EE150: Signals and Systems, Spring 2022
Comprehensive Problem Sets Answer

(Due Monday, May.23 at 11:59am (CST))

1. [20 points] For each of the following statements, judge if it is true, and give a justification or counterexample.

   (a) If $x(t), t \in \mathbf{R}$ is a real-valued signal, then its Fourier transform $X(f)$, $f \in \mathbf{R}$, is also real-valued.

   (b) A linear causal continuous-time system is always time-invariant.

   (c) The inverse of a causal linear and time-invariant(LTI) system is always causal.

   (d) The system with real-valued input $x(t)$ and output

   $$y(t) = (1 + x^4(t))^{(\cos^2(5t) - \sin^2(5t))} \tag{1}$$

   is stable.

   (e) The discrete-time signal $x[n] = sin[\frac{3}{2}n]$ is a periodic signal.

   (f) The following two signals $x_1(t)$ and $x_2(t)$ are periodic with period $T = 1$, as shown in Figure 1.

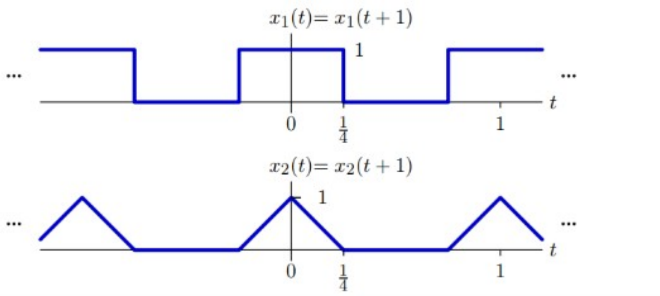

Figure 1: $x_1(t)$ and $x_2(t)$

   For the system shown in Figure 2, if $x(t) = x_1(t)$ and $y(t) = x_2(t)$, then this system cannot be a linear time-invariant system.

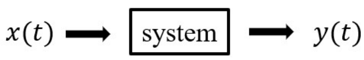

Figure 2: The system

   (g) If $f(t)$ and $h(t), t \in \mathbf{R}$ are real-valued signals, and the convolution satisfies $y(t) = f(t) * h(t)$, then $y(-t) = f(-t) * h(-t)$.

   Answer:

   (a) False. If $x(t) = \sin \omega_0 t$ is real-valued signal, $X(f) = -\pi j\,[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ is certainly not a real-valued signal.

   (b) False. If $y(t) = \sin(t) \cdot x(t)$. This system is a linear causal continuous-time system. (Since $ax(t) \to ay(t)$ and y(t) does not depend on the future input.) However, this system is not time-invariant, since $x(t) \to y(t)$ cannot interpret $x(t - t_0) \to y(t - t_0)$. ($y_1(t) = a(t)x(t - t_0), y_2(t) = a(t - t_0)x(t - t_0)$.certainly, $y_1(t) \neq y_2(t)$.)

   (c) False. If $y(t) = x(t-1)$ (causal LTI system), the inverse of the system is $y(t) = x(t+1)$. This is certainly not a causal system.

(d) True. Since $y(t) = (1+x^4(t))^{\cos(10t)}$, If $x(t) \leq M$, then $\frac{1}{1+M^4} \leq y(t) \leq 1+M^4$, since $-1 \leq \cos(10t) \leq 1$. $\because M^4 \geq 0$, $\frac{1}{1+M^4}$ is bounded and $1+M^4$ is also bounded. Therefore, this system is stable.

(e) False. $\sin[\frac{3}{2}n]$ signal is not a periodic signal since the base frequency is not multiples of π.

(f) True. First, we can calculate the Fourier series coefficients. Then ask if each Fourier series coefficient in the output is a scaled version of the corresponding coefficient in the input.

