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习题12.2

1. $a_0 = \frac{2}{\pi}\int_0^a 1\cdot dx = \frac{2a}{\pi}$

$a_n = \frac{2}{\pi}\int_0^a \cos nx\, dx = \frac{2}{n\pi}\sin na$

$$f(x) \sim \frac{a}{\pi} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin na\cos nx = \begin{cases} 1, & |x|<a \\ \frac{1}{2}, & |x|=a \\ 0, & a\le |x|<\pi \end{cases}$$

(1) $f\in L^2[-\pi,\pi]$，由Parseval等式：

$$\frac{1}{2}\left(\frac{2a}{\pi}\right)^2 + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\sin na\right)^2 = \frac{1}{\pi}\int_{-a}^{a} f^2(x)dx = \frac{2a}{\pi}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{\sin^2 na}{n^2} = \frac{a\pi}{2} - \frac{1}{2}a^2$$

(2) $$\sum_{n=1}^{\infty}\frac{\cos^2 na}{n^2} = \sum_{n=1}^{\infty}\frac{1-\sin^2 na}{n^2}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{\sin^2 na}{n^2}$$

$$= \frac{\pi^2}{6} - \frac{1}{2}a\pi + \frac{1}{2}a^2$$

2. $f\in L^2[-\pi,\pi]$，由Besel不等式：

$\sum_{n=1}^{\infty}(a_n^2+b_n^2)$ 收敛

又 $a_n^2\ge 0$，$b_n^2\ge 0$，故 $\sum_{n=1}^{\infty}a_n^2$，$\sum_{n=1}^{\infty}b_n^2$ 有界.

即 $\sum_{n=1}^{\infty}a_n^2\le M_1$，$\sum_{n=1}^{\infty}b_n^2\le M_2$. 其中 $0<M_1, M_2<+\infty$

由Cauchy不等式：

$$\left(\sum_{k=1}^{n}\left|\frac{a_k}{k}\right|\right)^2 \le \sum_{k=1}^{n}|a_k|^2\cdot\sum_{k=1}^{n}\left|\frac{1}{k^2}\right|^2 = \sum_{k=1}^{n}a_k^2\cdot\sum_{k=1}^{n}\frac{1}{k^2} < M_1\cdot\frac{\pi^2}{6}$$

$\sum_{k=1}^{\infty}\left|\frac{a_k}{k}\right|$ 为正项级数，部分和有上界，故收敛.

从而 $\sum_{n=1}^{\infty}\frac{a_n}{n}$ 收敛；同理可证得 $\sum_{n=1}^{\infty}\frac{b_n}{n}$ 收敛

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3. $b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx = \frac{2}{n\pi}(1-(-1)^n)$

$\Rightarrow f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n}\sin nx = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$

由 $f \in L^2[-\pi,\pi]$：

$$\frac{16}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}\right)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = 2$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

又 $\int_0^x f(x)\,dx = \int_0^x \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$.

由 Dirichlet 定理，Fourier 级数在 $(-\pi,\pi)$ 内闭一致收敛于 $f(x)$.

故 $\int_0^x f(x)\,dx = \int_0^x \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1} = \frac{4}{\pi}\sum_{n=1}^{\infty}\int_0^x \frac{\sin(2n-1)x}{2n-1}$

$$= \frac{4}{\pi}\left(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} - \sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}\right)$$

$$= x$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} = \frac{\pi^2}{8} - \frac{\pi x}{4},\quad 0<x<\pi.$$

再考虑端点，$x=0$ 或 $x=\pi$ 时上式仍成立

故有 $\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} = \frac{\pi^2}{8} - \frac{\pi x}{4}$，$0\le x\le\pi$

4(2) 记 $f_n(x) = \sin\frac{n\pi}{l}x$，$n=0,1,2,\cdots$

$\langle f_m, f_n\rangle = \int_0^l \sin\frac{m\pi}{l}x\,\sin\frac{n\pi}{l}x\,dx$

