1. (10 points) Let $P(x)$ = "x is a person", $L(x, y)$ = "x likes y" and $E(x, y)$ ="$x = y$". Translate the following statements into formulas:

(a) "Every person likes some other person."

(b) "There is a person who is liked by every other person."

(a) $\forall x\Big(P(x) \rightarrow \exists y\big(L(x,y) \wedge \neg E(x,y)\big)\Big)$

(b). $(\exists x\, P(x)) \wedge \Big(\forall y\big(L(x,y) \wedge \neg E(x,y)\big)\Big)$

2. (10 points) Let A be the formula $\forall x\,\big(\forall y\,(\,(x \neq y) \rightarrow \forall z\,((z = x) \vee (z = y))\,)\,\big)$

(a) Find a domain $D_1 \neq \emptyset$ such that A is true when x, y, z are taken over D_1.

(b) Find a domain D_2 such that A is false when x, y, z are taken over D_2.

(a) $D_1 = \{1, 2\}$

(b) $D_2 = R$ (the set of real numbers)

3. (10 points) Determine if the following formulas are logically valid, satisfiable or unsatisfiable.

(a) $(\exists x P(x) \leftrightarrow \exists x Q(x)) \rightarrow \exists x(P(x) \leftrightarrow Q(x))$

(b) $\exists x(\mathbf{T} \vee P(x) \rightarrow \mathbf{F})$

(c) $\forall x(P(x) \vee \neg \exists y(Q(y) \wedge \neg Q(y)))$

(a) $(\exists x P(x) \leftrightarrow \exists x Q(x)) \equiv (\neg \exists x P(x) \vee \exists x Q(x)) \wedge (\exists x P(x) \vee \neg \exists x Q(x))$

$\Rightarrow \neg \exists x P(x) \vee \exists x Q(x)$

$\equiv \neg(\exists x P(x) \wedge \neg \exists x Q(x))$

$\Rightarrow \neg \exists x P(x)$

$\exists x(P(x) \leftrightarrow Q(x)) \equiv \exists x((\neg P(x) \vee Q(x)) \wedge (P(x) \vee \neg Q(x))$

$\Rightarrow \exists x((\neg P(x) \vee Q(x))$

$\equiv \exists x(\neg(P(x) \wedge \neg Q(x)))$

$\Rightarrow \exists x \neg P(x)$

So (a) is satisfiable

(b) $T \vee P(x) \rightarrow F \equiv T \rightarrow F \equiv F$

So (b) is unsatisfiable

(c) $P(x) \vee \neg \exists y(Q(y) \wedge \neg Q(y)) \equiv P(x) \vee \neg \exists y F$

$\equiv P(x) \vee T$

$\equiv T$

So (c) is logically valid

4. (20 points) Show the following statements with interpretations of the formulas.

(a) $\forall x(P(x) \vee Q(x))$ and $\forall x P(x) \vee \forall x Q(x)$ are not logically equivalent.

(b) $\exists x(P(x) \wedge Q(x))$ and $\exists x P(x) \wedge \exists x Q(x)$ are not logically equivalent.

(a) Suppose that $\forall x (P(x) \vee Q(x))$ is T in an interpretation I

$P(x) \vee Q(x)$ is T for every x in I

but $P(x)$ is not T for every x in I and $Q(x)$ is not T for every x in I

So there is an x_0 such that $P(x_0) \equiv Q(x_0) \equiv F$

then $\forall x P(x_0) \vee \forall x Q(x_0) \equiv F$

so $\forall x P(x) \vee \forall x Q(x)$ is not T in I

So $\forall x (P(x) \vee Q(x)) \rightarrow \forall x P(x) \vee \forall x Q(x)$ is not logically valid.

So $\forall x (P(x) \vee Q(x))$ and $\forall x P(x) \vee \forall x Q(x)$ are not logically equivalent.

4. (20 points) Show the following statements with interpretations of the formulas.