$$x(t) \leftrightarrow a_k = \frac{1}{1}\int_{\frac{1}{4}}^{-\frac{1}{4}} e^{-j\frac{2\pi}{1}kt}dt = \frac{\sin\frac{k\pi}{2}}{k\pi} = \begin{cases} \frac{1}{2} & k=0 \\ \frac{1}{k\pi} & |k| = 1,5,9,13,... \\ -\frac{1}{k\pi} & |k| = 3,7,11,15,... \\ 0 & |k| = 2,4,6,8,... \end{cases}$$

$$y(t) \leftrightarrow b_k = 1 \times \frac{4\sin^2(\frac{k\pi}{4})}{k^2\pi^2} = \begin{cases} \frac{1}{4} & k=0 \\ \frac{2}{k^2\pi^2} & |k| = 1,3,5,7,9,11,13,... \\ \frac{4}{k^2\pi^2} & |k| = 2,6,10,14,... \\ 0 & |k| = 4,8,12,16,... \end{cases}$$

we can see that the Fourier series coefficients at $k = 2,6,10,...$ are zero in $x(t)$ but these are not zero in $y(t)$. Therefore, the system could not be LTI,

(g) True. $y(t) = f(t) * h(t) = \int_{+\infty}^{-\infty} f(\tau)h(t-\tau)d\tau$, $\therefore y(-t) = \int_{-\infty}^{+\infty} f(\tau)h(-t-\tau)d\tau = \int_{-\infty}^{+\infty} f(-u)h(-(t-u))du = f(-t) * h(-t)$.

2. [20 points]

(a) Consider a linear, time-invariant system with impulse response

$$h[n] = (\frac{1}{2})^{|n|}$$

Find the Fourier series representation of the output $y[n]$ for each of the following inputs.

(i) $x[n] = sin(\frac{3\pi n}{4})$

(ii) $x[n] = j^n + (-1)^n$

(b) Repeat (a) for

$$h[n] = \begin{cases} 1, & 1 \leq n \leq 2 \\ -1, & -2 \leq n \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

Answer:

$$\begin{aligned} H(\omega) &= \sum_{n=0}^{\infty}(\frac{1}{2})^n e^{-j\omega n} + \sum_{n=\infty}^{0}(\frac{1}{2})^{-n} e^{-j\omega n} - 1 \\ &= \frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{1}{1-\frac{1}{2}e^{j\omega}} - 1 \\ &= \frac{3}{5-4cos\omega} \end{aligned}$$

(a) (i) $x[n] = \frac{1}{2j}e^{j\frac{3\pi}{4}n} - \frac{1}{2j}e^{-j\frac{3\pi}{4}n}$

Assume period of x is N, then $\frac{3\pi N}{4} = 2\pi m$, the minimum value of N is 8, so that

$$x[n] = \sum_{k=0}^{7} a_k e^{jk\frac{2\pi}{8}n}$$

$a_3 = \frac{1}{2j}, a_{-3} = -\frac{1}{2j}$ $\omega = \frac{2\pi}{8}$, from the convolution property, $b_k = a_k H(\omega k)$, so that $b_3 = \frac{1}{2j}\frac{3}{5-4cos(\frac{3\pi}{4})}$,$b_{-3} = -\frac{1}{2j}\frac{3}{5-4cos(\frac{3\pi}{4})}$ otherwise zero in the period.

(ii) Period of x is 4, $\omega = \frac{\pi}{2}$, and $x[n] = [e^{j\frac{\pi}{2}}]^n + (e^{j\pi})^n$

So that $a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 0$, then $b_1 = \frac{3}{5-4cos(\frac{\pi}{2})} = \frac{3}{5}, b_2 = \frac{3}{5-4cos(\pi)} = \frac{1}{3}$, otherwise zero in the period.

(b) $H(\omega) = -e^{j2\omega} - e^{j\omega} - 1 + e^{-j\omega} + e^{-j2\omega} = -1 - 2jsin\omega - 2jsin2\omega$

(i) From a, period is 8 $b_3 = \frac{1}{2j}H(\frac{3\pi}{4}) = -\frac{1}{2j} - \frac{\sqrt{2}}{2} + 1$,$b_{-3} = -\frac{1}{2j}H(-\frac{3\pi}{4}) = \frac{1}{2j} - \frac{\sqrt{2}}{2} + 1$, otherwise zero in a period.

(ii) From a, period is 4, $b_1 = H(\frac{\pi}{2}) = -1 - 2j, b_2 = H(\pi) = -1$, otherwise zero in a period.

3. [15 points] Consider a periodic signal $s(t)$ with period $\frac{1}{2}$ and Fourier coefficients $a_1 = a_{-1} = \frac{1}{2}, a_2 = a_{-2} = 1$, and $a_k = 0$ otherwise.

   (a) Determine $s(t)$.

