^2(\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) + m_2 d_1 d_2\cos\theta_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) \\ P_2 = -m_2 g d_1\cos\theta_1 - m_2 g d_2\cos(\theta_1+\theta_2) \end{cases}$$

Rigid Body Dynamics

- System Dynamics Equation (Rotation Motion) Lagrange Approach

$$\begin{cases} K = K_1 + K_2 = \dfrac{1}{2}(m_1 + m_2)d_1^2\dot{\theta}_1^2 + \dfrac{1}{2}m_2 d_2^2(\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) + m_2 d_1 d_2 \cos\theta_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) \\ P = P_1 + P_2 = -(m_1 + m_2)g d_1 \cos\theta_1 - m_2 g d_2 \cos(\theta_1 + \theta_2) \end{cases}$$

✓ Lagrange Approach： *L*=*K*-*P*

$$L = K - P = \frac{1}{2}(m_1 + m_2)d_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 d_2^2(\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) + m_2 d_1 d_2 \cos\theta_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)$$
$$+ (m_1 + m_2)g d_1 \cos\theta_1 + m_2 g d_2 \cos(\theta_1 + \theta_2)$$

$$F_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}, \quad i = 1,2,\ldots,n$$

Now we need to figure out $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}_1}, \quad \dfrac{\partial L}{\partial \theta_1}, \quad \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}_2}, \quad \dfrac{\partial L}{\partial \theta_2}$

Rigid Body Dynamics

➢ System Dynamics Equation (Rotation Motion) Lagrange Approach

$$T_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = [(m_1+m_2)d_1^2 + m_2 d_2^2 + 2m_2 d_1 d_2 \cos\theta_2]\ddot{\theta}_1 + (m_2 d_2^2 + m_2 d_1 d_2 \cos\theta_2)\ddot{\theta}_2$$

$$-2m_2 d_1 d_2 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 d_1 d_2 \sin\theta_2 \dot{\theta}_2^2 + (m_1+m_2) g d_1 \sin\theta_1 + m_2 g d_2 \sin(\theta_1+\theta_2)$$

$$T_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = (m_2 d_2^2 + m_2 d_1 d_2 \cos\theta_2)\ddot{\theta}_1 + m_2 d_2^2 \ddot{\theta}_2 + m_2 d_1 d_2 \sin\theta_2 \dot{\theta}_1^2$$

$$+ m_2 g d_2 \sin(\theta_1+\theta_2)$$

$$\begin{cases} T_1 = D_{11}\ddot{\theta}_1 + D_{12}\ddot{\theta}_2 + D_{111}\dot{\theta}_1^2 + D_{122}\dot{\theta}_2^2 + D_{112}\dot{\theta}_1\dot{\theta}_2 + D_{121}\dot{\theta}_2\dot{\theta}_1 + D_1 \\ T_2 = D_{21}\ddot{\theta}_1 + D_{22}\ddot{\theta}_2 + D_{211}\dot{\theta}_1^2 + D_{222}\dot{\theta}_2^2 + D_{212}\dot{\theta}_1\dot{\theta}_2 + D_{221}\dot{\theta}_2\dot{\theta}_1 + D_2 \end{cases}$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} D_{111} & D_{122} \\ D_{211} & D_{222} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} D_{112} & D_{121} \\ D_{212} & D_{221} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

Torque Inertia Gravity

Rigid Body Dynamics

- System Dynamics Equation (Rotation Motion) Lagrange Approach

Effective Inertia:
$$\begin{cases} D_{11} = (m_1 + m_2)d_1^2 + m_2 d_2^2 + 2m_2 d_1 d_2 \cos\theta_2 \\ D_{22} = m_2 d_2^2 \end{cases}$$

Correlated Inertia: $D_{12} = D_{21} = m_2 d_2^2 + m_2 d_1 d_2 \cos\theta_2$

Centripetal Acceleration:
$$\begin{cases} D_{111} = 0 \\ D_{122} = -m_2 d_1 d_2 \sin\theta_2 \\ D_{211} = m_2 d_1 d_2 \sin\theta_2 \\ D_{222} = 0 \end{cases}$$

Coriolis Acceleration:
$$\begin{cases} D_{112} = D_{121} = -m_2 d_1 d_2 \sin\theta_2 \\ D_{212} = D_{221} = 0 \end{cases}$$

