

The Discrete-Time Fourier Transform (ch.5)

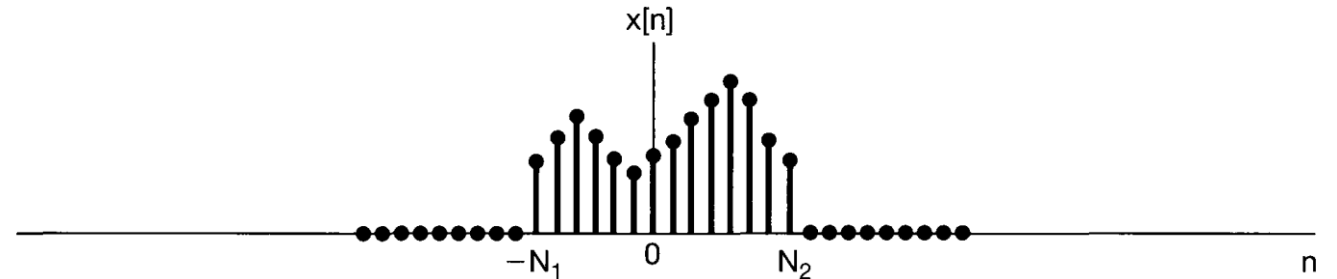
- ❑ Representation of aperiodic signals - Discrete Fourier transform
- ❑ Fourier transform for periodic signals
- ❑ Properties of discrete-time Fourier transform
- ❑ The convolution property
- ❑ The multiplication property
- ❑ Duality
- ❑ Systems characterized by difference equations

Discrete Fourier Transform

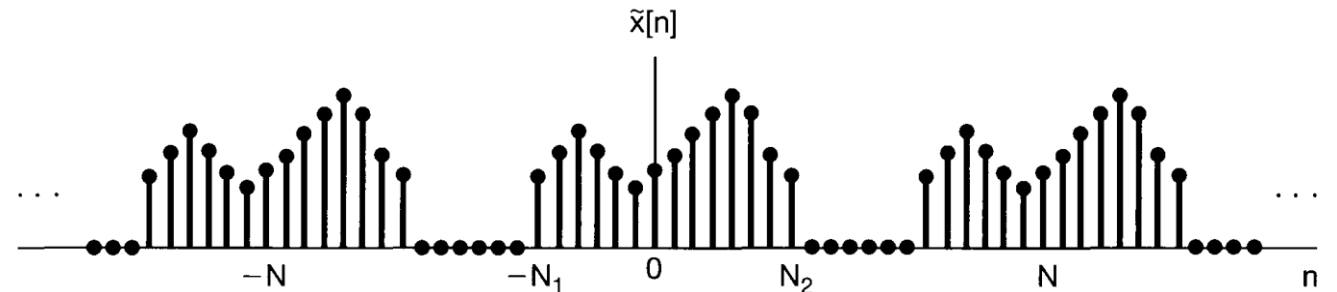


Representation of aperiodic signals

□ Consider a general sequence of finite duration: $x[n] = 0$ if $n < N_1$ or $n > N_2$



□ Periodic extension of $x[n]$ with N



□ FS representation of $\tilde{x}[n]$

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

Discrete Fourier Transform

Representation of aperiodic signals

□ FS coefficients of $\tilde{x}[n]$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(2\pi/N)n}$$

Define $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$

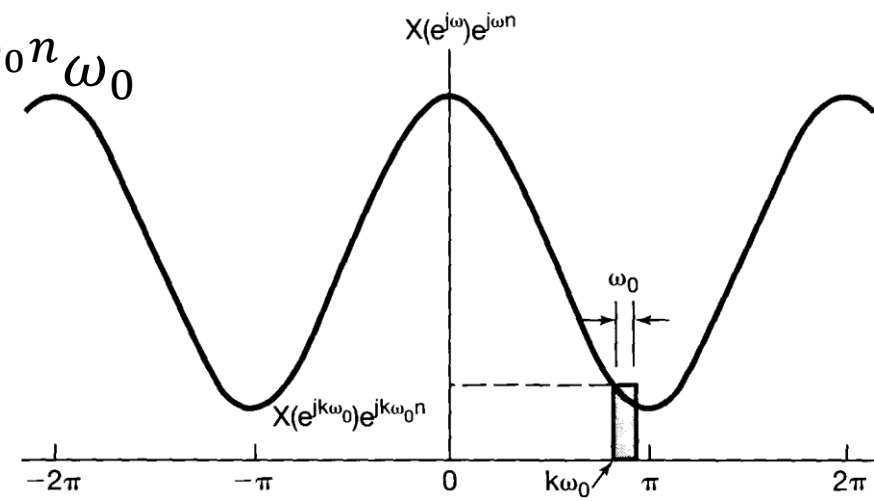
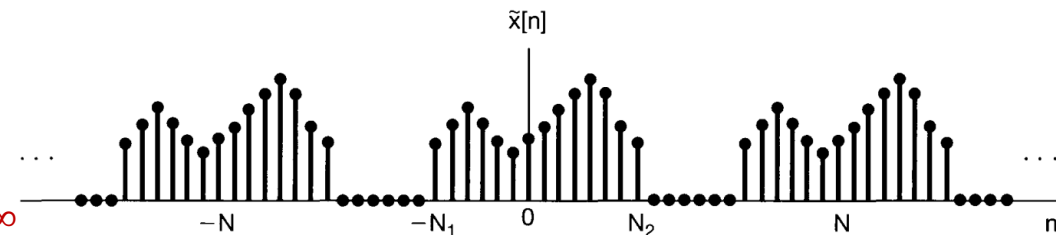
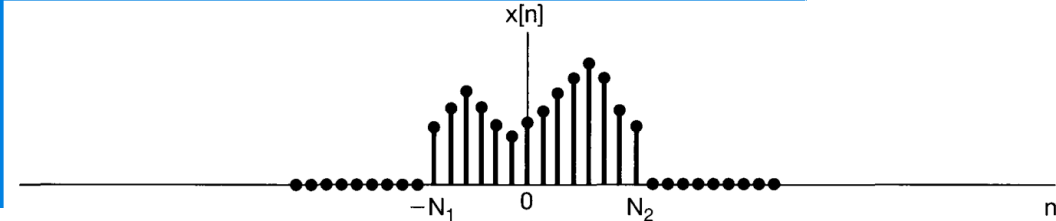
$$= \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N} X(e^{jk\omega_0})$$

□ FS of $\tilde{x}[n]$

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

□ $N \rightarrow \infty, \tilde{x}[n] \rightarrow x[n]$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$



Discrete Fourier Transform



FT pairs

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

Fourier transform (FT)

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Inverse Fourier transform

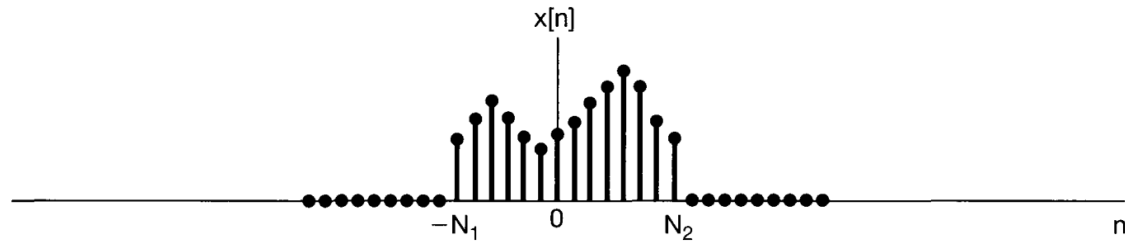
- ❑ $x[n]$ is a linear combination (specifically, an integral) of complex exponentials at different frequencies
- ❑ $X(e^{j\omega})(d\omega/2\pi)$ is the weight for different frequencies
- ❑ $X(e^{j\omega})$ is called the spectrum

Discrete Fourier Transform



FT vs. FS

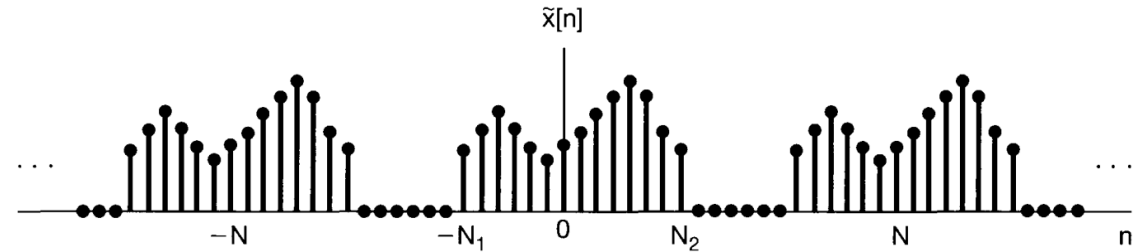
Fourier transform (FT)



$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

Fourier series (FS)



