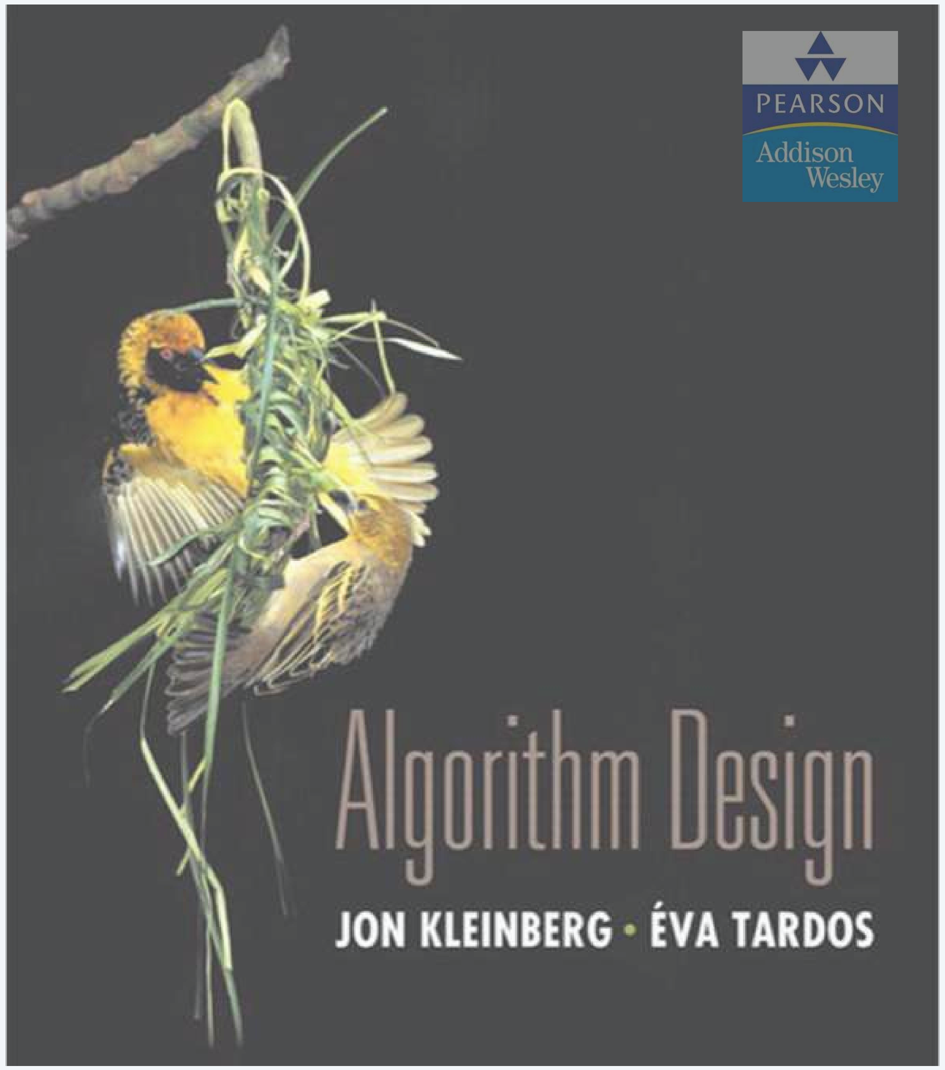
CS101 Algorithms and Data Structures

Reductions, P and NP



SECTION 8.1

REDUCTIONS, P AND NP

- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- *graph coloring*
- *P vs. NP*
- *NP-complete*

Algorithm design patterns and antipatterns

Algorithm design patterns.

- Greedy.
- Divide and conquer.
- Dynamic programming.
- Reductions.
- Duality.
- Local search.
- Randomization.

Algorithm design antipatterns.

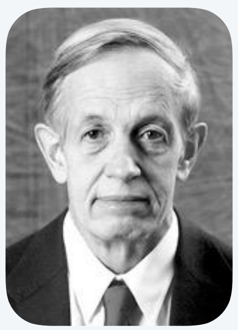
- NP-completeness. $O(n^k)$ algorithm unlikely.
- PSPACE-completeness. $O(n^k)$ certification algorithm unlikely.
- Undecidability. No algorithm possible.

Classify problems according to computational requirements

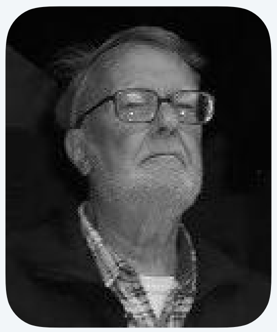
Q. Which problems will we be able to solve in practice?

A working definition. Those with poly-time algorithms.

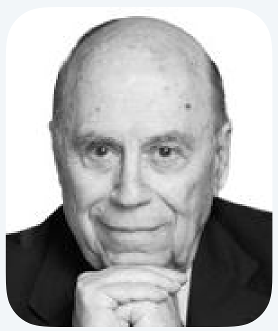

von Neumann
(1953)


Nash
(1955)


Gödel
(1956)


Cobham
(1964)


Edmonds
(1965)


Rabin
(1966)

Turing machine, word RAM, uniform circuits, …

Theory. Definition is broad and robust.

constants tend to be small, e.g., $3n^2$

Practice. Poly-time algorithms scale to huge problems.

Classify problems according to computational requirements

Q. Which problems will we be able to solve in practice?

A working definition. Those with poly-time algorithms.

yes	probably no
shortest path	longest path
min cut	max cut
2-satisfiability	3-satisfiability
planar 4-colorability	planar 3-colorability
bipartite vertex cover	vertex cover
matching	3d-matching
primality testing	factoring
linear programming	integer linear programming

Classify problems

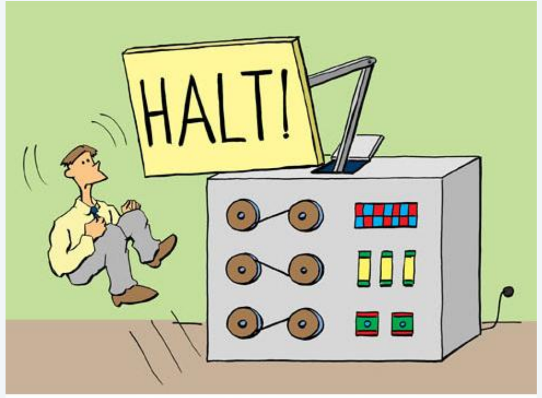
Desiderata. Classify problems according to those that can be solved in polynomial time and those that cannot.

Provably requires exponential time.

input size = $c + \log k$

- Given a constant-size program, does it halt in at most k steps?
- Given a board position in an n-by-n generalization of checkers, can black guarantee a win?

using forced capture rule



Alan designed the perfect computer

Frustrating news. Huge number of fundamental problems have defied classification for decades.

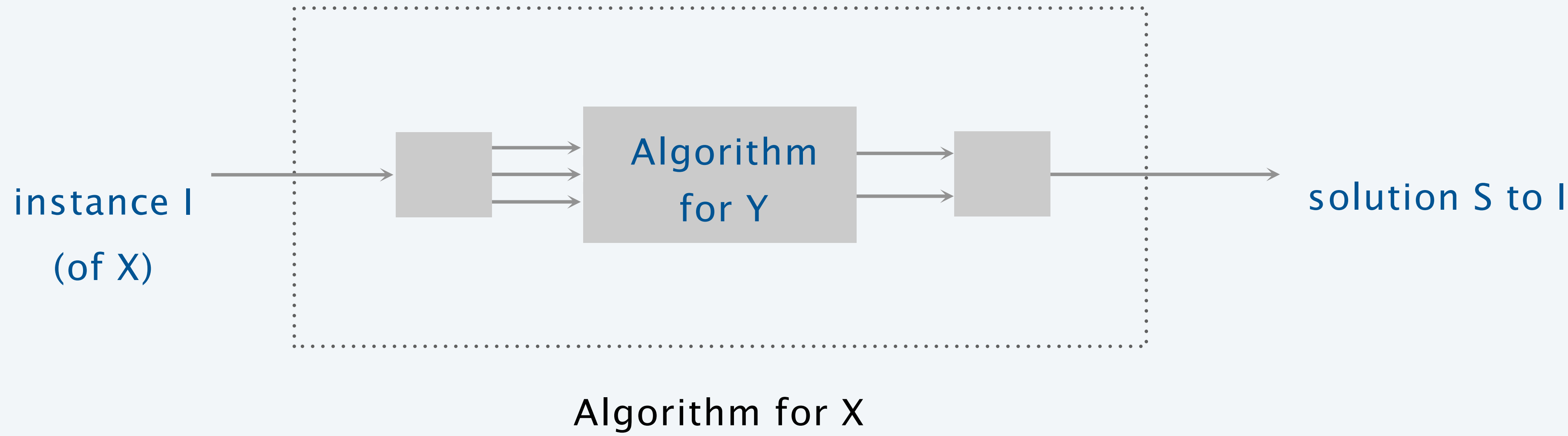
Poly-time reductions

Desiderata′. Suppose we could solve problem Y in polynomial time. What else could we solve in polynomial time?

Reduction. Problem X polynomial-time (Cook) reduces to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

computational model supplemented by special piece of hardware that solves instances of Y in a single step



Poly-time reductions

Desiderata′. Suppose we could solve problem Y in polynomial time. What else could we solve in polynomial time?

Reduction. Problem X polynomial-time (Cook) reduces to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

Notation. $X \leq_P Y$.

Note. We pay for time to write down instances of Y sent to oracle $\Rightarrow$ instances of Y must be of polynomial size.

Novice mistake. Confusing $X \leq_P Y$ with $Y \leq_P X$.

Intractability: quiz 1

Suppose that $X \leq_P Y$. Which of the following can we infer?

A. If X can be solved in polynomial time, then so can Y.

B. X can be solved in poly time iff Y can be solved in poly time.

C. If X cannot be solved in polynomial time, then neither can Y.

D. If Y cannot be solved in polynomial time, then neither can X.

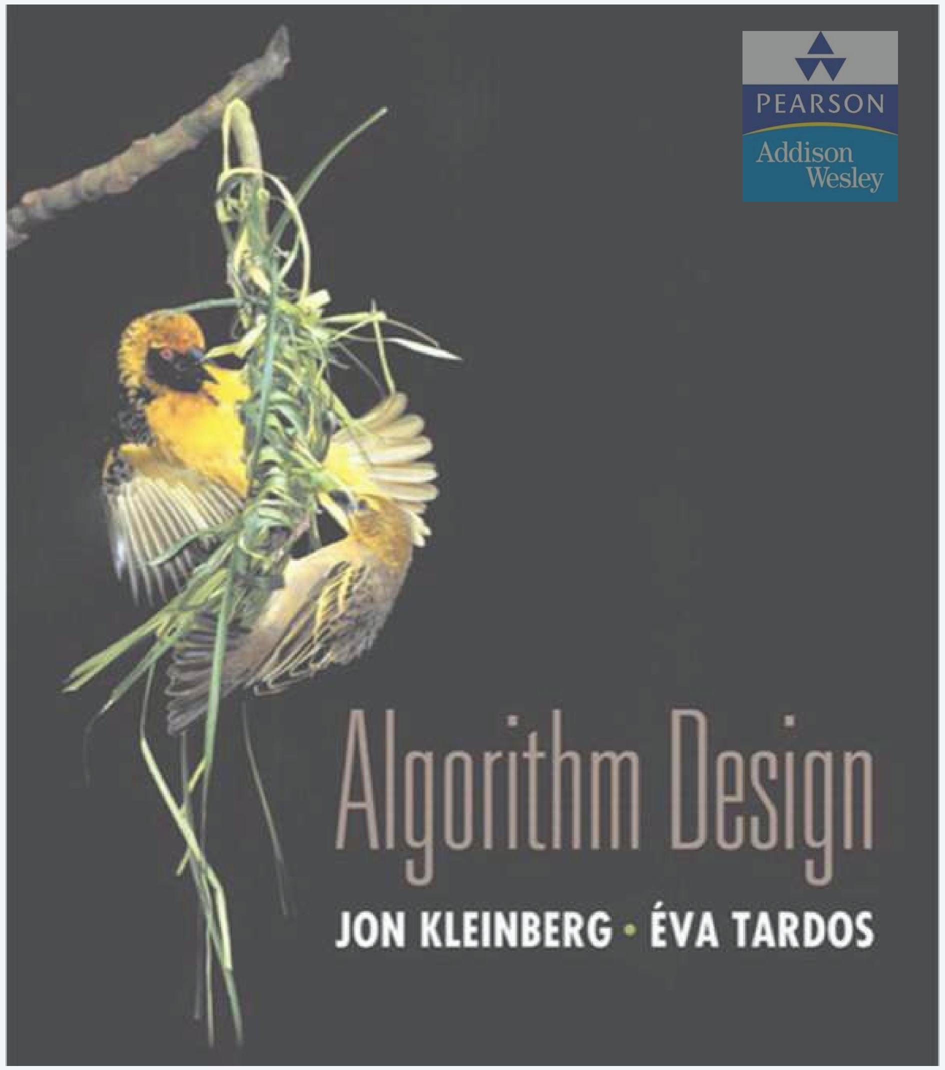
Poly-time reductions

Design algorithms. If $X \leq_P Y$ and Y can be solved in polynomial time, then X can be solved in polynomial time.

Establish intractability. If $X \leq_P Y$ and X cannot be solved in polynomial time, then Y cannot be solved in polynomial time.

Establish equivalence. If both $X \leq_P Y$ and $Y \leq_P X$, we use notation $X \equiv_P Y$. In this case, X can be solved in polynomial time iff Y can be.

Bottom line. Reductions classify problems according to relative difficulty.



SECTION 8.1

REDUCTIONS, P AND NP

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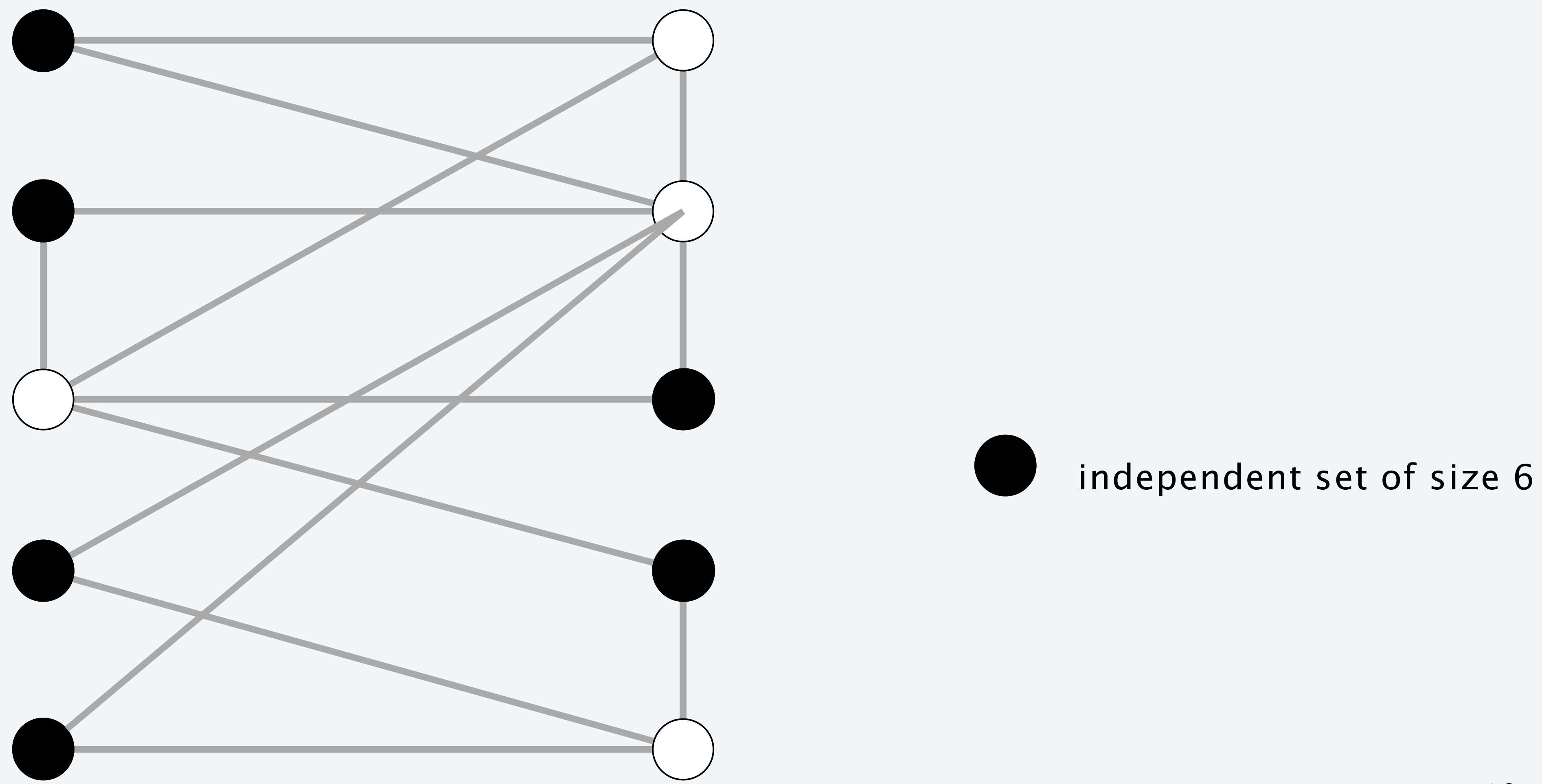
- *P vs. NP*
- *NP-complete*

Independent set

INDEPENDENT-SET. Given a graph $G = (V, E)$ and an integer k, is there a subset of k (or more) vertices such that no two are adjacent?

Ex. Is there an independent set of size ≥ 6 ?
Ex. Is there an independent set of size ≥ 7 ?

