CS101 Algorithms and Data Structures

Topological Sort

Textbook Ch 22.4

Outline

- Topological sorting
  - Definitions
  - Algorithm
- Finding the critical path

Motivation

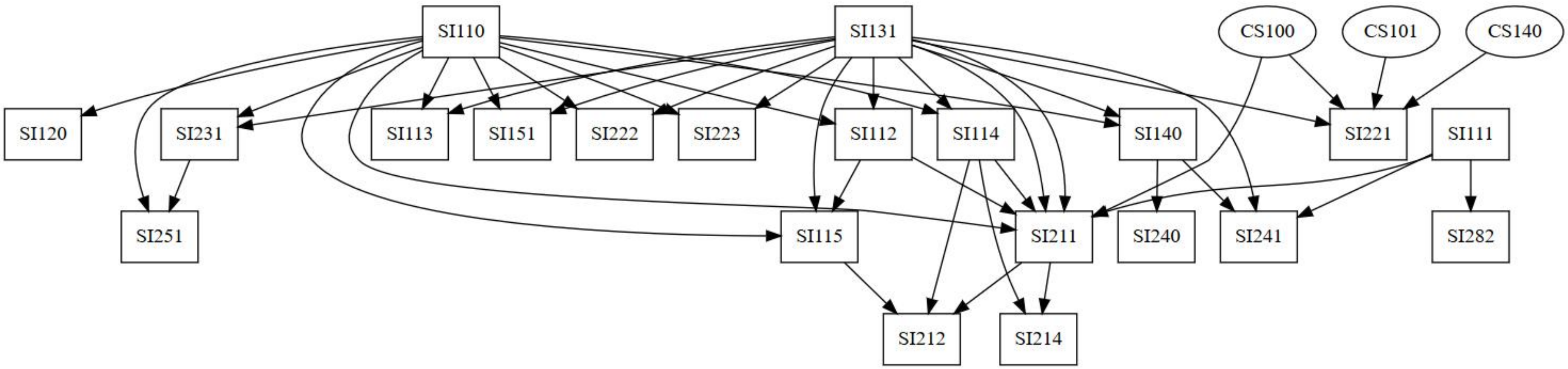
Dependency between tasks: one task is required to be done before the other task can be done

Dependencies form a partial ordering

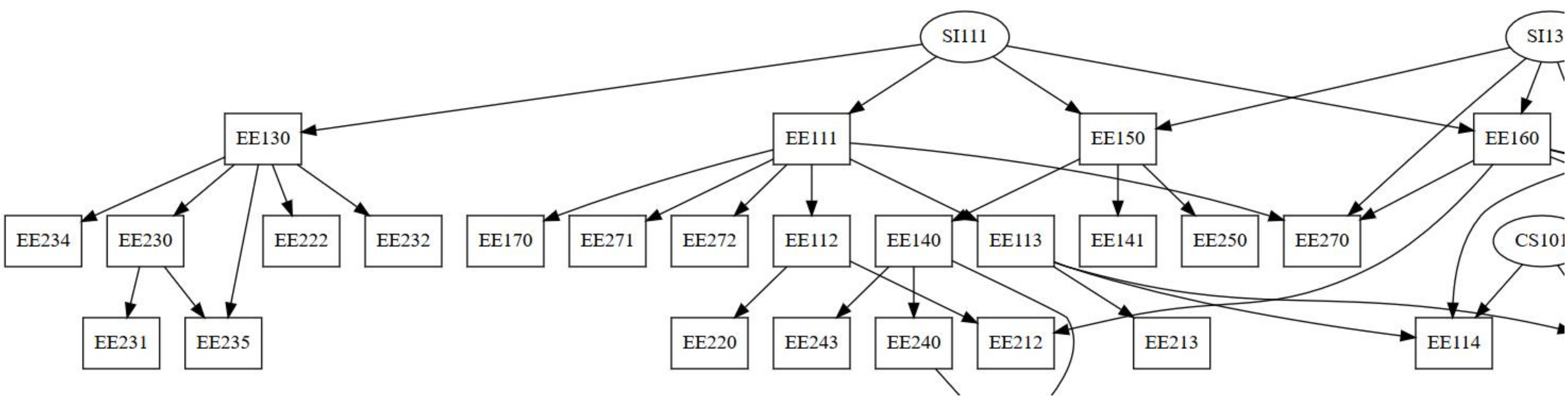
- A partial ordering on a finite number of objects can be represented as a directed acyclic graph (DAG)

Motivation: SIST course curriculum

SI courses

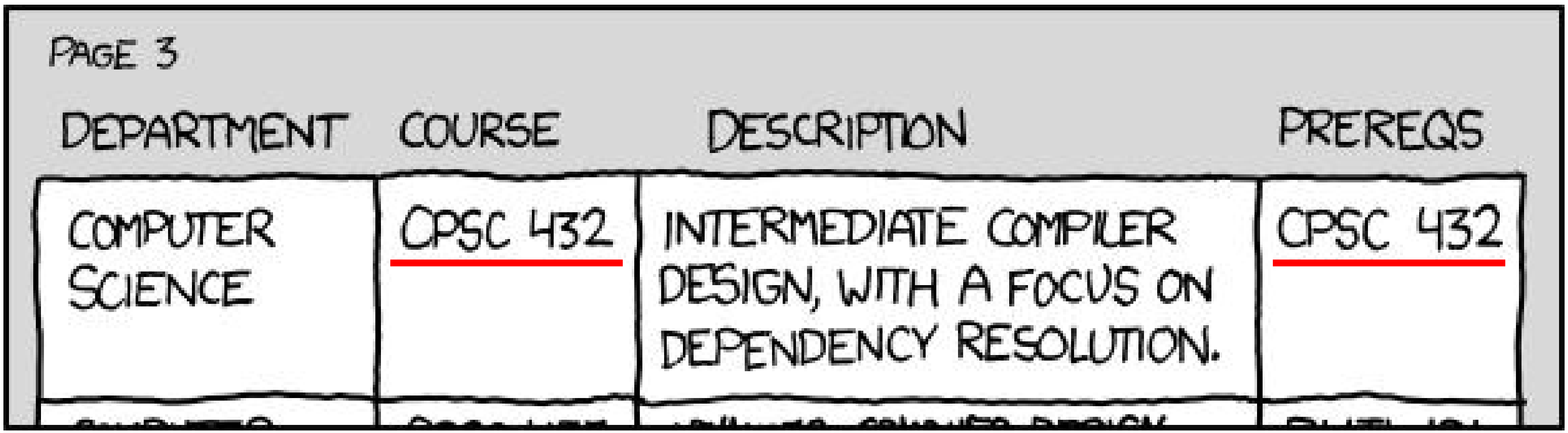


EE courses



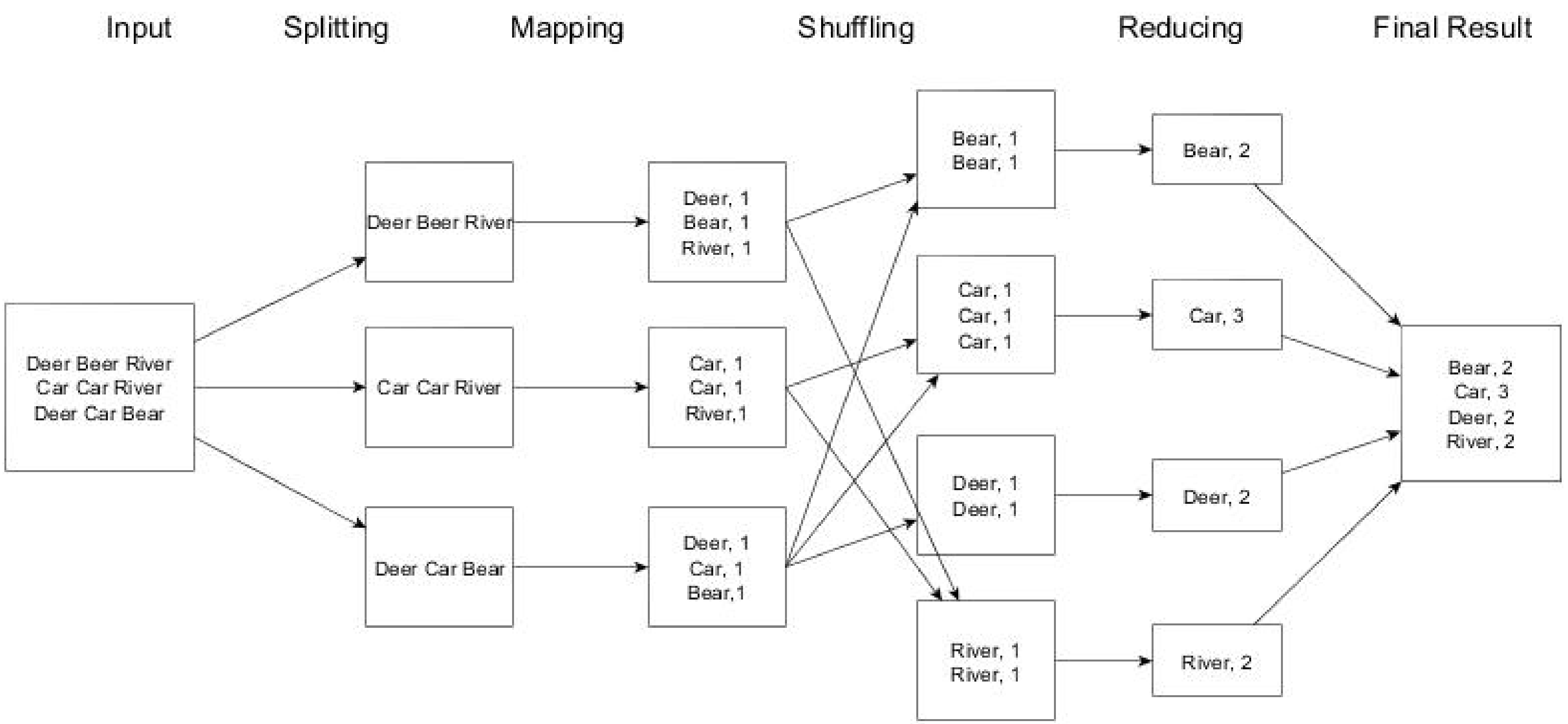
Motivation: SIST course curriculum

Cycles in dependencies can cause issues...



http://xkcd.com/754/

Motivation: word count in MapReduce



Topological sorting

Given a set of tasks with dependencies, is there an order in which we can complete the tasks?

A topological sorting of the vertices in a DAG is an ordering

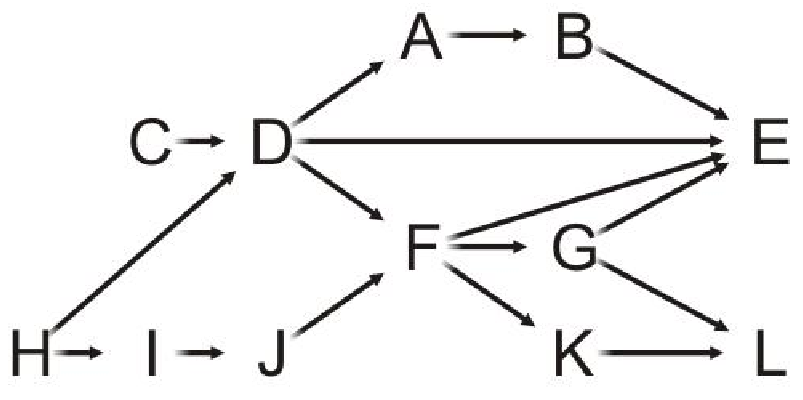
$$v_1, v_2, v_3, \ldots, v_{|V|}$$

such that v_j appears before v_k if there is a path from v_j to v_k

Example

Given this DAG, a topological sort is

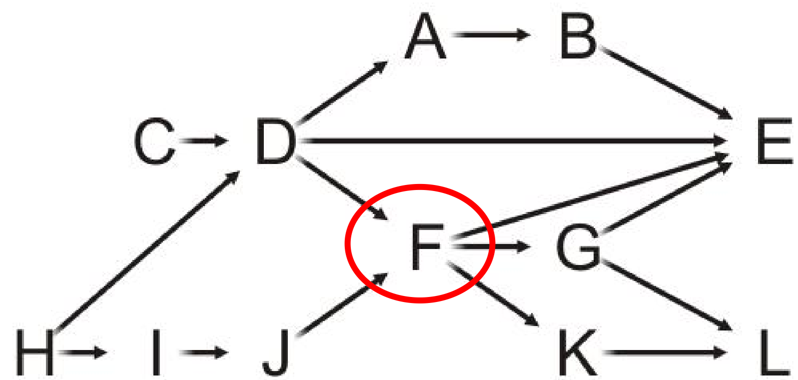
H, C, I, D, J, A, F, B, G, K, E, L



Example

For example, there are paths from H, C, I, D and J to F, so all these must come before F in a topological sort

H, **C**, **I**, **D**, **J**, A, **F**, B, G, K, E, L



Clearly, this sorting need not be unique

Applications

Taking courses

- The courses must be taken in an order such that the prerequisites of a course are taken before that course

Applications

Consider you getting ready for a dinner out

You must wear the following:

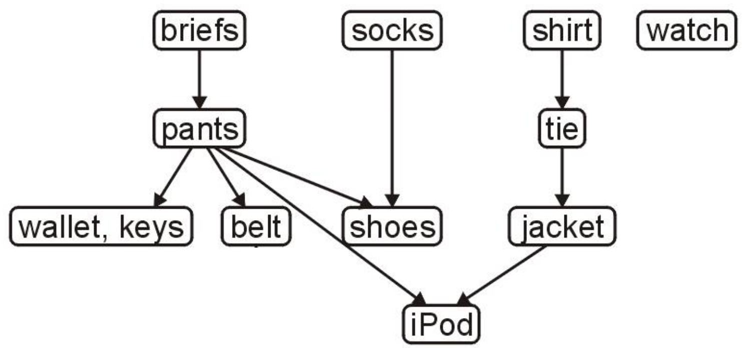
- jacket, shirt, briefs, socks, tie, etc.

There are certain constraints:

- the pants really should go on after the briefs,
- socks are put on before shoes

Applications

The following is a task graph for getting dressed:



Many people would go like this (a possible topological sort):

briefs, shirt, socks, pants, belt, tie, jacket, wallet, keys, iPod, watch, shoes

Another topological sort is:

briefs, pants, wallet, keys, belt, socks, shoes, shirt, tie, jacket, iPod, watch

Applications

C++ header and source files have #include statements

- A change to an included file requires a recompilation of the current file
- On a large project, it is desirable to recompile only those source files that depended on those files which changed
- For large software projects, full compilations may take hours

Topological Sort

Theorem:

A graph is a DAG if and only if it has a topological sorting

Proof strategy:

Such a statement is of the form $a \leftrightarrow b$ and this is equivalent to:

$a \rightarrow b$ and $b \rightarrow a$

Topological Sort

First, we need a two lemmas:

- A DAG always has at least one vertex with in-degree zero
  - That is, it has at least one *source*

Proof by contradiction:

- If we cannot find a vertex with in-degree zero, we will show there must be a cycle
- Start with any vertex and define a list $L = (v)$
- Then iterate this loop $|V|$ times:
  - The first vertex ℓ_1 in the list L does not have in-degree zero
  - So we can find a vertex w such that (w, ℓ_1) is an edge
  - Add w to the list: $L = (w, \ell_1, \ldots, \ell_k)$
- By the pigeon-hole principle, at least one vertex must appear twice
  - This forms a cycle; hence a contradiction, as this is a DAG

Topological Sort

First, we need a two lemmas:

– Any sub-graph of a DAG is a DAG

Proof:

– If a sub-graph has a cycle, that same cycle must appear in the super-graph
– We assumed the super-graph was a DAG
– This is a contradiction

∴ the sub-graph must be a DAG

Topological Sort

We will start with showing $a \rightarrow b$:

If a graph is a DAG, it has a topological sort

Proof by induction:

A graph with one vertex is a DAG and it has a topological sort

Assume a DAG with n vertices has a topological sort

A DAG with $n + 1$ vertices must have at least one vertex v of in-degree zero

Removing the vertex v and consider the vertex-induced sub-graph with the remaining n vertices

- If this sub-graph has a cycle, so would the original graph—contradiction
- Thus, the graph with n vertices is also a DAG, therefore it has a topological sort

Add the vertex v to the start of the topological sort to get one for the graph of size $n + 1$

Topological Sort

Next, we will show that $b \rightarrow a$:

If a graph has a topological ordering, it must be a DAG

We will show this by showing the contrapositive: $\neg a \rightarrow \neg b$:

If a graph is not a DAG, it does not have a topological sort

By definition, it has a cycle: $(v_1, v_2, v_3, \ldots, v_k, v_1)$

- In any topological sort, v_1 must appear before v_2, because (v_1, v_2) is a path
- However, there is also a path from v_2 to v_1: $(v_2, v_3, \ldots, v_k, v_1)$
- Therefore, v_2 must appear in the topological sort before v_1

This is a contradiction, therefore the graph cannot have a topological sort

$\therefore$ $a \leftrightarrow b$: A graph is a DAG if and only if it has a topological sorting

Outline

- Topological sorting
  - Definitions
  - Algorithm
- Finding the critical path

Topological Sort

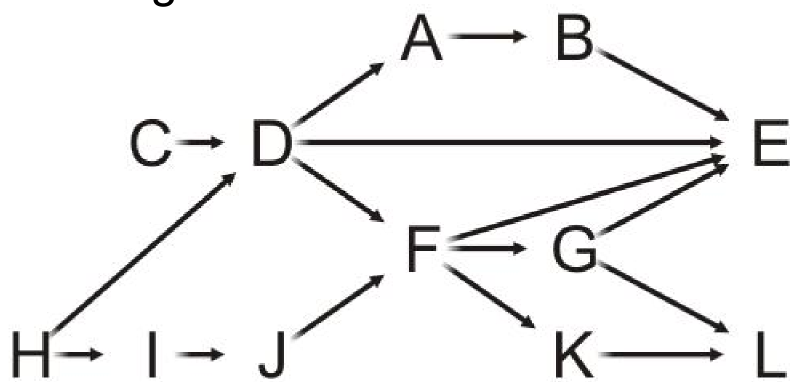
Idea:

- Given a DAG V, iterate:
  - Find a vertex v in V with in-degree zero
  - Let v be the next vertex in the topological sort
  - Continue iterating with the vertex-induced sub-graph $V \setminus \{v\}$

Example

On this graph, iterate the following $|V| = 12$ times

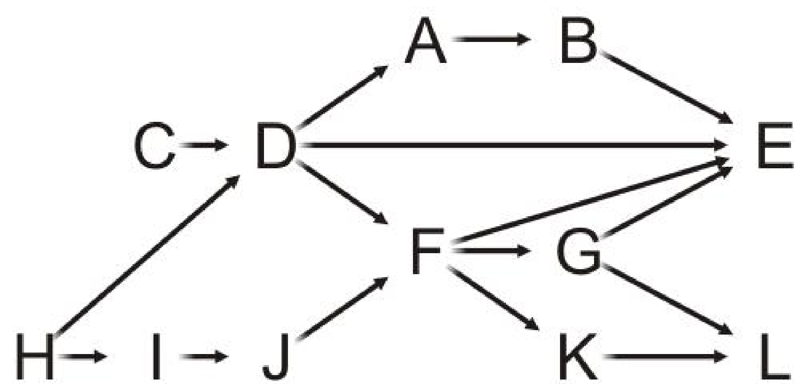
- Choose a vertex v that has in-degree zero
- Let v be the next vertex in our topological sort
- Remove v and all edges connected to it



Example

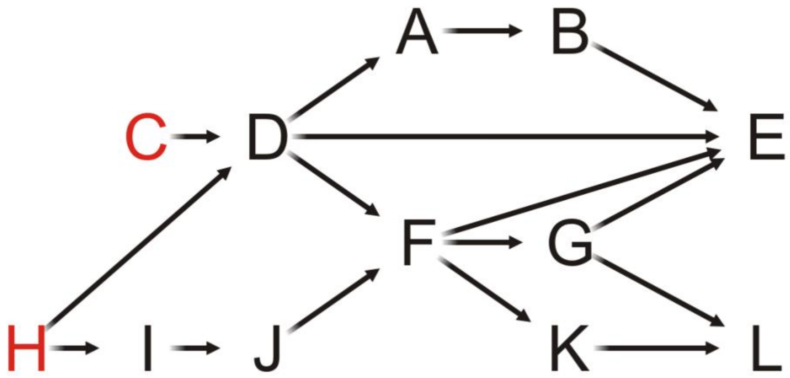
Let’s step through this algorithm with this example

– Which task can we start with?



