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§ 8.1 (Page 791)

10. Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where $\mathbf{v}_1 = (-2, 1)$ and $\mathbf{v}_2 = (1, 3)$, and let $T: R^2 \to R^3$ be the linear transformation such that

$$T(\mathbf{v}_1) = (-1, 2, 0) \quad \text{and} \quad T(\mathbf{v}_2) = (0, -3, 5)$$

Find a formula for $T(x_1, x_2)$, and use that formula to find $T(2, -3)$.

$(x_1, x_2) = c_1(-2, 1) + c_2(1, 3)$

$\begin{cases} -2c_1 + c_2 = x_1 \\ c_1 + 3c_2 = x_2 \end{cases} \Rightarrow \begin{cases} c_1 = -\frac{3}{7}x_1 + \frac{1}{7}x_2 \\ c_2 = \frac{1}{7}x_1 + \frac{2}{7}x_2 \end{cases}$

$(x_1, x_2) = (-\frac{3}{7}x_1 + \frac{1}{7}x_2)\vec{v_1} + (\frac{1}{7}x_1 + \frac{2}{7}x_2)\vec{v_2}$

$T(x_1, x_2) = (-\frac{3}{7}x_1 + \frac{1}{7}x_2)(-1, 2, 0) + (\frac{1}{7}x_1 + \frac{2}{7}x_2)(0, -3, 5)$

$= (\frac{3}{7}x_1 - \frac{1}{7}x_2, -\frac{9}{7}x_1 - \frac{4}{7}x_2, \frac{5}{7}x_1 + \frac{10}{7}x_2)$

$T(2, -3) = (\frac{9}{7}, -\frac{6}{7}, -\frac{20}{7})$

18. Let $T: P_2 \to P_3$ be the linear transformation defined by $T(p(x)) = xp(x)$. Which of the following are in $\ker(t)$?

(a) x^2

(b) 0

(c) $1 + x$

(b)

19. Let $T: P_2 \to P_3$ be the linear transformation in Exercise 18. Which of the following are in $R(t)$?

(a) $x + x^2$

(b) $1 + x$

(c) $3 - x^2$

(a) (b) (c).

§8.4 (Page 832)

4. Let $T: R^2 \to R^2$ be the linear operator defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ be the basis for which

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

(a) Find $[T]_B$.

$T(\vec{u_1}) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$T(\vec{u_2}) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$[T]_B = T(\vec{u_1}) + T(\vec{u_2}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(a) $[T(\vec{v_1})]_{B'} = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}$ $[T(\vec{v_2})]_{B'} = \begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix}$ $[T(\vec{v_3})]_{B'} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$ $[T(\vec{v_4})]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(b) $T(\vec{v_1}) = \begin{bmatrix} 0 \\ 24 \\ 24 \end{bmatrix} + \begin{bmatrix} -7 \\ 8 \\ 1 \end{bmatrix} + \begin{bmatrix} 18 \\ -27 \\ -3 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ 22 \end{bmatrix}$ $T(\vec{v_2}) = \begin{bmatrix} 0 \\ -16 \\ -16 \end{bmatrix} + \begin{bmatrix} -42 \\ 48 \\ 6 \end{bmatrix} = \begin{bmatrix} -42 \\ 32 \\ -10 \end{bmatrix}$

$T(\vec{v_3}) = \begin{bmatrix} 0 \\ 8 \\ 8 \end{bmatrix} + \begin{bmatrix} -14 \\ 16 \\ 2 \end{bmatrix} + \begin{bmatrix} -42 \\ 63 \\ 7 \end{bmatrix} = \begin{bmatrix} -56 \\ 87 \\ 17 \end{bmatrix}$ $T(\vec{v_4}) = \begin{bmatrix} -7 \\ 8 \\ 1 \end{bmatrix} + \begin{bmatrix} -6 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ 17 \\ 2 \end{bmatrix}$

10. Let $A = \begin{bmatrix} 3 & -2 & 1 & 0 \\ 1 & 6 & 2 & 1 \\ -3 & 0 & 7 & 1 \end{bmatrix}$ be the matrix for $T: R^4 \to R^3$ relative to the bases $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $B' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 6 \\ 9 \\ 4 \\ 2 \end{bmatrix}$$

$$\mathbf{w}_1 = \begin{bmatrix} 0 \\ 8 \\ 8 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -7 \\ 8 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} -6 \\ 9 \\ 1 \end{bmatrix}$$

(a) Find $[T(\mathbf{v}_1)]_{B'}$, $[T(\mathbf{v}_2)]_{B'}$, $[T(\mathbf{v}_3)]_{B'}$, and $[T(\mathbf{v}_4)]_{B'}$.

(b) Find $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, $T(\mathbf{v}_3)$, and $T(\mathbf{v}_4)$.

