1. (20 points) Show that if $n > 6$, then $p_3(n) = p_3(n-6) + n - 3$.

let $t = n-6$ so that $t \in Z^+$ because $n \in Z^+$ and $n > 6$

so we need to prove $p_3(t+6) = p_3(t) + t + 3$

$$p_3(t+6) = p_3[(t+3)+3] = p_1(t+3) + p_2(t+3) + p_3(t+3)$$

$$= 1 + p_2[(t+1)+2] + p_3(t+3)$$

$$= 1 + p_1(t+1) + p_2(t+1) + p_1(t) + p_2(t) + p_3(t)$$

$$= 1 + 1 + p_2(t+1) + 1 + p_2(t) + p_3(t)$$

$$= [p_2(t+1) + p_2(t)] + p_3(t) + 3$$

if t is even then $p_2(t+1) + p_2(t) = \frac{t}{2} + \frac{t}{2} = t$

if t is odd then $p_2(t+1) + p_2(t) = \frac{t+1}{2} + \frac{t-1}{2} = t$

so $p_2(t+1) + p_2(t) = t$ for any $t \in Z^+$

so $p_3(t+6) = p_3(t) + t + 3$

and so is $p_3(n) = p_3(n-6) + n - 3$

2. (20 points) Suppose that $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where $k \geq 2$, $e_i \geq 1$ for all $i \in [k]$, and $p_1, p_2, \ldots, p_k$ are k distinct primes. Show that $\phi(n) = n(1 - 1/p_1) \cdots (1 - 1/p_k)$ by using the principle of inclusion-exclusion.

(**Hint**: Calculate the number of integers in $[n] = \{1, 2, \ldots, n\}$ that can be divided by at least one of the primes.)

let s, t be common divisors of n.

then in $[n] = \{1, 2, \ldots, n\}$

1' the multiple of s is $s, 2s, 3s, \cdots, (\frac{n}{s})s$

the total number is $\frac{n}{s}$

2' the multiple of t is $t, 2t, 3t, \cdots, (\frac{n}{t})t$

the total number is $\frac{n}{t}$

so $\phi(n) = n - \frac{n}{s} - \frac{n}{t} + \frac{n}{st}$ because $\frac{n}{st}$ numbers have been subtracted twice (according to the principle of inclusion-exclusion)

$$\phi(n) = n - \frac{n}{s} - \frac{n}{t} + \frac{n}{st}$$

$$= n\left(1 - \frac{1}{s} - \frac{1}{t} + \frac{1}{st}\right)$$

$$= n\left(1 - \frac{1}{s}\right)\left(1 - \frac{1}{t}\right)$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

3. (20 points) Let $a \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Show that there exist $p, q \in \mathbb{Z}$ such that $p \in [n]$ and

$$\left|a - \frac{q}{p}\right| < \frac{1}{n}.$$

for $x_0, x_1, \cdots, x_n \in [0,1)$

divide $[0,1)$ into n compartments $[0,\frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), \cdots [\frac{n-1}{n}, 1)$

so there are definitely two numbers x_i, x_j $(0 \le i < j \le n)$

in $n+1$ numbers $x_0, x_1, \cdots, x_n$ such that x_i and x_j

are in the same interval. thus $|x_i - x_j| < \frac{1}{n}$

let $m_i = [ia]$. $i = 0, 1, 2, \cdots, n$

then $m_i \le ia < m_i + 1 \Rightarrow 0 \le i - m_i < 1$

according to the theorem above

there exist $0 \le k < l \le n$ such that

$|(la - m_l) - (ka - m_k)| < \frac{1}{n}$

that is $|(l-k) - (m_l - m_k)| < \frac{1}{n}$

let $p = l - k$. $q = m_l - m_k$. then $p, q \in \mathbb{Z}$

so $|pa - q| < \frac{1}{n}$ that is $\left|a - \frac{q}{p}\right| < \frac{1}{np} < \frac{1}{n}$

4. (20 points) Solve $a_n = 8a_{n-2} - 16a_{n-4}$ with $a_0 = 3, a_1 = 6, a_2 = 44$, and $a_3 = 56$.

characteristic equation : $r^4 - 8r^2 + 16 = 0$

$(r^2 - 4)^2 = 0 \quad r_1 = 2 \quad r_2 = -2$

(1) n is even. $a_n = \alpha_1 r_1^{\frac{n}{2}} + \alpha_2 r_2^{\frac{n}{2}}$

$$\begin{cases} a_0 = \alpha_1 + \alpha_2 = 3 \\ a_2 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-2) = 44 \end{cases} \Rightarrow \begin{cases} \alpha_1 = \frac{25}{2} \\ \alpha_2 = -\frac{19}{2} \end{cases}$$

$$a_n = \frac{25}{2} \cdot 2^n + \left(-\frac{19}{2}\right) \cdot (-2)^n = 25 \cdot 2^{\frac{n}{2}-1} - 19 \cdot (-2)^{\frac{n}{2}-1}$$

(2) n is odd $a_n = \alpha_1 r_1^{\frac{n-1}{2}} + \alpha_2 r_2^{\frac{n-1}{2}}$

$$\begin{cases} a_1 = \alpha_1 + \alpha_2 = 6 \\ a_3 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-2) = 56 \end{cases} \Rightarrow \begin{cases} \alpha_1 = 17 \\ \alpha_2 = -11 \end{cases}$$

$$a_n = 17 \cdot 2^{\frac{n-1}{2}} - 11 \cdot (-2)^{\frac{n-1}{2}}$$

Conclusion :

$$a_n = \begin{cases} 17 \cdot 2^{\frac{n-1}{2}} - 11 \cdot (-2)^{\frac{n-1}{2}} & (n \text{ is odd}) \\ 25 \cdot 2^{\frac{n}{2}-1} - 19 \cdot (-2)^{\frac{n}{2}-1} & (n \text{ is even}) \end{cases}$$

5. (20 points) Solve $a_n = 3a_{n-1} - 2a_{n-2} + n \cdot 2^n$ with $a_0 = 1$ and $a_1 = -1$.

characteristic equation: $r^2 - 3r + 2 = 0$, $r_1 = 1$ $r_2 = 2$

$$a_n^{(h)} = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n = \alpha_1 + \alpha_2 \cdot 2^n$$

$$a_n^{(p)} = c \cdot n \cdot 2^n$$

$$c \cdot n \cdot 2^n = 3c(n-1) \cdot 2^{n-1} - 2c(n-2) \cdot 2^{n-2} + n \cdot 2^n$$

$$4cn = 6c(n-1) - 2c(n-2) + 4n$$

$$c = 2n$$

So $a_n = \alpha_1 + \alpha_2 \cdot 2^n + n^2 \cdot 2^{n+1}$

$$\begin{cases} a_0 = \alpha_1 + \alpha_2 = 1 \\ a_1 = \alpha_1 + 2\alpha_2 + 4 = -1 \end{cases} \Rightarrow \begin{cases} \alpha_1 = 7 \\ \alpha_2 = -6 \end{cases}$$

$$a_n = 7 - 6 \cdot 2^n + 2n^2 \cdot 2^n$$

$$= 7 + (2n^2 - 6) \cdot 2^n$$