$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} X(e^{j\omega}) \text{ with } \omega = k(2\pi/N)$$

Discrete Fourier Transform



Discrete FT vs. continuous FT

Discrete FT

$$x[n] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(e^{j\omega}) e^{j\omega n} d\omega$$
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

Continuous FT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

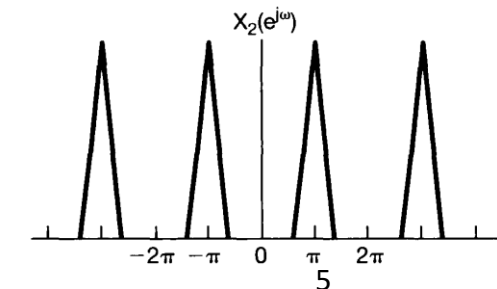
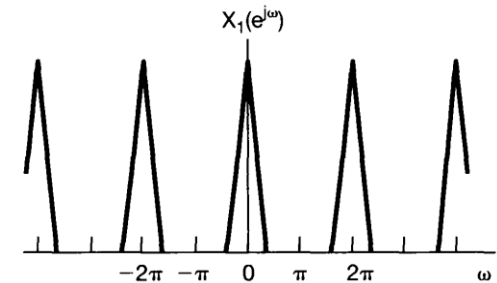
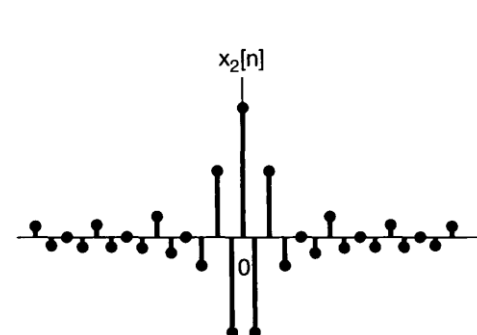
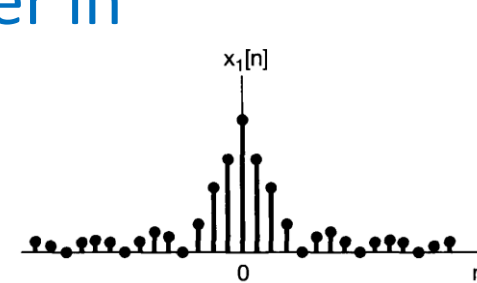
□ Discrete-time complex exponentials that differ in frequency by a multiple of 2π are identical

□ $X(e^{j\omega})$ is periodic

□ Finite interval of integration in the synthesis equation for $x[n]$

□ $\omega = 0, 2\pi, 4\pi, \dots \Rightarrow$ low-frequency

□ $\omega = \pi, 3\pi, 5\pi, \dots \Rightarrow$ high-frequency



Discrete Fourier Transform



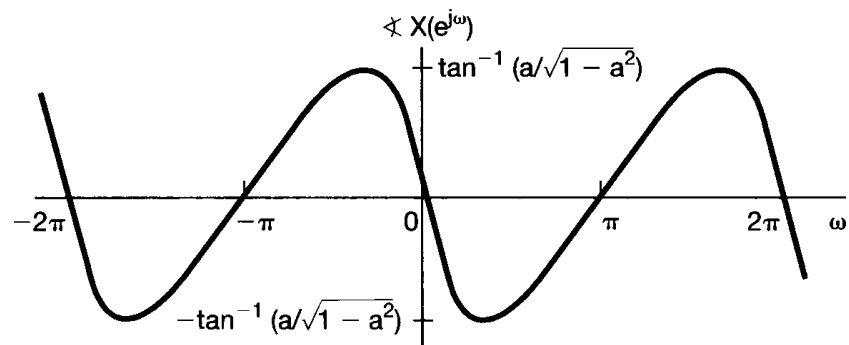
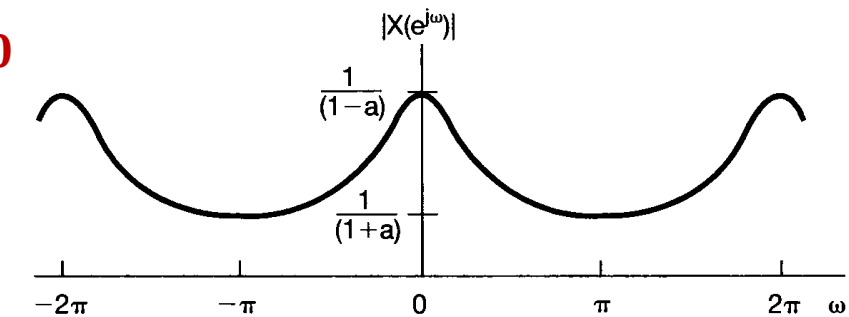
Examples

$$x[n] = a^n u[n], |a| < 1 \quad X(e^{j\omega}) = ?$$

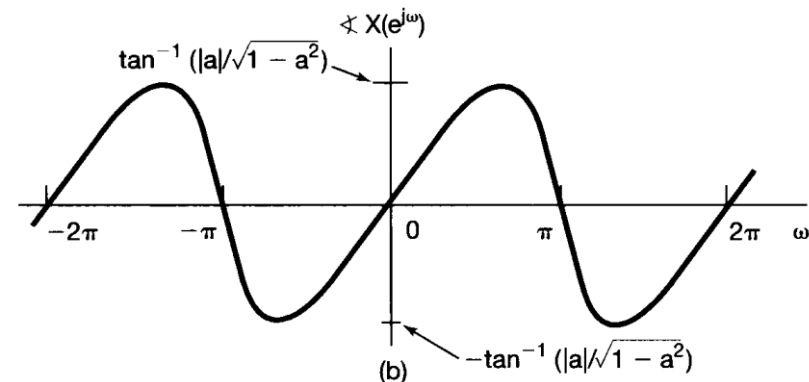
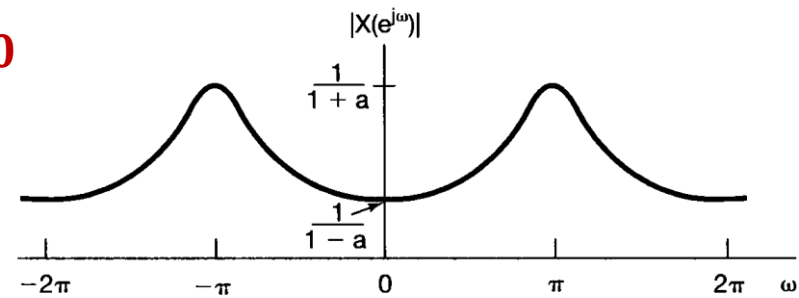
Solution

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

$a > 0$



$a < 0$



Discrete Fourier Transform



Examples

$$x[n] = a^{|n|}, |a| < 1$$

$$X(e^{j\omega}) = ?$$

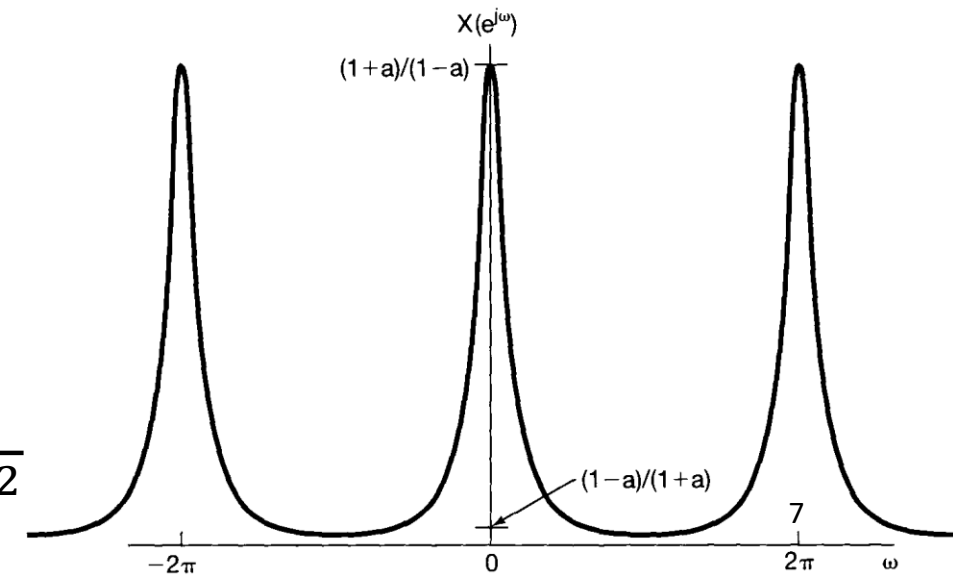
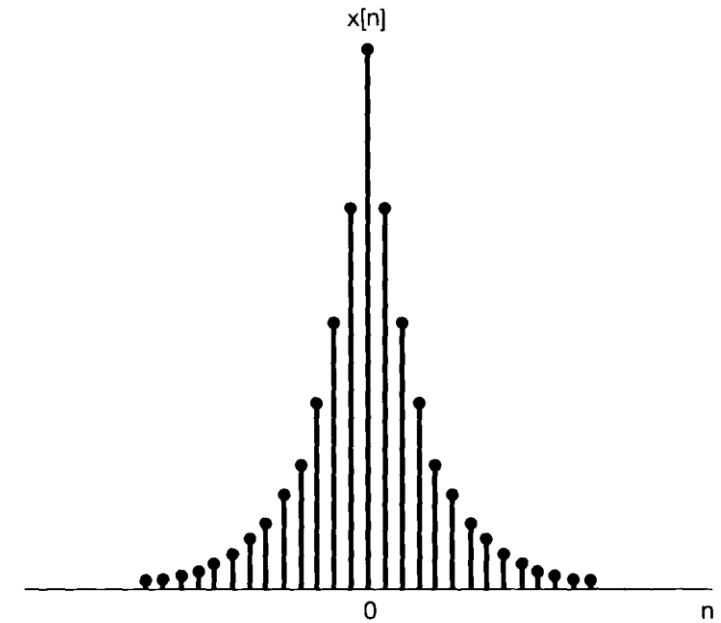
Solution