$= \frac{1}{2}\int_0^l \left(\cos\frac{(m+n)\pi}{l}x - \cos\frac{(m-n)\pi}{l}x\right)dx$

$$= \frac{1}{2}\left(\frac{l}{(m+n)\pi}\left(\sin(m+n)\pi - 0\right) - \frac{l}{(m-n)\pi}\left(\sin(m-n)\pi - 0\right)\right)$$

$$= 0,\quad m \neq n.$$

$$\langle f_n, f_n\rangle = \int_0^l \sin^2 \frac{n\pi}{l}x\,dx$$

$$= \frac{1}{2}\int_0^l \left(1 - \cos\frac{2n\pi}{l}x\right)dx$$

$$= \frac{1}{2}l$$

因此 $\{f_n\}$ 构成正交函数系，且：

$\left\{\tilde{f}_n(x) = \sqrt{\frac{2}{l}}\sin\frac{n\pi}{l}x\right\}$ 构成标准正交系.

5. $f(x) = a(1-\frac{x}{l}),\ 0 \le x \le l.$ 此处为标准正交系.

$$a_n = \int_0^l f(x)\varphi_n(x)\,dx$$

$$= \int_0^l a(1-\frac{x}{l})\sqrt{\frac{2}{l}}\sin\frac{n\pi}{l}x\,dx$$

$$= a\sqrt{\frac{2}{l}}\int_0^l \sin\frac{n\pi}{l}x\,dx - a\sqrt{\frac{2}{l}}\int_0^l x\sin\frac{n\pi}{l}x\,dx$$

$$= a\sqrt{\frac{2}{l}}\frac{l}{n\pi}(1-\cos n\pi) - \frac{a\sqrt{2l}}{n\pi}\frac{\cos n\pi}{\pi}$$

$$= \frac{a\sqrt{2l}}{n\pi}\left(1-(-1)^n - \frac{(-1)}{\pi}\right)$$

$$= \frac{a\sqrt{2l}}{n\pi}\left(1-(-1)^n\frac{\pi+1}{\pi}\right)$$

6. $f(x) = x,\ 0 \le x \le l.\quad \varphi_n(x) = \sqrt{\frac{2}{l}}\cos\frac{(2n+1)\pi}{2l}x.$

$$a_n = \int_0^l f(x)\varphi_n(x)\,dx$$

$$= \sqrt{\frac{2}{l}}\int_0^l x\cos\frac{(2n+1)\pi}{2l}x$$

$$= \sqrt{\frac{2}{l}}\left(\frac{2l^2\cos n\pi}{(1+2n)\pi} - \frac{4l^2}{(1+2n)^2\pi^2}\right)$$

7. 不妨令 $n>m$：

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{1}{2^{m+n}m!\,n!}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n\frac{d^m}{dx^m}(x^2-1)^m\,dx$$

$$= \frac{1}{2^{m+n}m!\,n!}\int_{-1}^{1}\frac{d^m}{dx^m}(x^2-1)^m\,d\left(\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right)$$

$$= \frac{1}{2^{m+n}m!\,n!}\left(\underbrace{\frac{d^m}{dx^m}(x^2-1)^m\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\Big|_{-1}^{1}}_{=0} - \int_{-1}^{1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m\,dx\right)$$

$$= \frac{-1}{2^{m+n}m!\,n!}\int_{-1}^{1}\frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

$\cdots$

$$= \frac{(-1)^m}{2^{m+n}m!\,n!}\int_{-1}^{1}\frac{d^{2m}}{dx^{m+1}}(x^2-1)^m\frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n\,dx$$

$$= \frac{(-1)^m}{2^{m+n}m!\,n!}(2m)!\int_{-1}^{1}\frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n\,dx$$

$$= \left[\frac{(-1)^m}{2^{m+n}m!\,n!}(2m)!\right]\cdot\frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n\Big|_{-1}^{1}$$

$$= 0$$

$\Rightarrow \langle P_n(x), P_m(x)\rangle = 0$，$m\neq n$. 得证正交系、

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8. (1). $\frac{d}{dx}\left((x^2-1)^n\right) = n(x^2-1)^{n-1}(2x)$