   (b) Assume a system $y(t) = x(s(t))$. Is this system Memoryless, Time Invariant, Linear, Causal, Stable? Explain why.

   (c) Consider an LTI system with impulse response

$$h(t) = \frac{sin(3(t-2))}{\pi(t-2)}$$

   Determine the output $y_1(t)$ if the input is $s(t)$.

Answer:
(a)T=$\frac{1}{2}$ so that $\omega = 4\pi$, then

$$s(t) = \frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t} + e^{j8\pi t} + e^{-j8\pi t} = cos(4\pi t) + 2cos(8\pi t)$$

(b)Linear: if $x_1(t) \longrightarrow y_1(t), x_2(t) \longrightarrow y_2(t)$ and $x_3(t) = ax_1(t) + bx_2(t)$ then $ay_1(t) + by_2(t) \longrightarrow ax_1(s(t)) + bx_2(s(t))$ so it is Linear
TI:$y_1(t + t_0) = x(cos4\pi(t + t_0) + 2cos8\pi(t + t_0)) \neq x(cos4\pi t + 2cos8\pi t + t_0)$, so not TI
Casual: if $t = 0$,then $y_1(0) = x(3)$, not casual
Memory: if $t = 4$,then $y_1(4) = x(3)$, not memoryless
Stable: if $|x(t)| < B$, then $y_1(t) = x(s(t)) < B$, so stable.

(c)$h(t + 2) = \frac{sin3t}{\pi t}$ so

$$H(j\omega)e^{2j\omega} = \begin{cases} 1, & |\omega| < 3 \\ 0, & otherwise \end{cases}$$

and $cos(4\pi t) \xrightarrow{F} \pi[\delta(\omega - 4\pi) + \delta(\omega + 4\pi)]$, $cos(8\pi t) \xrightarrow{F} \pi[\delta(\omega - 8\pi) + \delta(\omega + 8\pi)]$
$S(j\omega)$ only has non-zero value in $\omega = \pm 4\pi$ and $\pm 8\pi$, and $H(j\omega)$ only has non-zero value in $|\omega| < 3$
As $4\pi > 3, 8\pi > 3$, so $Y = S(j\omega)H(j\omega)$=0, output $y(t) = 0$.

4. [20 points] When the input of a LTI system is $f(t)$, the corresponding output is

$$y(t) = \frac{1}{a}\int_{-\infty}^{\infty} g(\frac{x-t}{b})f(x-c)dx$$

where a, b are non-zero constants and we know that the Fourier Transform of $g(t)$ is $G(j\omega)$.

(a) Determine the frequency response $H(j\omega)$ of the system.

(b) Let the Fourier Transform of $f(t)$ be $F(j\omega) = 2\pi|d|\delta(\omega^2 - d^2)$, where d is a non-zero constant. By setting $G(j\omega) = \frac{a}{|b|}\frac{bd+j\omega}{bd-j\omega}e^{-j\frac{c}{b}\omega}$, determine the output of the LTI system, $y(t)$, by using the answer in part(a).

Answer:
Part(a)

$$y(t) = \frac{1}{a}\int_{-\infty}^{\infty} g(\frac{x-t}{b})f(x-c)dx = \frac{1}{a}\int_{-\infty}^{\infty} g(-\frac{t-x}{b})f(x-c)dx = \frac{1}{a}g(-\frac{t}{b}) * f(t-c)$$

Time shift and scale

$$\frac{1}{a}g(-\frac{t}{b}) \leftrightarrow \frac{|b|}{a}G(-jb\omega), f(t-c) \leftrightarrow e^{-jc\omega}F(j\omega)$$

Then

$$Y(j\omega) = \frac{|b|}{a}G(-jb\omega)e^{-jc\omega}F(j\omega)$$

Get the frequency response

$$H(j\omega) = \frac{Y(j\omega)}{F(j\omega)} = \frac{|b|}{a}G(-jb\omega)e^{-jc\omega}$$

Part(b)
Since

$$G(j\omega) = \frac{a}{|b|}\frac{bd+j\omega}{bd-j\omega}e^{-j\frac{c}{b}\omega}$$

then

$$G(-jb\omega) = \frac{a}{|b|}\frac{d-j\omega}{d+j\omega}e^{jc\omega}$$

Substitute in $H(j\omega)$

$$H(j\omega) = \frac{|b|}{a}G(-jb\omega)e^{-jc\omega} = \frac{d-j\omega}{d+j\omega}$$