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n + \sum_{m=1}^{\infty} (ae^{j\omega})^m$$

$$= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} = \frac{1 - a^2}{1 - 2a \cos \omega + a^2}$$



Discrete Fourier Transform



Examples

$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases} \quad X(e^{j\omega}) = ?$$

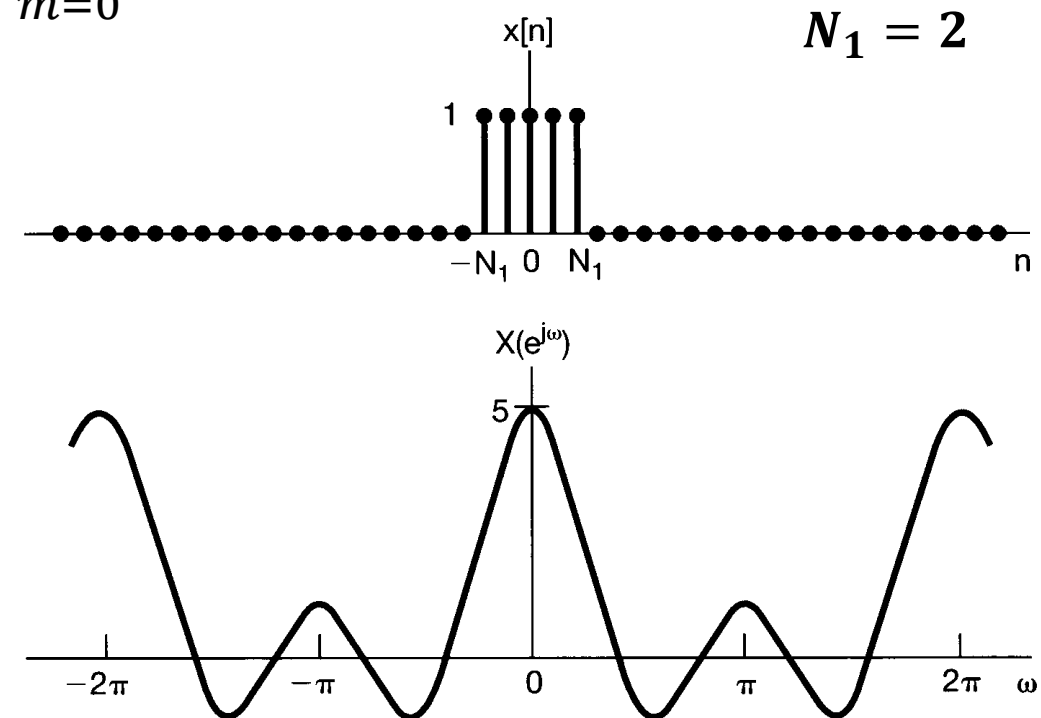
Solution

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \sum_{m=0}^{2N_1} e^{-j\omega(m-N_1)} = e^{j\omega N_1} \sum_{m=0}^{2N_1} e^{-j\omega m}$$

$$= e^{j\omega N_1} \left(\frac{1 - e^{-j\omega(2N_1+1)}}{1 - e^{-j\omega}} \right)$$

$$= \frac{e^{-j\omega/2} (e^{j\omega(N_1+1/2)} - e^{-j\omega(N_1+1/2)})}{e^{-j\omega/2} (e^{j\omega/2} - e^{-j\omega/2})}$$

$$= \frac{\sin[\omega(N_1 + 1/2)]}{\sin(\omega/2)}$$



Discrete Fourier Transform



Convergence of FT

□ For the analysis equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

- Finite energy condition
- Absolutely summable

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

□ For the synthesis equation $x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$

- No convergence issues (finite interval of integration)

Discrete Fourier Transform



Examples

$$x[n] = \delta[n]$$

$$X(e^{j\omega}) = ?$$

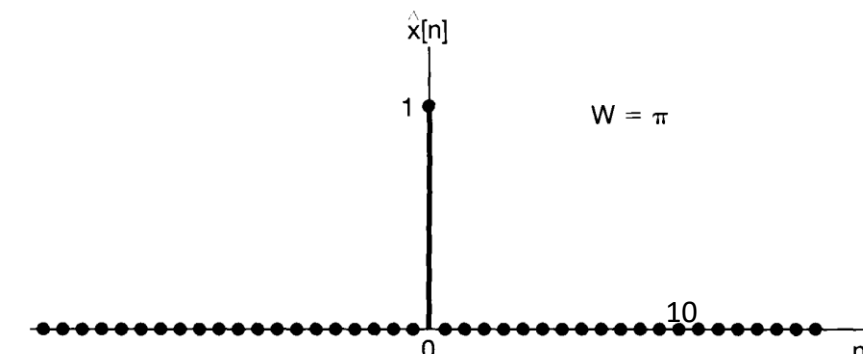
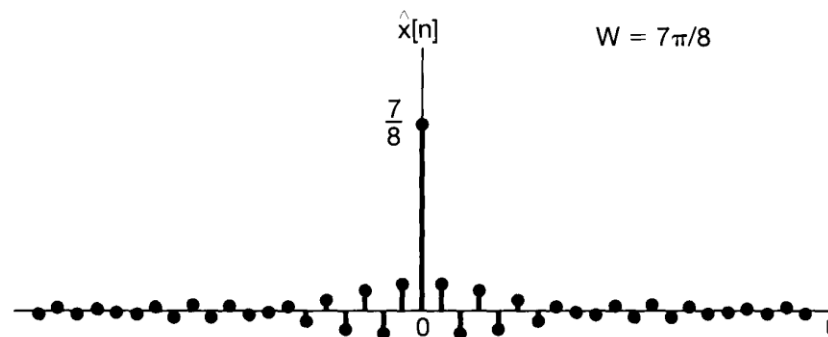
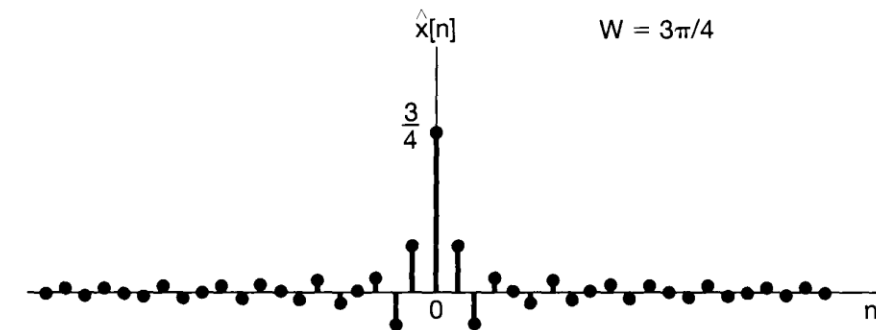
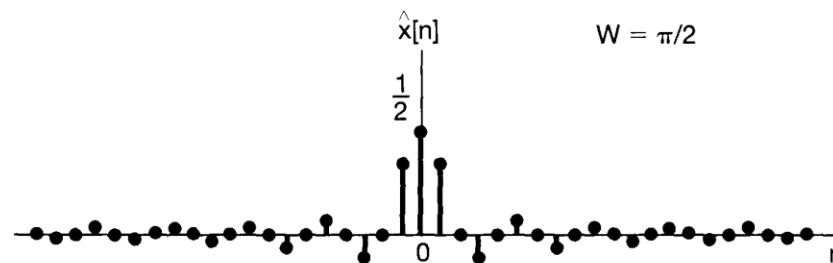
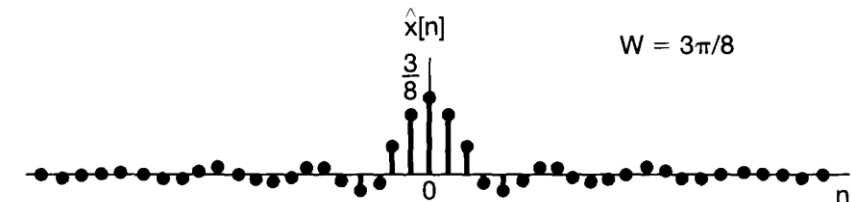
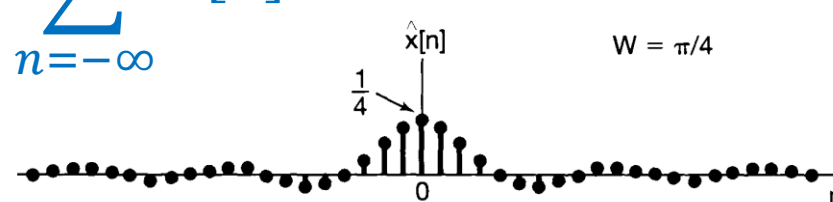
□ Solution 1 $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n] e^{-j\omega n} = 1$

