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§ 7.2 (Page 722)

20. Prove: If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis for R^n, and if A can be expressed as

$$A = c\mathbf{u}_1\mathbf{u}_1^T + c_2\mathbf{u}_2\mathbf{u}_2^T + \ldots + c_n\mathbf{u}_n\mathbf{u}_n^T$$

then A is symmetric and has eigenvalues $c_1, c_2, \ldots, c_n$.

if $V = [v_i]$ are orthogonal matrices

then $A = u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T + \cdots + u_n\sigma_n v_n^T$

where σ_i are singular values.

in the ~~qesti~~ question. $U = [U_i]$ is orthonormal

so $u_i \sigma_i V_i^T = c_i u_i u_i^T$

then $c_i =$

§ 9.5 (Page 906)

In Exercises 1–4, find the distinct singular values of A.

1. $A = [1 \quad 2 \quad 0]$

$$A^TA = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} [1 \ 2 \ 0] = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det(\lambda I - A^TA) = \begin{vmatrix} \lambda-1 & -2 & 0 \\ -2 & \lambda-4 & 0 \\ 0 & 0 & \lambda \end{vmatrix}$$

$$= (\lambda-1)\lambda(\lambda-4) + 2(-2\lambda)$$

$\lambda_1 = 0 \quad \lambda_2 = 5$ so singular values are $\sigma_1 = 0 \quad \sigma_2 = \sqrt{5}$

In Exercises 5–12, find a singular value decomposition of A.

10. $A = \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}$

$$A^TA = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & -8 \\ 4 & 2 & -4 \\ -8 & -4 & 8 \end{bmatrix}$$

$$\det(\lambda I - A^TA) = \begin{vmatrix} \lambda-8 & -4 & 8 \\ -4 & \lambda-2 & 4 \\ 8 & 4 & \lambda-8 \end{vmatrix} = (\lambda-8)(\lambda^2-10\lambda) + 4(-4\lambda) + 8(-8\lambda)$$

$$= \lambda^3 - 18\lambda^2 + 80\lambda - 80\lambda = \lambda^2(\lambda-18)$$

$\lambda_1 = 0 \quad \lambda_2 = 18 \quad V_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

$v_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ $v_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$ $u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{8} \begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{2} \begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

12.

$$A = \begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix}$$

$A^T A = \begin{bmatrix} 6 & 0 & 4 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix}$

$= \begin{bmatrix} 52 & 24 \\ 24 & 16 \end{bmatrix}$

$A = U \Sigma V^T$

$= \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}$

$\det(\lambda I - A^T A) = \begin{vmatrix} \lambda - 52 & -24 \\ -24 & \lambda - 16 \end{vmatrix}$

$= (\lambda - 52)(\lambda - 16) - 24^2 = 0$

$\lambda_1 = 64 \quad \lambda_2 = 4$

13. Prove: If A is an $m \times n$ matrix, then $A^T A$ and AA^T have the same rank.

17. Show that the singular values of $A^T A$ are the squares of the singular values of A.

every element in matrix $A^T A$ are the squares of exact element in matrix A

so from the solution $\det(\lambda_1 I - A^T A) = 0$ and $\det(\lambda_2 I - A) = 0$

we can get that $\lambda_1 = \lambda_2^2$

so the singulare values are.