Gravity:
$$\begin{cases} D_1 = (m_1 + m_2) g d_1 \sin\theta_1 + m_2 g d_2 \sin(\theta_1 + \theta_2) \\ D_2 = m_2 g d_2 \sin(\theta_1 + \theta_2) \end{cases}$$

Rigid Body Dynamics

➢ System Dynamics Equation (Rotation Motion) Newton-Euler Approach

First figure out K, P, D, and W

❖ K and P

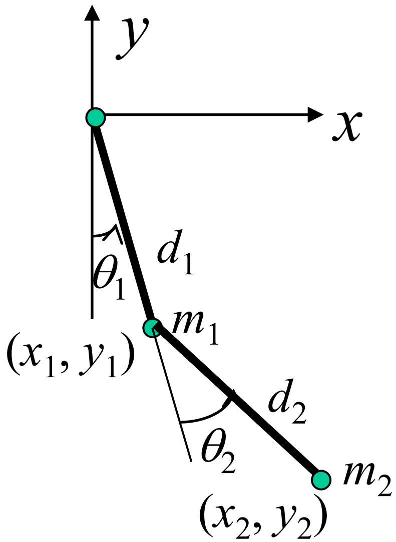
$$\begin{cases} K = K_1 + K_2 = \dfrac{1}{2}(m_1 + m_2)d_1^2\dot{\theta}_1^2 + \dfrac{1}{2}m_2 d_2^2(\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) + m_2 d_1 d_2 \cos\theta_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) \\ P = P_1 + P_2 = -(m_1 + m_2)gd_1\cos\theta_1 - m_2 g d_2 \cos(\theta_1 + \theta_2) \end{cases}$$

❖ D

$$D = \frac{1}{2}c_1\dot{\theta}_1^2 + \frac{1}{2}c_2\dot{\theta}_2^2$$

❖ W

$$W = T_1\theta_1 + T_2\theta_2$$



Rigid Body Dynamics

- System Dynamics Equation (Rotation Motion) Newton-Euler Approach

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_i}\right)+\frac{\partial D}{\partial \dot{\theta}_i}-\frac{\partial K}{\partial \theta_i}+\frac{\partial P}{\partial \theta_i}=\frac{\partial W}{\partial \theta_i}$$

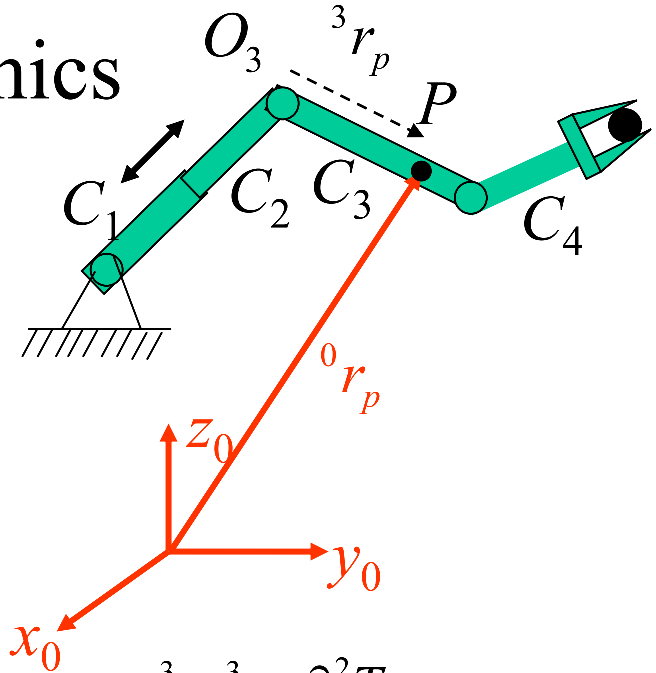
$$T_1=[(m_1+m_2)d_1^2+m_2d_2^2+2m_2d_1d_2\cos\theta_2]\ddot{\theta}_1+(m_2d_2^2+m_2d_1d_2\cos\theta_2)\ddot{\theta}_2+c_1\dot{\theta}_1$$
$$-2m_2d_1d_2\sin\theta_2\dot{\theta}_1\dot{\theta}_2-m_2d_1d_2\sin\theta_2\dot{\theta}_2^2+(m_1+m_2)gd_1\sin\theta_1+m_2gd_2\sin(\theta_1+\theta_2)$$

$$T_2=(m_2d_2^2+m_2d_1d_2\cos\theta_2)\ddot{\theta}_1+m_2d_2^2\ddot{\theta}_2+m_2d_1d_2\sin\theta_2\dot{\theta}_1^2+c_2\dot{\theta}_2+m_2gd_2\sin(\theta_1+\theta_2)$$

If energy dissipation is zero， $c_1=c_2=0$. In this circumstance, both approaches yield the same system dynamic equation.