□ Solution 2

$$\hat{x}[n] = \frac{1}{2\pi} \int_{-W}^W e^{j\omega n} d\omega$$

$$= \frac{\sin Wn}{\pi n}$$

$$\lim_{W \rightarrow \pi} \hat{x}[n] = \delta[n]$$



The Discrete-Time Fourier Transform

(ch.5)

- ❑ Representation of aperiodic signals- Discrete Fourier transform
- ❑ **Fourier transform for periodic signals**
- ❑ Properties of discrete-time Fourier transform
- ❑ The convolution property
- ❑ The multiplication property
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Fourier transform for periodic signals

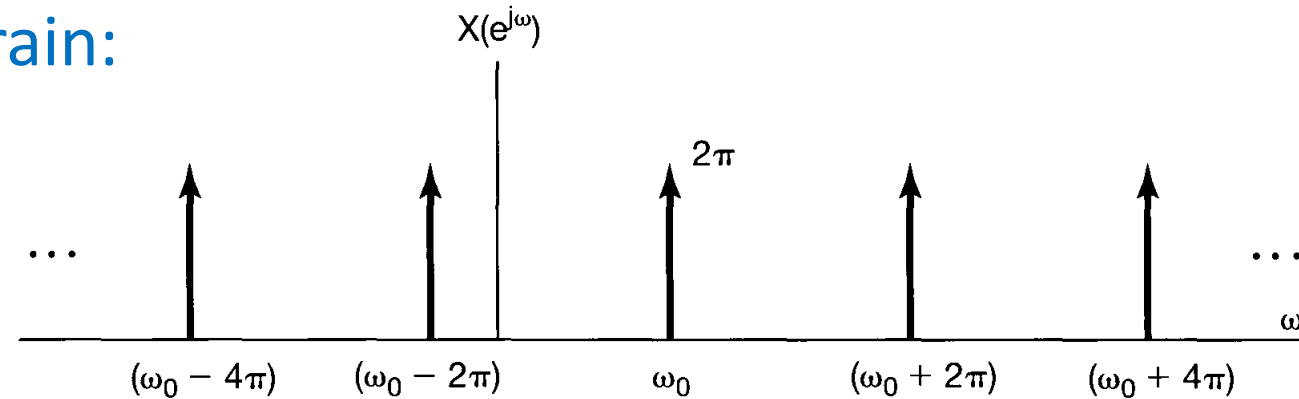


- Consider the sinusoidal signal

$$x[n] = e^{j\omega_0 n}$$

- The FT should be a periodic pulse train:

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$



- Check validity: evaluate the inverse transform

$$\begin{aligned} \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega &= \frac{1}{2\pi} \int_{2\pi} \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l) e^{j\omega n} d\omega \\ &= e^{j(\omega_0 + 2\pi r)n} \quad \text{Fixed in one period } l = r \text{ cause } \int_{2\pi} \\ &= e^{j\omega_0 n} \end{aligned}$$

Fourier transform for periodic signals



□ Consider a periodic sequence

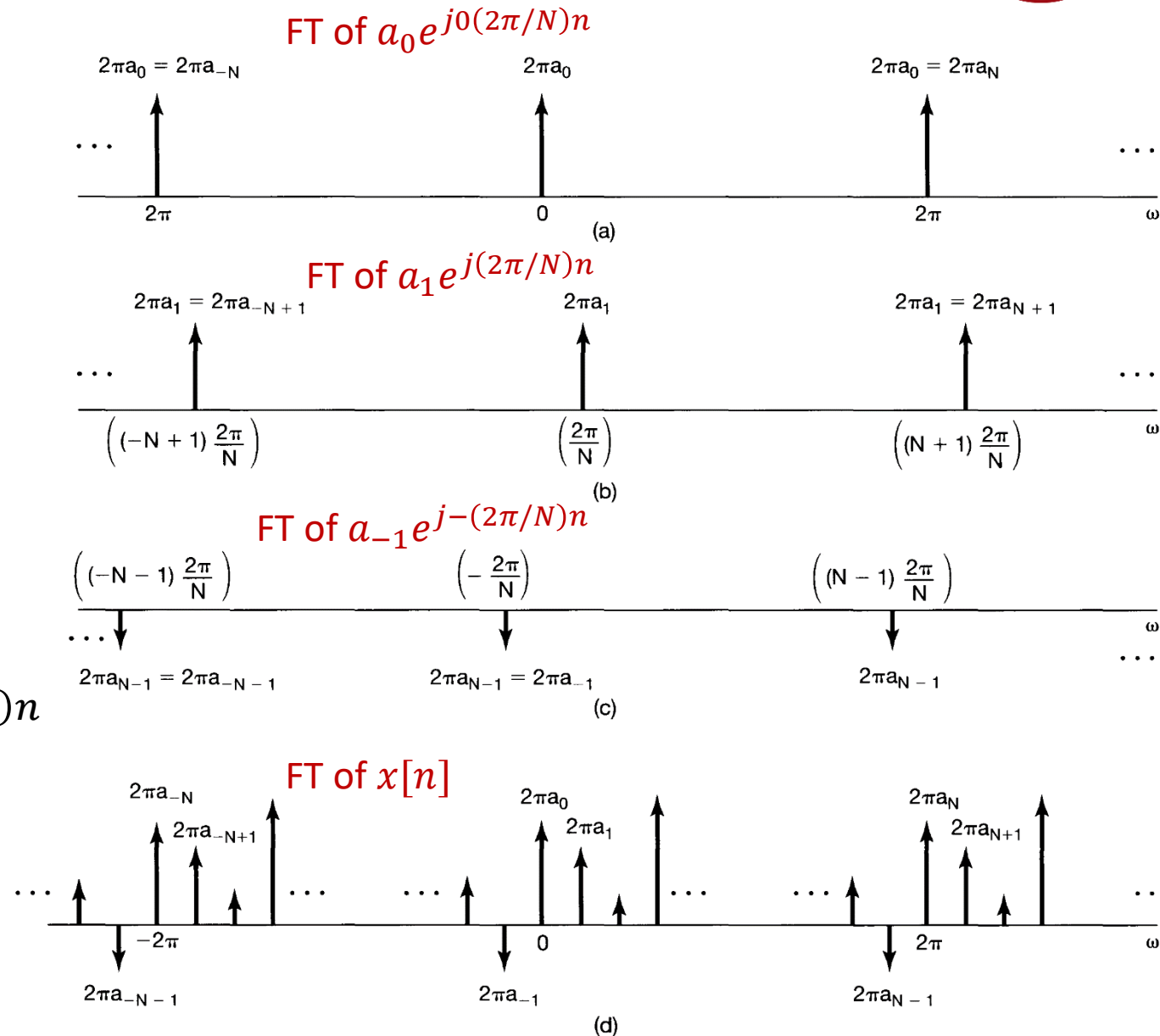
$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

□ FT

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - 2\pi k/N)$$

□ Verify

$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + a_2 e^{j2(2\pi/N)n} + \dots + a_{N-1} e^{j(N-1)(2\pi/N)n}$$



