

Lab 6 Laplace Transform

Objective

- Master the Laplace transform and inverse Laplace transform.
- Understand the poles and zeros of the system and their effects.
- Application of Laplace transform in system analysis.

Content

Laplace Transform and Inverse Laplace Transform

The definition of the Laplace Transform is:

- a) Bilateral Laplace transform

$$F_b(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{+\infty} f(t)e^{-st} dt, \quad s = \sigma + j\omega$$

- b) Unilateral Laplace transform

$$F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} f(t)e^{-st} dt, \quad s = \sigma + j\omega$$

And the inverse Laplace Transform is:

- a) Inverse bilateral Laplace transform

$$f(t) = \mathcal{L}^{-1}[F_b(s)] = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F_b(s)e^{st} ds$$

- b) Inverse unilateral Laplace transform

$$f(t) = \mathcal{L}^{-1}[F(s)] = \left[\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds \right] u(t)$$

In our lab, if there are no special instruction, unilateral Laplace transform is performed by default. In MATLAB symbolic toolbox, function **laplace** and **ilaplace** are used to do Laplace transform and inverse Laplace transform, as shown in Table 1.

Table 1 laplace and ilaplace

Function	Syntax	Description
laplace	laplace(f)	Returns the Laplace transform of f using the default independent variable t and the default transformation variable s.
	laplace(f, transVar)	Uses the specified transformation variable transVar instead of s.
	laplace(f, var, transVar)	Uses the specified independent variable var and transformation variable transVar instead of t and s respectively.
ilaplace	ilaplace(F)	Returns the inverse Laplace transform of F using the default independent variable s for the default transformation variable t.
	ilaplace(F, transVar)	Uses the specified transformation variable transVar instead of t.

	ilaplace(F, var, transVar)	Uses the specified independent variable var and transformation variable transVar instead of s and t respectively.
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Example 1: Find the Laplace transform of $f(t) = e^{-t} \sin(at) u(t)$.

```
syms t a
f = exp(-t)*sin(a*t)*heaviside(t);
L = laplace(f);
```

Example 2: Find the inverse Laplace transform of $F(s) = \frac{s^2}{s^2+1}$.

```
syms s
F = s^2/(s^2+1);
ft = ilaplace(F);
```

Relationship between Laplace Transform and Fourier Transform

The Laplace transform formula is similar to the Fourier transform formula in form, except that the exponential term of the Laplace transform is a complex number $s = \sigma + j\omega$ instead of an imaginary number $j\omega$. The exponential factor σ has the effect of forcing the signal to converge. That is why Laplace transform can be applied to a wider range of signal categories than the Fourier transform, including exponentially increasing signals. In the Fourier transform, the signal in the time domain and the spectrum in the frequency domain are both two-dimensional. However, the Laplace transform is a three-dimensional function defined on a two-dimensional complex plane (s plane). The s-plane consists of two variables, the horizontal real axis (σ) and the vertical imaginary axis ($j\omega$). And when $\sigma = 0$, Laplace transform simply becomes the Fourier transform.

Poles and Zeros

The Symbolic toolbox of MATLAB provides function **pole** and **zero** to find out the poles and zeroes of the linear system. The format of the two functions is **pole(sys)** and **zero(sys)**.

Function **pzmap** is used to draw the pole-zero distribution plot. The format is **pzmap(sys)**.

Sys is the system transfer function which can be generated by function **tf(b,a)** or **tf(num,den)**.

The distribution of zeroes and poles is of great importance. By analyzing zeroes and poles in s domain, the following information can be obtained:

1. The characteristics of the system impulse response ($h(t)$) in the time domain.
2. The stability of the system.
3. The frequency characteristics.

The relationship between poles and $h(t)$ is displayed in Table 1.