- Steps to get System Dynamic Equations:
  - Calculate the velocity of point mass
  - Calculate the kinetic energy of point mass
  - Calculate the potential energy of point mass
  - Calculate the derivative function to get dynamic function

Manipulator Dynamics



- Velocity Calculation
  - P's Position $\quad {}^0\vec{r}_p = T_3\, {}^3\vec{r}_p$
  - P's Velocity $\quad {}^0\vec{v}_p = \frac{d}{dt}({}^0\vec{r}_p) = \dot{T}_3\, {}^3\vec{r}_p$

$$\dot{T}_3 = \sum_{j=1}^{3} \frac{\partial T_3}{\partial q_j}\dot{q}_j \Rightarrow {}^0\vec{v}_p = (\sum_{j=1}^{3} \frac{\partial T_3}{\partial q_j}\dot{q}_j)\, {}^3\vec{r}_p$$

  - P's Acceleration

$${}^0\vec{a}_p = \frac{d}{dt}({}^0\vec{v}_p) = \frac{d}{dt}(\sum_{j=1}^{3} \frac{\partial T_3}{\partial q_j}\dot{q}_j)\, {}^3\vec{r}_p = (\sum_{j=1}^{3} \frac{\partial T_3}{\partial q_j}\ddot{q}_j)\, {}^3\vec{r}_p + (\sum_{k=1}^{3}\sum_{j=1}^{3} \frac{\partial^2 T_3}{\partial q_k \partial q_j}\dot{q}_k\dot{q}_j)\, {}^3\vec{r}_p$$

  - P's velocity square

$$({}^0v_p)^2 = {}^0v_p \cdot {}^0v_p = Trace[({}^0v_p)\cdot({}^0v_p)^T] = Trace\{[(\sum_{j=1}^{3} \frac{\partial T_3}{\partial q_j}\dot{q}_j)\, {}^3\vec{r}_p]\cdot[(\sum_{j=1}^{3} \frac{\partial T_3}{\partial q_j}\dot{q}_j)\, {}^3\vec{r}_p]^T\}$$

$$= Trace[\sum_{j=1}^{3}\sum_{k=1}^{3} \frac{\partial T_3}{\partial q_j}({}^3\vec{r}_p)({}^3\vec{r}_p)^T \frac{\partial T_3}{\partial q_k}^T \dot{q}_j\dot{q}_k]$$

$$v_p \cdot v_p^T = \begin{bmatrix} v_{px} \\ v_{py} \\ v_{pz} \end{bmatrix} \begin{bmatrix} v_{px} & v_{py} & v_{pz} \end{bmatrix} = \begin{bmatrix} v_{px}^2 & v_{px}v_{py} & v_{px}v_{pz} \\ v_{py}v_{px} & v_{py}^2 & v_{py}v_{pz} \\ v_{pz}v_{px} & v_{pz}v_{py} & v_{pz}^2 \end{bmatrix}$$

Manipulator Dynamics

- Velocity on arbitrary point
  - ✓ Position of arbitrary point on Link *i* $\quad {}^0r = T_i\ {}^i r$
  - ✓ Velocity of arbitrary point on Link *i*