Fourier transform for periodic signals

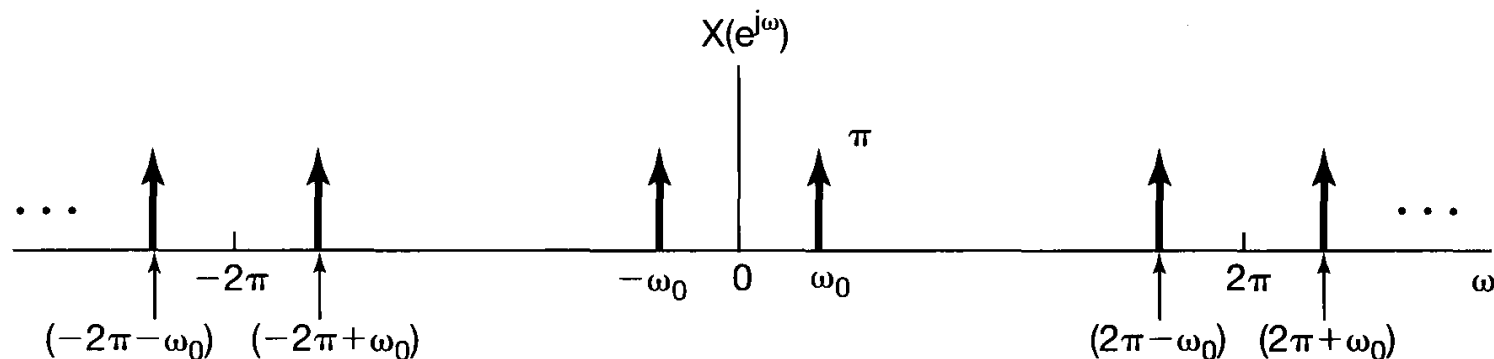


Examples

$$x[n] = \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}, \omega_0 = \frac{2\pi}{5} \quad X(e^{j\omega}) = ?$$

Solution

$$\begin{aligned} X(e^{j\omega}) &= \sum_{l=-\infty}^{+\infty} \pi \delta \left(\omega - \frac{2\pi}{5} - 2\pi l \right) + \sum_{l=-\infty}^{+\infty} \pi \delta \left(\omega + \frac{2\pi}{5} - 2\pi l \right) \\ &= \pi \delta \left(\omega - \frac{2\pi}{5} \right) + \pi \delta \left(\omega + \frac{2\pi}{5} \right), -\pi < \omega < \pi \end{aligned}$$



Fourier transform for periodic signals



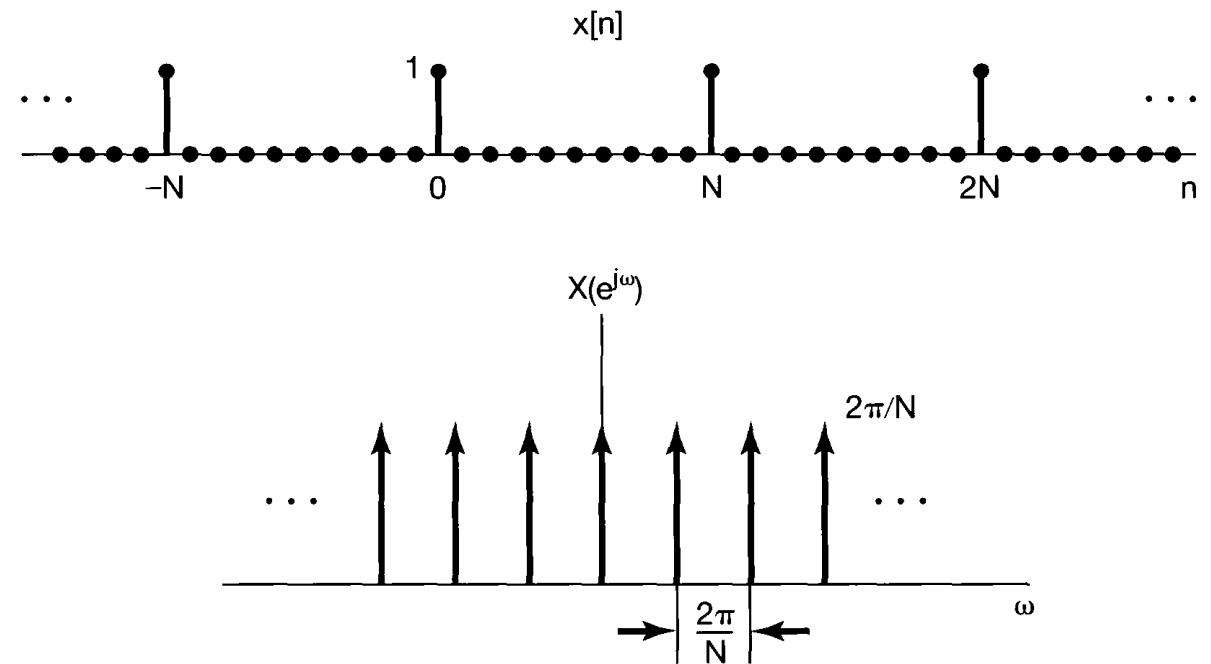
Examples

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN] \quad X(e^{j\omega}) = ?$$

Solution

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N}$$

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$



The Discrete-Time Fourier Transform (ch.5)

- ☐ Representation of aperiodic signals- Discrete Fourier transform
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Properties of discrete-time Fourier Transform



Short notation for FT pairs

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\}$$

$$x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$$



Properties of discrete-time Fourier Transform

Periodicity In contrast to continuous FT

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

Linearity

$$x_1[n] \xleftrightarrow{\mathcal{F}} X_1(e^{j\omega})$$

$$x_2[n] \xleftrightarrow{\mathcal{F}} X_2(e^{j\omega})$$

$$\Rightarrow ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{F}} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$



Properties of discrete-time Fourier Transform

Time shifting and frequency shifting

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \Rightarrow \begin{cases} x[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X(e^{j\omega}) \\ e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(e^{j(\omega - \omega_0)}) \end{cases}$$

Examples

Low-pass filter \Rightarrow

$$H_{lp}(e^{j\omega})$$

$$\updownarrow \mathcal{F}$$

$$h_{lp}[n]$$

shift by π

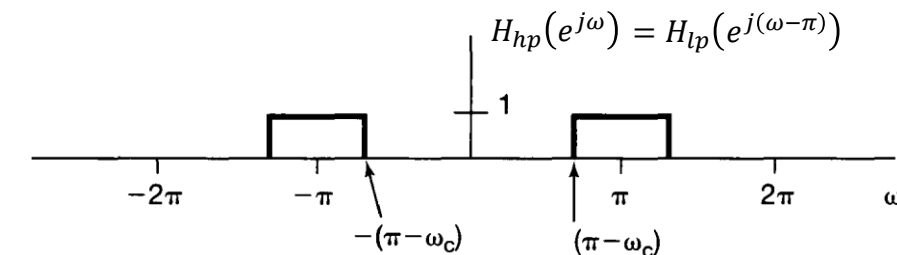
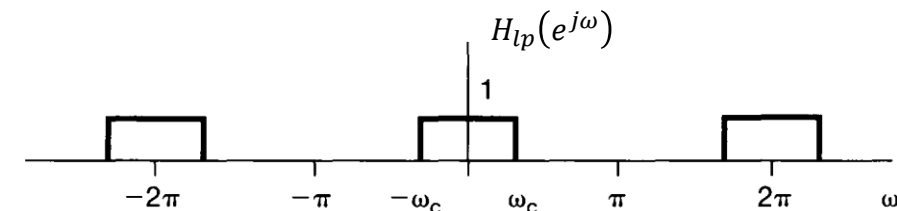


High-pass filter

$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega - \pi)})$$

$$\updownarrow \mathcal{F}$$

$$\begin{aligned} h_{hp} &= e^{j\pi n} h_{lp}[n] \\ &= (-1)^n h_{lp}[n] \end{aligned}$$



Properties of discrete-time Fourier Transform



Conjugation and Conjugate Symmetry

□ Conjugation property

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \implies x^*[n] \xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega})$$

□ Conjugation Symmetry

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \quad [x[n] \text{ real}]$$

$\mathcal{Re}\{X(e^{j\omega})\}$ is even, $\mathcal{Im}\{X(e^{j\omega})\}$ is odd.



Properties of discrete-time Fourier Transform

Time reversal

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \quad \Rightarrow \quad x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-j\omega})$$

□ $x[n]$ even, $X(e^{j\omega})$ even; $x[n]$ odd, $X(e^{j\omega})$ odd

Recall: $x[n]$ real: $X(e^{j\omega}) = X^*(e^{-j\omega})$



□ $x[n]$ real and even $\Rightarrow X(e^{j\omega})$ real and even

$x[n]$ real and odd $\Rightarrow X(e^{j\omega})$ odd and purely imaginary

Properties of discrete-time Fourier Transform



Time reversal

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \Rightarrow x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-j\omega})$$

□ If $x[n]$ real

$$\begin{aligned} \mathcal{F}\{x[n]\} &= \mathcal{F}\{\text{Ev}\{x[n]\}\} + \mathcal{F}\{\text{Od}\{x[n]\}\} \\ &= \mathcal{R}e\{X(e^{j\omega})\} + j\mathcal{I}m\{X(e^{j\omega})\} \end{aligned} \Rightarrow \begin{cases} \text{Ev}\{x[n]\} \xleftrightarrow{\mathcal{F}} \mathcal{R}e\{X(e^{j\omega})\} \\ \text{Od}\{x[n]\} \xleftrightarrow{\mathcal{F}} j\mathcal{I}m\{X(e^{j\omega})\} \end{cases}$$