Since

$$\delta(\omega^2 - d^2) = \frac{1}{2|d|}[\delta(\omega+d) + \delta(\omega-d)]$$

Calculate $F(j\omega)$

$$F(j\omega) = \pi[\delta(\omega+d) + \delta(\omega-d)]$$

Then calculate $Y(j\omega)$

$$\begin{aligned} Y(j\omega) = F(j\omega)H(j\omega) &= \pi[\delta(\omega+d) + \delta(\omega-d)]\frac{d-j\omega}{d+j\omega} \\ &= \pi[\delta(\omega+d)\frac{d+jd}{d-jd} + \delta(\omega-d)\frac{d-jd}{d+jd}] \\ &= j\pi[\delta(\omega+d) - \delta(\omega-d)] \end{aligned}$$

Therefore

$$y(t) = sin(dt)$$

Verify

$$\delta(\omega^2 - d^2) = \frac{1}{2|d|}[\delta(\omega+d) + \delta(\omega-d)]$$

First, note that $\theta(\omega^2 - d^2)$ takes the values

$$\theta(\omega^2 - d^2) = \begin{cases} 1 & \text{for } \omega < -d \\ 0 & \text{for } -d < \omega < d \\ 1 & \text{for } \omega > d \end{cases}$$

and can be written as

$$\theta(\omega^2 - d^2) = 1 - (\theta(\omega + |d|) - \theta(\omega - |d|))$$

Hence, $\frac{d}{d\omega}\theta(\omega \pm d) = \delta(\omega \pm d)$, so

$$\frac{d}{d\omega}\theta(\omega^2 - d^2) = -\delta(\omega + d) + \delta(\omega - d)$$

Letting $u = \omega^2$ and taking the derivative of the left hand side of the last equation yields

$$\frac{d}{d\omega}\theta(\omega^2 - d^2) = \frac{du}{d\omega}\frac{d}{du}\theta(u - d^2) = 2\omega\delta(u - d^2) = 2\omega\delta(\omega^2 - d^2)$$

We see that $\delta(\omega^2 - d^2) = \frac{1}{2\omega}[\delta(\omega - |d|) - \delta(\omega + |d|)]$ $\omega = \pm d$,so

$$\delta(\omega^2 - d^2) = \frac{1}{2|d|}[\delta(\omega + d) + \delta(\omega - d)]$$

5. [25 points] In this problem, we will discuss two kinds of filters: RC filter and Gaussian filter.
Part 1. RC circuit
RC circuit is the most common low-pass filter.

(a) Determine the frequency response $H(j\omega)$ of the RC circuit below, which can be governed by

$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

(Hint: You can substitute $x(t) = e^{j\omega t}$ and $y(t) = H(j\omega)e^{j\omega t}$ in the differential equation and then you can obtain $H(j\omega)$)

(b) Explain why $H(j\omega)$ is a low-pass filter.

(c) Derive the continuous-time Fourier transform of the unit step function $u(t)$. And find the corresponding $Y(j\omega)$ when $x(t) = u(t)$.

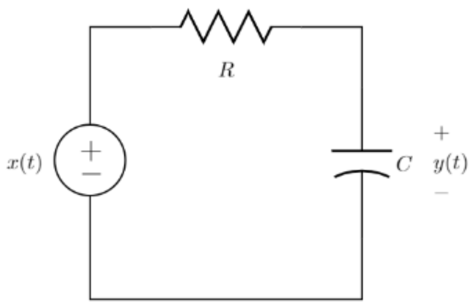

Figure 3: RC circuit

Part 2. Gaussian filter
Gaussian filter is widely used in computer vision. There are blurs under many natural situations and we can interpret them as Gaussian blur.

(a) Please find the continuous-time Fourier transform of $g(t) = e^{-t^2}$. (Hint: $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$)

(b) Now we define the one-dimensional Gaussian filter as $g(t) = \frac{1}{\sigma\sqrt{\pi}}e^{-\frac{t^2}{\sigma^2}}$. We also define the error function $erf(t) = \frac{1}{\sqrt{\pi}}\int_{-t}^{t} e^{-\tau^2} d\tau$. Error function is widely used in probability and statistics. $erf(t)$ can be seen in the graph below. It has the following property:

$$\int_{-\infty}^{t} e^{-\frac{\tau^2}{\sigma^2}} d\tau = \frac{\sigma\sqrt{\pi}}{2} + \frac{\sigma\sqrt{\pi}}{2} erf(\frac{t}{\sigma})$$

Please find and sketch $f(t) = u(t) * g(t)$ when $\sigma = 1$, where $u(t)$ is unit step function.