Table 1 Relationship between Poles and $h(t)$

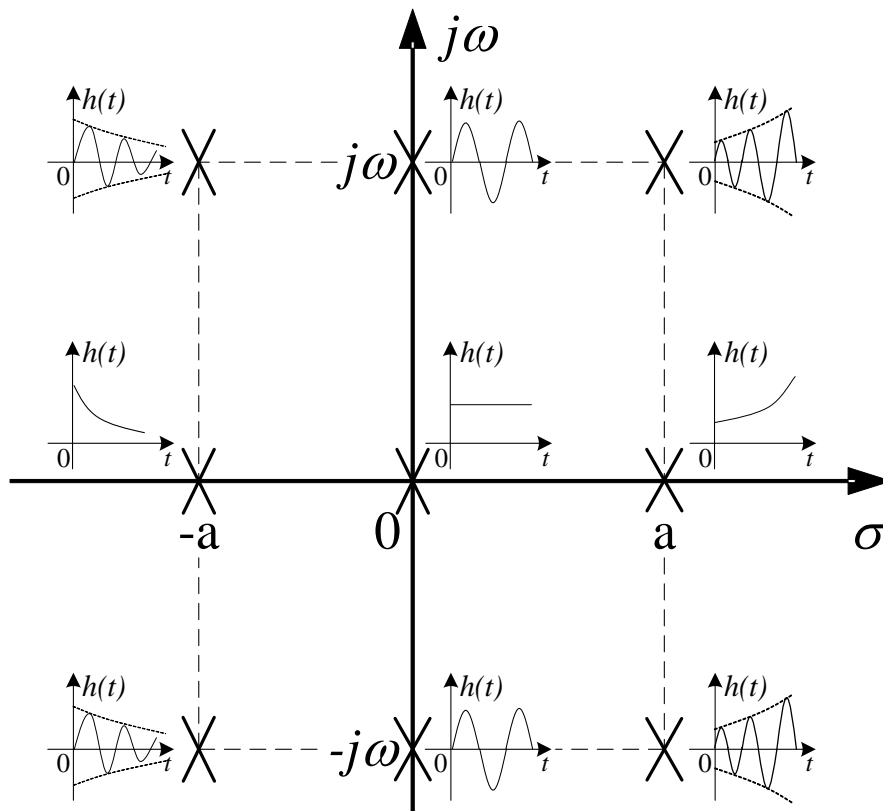
H(s)	Pole Position	h(t)
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$H(s)=1/s$	$p1=0$	$h(t)=u(t)$
$H(s)=1/(s+a)$	$p1=-a$	
	$a>0$, left real axis	$h(t)=e^{-at}u(t)$, exponential decay
	$a<0$, right real axis	$h(t)=e^{-at}u(t)$, $-a>0$, exponential increase
$H(s)=w/(s^2+w^2)$	$p1=\pm jw$	$h(t)=\sin wt*u(t)$, continuous oscillation
$H(s)=w/((s+a)^2+w^2)$	$p1=-a\pm jw$	
	$a>0$, left half-plane	$h(t)=e^{-at}\sin wt u(t)$, oscillation attenuation
	$a<0$, right half-plane	$h(t)=e^{-at}\sin wt u(t)$, $-a>0$, oscillation increase
$H(s)=1/s^2$	$p1=p2=0$	$h(t)=tu(t)$ $t\rightarrow\infty$, $h(t)\rightarrow\infty$
$H(s)=1/(s+a)^2$	$p1=p2=-a$	
	$a>0$, left real axis	$h(t)=te^{-at}u(t)$, $a>0$, $t\rightarrow\infty$, $h(t)\rightarrow 0$
	$a<0$, right real axis	$h(t)=te^{-at}u(t)$, $a>0$, $t\rightarrow\infty$, $h(t)\rightarrow\infty$
$H(s)=2ws/(s^2+w^2)^2$	$p1=p2=\pm jw$	$h(t)=t\sin wt*u(t)$, oscillation increase
$H(s)=2w(s+a)/((s+a)^2+w^2)^2$	$p1=p2=-a\pm jw$	
	$a>0$, left half-plane	$h(t)=te^{-at}\sin wt*u(t)$, $a>0$, $t\rightarrow\infty$, $h(t)\rightarrow 0$
	$a<0$, right half-plane	$h(t)=te^{-at}\sin wt*u(t)$, $a>0$, $t\rightarrow\infty$, $h(t)\rightarrow\infty$

