LA homework Dec.10 name: 王柯皓 id: 2021533025

§ 5.3 (Page 589)

In Exercises 15–18, find the eigenvalues and bases for the eigenspaces of A.

16. $A=\begin{bmatrix}-1 & -5\\ 4 & 7\end{bmatrix}$

$\det(\lambda I-A)=\begin{vmatrix}\lambda+1 & 5\\ -4 & \lambda-7\end{vmatrix}=\lambda^2-6\lambda+13=0$

$\lambda_1=3+2i \quad X_1=\begin{bmatrix}1-\frac{1}{2}i\\ 1\end{bmatrix}$

$\lambda_2=3-2i \quad X_1=\begin{bmatrix}1+\frac{1}{2}i\\ 1\end{bmatrix}$

27. Find all complex scalars k, if any, for which $\mathbf{u}$ and $\mathbf{v}$ are orthogonal in C^3.

(a) $\mathbf{u}=(2i, i, 3i)$, $\mathbf{v}=(i, 6i, k)$

(b) $\mathbf{u}=(k, k, 1+i)$, $\mathbf{v}=(1, -1, 1-i)$

(a) $-2-6+3ki=0.$ (b) $k-k+2=0$

$k=\frac{8}{3i}=-\frac{8}{3}i$ no solution.

29. The matrices

$$\sigma_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \sigma_2=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\quad \sigma_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

called ***Pauli spin matrices***, are used in quantum mechanics to study particle spin. The ***Dirac matrices***, which are also used in quantum mechanics, are expressed in terms of the Pauli spin matrices and the 2×2 identity matrix I_2 as

$$\beta=\begin{bmatrix}I_2 & 0\\ 0 & -I_2\end{bmatrix},\quad \alpha_x=\begin{bmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{bmatrix},$$
$$\alpha_y=\begin{bmatrix}0 & \sigma_2\\ \sigma_2 & 0\end{bmatrix},\quad \alpha_z=\begin{bmatrix}0 & \sigma_3\\ \sigma_3 & 0\end{bmatrix}$$

(a) Show that $\beta^2=\alpha_x^2=\alpha_y^2=\alpha_z^2$.

(b) Matrices A and B for which $AB=-BA$ are said to be ***anticommutative***. Show that the Dirac matrices are anticommutative.

(a) $\beta^2=\begin{bmatrix}I_2 & 0\\ 0 & -I_2\end{bmatrix}\begin{bmatrix}I_2 & 0\\ 0 & -I_2\end{bmatrix}=\begin{bmatrix}I_2 & 0\\ 0 & I_2\end{bmatrix}$

$\alpha_x^2=\begin{bmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{bmatrix}=\begin{bmatrix}\sigma_1^2 & 0\\ 0 & \sigma_1^2\end{bmatrix}$

$\because \sigma_1^2=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I_2 \quad \therefore \beta^2=\alpha_x^2$

同理 $\beta^2=\alpha_x^2=\alpha_y^2=\alpha_z^2$

(b). $\beta\alpha_x=\begin{bmatrix}I_2 & 0\\ 0 & -I_2\end{bmatrix}\begin{bmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{bmatrix}=\begin{bmatrix}0 & I_2\sigma_1\\ -I_2\sigma_1 & 0\end{bmatrix}$

$-\alpha_x\beta=-\begin{bmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{bmatrix}\begin{bmatrix}I_2 & 0\\ 0 & -I_2\end{bmatrix}=-\begin{bmatrix}0 & -I_2\sigma_1\\ I_2\sigma_1 & 0\end{bmatrix}=\begin{bmatrix}0 & I_2\sigma_1\\ I_2\sigma_1 & 0\end{bmatrix}$

$\therefore \beta\alpha_x=-\alpha_x\beta$

同理 任意2个 $\beta.\alpha_x.\alpha_y.\alpha_z$ 中的矩阵均有 $AB=-BA$ 成立

§ 6.1 (Page 618)

4. Repeat Exercise 3 for the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + 5u_2v_2$.

exercise3 for reference:

3. Let $\langle \mathbf{u}, \mathbf{v} \rangle$ be the Euclidean inner product on R^2, and let $\mathbf{u} = (3, -2)$, $\mathbf{v} = (4, 5)$, $\mathbf{w} = (-1, 6)$, and $k = -4$. Verify the following.