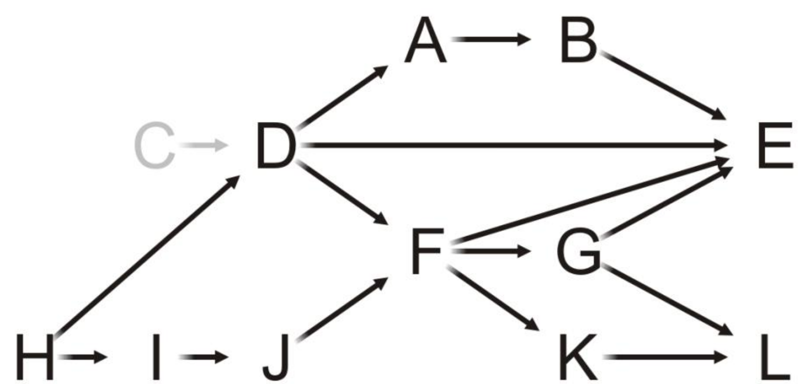
Example

Of Tasks C or H, choose Task C



Example

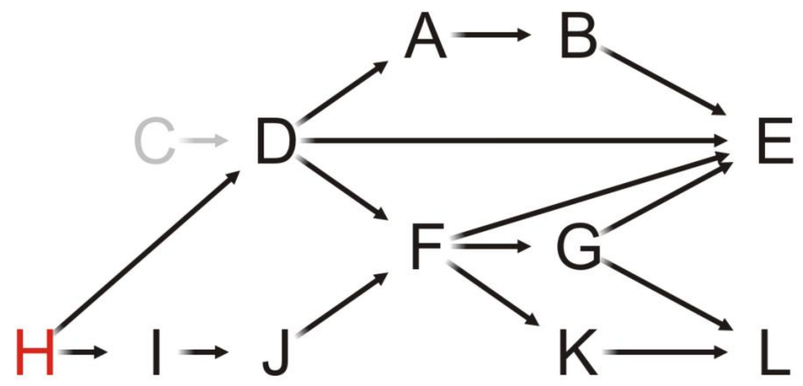
Having completed Task C, which vertices have in-degree zero?



C

Example

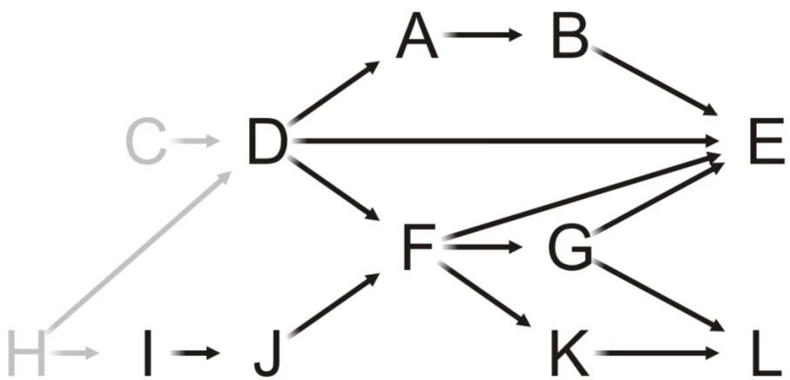
Only Task H can be completed, so we choose it



C

Example

Having removed H, what is next?

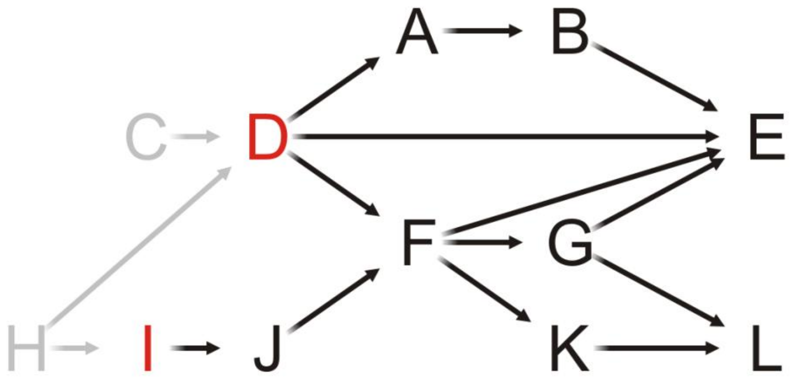


C, H

Example

Both Tasks D and I have in-degree zero

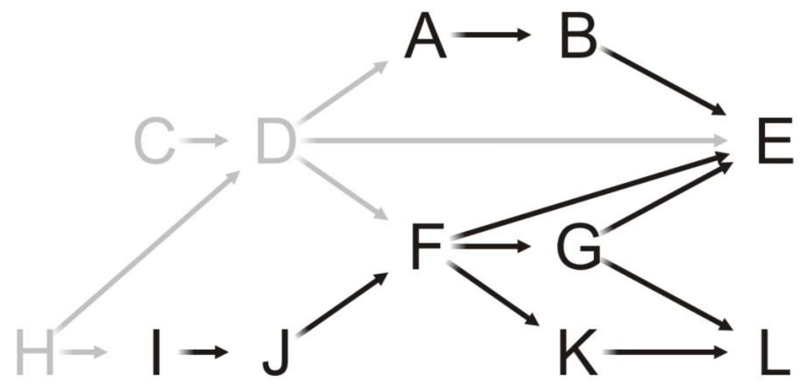
– Let us choose Task D



C, H

Example

We remove Task D, and now?

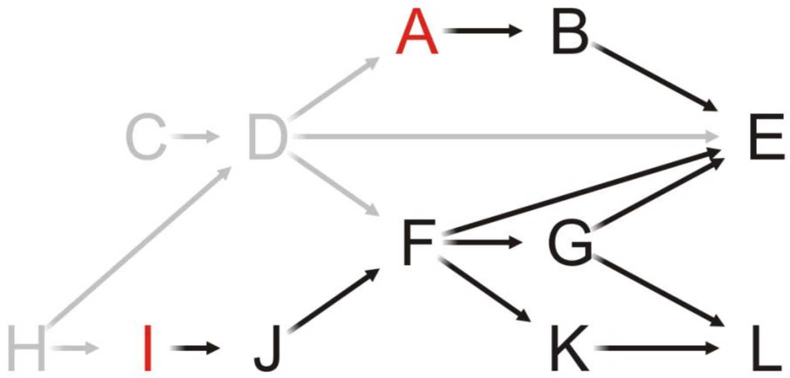


C, H, D

Example

Both Tasks A and I have in-degree zero

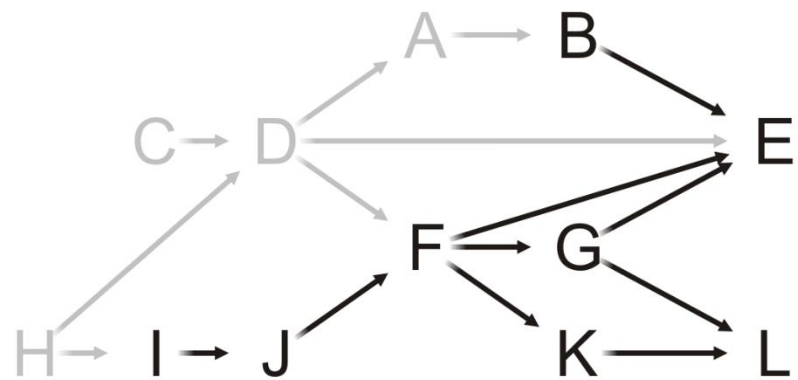
- Let's choose Task A



C, H, D

Example

Having removed A, what now?

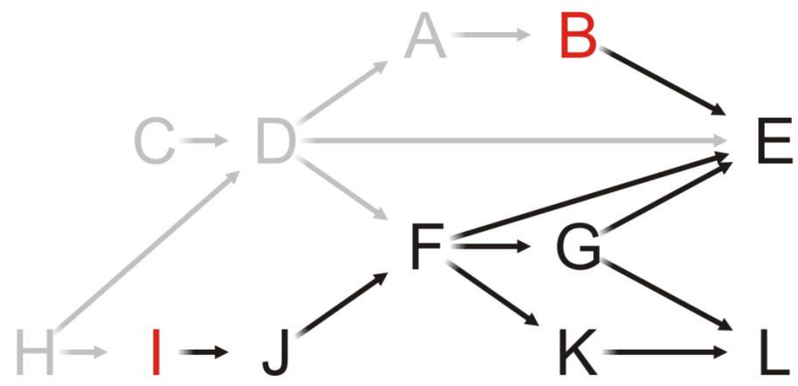


C, H, D, A

Example

Both Tasks B and I have in-degree zero

– Choose Task B

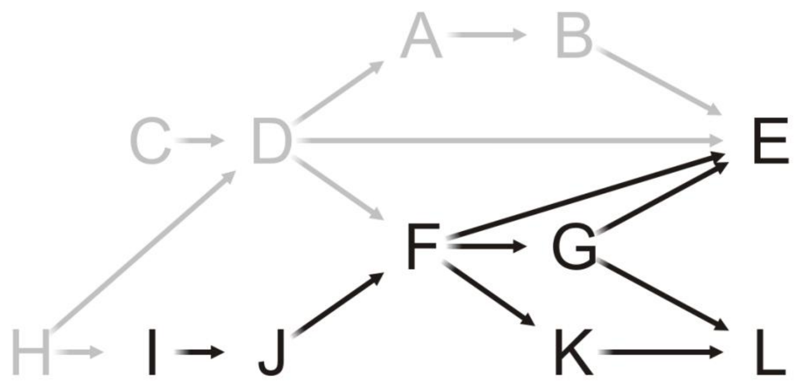


C, H, D, A

Example

Removing Task B, we note that Task E still has an in-degree of two

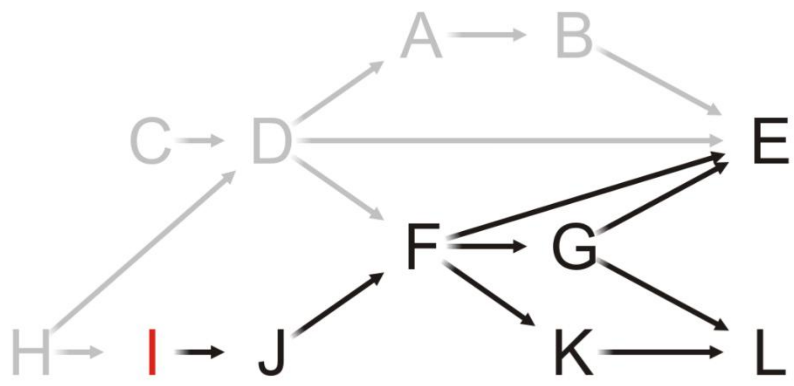
- Next?



C, H, D, A, B

Example

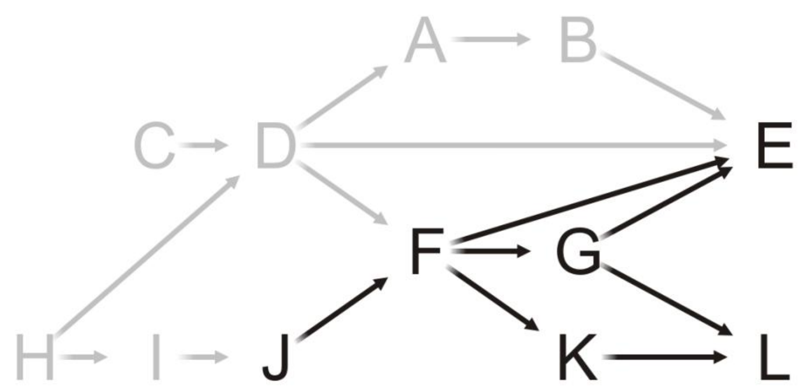
As only Task I has in-degree zero, we choose it



C, H, D, A, B

Example

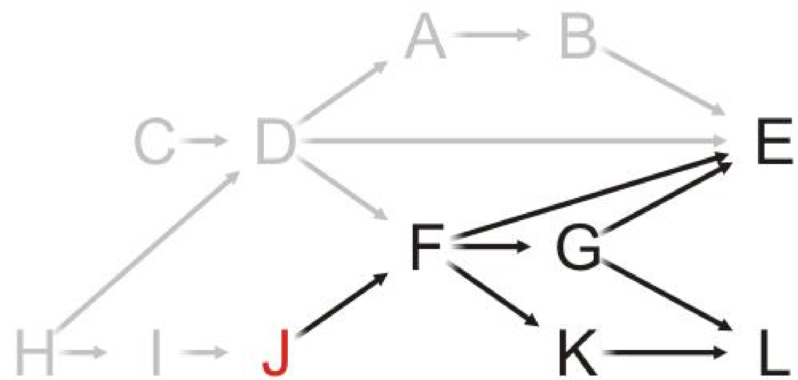
Having completed Task I, what now?



C, H, D, A, B, I

Example

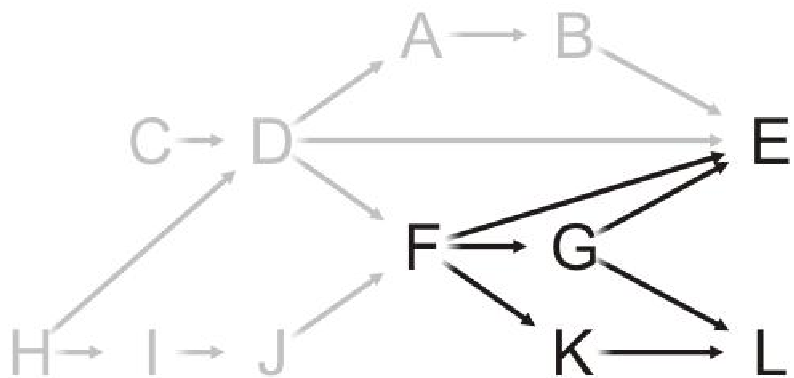
Only Task J has in-degree zero: choose it



C, H, D, A, B, I

Example

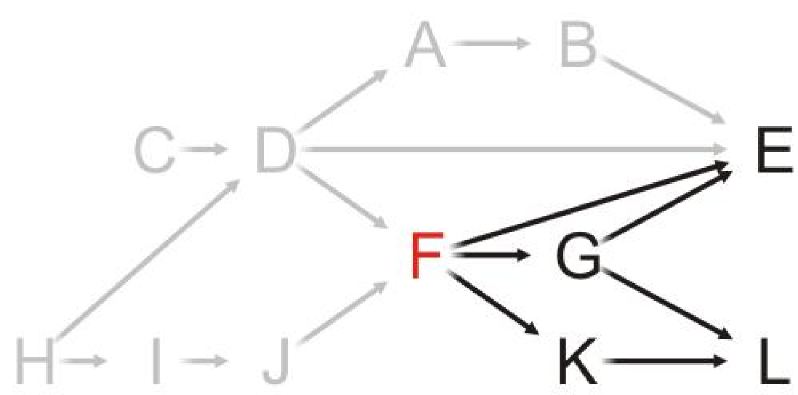
Having completed Task J, what now?



C, H, D, A, B, I, J

Example

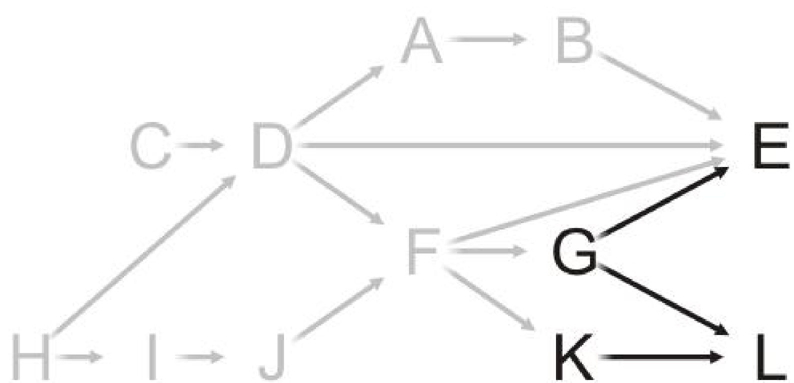
Only Task F can be completed, so choose it



C, H, D, A, B, I, J

Example

What choices do we have now?

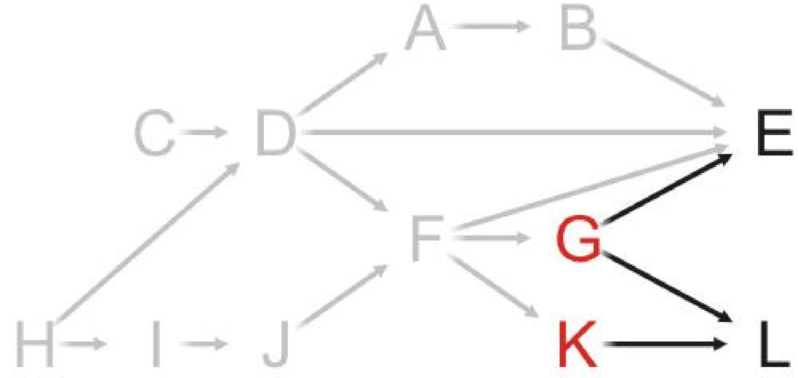


C, H, D, A, B, I, J, F

Example

We can perform Tasks G or K

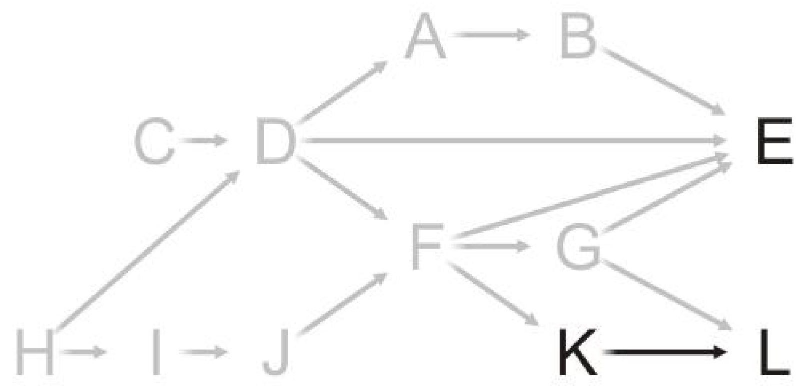
- Choose Task G



C, H, D, A, B, I, J, F

Example

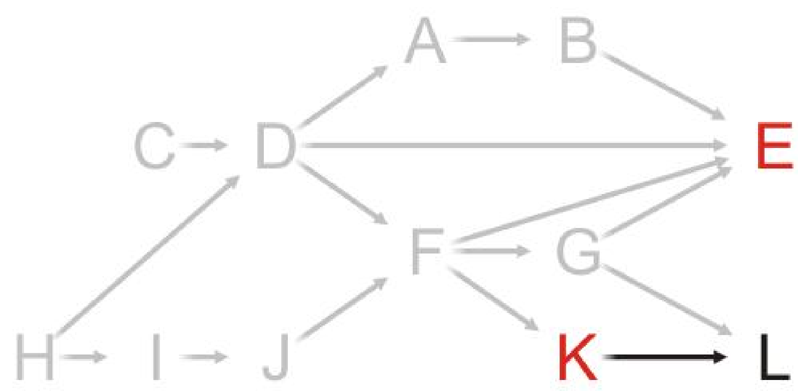
Having removed Task G from the graph, what next?



