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8. Find bases for the eigenspaces of the matrices in Exercise 6.

(e) $\begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}$

$\lambda = 2$ $\begin{bmatrix} -3 & 0 & -1 \\ -1 & 1 & 0 \\ 7 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $x_1 = t$, $x_2 = t$, $x_3 = -3t$

$\det(\lambda I_3 - A) = 0$

$$\begin{vmatrix} \lambda-5 & 0 & -1 \\ -1 & \lambda-1 & 0 \\ 7 & -1 & \lambda \end{vmatrix} = (-1)[1-7(\lambda-1)] + \lambda(\lambda-1)(\lambda-5) = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

so $\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 2$

10. Find the eigenvalues of the matrices in Exercise 9.

(a) $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\lambda_1 = 1$ $\lambda_2 = -1$ $\lambda_3 = -2$

$= (\lambda-1)^2(\lambda+1)(\lambda+2)$

$\det(\lambda I_4 - A) = 0$

$$\begin{vmatrix} \lambda & 0 & -2 & 0 \\ -1 & \lambda & -1 & 0 \\ 0 & -1 & \lambda+2 & 0 \\ 0 & 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda & 0 & -2 \\ -1 & \lambda & -1 \\ 0 & -1 & \lambda+2 \end{vmatrix} = (\lambda-1)\left[\lambda[\lambda(\lambda+2)-1] - 2\right]$$

14. Find the eigenvalues and bases for the eigenspaces of A^{25} for

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

(2) $\lambda_2 = 1$ $\begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

(1) $\lambda_1 = -1$ $\begin{bmatrix} 0 & 2 & 2 \\ -1 & -3 & -1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

$\det(\lambda I_3 - A) = 0$

$$\begin{vmatrix} \lambda+1 & 2 & 2 \\ -1 & \lambda-2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} = (\lambda+1)[\lambda(\lambda-2)+1] + 2\lambda - 2 + [-2 - 2(\lambda-2)]$$

$= (\lambda+1)(\lambda^2 - 2\lambda + 1) + 2\lambda - 2 - 2 - 2\lambda + 4$

$= (\lambda+1)(\lambda-1)^2$ $\lambda_1 = -1$ $\lambda_2 = 1$

so the eigenvalues are -1 and 1 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ is a basis corresponding to $\lambda=-1$ $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are bases corresponding to $\lambda=1$

16. Find $\det(A)$ given that A has $p(\lambda)$ as its characteristic polynomial.

(a) $p(\lambda)=\lambda^3-2\lambda^2+\lambda+5$

Setting $\lambda=0$.

$\det(-A)=5$

that is $(-1)^3\det(A)=5$

so $\det(A)=-5$

18. Show that the characteristic equation of a 2×2 matrix A can be expressed as $\lambda^2-\operatorname{tr}(A)\lambda+\det(A)=0$, where $\operatorname{tr}(A)$ is the trace of A.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det(\lambda I_2-A)=\begin{vmatrix} \lambda-a & -b \\ -c & \lambda-d \end{vmatrix}=(\lambda-a)(\lambda-d)-bc$

$=\lambda^2-(a+d)\lambda+(ad-bc)$

$=0$

since $\operatorname{tr}(A)=a+d$ $\det(A)=ad-bc$

so $\det(\lambda I_2-A)=\lambda^2-\operatorname{tr}(A)\lambda+\det(A)=0$.

23. Prove: If λ is an eigenvalue of an invertible matrix A, and **x** is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1}, and **x** is a corresponding eigenvector.

$(\frac{1}{\lambda}I-A^{-1})x$

$=(I-\lambda A^{-1})x$

$=(A-\lambda I)x$

$=-(\lambda I-A)x$

since λ is an eigenvalue of an invertible matrix A and x is a corresponding eigenvector.

then $(A-\lambda I)x=0$ so $(\frac{1}{\lambda}I-A^{-1})x=0$.

(a) $1, \frac{1}{2}, \frac{1}{3}$ (c) $3, 4, 5$

(b) $-2, -1, 0$ all the bases are $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ respectively

26. Find the eigenvalues and bases for the eigenspaces of

$$A = \begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

and then use Exercises 23 and 24 to find the eigenvalues and bases for the eigenspaces of

(a) A^{-1}

(b) $A - 3I$

(c) $A + 2I$

$\det(\lambda I_3 - A) = \begin{vmatrix} \lambda+2 & -2 & -3 \\ 2 & \lambda-3 & -2 \\ 4 & -2 & \lambda-5 \end{vmatrix} = (\lambda+2)[(\lambda-3)(\lambda-5)-4] + 2[2(\lambda-5)+6] + 4[4+3(\lambda-3)] = 0$ $\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 3$

$\begin{bmatrix} 3 & -2 & -3 \\ 2 & -2 & -2 \\ 4 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 4 & -2 & -3 \\ 2 & -1 & -2 \\ 4 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 5 & -2 & -3 \\ 2 & 0 & -2 \\ 4 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

27. (a) Prove that if A is a square matrix, then A and A^T have the same eigenvalues. [*Hint:* Look at the characteristic equation $\det(\lambda I - A) = 0$.]

$\det(\lambda I - A) = \det(\lambda I) - \det(A)$

$\det(\lambda I - A^T) = \det(\lambda I) - \det(A^T)$

since $\det(A) = \det(A^T)$

the $\det(\lambda I - A)$, $\det(\lambda I - A^T)$ has the same solution of λ.

28. Suppose that the characteristic polynomial of some matrix A is found to be $p(\lambda) = (\lambda-1)(\lambda-3)^2(\lambda-4)^3$. In each part, answer the question and explain your reasoning.

(a) What is the size of A?

(b) Is A invertible?

(c) How many eigenspaces does A have?

(a) 6×6

(b) Yes

(c) 3