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

(b) $\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

(c) $\langle \mathbf{u}, \mathbf{v}+\mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

(d) $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, k\mathbf{v} \rangle$

(e) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

(d) $<ku,v> = 3k\times 4 + (-2k)\times 5 = 2k$

$k<u,v> = k\times 2 = 2k$

$<u,kv> = 3\times 4k + (-2)\times 5k = 2k$

(e) $<0,v> = 0\times 4 + 0\times 5 = 0$

$<v,0> = 4\times 0 + 5\times 0 = 0$

(a) $<u,v> = 3\times 4 + (-2)\times 5 = 2$

$<v,u> = 4\times 3 + 5\times(-2) = 2$

(b) $<u+v,w> = (3+4)\times(-1) + (-2+5)\times 6 = 11$

$<u,w> + <v,w> = 3\times(-1) + (-2)\times 6 + 4\times(-1) + 5\times 6 = 11$

(c) $<u,v+w> = 3\times(4-1) + (-2)\times(5+6) = -13$

$<u,v> + <u,w> = 3\times 4 - 2\times 5 + 3\times(-1) - 2\times 6 = -13$

6. Repeat Exercise 5 for the inner product on R^2 generated by $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

exercise 5 for reference:

5. Let $\langle \mathbf{u}, \mathbf{v} \rangle$ be the inner product on R^2 generated by $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, and let $\mathbf{u} = (2, 1)$, $\mathbf{v} = (-1, 1)$, $\mathbf{w} = (0, -1)$. Compute the following.

(a) $\langle \mathbf{u}, \mathbf{v} \rangle$

(b) $\langle \mathbf{v}, \mathbf{w} \rangle$

(c) $\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle$

(d) $\|\mathbf{v}\|$

(e) $d(\mathbf{v}, \mathbf{w})$

(f) $\|\mathbf{v}-\mathbf{w}\|^2$

$A^T = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

(a) $<u,v> = v^TA^TAu$

$= \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -7 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = -11$

(b) $<v,w> = w^TA^TAv$

$= \begin{bmatrix} 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3$

(c) $<u+v,w> = w^TA^TA\cdot(u+v)$

$= \begin{bmatrix} 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$

(d) $<v,v> = v^TA^TAv = \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 10$ $\quad \|v\| = \sqrt{<v,v>} = \sqrt{10}$

(e) $<v-w, v-w> = (v-w)^TA^TA(v-w)$

$= \begin{bmatrix} -1 & 2 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -9 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = 17$

$d(v,w) = \|v-w\| = \sqrt{<v-w, v-w>} = \sqrt{17}$

(f) $\|v-w\|^2 = (d(v,w))^2 = 17$

10. (a) Use Formula 4 to show that

$$\langle \mathbf{u}, \mathbf{v} \rangle = 5u_1v_1 - u_1v_2 - u_2v_1 + 10u_2v_2$$

is the inner product on R^2 generated by

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

formula 4: $\langle \mathbf{u}, \mathbf{v} \rangle = A\mathbf{u} \cdot A\mathbf{v}$

(b) Use the inner product in part (a) to compute $\langle \mathbf{u}, \mathbf{v} \rangle$ if $\mathbf{u} = (0, -3)$ and $\mathbf{v} = (6, 2)$.

(a) $A^TA = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 10 \end{bmatrix}$

For $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$<u,v> = v^TA^TAu$

$= \begin{bmatrix} v_1 & v_2 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ -1 & 10 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$= \begin{bmatrix} 5v_1 - v_2 & -v_1 + 10v_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$= (5v_1 - v_2)u_1 + (-v_1 + 10v_2)u_2$

$= 5u_1v_1 - u_1v_2 - u_2v_1 + 10u_2v_2$

(b) $<u,v> = 5\times 0\times 6 - 0\times 2$

$- (-3)\times 6 + 10\times(-3)\times 2$

$= -42$