$$= 2nx\,(x^2-1)^{n-1}$$

$$\Rightarrow \quad (x^2-1)\frac{d}{dx}\left((x^2-1)^n\right) = 2nx\,(x^2-1)^n$$

两边同时求n+1阶导数;　　$(uv)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}u^{(k)}v^{(n-k)}$

$$LHS = \frac{d^{n+1}}{dx^{n+1}}\left[(x^2-1)\frac{d}{dx}\left((x^2-1)^n\right)\right]$$

$$= \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{d^k}{dx^k}(x^2-1)\,\frac{d^{n-k+2}}{dx^{n-k+2}}(x^2-1)^n$$

$$= (x^2-1)\frac{d^{n+2}}{dx^{n+2}}(x^2-1)^n + 2(n+1)x\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n$$

$$+ n(n+1)\frac{d^n}{dx^n}(x^2-1)^n$$

记 $y(x) = P_n(x)\cdot 2^n\cdot n!$

有 $LHS = (x^2-1)y'' + 2(n+1)xy' + n(n+1)y$

$RHS = (2nx)y' + 2n(n+1)y$

$\therefore\ (x^2-1)y'' + 2(n+1)xy' + n(n+1)y = (2nx)y' + 2n(n+1)y$

$\Rightarrow (1-x^2)y'' - 2xy' + n(n+1)y = 0$

即 $(1-x^2)\frac{d^2P_n(x)}{dx^2} - 2x\frac{dP_n(x)}{dx} + n(n+1)P_n(x) = 0$

(2). $P_n'(x) = \frac{d^{n+1}}{dx^{n+1}}\left(\frac{(x^2-1)^n}{2^n\cdot n!}\right) = \frac{d^n}{dx^n}\left(\frac{2nx\,(x^2-1)^{n-1}}{2^n\cdot n!}\right)$

$$= \frac{d^n}{dx^n}\frac{x(x^2-1)^{n-1}}{2^{n-1}(n-1)!}$$

$$= x P_{n-1}'(x) + n P_{n-1}(x)$$

即 $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$ (*)

求导 $\Rightarrow$

$$P_n''(x) = P_{n-1}'(x) + xP_{n-1}''(x) + nP_{n-1}'(x)$$

$$= xP_{n-1}''(x) + (1+n)P_{n-1}'(x)$$

代入 (1) 微分方程，消 $P_n''(x)$，$P_{n-1}''(x)$.

$$2xP_n' - n(n+1)P_n = x\left[2xP_{n-1}' - n(n-1)P_{n-1}\right] + (1-x^2)(1+n)P_{n-1}'$$

$$= 2x\cdot xP_{n-1}' - n(n-1)xP_{n-1} + (1-x^2)(1+n)P_{n-1}'$$

代入 (*) 消 P_{n-1}'：

$$2xP_n' - n(n+1)P_n = \left(\frac{1+x^2+n(1-x^2)}{x}\right)\left(P_n' - nP_{n-1}\right) - n(n-1)xP_{n-1}$$

$$\Rightarrow (1-x^2)(n+1)P_n' = \left[1+x^2+n(1-x^2)\right]nP_{n-1} + n(n-1)xP_{n-1} - n(n+1)P_n$$

$$\Rightarrow \boxed{\frac{(x^2-1)}{n}\frac{dP_n}{dx} = xP_n - P_{n-1}} \quad (\text{II})$$

由 (II)，消 P_n'，P_{n-1}' 代入 (*).

$$\Rightarrow nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}.$$

令 $n = n+1 \Rightarrow \boxed{(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}}$ (I)

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(4) (12) (*) $\Rightarrow$ $\boxed{(2n+1)P_n=\frac{d}{dx}(P_{n+1}-P_{n-1})}$ (12.2)

12.3

1. $f(x)=\operatorname{sgn}x=\begin{cases}1, & 0<x<\pi\\ 0, & x=0\\ -1, & -\pi<x<0\end{cases}$

由 12.2.3 有: $f(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}, \quad -\pi<x<\pi$