(c)

Find a formula for $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right)$.

(d)

Use the formula obtained in (c) to compute $T\left(\begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}\right)$.

(c) $\begin{cases} 2c_2 + c_3 + 6c_4 = x_1 \\ c_1 + c_2 + 4c_3 + 9c_4 = x_2 \\ c_1 - c_2 - c_3 + 4c_4 = x_3 \\ c_1 - c_2 + 2c_3 + 2c_4 = x_4 \end{cases}$

$c_1 =$

$\Rightarrow$

$c_3 = -\frac{2}{5}x_1 + \frac{2}{5}x_2 - \frac{1}{5}x_3 - \frac{1}{5}x_4$ $c_4 = -\frac{3}{5}x_1$

14. Show that if $T: V \to W$ is the zero transformation, then the matrix for T with respect to any bases for V and W is a zero matrix.

if $T: V \to W$ is the zero transformation.

then T maps basis of V to zero $\{k\vec{v_1}, k\vec{v_2}, \cdots, k\vec{v_n}\}$

where $\vec{v_n} = \vec{0}$, k is a nonzero constant.

thus the matrix for T with respect to any bases ~~for~~ for V and W is a zero matrix

15. Show that if $T: V \to V$ is a contraction or a dilation of V (Example 4) of Section 8.1), then the matrix for T relative to any basis for V is a positive scalar multiple of the identity matrix.

If T is a contraction or dilation of V then T maps any basis $B = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ of V to $\{k\vec{v_1}, k\vec{v_2}, \cdots, k\vec{v_n}\}$ where k is a nonzero constant. Thus the matrix for T relative to B is

$$\begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & k \end{bmatrix}$$

16. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis for a vector space V. Find the matrix with respect to B of the linear operator $T: V \to V$ defined by $T(\mathbf{v}_1) = \mathbf{v}_2$, $T(\mathbf{v}_2) = \mathbf{v}_3$, $T(\mathbf{v}_3) = \mathbf{v}_4$, $T(\mathbf{v}_4) = \mathbf{v}_1$.

$$[T(\vec{v_1})]_B = \vec{V_2} \cdot \vec{V_1}$$

$$[T(\vec{v_2})]_B = \vec{V_3} \cdot \vec{V_2}$$

$$[T(\vec{v_3})]_B = \vec{V_4} \cdot \vec{V_3}$$

$$[T(\vec{v_4})]_B = \vec{V_1} \cdot \vec{V_4}$$

So the matrix is $\left[\vec{V_2} \cdot \vec{V_1} \mid \vec{V_3} \cdot \vec{V_2} \mid \vec{V_4} \cdot \vec{V_3} \mid \vec{V_1} \cdot \vec{V_4} \right]$.

18. ***(Calculus required)*** Let $D\colon P_2 \to P_2$ be the differentiation operator $D(\mathbf{p}) = p'(x)$. In parts (a) and (b), find the matrix of D relative to the basis $B = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

(a) $\mathbf{p}_1 = 1,\ \mathbf{p}_2 = x,\ \mathbf{p}_3 = x^2$

(b) $\mathbf{p}_1 = 2,\ \mathbf{p}_2 = 2 - 3x,\ \mathbf{p}_3 = 2 - 3x + 8x^2$

(c) Use the matrix in part (a) to compute $D(6 - 6x + 24x^2)$.

(d) Repeat the directions for part (c) for the matrix in part (b).

(a) $D(p_1) = 0 \quad D(p_2) = 1 \quad D(p_3) = 2x.$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) $D(p_1) = 0 \quad D(p_2) = -3 \quad D(p_3) = -3 + 16x$

$$\begin{bmatrix} 0 & -3 & -3 \\ 0 & 0 & 16 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) $D(6 - 6x + 24x^2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} -6 \\ 48 \\ 0 \end{bmatrix} (1, x, x^2) = -6 + 48x$

(d) $D(6 - 6x + 24x^2) = \begin{bmatrix} 0 & -3 & -3 \\ 0 & 0 & 16 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} -54 \\ 384 \\ 0 \end{bmatrix} (2, 2-3x, 2-3x+8x^2)$

$= -108 + 384(2 - 3x)$

$= -1152x + 660$

19. ***(Calculus required)*** In each part, suppose that $B = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is a basis for a subspace V of the vector space of real-valued functions defined on the real line. Find the matrix with respect to B for differentiation operator $D\colon V \to V$.

(a) $\mathbf{f}_1 = 1,\ \mathbf{f}_2 = \sin x,\ \mathbf{f}_3 = \cos x$

$D(f_1) = D(1) = 0$

$D(f_2) = D(\sin x) = \cos x$

$D(f_3) = D(\cos x) = -\sin x$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$