QUIZ 2022/5/12

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1. (15 points) Suppose that $f(x,y)$ is defined on a domain $G=\{(x,y)\,|x^2+y^2<1\}$, and $f(x,0)$ is continuous(连续的) at $x=0$, $f'_y(x,y)$ is bounded(有界的) on domain G. Prove that $f(x,y)$ is continuous at $(0,0)$.

证明：由题目条件可知，$\exists M>0$. s.t $|f'_y(x,y)|\le M$ 且 $(x,y)\in G$

根据微分中值定理，$\exists\lambda\in(0,y)$ s.t $|f(x,y)-f(x,0)|=|f'_y(x,\lambda)|\cdot|y|\le M\|y\|$

$\therefore\ \forall\varepsilon>0$，$\exists\,\delta=\frac{\varepsilon}{2M}$ 当 $|y-0|<\delta$ 时 有 $|f(x,y)-f(x,0)|<\frac{\varepsilon}{2}$

又 $\because f(x,0)$ 在 $x=0$ 处连续 $\therefore\exists\ 0<\delta'<\delta$

当 $|x-0|<\delta'$ 时 $|f(x,0)-f(0,0)|<\frac{\varepsilon}{2}$

$\therefore f(x,y)$ 在 $(0,0)$ 处连续

2. (15 points) Suppose that $x>0,y>0,z>0$. Calculate the maximum of the function

$$f(x,y,z)=\ln x+2\ln y+3\ln z$$

on sphere(球面) $x^2+y^2+z^2=6r^2$, and prove that

$$ab^2c^3<108\left(\frac{a+b+c}{6}\right)^6,$$

when a,b,c are positive real numbers.

解：记 $f(x,y,z,\lambda)=\ln x+2\ln y+3\ln z+\lambda(x^2+y^2+z^2-6r^2)$

$$\begin{cases}\frac{1}{x}+2\lambda x=0\\ \frac{2}{y}+2\lambda y=0\\ \frac{3}{z}+2\lambda z=0\\ x^2+y^2+z^2-6r^2=0.\end{cases}$$

解得 $\lambda=-\frac{1}{2r^2}$ $x=r$

$y=\sqrt{2}r$ $z=\sqrt{3}r$

So $\operatorname{Max}f(x,y,z)=f(r,\sqrt{2}r,\sqrt{3}r)=\ln r+2\ln\sqrt{2}r+3\ln\sqrt{3}r$

$=\ln r+\ln 2r^2+\ln 3\sqrt{3}r^3$

$=\ln(6\sqrt{3}r^6)$

so $\ln ab^2c^3=\ln a+2\ln b+3\ln c$

$=f(a,b,c)\le\ln\left[6\sqrt{3}\cdot\left(\frac{a^2+b^2+c^2}{6}\right)^3\right]$

$\therefore ab^2c^3\le 6\sqrt{3}\cdot\left(\frac{a^2+b^2+c^2}{6}\right)^3<108\left(\frac{a+b+c}{6}\right)^6$

3. (15 points) Suppose that the function $u = u(x, y, z)$ is given by the equation

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1,$$

where a, b, c are constants $(a^2 \neq b^2 \neq c^2)$.

Prove that

$$|\nabla u|^2 = 2\vec{r} \cdot \nabla u,$$

where $\vec{r} = (x, y, z)$.

令 $f(x,y,z) = \frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}$

等式两边对 x 求偏导 $\frac{2x(a^2+u) - x^2(a^2+u'_x)}{(a^2+u)^2} - \frac{u'_x y^2}{(b^2+u)^2} - \frac{u'_x z^2}{(c^2+u)^2} = 0.$ ①

同理 对 y ... $\frac{2y(b^2+u) - y^2(b^2+u'_y)}{(b^2+u)^2} - \frac{u'_y x^2}{(a^2+u)^2} - \frac{u'_y z^2}{(c^2+u)^2} = 0$ ②

对 z ... $\frac{2z(c^2+u) - z^2(c^2+u'_z)}{(c^2+u)^2} - \frac{u'_z x^2}{(a^2+u)^2} - \frac{u'_z y^2}{(b^2+u)^2} = 0.$ ③

将 ① ② ③ 式中后两项移至"="右侧

则 ①+②+③ 即为 $|\nabla u|^2 = 2\vec{r} \cdot \nabla u$

4. (15points) Suppose that $f(x, y)$ is differentiable(可微的) on $\mathbb{R}^2$, and satisfies that

$$\lim_{\rho\to\infty} \frac{f(x,y)}{\rho} = +\infty,$$

where $\rho = \sqrt{x^2+y^2}$.

Prove that for any $\boldsymbol{v} = (v_1, v_2)$, there exists a point (x_0, y_0), which satisfies

$$\nabla f(x_0, y_0) = \boldsymbol{v}$$

$\because \lim\limits_{\rho\to\infty} \frac{f(x,y)}{\rho} = +\infty,$

$\therefore f(x,y)$ 是比 $\sqrt{x^2+y^2}$ 更高阶的无穷大量.

$\therefore \lim\limits_{\substack{x\to+\infty \\ y\to+\infty}} f'_x(x,y) = +\infty \quad \lim\limits_{\substack{x\to+\infty \\ y\to+\infty}} f'_y(x,y) = +\infty.$

$\therefore$ 对于 $\forall v_1, v_2 \in R \quad \exists\ x_0, y_0 \in R.$ s.t $\frac{\partial f}{\partial x} = v_1 \quad \frac{\partial f}{\partial y} = v_2$

$\therefore \forall \vec{v} = (v_1, v_2) \quad \exists\ (x_0, y_0)$ s.t $\nabla f(x_0, y_0) = \vec{v}$