C, H, D, A, B, I, J, F, G

Example

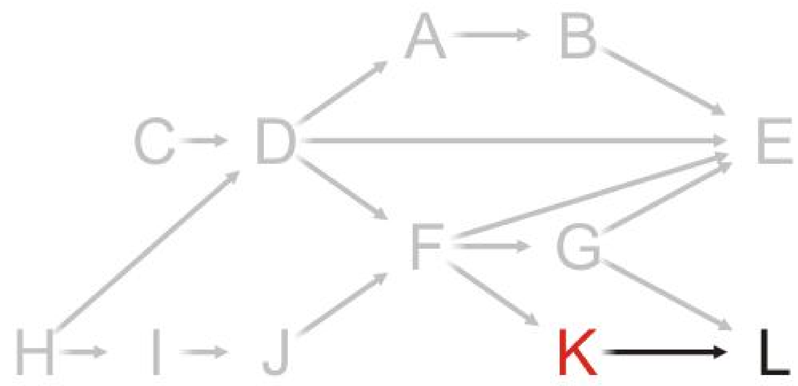
Choosing between Tasks E and K, choose Task E



C, H, D, A, B, I, J, F, G

Example

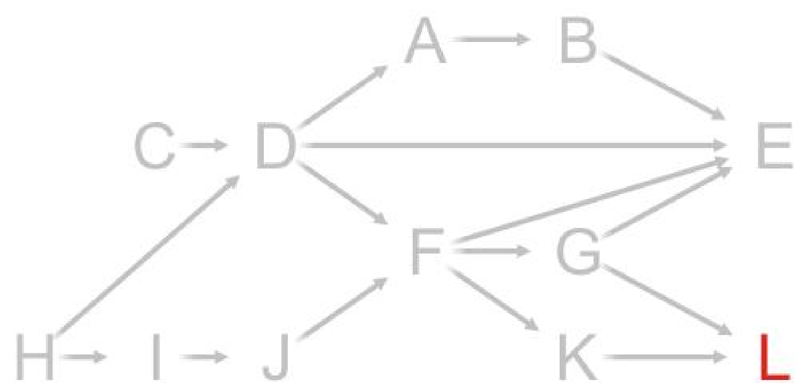
At this point, Task K is the only one that can be run



C, H, D, A, B, I, J, F, G, E

Example

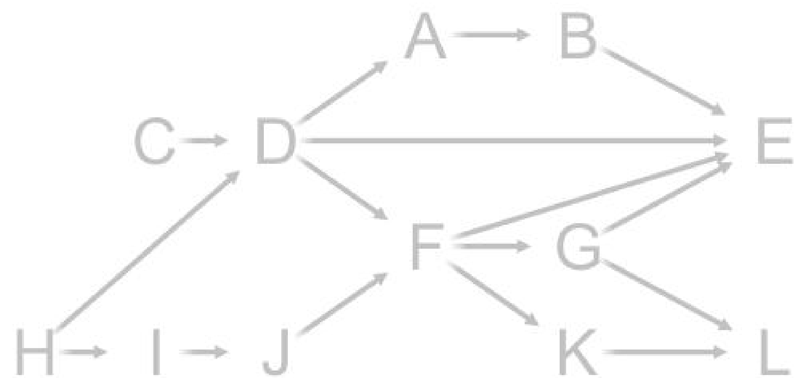
And now that both Tasks G and K are complete, we can complete Task L



C, H, D, A, B, I, J, F, G, E, K

Example

There are no more vertices left

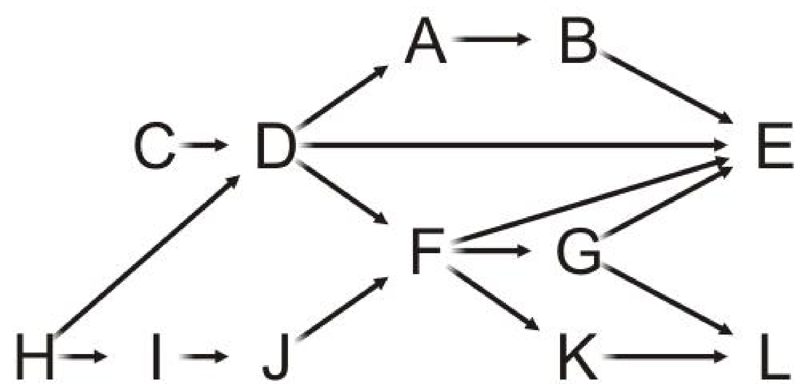


C, H, D, A, B, I, J, F, G, E, K, L

Example

Thus, one possible topological sort would be:

C, H, D, A, B, I, J, F, G, E, K, L

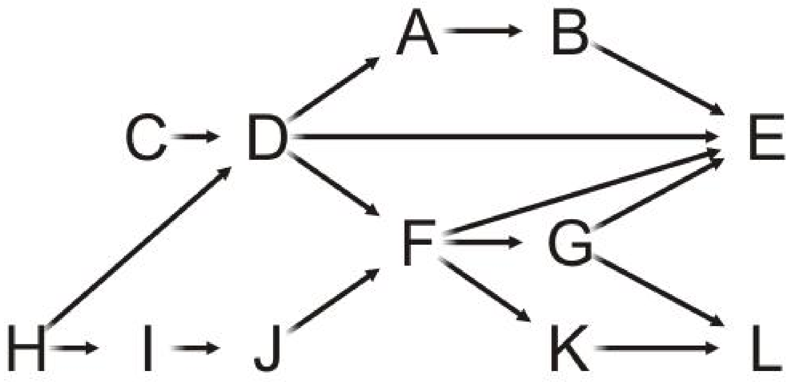


Example

Note that topological sorts need not be unique:

C, H, D, A, B, I, J, F, G, E, K, L

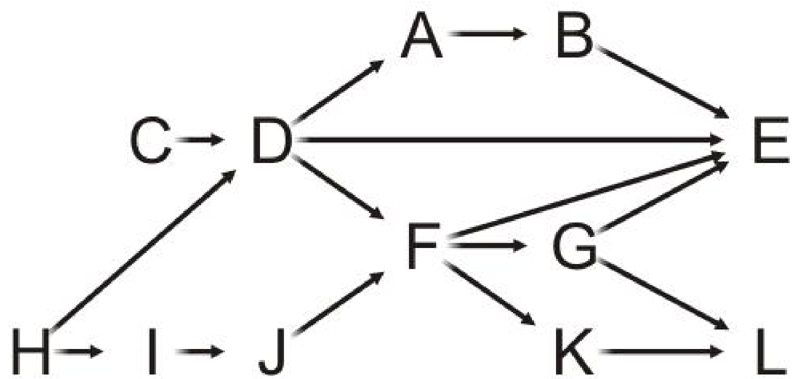
H, I, J, C, D, F, G, K, L, A, B, E



Analysis

What are the tools necessary for a topological sort?

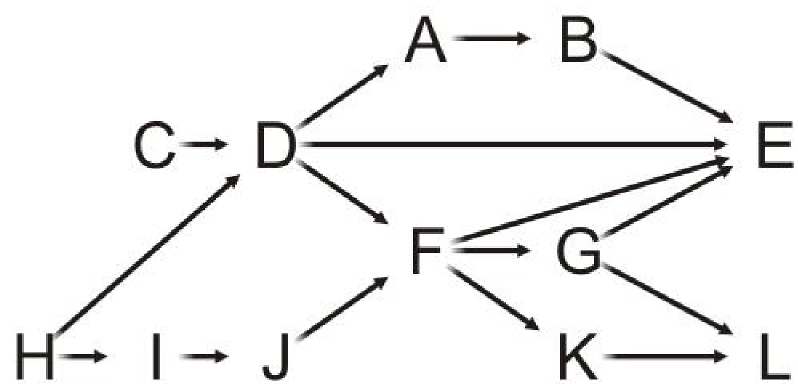
- We must know and be able to update the in-degrees of each of the vertices
- We could do this with a table of the in-degrees of each of the vertices
- This requires $\Theta(|V|)$ memory



A	1
B	1
C	0
D	2
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	2

Analysis

We must iterate at least $|V|$ times, so the run-time must be $\Omega(|V|)$

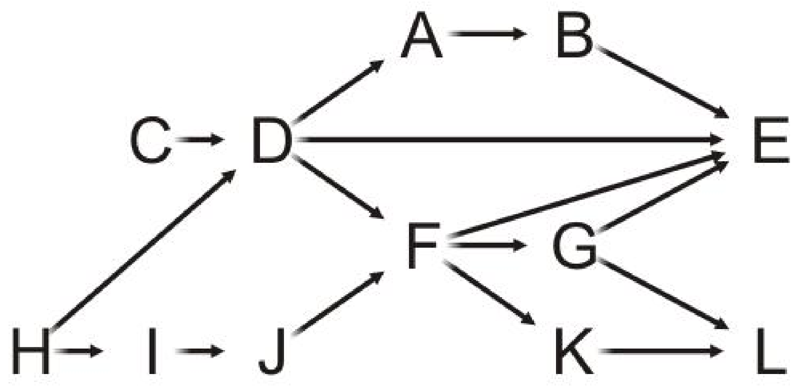


A	1
B	1
C	0
D	2
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	2

Analysis

We need to find vertices with in-degree zero

- We could loop through the table with each iteration
- The run time would be $O(|V|^2)$

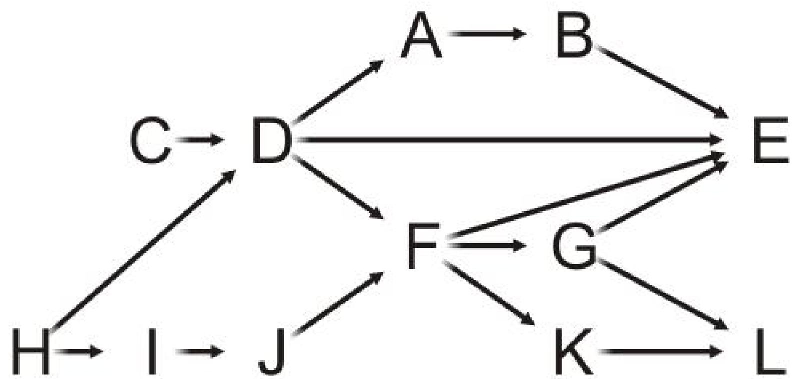


A	1
B	1
C	0
D	2
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	2

Analysis

A better approach

- Use a queue (or other container) to temporarily store those vertices with in-degree zero
- Each time the in-degree of a vertex is decremented to zero, push it onto the queue

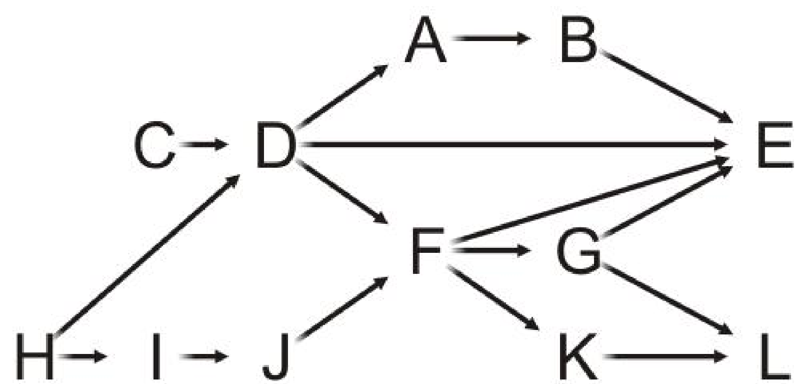


A	1
B	1
C	0
D	2
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	2

Analysis

What are the run times associated with the queue?

- Initially, we must scan through each of the vertices: $\Theta(|V|)$
- For each vertex, we will have to push onto and pop off the queue once, also $\Theta(|V|)$

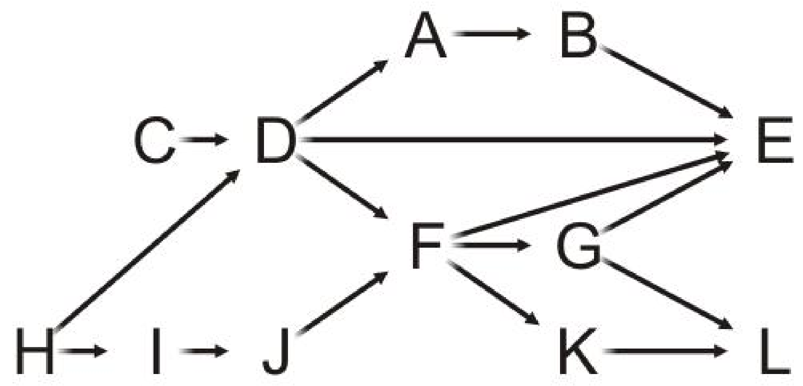


A	1
B	1
C	0
D	2
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	2

Analysis

Finally, every time we remove a vertex v, all its edges shall also be removed and the in-degree table be updated

- The run time of these operations is $\Omega(|E|)$
- If we are using an adjacency matrix: $\Theta(|V|^2)$
- If we are using an adjacency list: $\Theta(|E|)$



Here, $|E| = 16$

A	1
B	1
C	0
D	2
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	+ 2
	16

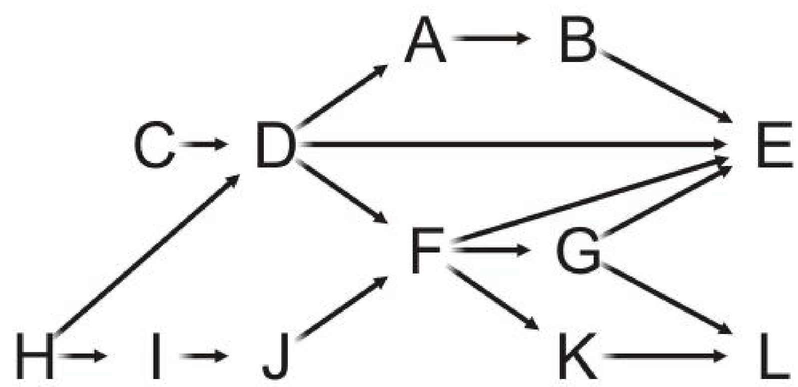
Analysis

Therefore, the run time of a topological sort is:

$\Theta(|V| + |E|)$ if we use an adjacency list

$\Theta(|V|^2)$ if we use an adjacency matrix

and the memory requirements is $\Theta(|V|)$



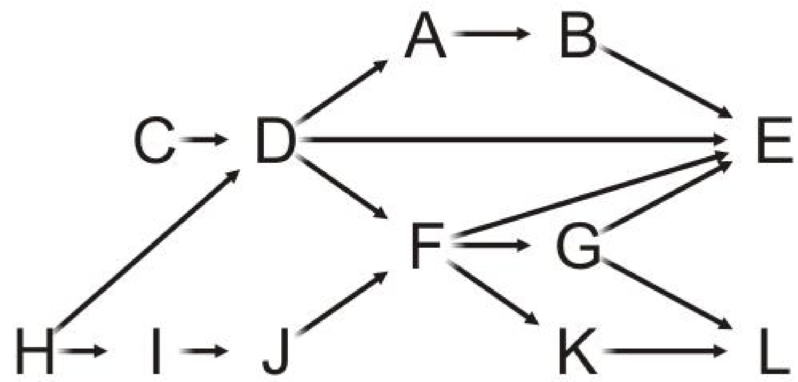
A	1
B	1
C	0
D	2
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	2

Analysis

What happens if at some step, all remaining vertices have an in-degree greater than zero?

- There must be at least one cycle within that sub-set of vertices

Consequence: we now have an $\Theta(|V| + |E|)$ algorithm for determining if a graph has a cycle



A	1
B	1
C	0
D	2
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	2

Implementation

Thus, to implement a topological sort:

- Allocate memory for and initialize an array of in-degrees
- Create a queue and initialize it with all vertices that have in-degree zero

While the queue is not empty:

- Pop a vertex from the queue
- Decrement the in-degree of each neighbor
- Those neighbors whose in-degree was decremented to zero are pushed onto the queue

Implementation

We will use an array implementation of our queue

Because we place each vertex into the queue exactly once

- We must never resize the array
- We do not have to worry about the queue cycling

Most importantly, however, because of the properties of a queue

- When we finish, the underlying array stores the topological sort

Implementation

The operations with our queue

– Initialization

```
Type array[vertex_size()];
int ihead = 0, itail = –1;
```

– Testing if empty:

```
ihead == itail + 1
```

– For push

```
++itail;
array[itail] = next vertex;
```

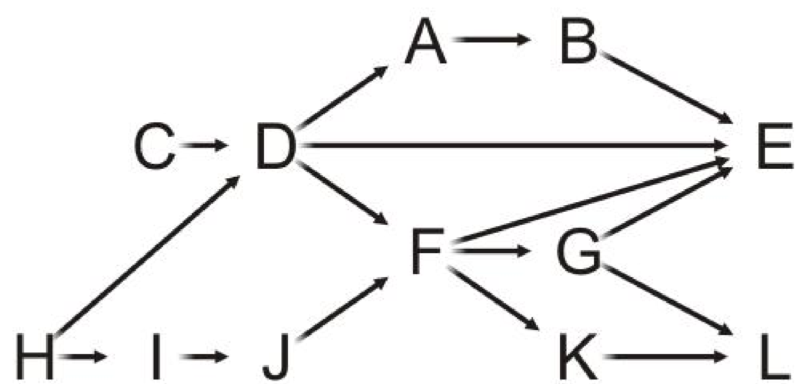
– For pop

```
Type current_top = array[ihead];
++ihead;
```

Example

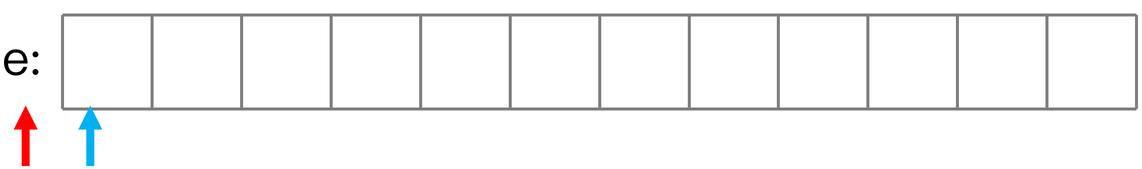
With the previous example, we initialize:

- The array of in-degrees
- The queue



A	1
B	1
C	0
D	2
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	2

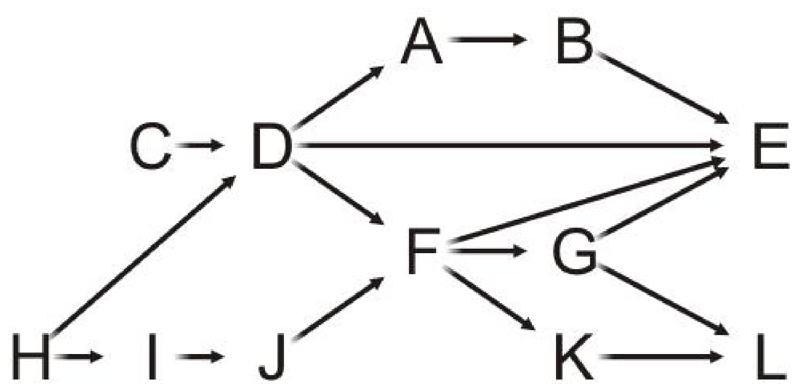
Queue:



