习题 11.1

1.(4) $\begin{cases} z^2=2ax \\ 9y^2=16xz \end{cases}$ 留z: $\begin{cases} x=\dfrac{z^2}{2a} \\ y=\dfrac{4}{3}\dfrac{z^{\frac{3}{2}}}{\sqrt{2a}} \end{cases}$

$$S=\int_0^{2a}\sqrt{(x_z')^2+(y_z')+1}\,dz=\int_0^{2a}\left(\frac{z}{a}+1\right)dz=4a.$$

(5). $\begin{cases} 4ax=(y+z)^2 \\ 4x^2+3y^2=3z^2 \end{cases} \Rightarrow \begin{cases} y=\sqrt{ax}-\dfrac{1}{3}\dfrac{x^{\frac{3}{2}}}{\sqrt{a}} \\ z=\sqrt{ax}+\dfrac{1}{3}\dfrac{x^{\frac{3}{2}}}{\sqrt{a}} \end{cases}$ 避免字母重复，令 $x=t$:

$$S=\int_0^x\sqrt{(y_x')^2+(z_x')^2+1}\,dt$$

$$=\int_0^x\sqrt{\left[\frac{1}{2}\sqrt{\frac{a}{t}}-\sqrt{\frac{t}{a}}\right]^2+\frac{1}{2}\left(\sqrt{\frac{a}{t}}+\sqrt{\frac{t}{a}}\right)^2+1}\,dt$$

$$=\int_0^x\sqrt{\frac{a}{2t}+\frac{t}{2a}+1}\,dt$$

$$=\int_0^x\frac{(a+t)}{\sqrt{2at}}dt=\sqrt{2ax}+\frac{\sqrt{2}}{3}\frac{x^{\frac{3}{2}}}{\sqrt{a}}$$

$$=\sqrt{2}z.$$

(注：此交线有两条，一条 $x\neq0$，为上述做法。另一条为 $\begin{cases} x=0 \\ y=-z \end{cases}$ 的直线。)

2.(4) $L: y=\frac{1}{2}x-2$.

$$\int_L\frac{ds}{x-y}=\int_0^4\frac{1}{x-y}\sqrt{1+(y_x')^2}\,dx=\sqrt{5}\int_0^4\frac{dx}{x+4}=\sqrt{5}\ln2$$

(5). AB 为平行于y轴直线：$\begin{cases} x=1 \\ z=0 \end{cases}$

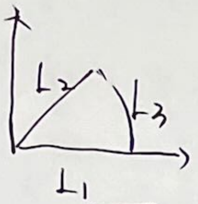
故 $\displaystyle\int_L(x+y+z)ds=\int_{L_{AB}}(x+y+z)ds+\int_{L_{BC}}(x+y+z)ds$

$$=\int_0^1(1+y+0)dy+\int_0^{2\pi}(\cos t+\sin t+t)\sqrt{(-\sin t)^2+(\cos t)^2+1}\,dt$$

$$=1+\frac{1}{2}+\sqrt{2}\int_0^{2\pi}(\cos t+\sin t+t)dt$$

$$=\frac{3}{2}+2\sqrt{2}\pi^2$$

(6).



$L_1: r\in[0,a],\ \varphi=0$

$L_2: r\in[0,a],\ \varphi=\frac{\pi}{4}$

$L_3: r=a,\ \varphi=\frac{\pi}{4}$.

$$\int_{L_1} e^{\sqrt{x^2+y^2}}ds=\int_{L_2} e^{\sqrt{x^2+y^2}}ds= e^{a}-1$$

$$\int_{L_3} e^{\sqrt{x^2+y^2}}ds=\int_0^{\frac{\pi}{4}} e^{a}\sqrt{(a\sin\varphi)^2+(a\cos\varphi)^2}\,d\varphi = a\cdot e^{a}\cdot\frac{\pi}{4}$$