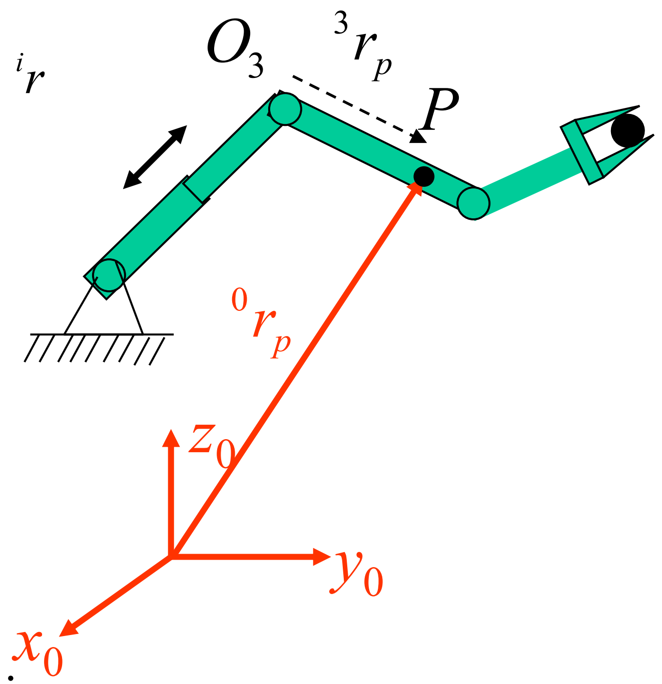
$$v = \left(\sum_{j=1}^{i} \frac{\partial T_i}{\partial q_j} \dot{q}_j\right) {}^i r$$

  - ✓ Acceleration of arbitrary point on Link *i*

$$a = \left(\sum_{j=1}^{i} \frac{\partial T_i}{\partial q_j} \ddot{q}_j\right) {}^i r + \left(\sum_{k=1}^{i}\sum_{j=1}^{i} \frac{\partial^2 T_i}{\partial q_k \partial q_j} \dot{q}_k \dot{q}_j\right) {}^i r$$

  - ✓ Velocity Square of arbitrary point on Link *i*

$$v^2 = Trace\left[\sum_{k=1}^{i}\sum_{j=1}^{i} \frac{\partial T_i}{\partial q_j}\ {}^i r\ {}^i r^T \left(\frac{\partial T_i}{\partial q_k}\right)^T \dot{q}_j \dot{q}_k\right]$$



Manipulator Dynamics

- **Kinetic Energy K_t**
  - ✓ Kinetic Energy of P： Supposing the mass on point P is dm：

$$dK_3 = \frac{1}{2} dm^0 v_p{}^2 = \frac{1}{2} dm\, Trace\left[\sum_{k=1}^{3} \sum_{j=1}^{3} \frac{\partial T_3}{\partial q_j} {}^3 r_p\ {}^3 r_p{}^T \left(\frac{\partial T_3}{\partial q_k} \right)^T \dot{q}_j \dot{q}_k \right]$$

$$= \frac{1}{2} Trace\left[\sum_{k=1}^{3} \sum_{j=1}^{3} \frac{\partial T_3}{\partial q_j} ({}^3 r_p dm\ {}^3 r_p{}^T) \left(\frac{\partial T_3}{\partial q_k} \right)^T \dot{q}_j \dot{q}_k \right]$$

  - ✓ Kinetic Energy of arbitrary point on link i

$$dK_i = \frac{1}{2} dm v^2 = \frac{1}{2} dm\, Trace\left[\sum_{k=1}^{i} \sum_{j=1}^{i} \frac{\partial T_i}{\partial q_j} {}^i r\ {}^i r^T \left(\frac{\partial T_i}{\partial q_k} \right)^T \dot{q}_j \dot{q}_k \right]$$

$$= \frac{1}{2} Trace\left[\sum_{k=1}^{i} \sum_{j=1}^{i} \frac{\partial T_i}{\partial q_j} ({}^i r dm\ {}^i r^T) \left(\frac{\partial T_i}{\partial q_k} \right)^T \dot{q}_j \dot{q}_k \right]$$

Manipulator Dynamics

- Kinetic Energy of Link i: K_i

$$K_i = \int_{\text{Link } i} dK_i = \frac{1}{2} Trace\left[\sum_{k=1}^{i}\sum_{j=1}^{i}\frac{\partial T_i}{\partial q_j}\left(\int_{\text{Link } i} {}^i\vec{r}\,{}^i\vec{r}^T dm\right)\left(\frac{\partial T_i}{\partial q_k}\right)^T \dot{q}_j\dot{q}_k\right]$$

$$= \frac{1}{2} Trace\left[\sum_{k=1}^{i}\sum_{j=1}^{i}\frac{\partial T_i}{\partial q_j} I_i \left(\frac{\partial T_i}{\partial q_k}\right)^T \dot{q}_j\dot{q}_k\right]$$