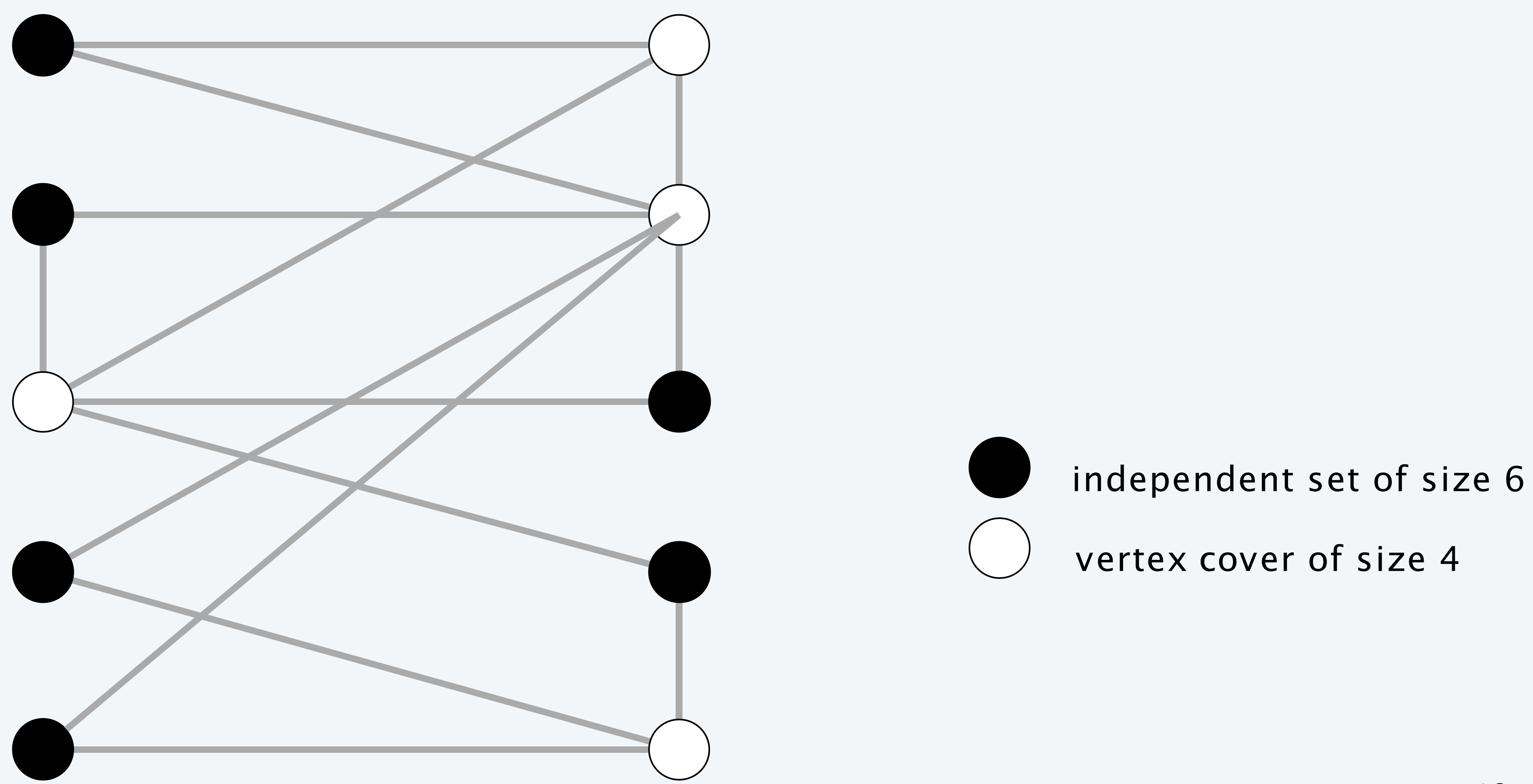


Vertex cover

VERTEX-COVER. Given a graph $G = (V, E)$ and an integer k, is there a subset of k (or fewer) vertices such that each edge is incident to at least one vertex in the subset?

Ex. Is there a vertex cover of size ≤ 4 ?

Ex. Is there a vertex cover of size ≤ 3 ?



Intractability: quiz 2

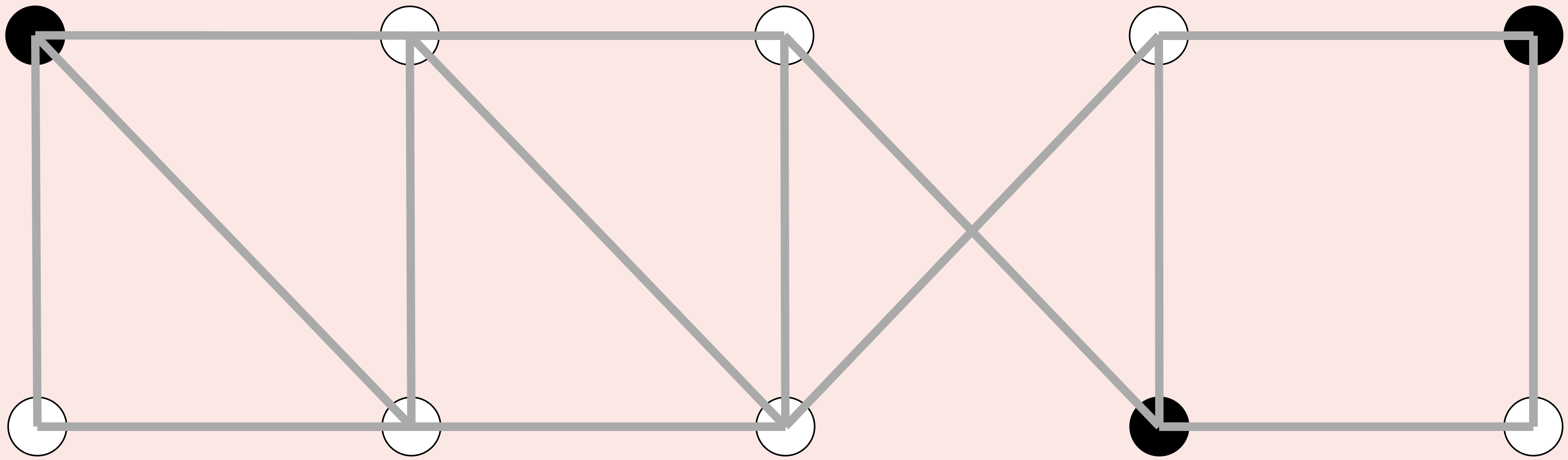
Consider the following graph G. Which are true?

A. The white vertices are a vertex cover of size 7.

B. The black vertices are an independent set of size 3.

C. Both A and B.

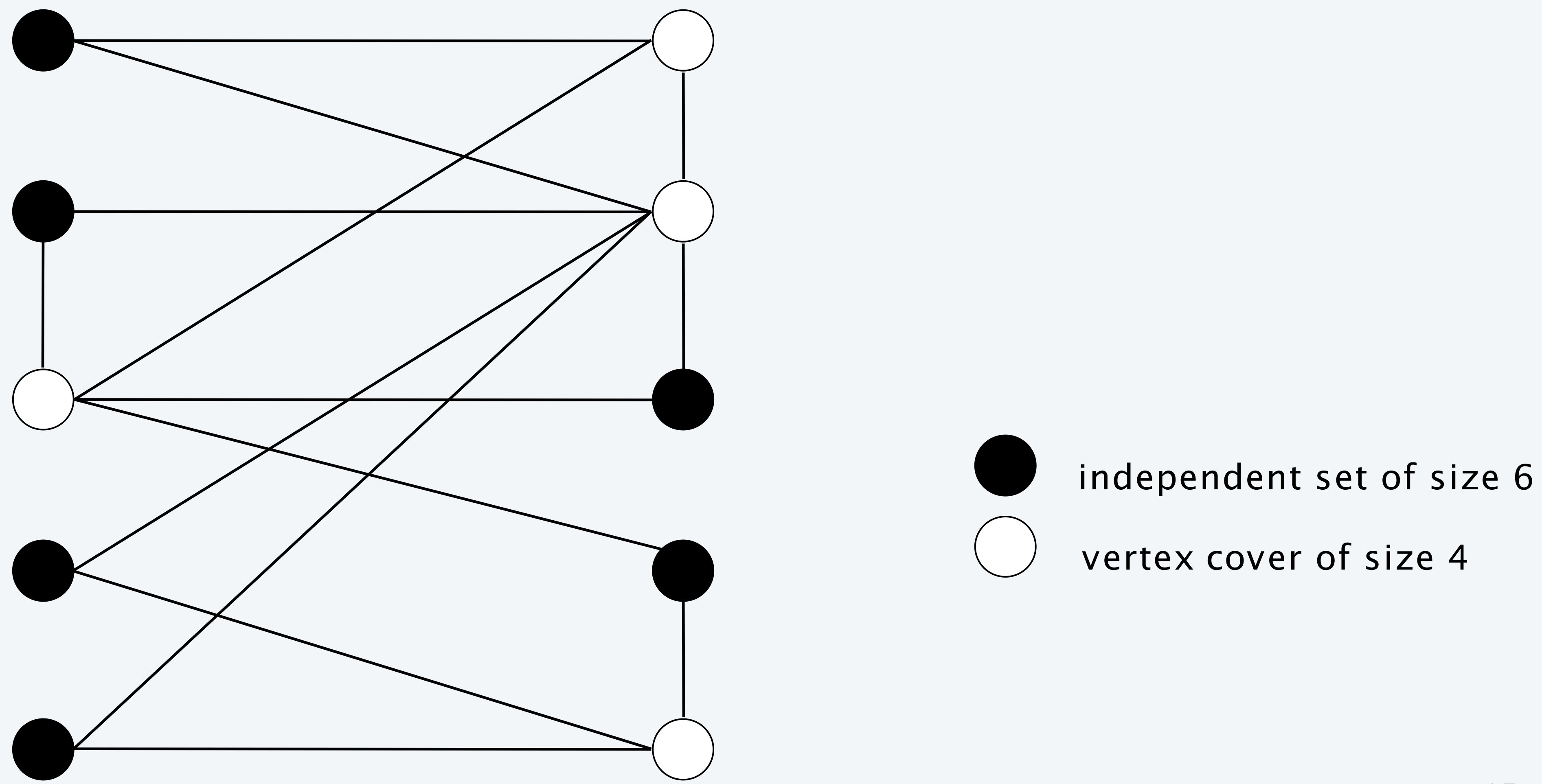
D. Neither A nor B.



Vertex cover and independent set reduce to one another

Theorem. INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.

Pf. We show S is an independent set of size k iff $V - S$ is a vertex cover of size $n - k$.



Vertex cover and independent set reduce to one another

Theorem. INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.

Pf. We show S is an independent set of size k iff $V - S$ is a vertex cover of size $n - k$.

$\Rightarrow$

- Let S be any independent set of size k.
- $V - S$ is of size $n - k$.
- Consider an arbitrary edge $(u, v) \in E$.
- S independent $\Rightarrow$ either $u \notin S$, or $v \notin S$, or both.
  $\Rightarrow$ either $u \in V - S$, or $v \in V - S$, or both.
- Thus, $V - S$ covers (u, v). ▪

Vertex cover and independent set reduce to one another

Theorem. INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.

Pf. We show S is an independent set of size k iff $V - S$ is a vertex cover of size $n - k$.

$\Leftarrow$

- Let $V - S$ be any vertex cover of size $n - k$.
- S is of size k.
- Consider an arbitrary edge $(u, v) \in E$.
- $V - S$ is a vertex cover $\Rightarrow$ either $u \in V - S$, or $v \in V - S$, or both.
  $\Rightarrow$ either $u \notin S$, or $v \notin S$, or both.
- Thus, S is an independent set. ▪

Set cover

SET-COVER. Given a set U of elements, a collection S of subsets of U, and an integer k, are there $\leq k$ of these subsets whose union is equal to U ?

Sample application.

- m available pieces of software.
- Set U of n capabilities that we would like our system to have.
- The i^{th} piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using fewest pieces of software.

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad$ $S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad$ $S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad$ $S_f = \{ 1, 2, 6, 7 \}$

$k = 2$

a set cover instance

Intractability: quiz 3

Given the universe U = { 1, 2, 3, 4, 5, 6, 7 } and the following sets, which is the minimum size of a set cover?

A. 1

B. 2

C. 3

D. None of the above.

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 1, 4, 6 \}$ $S_b = \{ 1, 6, 7 \}$

$S_c = \{ 1, 2, 3, 6 \}$ $S_d = \{ 1, 3, 5, 7 \}$

$S_e = \{ 2, 6, 7 \}$ $S_f = \{ 3, 4, 5 \}$

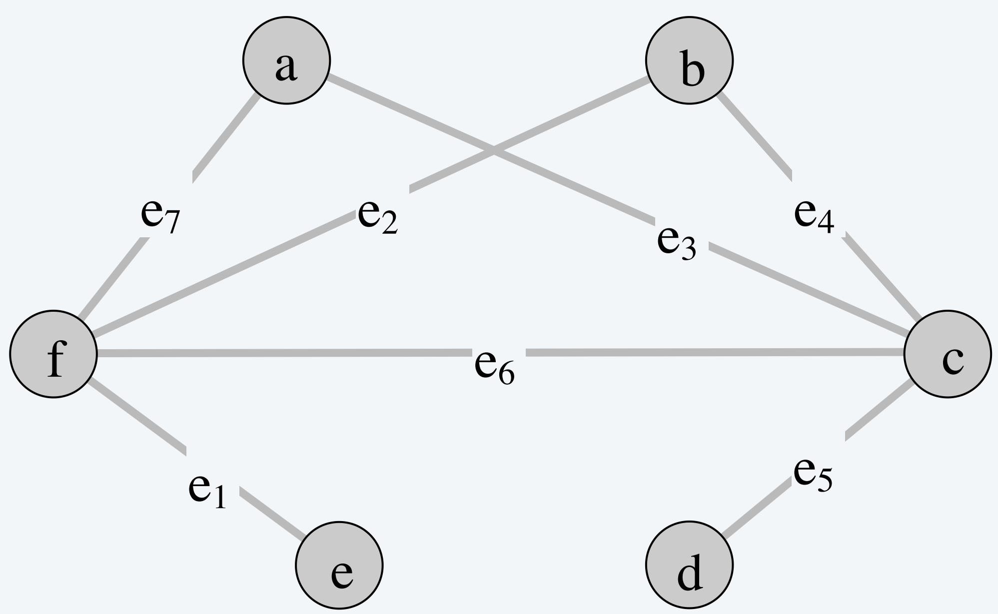
Vertex cover reduces to set cover

Theorem. VERTEX-COVER $\le_P$ SET-COVER.

Pf. Given a VERTEX-COVER instance $G = (V, E)$ and k, we construct a SET-COVER instance (U, S, k) that has a set cover of size k iff G has a vertex cover of size k.

Construction.

- Universe $U = E$.
- Include one subset for each node $v \in V$: $S_v = \{e \in E : e \text{ incident to } v\}$.



vertex cover instance
(k = 2)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad S_d = \{ 5 \}$

$S_e = \{ 1 \}$ $\quad S_f = \{ 1, 2, 6, 7 \}$

set cover instance
(k = 2)

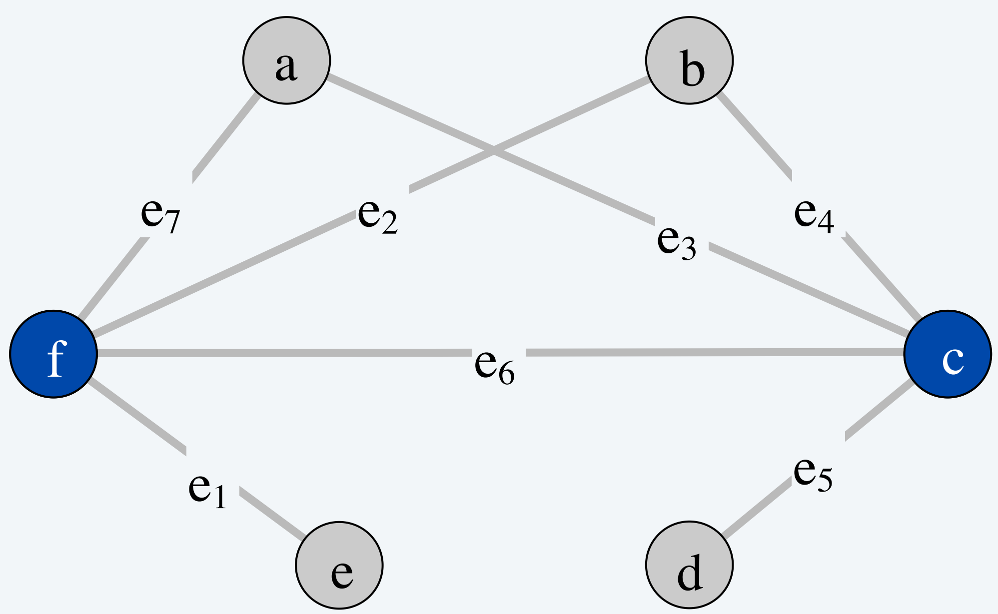
Vertex cover reduces to set cover

Lemma. $G = (V, E)$ contains a vertex cover of size k iff (U, S, k) contains a set cover of size k.

Pf. $\Rightarrow$ Let $X \subseteq V$ be a vertex cover of size k in G.

- Then $Y = \{ S_v : v \in X \}$ is a set cover of size k. ▪

"yes" instances of VERTEX-COVER are solved correctly



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(k = 2)

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set cover instance
(k = 2)

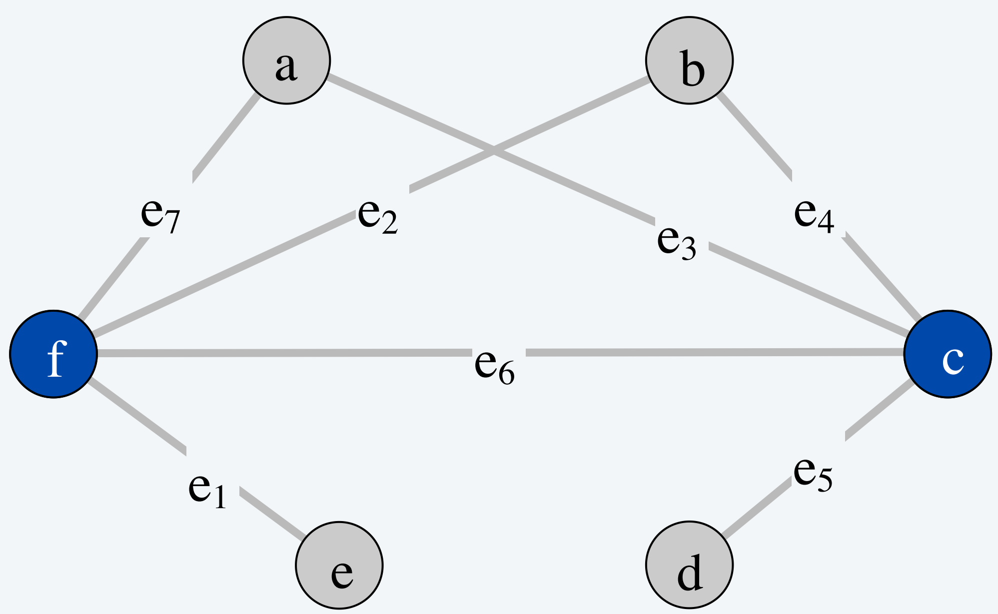
Vertex cover reduces to set cover

Lemma. $G = (V, E)$ contains a vertex cover of size k iff (U, S, k) contains a set cover of size k.

Pf. $\Leftarrow$ Let $Y \subseteq S$ be a set cover of size k in (U, S, k).

