数分QUIZ 2022/5/26

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1. (15 points) Calculate

$$\iint\limits_S x\mathrm{d}S,$$

where S is the surface bounded by（围成）cylindrical surface $x^2+y^2=1$,and plane $z=0$,$z=2+x$.

$x=\cos\theta,\ y=\sin\theta,\ z=z\quad (0\le\theta\le 2\pi,\ z\le 2+x=2+\cos\theta)$

$$r_\theta\times r_x=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -\sin\theta & \cos\theta & 0\\ 1 & 0 & 0\end{vmatrix}=-\cos\theta\vec{k}\qquad |r_\theta\times r_x|=|\cos\theta|$$

$$\iint_{S_1} x\,dS=\iint \cos^2\theta\,d\theta\,dz=\int_0^{2\pi}\frac{1}{2}\cos 2\theta+\frac{1}{2}d\theta d\theta\int_2^3 dz=\pi$$

$$\iint_{S_2} x\,dS=\int_0^1 x\,dx\int_0^{\sqrt{1-x^2}}dy=\frac{2\pi}{3}$$

$$\iint_{S_3} x\,dS=\iint x\sqrt{1+1+0}\,dx\,dy=\sqrt{2}\pi$$

$$\therefore \iint_S x\,dS=\left(\frac{5}{3}+\sqrt{2}\right)\pi$$

2. (15 points) Calculate

$$\iint\limits_S \frac{x\mathrm{d}y\mathrm{d}z+z^2\mathrm{d}x\mathrm{d}y}{x^2+y^2+z^2},$$

where S is the outside surface bounded by（围成）cylindrical surface $x^2+y^2=R^2$,and plane $z=-R$,$z=R$.

设 S 的上.下底和圆柱面分别为 S_1, S_2, S_3

$$\iint_{S_1}\frac{x\,dy\,dz}{x^2+y^2+z^2}=\iint_{S_2}\frac{x\,dy\,dz}{x^2+y^2+z^2}=0$$

设 S_1, S_2 在 xOy 平面上的投影区域为 D

$$\iint_{S_1+S_2}\frac{z^2dx\,dy}{x^2+y^2+z^2}=\iint_D\frac{R^2dx\,dy}{x^2+y^2+R^2}-\iint_D\frac{(-R)^2dx\,dy}{x^2+y^2+R^2}=0$$

记 S_3 在 yOz 平面上的投影区域为 D'

$$\iint_{S_3}\frac{x\,dy\,dz}{x^2+y^2+z^2}=\iint_{D'}\frac{\sqrt{R^2-y^2}\,dy\,dz}{R^2+z^2}-\iint_{D'}\frac{-\sqrt{R^2-y^2}\,dy\,dz}{R^2+z^2}$$

$$=2\iint_{D'}\frac{\sqrt{R^2-y^2}\,dy\,dz}{R^2+z^2}=2\int_{-R}^{R}\sqrt{R^2-y^2}\,dy\int_{-R}^{R}\frac{dz}{R^2+z^2}=\frac{\pi^2R}{2}$$

$$\therefore \iint_S\frac{x\,dy\,dz+z^2dx\,dy}{x^2+y^2+z^2}=\frac{\pi^2R}{2}$$

3. (15 points) Calculate

$$\iiint\limits_V |y| \,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

$$V = \left\{ (x,y,z) \middle| \begin{array}{c} x^2+y^2+z^2 \le 1 \\ z \ge 0 \\ y^2 \ge 2xz \end{array} \right\}.$$

$x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

V: $0 \le r \le 1$, $0 \le \theta \le \frac{\pi}{2}$ $\varphi \in [0, 2\pi]$

$$\iiint\limits_V |y| dxdydz = 2\iiint y\, dxdydz$$

$$= 2\int_0^1 r dr \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{\pi} \sin\varphi\, d\varphi$$

$$= 2 \times \frac{1}{2} \times 1 \times 2$$

$$= 2$$

4. (15 points) Suppose $f(x,y)$ is not constant to（不恒为）0 and has continous partial derivative on $D = \{(x,y)\,| x^2+y^2 \le a^2\}$, and $f(x,y)|_{\partial D} = 0$.
Prove that:

$$\iint\limits_D f^2(x,y)\,\mathrm{d}x\mathrm{d}y \le a^2 \iint\limits_D \|\nabla f\|^2 \mathrm{d}x\mathrm{d}y$$

令 $P = \frac{1}{a} x f(x,y)$ $Q = \frac{1}{a} y f(x,y)$

则 $\frac{\partial P}{\partial x} = \frac{1}{a} f(x,y) + \frac{1}{2} x f_x'(x,y)$ $\frac{\partial Q}{\partial y} = \frac{1}{a} f(x,y) + \frac{1}{2} y f_y'(x,y)$

$$\oint_{\partial D} P dy + Q dx = \iint_D f(x,y) + \frac{1}{a}(x f_x' + y f_y')\, dxdy$$

又 $f(x,y)|_{\partial D} = 0$ $\therefore \iint_D f(x,y) + \frac{1}{a}(x f_x' + y f_y')\, dxdy = 0$

$$\Rightarrow \iint_D f^2(x,y) dxdy = \left[-\frac{1}{a}\iint_D (x f_x' + y f_y')\, dxdy\right]^2$$

$$= \frac{1}{a^2}\iint_D (x f_x' + y f_y')^2 dxdy$$

$$\le \frac{1}{a^2}\iint_D (x^2+y^2)(f_x'^2 + f_y'^2) dxdy = a^2 \iint_D \|\nabla f\|^2 dxdy$$