$$I_i = \int_{\text{Link } i} {}^i\vec{r}\,{}^i\vec{r}^T dm, \qquad {}^i\vec{r}\,{}^i\vec{r}^T = \begin{bmatrix} {}^ix \\ {}^iy \\ {}^iz \\ 1 \end{bmatrix}\begin{bmatrix} {}^ix & {}^iy & {}^iz & 1 \end{bmatrix} = \begin{bmatrix} {}^ix^2 & {}^ix{}^iy & {}^ix{}^iz & {}^ix \\ {}^iy{}^ix & {}^iy^2 & {}^iy{}^iz & {}^iy \\ {}^iz{}^ix & {}^iz{}^iy & {}^iz^2 & {}^iz \\ {}^ix & {}^iy & {}^iz & 1 \end{bmatrix}$$

$$I_i = \int_{\text{Link } i} {}^i\vec{r}\,{}^i\vec{r}^T dm = \begin{bmatrix} \int {}^ix^2 dm & \int {}^ix{}^iy dm & \int {}^ix{}^iz dm & \int {}^ix dm \\ \int {}^iy{}^ix dm & \int {}^iy^2 dm & \int {}^iy{}^iz dm & \int {}^iy dm \\ \int {}^iz{}^ix dm & \int {}^iz{}^iy dm & \int {}^iz^2 dm & \int {}^iz dm \\ \int {}^ix dm & \int {}^iy dm & \int {}^iz dm & \int dm \end{bmatrix}$$

I_i: pseudo moment of inertia

Manipulator Dynamics

- Kinetic Energy of Link i: K_i

$$I_{xx}=\int(y^2+z^2)dm,\quad I_{yy}=\int(x^2+z^2)dm,\quad I_{zz}=\int(x^2+y^2)dm$$

$$I_{xy}=I_{yx}=\int xydm,\quad I_{xz}=I_{zx}=\int xzdm,\quad I_{yz}=I_{zy}=\int yzdm$$

$$mx=\int xdm,\quad my=\int ydm,\quad mz=\int zdm$$

$$\int x^2dm=-\frac{1}{2}\int(y^2+z^2)dm+\frac{1}{2}\int(x^2+z^2)dm+\frac{1}{2}\int(x^2+y^2)dm$$

$$=(-I_{xx}+I_{yy}+I_{zz})/2$$

$$\int y^2dm=\frac{1}{2}\int(y^2+z^2)dm-\frac{1}{2}\int(x^2+z^2)dm+\frac{1}{2}\int(x^2+y^2)dm$$

$$=(I_{xx}-I_{yy}+I_{zz})/2$$

$$\int z^2dm=\frac{1}{2}\int(y^2+z^2)dm+\frac{1}{2}\int(x^2+z^2)dm-\frac{1}{2}\int(x^2+y^2)dm$$

$$=(I_{xx}+I_{yy}-I_{zz})/2$$

Manipulator Dynamics

- System Kinetic Energy K_t

$$I_i = \begin{bmatrix} (-I_{ixx}+I_{iyy}+I_{izz})/2 & I_{ixy} & I_{ixz} & m_i\bar{x}_i \\ I_{ixy} & (I_{ixx}-I_{iyy}+I_{izz})/2 & I_{iyz} & m_i\bar{y}_i \\ I_{ixz} & I_{iyz} & (I_{ixx}+I_{iyy}-I_{izz})/2 & m_i\bar{z}_i \\ m_i\bar{x}_i & m_i\bar{y}_i & m_i\bar{z}_i & m_i \end{bmatrix}$$

✓ Kinetic Energy of n links:

$$K = \sum_{i=1}^{n} K_i = \frac{1}{2}\sum_{i=1}^{n} Trace\left[\sum_{k=1}^{i}\sum_{j=1}^{i}\frac{\partial T_i}{\partial q_j} I_i \left(\frac{\partial T_i}{\partial q_k}\right)^T \dot{q}_j \dot{q}_k\right] = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{j=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_j} I_i \frac{\partial T_i^T}{\partial q_k}\right)\dot{q}_j \dot{q}_k$$

✓ Kinetic Energy of driving mechanism on Link i: $K_{ai} = \frac{1}{2} I_{ai}\dot{q}_i^2$

✓ Kinetic Energy of driving mechanisms on all links: $K_a = \frac{1}{2}\sum_{i=1}^{n} I_{ai}\dot{q}_i^2$