The queue is empty

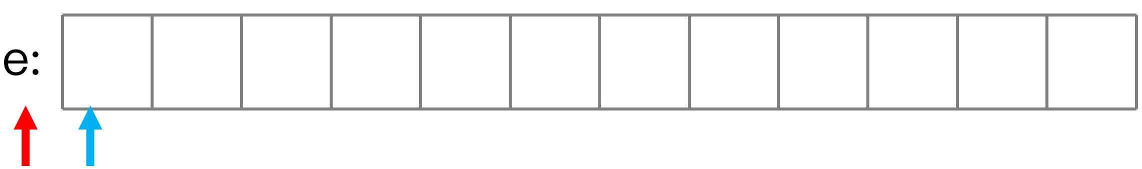
Example

Stepping through the array, push all source vertices into the queue



A	1
B	1
C	0
D	2
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	2

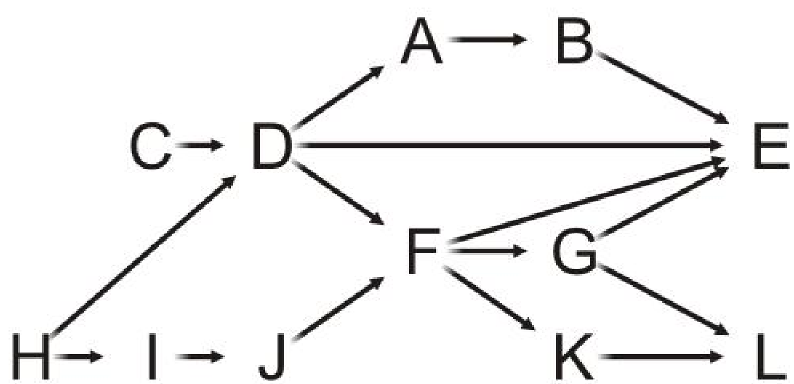
Queue:



The queue is empty

Example

Stepping through the table, push all source vertices into the queue



A	1
B	1
C	**0**
D	2
E	4
F	2
G	1
H	**0**
I	1
J	1
K	1
L	2

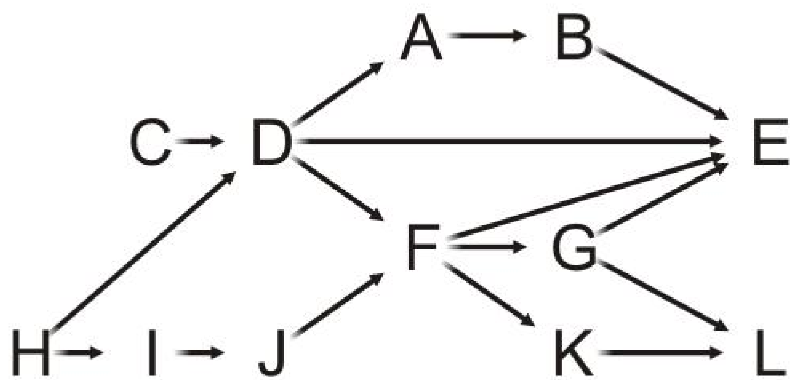
Queue:

C	**H**										

The queue is empty

Example

Pop the front of the queue



A	1
B	1
C	0
D	2
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	2

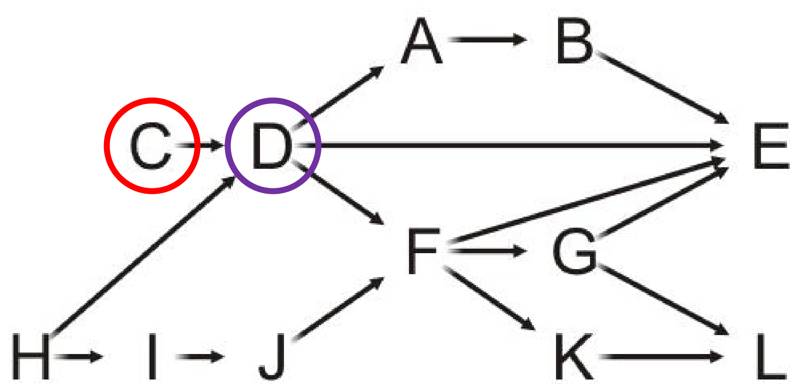
Queue:

C	H										

Example

Pop the front of the queue

- C has one neighbor: D



Queue:

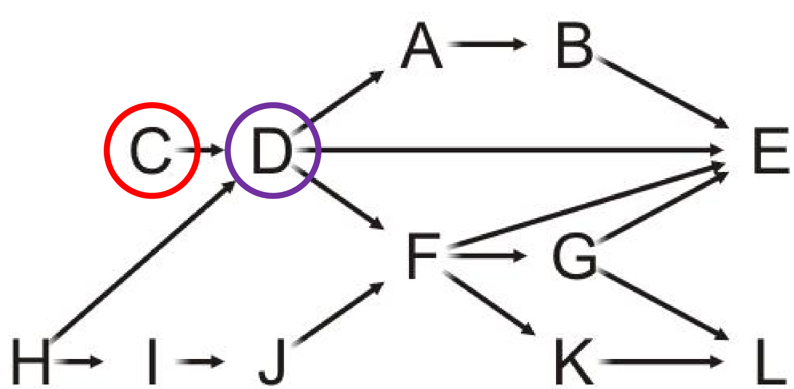
C	H										

A	1
B	1
C	**0**
D	**2**
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	2

Example

Pop the front of the queue

- C has one neighbor: D
- Decrement its in-degree



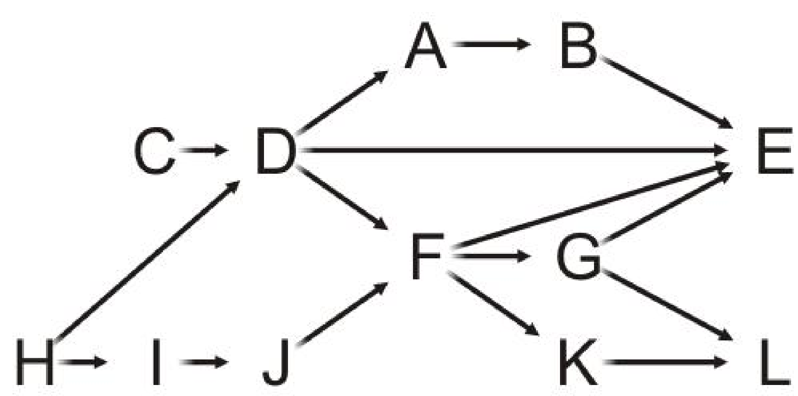
A	1
B	1
C	**0**
D	**1**
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	2

Queue:

C	H										

Example

Pop the front of the queue



A	1
B	1
C	0
D	1
E	4
F	2
G	1
H	0
I	1
J	1
K	1
L	2

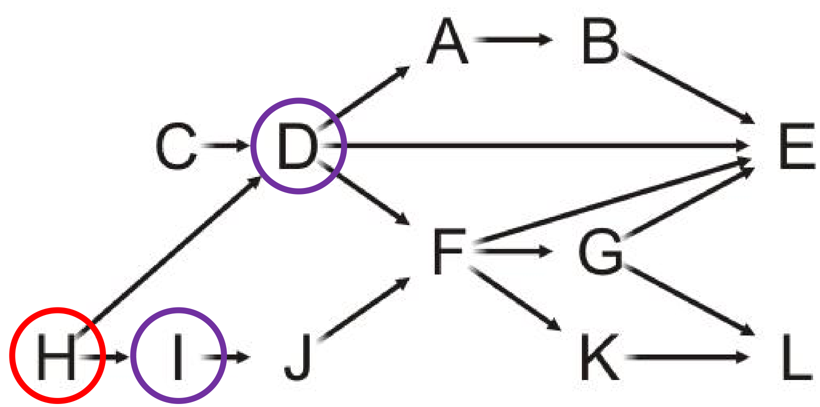
Queue:

C	H										

Example

Pop the front of the queue

- H has two neighbors: D and I



Queue:

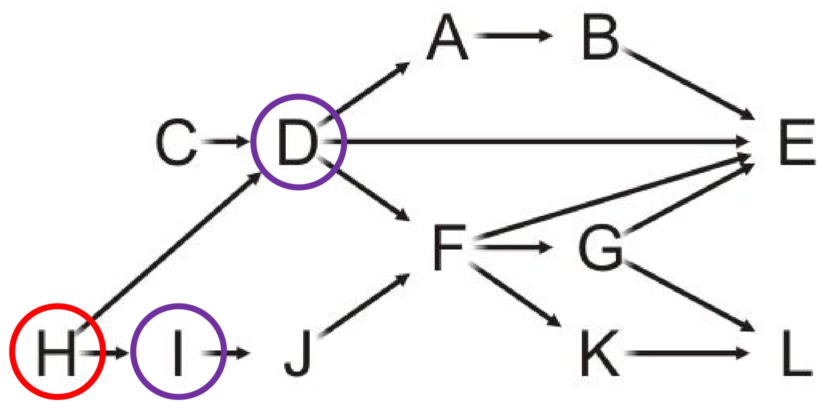
C	H										

A	1
B	1
C	0
D	**1**
E	4
F	2
G	1
H	**0**
I	**1**
J	1
K	1
L	2

Example

Pop the front of the queue

- H has two neighbors: D and I
- Decrement their in-degrees



Queue:

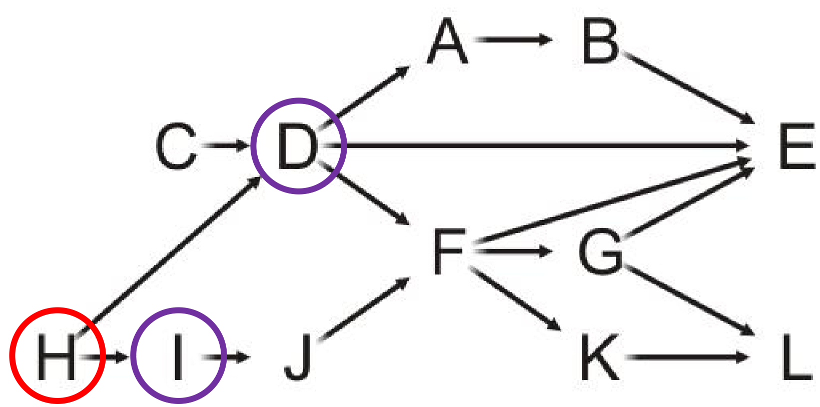
C	H										

A	1
B	1
C	0
D	**0**
E	4
F	2
G	1
H	**0**
I	**0**
J	1
K	1
L	2

Example

Pop the front of the queue

- H has two neighbors: D and I
- Decrement their in-degrees
  - Both are decremented to zero, so push them onto the queue



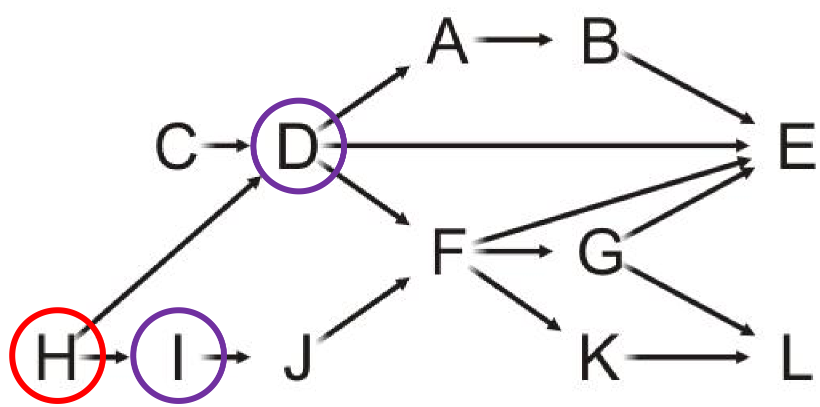
Queue:	C	H										

A	1
B	1
C	0
D	**0**
E	4
F	2
G	1
H	**0**
I	**0**
J	1
K	1
L	2

Example

Pop the front of the queue

- H has two neighbors: D and I
- Decrement their in-degrees
    - Both are decremented to zero, so push them onto the queue



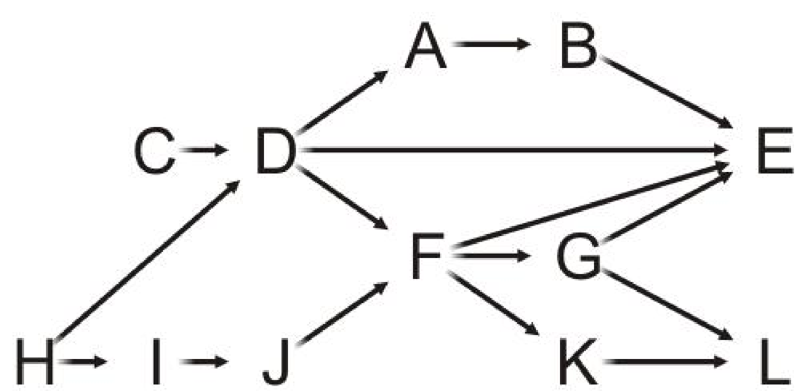
Queue:

C	H	D	I								

A	1
B	1
C	0
D	**0**
E	4
F	2
G	1
H	**0**
I	**0**
J	1
K	1
L	2

Example

Pop the front of the queue



Queue:

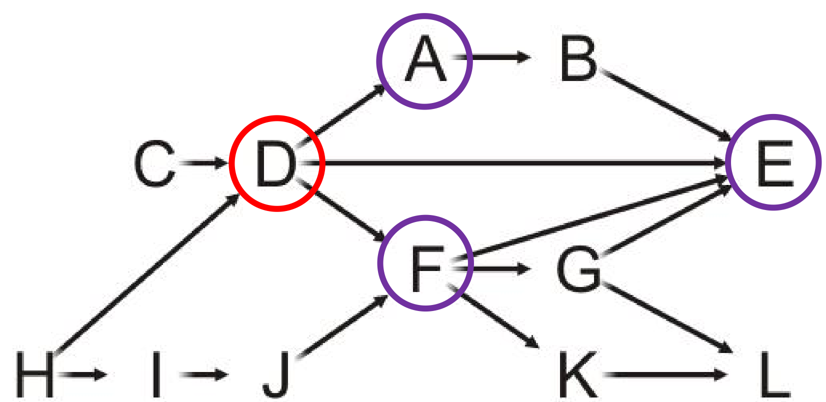
C	H	D	I								

A	1
B	1
C	0
D	0
E	4
F	2
G	1
H	0
I	0
J	1
K	1
L	2

Example

Pop the front of the queue

– D has three neighbors: A, E and F



A	1
B	1
C	0
D	0
E	4
F	2
G	1
H	0
I	0
J	1
K	1
L	2

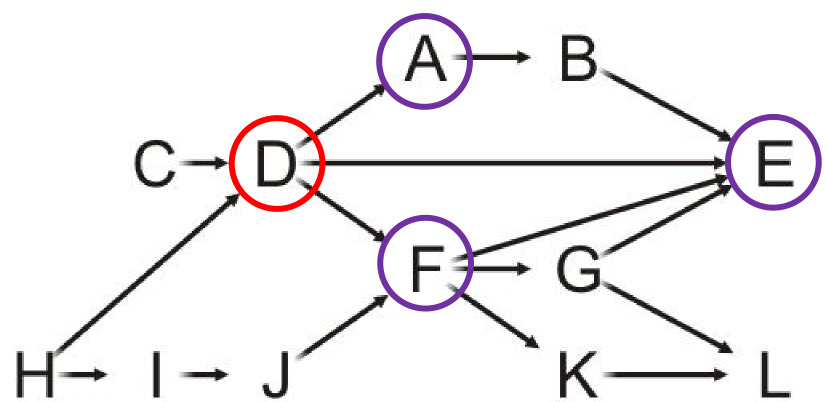
Queue:

C	H	D	I								

Example

Pop the front of the queue

- D has three neighbors: A, E and F
- Decrement their in-degrees



A	**0**
B	1
C	0
D	**0**
E	**3**
F	**1**
G	1
H	0
I	0
J	1
K	1
L	2

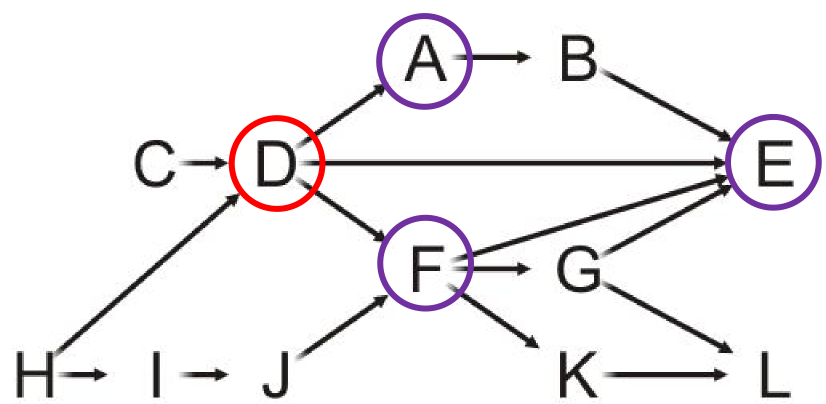
Queue:

C	H	D	I								

Example

Pop the front of the queue

- D has three neighbors: A, E and F
- Decrement their in-degrees
  - A is decremented to zero, so push it onto the queue



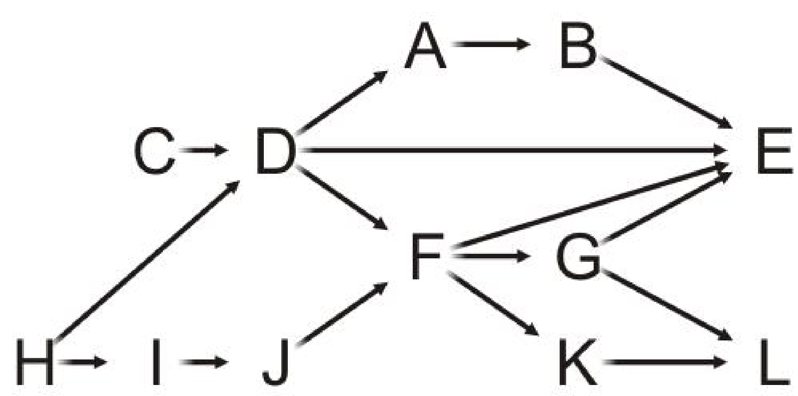
Queue:

C	H	D	I	A							

A	**0**
B	1
C	0
D	**0**
E	**3**
F	**1**
G	1
H	0
I	0
J	1
K	1
L	2

Example

Pop the front of the queue



A	0
B	1
C	0
D	0
E	3
F	1
G	1
H	0
I	0
J	1
K	1
L	2

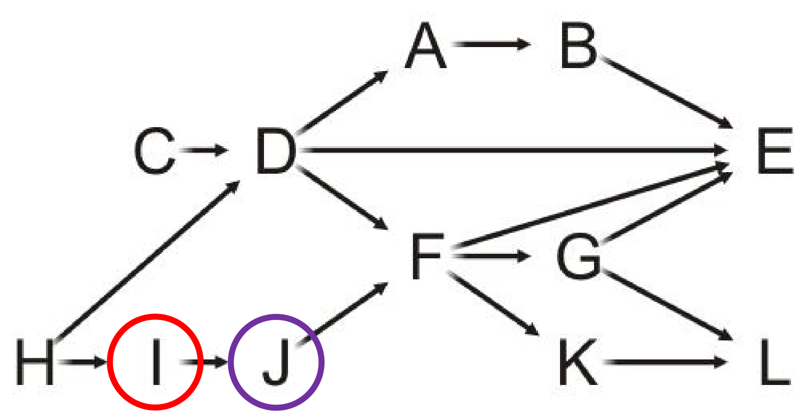
Queue:

C	H	D	I	A							

Example

Pop the front of the queue

- I has one neighbor: J



Queue:

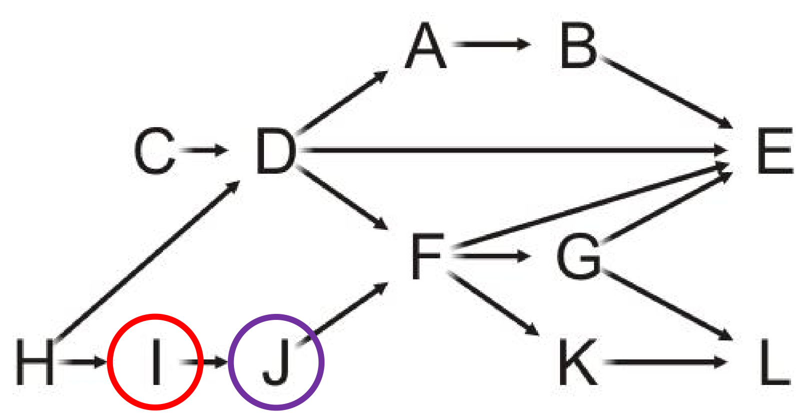
C	H	D	I	A							

A	0
B	1
C	0
D	0
E	3
F	1
G	1
H	0
I	**0**
J	**1**
K	1
L	2

Example

Pop the front of the queue

– I has one neighbor: J
– Decrement its in-degree



Queue:

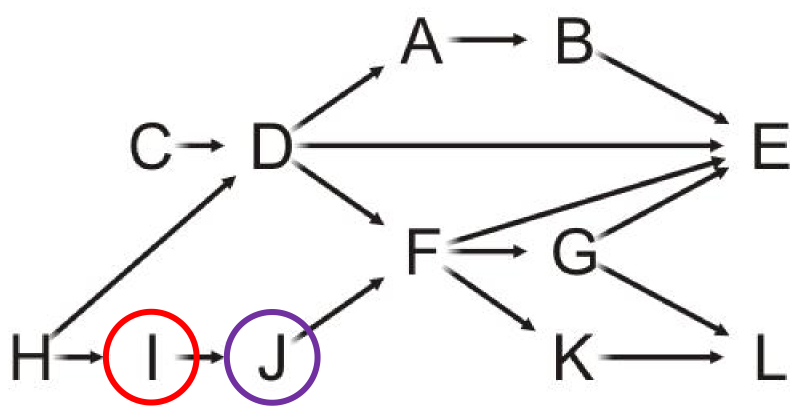
C	H	D	I	A							

A	0
B	1
C	0
D	0
E	3
F	1
G	1
H	0
I	**0**
J	**0**
K	1
L	2

Example

Pop the front of the queue

- I has one neighbor: J
- Decrement its in-degree
  - J is decremented to zero, so push it onto the queue



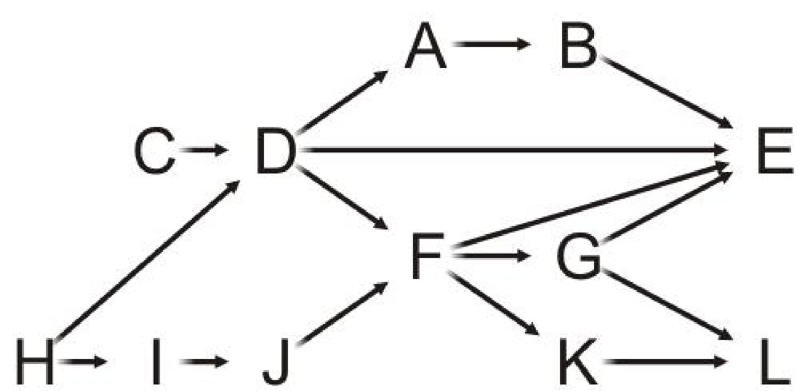
Queue:

C	H	D	I	A	J						

A	0
B	1
C	0
D	0
E	3
F	1
G	1
H	0
I	**0**
J	**0**
K	1
L	2

Example

Pop the front of the queue



Queue:

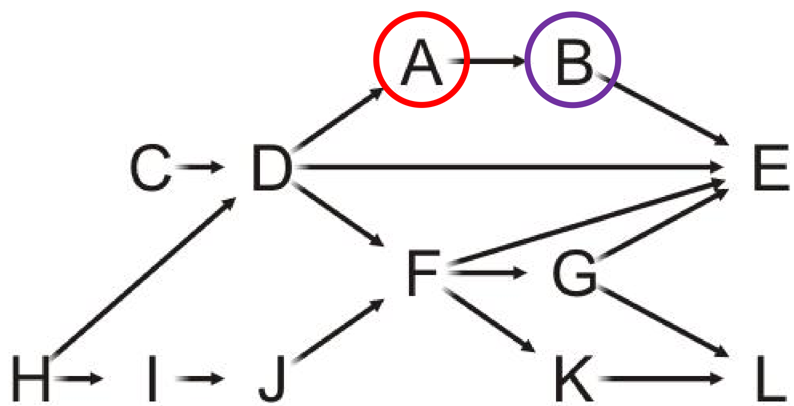
C	H	D	I	A	J						

A	0
B	1
C	0
D	0
E	3
F	1
G	1
H	0
I	0
J	0
K	1
L	2

Example

Pop the front of the queue

- A has one neighbor: B



Queue:

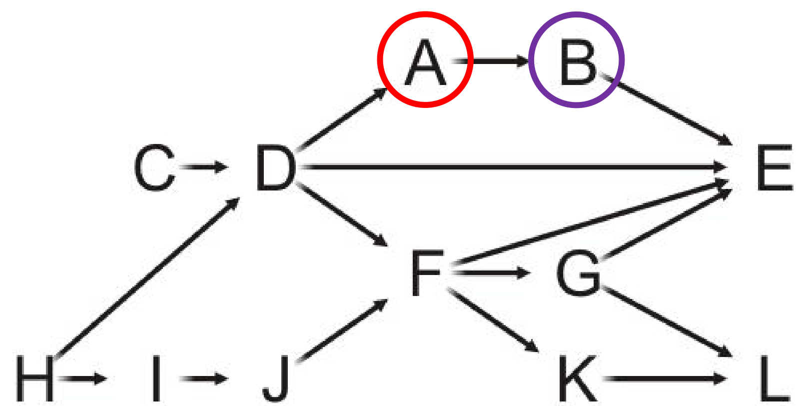
C	H	D	I	A	J						

A	**0**
B	**1**
C	0
D	0
E	3
F	1
G	1
H	0
I	0
J	0
K	1
L	2

Example

Pop the front of the queue

- A has one neighbor: B
- Decrement its in-degree



A	0
B	0
C	0
D	0
E	3
F	1
G	1
H	0
I	0
J	0
K	1
L	2

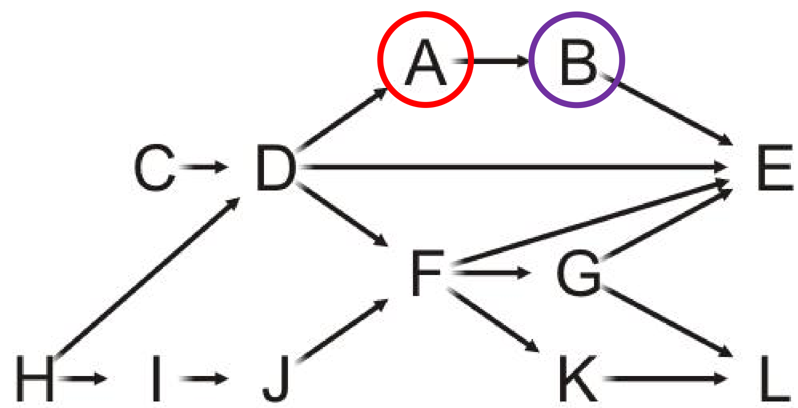
Queue:

C	H	D	I	A	J						

Example

Pop the front of the queue

- A has one neighbor: B
- Decrement its in-degree
  - B is decremented to zero, so push it onto the queue



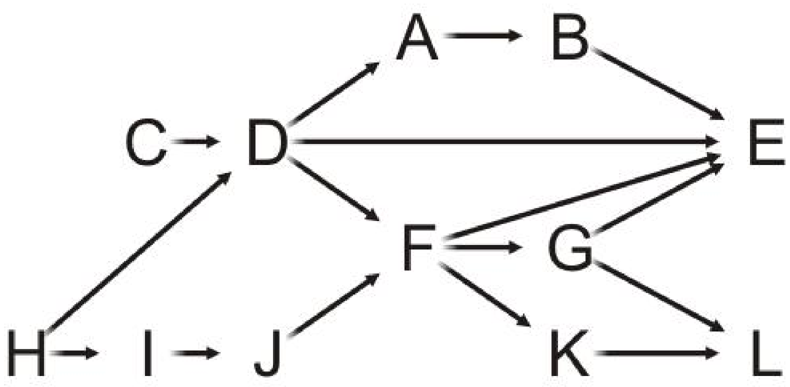
Queue:

C	H	D	I	A	J	B					

A	**0**
B	**0**
C	0
D	0
E	3
F	1
G	1
H	0
I	0
J	0
K	1
L	2

Example

Pop the front of the queue



Queue:

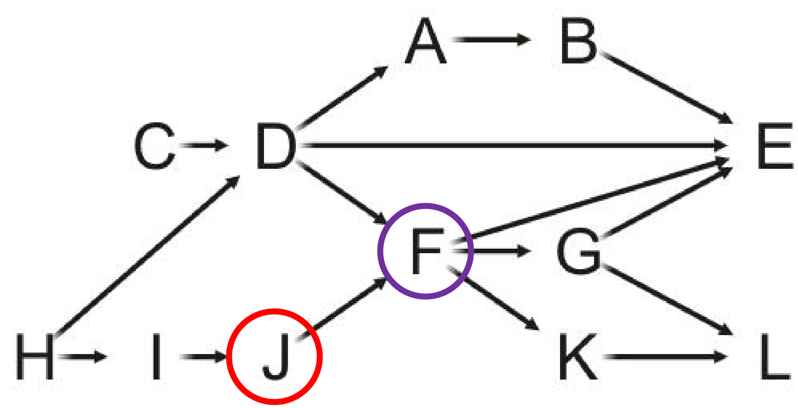
C	H	D	I	A	J	B					

A	0
B	0
C	0
D	0
E	3
F	1
G	1
H	0
I	0
J	0
K	1
L	2

Example

Pop the front of the queue

– J has one neighbor: F



Queue:

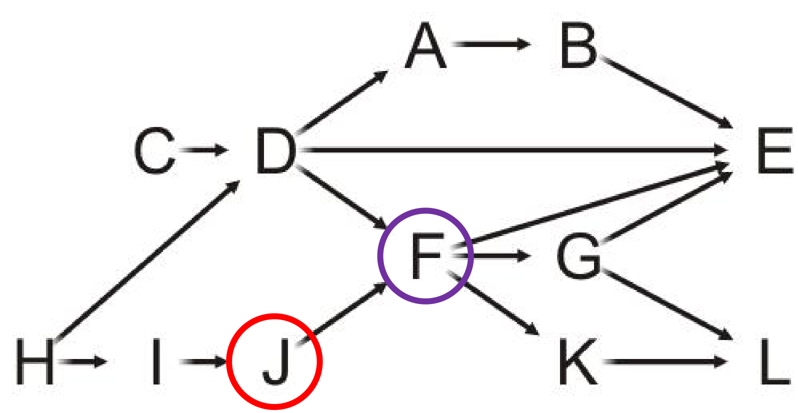
C	H	D	I	A	J	B					

A	0
B	0
C	0
D	0
E	3
F	**1**
G	1
H	0
I	0
J	**0**
K	1
L	2

Example

Pop the front of the queue

- J has one neighbor: F
- Decrement its in-degree



Queue:

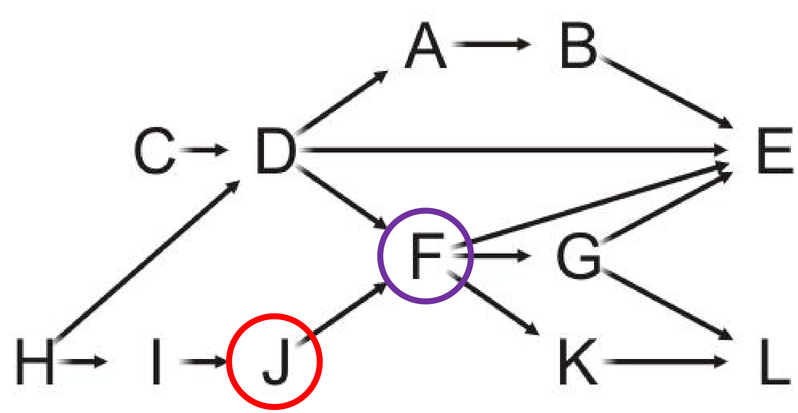
C	H	D	I	A	J	B					

A	0
B	0
C	0
D	0
E	3
F	**0**
G	1
H	0
I	0
J	**0**
K	1
L	2

Example

Pop the front of the queue

- J has one neighbor: F
- Decrement its in-degree
  - F is decremented to zero, so push it onto the queue



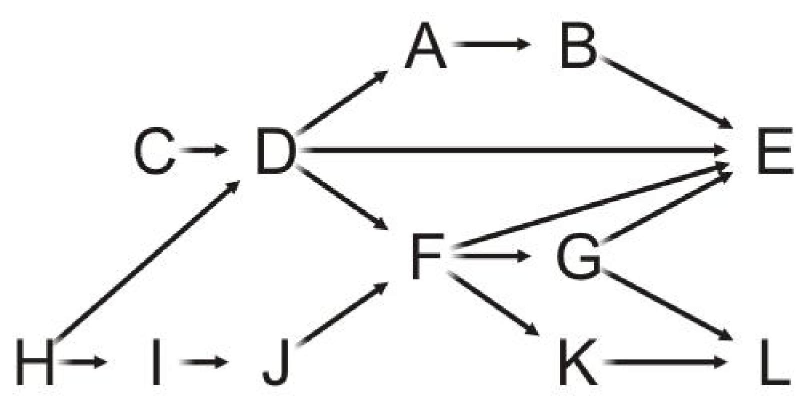
Queue:

C	H	D	I	A	J	B	F				

A	0
B	0
C	0
D	0
E	3
F	**0**
G	1
H	0
I	0
J	**0**
K	1
L	2

Example

Pop the front of the queue



Queue:

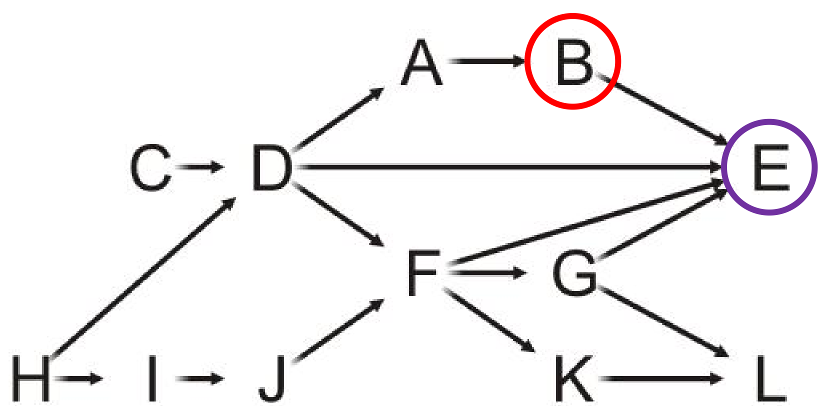
C	H	D	I	A	J	B	F				

A	0
B	0
C	0
D	0
E	3
F	0
G	1
H	0
I	0
J	0
K	1
L	2

Example

Pop the front of the queue

– B has one neighbor: E



Queue:

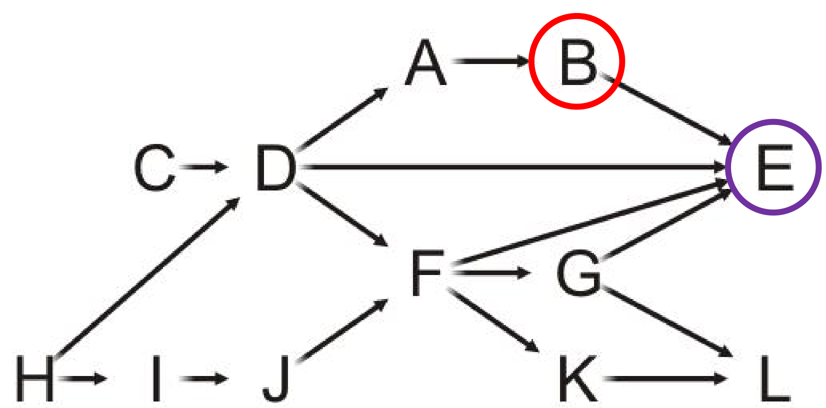
C	H	D	I	A	J	B	F				

A	0
B	**0**
C	0
D	0
E	**3**
F	0
G	1
H	0
I	0
J	0
K	1
L	2

Example

Pop the front of the queue

- B has one neighbor: E
- Decrement its in-degree



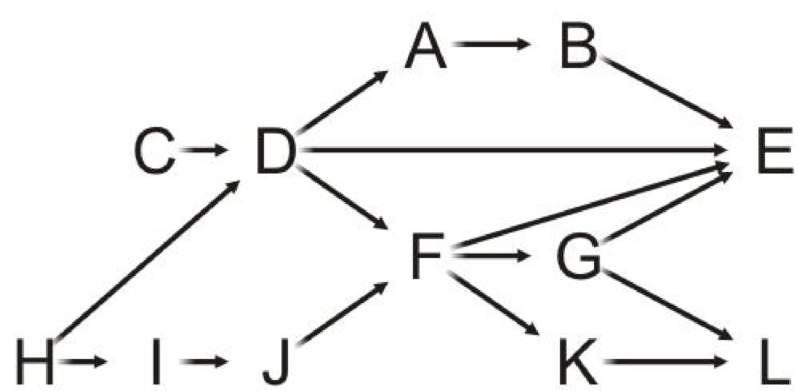
Queue:

C	H	D	I	A	J	B	F				

A	0
B	**0**
C	0
D	0
E	**2**
F	0
G	1
H	0
I	0
J	0
K	1
L	2

Example

Pop the front of the queue



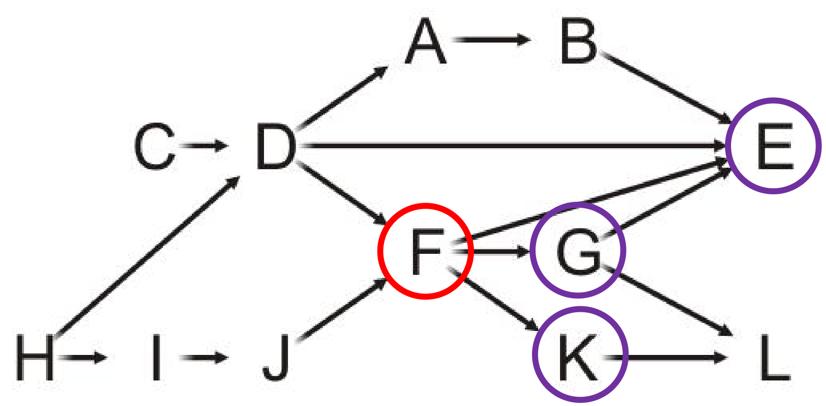
Queue:	C	H	D	I	A	J	B	F				

A	0
B	0
C	0
D	0
E	2
F	0
G	1
H	0
I	0
J	0
K	1
L	2

Example

Pop the front of the queue

- F has three neighbors: E, G and K



Queue:

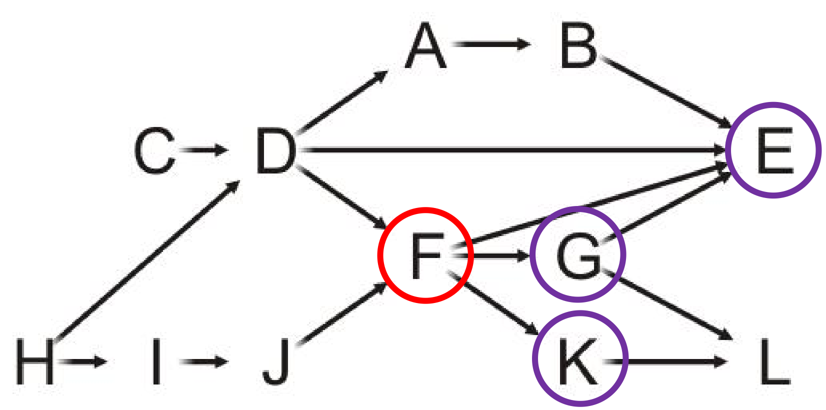
C	H	D	I	A	J	B	F				

A	0
B	0
C	0
D	0
E	**2**
F	**0**
G	**1**
H	0
I	0
J	0
K	**1**
L	2

Example

Pop the front of the queue

- F has three neighbors: E, G and K
- Decrement their in-degrees



Queue:

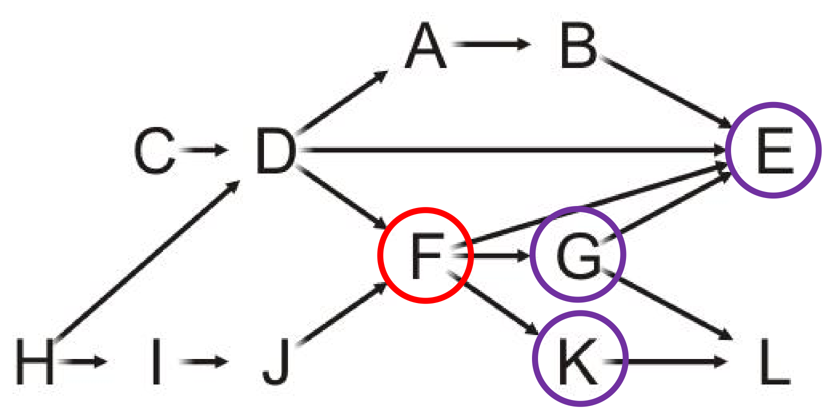
C	H	D	I	A	J	B	F				

A	0
B	0
C	0
D	0
E	**1**
F	**0**
G	**0**
H	0
I	0
J	0
K	**0**
L	2

Example

Pop the front of the queue

- F has three neighbors: E, G and K
- Decrement their in-degrees
  - G and K are decremented to zero,
    so push them onto the queue



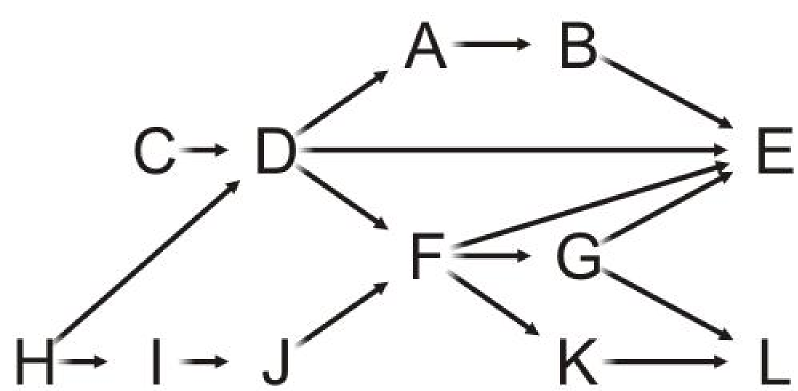
Queue:

C	H	D	I	A	J	B	F	G	K		

A	0
B	0
C	0
D	0
E	**1**
F	**0**
G	**0**
H	0
I	0
J	0
K	**0**
L	2

Example

Pop the front of the queue



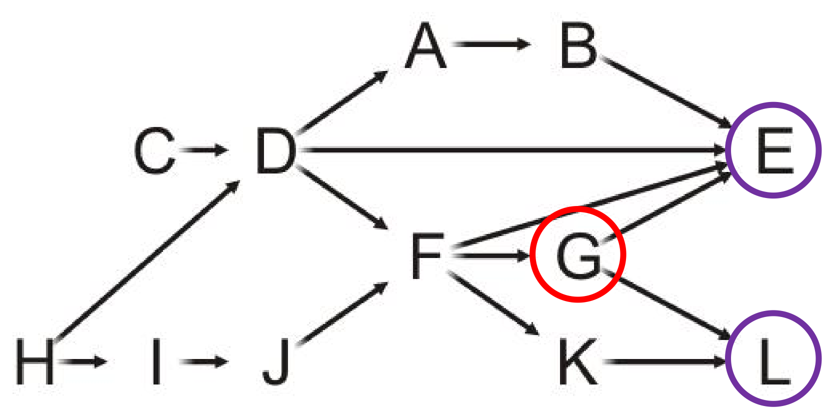
Queue:												
	C	H	D	I	A	J	B	F	G	K		

A	0
B	0
C	0
D	0
E	1
F	0
G	0
H	0
I	0
J	0
K	0
L	2

Example

Pop the front of the queue

- G has two neighbors: E and L



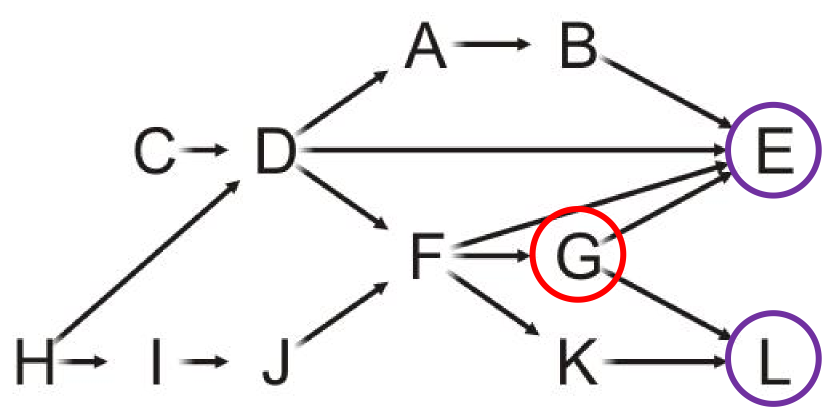
Queue:	C	H	D	I	A	J	B	F	G	K		

A	0
B	0
C	0
D	0
E	**1**
F	0
G	**0**
H	0
I	0
J	0
K	0
L	**2**

Example

Pop the front of the queue

- G has two neighbors: E and L
- Decrement their in-degrees



Queue:

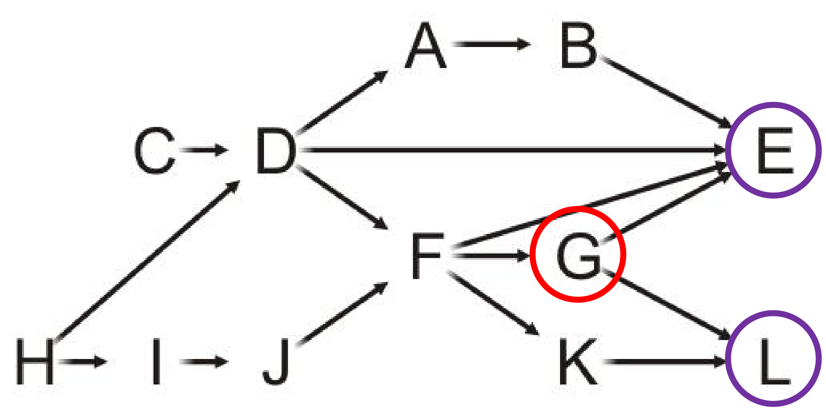
C	H	D	I	A	J	B	F	G	K		

A	0
B	0
C	0
D	0
E	0
F	0
G	0
H	0
I	0
J	0
K	0
L	1

Example

Pop the front of the queue

- G has two neighbors: E and L
- Decrement their in-degrees
  - E is decremented to zero, so push it onto the queue



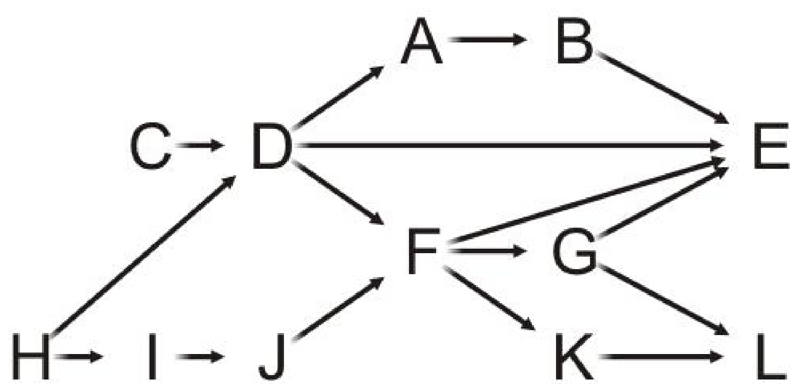
A	0
B	0
C	0
D	0
E	**0**
F	0
G	**0**
H	0
I	0
J	0
K	0
L	**1**

Queue:

C	H	D	I	A	J	B	F	G	K	E	

Example

Pop the front of the queue



Queue:

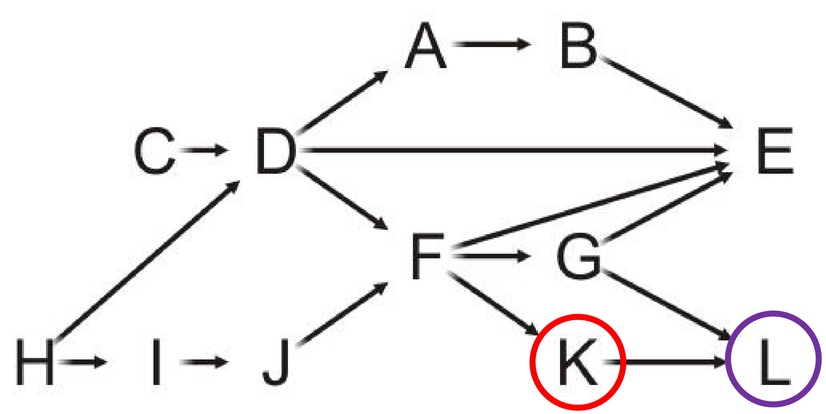
C	H	D	I	A	J	B	F	G	K	E	

A	0
B	0
C	0
D	0
E	0
F	0
G	0
H	0
I	0
J	0
K	0
L	1

Example

Pop the front of the queue

- K has one neighbors: L



Queue:

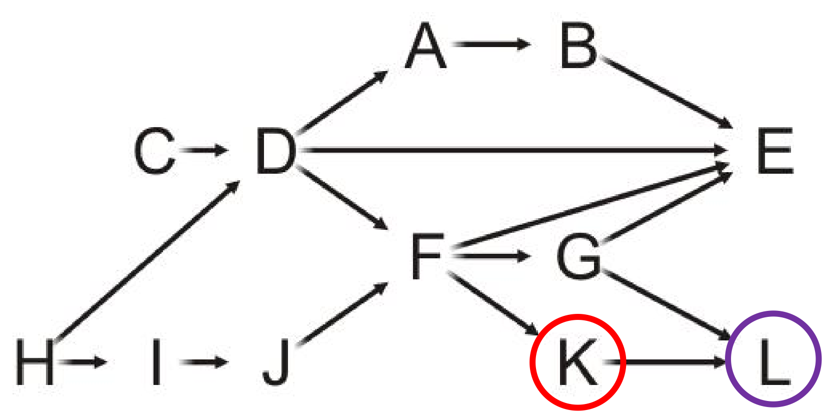
C	H	D	I	A	J	B	F	G	K	E	

A	0
B	0
C	0
D	0
E	0
F	0
G	0
H	0
I	0
J	0
K	**0**
L	**1**

Example

Pop the front of the queue

- K has one neighbors: L
- Decrement its in-degree



Queue:

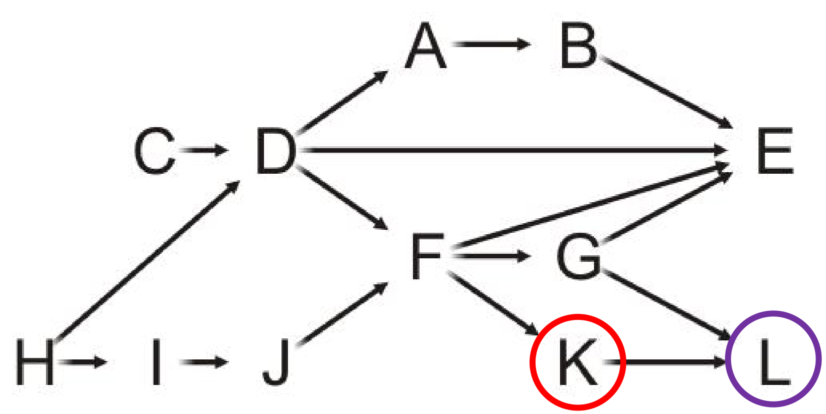
C	H	D	I	A	J	B	F	G	K	E	

A	0
B	0
C	0
D	0
E	0
F	0
G	0
H	0
I	0
J	0
K	**0**
L	**0**

Example

Pop the front of the queue

- K has one neighbors: L
- Decrement its in-degree
  - L is decremented to zero, so push it onto the queue



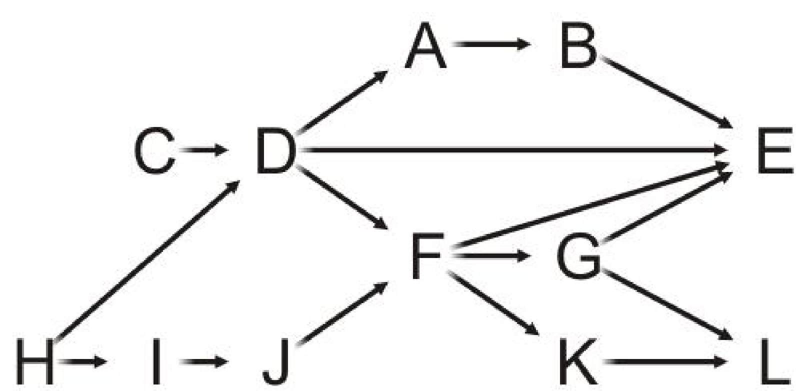
Queue:

C	H	D	I	A	J	B	F	G	K	E	L

A	0
B	0
C	0
D	0
E	0
F	0
G	0
H	0
I	0
J	0
K	**0**
L	**0**

Example

Pop the front of the queue



Queue:

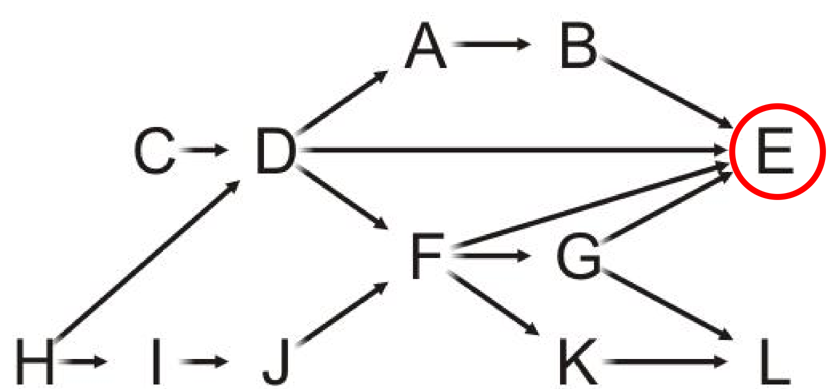
C	H	D	I	A	J	B	F	G	K	E	L

A	0
B	0
C	0
D	0
E	0
F	0
G	0
H	0
I	0
J	0
K	0
L	0

Example

Pop the front of the queue

– E has no neighbors—it is a *sink*

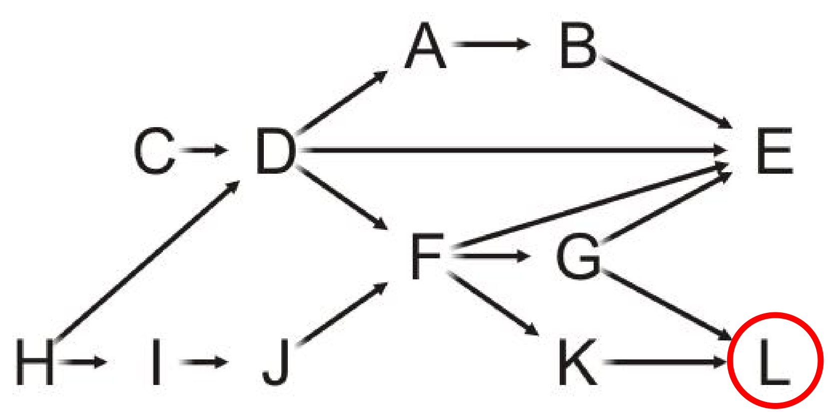


Queue:	C	H	D	I	A	J	B	F	G	K	E	L

A	0
B	0
C	0
D	0
E	**0**
F	0
G	0
H	0
I	0
J	0
K	0
L	0

Example

Pop the front of the queue



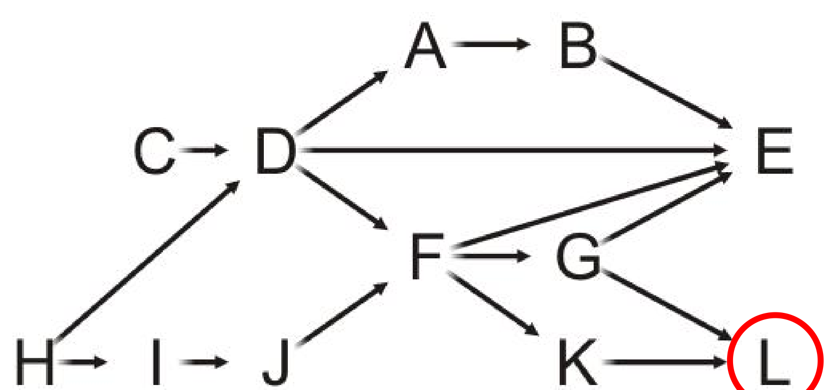
Queue:	C	H	D	I	A	J	B	F	G	K	E	L

A	0
B	0
C	0
D	0
E	0
F	0
G	0
H	0
I	0
J	0
K	0
L	0

Example

Pop the front of the queue

- L has no neighbors—it is also a *sink*



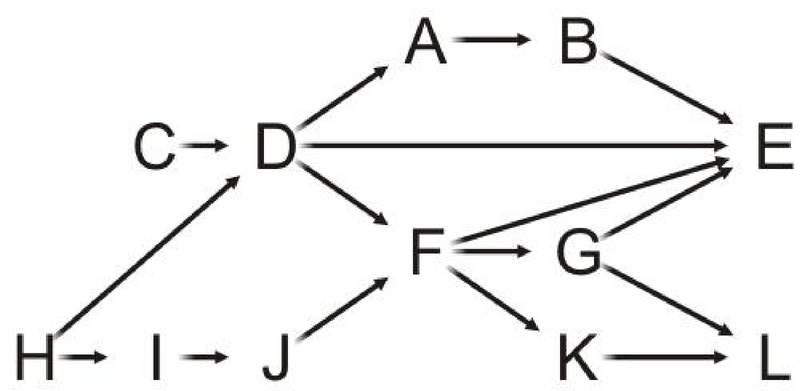
Queue:

C	H	D	I	A	J	B	F	G	K	E	L

A	0
B	0
C	0
D	0
E	0
F	0
G	0
H	0
I	0
J	0
K	0
L	**0**

Example

The queue is empty, so we are done



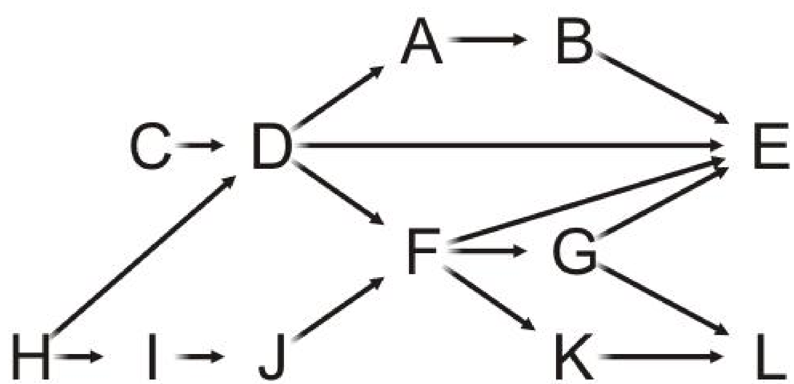
Queue:

C	H	D	I	A	J	B	F	G	K	E	L

A	0
B	0
C	0
D	0
E	0
F	0
G	0
H	0
I	0
J	0
K	0
L	0

Example

The array used for the queue stores the topological sort



C	H	D	I	A	J	B	F	G	K	E	L

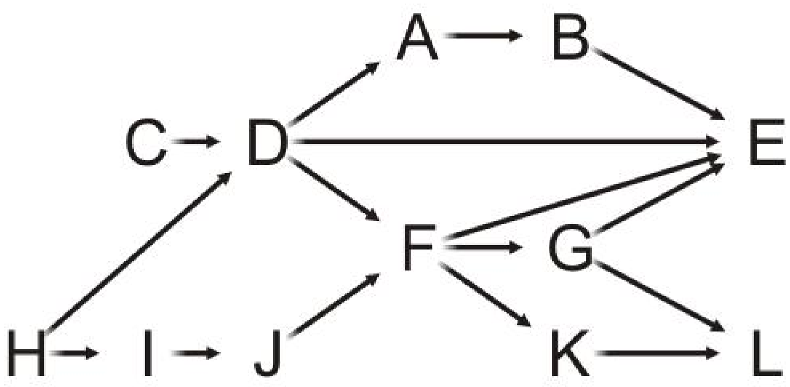
A	0
B	0
C	0
D	0
E	0
F	0
G	0
H	0
I	0
J	0
K	0
L	0

Example

The array used for the queue stores the topological sort

– Note the difference in order from our previous sort?