故 $\int_{L} e^{\sqrt{x^2+y^2}}ds = 2(e^{a}-1)+\frac{\pi}{4}ae^{a}$

(7)
$$\begin{cases} x=ae^{k\varphi}\cos\varphi \\ y=ae^{k\varphi}\sin\varphi \end{cases}\quad \varphi\in(-\infty,0].$$

$$\int_L x\,ds=\int_{-\infty}^{0} ae^{k\varphi}\cos\varphi\cdot\sqrt{a^2(e^{k\varphi}(-\sin\varphi)+ke^{k\varphi}\cos\varphi)^2+a^2(e^{k\varphi}\cos\varphi+ke^{k\varphi}\sin\varphi)^2}\,d\varphi$$

$$=\int_{-\infty}^{0} ae^{k\varphi}\cos\varphi\cdot ae^{k\varphi}\cdot\sqrt{1+k^2}\,d\varphi$$

$$=a^2\sqrt{1+k^2}\int_{-\infty}^{0}e^{2k\varphi}\cos\varphi\,d\varphi$$

$$=\frac{2ka^2\sqrt{1+k^2}}{1+4k^2}$$

(注: $I=\int_{-\infty}^{0}e^{2k\varphi}\cos\varphi\,d\varphi=\frac{1}{2k}\left[e^{2k\varphi}\cos\varphi\Big|_{-\infty}^{0}+\int_{-\infty}^{0}e^{2k\varphi}\sin\varphi\,d\varphi\right]$

$=\frac{1}{2k}\left\{1+\frac{1}{2k}[0-I]\right\}$

$\Rightarrow I=\frac{2k}{4k^2+1}$

(9) 双纽线: $(x^2+y^2)^2=a^2(x^2-y^2)$, $x\geq0$

$\Rightarrow \rho^2=a^2\cos2\theta\geq0 \Rightarrow \theta\in[-\frac{\pi}{4},\frac{\pi}{4}]\cup[\frac{3}{4}\pi,\frac{5}{4}\pi]$.

又 $x\geq0 \Rightarrow \theta\in[-\frac{\pi}{4},\frac{\pi}{4}]$

$$\int_L x\sqrt{x^2-y^2}\,ds=2\int_0^{\frac{\pi}{4}}(a\sqrt{\cos2\theta}\cos\theta)(a\sqrt{\cos2\theta})\sqrt{\cos2\theta}\cdot\sqrt{\frac{a^2\sin^2 2\theta}{\cos2\theta}+a^2\cos2\theta}\,d\theta$$

$$=2\int_0^{\frac{\pi}{4}}a^3\cos2\theta\cos\theta\,d\theta=2a^3\int_0^{\frac{\pi}{4}}(1-2\sin^2\theta)\cos\theta\,d\theta$$

$$\cancel{=2a^3\int_0^{\frac{\pi}{4}}\left(\cos^3\theta-\frac{\sin^2\theta\cos\theta}{\cos\theta}\right)d\theta}$$

$$=\frac{2\sqrt{2}}{3}a^3$$

(10) 令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$. $L: \begin{cases} r = a \\ z = 0 \end{cases}, \theta \in [0, 2\pi]$

$$\int_L (x^2+y^2+z^2)^n ds = \int_0^{2\pi} (a^2)^n \cdot \sqrt{[a(-\sin\theta)]^2 + (a\cos\theta)^2 + 0}\, d\theta$$

$$= a^{2n} \cdot a \cdot 2\pi$$

$$= 2\pi \cdot a^{2n+1}$$

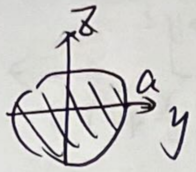
(11) $\int_L x^2 ds = \frac{1}{3}\int_L (x^2+y^2+z^2) ds = \frac{a^2}{3}\int_L ds = \frac{a^2}{3}\cdot(2\pi a) = \frac{2}{3}\pi a^3$

3. $\rho(t) = \frac{2}{x^2+y^2+z^2} = \frac{2}{2e^{2t}} = \frac{1}{e^{2t}} = e^{-2t}$

$$m = \int_0^{t_0} \rho(t)\cdot ds = \int_0^{t_0} e^{-2t} \cdot \sqrt{3e^{2t}}\, dt$$

$$= \sqrt{3}\int_0^{t_0} e^{-t} dt = \sqrt{3}(1 - e^{t_0})$$

习题 11.2

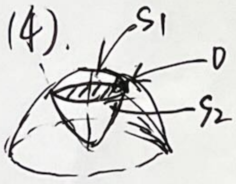
1.(3) 设 $S_1: x=\sqrt{a^2-y^2}$, $(y,z)\in D$, D: 

$$S=2S_1=2\iint_D\sqrt{1+\left(\frac{-y}{\sqrt{a^2-y^2}}\right)^2+0}\,dydz$$

$$=2\iint_D\frac{a}{\sqrt{a^2-y^2}}dydz=2a\int_{-a}^{a}\frac{1}{\sqrt{a^2-y^2}}dy\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}dz$$

$$\cancel{=2\iint_{D'}\frac{a}{\sqrt{a^2-r}}}$$

$$=4a\int_{-a}^{a}dy=8a^2$$

(4). 