Properties of discrete-time Fourier Transform

Differencing and accumulation

□ If $x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$

□ Then

$$x[n] - x[n-1] \xleftrightarrow{\mathcal{F}} (1 - e^{-j\omega})X(e^{j\omega})$$

$$\sum_{m=-\infty}^n x[m] \xleftrightarrow{\mathcal{F}} \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \underbrace{\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)}_{\text{DC component}}$$

DC component



Properties of discrete-time Fourier Transform

Differencing and accumulation

□ Examples Determine FT of unit step $x[n] = u[n]$

Solution

$$\begin{aligned} g[n] = \delta[n] &\xleftrightarrow{\mathcal{F}} G(e^{j\omega}) = 1 & x[n] &= \sum_{m=-\infty}^n g[m] \\ X(e^{j\omega}) &= \frac{1}{1 - e^{-j\omega}} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k) \\ &= \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k) \end{aligned}$$

Properties of discrete-time Fourier Transform

Time expansion

□ Recall the continuous time property

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

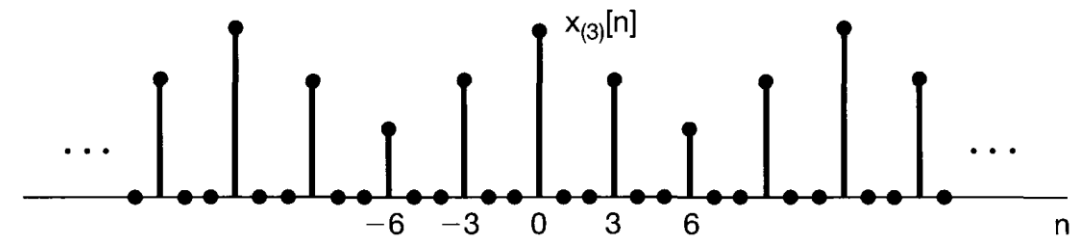
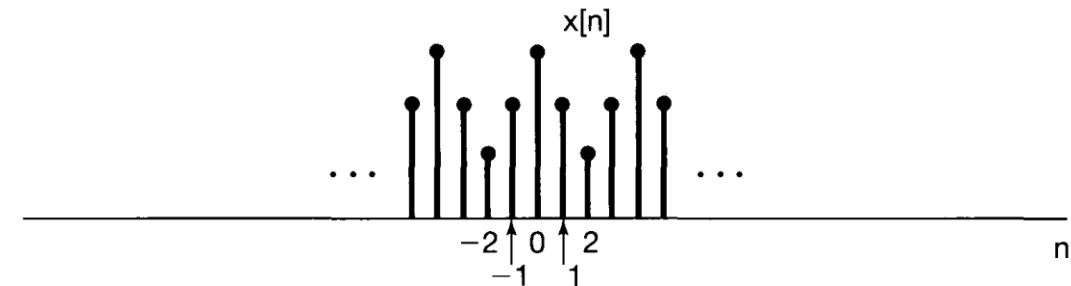
□ Try to define $x[an]$

□ a should be an integer and $a > 1$

□ not merely speed up, but also resample $x[n]$

□ Define instead

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$



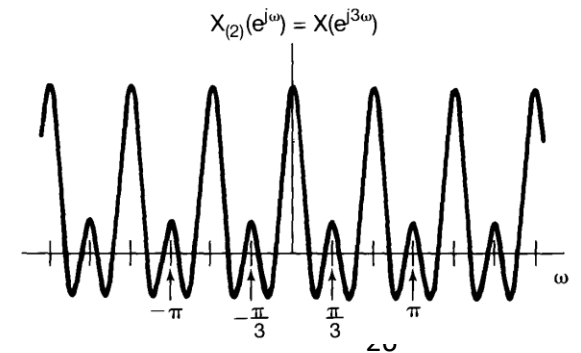
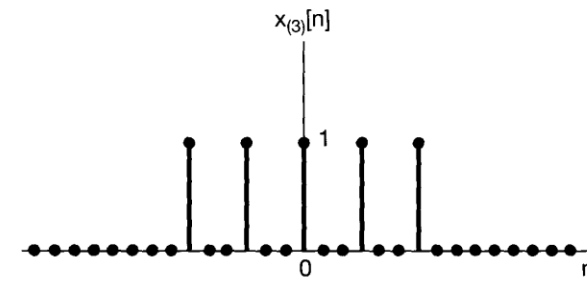
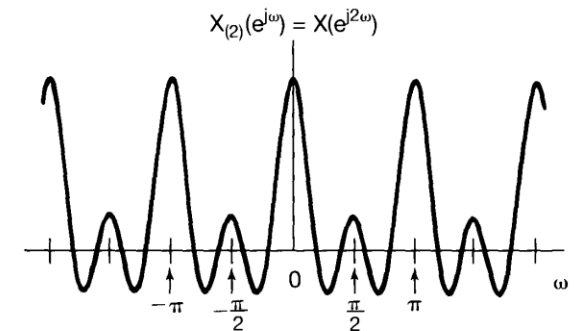
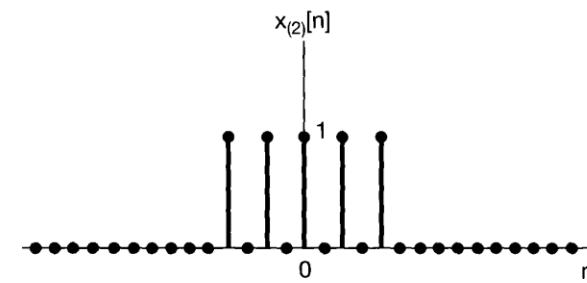
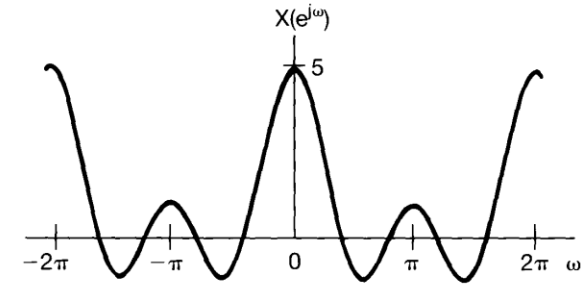
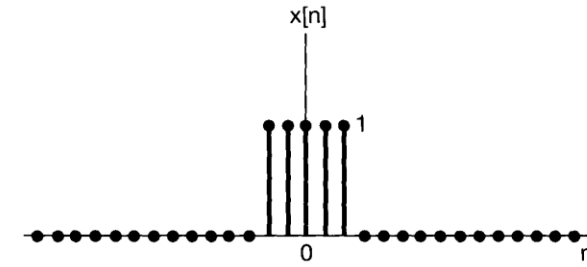


Properties of discrete-time Fourier Transform

Time expansion

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

$$\begin{aligned} x[n] &\xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \\ x_{(k)}[n] &\xleftrightarrow{\mathcal{F}} X_{(k)}(e^{j\omega}) = X(e^{jk\omega}) \end{aligned}$$



$$\begin{aligned} X_{(k)}(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x_{(k)}[n] e^{-j\omega n} \\ &= \sum_{r=-\infty}^{+\infty} x_{(k)}[rk] e^{-j\omega rk} \quad n = rk \\ &= \sum_{r=-\infty}^{+\infty} x[r] e^{-j(k\omega)r} \quad x_{(k)}[rk] = x[r] \\ &= X(e^{jk\omega}) \end{aligned}$$



Properties of discrete-time Fourier Transform

Examples

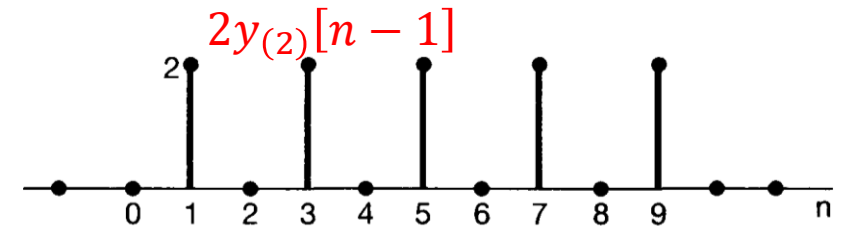
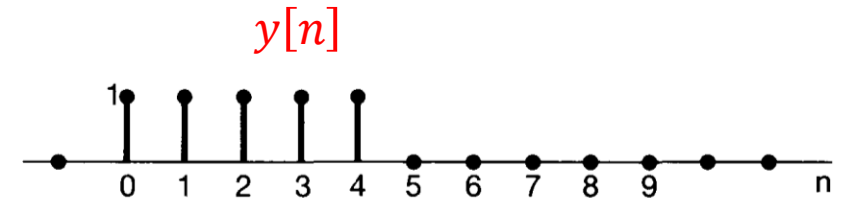
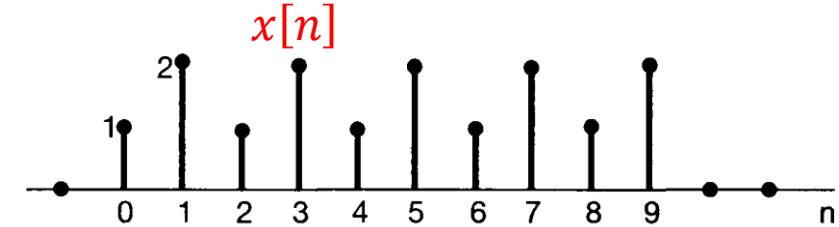
$$X(e^{j\omega}) = ?$$