$\Rightarrow \sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}=\frac{\pi}{4}f(x)=\frac{\pi}{4}, \quad 0<x<\pi.$

令 $x=\frac{\pi}{2}$, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}=\frac{\pi}{4}$

2. (1) $a_0=\frac{2}{\pi}\int_0^{\pi}x\,dx=\pi$

$a_n=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx$

$=\frac{2}{\pi}\quad\frac{-1+(-1)^n}{n^2}$

$\Rightarrow f(x)\sim\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x$

(2) $b_n=\frac{2}{\pi}\int_0^{\pi}\sin ax\sin nx\,dx$

$=\frac{2}{\pi}\quad\frac{n\cos n\pi\sin a\pi}{a^2-n^2}$

$=(-1)^n\frac{2}{\pi}\quad\frac{n\sin a\pi}{a^2-n^2}$

$$f(x) \sim \frac{2}{\pi}\sin a\pi \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2-n^2} \sin nx$$

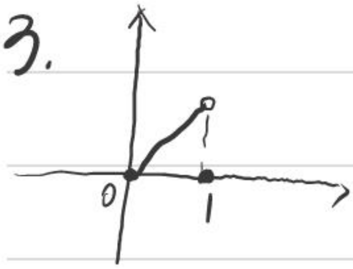
(3) $$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[-\frac{2n + 2n\cos n\pi}{(-1+n^2)^2}\right]$$

$$= -\frac{4n\,[1+(-1)^n]}{(n^2-1)^2\pi}$$

$$f(x) \sim -\frac{16}{\pi} \sum_{n=1}^{\infty} \frac{n}{(4n^2-1)^2\pi} \sin 2nx$$

3.



$$a_0 = \frac{1}{\left(\frac{1}{2}\right)} \int_0^1 f(x)\,dx = 1$$

$$a_n = 2\int_0^1 f(x)\cos 2n\pi x \, dx$$

$$= 2\int_0^1 x\cos 2n\pi x \, dx$$

$$= 2\left[\frac{x}{2n\pi}\sin 2n\pi x + \frac{1}{(2n\pi)^2}\cos 2n\pi x\right]_0^1$$

$$= 0$$

$$b_n = 2\int_0^1 x \sin 2n\pi x \, dx$$

$$= 2\left[-\frac{x}{2n\pi}\cos 2n\pi x + \frac{1}{(2n\pi)^2}\sin 2n\pi x\right]_0^1$$

$$= -\frac{1}{n\pi}$$

$$f(x) \sim \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n} = \begin{cases} x-[x], & x \neq k \\ \frac{1}{2}, & x = k \end{cases}, \quad k \in \mathbb{Z}$$

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4、 $f(x) = \cos ax$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \cos ax\,dx = \frac{2}{a\pi}\sin a\pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos ax \cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(\cos(a+n)x + \cos(a-n)x)\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{a+n}\sin(a+n)\pi + \frac{1}{a-n}\sin(a-n)\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin a\pi\,(-1)^n}{a+n} + \frac{\sin a\pi\,(-1)^n}{a-n}\right]$$

$$= \frac{(-1)^n \sin a\pi\,(2a)}{(a^2-n^2)\pi}$$

$$\Rightarrow \cos ax = \frac{\sin a\pi}{a\pi} + \frac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2-n^2}\cos nx$$

(1) 令 $x=\pi$,

$$\cos a\pi = \frac{\sin a\pi}{a\pi} + \frac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{1}{a^2-n^2}$$

$$\Rightarrow \cot a\pi = \frac{1}{a\pi} + \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{a^2-n^2}$$

$$= \frac{1}{a\pi} + \sum_{n=1}^{\infty}\frac{2a\pi}{a^2\pi^2-n^2\pi^2}$$

令 $a\pi = x$

$$\Rightarrow \cot x = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{2x}{x^2-n^2\pi^2}$$