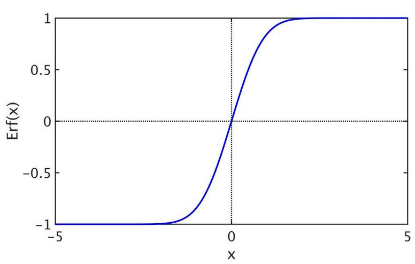

Figure 4: Error function

Answer:
Part 1: RC circuits

(a) First we substitute $x(t) = e^{j\omega t}$ and $y(t) = H(j\omega)e^{j\omega t}$ in the differential equation, and we can get

$$RCj\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

then, we can get $H(j\omega) = \frac{1}{1+RCj\omega}$

(b) When $\omega \to \infty$, $|H(j\omega)| \to 0$, so, this is a low pass filter.

(c)

$$sgn(t) = \lim_{a\to 0}[e^{-at}u(t) - e^{at}u(-t)]$$

The continuous time Fourier transform of $sgn(t)$ is

$$\mathcal{F}(sgn(t)) = \lim_{a\to 0}\frac{-2j\omega}{a^2+\omega^2} = \frac{2}{j\omega}$$

And $u(t) = \frac{1+sgn(t)}{2}$, then we can get the continuous time Fourier transform of $u(t)$

$$\mathcal{F}(u(t)) = \mathcal{F}(\frac{1}{2}) + \frac{1}{2}\mathcal{F}(sgn(t)) = \pi\delta(\omega) + \frac{1}{j\omega}$$

Then we can easily get $Y(j\omega)$ by multiply them on the frequency domain

$$Y(j\omega) = \frac{\pi\delta(\omega) + \frac{1}{j\omega}}{RCj\omega + 1}$$

Part 2: Gaussian filter

(a)

$$\begin{aligned}
G(j\omega) &= \int_{-\infty}^{\infty} e^{-t^2}e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} e^{-(t^2+j\omega t)}dt \\
&= \int_{-\infty}^{\infty} e^{-[(t+\frac{1}{2}j\omega)^2+\frac{1}{4}\omega^2]}dt \\
&= \int_{-\infty}^{\infty} e^{-(t+\frac{1}{2}j\omega)^2}dte^{-\frac{1}{4}\omega^2} \\
&= \sqrt{\pi}e^{-\frac{1}{4}\omega^2}
\end{aligned}$$

(b)

$$\begin{aligned}
f(t) &= u(t) * g(t) \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\tau^2}u(t-\tau)d\tau \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t} e^{-\tau^2}d\tau \\
&= \frac{1}{2} + \frac{1}{2}erf(t)
\end{aligned}$$

In the last equation, we use the property of the $erf(t)$. So, as we can see, the difference between $f(t)$ and $u(t)$ is that $f(t)$ is more smooth, which can be seen as a kind of blur.

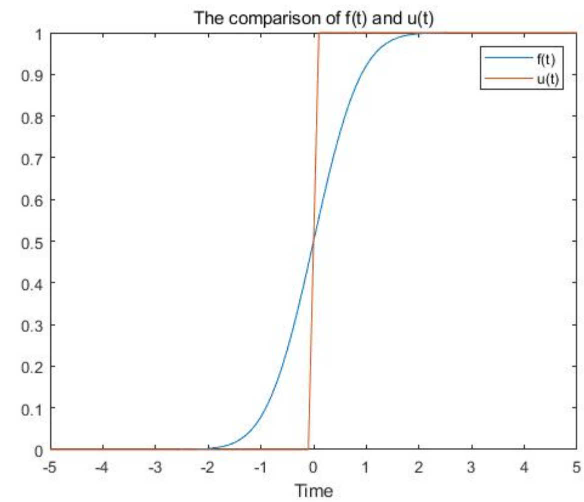

Figure 5: Comparison of f(t) and u(t)