(a) $\forall x(P(x) \vee Q(x))$ and $\forall x P(x) \vee \forall x Q(x)$ are not logically equivalent.

(b) $\exists x(P(x) \wedge Q(x))$ and $\exists x P(x) \wedge \exists x Q(x)$ are not logically equivalent.

(b) Suppose that $\exists x P(x) \wedge \exists x Q(x)$ is T in an interpretation I

so there are x_1, x_2 such that $\exists x P(x_1) \equiv \exists x Q(x_2) \equiv T$

but for x in I, $\exists x P(x)$ and $\exists x Q(x)$ may not be both T at the same time.

So $\exists x P(x) \wedge \exists x Q(x) \rightarrow \exists x (P(x) \wedge Q(x))$ is not logically valid

So $\exists x (P(x) \wedge Q(x))$ and $\exists x P(x) \wedge \exists x Q(x)$ are not logically equivalent.

5. (10 points) Show that $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$.

(1) Suppose that $\exists x (P(x) \vee Q(x))$ is T in an interpretation I

there is an x_0 such that $\exists x(P(x_0) \vee Q(x_0))$ is T in I

there is an x_0 such that $P(x_0) \vee Q(x_0)$ is T in I

there is an x_0 such that $P(x_0) \equiv Q(x_0) \equiv T$ or $P(x_0) \equiv T, Q(x_0) \equiv F$ or $P(x_0) \equiv F, Q(x_0) \equiv T$ in I

there is an x_0 such that $\exists x P(x_0) \vee \exists x Q(x_0) \equiv T$ in I

So $\exists x P(x) \vee \exists x Q(x)$ is T in I

and $\exists x (P(x) \vee Q(x)) \rightarrow \exists x P(x) \vee \exists x Q(x)$ is logically valid

(2) Suppose that $\exists x P(x) \vee \exists x Q(x)$ is T in an interpretation I

there is an x_0 such that $\exists x P(x_0) \equiv \exists x Q(x_0) \equiv T$ or $\exists x P(x_0) \equiv T$, $\exists x Q(x) \equiv F$ or $\exists x P(x_0) \equiv F$, $\exists x Q(x_0) \equiv T$ in I

there is an x_0 such that $P(x_0) \equiv Q(x_0) \equiv T$ or $P(x_0) \equiv T, Q(x_0) \equiv F$ or $P(x_0) \equiv F, Q(x_0) \equiv T$ in I

there is an x_0 such that $P(x_0) \vee Q(x_0)$ is T in I

there is an x_0 such that $\exists x (P(x_0) \vee Q(x_0))$ is T in I

So $\exists x (P(x) \vee Q(x))$ is T in I

and $\exists x P(x) \vee \exists x Q(x) \rightarrow \exists x (P(x) \vee Q(x))$ is logically valid

Conclusion. $\exists x (P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$

6. (20 points) Show that $\forall x\,(P(x) \to Q(x)) \Rightarrow \forall x P(x) \to \forall x\, Q(x)$.

(1) $\forall x\,(P(x) \to Q(x))$ premise

(2) $P(a) \to Q(a)$ Universal Instantiation from (1)

(3) $P(a)$ from (2)

(4) $\forall x P(x)$ Universal Generalization from (3)

(5) $Q(a)$ from (2)

(6) $\forall x\, Q(x)$ Universal Generalization from (5)

(7) $\forall x P(x) \to \forall x\, Q(x)$ from (4) and (6)

7. (20 points) Show that $\exists x P(x) \land \forall x Q(x) \Rightarrow \exists x(P(x) \land Q(x))$.

(1) $\exists x P(x)$ premise

(2) $P(a)$ Existential Instantiation from (1)

(3) $\forall x Q(x)$ premise

(4) $Q(a)$ Universal Instantiation from (3)

(5) $P(a) \land Q(a)$ from (2) and (4)

(6) $\exists x (P(x) \land Q(x))$ Existential Generalization from (5)