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§ 5.2 (Page 570)

6. Let

$$A=\begin{bmatrix}4&0&1\\2&3&2\\1&0&4\end{bmatrix}$$

(a) Find the eigenvalues of A.

(b) For each eigenvalue λ, find the rank of the matrix $\lambda I - A$.

(c) Is A diagonalizable? Justify your conclusion.

(a) $\det(\lambda I_3 - A) = \begin{vmatrix}\lambda-4 & 0 & -1\\ -2 & \lambda-3 & -2\\ -1 & 0 & \lambda-4\end{vmatrix}$

$= (\lambda-3)[\lambda^2 - 8\lambda + 15]$

$= (\lambda-3)^2(\lambda-5)$

$\lambda_1 = 3 \quad \lambda_2 = 5$

(b). $\lambda_1 = 3 \quad \lambda I - A = \begin{bmatrix}-1&0&-1\\-2&0&-2\\-1&0&-1\end{bmatrix} \quad \text{rank}(\lambda I - A) = 1$

$\lambda_2 = 5 \quad \lambda I - A = \begin{bmatrix}1&0&-1\\-2&2&-2\\-1&0&1\end{bmatrix} \quad \text{rank}(\lambda I - A) = 2$

(c) $\lambda_1 = 3 \quad \begin{bmatrix}-1&0&-1\\-2&0&-2\\-1&0&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \quad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$\lambda_2 = 5 \quad \begin{bmatrix}1&0&-1\\-1&2&-2\\-1&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \quad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = t\begin{bmatrix}1\\\frac{3}{2}\\2\end{bmatrix}$

so A is diagonalizable

In Exercises 7–11, use the method of Exercise 6 to determine whether the matrix is diagonalizable.

11. $\begin{bmatrix}2&-1&0&1\\0&2&1&-1\\0&0&3&2\\0&0&0&3\end{bmatrix}$

(1) $\lambda_1 = 2 \quad \begin{bmatrix}0&1&0&-1\\0&0&-1&1\\0&0&-1&-2\\0&0&0&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = 0 \quad \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

$\det(\lambda I_4 - A) = \begin{vmatrix}\lambda-2 & 1 & 0 & -1\\ 0 & \lambda-2 & -1 & 1\\ 0 & 0 & \lambda-3 & -2\\ 0 & 0 & 0 & \lambda-3\end{vmatrix}$

so it is not diagonalizable.

$= (\lambda-2)^2(\lambda-3)^2 = 0$

$\lambda_1 = 2 \quad \lambda_2 = 3$

In Exercises 12–15, find a matrix P that diagonalizes A, and compute $P^{-1}AP$.

12. $A=\begin{bmatrix}-14&12\\-20&17\end{bmatrix}$

(1) $\lambda_1 = 1 \quad \begin{bmatrix}15&-12\\20&-16\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = 0 \quad \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\\frac{5}{4}\end{bmatrix}$

$\det(\lambda I_2 - A) = \begin{vmatrix}\lambda+14 & -12\\ 20 & \lambda-17\end{vmatrix}$

(2) $\lambda_2 = 2 \quad \begin{bmatrix}16&-12\\20&-15\end{bmatrix} = 0 \quad \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\\frac{4}{3}\end{bmatrix}$

$= (\lambda+14)(\lambda-17) + 240 = 0.$

$\lambda_1 = 1 \quad \lambda_2 = 2$

$P = \begin{bmatrix}1&1\\\frac{5}{4}&\frac{4}{3}\end{bmatrix} \quad P^{-1}AP = \begin{bmatrix}16&-12\\-15&12\end{bmatrix}\begin{bmatrix}-14&12\\-20&17\end{bmatrix}\begin{bmatrix}1&1\\\frac{5}{4}&\frac{4}{3}\end{bmatrix}$

$= \begin{bmatrix}16&-12\\-30&24\end{bmatrix}\begin{bmatrix}1&1\\\frac{5}{4}&\frac{4}{3}\end{bmatrix}$

$= \begin{bmatrix}1&0\\0&2\end{bmatrix}$

In Exercises 12–15, find a matrix P that diagonalizes A, and compute $P^{-1}AP$.

14. $A=\begin{bmatrix}1&0&0\\0&1&1\\0&1&1\end{bmatrix}$

(1) $\lambda_1 = 0 \quad \begin{bmatrix}-1&0&0\\0&-1&-1\\0&-1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \quad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\1\\-1\end{bmatrix}$

$\det(\lambda I_3 - A) = \begin{vmatrix}\lambda-1 & 0 & 0\\ 0 & \lambda-1 & -1\\ 0 & -1 & \lambda-1\end{vmatrix}$

(2) $\lambda_2 = 1 \quad \begin{bmatrix}0&0&0\\0&0&-1\\0&-1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \quad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$= (\lambda-1)(\lambda^2 - 2\lambda)$

$= \lambda(\lambda-1)(\lambda-2)$

(3) $\lambda_3 = 2 \quad \begin{bmatrix}1&0&0\\0&1&-1\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \quad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}$

$\lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_3 = 2$

A is not diagonalizable

$$P=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix} \quad P^{-1}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$

(d) $$A^{-2301}=PD^{-2301}P^{-1}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&2^{-2301}&-8^{-2301}\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$

$$=\begin{bmatrix}1&2^{-2301}&-8^{-2301}\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}=\begin{bmatrix}1&-2^{-2301}&8^{-2301}\\0&-1&0\\0&0&-1\end{bmatrix}$$

24. In each part, compute the stated power of

$$A=\begin{bmatrix}1&-2&8\\0&-1&0\\0&0&-1\end{bmatrix}$$

$\boxed{\text{(a) } A^{1000}}$ (b) A^{-1000} (c) A^{2301} $\boxed{\text{(d) } A^{-2301}}$

$$D=P^{-1}AP=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&-2&8\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}=\begin{bmatrix}1&-2&8\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$

$$=\begin{bmatrix}1&2&-8\\0&-1&0\\0&0&-1\end{bmatrix}$$

(a) $$A^{1000}=PD^{1000}P^{-1}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&2^{1000}&8^{1000}\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$

$$=\begin{bmatrix}1&2^{1000}&8^{1000}\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}=\begin{bmatrix}1&-2^{1000}&-8^{1000}\\0&1&0\\0&0&1\end{bmatrix}$$

26. Let

$$A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$$

Show that

(a) A is diagonalizable if $(a-d)^2+4bc>0$.

(b) A is not diagonalizable if $(a-d)^2+4bc<0$.

[*Hint:* See Exercise 19 of Section 5.1.]

since $\lambda=\frac{1}{2}\left[(a+d)\pm\sqrt{(a-d)^2+4bc}\right]$,

(a) if $(a-d)^2+4bc>0$ then there exists two distinct real eigenvalues so A is diagonalizable

(b) if $(a-d)^2+4bc<0$ then there only exists complex conjugate eigenvalues so A is not diagonalizable.

33. Suppose that the characteristic polynomial of some matrix A is found to be $p(\lambda)=(\lambda-1)(\lambda-3)^2(\lambda-4)^3$. In each part, answer the question and explain your reasoning.

(a) What can you say about the dimensions of the eigenspaces of A?

(b) What can you say about the dimensions of the eigenspaces if you know that A is diagonalizable?

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of eigenvectors of A all of which correspond to the same eigenvalue of A, what can you say about the eigenvalue?

(a) $\lambda=1$, dimension $=1$; $\lambda=3$, dimension ≤ 2, $\lambda=4$, dimension ≤ 3

(b) 1, 2, 3

(c) $\lambda=4$