1. (15 points) Show that $\log_5 7$ is an irrational number.

Suppose $\log_5 7$ is a rational number

then $\log_5 7$ can be expressed as $\frac{b}{a}$ (a, b are all integers)

that is $\log_5 7 = \frac{b}{a}$

then $5^a = 7^b$

since 5 and 7 are all primes

a, b have only one value, namely $a=b=0$

so $\frac{b}{a}$ make no sense

so the hypothesis doesn't work

$\log_5 7$ is an irrational number

2. (20 points) Let p be a prime and let k be an integer such that $0 < k < p$. We know that the binomial coefficient

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$

is an integer. Show that $\binom{p}{k}$ is a multiple of p.

$\frac{p!}{k!(p-k)!} \cdot k! = \frac{p!}{(p-k)!}$ it's obvious that $p \mid \frac{p!}{(p-k)!}$

and since p is a prime, $\gcd(p, k!) = 1$

according to FTA Theorem.

$p \mid \frac{p!}{k!(p-k)!}$ so $\binom{p}{k}$ is a multiple of p.

3. (20 points) Let $a, b > 1$ be relatively prime integers. Show that if $a|n$ and $b|n$, then $ab|n$.

since $a|n$, then there exits x, such that $n = ax$

since $b|n$, then there exits y, such that $n = by$.

so $ax = by$, namely $\frac{a}{b} = \frac{y}{x}$

since a, b are relatively prime integers, then $b|x$.

then there exits z, such that $x = bz$

so $n = ax = abz$

which shows $ab|n$.

4. (25 points) Let $a, b, c \in \mathbb{Z}^+$. Show that $\gcd(a, bc) = 1$ if and only if $\gcd(a, b) = \gcd(a, c) = 1$.

(1) since $\gcd(a, b) = \gcd(a, c) = 1$

then a, b and a, c are relatively prime integers

so a and bc are relatively prime integers

so $\gcd(a, bc) = 1$

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(2) since $\gcd(a, bc) = 1$

then a and bc are relatively prime integers

if $a|b$ or $a|c$, then it contradicts with $a \nmid bc$

so $a \nmid b$ and $a \nmid c$

that is $\gcd(a, b) = \gcd(a, c) = 1$

5. (20 points) Let $\mathbb{R}$ be the set of real numbers. Let $S = (\mathbb{R} \times \mathbb{R}) \setminus \{(0,0)\}$. Let

$$R = \Big\{((a,b),(c,d)) : (a,b),(c,d) \in S \text{ and } \exists \lambda \in \mathbb{R} \setminus \{0\} \text{ such that } (a,b) = (\lambda c, \lambda d)\Big\}$$

Show that R is an equivalence relation.

Reflexive: for any $(a,b) \in S$. let $\lambda = 1$

then $((a,b),(a,b)) \in R$ since $(a,b) = (a,b)$

Symmetric: for any $((a,b),(c,d)) \in R$.

there exists $\lambda = \lambda_0$ such that $(a,b) = (\lambda_0 c, \lambda_0 d)$

let $\lambda = \frac{1}{\lambda_0}$ $\quad (c,d) = (\frac{1}{\lambda_0} a, \frac{1}{\lambda_0} b)$,

so $((c,d),(a,b)) \in R$

Transitive: for any $((a,b),(c,d))$, $((c,d),(e,f)) \in R$.

we know that $(a,b) = (\lambda_1 c, \lambda_1 d)$ $(c,d) = (\lambda_2 e, \lambda_2 f)$

so $(a,b) = (\lambda_1 \lambda_2 e, \lambda_1 \lambda_2 f)$

it means $((a,b),(e,f)) \in R$.

In conclusion. R is an equivalence relation.