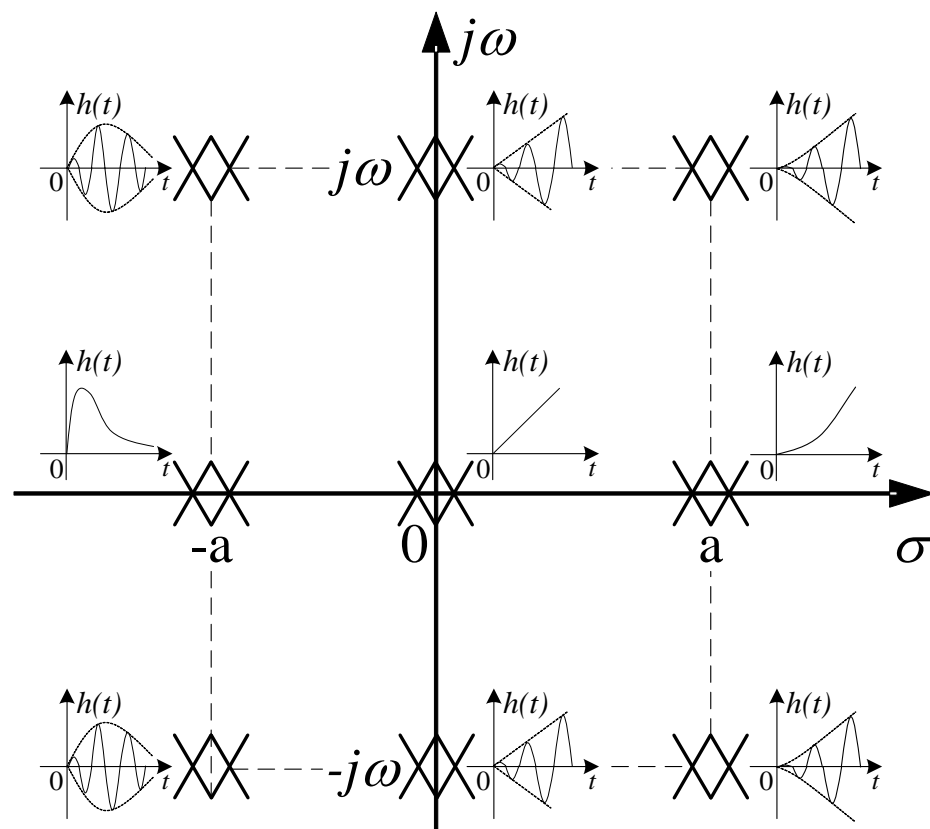
The details of the table are shown in Figure 1. (a) is for first-order poles and (b) is for second-order poles.

If a continuous system is stable, all the poles of $H(s)$ should be on the left half-plane of s plane.

The distribution of zeroes affects the amplitude and phase of $h(t)$, which we will not discuss here in detail.



(a) First-order Poles



(b) Second-order Poles

Figure 1 Relationship between $H(s)$ Poles and $h(t)$ Waveform Characteristics

Use function pzplot(sys) to draw the poles and zeros.

Eg: $F(s) = \frac{2(s-3)(s+3)}{(s-5)(s^2+10)}$, find out the poles and zeros.

```
syms s
b = [2 0 -18];
a = [1 -5 10 -50];
% draw the poles and zeros
pzplot(tf(b,a))
% draw the impulse response
t = 0:0.01:1;
plot(t,impz(tf(b,a),t));
xlabel('t(s)');ylabel('h(t)');title('impulse response');
```

System Analysis with Laplace Transform

Laplace transform is often used to analyze the system. Here we will discuss two applications of Laplace transform used in system analysis. They are the solution of difference equations and the analysis of circuits.