✓ System Kinetic Energy is:

$$K_t = K + K_a = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{j=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_j} I_i \frac{\partial T_i^T}{\partial q_k}\right)\dot{q}_j \dot{q}_k + \frac{1}{2}\sum_{i=1}^{n} I_{ai}\dot{q}_i^2$$

Manipulator Dynamics

➢ Potential Energy P

- ✓ Link i, position ${}^i r$, point mass dm:

$$dP_i = -dm\vec{g}^{T}\,{}^{0}\vec{r} = -\vec{g}^{T} T_i\,{}^{i}\vec{r}\,dm \qquad \vec{g}^{T} = \begin{bmatrix} g_x & g_y & g_z & 1 \end{bmatrix}$$

Centroid position

- ✓ Potential Energy of Link i:

$$P_i = \int dP_i = -\int \vec{g}^{T} T_i\,{}^{i}\vec{r}\,dm = -m_i \vec{g}^{T} T_i\,{}^{i}\vec{r}$$

- ✓ Potential Energy of n links, (Neglecting potential energy of driving mechanism)

$$P = \sum_{i=1}^{n} (P_i - P_{ai}) \approx \sum_{i=1}^{n} P_i = -\sum_{i=1}^{n} m_i \vec{g}^{T} T_i\,{}^{i}\vec{r}$$

➢ Lagrangian

$$L = K_t - P = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{j=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_j} I_i \frac{\partial T_i^{T}}{\partial q_k}\right)\dot{q}_j \dot{q}_k + \frac{1}{2}\sum_{i=1}^{n} I_{ai}\dot{q}_i^2 + \sum_{i=1}^{n} m_i \vec{g}^{T} T_i\,{}^{i}\vec{r}$$

Manipulator Dynamics

➢ System Dynamics Equation

$$\frac{\partial L}{\partial \dot{q}_p} = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_p} I_i \frac{\partial T_i^T}{\partial q_k}\right)\dot{q}_k + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_j} I_i \frac{\partial T_i^T}{\partial q_p}\right)\dot{q}_j + I_{ap}\dot{q}_p$$

$$p = 1,2,\ldots,n$$

Since I_i is symmetric, we have

$$Trace\left(\frac{\partial T_i}{\partial q_j} I_i \frac{\partial T_i^T}{\partial q_k}\right) = Trace\left(\frac{\partial T_i}{\partial q_k} I_i^T \frac{\partial T_i^T}{\partial q_j}\right) = Trace\left(\frac{\partial T_i}{\partial q_k} I_i \frac{\partial T_i^T}{\partial q_j}\right) \Rightarrow$$

$$\frac{\partial L}{\partial \dot{q}_p} = \sum_{i=1}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_p} I_i \frac{\partial T_i^T}{\partial q_k}\right)\dot{q}_k + I_{ap}\dot{q}_p$$

$$= \sum_{i=p}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_p} I_i \frac{\partial T_i^T}{\partial q_k}\right)\dot{q}_k + I_{ap}\dot{q}_p$$

Manipulator Dynamics

➢ System Dynamics Equation

$$\frac{d}{dt}\left(\frac{\partial T_i}{\partial q_j}\right)=\sum_{k=1}^{i}\frac{\partial}{\partial q_k}\left(\frac{\partial T_i}{\partial q_j}\right)\dot{q}_k=\sum_{k=1}^{i}\frac{\partial^2 T_i}{\partial q_k \partial q_j}\dot{q}_k$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_p}\right)=\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_j \partial q_k} I_i \frac{\partial T_i^T}{\partial q_p}\right)\dot{q}_j\dot{q}_k+\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_p \partial q_k} I_i \frac{\partial T_i^T}{\partial q_j}\right)\dot{q}_j\dot{q}_k$$

$$+\sum_{i=p}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_p} I_i \frac{\partial T_i^T}{\partial q_k}\right)\ddot{q}_k + I_{ap}\ddot{q}_p$$

$$=\sum_{i=p}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_p} I_i \frac{\partial T_i^T}{\partial q_k}\right)\ddot{q}_k + I_{ap}\ddot{q}_p + 2\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_j \partial q_k} I_i \frac{\partial T_i^T}{\partial q_p}\right)\dot{q}_j\dot{q}_k$$

$$\frac{\partial L}{\partial q_p}=\frac{1}{2}\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_j \partial q_k} I_i \frac{\partial T_i^T}{\partial q_p}\right)\dot{q}_j\dot{q}_k+\frac{1}{2}\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_p \partial q_k} I_i \frac{\partial T_i^T}{\partial q_j}\right)\dot{q}_j\dot{q}_k$$

$$+\sum_{i=p}^{n} m_i \vec{g}^T \frac{\partial T_i}{\partial q_p} {}^i\vec{r}_i$$

$$=\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_p \partial q_k} I_i \frac{\partial T_i^T}{\partial q_j}\right)\dot{q}_j\dot{q}_k+\sum_{i=p}^{n} m_i \vec{g}^T \frac{\partial T_i}{\partial q_p} {}^i\vec{r}_i$$