$$D:\begin{cases}x^2+y^2+z^2=3a^2\\ x^2+y^2=2az\end{cases}\Rightarrow\begin{cases}z=a\\ x^2+y^2=2a^2\end{cases}$$

$S_1: z=\sqrt{3a^2-x^2-y^2}$, $(x,y)\in D$

$S_2: z=\dfrac{x^2+y^2}{2a}$, $(x,y)\in D$

$$S_1=\iint_D\sqrt{1+\left(\frac{-x}{\sqrt{3a^2-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{3a^2-x^2-y^2}}\right)^2}\,dxdy$$

$$=\iint_D\frac{\sqrt{3}a}{\sqrt{3a^2-x^2-y^2}}dxdy$$

$$=\iint_{D'}\frac{\sqrt{3}a}{\sqrt{3a^2-r^2}}rdrd\theta=\sqrt{3}a\cdot\frac{1}{2}\cdot\int_0^{\sqrt{2}a}\frac{1}{\sqrt{3a^2-r^2}}dr^2\cdot\int_0^{2\pi}d\theta$$

$$=(6-2\sqrt{3})\pi a^2$$

$$S_2=\iint_D\sqrt{1+\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2}\,dxdy=\iint_D\sqrt{1+\frac{x^2+y^2}{a^2}}\,dxdy$$

$$=\iint_{D'}\sqrt{1+\frac{r^2}{a^2}}\,rdrd\theta=\frac{1}{2}\int_0^{\sqrt{2}a}\sqrt{1+\frac{r^2}{a^2}}\,d(r^2)\cdot\int_0^{2\pi}d\theta$$

$$=\frac{1}{2}\cdot\frac{2}{3}a^2(3\sqrt{3}-1)\cdot 2\pi$$

$$=\left(2\sqrt{3}-\frac{2}{3}\right)\pi a^2$$

$$S=S_1+S_2=\frac{16}{3}\pi a^2$$

(16). $\begin{cases} z^2=x^2+y^2 \\ z=\sqrt{2}\left(\frac{x}{2}+1\right) \end{cases} \Rightarrow (x-2)^2+2y^2=8 \Rightarrow \frac{(x-2)^2}{8}+\frac{y^2}{4}=1$

$S: \ z=\sqrt{x^2+y^2}, \ (x,y)\in D.$

$S=\iint\limits_D \sqrt{1+\frac{x^2+y^2}{x^2+y^2}}\,dxdy$

$=\sqrt{2}\cdot\iint\limits_D dxdy$

$=\sqrt{2}\cdot\pi\cdot 2\sqrt{2}\cdot 2$

$=8\pi$

(18) 令 $\begin{cases} x=r\sin\theta\cos\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\theta \end{cases}$ $\quad \begin{cases} x=a\sin^2\theta\cos\varphi\sqrt{\sin 2\varphi} \\ y=a\sin^2\theta\sin\varphi\sqrt{\sin 2\varphi} \\ z=a\sin\theta\cos\theta\sqrt{\sin 2\varphi} \end{cases}$

曲面: $r=a\sin\theta\sqrt{\sin 2\varphi}$ $\quad \theta\in[0,\pi]$.

$\sin 2\varphi\geq 0 \Rightarrow \varphi\in[0,\frac{\pi}{2}]\cup[\pi,\frac{3}{2}\pi]$

令 $D_1=\{(\theta,\varphi)\mid\theta\in[0,\frac{\pi}{2}],\varphi\in[0,\frac{\pi}{2}]\}$.

$E=x_\theta'^2+y_\theta'^2+z_\theta'^2$

$=(a\sin 2\theta\cos\varphi\sqrt{\sin 2\varphi})^2+(a\sin 2\theta\sin\varphi\sqrt{\sin 2\varphi})^2+(a\cos 2\theta\sqrt{\sin 2\varphi})^2=a^2\sin 2\varphi$

$F=x_\varphi'x_\theta'+y_\varphi'y_\theta'+z_\varphi'z_\theta'=a^2\sin 2\theta\sin^2\theta\cos\varphi\cos 3\varphi+a^2\sin 2\theta\sin^2\theta\sin\varphi\sin 3\varphi$
$+a^2\cos 2\theta\sin\theta\cos\theta\cos 2\varphi$