Solving Difference Equations with Laplace Transform

Laplace transform provides a method for solving differential equations. When using this method, differential equations in the time domain are first converted to algebraic equations in the frequency domain. After solving the algebraic equations in the frequency domain, the results are converted to the time domain again to obtain the final solution of the differential equations. The operations include Laplace transform and inverse Laplace transform.

Take the example we talked about in Lab2:

$y''(t) + 3y'(t) + 2y(t) = f(t)$, $f(t) = e^{-t}u(t)$, $y(0_-) = 1$, $y'(0_-) = 2$, find out $y(t)$

To find out the $y(t)$, we can

- Do unilateral Laplace transform to both side of the differential equation
 $[s^2Y(s) - sy(0_-) - y'(0_-)] + 3 \cdot [sY(s) - y(0_-)] + 2Y(s) = F(s)$
- Solve the above algebraic equation

$$Y(s) = \underbrace{\frac{s+5}{s^2+3s+2}}_{Y_{zi}} + \underbrace{\frac{1}{s^2+3s+2}}_{H(s)} F(s)$$

$$F(s) = \frac{1}{s+1}$$

Obviously the first part is only related to the characteristics of the system and the initial state. This part corresponds to the zero-input response.

The second part is related to system characteristics and the inputs, and has no relation with the initial

state. This part corresponds to the zero-state response.

c) Do the inverse Laplace transform:

$$\begin{aligned}
 Y(s) &= \frac{s+5}{(s+2)(s+1)} + \frac{1}{(s+2)(s+1)} \bullet \frac{1}{(s+1)} \\
 &= \left[\frac{-3}{s+2} + \frac{4}{s+3} \right] + \left[\frac{1}{(s+1)^2} + \frac{-1}{s+1} + \frac{1}{s+2} \right] \\
 \therefore y(t) &= [-3e^{-2t} + 4e^{-3t}]u(t) + [te^{-t} - e^{-t} + e^{-2t}]u(t)
 \end{aligned}$$

When the initial state is Zero, we get:

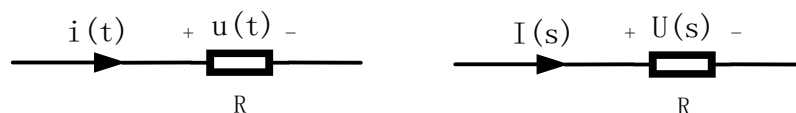
$$H(s) = \frac{Y(s)}{F(s)} = \frac{1}{s^2 + 3s + 2}$$

H(s) is called the transfer function, and it is a Laplace transform pair with the impulse response h(t) in time domain.

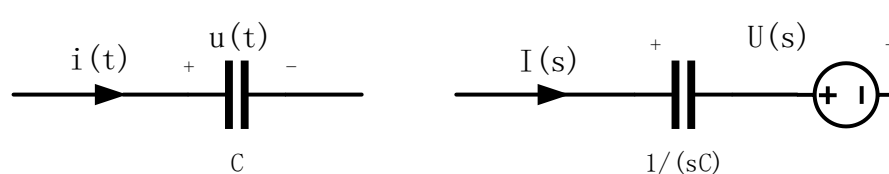
Analysis of Circuits with Laplace Transform

Based on the KCL and KVL theorems, we can equate devices such as resistors, capacitors and inductors with impedance, capacitive reactance, and inductive reactance in the complex frequency domain, thereby simplifying the analysis of the circuit.