Manipulator Dynamics

➢ System Dynamics Equation

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_p}) - \frac{\partial L}{\partial q_p} = \sum_{i=p}^{n}\sum_{k=1}^{i} Trace(\frac{\partial T_i}{\partial q_p} I_i \frac{\partial T_i^T}{\partial q_k})\ddot{q}_k + I_{ap}\ddot{q}_p + 2\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace(\frac{\partial^2 T_i}{\partial q_j \partial q_k} I_i \frac{\partial T_i^T}{\partial q_p})\dot{q}_j\dot{q}_k$$

$$-\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace(\frac{\partial^2 T_i}{\partial q_p \partial q_k} I_i \frac{\partial T_i^T}{\partial q_j})\dot{q}_j\dot{q}_k - \sum_{i=p}^{n} m_i \vec{g}^T \frac{\partial T_i}{\partial q_p} {}^i\vec{r}_i$$

$$= \sum_{i=p}^{n}\sum_{k=1}^{i} Trace(\frac{\partial T_i}{\partial q_p} I_i \frac{\partial T_i^T}{\partial q_k})\ddot{q}_k + I_{ap}\ddot{q}_p + \sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace(\frac{\partial^2 T_i}{\partial q_j \partial q_k} I_i \frac{\partial T_i^T}{\partial q_p})\dot{q}_j\dot{q}_k - \sum_{i=p}^{n} m_i \vec{g}^T \frac{\partial T_i}{\partial q_p} {}^i\vec{r}_i$$

$$\tau_i = \sum_{j=i}^{n}\sum_{k=1}^{j} Trace(\frac{\partial T_j}{\partial q_i} I_j \frac{\partial T_j^T}{\partial q_k})\ddot{q}_k + I_{ai}\ddot{q}_i + \sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} Trace(\frac{\partial^2 T_j}{\partial q_k \partial q_m} I_j \frac{\partial T_j^T}{\partial q_i})\dot{q}_k\dot{q}_m - \sum_{j=i}^{n} m_j \vec{g}^T \frac{\partial T_j}{\partial q_i} {}^j\vec{r}_j$$

➢ Further Representation

$$\tau_i = \sum_{j=1}^{n} D_{ij}\ddot{q}_k + I_{ai}\ddot{q}_i + \sum_{j=i}^{n}\sum_{k=1}^{j} D_{ijk}\dot{q}_j\dot{q}_k + D_i$$

$$D_{ij} = \sum_{p=\max i,j}^{j} Trace(\frac{\partial T_p}{\partial q_j} I_p \frac{\partial T_p^T}{\partial q_i}), D_{ijk} = \sum_{p=\max i,j,k}^{n} Trace(\frac{\partial^2 T_p}{\partial q_k \partial q_m} I_p \frac{\partial T_p^T}{\partial q_i}), D_i = -\sum_{p=i}^{n} m_p \vec{g}^T \frac{\partial T_p}{\partial q_i} {}^p\vec{r}_p$$

Kinematics vs Dynamics

Kinematics and Dynamics

➢ Kinematics

$X = h(q)$ X: Generalized position of robot end-effector, $X \in R^m$; q is joint vector, $q=[q_1, q_2, \ldots, q_n]^T \in R^n$; h: forward kinematics

$\dot{X} = J(q)\dot{q}$ $J(q)$ defines relationship between Joint velocity and end velocity

➢ Dynamics

$$M(q)\ddot{q} + \left(\frac{1}{2}\dot{M}(q) + S(q,\dot{q})\right)\dot{q} + G(q) = \tau$$

$M(q)$: Inertia, $M(q) \in R^{n\times n}$; $G(q)$: *Gravity*; τ : Force or Torque, $\tau \in R^n$

$$S(q,\dot{q})\dot{q} = \frac{1}{2}\dot{M}(q)\dot{q} - \frac{1}{2}\left(\frac{\partial}{\partial q}\dot{q}^T M(q)\dot{q}\right)^T$$