- Then $X = \{ v : S_v \in Y \}$ is a vertex cover of size k in G. ▪

"no" instances of VERTEX-COVER are solved correctly



vertex cover instance
($k = 2$)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{ 3, 7 \}$ $\quad S_b = \{ 2, 4 \}$

$S_c = \{ 3, 4, 5, 6 \}$ $\quad S_d = \{ 5 \}$

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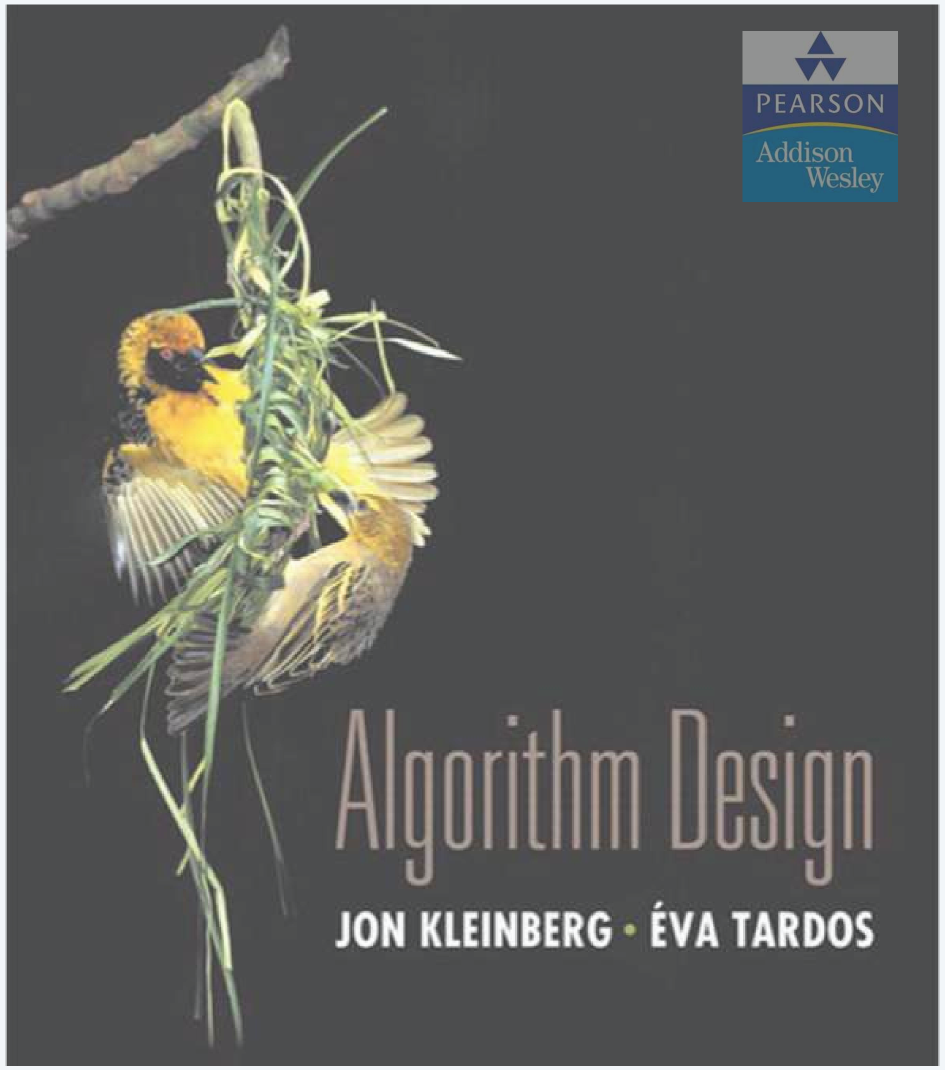
set cover instance
($k = 2$)



Section 8.2

Reductions, P and NP

- *poly-time reductions*
- *packing and covering problems*
- ***constraint satisfaction problems***
- *graph coloring*
- *P vs. NP*
- *NP-complete*

Satisfiability

Literal. A Boolean variable or its negation. x_i or $\overline{x_i}$

Clause. A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form (CNF). A propositional formula Φ that is a conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

SAT. Given a CNF formula Φ, does it have a satisfying truth assignment?
3-SAT. SAT where each clause contains exactly 3 literals
(and each literal corresponds to a different variable).

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

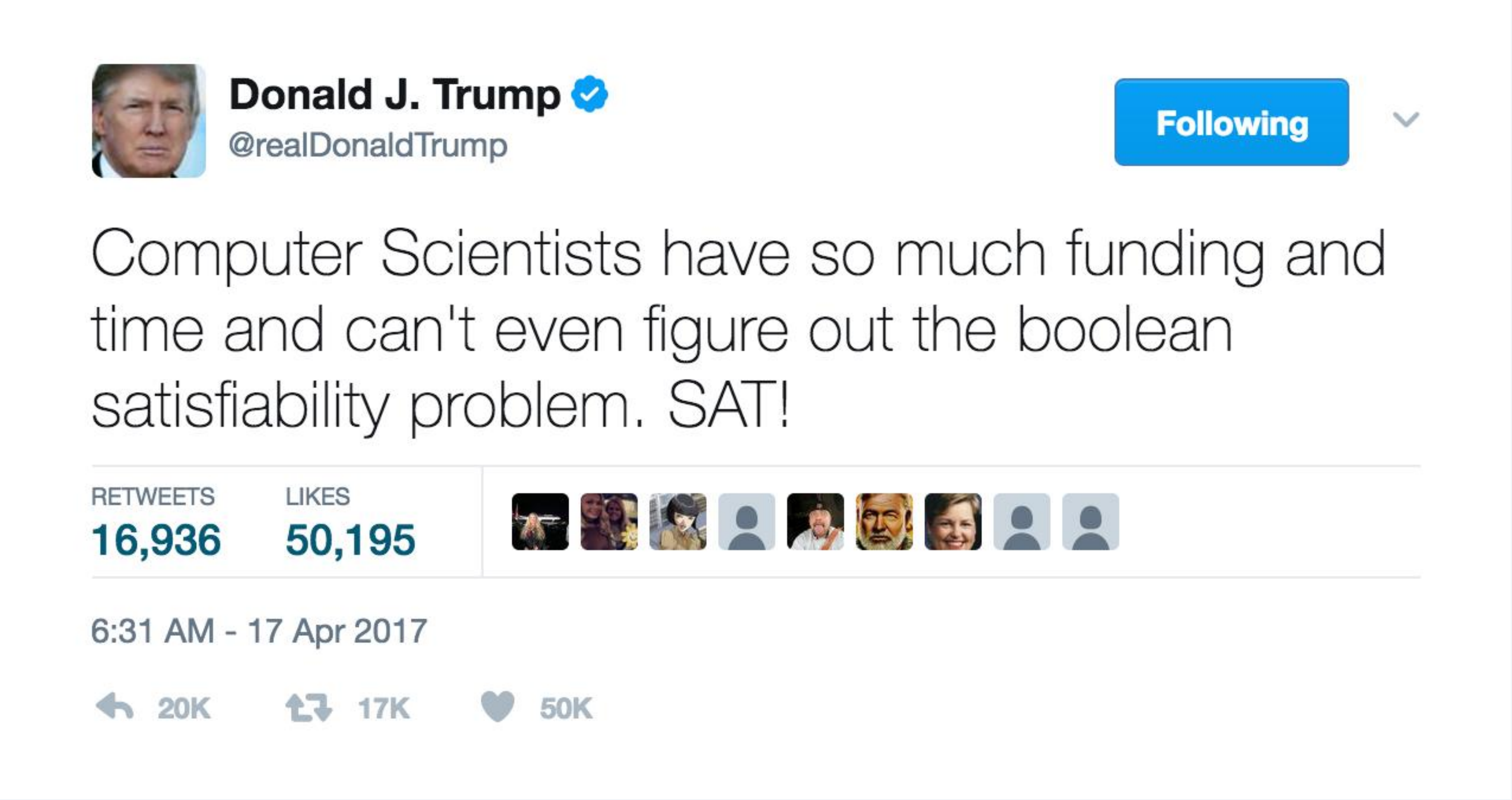
yes instance: x_1 = true, x_2 = true, x_3 = false, x_4 = false

Key application. Electronic design automation (EDA).

Satisfiability is hard

Scientific hypothesis. There does not exists a poly-time algorithm for 3-SAT.

P vs. NP. This hypothesis is equivalent to P ≠ NP conjecture.



https://www.facebook.com/pg/npcompleteteens

3-satisfiability reduces to independent set

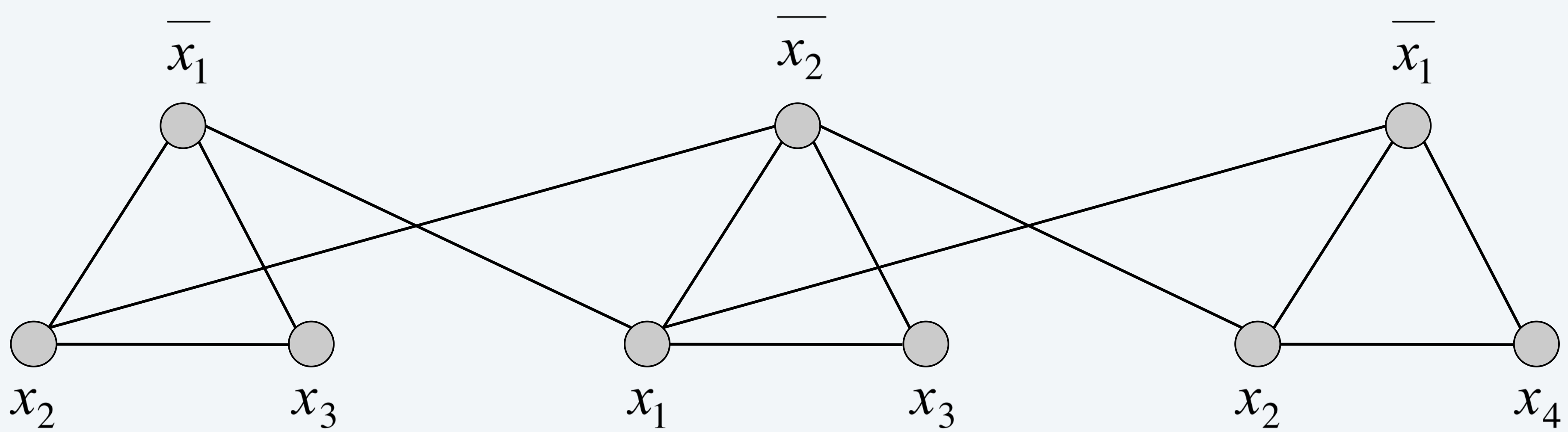
Theorem. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size $k = |\Phi|$ iff Φ is satisfiable.

Construction.

- G contains 3 nodes for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

3-satisfiability reduces to independent set

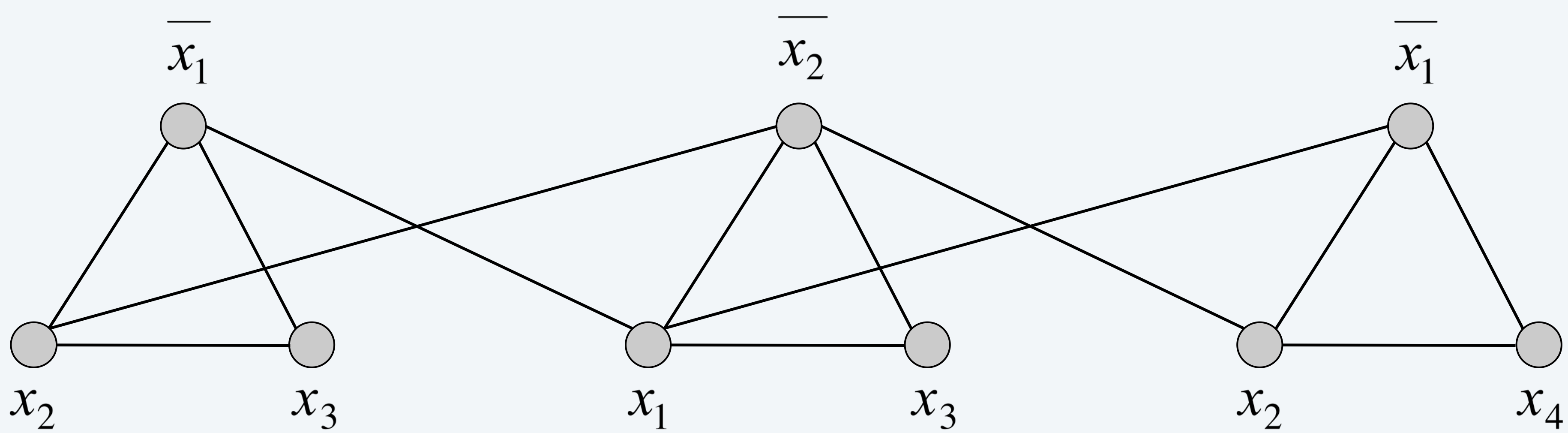
Lemma. Φ is satisfiable iff G contains an independent set of size $k = |\Phi|$.

Pf. $\Rightarrow$ Consider any satisfying assignment for Φ.

- Select one true literal from each clause/triangle.
- This is an independent set of size $k = |\Phi|$. ▪

"yes" instances of 3-SAT are solved correctly

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

3-satisfiability reduces to independent set

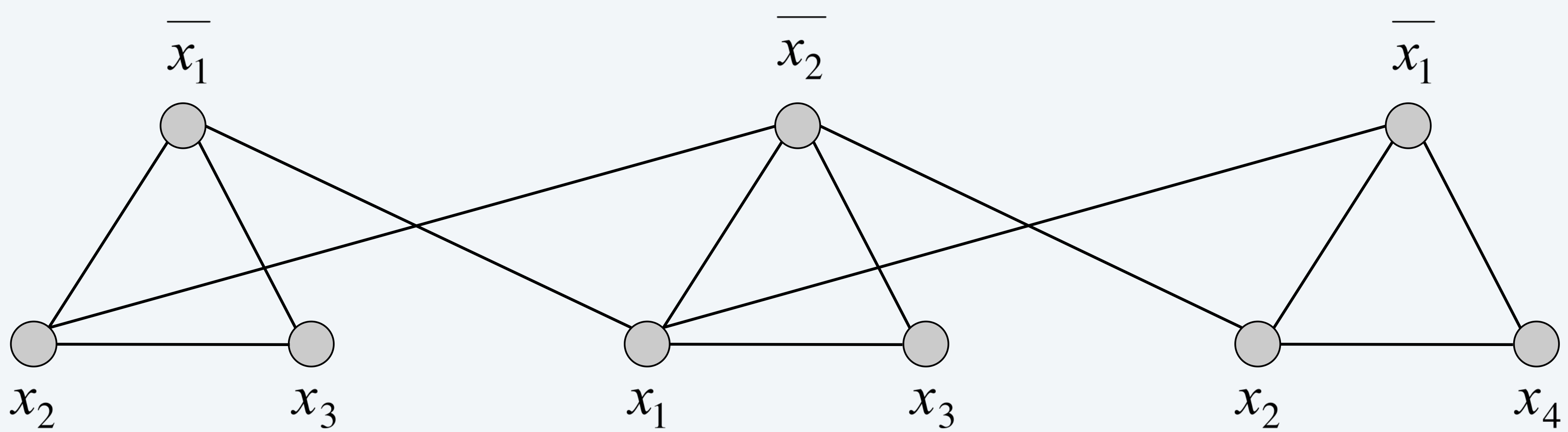
Lemma. Φ is satisfiable iff G contains an independent set of size $k = |\Phi|$.

Pf. $\Leftarrow$ Let S be independent set of size k.

- S must contain exactly one node in each triangle.
- Set these literals to $true$ (and remaining literals consistently).
- All clauses in Φ are satisfied. ▪

“no” instances of 3-SAT are solved correctly

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

Review

Basic reduction strategies.

- Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
- Special case to general case: VERTEX-COVER $\leq_P$ SET-COVER.
- Encoding with gadgets: 3-SAT $\leq_P$ INDEPENDENT-SET.

Transitivity. If $X \leq_P Y$ and $Y \leq_P Z$, then $X \leq_P Z$.
Pf idea. Compose the two algorithms.

Ex. 3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER.

Decision, search, and optimization problems

Decision problem. Does there exist a vertex cover of size $\leq k$?

Search problem. Find a vertex cover of size $\leq k$.

Optimization problem. Find a vertex cover of minimum size.

Goal. Show that all three problems poly-time reduce to one another.

SEARCH PROBLEMS VS. DECISION PROBLEMS

VERTEX-COVER. Does there exist a vertex cover of size $\le k$?

FIND-VERTEX-COVER. Find a vertex cover of size $\le k$.

Theorem. VERTEX-COVER $\equiv_P$ FIND-VERTEX-COVER.

Pf. $\le_P$ Decision problem is a special case of search problem. ▪

Pf. $\ge_P$

To find a vertex cover of size $\le k$:

- Determine if there exists a vertex cover of size $\le k$.
- Find a vertex v such that $G - \{v\}$ has a vertex cover of size $\le k - 1$. (any vertex in any vertex cover of size $\le k$ will have this property)
- Include v in the vertex cover.
- Recursively find a vertex cover of size $\le k - 1$ in $G - \{v\}$. ▪

delete v and all incident edges

OPTIMIZATION PROBLEMS VS. SEARCH PROBLEMS

FIND-VERTEX-COVER. Find a vertex cover of size $\leq k$.

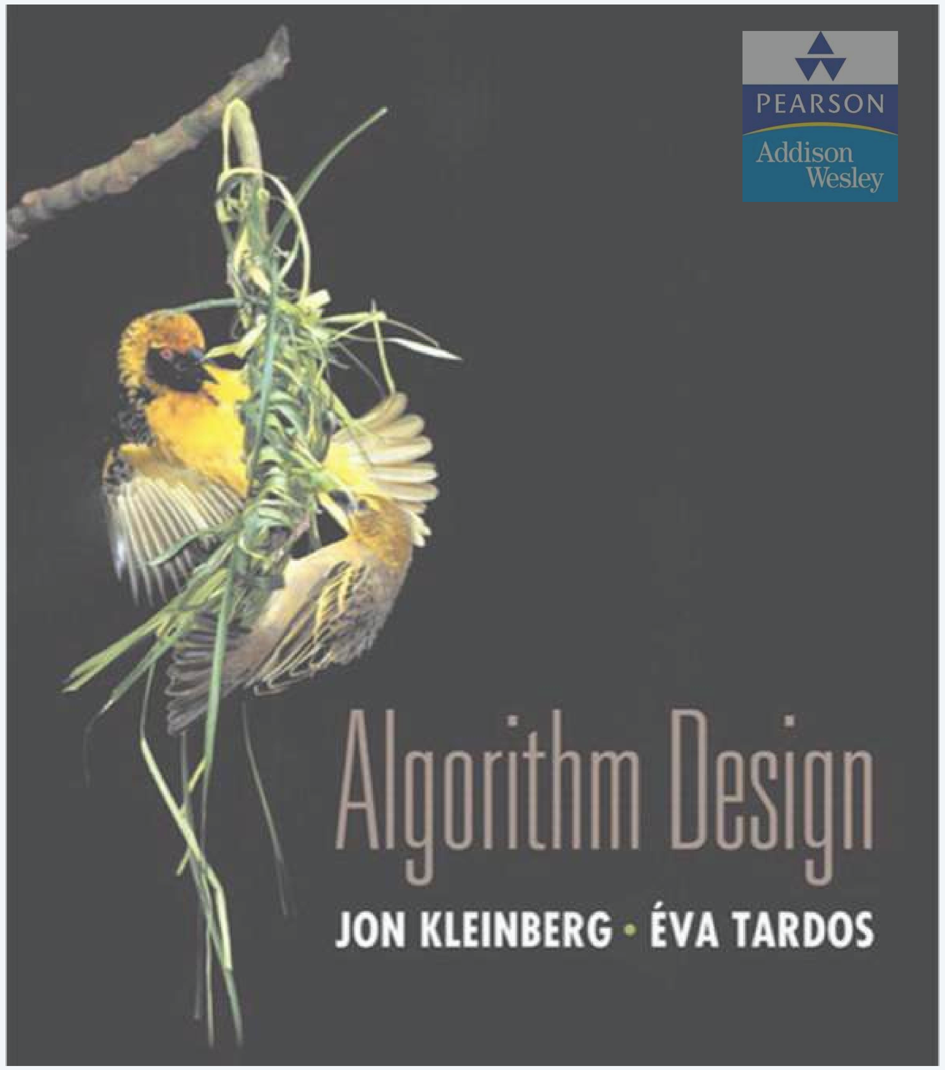
FIND-MIN-VERTEX-COVER. Find a vertex cover of minimum size.