C, H, D, A, B, I, J, F, G, E, K, L



C	H	D	I	A	J	B	F	G	K	E	L

A	0
B	0
C	0
D	0
E	0
F	0
G	0
H	0
I	0
J	0
K	0
L	0

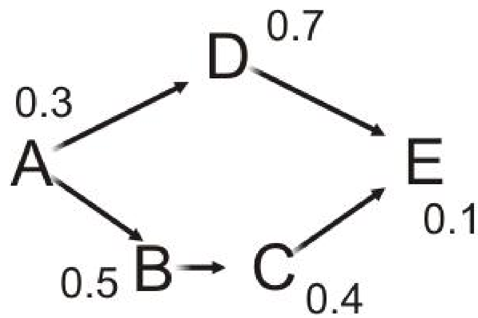
Outline

- Topological sorting
  - Definitions
  - Algorithm
- Finding the critical path

Critical path

Suppose each task has a performance time associated with it

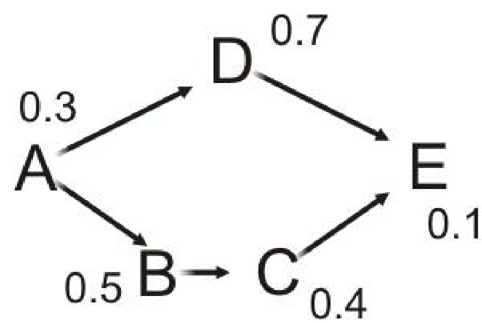
- If the tasks are performed serially, the time required to complete the last task equals to the sum of the individual task times



- These tasks require $0.3 + 0.7 + 0.5 + 0.4 + 0.1 = 2.0$ s to execute serially

Critical path

In many cases, however, we could perform tasks in parallel

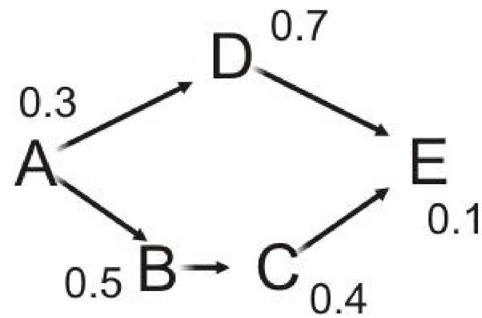


– Computer tasks can be executed in parallel (multi-processing)
– Different tasks can be completed by different teams in a company

Critical path

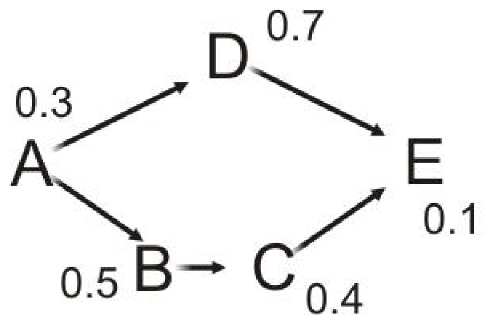
Suppose Task A completes

- We can now execute Tasks B and D in parallel



Critical path

Note that, Task E cannot execute until Task C completes, and Task C cannot execute until Task B completes



- The least time in which these five tasks can be completed is
  $$0.3 + 0.5 + 0.4 + 0.1 = 1.3 \text{ s}$$
- This is called the *critical time of all tasks*
- The path (A, B, C, E) is said to be the *critical path*

Critical path

The *critical time* of each task is the earliest time that it could be completed after the start of execution

The *critical path* is the sequence of tasks determining the minimum time needed to complete the project

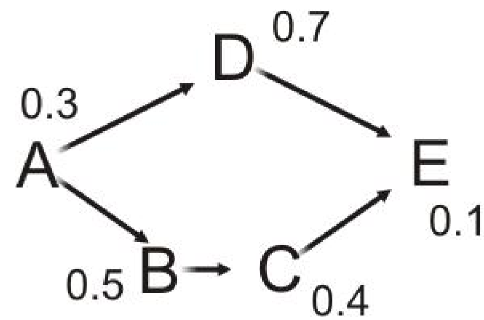
- If a task on the critical path is delayed, the entire project will be delayed

Finding the critical path

Tasks that have no prerequisites have a critical time equal to the time it takes to complete that task

For tasks that depend on others, the critical time will be:

- The maximum critical time that it takes to complete a prerequisite
- Plus the time it takes to complete this task



In this example, the critical times are:

- Task A completes in 0.3 s
- Task B must wait for A and completes after 0.8 s
- Task D must wait for A and completes after 1.0 s
- Task C must wait for B and completes after 1.2 s
- Task E must wait for both C and D, and completes after $\max(1.0, 1.2) + 0.1 = 1.3$ s

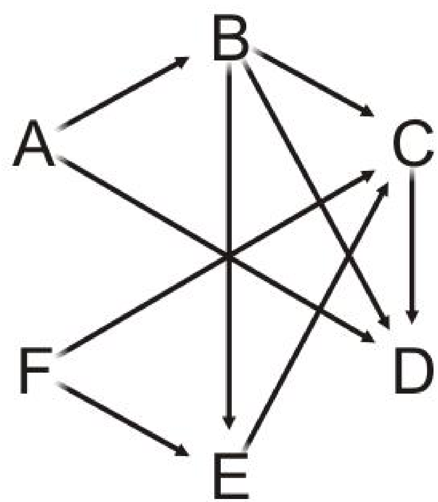
Finding the critical path

To find the critical time/path, we run topological sorting and require the following additional information:

- We must know the execution time of each task
- We will have to record the critical time for each task
  - Initialize these to zero
- We will need to know the previous task with the longest critical time to determine the critical path
  - Set these to null

Finding the critical path

Suppose we have the following times for the tasks



Queue

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	0.0	Ø
B	1	6.1	0.0	Ø
C	3	4.7	0.0	Ø
D	3	8.1	0.0	Ø
E	2	9.5	0.0	Ø
F	0	17.1	0.0	Ø

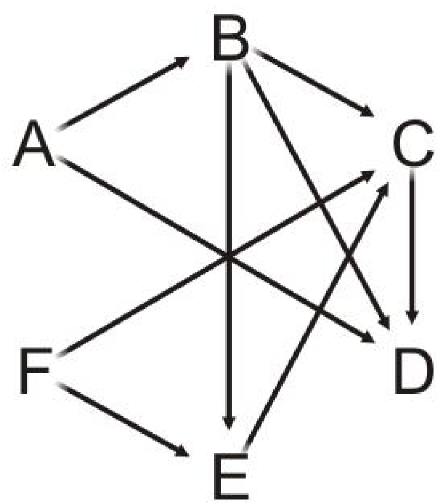
Finding the critical path

Each time we pop a vertex v, in addition to what we already do:

- For v, add the task time onto the critical time for that vertex:
  - That is the critical time for v
- For each adjacent vertex w:
  - If the critical time for v is greater than the currently stored critical time for w
    - Update the critical time with the critical time for v
    - Set the previous pointer to the vertex v

Finding the critical path

So we initialize the queue with those vertices with in-degree zero



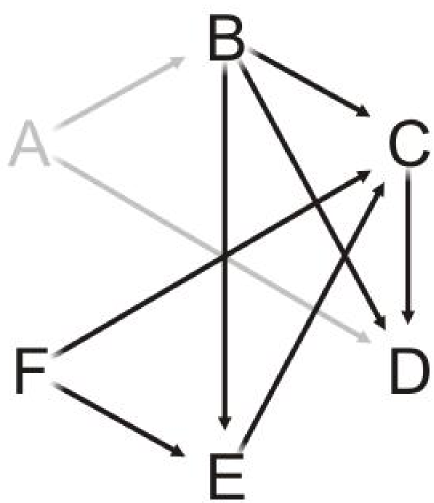
Queue

A	F		

Task	In-degree	Task Time	Critical Time	Previous Task
A	**0**	5.2	0.0	Ø
B	1	6.1	0.0	Ø
C	3	4.7	0.0	Ø
D	3	8.1	0.0	Ø
E	2	9.5	0.0	Ø
F	**0**	17.1	0.0	Ø

Finding the critical path

Pop Task A and update its critical time 0.0 + 5.2 = 5.2



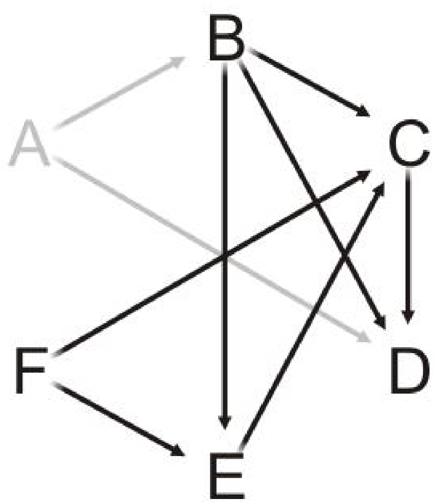
Queue

F			

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	0.0	Ø
B	1	6.1	0.0	Ø
C	3	4.7	0.0	Ø
D	3	8.1	0.0	Ø
E	2	9.5	0.0	Ø
F	0	17.1	0.0	Ø

Finding the critical path

Pop Task A and update its critical time 0.0 + 5.2 = 5.2



Queue

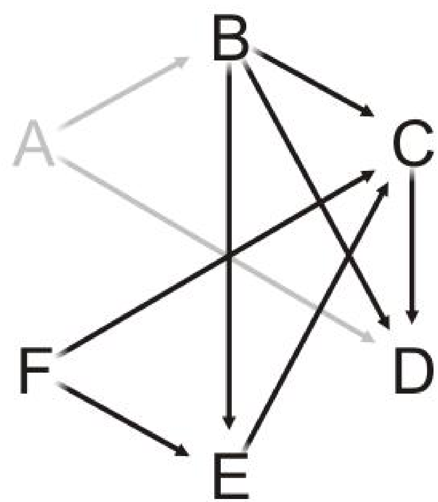
F			

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	**5.2**	Ø
B	1	6.1	0.0	Ø
C	3	4.7	0.0	Ø
D	3	8.1	0.0	Ø
E	2	9.5	0.0	Ø
F	0	17.1	0.0	Ø

Finding the critical path

For each neighbor of Task A:

– Decrement the in-degree, push if necessary, and check if we must update the critical time



Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	1	6.1	0.0	Ø
C	3	4.7	0.0	Ø
D	3	8.1	0.0	Ø
E	2	9.5	0.0	Ø
F	0	17.1	0.0	Ø

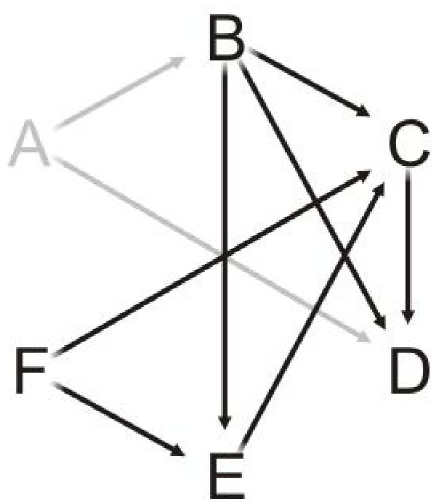
Queue

F			

Finding the critical path

For each neighbor of Task A:

– Decrement the in-degree, push if necessary, and check if we must update the critical time



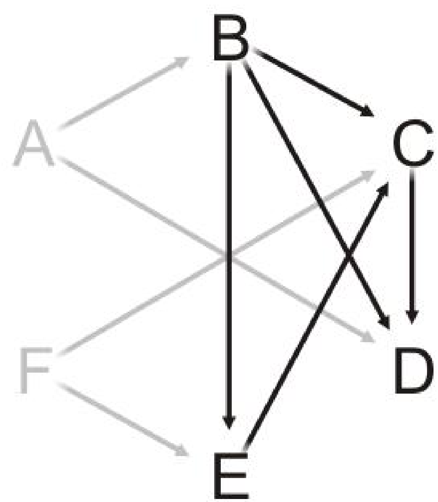
Queue

F	**B**		

Task	**In-degree**	**Task Time**	**Critical Time**	**Previous Task**
A	0	5.2	5.2	Ø
B	**0**	6.1	**5.2**	**A**
C	3	4.7	0.0	Ø
D	**2**	8.1	**5.2**	**A**
E	2	9.5	0.0	Ø
F	0	17.1	0.0	Ø

Finding the critical path

Pop Task F and update its critical time 0.0 + 17.1 = 17.1



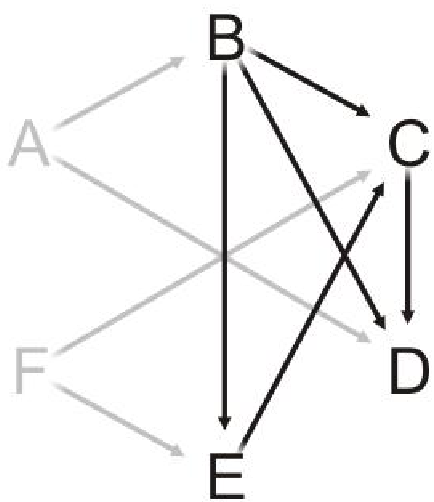
Queue

B			

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	5.2	A
C	3	4.7	0.0	Ø
D	2	8.1	5.2	A
E	2	9.5	0.0	Ø
F	0	17.1	0.0	Ø

Finding the critical path

Pop Task F and update its critical time 0.0 + 17.1 = 17.1



Queue

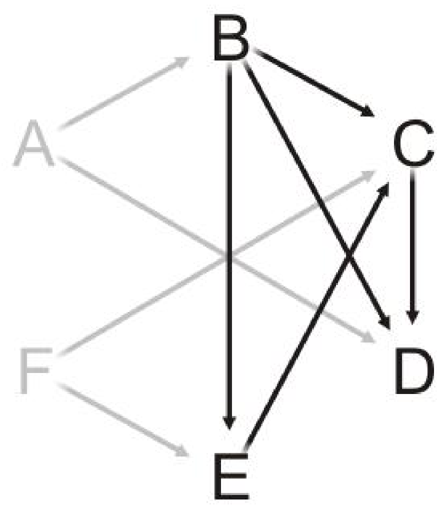
B			

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	5.2	A
C	3	4.7	0.0	Ø
D	2	8.1	5.2	A
E	2	9.5	0.0	Ø
F	0	17.1	**17.1**	Ø

Finding the critical path

For each neighbor of Task F:

- Decrement the in-degree, push if necessary, and check if we must update the critical time



Queue

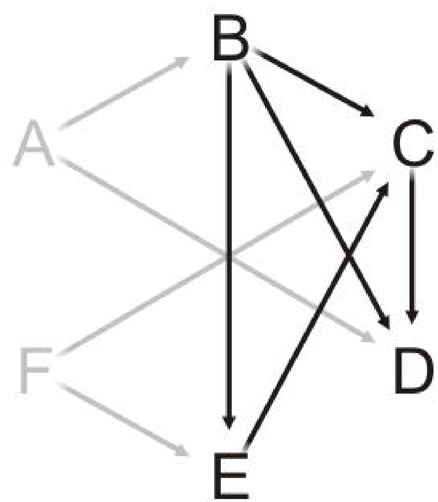
B			

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	5.2	A
C	3	4.7	0.0	Ø
D	2	8.1	5.2	A
E	2	9.5	0.0	Ø
F	0	17.1	17.1	Ø

Finding the critical path

For each neighbor of Task F:

– Decrement the in-degree, push if necessary, and check if we must update the critical time



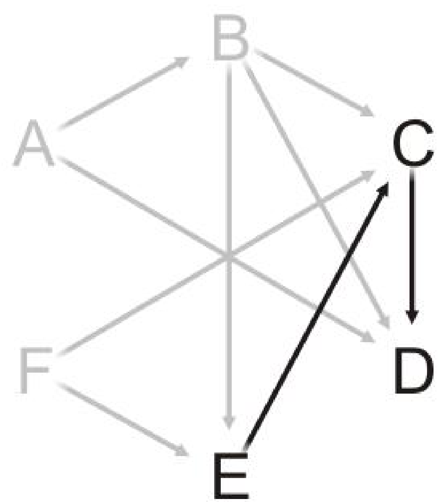
Queue

B			

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	5.2	A
C	**2**	4.7	**17.1**	**F**
D	2	8.1	5.2	A
E	**1**	9.5	**17.1**	**F**
F	0	17.1	17.1	Ø

Finding the critical path

Pop Task B and update its critical time 5.2 + 6.1 = 11.3

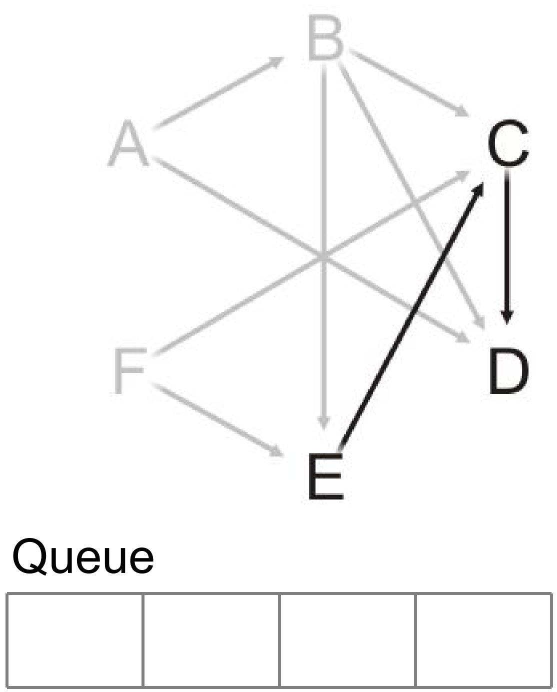


Queue

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	5.2	A
C	2	4.7	17.1	F
D	2	8.1	5.2	A
E	1	9.5	17.1	F
F	0	17.1	17.1	Ø