Solution

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

where $y[n] = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{else} \end{cases}$

$$y_2[n] = \begin{cases} y[n/2], & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$$



Properties of discrete-time Fourier Transform

Examples

$$X(e^{j\omega}) = ?$$

Solution

$$Y(e^{j\omega}) = e^{-j2\omega} \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

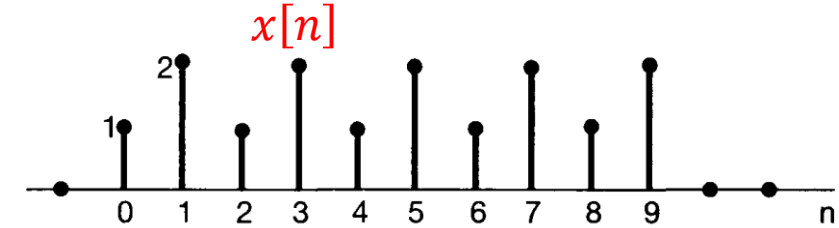
□ Using the time expansion property

$$y_{(2)}[n] \xleftrightarrow{\mathcal{F}} Y_2(e^{j\omega}) = Y(e^{j2\omega}) = e^{-j4\omega} \frac{\sin(5\omega)}{\sin(\omega)}$$

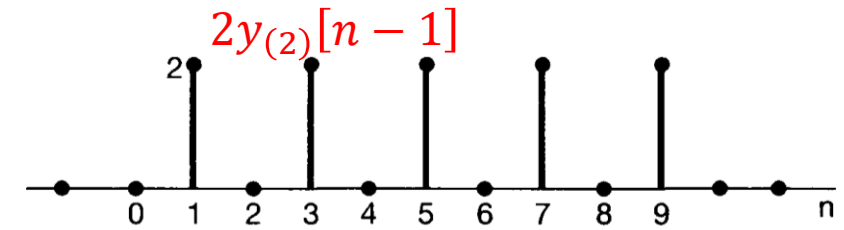
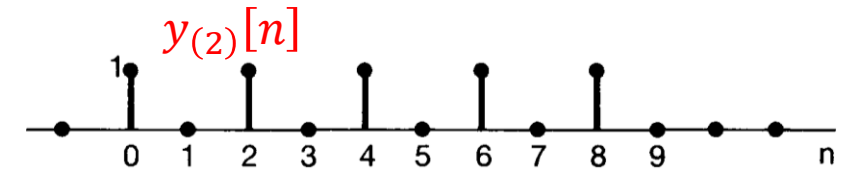
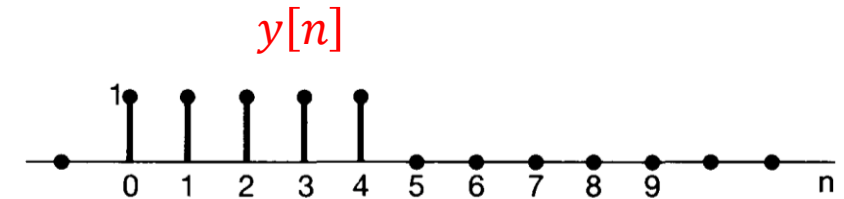
□ Using the linearity and time-shifting properties

$$2y_{(2)}[n-1] \xleftrightarrow{\mathcal{F}} 2e^{-j5\omega} \frac{\sin(5\omega)}{\sin(\omega)}$$

$$X(e^{j\omega}) = e^{-j4\omega} (1 + 2e^{-j\omega}) \left(\frac{\sin(5\omega)}{\sin(\omega)} \right)$$



$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$





Properties of discrete-time Fourier Transform

Differentiation in frequency

$$nx[n] \xleftrightarrow{\mathcal{F}} j \frac{dX(e^{j\omega})}{d\omega}$$

□ Consider

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} \boxed{-jnx[n]} e^{-j\omega n}$$

$$\Rightarrow -jnx[n] \xleftrightarrow{\mathcal{F}} \frac{dX(e^{j\omega})}{d\omega}$$

$$\Rightarrow nx[n] \xleftrightarrow{\mathcal{F}} j \frac{dX(e^{j\omega})}{d\omega}$$

Parseval's relation

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Properties of discrete-time Fourier Transform

Examples

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

□ $x[n]$ is

- Periodic?

No

- Real?

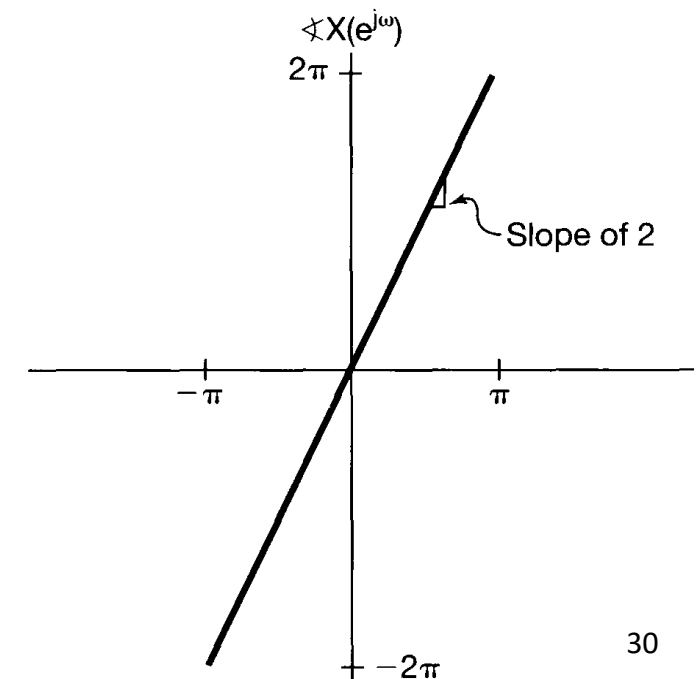
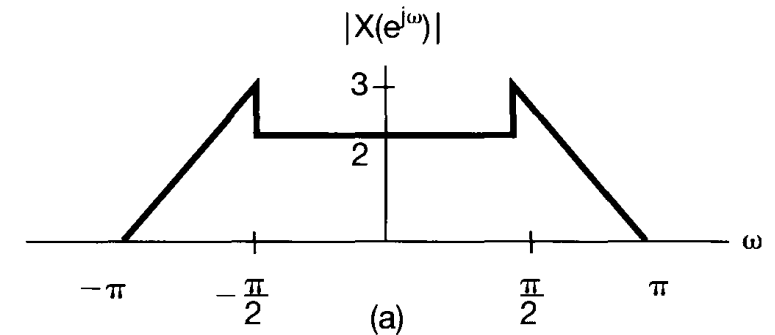
Yes

- Even?

No

- Of finite energy?

Yes

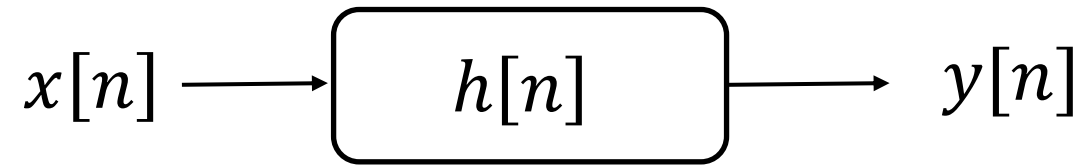


The Discrete-Time Fourier Transform

(ch.5)

- ❑ Representation of aperiodic signals- Discrete Fourier transform
- ❑ Fourier transform for periodic signals
- ❑ Properties of discrete-time Fourier transform
- ❑ **The convolution property**
- ❑ The multiplication property
- ❑ Duality
- ❑ Systems characterized by difference equations

The convolution property



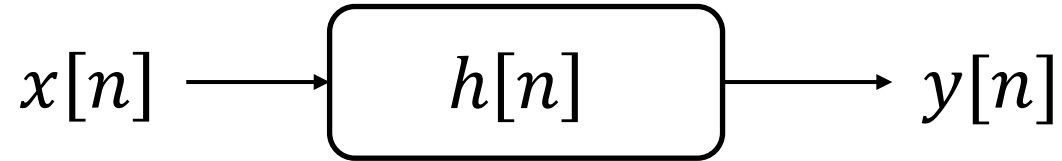
$$y[n] = x[n] * h[n] \quad \xleftrightarrow{\mathcal{F}} \quad Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