$=a^2\sin\theta\cos\theta\cos 2\varphi$

~~$G=x_\theta'x_\varphi'+y_\theta'y_\varphi'+z_\theta'z_\varphi'$~~

$G=x_\varphi'^2+y_\varphi'^2+z_\varphi'^2$

$=\left(\frac{a\sin^2\theta\cos 3\varphi}{\sqrt{\sin 2\varphi}}\right)^2+\left(\frac{a\sin^2\theta\sin 3\varphi}{\sqrt{\sin 2\varphi}}\right)^2+\left(\frac{a\sin\theta\cos\theta\cos 2\varphi}{\sqrt{\sin 2\varphi}}\right)^2$

$=\frac{a^2}{\sin 2\varphi}(\sin^4\theta+\sin^2\theta\cos^2\theta\cos^2 2\varphi)$

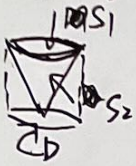
$\sqrt{EG-F^2}=\sqrt{a^4(\sin^4\theta+\sin^2\theta\cos^2\theta\cos^2 2\varphi)-a^4\sin^2\theta\cos^2\theta\cos^2 2\varphi}$

$=a^2\sin^2\theta$

故 $S=4\iint\limits_{D_1}d\sigma=4a^2\iint\limits_{D_1}\sin^2\theta\,d\theta d\varphi=4a^2\int_0^{\frac{\pi}{2}}\sin^2\theta\,d\theta\int_0^{\frac{\pi}{2}}d\varphi=4a^2\cdot\frac{\pi}{4}\cdot\frac{\pi}{2}$

$=\frac{\pi^2a^2}{2}$

2. (3). 解: D: $x^2+y^2\le 1$. $S_1: z=1$, $S_2: z=\sqrt{x^2+y^2}$.

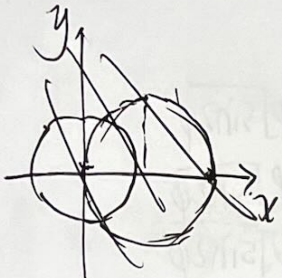


$$\iint_S (x^2+y^2)\,dS = \iint_D (x^2+y^2)\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dxdy + \iint_{S_1}(x^2+y^2)\,dS$$

$$=\sqrt{2}\iint_D (x^2+y^2)\,dxdy + \iint_D (x^2+y^2)\,dxdy$$

$$=(1+\sqrt{2})\iint_{D'} r^2\cdot r\,dr\,d\theta$$

$$=(1+\sqrt{2})\cdot\int_0^1 r^3\,dr\cdot\int_0^{2\pi}d\theta$$



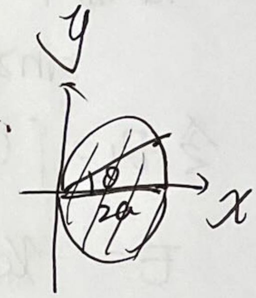
$$=\frac{1}{2}(1+\sqrt{2})\cdot\pi$$

(4): D: $(x-a)^2+y^2\le a^2$.

S: $z=\sqrt{x^2+y^2}$, $(x,y)\in D$.

~~x = a + r cosθ~~

~~y = a + r sinθ~~



$$\iint_S (xy+yz+zx)\,dS = \iint_D \left[xy+(x+y)\sqrt{x^2+y^2}\right]\cdot\sqrt{2}\cdot dxdy$$

对称性

$$=\sqrt{2}\iint_D (x+y)\sqrt{x^2+y^2}\,dxdy$$

D': $0\le r\le 2a\cos\theta$

$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$$=\sqrt{2}\iint_{D'} r(\sin\theta+\cos\theta)\,r\cdot r\,dr\,d\theta$$

$$=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin\theta+\cos\theta)\int_0^{2a\cos\theta} r^3\,dr\,d\theta$$

$$=4\sqrt{2}a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta(\sin\theta+\cos\theta)\,d\theta$$

$$=4\sqrt{2}a^4\cdot 2\int_0^{\frac{\pi}{2}}\cos^5\theta\,d\theta$$

$$=4\sqrt{2}\cdot a^4\cdot 2\cdot\frac{4!!}{5!!}$$

$$=\frac{64\sqrt{2}}{15}a^4$$

(5). $D: (x-1)^2+y^2\le 1$.