Finding the critical path

Pop Task B and update its critical time 5.2 + 6.1 = 11.3

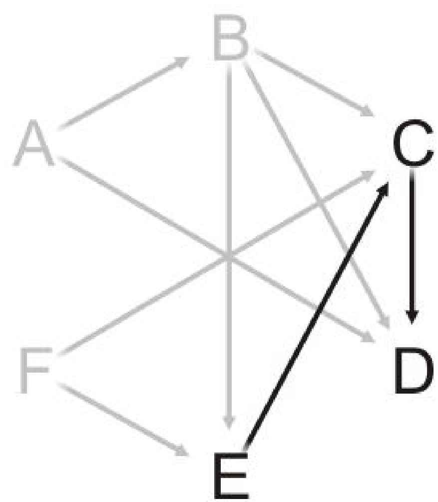


Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	**11.3**	A
C	2	4.7	17.1	F
D	2	8.1	5.2	A
E	1	9.5	17.1	F
F	0	17.1	17.1	Ø

Finding the critical path

For each neighbor of Task B:

– Decrement the in-degree, push if necessary, and check if we must update the critical time



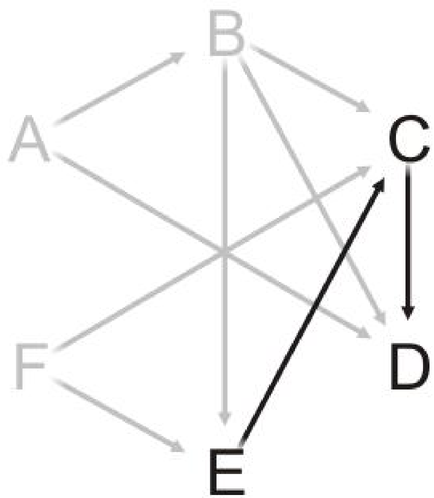
Queue

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	2	4.7	17.1	F
D	2	8.1	5.2	A
E	1	9.5	17.1	F
F	0	17.1	17.1	Ø

Finding the critical path

For each neighbor of Task F:

- Decrement the in-degree, push if necessary, and check if we must update the critical time
- Both C and E are waiting on F



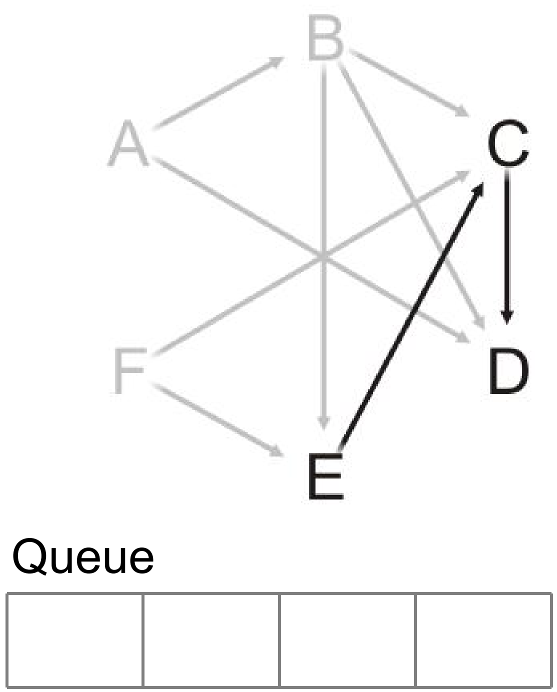
Queue

E			

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	**1**	4.7	17.1	F
D	**1**	8.1	**11.3**	**B**
E	**0**	9.5	17.1	F
F	0	17.1	17.1	Ø

Finding the critical path

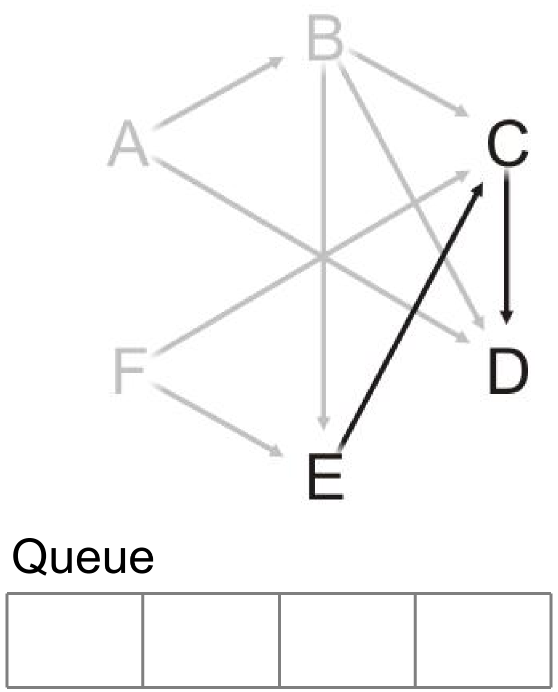
Pop Task E and update its critical time 17.1 + 9.5 = 26.6



Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	1	4.7	17.1	F
D	1	8.1	11.3	B
E	0	9.5	17.1	F
F	0	17.1	17.1	Ø

Finding the critical path

Pop Task E and update its critical time 17.1 + 9.5 = 26.6

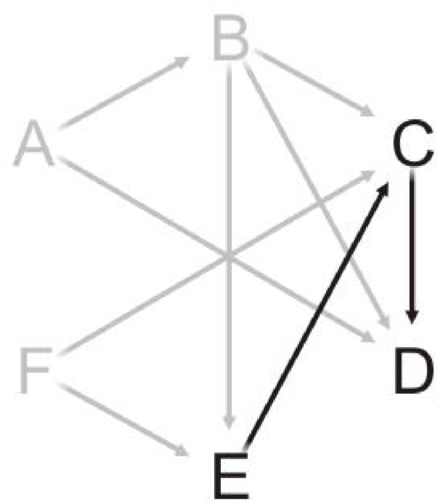


Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	1	4.7	17.1	F
D	1	8.1	11.3	B
E	0	9.5	**26.6**	F
F	0	17.1	17.1	Ø

Finding the critical path

For each neighbor of Task E:

– Decrement the in-degree, push if necessary, and check if we must update the critical time



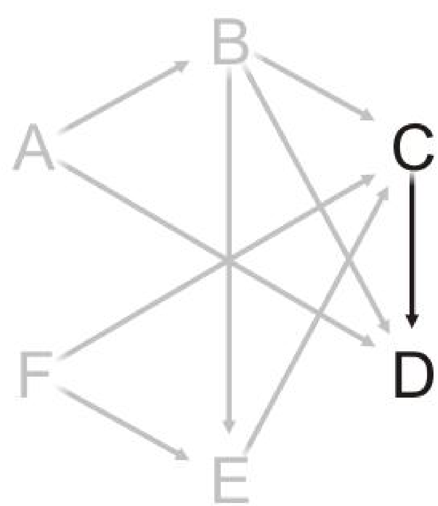
Queue

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	1	4.7	17.1	F
D	1	8.1	11.3	B
E	0	9.5	26.6	F
F	0	17.1	17.1	Ø

Finding the critical path

For each neighbor of Task E:

- Decrement the in-degree, push if necessary, and check if we must update the critical time



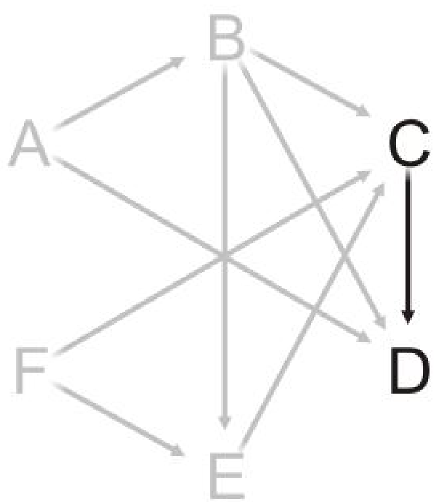
Queue

C			

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	**0**	4.7	**26.6**	**E**
D	1	8.1	11.3	B
E	0	9.5	26.6	F
F	0	17.1	17.1	Ø

Finding the critical path

Pop Task C and update its critical time 26.6 + 4.7 = 31.3

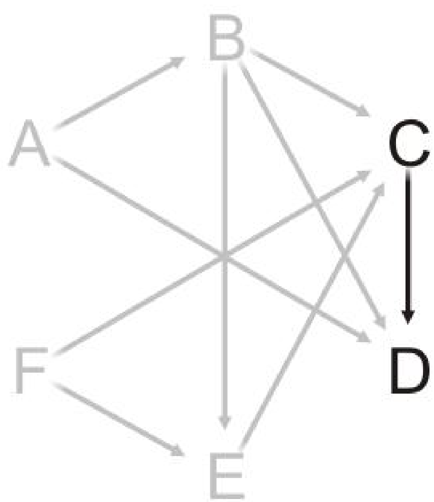


Queue

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	0	4.7	26.6	E
D	1	8.1	11.3	B
E	0	9.5	26.6	F
F	0	17.1	17.1	Ø

Finding the critical path

Pop Task C and update its critical time 26.6 + 4.7 = 31.3



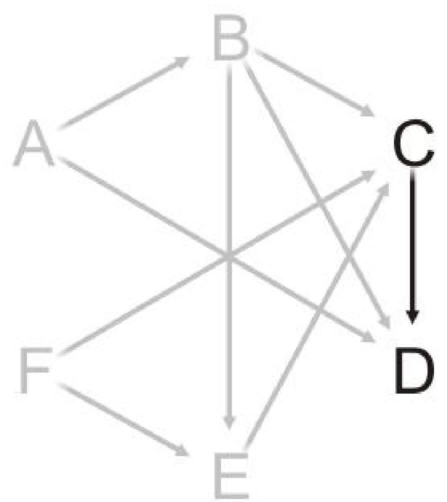
Queue

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	0	4.7	**31.3**	E
D	1	8.1	11.3	B
E	0	9.5	26.6	F
F	0	17.1	17.1	Ø

Finding the critical path

For each neighbor of Task C:

- Decrement the in-degree, push if necessary, and check if we must update the critical time



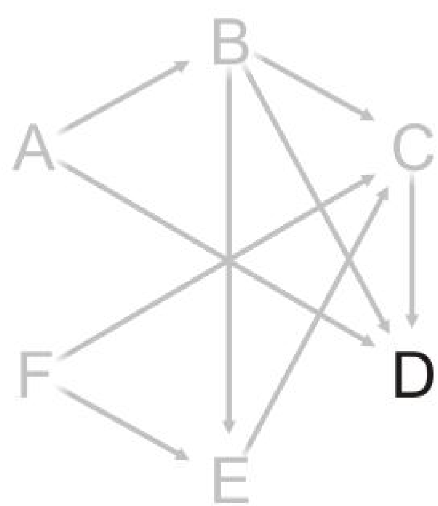
Queue

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	0	4.7	31.3	E
D	1	8.1	11.3	B
E	0	9.5	26.6	F
F	0	17.1	17.1	Ø

Finding the critical path

For each neighbor of Task C:

– Decrement the in-degree, push if necessary, and check if we must update the critical time



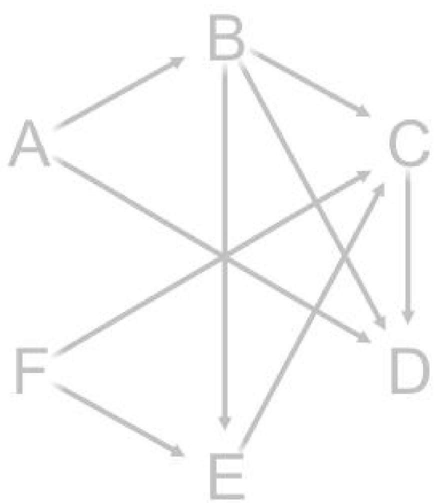
Queue

D			

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	0	4.7	31.3	E
D	**0**	8.1	**31.3**	**C**
E	0	9.5	26.6	F
F	0	17.1	17.1	Ø

Finding the critical path

Pop Task D and update its critical time 31.3 + 8.1 = 39.4

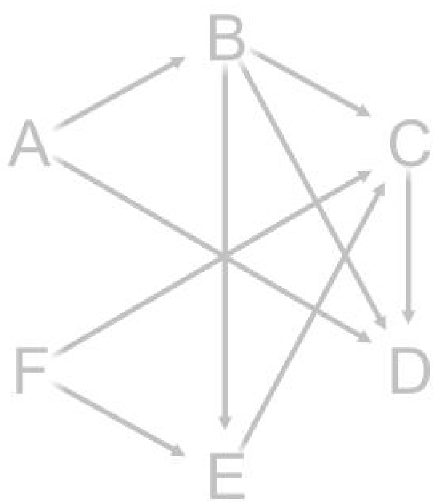


Queue

Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	0	4.7	31.3	E
D	0	8.1	31.3	C
E	0	9.5	26.6	F
F	0	17.1	17.1	Ø

Finding the critical path

Pop Task D and update its critical time 31.3 + 8.1 = 39.4



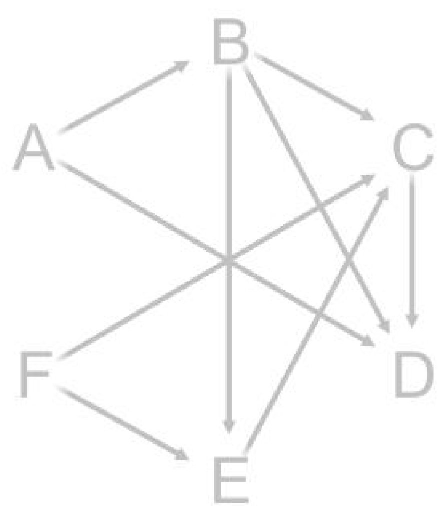
Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	0	4.7	31.3	E
D	0	8.1	**39.4**	C
E	0	9.5	26.6	F
F	0	17.1	17.1	Ø

Queue

Finding the critical path

Task D has no neighbors and the queue is empty

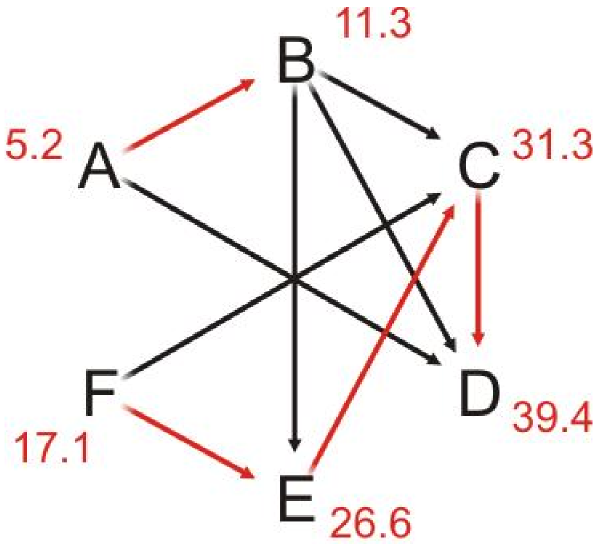
- We are done



Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	0	4.7	31.3	E
D	0	8.1	39.4	C
E	0	9.5	26.6	F
F	0	17.1	17.1	Ø

Queue

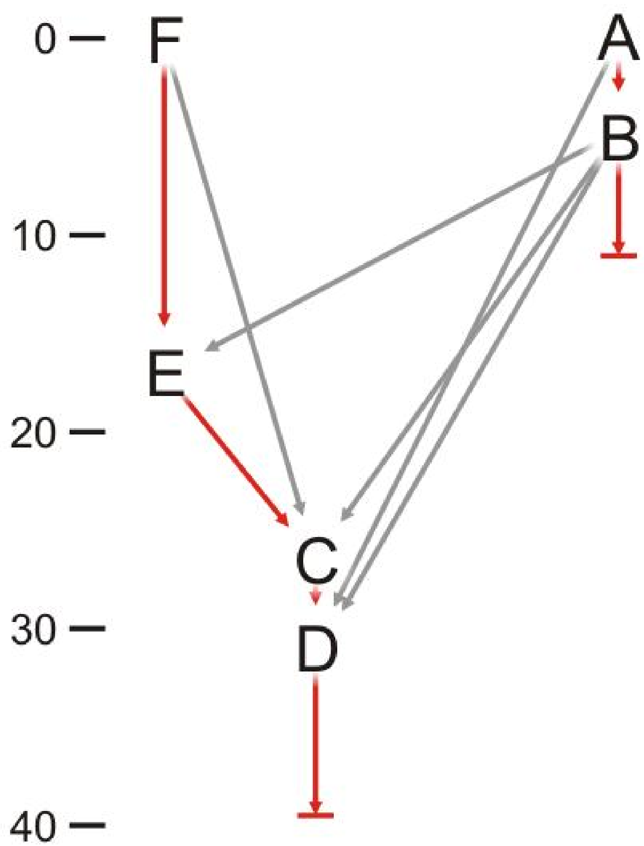
Finding the critical path



Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	0	4.7	31.3	E
D	0	8.1	39.4	C
E	0	9.5	26.6	F
F	0	17.1	17.1	Ø

Finding the critical path

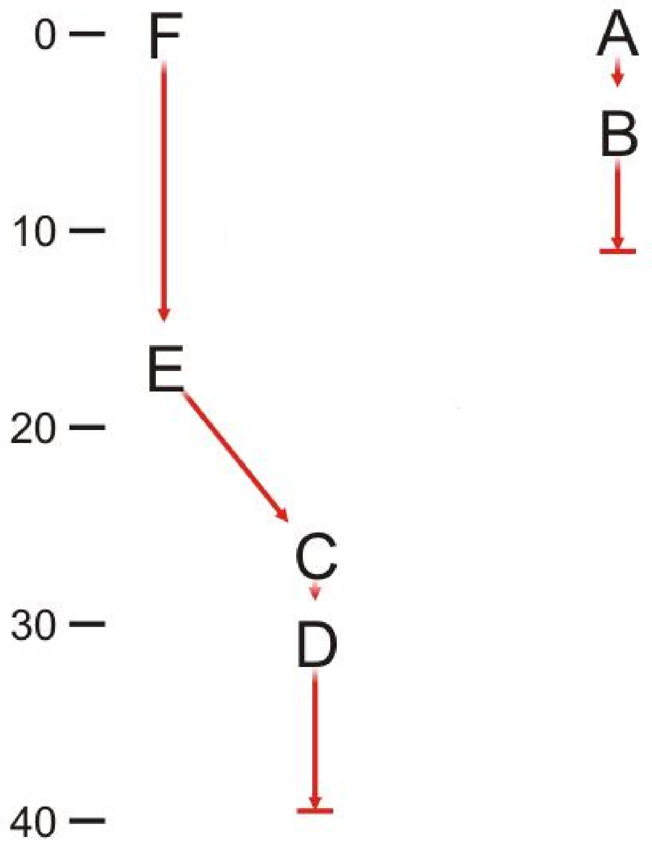
We can also plot the completing of the tasks in time



Task	In-degree	Task Time	Critical Time	Previous Task
A	0	5.2	5.2	Ø
B	0	6.1	11.3	A
C	0	4.7	31.3	E
D	0	8.1	39.4	C
E	0	9.5	26.6	F
F	0	17.1	17.1	Ø

Finding the critical path

Incidentally, the task and previous task defines a forest using the parental tree data structure



Task	Previous Task
A	Ø
B	A
C	E
D	C
E	F
F	Ø

Summary

In this topic, we have discussed topological sorts

- Sorting of elements in a DAG
- Implementation
  - A table of in-degrees
  - Select that vertex which has current in-degree zero
- We defined critical paths
  - The implementation requires only a few more table entries