(2) 令 $x=0$,

$$1 = \frac{\sin a\pi}{a\pi} + \frac{2a\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2-n^2}$$

$$\Rightarrow \frac{1}{\sin a\pi} = \frac{1}{a\pi} + \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2-n^2} = \frac{1}{a\pi} + \sum_{n=1}^{\infty}\frac{(-1)^n 2a\pi}{a^2\pi^2-n^2\pi^2}$$

令 $a\pi = x$, $\frac{1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{(-1)^n\,2x}{x^2-n^2\pi^2}$

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5. $f(x) = |\cos x|$,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |\cos x| dx = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos x dx = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x \cos nx dx - \int_{\frac{\pi}{2}}^{\pi} \cos x \cos nx dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x \cos nx dx - \int_0^{\frac{\pi}{2}} \cos(\pi - t) \cos n(\pi - t) dt\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x \cos nx dx + \int_0^{\frac{\pi}{2}} \cos t \cdot (-1)^n \cos nt \, dt\right]$$

$$= \frac{2}{\pi}(1+(-1)^n)\int_0^{\frac{\pi}{2}} \cos x \cos nx dx$$

$$= \frac{1+(-1)^n}{\pi}\left(\frac{\sin\frac{(n-1)\pi}{2}}{n-1} + \frac{\sin\frac{(n+1)\pi}{2}}{n+1}\right)$$

$$= \frac{4}{\pi}\frac{(-1)^{k-1}}{4k^2-1}$$

$$\Rightarrow |\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}\cos 2nx$$

令 $x = \frac{\pi}{2} - x$

$$\Rightarrow |\sin x| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}\cos(n\pi - 2nx)$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx$$

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6. $f(x) = e^{ax}, \quad x \in (0, 2\pi)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{ax}dx = \frac{1}{a\pi}(e^{2a\pi}-1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cos nx\,dx$$

$$= \frac{(e^{2a\pi}-1)a}{\pi(a^2+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\sin nx\,dx$$

$$= \frac{(1-e^{2ax})n}{\pi(a^2+n^2)}$$

$$e^{ax} = \frac{e^{2a\pi}-1}{\pi}\left(\frac{1}{2a}+\sum_{n=1}^{\infty}\frac{a\cos nx - n\sin nx}{n^2+a^2}\right)$$

7.

$$x = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n}$$

$$x^2 = 2\int_0^x t\,dt$$

$$= 2\int_0^x 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nt}{n}dt$$

$$= 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^x \sin nt\,dt$$

$$= 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}(1-\cos nx)$$

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$$= 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

其中 $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2} - 2\sum_{n=1}^{\infty}\frac{1}{(2n)^2}$$

$$= \frac{1}{2}\frac{\pi^2}{6} = \frac{\pi^2}{12}$$

故 $\boxed{x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx}$ ①

$$x^3 = 3\int_0^x t^2 dt = \pi^2 x + 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^x \cos nt\,dt$$

$$= \pi^2 x + 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nx$$

$$= 2\pi^2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n} + 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nx$$

$$\boxed{x^3 = 2\sum_{n=1}^{\infty}(-1)^{n-1}\left(\pi^2 - \frac{6}{n^2}\right)\frac{\sin nx}{n}}$$ ②

$$x^4 = 4\int_0^x t^3 dt = 8\sum_{n=1}^{\infty}(-1)^{n-1}\left(\pi^2 - \frac{6}{n^2}\right)\int_0^x \frac{\sin nt}{n}dt$$

$$= 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\left(\pi^2 - \frac{6}{n^2}\right)(1 - \cos nx)$$

③

$$\boxed{= 8\sum_{n=1}^{\infty} \pi^2 \frac{(-1)^{n-1}}{n^2} + 6\frac{(-1)^n}{n^4} + \frac{(-1)^n}{n^2}\left(\pi^2 - \frac{6}{n^2}\right)\cos nx}$$