$S: z=\sqrt{x^2+y^2}, (x,y)\in D$.

$$\iint_S (x^4-y^4+y^2z^2-x^2z^2+1)\,dS$$

$$=\iint_S\left[(x^2+y^2)(x^2-y^2)-(x^2+y^2)(x^2-y^2)+1\right]dS$$

$$=\iint_S dS$$

$$=\sqrt{2}\iint_D dS=\sqrt{2}\pi$$

(6) $\cancel{\iint D: x^2+y^2\le R^2}$.

$$\iint_S \frac{dS}{r^2}=\iint_S\frac{dS}{x^2+y^2+z^2}=\iint_S\frac{dS}{R^2+z^2}=\int_0^H\frac{(2\pi R)dz}{R^2+z^2}$$

$$=(2\pi R)\cdot\frac{1}{R}\arctan\frac{H}{R}$$

$$=2\pi\arctan\frac{H}{R}$$

(7) $S: z=x^2+y^2, (x,y)\in D$.

D_1为本圆.

$D: x^2+y^2\le 1$.

$$\iint_S |xyz|\,dS=4\iint_{D_1}xy(x^2+y^2)\sqrt{1+(2x)^2+(2y)^2}\,dxdy$$

$$=4\iint_{D_1}xy(x^2+y^2)\sqrt{1+4(x^2+y^2)}\,dxdy$$

$$=4\iint_{D_1'}r^2\sin\theta\cos\theta\cdot r^2\cdot\sqrt{1+4r^2}\,r\,dr\,d\theta$$

$$=4\int_0^1 r^5\sqrt{1+4r^2}\,dr\cdot\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta$$

$$=4\cdot\frac{125\sqrt{5}-1}{840}\cdot\frac{1}{2}=\frac{125\sqrt{5}-1}{420}$$

注:

$$\int_0^1 r^5\sqrt{1+4r^2}\,dr$$

$$\xlongequal{t=\sqrt{1+4r^2}}\frac{1}{64}\int_1^{\sqrt{5}}t^2(t^2-1)^2\,dt$$

$$=\frac{1}{64}\int_1^{\sqrt{5}}(t^6-2t^4+t^2)\,dt$$

$$=\frac{25}{168}\sqrt{5}-\frac{1}{840}$$

$$=\frac{125\sqrt{5}-1}{840}.$$

3(1): $\iint\limits_S (x^2+y^2)ds$

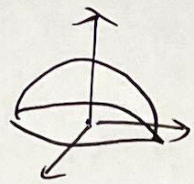
$= \frac{2}{3}\iint\limits_S (x^2+y^2+z^2)ds$

$= \frac{2}{3}R^2\cdot \iint\limits_S ds$

$= \frac{2}{3}R^2\cdot 4\pi R^2$

$= \frac{8}{3}\pi R^4$

(2). $\iint\limits_S (x+y+z)ds \overset{对称性}{=} \iint\limits_S z\,ds$



$= \iint\limits_D \sqrt{a^2-x^2-y^2}\cdot\sqrt{1+\left(\frac{x}{\sqrt{a^2-x^2-y^2}}\right)^2+\left(\frac{y}{\sqrt{a^2-x^2-y^2}}\right)^2}\,dxdy$

$= a\iint\limits_D dxdy$

$= \pi a^3$

4. $G_1: z = -\frac{1}{c}(Ax+By+D)$

$S_G = \iint\limits_{D_{xy}} \cancel{\frac{1}{c}}\sqrt{1+z_x'^2+z_y'^2}\,dxdy = \iint\limits_{D_{xy}} \frac{\sqrt{A^2+B^2+C^2}}{C}dxdy$

$= \sqrt{\frac{A^2+B^2+C^2}{C^2}}\iint\limits_{D_{xy}} dxdy$

$= \sqrt{\frac{A^2+B^2+C^2}{C^2}} \times S_{G_1}$

得证.

5. $m = \iint\limits_S \rho\,ds = \iint\limits_S z\,ds = \frac{1}{2}\iint\limits_D (x^2+y^2)\sqrt{1+x^2+y^2}\,dxdy$

$= \frac{1}{2}\int_0^{\sqrt{2}} r^3\sqrt{1+r^2}\,dr\cdot\int_0^{2\pi}d\theta$

$= \frac{\pi}{2}\int_0^{\sqrt{2}} r^2\sqrt{1+r^2}\,dr^2$

$= \frac{\pi}{\cancel{2}}\int_1^{\sqrt{3}}(t^2-1)t(\cancel{2}t)dt$

$= \pi\int_1^{\sqrt{3}}(t^4-t^2)dt$

$= \frac{12\sqrt{3}+2}{15}\pi$