Theorem. FIND-VERTEX-COVER $\equiv_P$ FIND-MIN-VERTEX-COVER.

Pf. $\leq_P$ Search problem is a special case of optimization problem. ▪

Pf. $\geq_P$ To find vertex cover of minimum size:

- Binary search (or linear search) for size k^* of min vertex cover.
- Solve search problem for given k^*. ▪



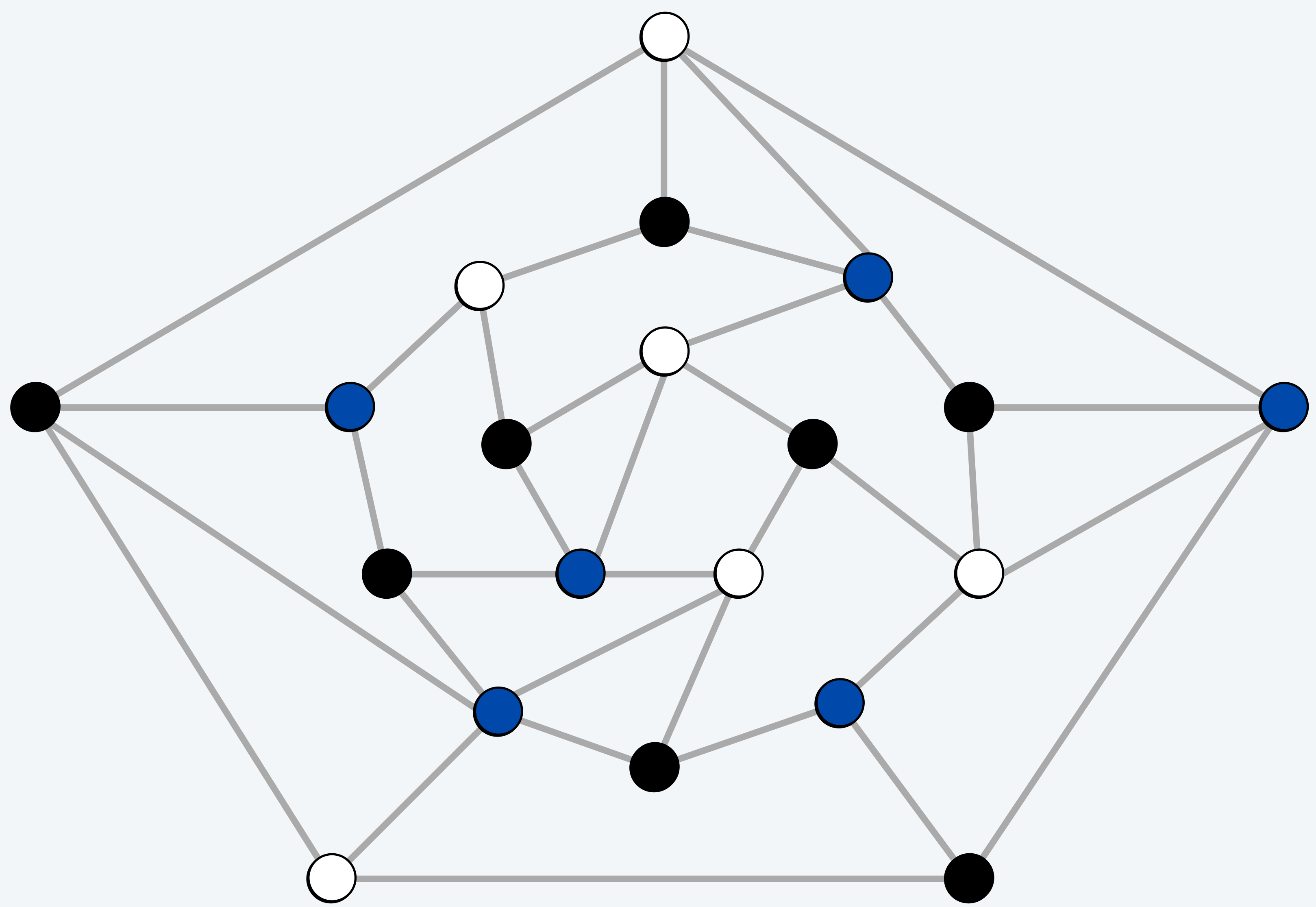
SECTION 8.7

REDUCTIONS, P AND NP

- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- *graph coloring*
- *P vs. NP*
- *NP-complete*

3-colorability

3-COLOR. Given an undirected graph G, can the nodes be colored black, white, and blue so that no adjacent nodes have the same color?



yes instance

Intractability: quiz 4

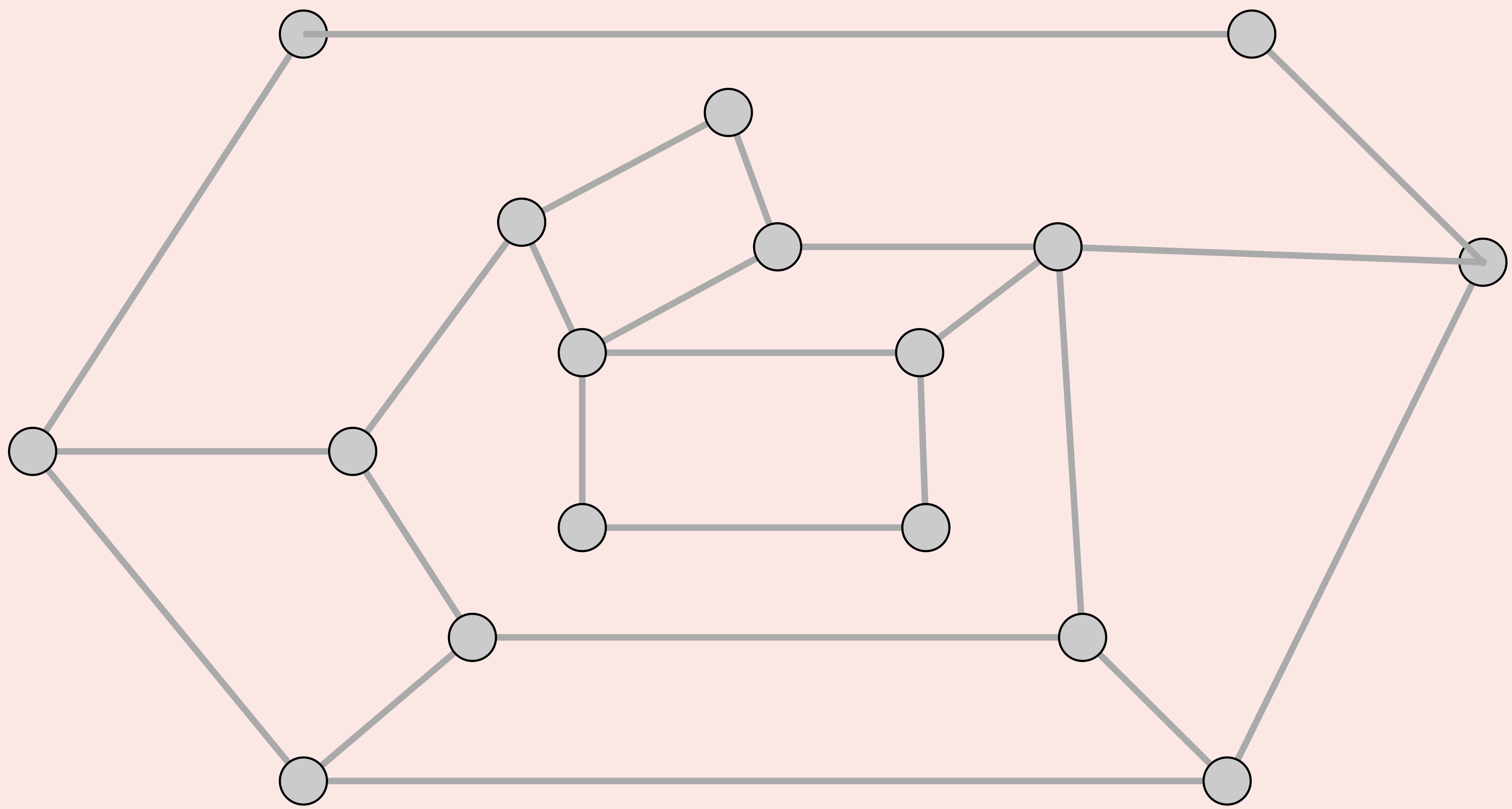
How difficult to solve 2-COLOR?

A. $O(m + n)$ using BFS or DFS.

B. $O(mn)$ using maximum flow.

C. $\Omega(2^n)$ using brute force.

D. Not even Tarjan knows.



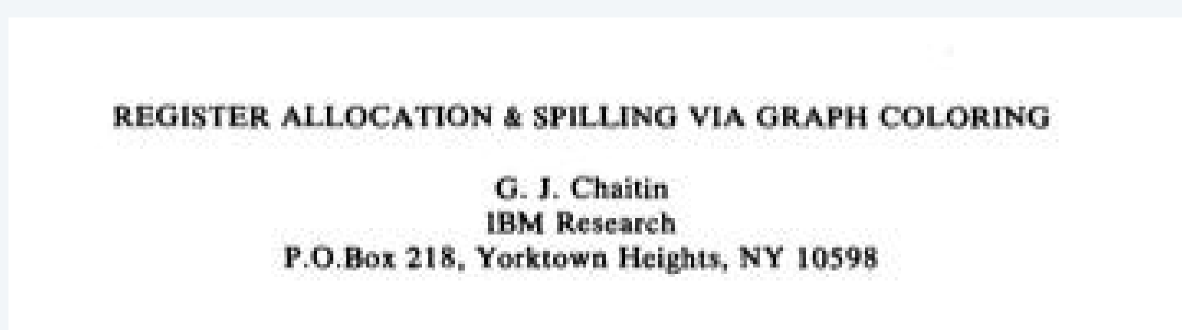
Application: register allocation

Register allocation. Assign program variables to machine registers so that no more than k registers are used and no two program variables that are needed at the same time are assigned to the same register.

Interference graph. Nodes are program variables; edge between u and v if there exists an operation where both u and v are "live" at the same time.

Observation. [Chaitin 1982] Can solve register allocation problem iff interference graph is k-colorable.

Fact. 3-COLOR $\le_P$ K-REGISTER-ALLOCATION for any constant $k \ge 3$.


REGISTER ALLOCATION & SPILLING VIA GRAPH COLORING

G. J. Chaitin
IBM Research
P.O.Box 218, Yorktown Heights, NY 10598

3-satisfiability reduces to 3-colorability

Theorem. 3-SAT $\le_P$ 3-COLOR.

Pf. Given 3-SAT instance Φ, we construct an instance of 3-COLOR that is 3-colorable iff Φ is satisfiable.

3-satisfiability reduces to 3-colorability

Construction.

(i) Create a graph G with a node for each literal.

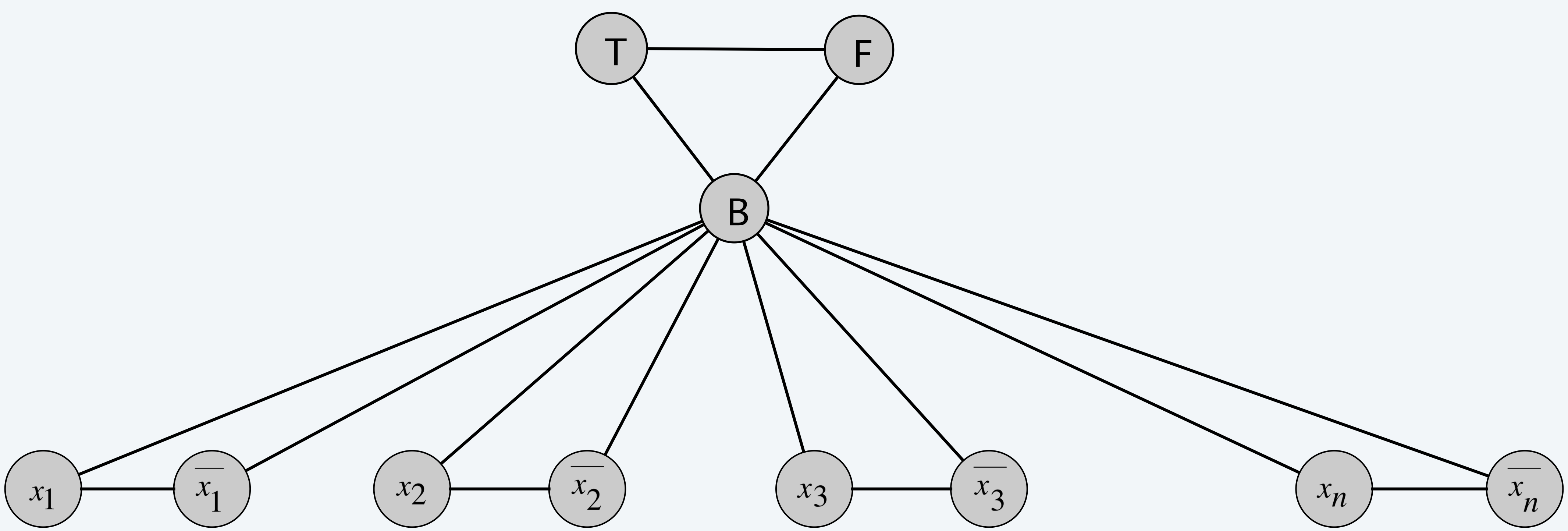
(ii) Connect each literal to its negation.

(iii) Create 3 new nodes T, F, and B; connect them in a triangle.

(iv) Connect each literal to B.

(v) For each clause C_j, add a gadget of 6 nodes and 13 edges.

to be described later

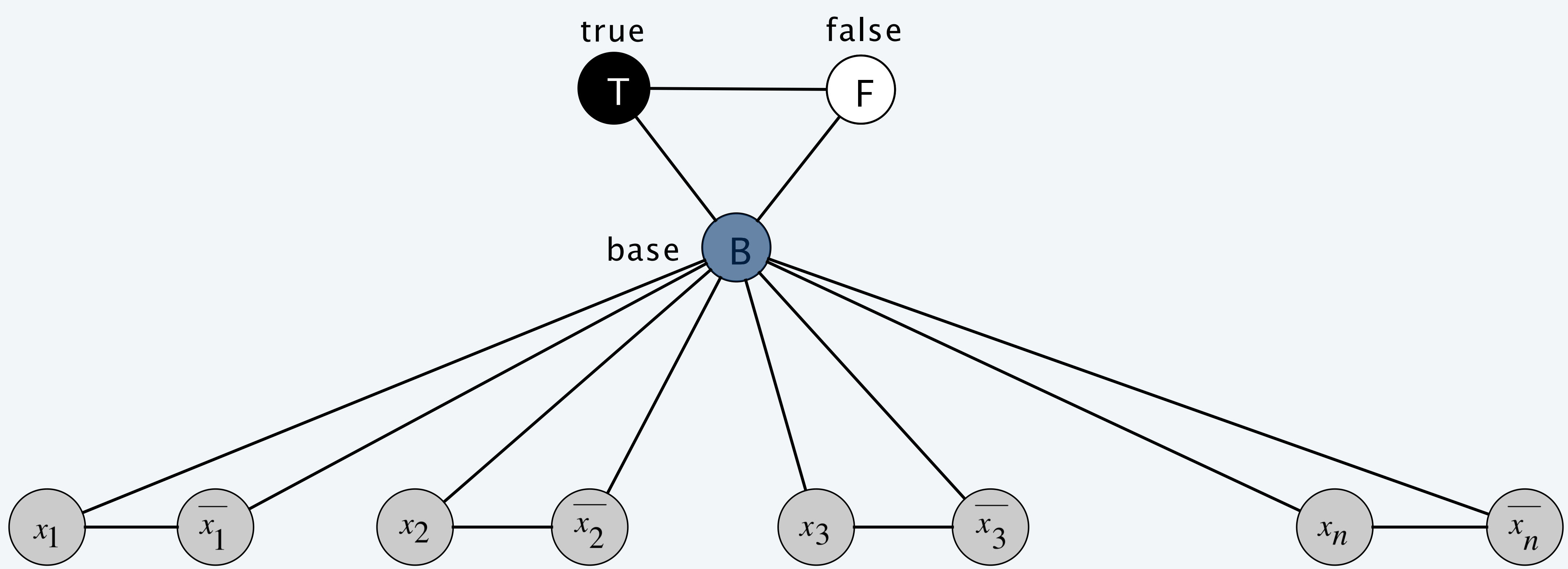


3-satisfiability reduces to 3-colorability

Lemma. Graph G is 3-colorable iff Φ is satisfiable.

Pf. $\Rightarrow$ Suppose graph G is 3-colorable.

- WLOG, assume that node T is colored black, F is white, and B is blue.
- Consider assignment that sets all black literals to true (and white to false).
- (iv) ensures each literal is colored either black or white.
- (ii) ensures that each literal is white if its negation is black (and vice versa).