For resistance:

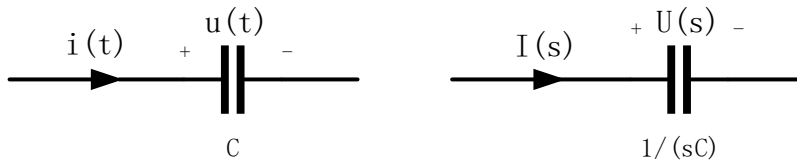
$$\begin{aligned}
 u(t) &= R \cdot i(t) \\
 \mathcal{L}[u(t)] &= \mathcal{L}[R \cdot i(t)] \\
 \frac{U(s)}{I(s)} &= R
 \end{aligned}$$


For capacitor:

$$\begin{aligned}
 i(t) &= C \frac{du(t)}{dt} \\
 \mathcal{L}[i(t)] &= \mathcal{L}\left[C \frac{du(t)}{dt}\right] \\
 I(s) &= C[sU(s) - u(0_-)]
 \end{aligned}$$


If we are only interested in the steady-state response, we can ignore $u(0_-)$, since it only effect transient response. For simplicity, we will take $u(0_-) = 0$. Thus we have:

$$\frac{U(s)}{I(s)} = \frac{1}{sC}$$

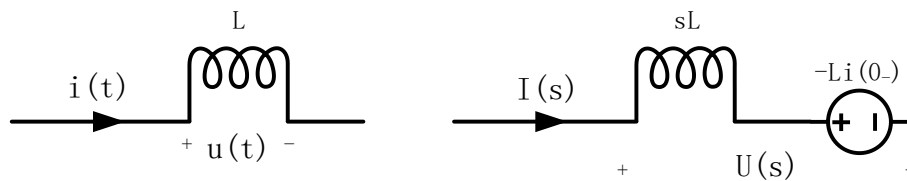


For inductor:

$$u(t) = L \frac{di(t)}{dt}$$

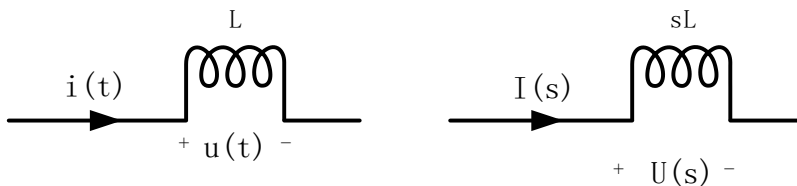
$$\mathcal{L}[u(t)] = \mathcal{L}\left[L \frac{di(t)}{dt}\right]$$

$$U(s) = L[sI(s) - i(0_-)]$$



Similarly, we take $i(0_-) = 0$ in this chapter also. Thus we have:

$$\frac{U(s)}{I(s)} = sL$$



Surface Plot for Laplace Transform

With Laplace transform, we have a complex-valued function $F(s)$ with a complex variable s ($s = \sigma + j\omega$). The variable s of the function is a plane composed of the real part σ and the imaginary part ω . So in order to draw $F(s)$, we need to introduce a three-dimensional graph.

Follow the steps below to get the surface graph of Laplace transform:

- Define vectors x (σ) and y (ω) as the real and imaginary axes of the complex s plane.
- Use the function `[X,Y]=meshgrid(x,y)` to generate the two-dimensional grid of x and y .
- Get s , where $s = X + 1j * Y$. Write the expression of $F(s)$ with s .
- Use function **abs**, **angle**, **real** or **imag** to get the magnitude, phase, real part or imaginary part of $F(s)$ in complex s plane.
- Use function **mesh** or **surf** to get the surface plot of $F(s)$.

The functions used here are listed in Table 2.