Approximate Jacobian Control

- For kinematics-based robot control, we cannot achieve high precision trajectory control if forward kinematics is not precise.
- Accuracy of Jacobian Matrix based robot control also depends on the precision of Jacobian matrix.
- Cheah et al proposed robot control based on estimated Jacobian Matrix

[1] C. C. Cheah, M. Hirano, S. Kawamura, S. Arimoto, Approximate Jacobian Control for Robots With Uncertain Kinematics and Dynamics, IEEE Transactions on Robotics and Automation, vol. 19, no. 4, pp. 692-702, 2003.
[2] C. C. Cheah, K. Li, S. Kawamura, S. Arimoto, Approximate Jacobian Feedback Control of Robots with Kinematic Uncertainty and its Application to Visual Servoing, Proceedings of the 2001 IEEE International Conference on Robotics 8, Automation, pp. 2535-2540, Seoul, Korea . May 21-26, 2001.

Approximate Jacobian Control Considering Full Gravity Compensation

➢ Control Law

$$\tau = -\hat{J}^T(q)K_p s(e) - B_v\dot{q} + G(q)$$

$e = X - X_d = [e_1, e_2, \ldots, e_m]^T$, $s(e) = [s_1(e_1), s_2(e_2), \ldots, s_m(e_m)]^T$, $K_p = k_p I_{m\times m}$, $B_v = b_v I_{n\times n}$, k_p and b_v are positive constants. K_p is coefficient for position error feedback in task space.;B_v is coefficient for velocity feedback in task space.

$\hat{J}^T(q)$ is estimated $J^T(q)$

$$\left\| J^T(q) - \hat{J}^T(q) \right\| \le p$$

P is positive number.

In first item, $K_p s(e)$ converts position error into force in task space, then $J^T(q)$ converts force in task space into joint space.
The second item concerns velocity feedback.
The third item concerns gravity.

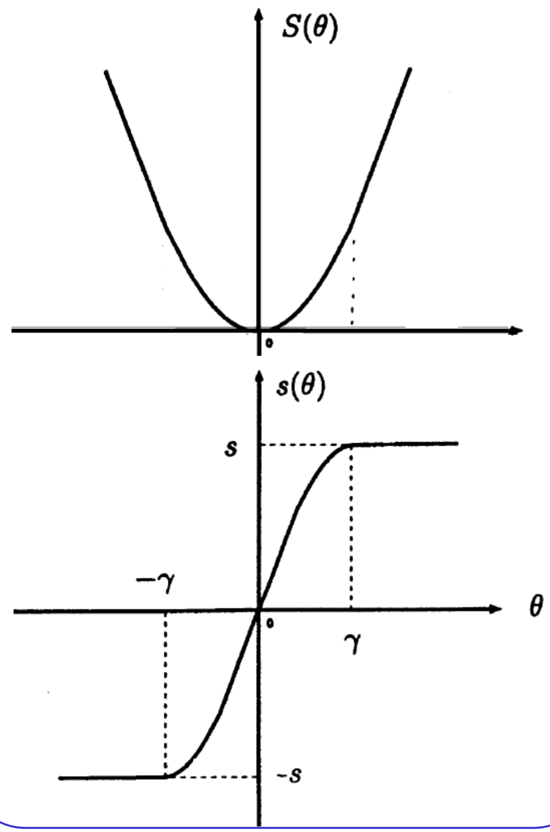
Approximate Jacobian Control Considering Full Gravity Compensation

- Define $S(\theta)$ and its first derivative $s(\theta)$

(1) *If* $\theta \neq 0$， $S(\theta)>0$； $S(0)=0$。

(2) $S(\theta)$ has second order derivative and $s(\theta)=dS(\theta)/d\theta$。 if $|\theta|<\gamma$， $s(\theta)$ is monotone increasing function; if $|\theta|\geq\gamma$， $s(\theta)$ is saturated, e.g., if $\theta\geq\gamma$， $s(\theta)=s$; if $\theta\leq-\gamma$， $s(\theta)=-s$. *The* s and γ are positive.

(3) There exists a constant $c>0$， if $\theta\neq0$， $S(\theta)\geq c\, s^2(\theta)$.



THANK YOU