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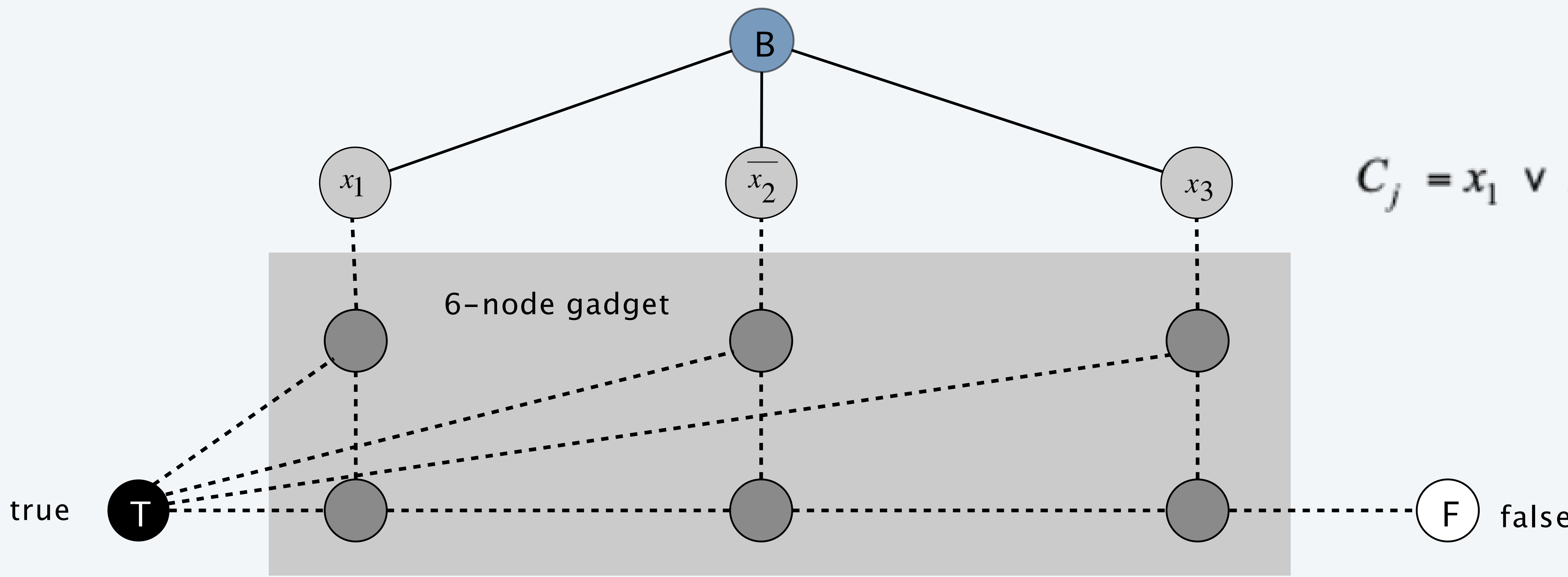
- Consider assignment that sets all black literals to true (and white to false).
- (iv) ensures each literal is colored either black or white.
- (ii) ensures that each literal is white if its negation is black (and vice versa).
- (v) ensures at least one literal in each clause is black.



$$C_j = x_1 \vee \overline{x_2} \vee x_3$$

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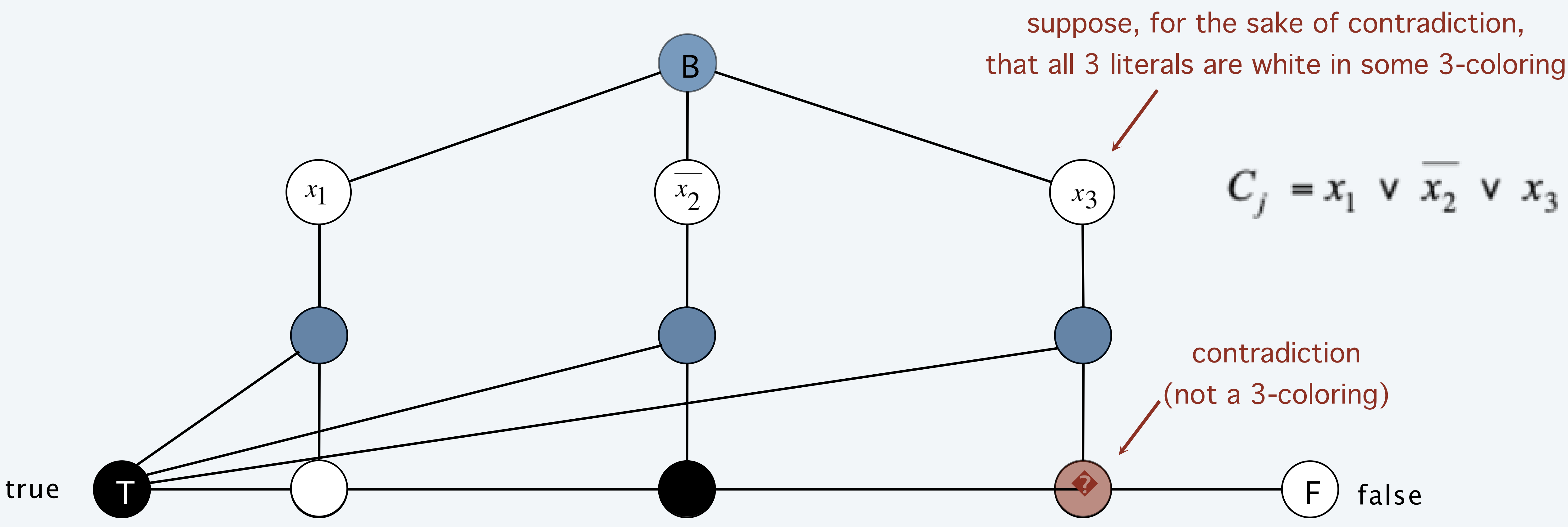
Pf. $\Rightarrow$ Suppose graph G is 3-colorable.

- WLOG, assume that node T is colored black, F is white, and B is blue.
- Consider assignment that sets all black literals to true (and white to false).
- (iv) ensures each literal is colored either black or white.
- (ii) ensures that each literal is white if its negation is black (and vice versa).
- (v) ensures at least one literal in each clause is black. ▪

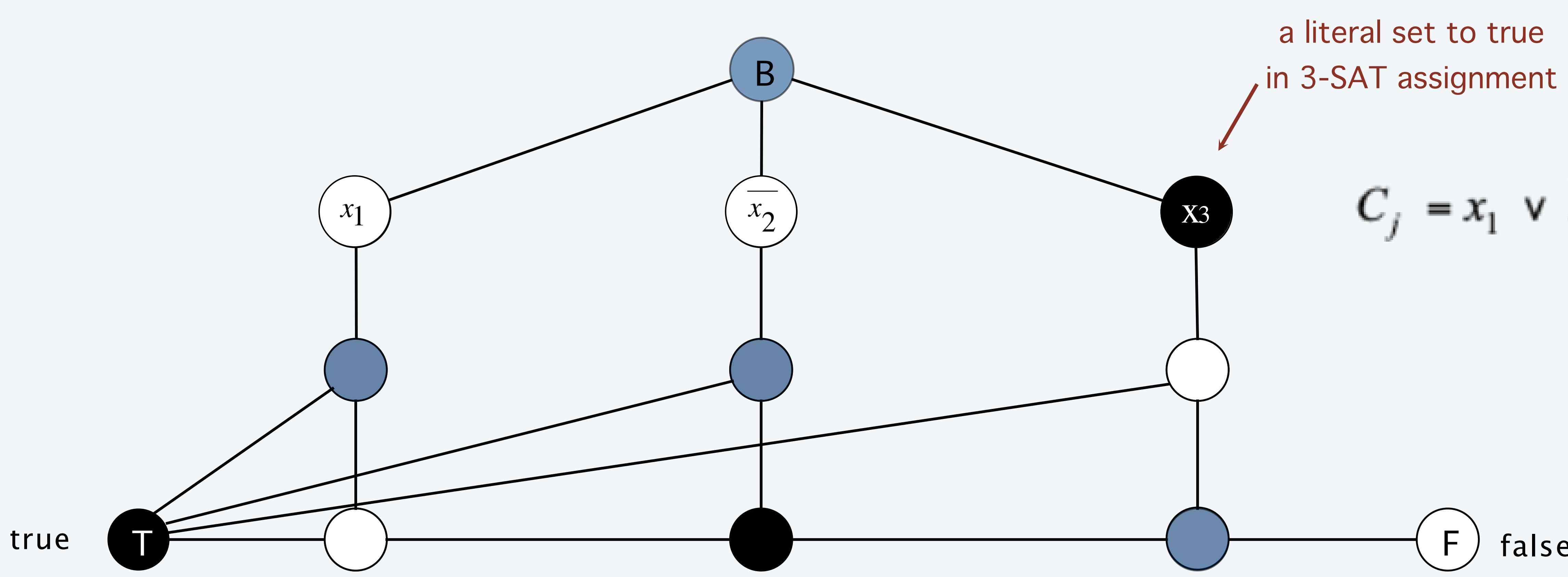


3-satisfiability reduces to 3-colorability

Lemma. Graph G is 3-colorable iff Φ is satisfiable.

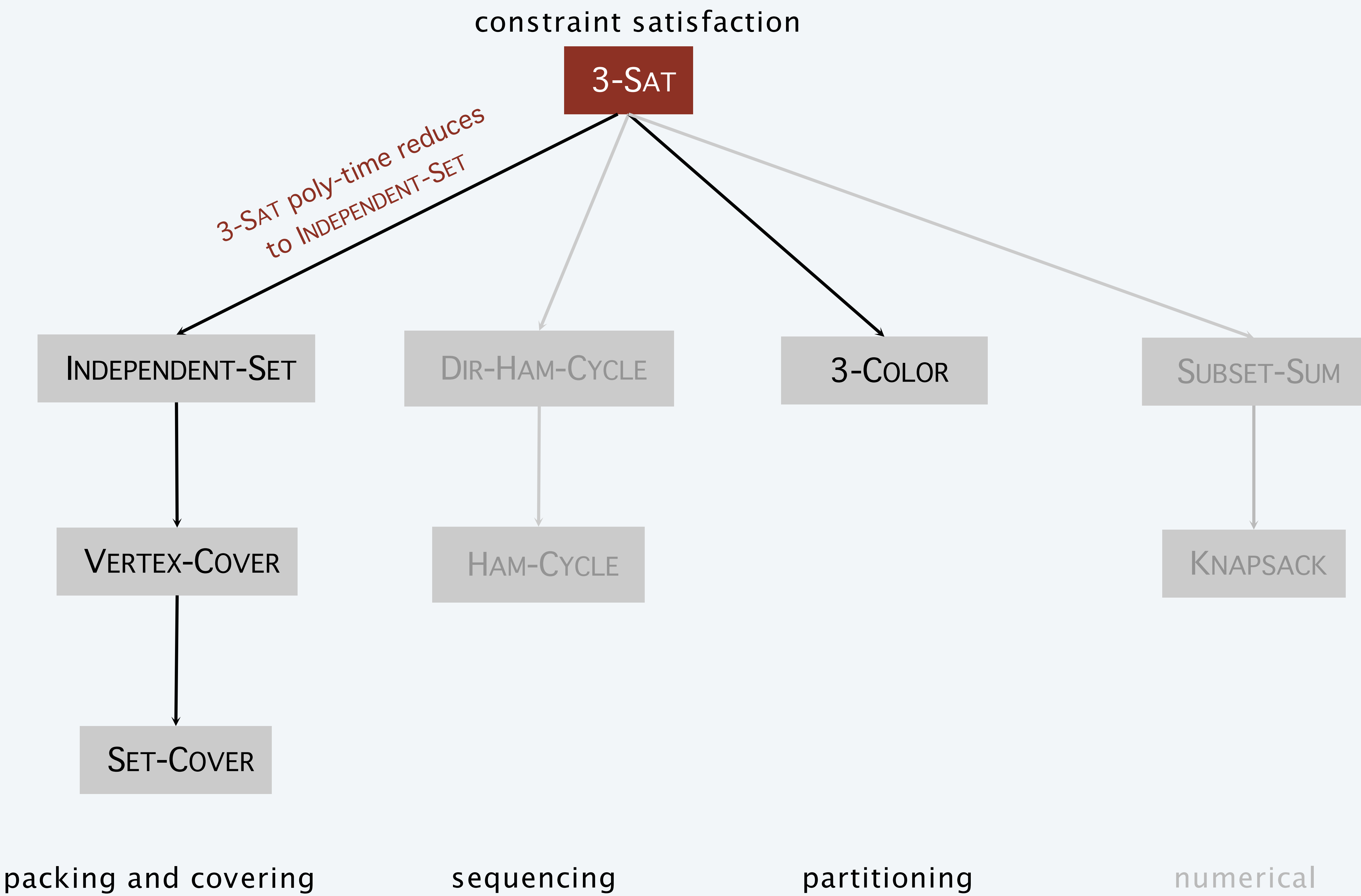
Pf. $\Leftarrow$ Suppose 3-SAT instance Φ is satisfiable.

- Color all true literals black and all false literals white.
- Pick one true literal; color node below that node white, and node below that blue.
- Color remaining middle row nodes blue.
- Color remaining bottom nodes black or white, as forced. ▪



$C_j = x_1 \vee \overline{x_2} \vee x_3$

Poly-time reductions



Karp's 20 poly-time reductions from satisfiability

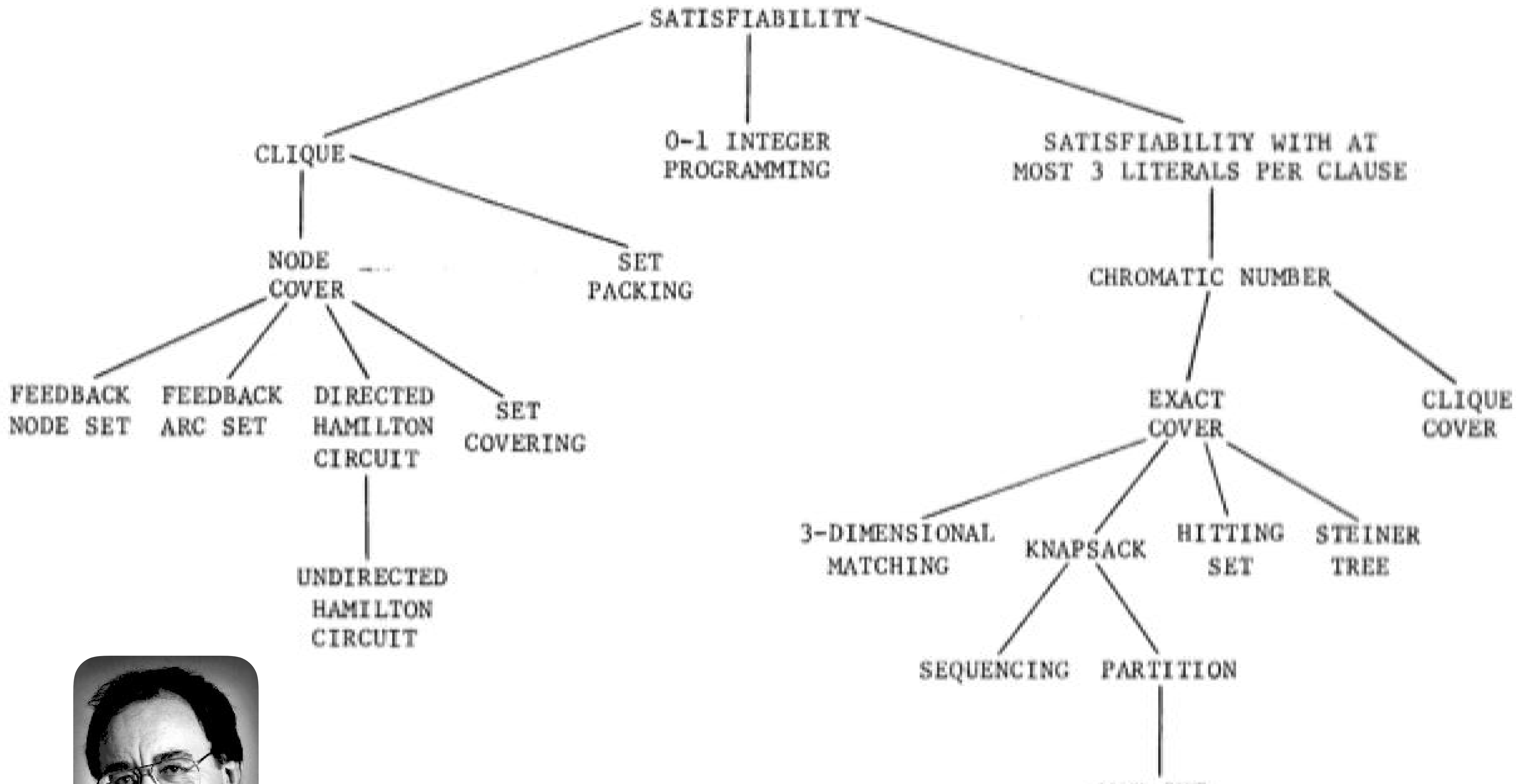


FIGURE 1 - Complete Problems

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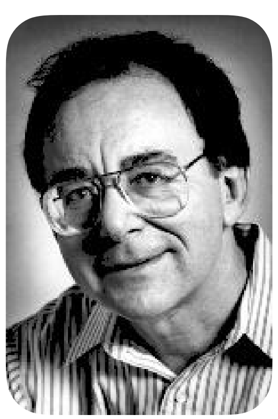
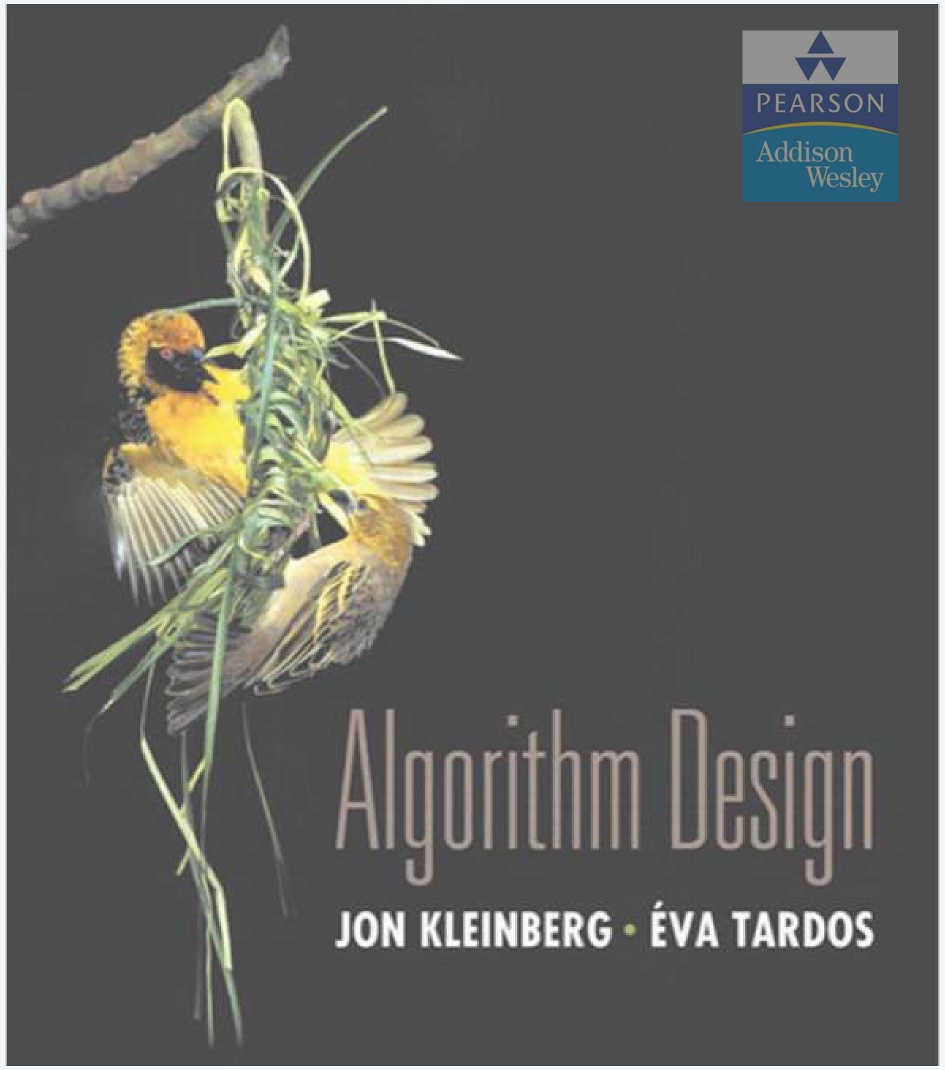
RICHARD M. KARP



Dick Karp (1972)
1985 Turing Award



SECTION 8.3

REDUCTIONS, P AND NP

- *poly-time reductions*
- *packing and covering problems*
- *constraint satisfaction problems*
- *graph coloring*
- ***P vs. NP***
- *NP-complete*

P

Decision problem.

