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§ 6.1 (Page 618)

8. Compute $\langle \mathbf{p}, \mathbf{q} \rangle$ using the inner product in Example 7.

(a) $\mathbf{p} = -2 + x + 3x^2$, $\mathbf{q} = 4 - 7x^2$

(b) $\mathbf{p} = -5 + 2x + x^2$, $\mathbf{q} = 3 + 2x - 4x^2$

$\langle p, q \rangle = (-5) \times 3 + 2 \times 2 + 1 \times (-4)$

$= -15$

24. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Determine which of the following are inner products on R^3. For those that are not, list the axioms that do not hold.

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_3 v_3$

(b) $\langle \mathbf{u}, \mathbf{v} \rangle = u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2$

(c) $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1 v_1 + u_2 v_2 + 4u_3 v_3$

(d) $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 - u_2 v_2 + u_3 v_3$

(a) is inner product

(b) Axiom 2, 3

(c) is inner product

(d) Axiom 4

25. Show that the following identity holds for vectors in any inner product space.

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

$\|u+v\|^2 + \|u-v\|^2 = \langle u+v, u+v \rangle + \langle u-v, u-v \rangle$

$= \langle u, u+v \rangle + \langle v, u+v \rangle + \langle u, u-v \rangle - \langle v, u-v \rangle$

$= \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle + \langle u,u \rangle - \langle u,v \rangle - \langle v,u \rangle + \langle v,v \rangle$

$= 2\langle u,u \rangle + 2\langle v,v \rangle$

$= 2\|u\|^2 + 2\|v\|^2$

30. ***Calculus required*** In each part, use the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x) g(x)\, dx$$

on $C[0,1]$ to compute $\langle \mathbf{f}, \mathbf{g} \rangle$.

(a) $\mathbf{f} = \cos 2\pi x$, $\mathbf{g} = \sin 2\pi x$

(b) $\mathbf{f} = x$, $\mathbf{g} = e^x$

(c) $\mathbf{f} = \tan \frac{\pi}{4} x$, $\mathbf{g} = 1$

(b) $\langle f, g \rangle = \int_0^1 x e^x dx$

$= (x-1) e^x \Big|_0^1$

$= 0 - (0-1) \times 1$

$= 1$

§ 6.2 (Page 633)

3. Let M_{22} have the inner product in Example 6 of Section 6.1 . Find the cosine of the angle between A and B.

(a) $A=\begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}, B=\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

(b) $A=\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}, B=\begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$

$\langle A, B\rangle = 2\times(-3)+4\times1+(-1)\times4+3\times2 = 0$

$\therefore \cos\theta = \dfrac{\langle A, B\rangle}{\|A\|\cdot\|B\|} = 0$

5. Show that $\mathbf{p}=1-x+2x^2$ and $\mathbf{q}=2x+x^2$ are orthogonal with respect to the inner product in Exercise 2.

$\langle p, q\rangle = 1\times0+(-1)\times2+2\times1 = 0$

so $\cos\theta = \dfrac{\langle p, q\rangle}{\|p\|\,\|q\|} = 0$

which shows that $\theta = 90^\circ$

So p and q are orthogonal

In Exercises 14–15, assume that R^n has the Euclidean inner product.

14. Let W be the line in R^2 with equation $y=2x$. Find an equation for $W^\perp$.

$y = -\frac{1}{2}x$

16. Find a basis for the orthogonal complement of the subspace of R^n spanned by the vectors.

(a) $\mathbf{v}_1=(1,-1,3), \mathbf{v}_2=(5,-4,-4), \mathbf{v}_3=(7,-6,2)$

(b) $\mathbf{v}_1=(2,0,-1), \mathbf{v}_2=(4,0,-2)$

$\begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$x_1 = s$

$x_2 = t$

$x_3 = 2s$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

So the basis for the orthogonal complement of the subspace of R^n

is $w_1 = (0,1,0)$ $w_2 = (1,0,2)$

17. Let V be an inner product space. Show that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors in V, then $\|\mathbf{u}-\mathbf{v}\| = \sqrt{2}$

$$\|u-v\| = \sqrt{\langle u-v, u-v\rangle}$$
$$= \sqrt{\langle u, u-v\rangle - \langle v, u-v\rangle}$$
$$= \sqrt{\langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle}$$
$$= \sqrt{\langle u,u\rangle + \langle v,v\rangle}$$
$$= \sqrt{\|u\|^2 + \|v\|^2} = \sqrt{2}$$

21. Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ be a basis for a subspace W of V. Show that $W^{\perp}$ consists of all vectors in V that are orthogonal to every basis vector.

21. Suppose that $\mathbf{v}$ is orthogonal to every basis vector. Then, as in exercise 19, $\mathbf{v}$ is orthogonal to the span of the set of basis vectors, which is all of W, hence $\mathbf{v}$ is in $W^{\perp}$. If $\mathbf{v}$ is not orthogonal to every basis vector, then $\mathbf{v}$ clearly cannot be in $W^{\perp}$. Thus $W^{\perp}$ consists of all vectors orthogonal to every basis vector.

23. Prove: If $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ matrices and A is an $n \times n$ matrix, then

$$\left(\mathbf{v}^T A^T A\mathbf{u}\right)^2 \le \left(\mathbf{u}^T A^T A\mathbf{u}\right)\left(\mathbf{v}^T A^T A\mathbf{v}\right)$$

since $|\langle u, v\rangle| \le \|u\|\|v\|$

then $|\langle u,v\rangle| = |v^T A^T A u|$

$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{u^T A^T A u}$

$\|v\| = \sqrt{\langle v,v\rangle} = \sqrt{v^T A^T A v}$

so $|v^T A^T A u| \le \sqrt{(u^T A^T A u)(v^T A^T A v)}$

thus $(v^T A^T A u)^2 \le (u^T A^T A u)(v^T A^T A v)$

29. ***Calculus required*** Let $f(x)$ and $g(x)$ be continuous functions on $[0, 1]$. Prove:

(a) $$\left[\int_0^1 f(x)g(x)\,dx\right]^2 \le \left[\int_0^1 f^2(x)\,dx\right]\left[\int_0^1 g^2(x)\,dx\right]$$

(b) $$\left[\int_0^1 [f(x)+g(x)]^2\,dx\right]^{1/2} \le \left[\int_0^1 f^2(x)\,dx\right]^{1/2} + \left[\int_0^1 g^2(x)\,dx\right]^{1/2}$$

[*Hint:* Use the Cauchy–Schwarz inequality.]

(a) $$\left[\int_0^1 f(x)\,g(x)\,dx\right]^2 = \left|\left\langle \int_0^1 f(x)\,dx, \int_0^1 g(x)\,dx\right\rangle\right|^2$$
$$\le \left\|\int_0^1 f(x)\,dx\right\|^2 \left\|\int_0^1 g(x)\,dx\right\|^2$$
$$= \left[\int_0^1 f^2(x)\,dx\right]\left[\int_0^1 g^2(x)\,dx\right]$$

(b) $$\left[\int_0^1 [f(x)+g(x)]^2\,dx\right]^{\frac{1}{2}} = \left\|\int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx\right\|$$
$$\le \left\|\int_0^1 f(x)\,dx\right\| + \left\|\int_0^1 g(x)\,dx\right\|$$
$$= \left[\int_0^1 f^2(x)\,dx\right]^{\frac{1}{2}} + \left[\int_0^1 g^2(x)\,dx\right]^{\frac{1}{2}}$$