Table 2 Functions Used to Draw Surface Plot

Function	Format	Description
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meshgrid	[X,Y]=meshgrid(x,y)	Returns 2-D grid coordinates based on the coordinates contained in vectors x and y. X is a matrix where each row is a copy of x, and Y is a matrix where each column is a copy of y. The grid represented by the coordinates X and Y has length(y) rows and length(x) columns.
mesh	mesh(X,Y,Z)	Draws a wireframe mesh with color determined by Z, so color is proportional to surface height. If X and Y are vectors, length(X) = n and length(Y) = m, where [m,n] = size(Z). In this case, (X(j), Y(i), Z(i,j)) are the intersections of the wireframe gridlines; X and Y correspond to the columns and rows of Z, respectively. If X and Y are matrices, (X(i,j), Y(i,j), Z(i,j)) are the intersections of the wireframe grid lines. The values in X, Y, or Z can be numeric, datetime, duration, or categorical values.
surf	surf(X,Y,Z)	Create a three-dimensional surface plot. The function plots the values in matrix Z as heights above a grid in the x-y plane defined by X and Y. The function also uses Z for the color data, so color is proportional to height.
colormap	colormap(map)	Sets the colormap for the current figure to the colormap specified by map .
rotate3d	rotate3d on	Enable mouse-base rotation on all axes within the current figure.

The parameter **map** of function **colormap** decides the color scale of the plot as shown in Figure 2.


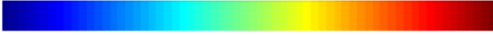
















Colormap Name	Color Scale
parula	
jet	
hsv	
hot	
cool	
spring	
summer	
autumn	
winter	
gray	
bone	
copper	
pink	
lines	
colorcube	
prism	
flag	
white	

Figure 2 the Value of map

Eg: $F(s) = \frac{2(s-3)(s+3)}{(s-5)(s^2+10)}$, do the surface plot of $F(s)$.

```
x = -6:0.48:6;y=x;
[x,y] = meshgrid(x,y);
s = x+1j*y;
Fs = (2*(s-3).*(s+3))./((s-5).*(s.*s+10));
Fsabs = abs(Fs);
surf(x,y,Fsabs);
axis([-6,6,-6,6,0,4.5]);
title('Surface Plot of Laplace Transform');
colormap(hsv);
rotate3d on;
```

The surface plot is show in Figure 3.

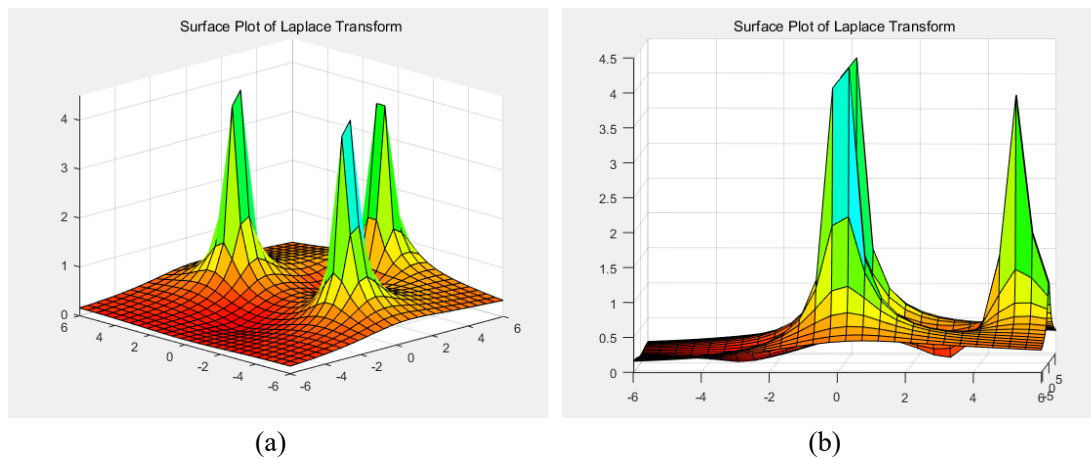


Figure 3 Surface Plot of F_s

From Figure 2, it is easy to identify that the three peaks (where $s = \pm j3.1623, s = 5$) on the surface plot correspond to the three poles of F_s , and the two valleys (where $s = \pm 3$ correspond to the zeros.