- Problem X is a set of strings.
- Instance s is one string.
- Algorithm A solves problem X: $A(s) = \begin{cases} yes & \text{if } s \in X \\ no & \text{if } s \notin X \end{cases}$

Def. Algorithm A runs in polynomial time if for every string s, $A(s)$ terminates in $\leq p(|s|)$ "steps," where $p(\cdot)$ is some polynomial function.

length of s

Def. P = set of decision problems for which there exists a poly-time algorithm.

on a deterministic Turing machine

problem PRIMES:	$\{ 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots \}$
instance s:	592335744548702854681
algorithm:	Agrawal–Kayal–Saxena (2002)

NP

Def. Algorithm $C(s, t)$ is a certifier for problem X if for every string s :
$s \in X$ iff there exists a string t such that $C(s, t) = yes$.

Def. NP = set of decision problems for which there exists a poly-time certifier.

- $C(s, t)$ is a poly-time algorithm.
- Certificate t is of polynomial size: $|t| \leq p(|s|)$ for some polynomial $p(\cdot)$.

"certificate" or "witness"

problem COMPOSITES:	$\{ 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, \ldots. \}$
instance s:	437669
certificate t:	541 ← 437,669 = 541 × 809
certifier C(s, t):	grade-school division

Certifiers and certificates: satisfiability

SAT. Given a CNF formula Φ, does it have a satisfying truth assignment?
3-SAT. SAT where each clause contains exactly 3 literals.

Certificate. An assignment of truth values to the Boolean variables.

Certifier. Check that each clause in Φ has at least one true literal.

instance s $\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$

certificate t $x_1 = \text{true},\ x_2 = \text{true},\ x_3 = \text{false},\ x_4 = \text{false}$

Conclusions. SAT $\in$ NP, 3-SAT $\in$ NP.

Which of the following graph problems are known to be in NP?

A. Is the length of the longest simple path $\le k$?

B. Is the length of the longest simple path $\ge k$?

C. Is the length of the longest simple path $= k$?

D. Find the length of the longest simple path.

E. All of the above.

Intractability: quiz 6

In complexity theory, the abbreviation NP stands for…

A. Nope.

B. No problem.

C. Not polynomial time.

D. Not polynomial space.

E. Nondeterministic polynomial time.

Significance of NP

NP. Decision problems for which there exists a poly-time certifier.

" NP captures vast domains of computational, scientific, and mathematical endeavors, and seems to roughly delimit what mathematicians and scientists have been aspiring to compute feasibly. " — Christos Papadimitriou

" In an ideal world it would be renamed P vs VP. " — Clyde Kruskal

P, NP, and EXP

P. Decision problems for which there exists a poly-time algorithm.
NP. Decision problems for which there exists a poly-time certifier.
EXP. Decision problems for which there exists an exponential-time algorithm.

Proposition. P $\subseteq$ NP.
Pf. Consider any problem $X \in$ P.
- By definition, there exists a poly-time algorithm $A(s)$ that solves X.
- Certificate $t = \varepsilon$, certifier $C(s, t) = A(s)$. ▪

Proposition. NP $\subseteq$ EXP.
Pf. Consider any problem $X \in$ NP.
- By definition, there exists a poly-time certifier $C(s, t)$ for X, where certificate t satisfies $|t| \leq p(|s|)$ for some polynomial $p(\cdot)$.
- To solve instance s, run $C(s, t)$ on all strings t with $|t| \leq p(|s|)$.
- Return yes iff $C(s, t)$ returns yes for any of these potential certificates. ▪

Fact. P $\neq$ EXP $\Rightarrow$ either P $\neq$ NP, or NP $\neq$ EXP, or both.

The main question: P vs. NP

Q. How to solve an instance of 3-SAT with n variables?

A. Exhaustive search: try all 2^n truth assignments.

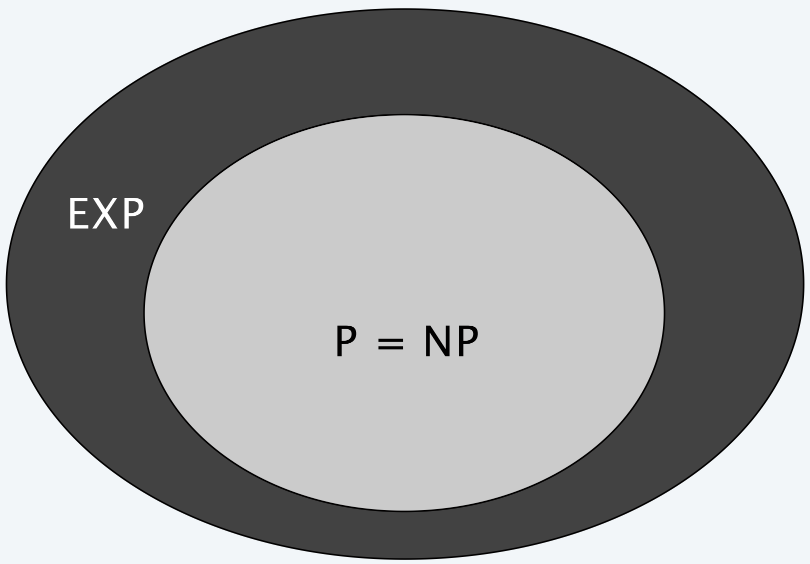
Q. Can we do anything substantially more clever?

Conjecture. $\underbrace{\text{No poly-time algorithm}}_{\text{“intractable”}}$ for 3-SAT.

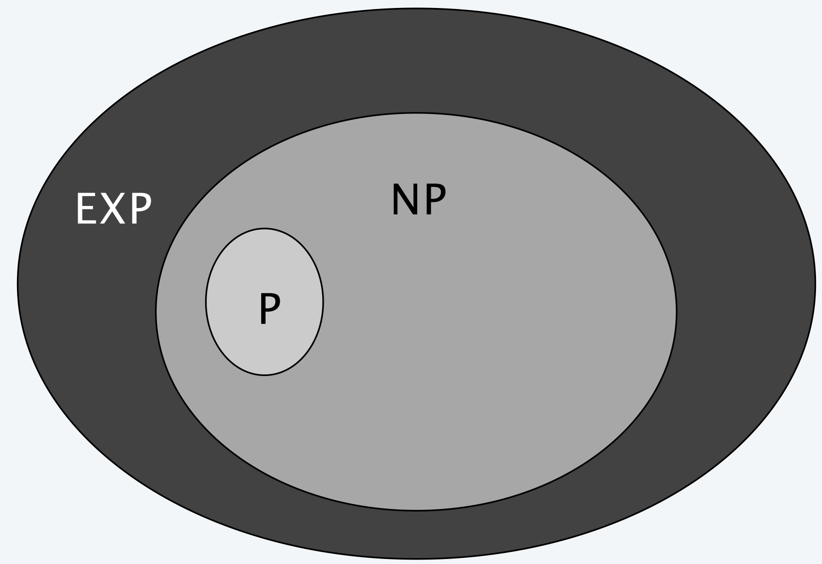
The main question: P vs. NP

Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]

Is the decision problem as easy as the certification problem?



If P = NP



If P ≠ NP

If yes... Efficient algorithms for 3-SAT, TSP, VERTEX-COVER, FACTOR, ...

If no... No efficient algorithms possible for 3-SAT, TSP, VERTEX-COVER, ...

Consensus opinion. Probably no.

Possible outcomes

P ≠ NP

" I conjecture that there is no good algorithm for the traveling salesman problem. My reasons are the same as for any mathematical conjecture: (i) It is a legitimate mathematical possibility and (ii) I do not know."

— Jack Edmonds 1966

" In my view, there is no way to even make intelligent guesses about the answer to any of these questions. If I had to bet now, I would bet that P is not equal to NP. I estimate the half-life of this problem at 25–50 more years, but I wouldn't bet on it being solved before 2100. "

— Bob Tarjan (2002)

Possible outcomes

$P \neq NP$

" We seem to be missing even the most basic understanding of the nature of its difficulty…. All approaches tried so far probably (in some cases, provably) have failed. In this sense P = NP is different from many other major mathematical problems on which a gradual progress was being constantly done (sometimes for centuries) whereupon they yielded, either completely or partially. "

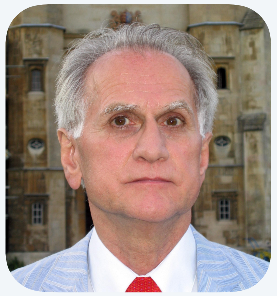
— Alexander Razborov (2002)

Possible outcomes

P = NP

“ I think that in this respect I am on the loony fringe of the mathematical
community: I think (not too strongly!) that P=NP and this will be proved within twenty years. Some years ago, Charles Read and I worked on it quite bit, and we even had a celebratory dinner in a good restaurant before we found an absolutely fatal mistake. ”

— Béla Bollobás (2002)



“ In my opinion this shouldn’t really be a hard problem; it’s just that we came late to this theory, and haven’t yet developed any techniques for proving computations to be hard. Eventually, it will just be a footnote in the books. ” — John Conway

Other possible outcomes

P = NP, but only $\Omega(n^{100})$ algorithm for 3-SAT.

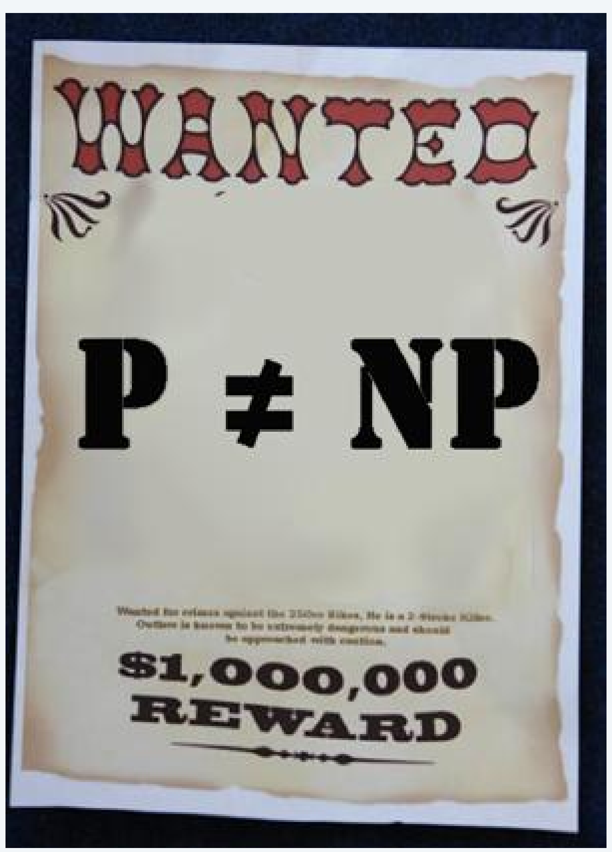
P ≠ NP, but with $O(n^{\log^* n})$ algorithm for 3-SAT.

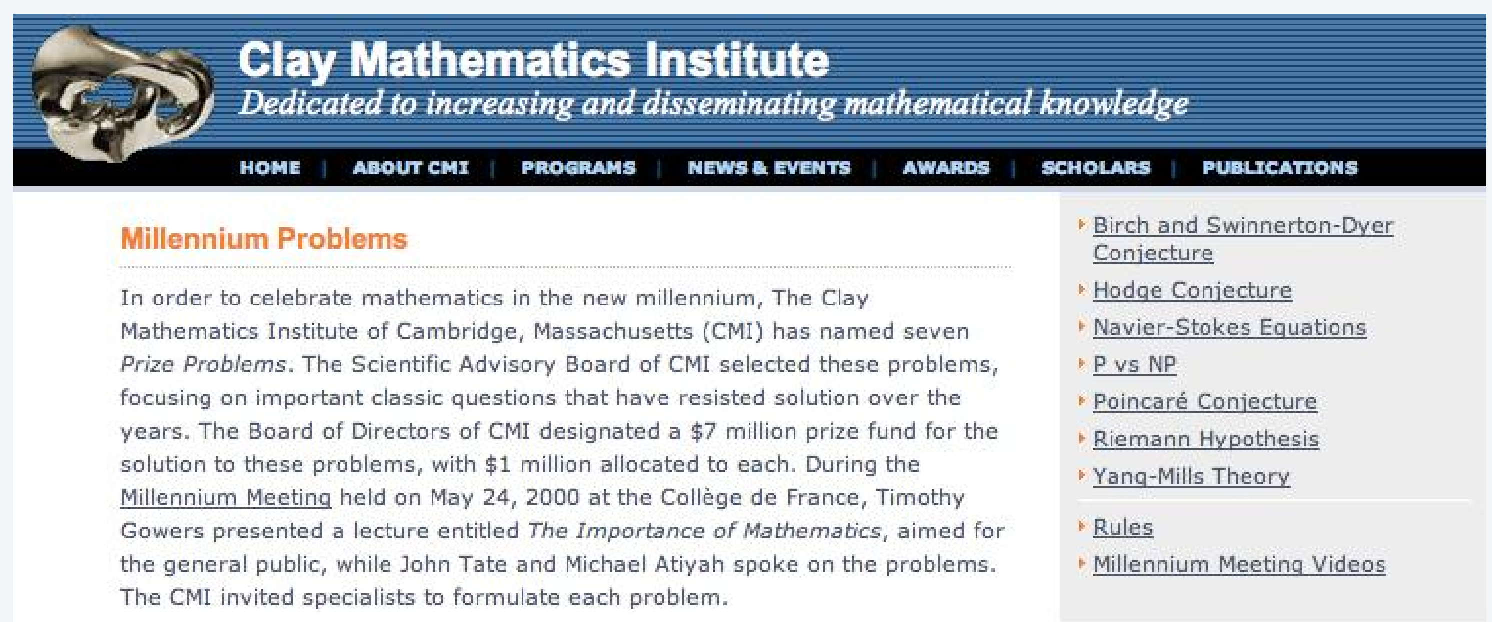
P = NP is independent (of ZFC axiomatic set theory).

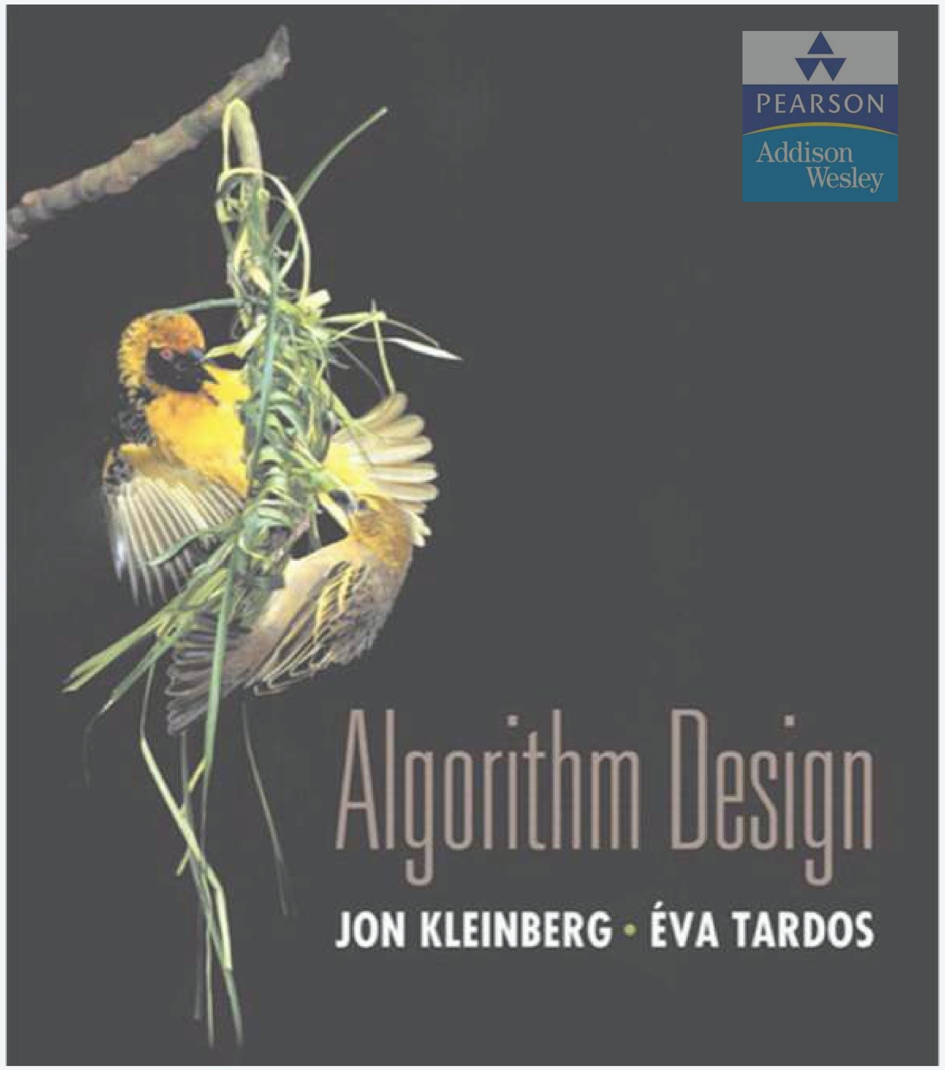
" It will be solved by either 2048 or 4096. I am currently somewhat pessimistic. The outcome will be the truly worst case scenario: namely that someone will prove P = NP because there are only finitely many obstructions to the opposite hypothesis; hence there exists a polynomial time solution to SAT but we will never know its complexity! " — Donald Knuth

Millennium prize

Millennium prize. $1 million for resolution of P ≠ NP problem.







SECTION 8.4

REDUCTIONS, P AND NP

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- *packing and covering problems*
- *constraint satisfaction problems*
- *graph coloring*
- *P vs. NP*
- *NP-complete*

Polynomial transformations

Def. Problem X polynomial (Cook) reduces to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

Def. Problem X polynomial (Karp) transforms to problem Y if given any instance x of X, we can construct an instance y of Y such that x is a yes instance of X iff y is a yes instance of Y.

we require $|y|$ to be of size polynomial in $|x|$

Note. Polynomial transformation is polynomial reduction with just one call to oracle for Y, exactly at the end of the algorithm for X. Almost all previous reductions were of this form.