其中 $\sum_{n=1}^{\infty} \pi^2 \frac{(-1)^{n-1}}{n^2} = \pi^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \pi^2 \frac{\pi^2}{12} = \frac{\pi^4}{12}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} = -\left(\sum_{n=1}^{\infty} \frac{1}{n^4} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^4}\right)$$

$$= -\frac{7}{8}\sum_{n=1}^{\infty} \frac{1}{n^4}$$

由③，代入 $x = \pi$

$$\Rightarrow \pi^4 = \frac{2}{3}\pi^4 - 42\sum_{n=1}^{\infty} \frac{1}{n^4} + \sum_{n=1}^{\infty} \frac{8}{n^2}\left(\pi^2 - \frac{6}{n^2}\right)$$

$$\Rightarrow 90\sum_{n=1}^{\infty} \frac{1}{n^4} = 8\pi^2 \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{3}\pi^4$$

$$= 8\pi^2 \cdot \frac{\pi^2}{6} - \frac{1}{3}\pi^4$$

$$= \pi^4$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{90}\pi^4$. 代回 x^4 展开式

$$\boxed{x^4 = \frac{1}{5}\pi^4 + 8\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\left(\pi^2 - \frac{6}{n^2}\right)\cos nx}$$

利用 Parseval 等式:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |x^3|^2 dx = 4\sum_{n=1}^{\infty} \left(\pi^2 - \frac{6}{n^2}\right)^2 \frac{1}{n^2}$$

$$\Rightarrow \frac{2}{7}\pi^6 = 4\pi^4 \cdot \sum_{n=1}^{\infty}\frac{1}{n^2} - 48\pi^2\sum_{n=1}^{\infty}\frac{1}{n^4} + 144\sum_{n=1}^{\infty}\frac{1}{n^6}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{1}{144}\left(\frac{2}{7}\pi^6 - 4\pi^4\cdot\frac{\pi^2}{6} + 48\pi^2\frac{\pi^4}{90}\right)$$

$$= \frac{\pi^6}{945}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(x^4)^2dx = \frac{1}{2}\cdot\left(\frac{2}{5}\pi^4\right)^2 + 64\sum_{n=1}^{\infty}\left(\pi^2-\frac{6}{n^2}\right)^2\frac{1}{n^4}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^8} = \frac{1}{2304}\left(\frac{2}{9}\pi^8 - \frac{2}{25}\pi^8 - 64\pi^4\sum_{n=1}^{\infty}\frac{1}{n^4} + 768\pi^2\sum_{n=1}^{\infty}\frac{1}{n^6}\right)$$

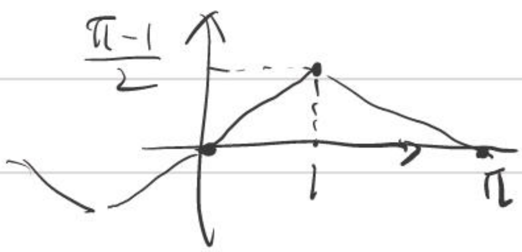
$$= \frac{1}{9450}\pi^8$$

8. $$b_n = \frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^1\left(\frac{\pi-1}{2}\right)x\sin nx\,dx + \int_1^{\pi}\left(\frac{\pi-x}{2}\right)\sin nx\,dx\right]$$

$$= \frac{-(-1+\pi)(n\cos n-\sin n) + n(-1+\pi)\cos n + \sin n}{\pi n^2}$$

$$= \frac{\sin n}{n^2}$$



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$$f(x) \sim \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \sin nx = f(x). \quad (0 \le x \le \pi)$$

9. (1) $f(x) \in C[-\pi, \pi] \Rightarrow$ 級數一致收斂于 f.

$$f'(0) = \frac{\pi-1}{2} = \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \cdot n \cos nx \Big|_{x=0}$$

$$= \sum_{n=1}^{\infty} \frac{\sin n}{n}$$

$$f(1) = \frac{\pi-1}{2} = \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \cdot \sin n = \sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^2$$

(2) 运用 Parseval 等式:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx = \sum_{n=1}^{\infty} \left(\frac{\sin n}{n^2}\right)^2 = \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^4}$$

$$= \frac{1}{6\pi}\left(\pi(-1+\pi)^2\right)$$

$$= \frac{(\pi-1)^2}{6}$$

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