Open question. Are these two concepts the same with respect to NP?

we abuse notation $\leq_P$ and blur distinction

NP-complete

NP-complete. A problem $Y \in$ NP with the property that for every problem $X \in$ NP, $X \leq_P Y$.

Proposition. Suppose $Y \in$ NP-complete. Then, $Y \in$ P iff P = NP.

Pf. $\Leftarrow$ If P = NP, then $Y \in$ P because $Y \in$ NP.

Pf. $\Rightarrow$ Suppose $Y \in$ P.

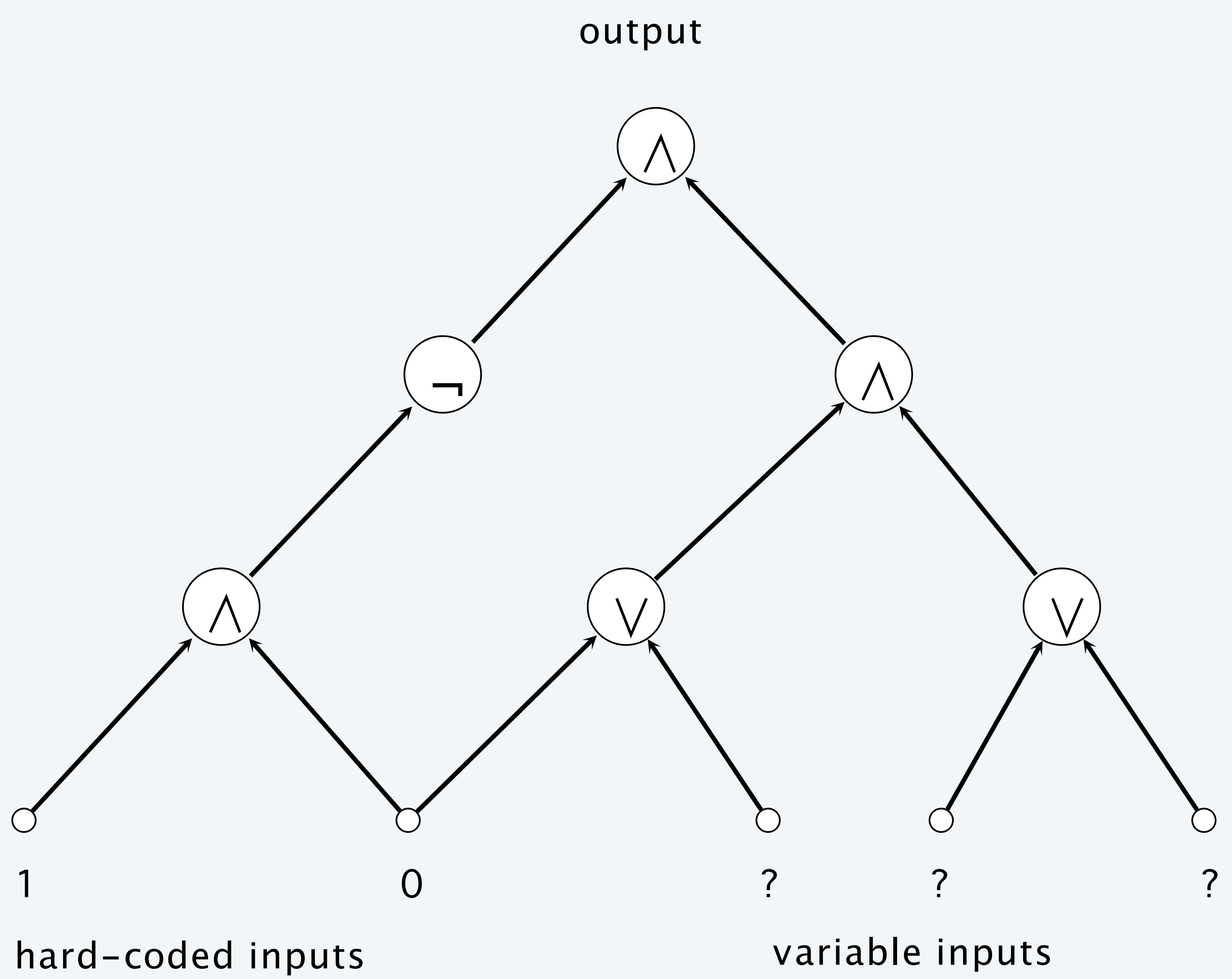
- Consider any problem $X \in$ NP. Since $X \leq_P Y$, we have $X \in$ P.
- This implies NP $\subseteq$ P.
- We already know P $\subseteq$ NP. Thus P = NP. ▪

Fundamental question. Are there any "natural" NP-complete problems?

Circuit satisfiability

CIRCUIT-SAT. Given a combinational circuit built from AND, OR, and NOT gates, is there a way to set the circuit inputs so that the output is 1?

yes: 1 0 1



The "first" NP-complete problem

Theorem. CIRCUIT-SAT ∈ NP-complete. [Cook 1971, Levin 1973]

The Complexity of Theorem-Proving Procedures

Stephen A. Cook

University of Toronto

Summary

It is shown that any recognition problem solved by a polynomial time-bounded nondeterministic Turing machine can be "reduced" to the problem of determining whether a given propositional formula is a tautology. Here "reduced" means, roughly speaking, that the first problem can be solved deterministically in polynomial time provided an oracle is available for solving the second. From this notion of reducible, polynomial degrees of difficulty are defined, and it is shown that the problem of determining tautologyhood has the same polynomial degree as the problem of determining whether the first of two given graphs is isomorphic to a subgraph of the second. Other examples are discussed. A method of measuring the complexity of proof procedures for the predicate calculus is introduced and discussed.

Throughout this paper, a set of strings means a set of strings on some fixed, large, finite alphabet Σ. This alphabet is large enough to include symbols for all sets described here. All Turing machines are deterministic recognition devices, unless the contrary is explicitly stated.

1. Tautologies and Polynomial Re-Reducibility.

Let us fix a formalism for the propositional calculus in which formulas are written as strings on Σ. Since we will require infinitely many proposition symbols (atoms), each such symbol will consist of a member of Σ followed by a number in binary notation to distinguish that symbol. Thus a formula of length n can only have about n/logn distinct function and predicate symbols. The logical connectives are & (and), ∨ (or), and ¬(not).

The set of tautologies (denoted by {tautologies}) is a certain recursive set of strings on this alphabet, and we are interested in the problem of finding a good lower bound on its possible recognition times. We provide no such lower bound here, but theorem 1 will give evidence that {tautologies} is a difficult set to recognize, since many apparently difficult problems can be reduced to determining tautologyhood. By reduced we mean, roughly speaking, that if tautologyhood could be decided instantly (by an "oracle") then these problems could be decided in polynomial time. In order to make this notion precise, we introduce query machines, which are like Turing machines with oracles in [1].

A query machine is a multitape Turing machine with a distinguished tape called the query tape, and three distinguished states called the query state, yes state, and no state, respectively. If M is a query machine and T is a set of strings, then a T-computation of M is a computation of M in which initially M is in the initial state and has an input string w on its input tape, and each time M assumes the query state there is a string u on the query tape, and the next state M assumes is the yes state if u∈T and the no state if u∉T. We think of an "oracle", which knows T, placing M in the yes state or no state.

Definition

A set S of strings is P-reducible (P for polynomial) to a set T of strings iff there is some query machine M and a polynomial Q(n) such that for each input string w, the T-computation of M with input w halts within Q(|w|) steps (|w| is the length of w), and ends in an accepting state iff w∈S.

It is not hard to see that P-reducibility is a transitive relation. Thus the relation E on

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УДК 519.14

УНИВЕРСАЛЬНЫЕ ЗАДАЧИ ПЕРЕБОРА

Л. А. Левин

В статье рассматривается несколько известных массовых задач «переборного типа» и доказывается, что эти задачи можно решать лишь за такое время, за которое можно решать вообще любые задачи указанного типа.

После уточнения понятия алгоритма была доказана алгоритмическая неразрешимость ряда классических массовых проблем (например, проблем тождества элементов групп, гомеоморфности многообразий, разрешимости диофантовых уравнений и других). Тем самым был снят вопрос о нахождении практического способа их решения. Однако существование алгоритмов для решения других задач не снимает для них аналогичного вопроса из-за фантастически большого объема работы, предписываемого этими алгоритмами. Такова ситуация с так называемыми переборными задачами: минимизации булевых функций, поиска доказательств ограниченной длины, выяснения изоморфности графов и другими. Все эти задачи решаются тривиальными алгоритмами, состоящими в переборе всех возможностей. Однако эти алгоритмы требуют экспоненциального времени работы и у математиков сложилось убеждение, что более простые алгоритмы для них невозможны. Был получен ряд серьезных аргументов в пользу его справедливости (см.[1, 2]), однако доказать это утверждение не удалось никому. (Например, до сих пор не доказано, что для нахождения математических доказательств нужно больше времени, чем для их проверки.)

Однако если предположить, что вообще существует какая-нибудь (хотя бы искусственно построенная) массовая задача переборного типа, неразрешимая простыми (в смысле объема вычислений) алгоритмами, то можно показать, что этим же свойством обладают и многие «классические» переборные задачи (в том числе задача минимизации, задача поиска доказательств и др.). В этом и состоят основные результаты статьи.

Функции $f(n)$ и $g(n)$ будем называть сравнимыми, если при некотором k

$$f(n) \leqslant (g(n)+2)^k \quad \text{и} \quad g(n) \leqslant (f(n)+2)^k.$$

Аналогично будем понимать термин «меньше или сравнимо».

Определение. Задачей переборного типа (или просто переборной задачей) будем называть задачу вида «по данному x найти какое-нибудь y длины, сравнимой с длиной x, такое, что выполняется $A(x, y)$», где $A(x, y)$ — какое-нибудь свойство, проверяемое алгоритмом, время работы которого сравнимо с длиной x. (Под алгоритмом здесь можно понимать, например, алгоритмы Колмогорова — Успенского или машины Тьюринга, или нормальные алгоритмы; x, y — двоичные слова). Квазипереборной задачей будем называть задачу выяснения, существует ли такое y.

Мы рассмотрим шесть задач этих типов. Рассматриваемые в них объекты кодируются естественным образом в виде двоичных слов. При этом выбор естественной кодировки не существен, так как все они дают сравнимые длины кодов.

Задача 1. Заданы списком конечное множество и покрытие его 500-элементными подмножествами. Найти подпокрытие заданной мощности (соответственно выяснить существует ли оно).

Задача 2. Таблично задана частичная булева функция. Найти заданного размера дизъюнктивную нормальную форму, реализующую эту функцию в области определения (соответственно выяснить существует ли она).

Задача 3. Выяснить, выводима или опровержима данная формула исчисления высказываний. (Или, что то же самое, равна ли константе данная булева формула.)

Задача 4. Даны два графа. Найти гомоморфизм одного на другой (выяснить его существование).

Задача 5. Даны два графа. Найти изоморфизм одного в другой (на его часть).

Задача 6. Рассматриваются матрицы из целых чисел от 1 до 100 и некоторое условие о том, какие числа в них могут соседствовать по вертикали и какие по горизонтали. Заданы числа на границе и требуется продолжить их на всю матрицу с соблюдением условия.

The "first" NP-complete problem

Theorem. [Cook 1971, Levin 1973] SAT ∈ NP-complete.

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Л. А. Левин

В статье рассматривается несколько известных массовых задач «переборного типа» и доказывается, что эти задачи можно решать лишь за такое время, за которое можно решать вообще любые задачи указанного типа.

После уточнения понятия алгоритма была доказана алгоритмическая неразрешимость ряда классических массовых проблем (например, проблем тождества элементов групп, гомеоморфности многообразий, разрешимости диофантовых уравнений и других). Тем самым был снят вопрос о нахождении практического способа их решения. Однако существование алгоритмов для решения других задач не снимает для них аналогичного вопроса из-за фантастически большого объема работы, предписываемого этими алгоритмами. Такова ситуация с так называемыми переборными задачами: минимизации булевых функций, поиска доказательств ограниченной длины, выяснения изоморфности графов и другими. Все эти задачи решаются тривиальными алгоритмами, состоящими в переборе всех возможностей. Однако эти алгоритмы требуют экспоненциального времени работы и у математиков сложилось убеждение, что более простые алгоритмы для них невозможны. Был получен ряд серьезных аргументов в пользу его справедливости (см.[1, 2]), однако доказать это утверждение не удалось никому. (Например, до сих пор не доказано, что для нахождения математических доказательств нужно больше времени, чем для их проверки.)

Однако если предположить, что вообще существует какая-нибудь (хотя бы искусственно построенная) массовая задача переборного типа, неразрешимая простыми (в смысле объема вычислений) алгоритмами, то можно показать, что этим же свойством обладают и многие «классические» переборные задачи (в том числе задача минимизации, задача поиска доказательств и др.). В этом и состоят основные результаты статьи.

Функции $f(n)$ и $g(n)$ будем называть сравнимыми, если при некотором k

$$f(n) \leqslant (g(n)+2)^k \quad \text{и} \quad g(n) \leqslant (f(n)+2)^k.$$

Аналогично будем понимать термин «меньше или сравнимо».

Определение. Задачей переборного типа (или просто переборной задачей) будем называть задачу вида «по данному x найти какое-нибудь y длины, сравнимой с длиной x, такое, что выполняется $A(x, y)$», где $A(x, y)$ – какое-нибудь свойство, проверяемое алгоритмом, время работы которого сравнимо с длиной x. (Под алгоритмом здесь можно понимать, например, алгоритмы Колмогорова — Успенского или машины Тьюринга, или нормальные алгоритмы; x, y – двоичные слова). Квазипереборной задачей будем называть задачу выяснения, существует ли такое y.

Мы рассмотрим шесть задач этих типов. Рассматриваемые в них объекты кодируются естественным образом в виде двоичных слов. При этом выбор естественной кодировки не существен, так как все они дают сравнимые длины кодов.

Задача 1. Заданы списком конечное множество и покрытие его 500-элементными подмножествами. Найти подпокрытие заданной мощности (соответственно выяснить существует ли оно).

Задача 2. Таблично задана частичная булева функция. Найти заданного размера дизъюнктивную нормальную форму, реализующую эту функцию в области определения (соответственно выяснить существует ли она).

Задача 3. Выяснить, выводима или опровержима данная формула исчисления высказываний. (Или, что то же самое, равна ли константе данная булева формула.)

Задача 4. Даны два графа. Найти гомоморфизм одного на другой (выяснить его существование).

Задача 5. Даны два графа. Найти изоморфизм одного в другой (на его часть).

Задача 6. Рассматриваются матрицы из целых чисел от 1 до 100 и некоторое условие о том, какие числа в них могут соседствовать по вертикали и какие по горизонтали. Заданы числа на границе и требуется продолжить их на всю матрицу с соблюдением условия.

The "first" NP-complete problem

Theorem. CIRCUIT-SAT ∈ NP-complete.

Pf sketch.

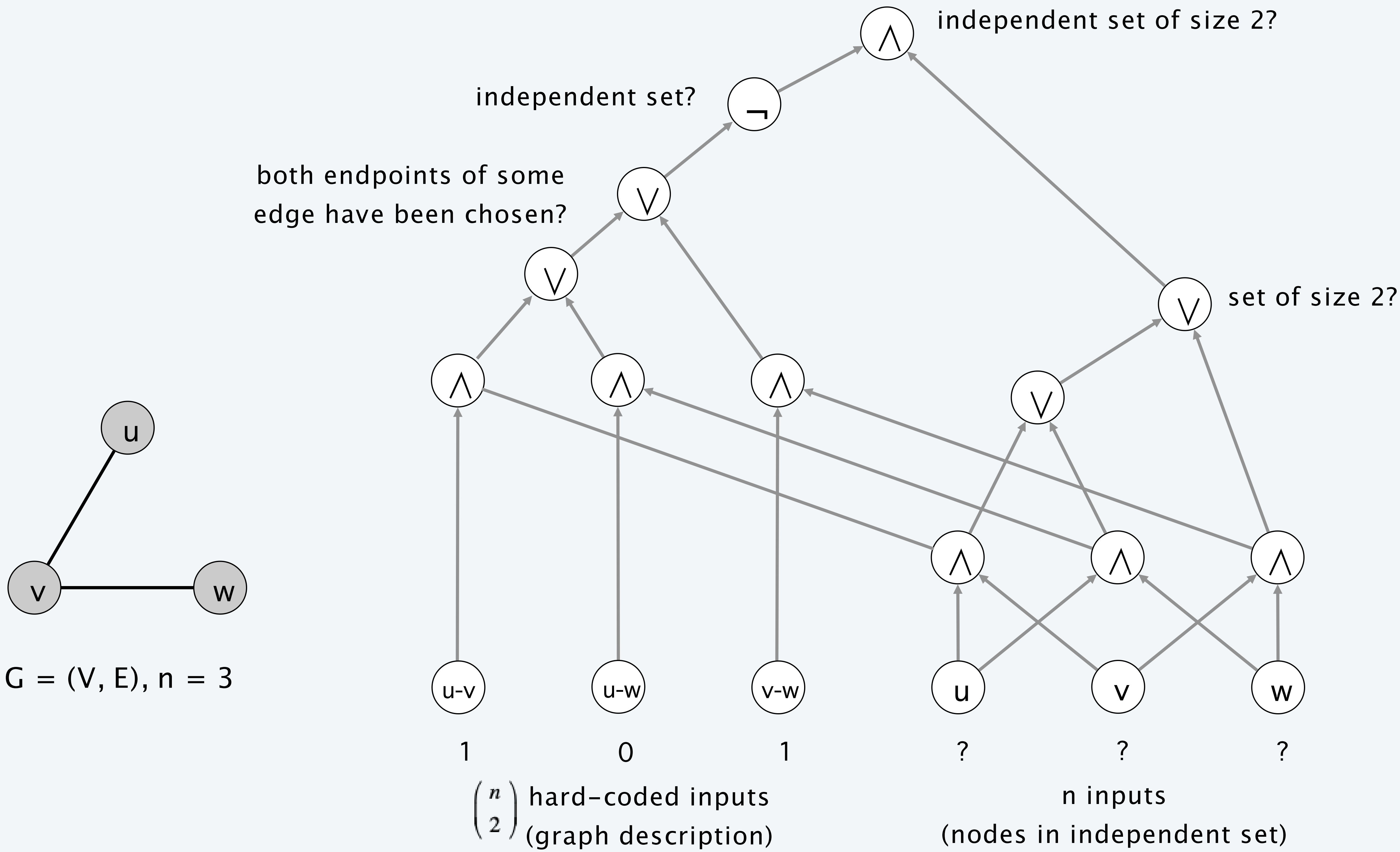
- Clearly, CIRCUIT-SAT ∈ NP.
- Any algorithm that takes a fixed number of bits n as input and produces a yes or no answer can be represented by such a circuit.
- Moreover, if algorithm takes poly-time, then circuit is of poly-size.

sketchy part of proof; fixing the number of bits is important, and reflects basic distinction between algorithms and circuits

- Consider any problem X ∈ NP. It has a poly-time certifier $C(s, t)$, where certificate t satisfies $|t| \leq p(|s|)$ for some polynomial $p(\cdot)$.
- View $C(s, t)$ as an algorithm with $|s| + p(|s|)$ input bits and convert it into a poly-size circuit K.
  - first $|s|$ bits are hard-coded with s
  - remaining $p(|s|)$ bits represent (unknown) bits of t
- Circuit K is satisfiable iff $C(s, t) = yes$.

Example

Ex. Construction below creates a circuit K whose inputs can be set so that it outputs 1 iff graph G has an independent set of size 2.



Establishing NP-completeness

Remark. Once we establish first "natural" NP-complete problem, others fall like dominoes.

Recipe. To prove that $Y \in$ NP-complete:

- Step 1. Show that $Y \in$ NP.
- Step 2. Choose an NP-complete problem X.
- Step 3. Prove that $X \leq_P Y$.

Proposition. If $X \in$ NP-complete, $Y \in$ NP, and $X \leq_P Y$, then $Y \in$ NP-complete.

Pf. Consider any problem $W \in$ NP. Then, both $\underbrace{W \leq_P X}_{\text{by definition of NP-complete}}$ and $\underbrace{X \leq_P Y}_{\text{by assumption}}$.

- By transitivity, $W \leq_P Y$.
- Hence $Y \in$ NP-complete. ▪

Reductions: quiz 7

Suppose that $\mathbf{X} \in$ NP-COMPLETE, $\mathbf{Y} \in$ NP, and $\mathbf{X} \leq_P \mathbf{Y}$. Which can you infer?

A. Y is NP-complete.

B. If $Y \notin$ P, then P $\neq$ NP.

C. If P $\neq$ NP, then neither X nor Y is in P.

D. All of the above.

3-satisfiability is NP-complete

Theorem. 3-SAT $\in$ NP-complete.

Pf.

- Suffices to show that CIRCUIT-SAT $\le_P$ 3-SAT since 3-SAT $\in$ NP.
- Given a combinational circuit K, we construct an instance Φ of 3-SAT that is satisfiable iff the inputs of K can be set so that it outputs 1.

3-satisfiability is NP-complete

Construction. Let K be any circuit.

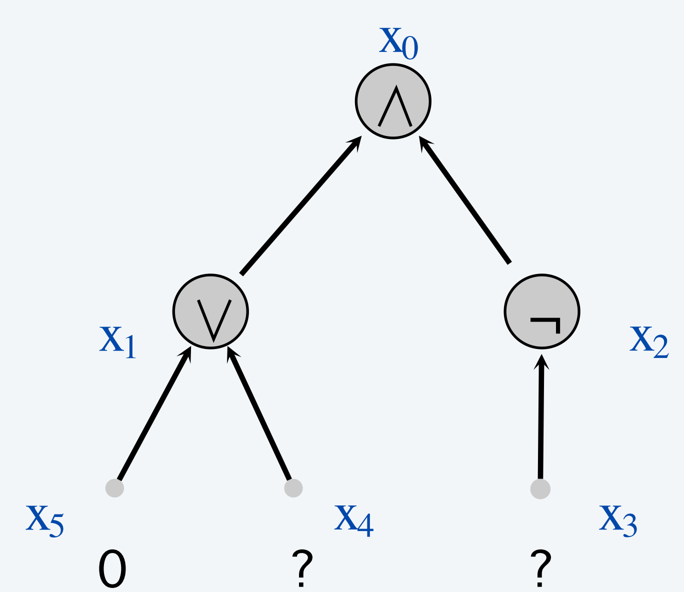
Step 1. Create a 3-SAT variable x_i for each circuit element i.

Step 2. Make circuit compute correct values at each node:

- $x_2 = \neg\, x_3$ ⇒ add 2 clauses: $x_2 \vee x_3$, $\overline{x_2} \vee \overline{x_3}$
- $x_1 = x_4 \vee x_5$ ⇒ add 3 clauses: $x_1 \vee \overline{x_4}$, $x_1 \vee \overline{x_5}$, $\overline{x_1} \vee x_4 \vee x_5$
- $x_0 = x_1 \wedge x_2$ ⇒ add 3 clauses: $\overline{x_0} \vee x_1$, $\overline{x_0} \vee x_2$, $x_0 \vee \overline{x_1} \vee \overline{x_2}$

Step 3. Hard-coded input values and output value.

- $x_5 = 0$ ⇒ add 1 clause: $\overline{x_5}$
- $x_0 = 1$ ⇒ add 1 clause: x_0



3-satisfiability is NP-complete

Construction. [continued]

Step 4. Turn clauses of length 1 or 2 into clauses of length 3.

- Create four new variables z_1, z_2, z_3, and z_4.
- Add 8 clauses to force $z_1 = z_2 = 0$:

$$(\overline{z_1} \vee z_3 \vee z_4),\ (\overline{z_1} \vee z_3 \vee \overline{z_4}),\ (\overline{z_1} \vee \overline{z_3} \vee z_4),\ (\overline{z_1} \vee \overline{z_3} \vee \overline{z_4})$$
$$(\overline{z_2} \vee z_3 \vee z_4),\ (\overline{z_2} \vee z_3 \vee \overline{z_4}),\ (\overline{z_2} \vee \overline{z_3} \vee z_4),\ (\overline{z_2} \vee \overline{z_3} \vee \overline{z_4})$$

- Replace any clause with a single term (t_i) with $(t_i \vee z_1 \vee z_2)$.
- Replace any clause with two terms $(t_i \vee t_j)$ with $(t_i \vee t_j \vee z_1)$.

3-satisfiability is NP-complete

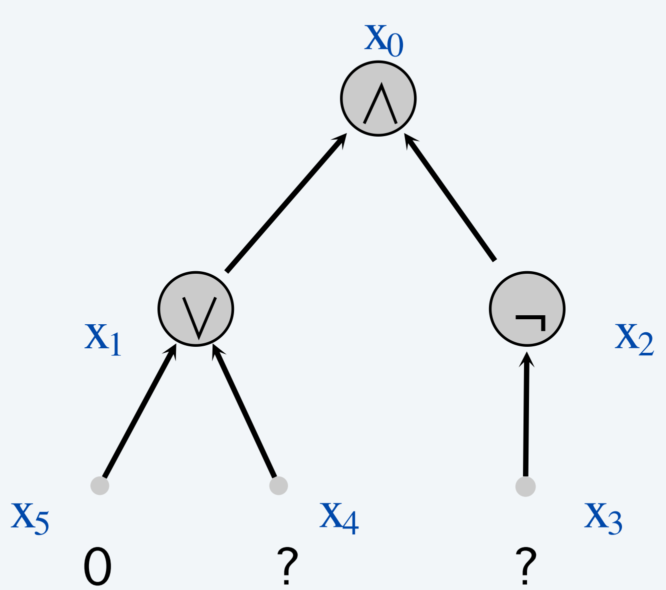
Lemma. Φ is satisfiable iff the inputs of K can be set so that it outputs 1.

Pf. $\Leftarrow$ Suppose there are inputs of K that make it output 1.

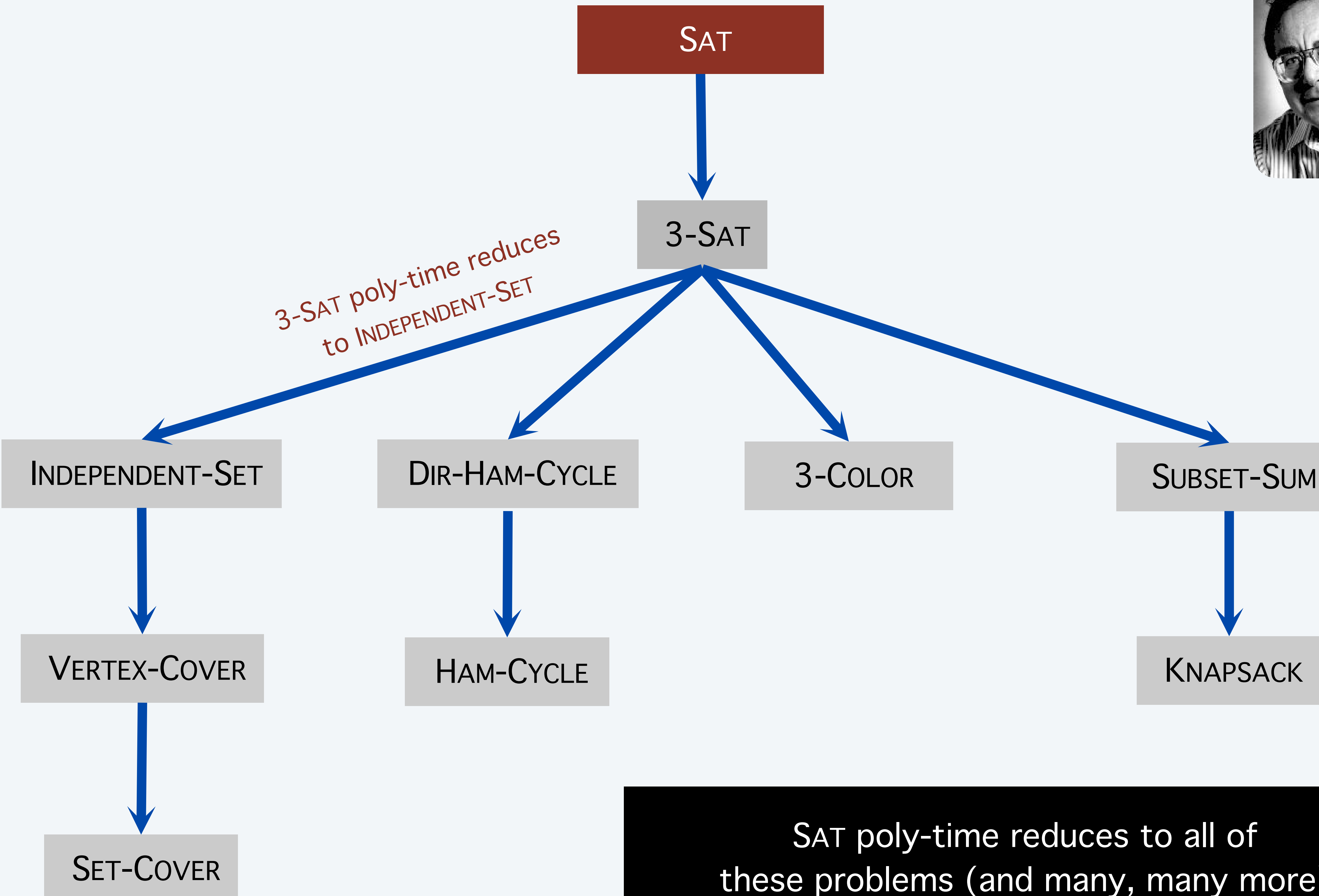
- Can propagate input values to create values at all nodes of K.
- This set of values satisfies Φ.

Pf. $\Rightarrow$ Suppose Φ is satisfiable.

- We claim that the set of values corresponding to the circuit inputs constitutes a way to make circuit K output 1.
- The 3-SAT clauses were designed to ensure that the values assigned to all node in K exactly match what the circuit would compute for these nodes. ▪

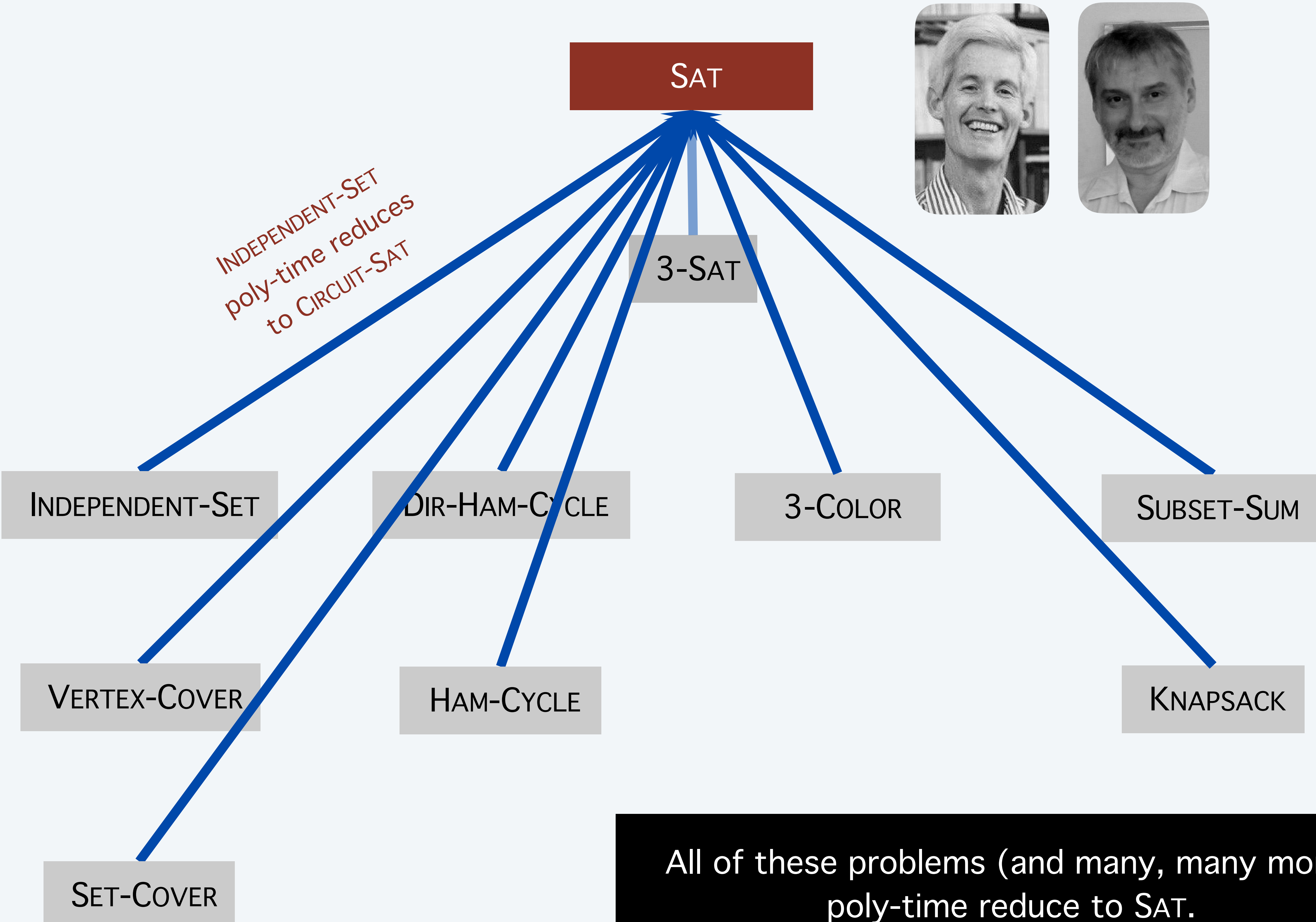


Implications of Karp



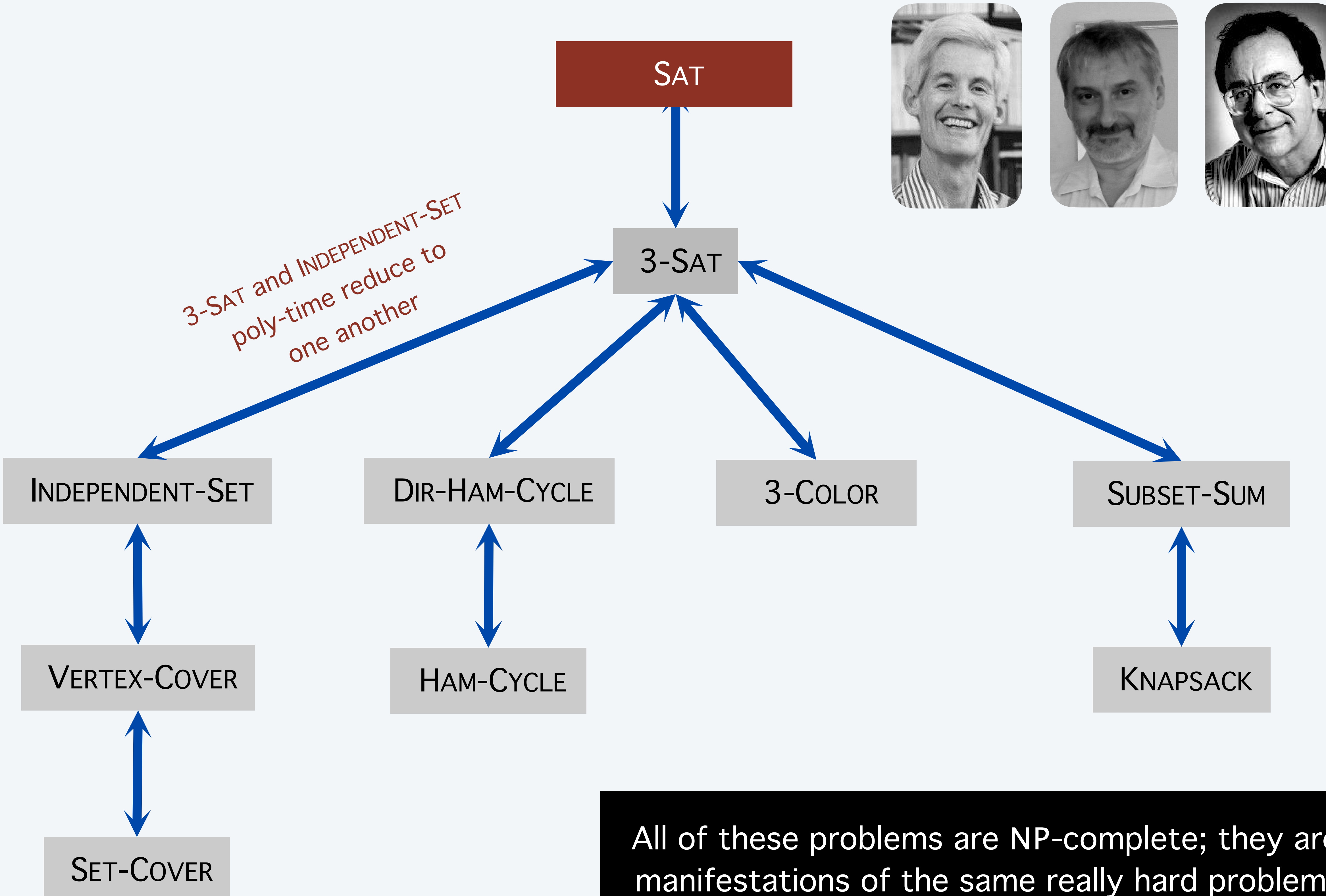
SAT poly-time reduces to all of these problems (and many, many more)

Implications of Cook–Levin



All of these problems (and many, many more) poly-time reduce to SAT.

Implications of Karp + Cook–Levin



All of these problems are NP-complete; they are manifestations of the same really hard problem.