□ $H(j\omega)$: Frequency response; FT of the impulse response of the system

The convolution property



Examples



$$h[n] = \delta[n - n_0] \text{ and } X(e^{j\omega}) = \mathcal{F}\{x[n]\} \quad Y(e^{j\omega}) = ?$$

Solution

$$h[n] = \delta[n - n_0]$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0] e^{-j\omega n}$$

$$= e^{-j\omega n_0}$$

$$y[n] = x[n - n_0]$$

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

The convolution property

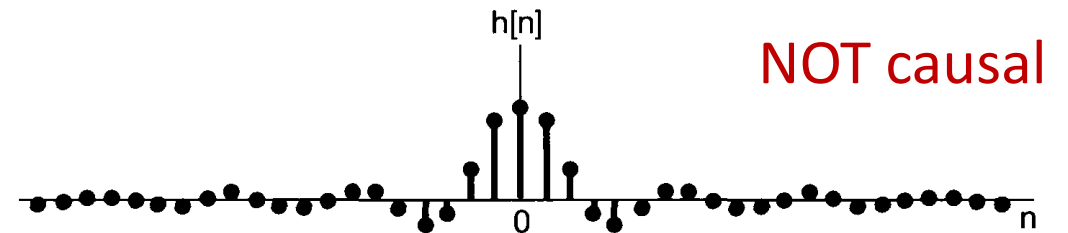
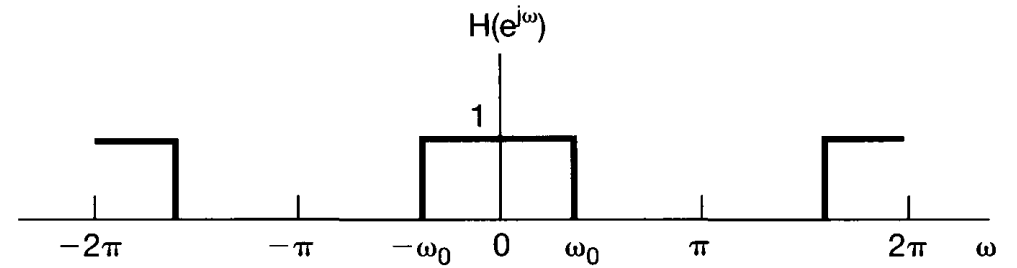


Examples

Determine the impulse response of an ideal low-pass filter

Solution

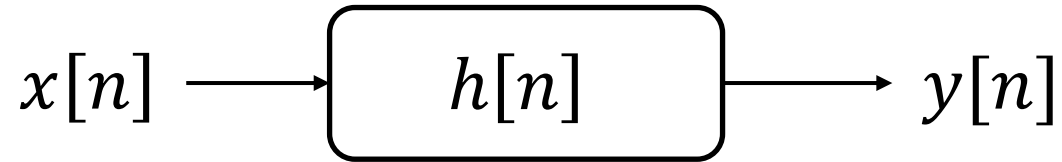
$$\begin{aligned} h[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega n} d\omega \\ &= \frac{\sin \omega_0 n}{\pi n} \end{aligned}$$





The convolution property

Examples



$$h[n] = \alpha^n u[n], (|\alpha| < 1) \quad x[n] = \beta^n u[n], (|\beta| < 1) \quad y[n] = ?$$

Solution

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \quad X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}} \quad Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}$$

When $\alpha \neq \beta$

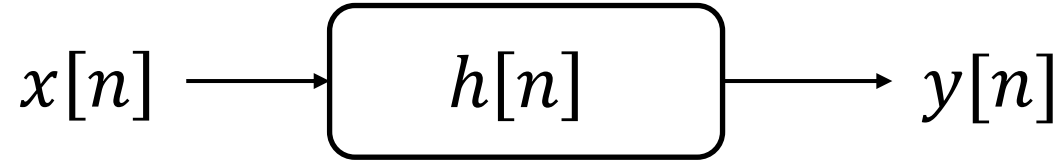
$$Y(e^{j\omega}) = \frac{A}{1 - \alpha e^{-j\omega}} - \frac{B}{1 - \beta e^{-j\omega}} \quad A = \frac{\alpha}{\alpha - \beta} \quad B = \frac{\beta}{\alpha - \beta}$$

$$y[n] = \frac{\alpha}{\alpha - \beta} \alpha^n u[n] - \frac{\beta}{\alpha - \beta} \beta^n u[n] = \frac{1}{\alpha - \beta} (\alpha^{n+1} u[n] - \beta^{n+1} u[n])$$

The convolution property



Examples



$$h[n] = \alpha^n u[n], (|\alpha| < 1) \quad x[n] = \beta^n u[n], (|\beta| < 1) \quad y[n] = ?$$

Solution

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \quad X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}} \quad Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}$$

When $\alpha = \beta$

$$Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^2} = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega} \left(\frac{1}{1 - \alpha e^{-j\omega}} \right)$$

$$y[n] = (n + 1) \alpha^n u[n + 1] = (n + 1) \alpha^n u[n]$$

The convolution property



Examples

Consider the ideal band-stop filter, $Y(e^{j\omega}) = ?$

$$w_1[n] = e^{j\pi n} x[n]$$

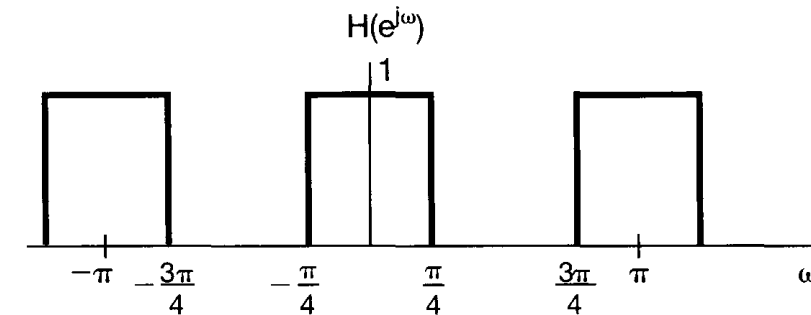
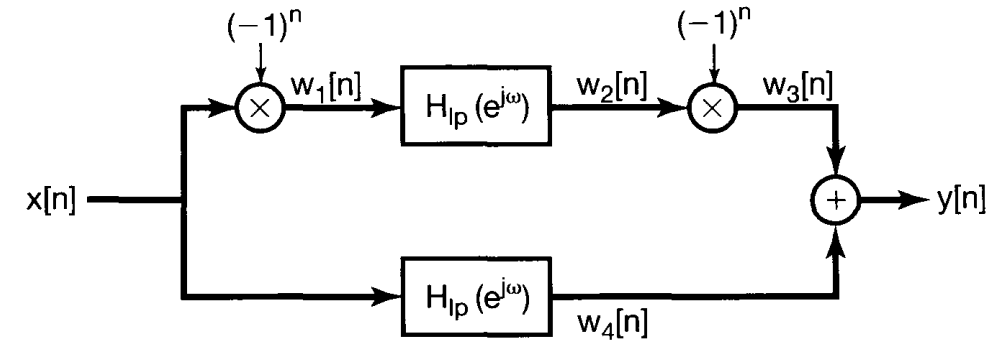
$$W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega}) X(e^{j(\omega-\pi)})$$

$$\begin{aligned} W_3(e^{j\omega}) &= W_2(e^{j(\omega-\pi)}) \\ &= H_{lp}(e^{j(\omega-\pi)}) X(e^{j(\omega-2\pi)}) \\ &= H_{lp}(e^{j(\omega-\pi)}) X(e^{j\omega}) \end{aligned}$$

$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega}) X(e^{j\omega})$$

$$Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega}) = [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})] X(e^{j\omega})$$



The Discrete-Time Fourier Transform (ch.5)

- ☐ Representation of aperiodic signals- Discrete Fourier transform
- ☐ Fourier transform for periodic signals
- ☐ Properties of discrete-time Fourier transform
- ☐ The convolution property
- ☒ **The multiplication property**
- ☐ Duality
- ☐ Systems characterized by difference equations

The multiplication property



$$y[n] = x_1[n]x_2[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = \underbrace{\frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta}_{\text{Periodic convolution}}$$

Proof

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x_1[n]x_2[n]e^{-j\omega n}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_2[n] \left\{ \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta \right\} e^{-j\omega n}$$

$$\curvearrowright x_1[n] = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta$$

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) \left[\sum_{n=-\infty}^{+\infty} x_2[n]e^{-j(\omega-\theta)n} \right] d\theta$$

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

The multiplication property



Examples

$$x_1[n] = \frac{\sin(\pi n/2)}{\pi n} \quad x_2[n] = \frac{\sin(3\pi n/4)}{\pi n}$$

$$x[n] = x_1[n]x_2[n] \quad X(e^{j\omega}) = ?$$

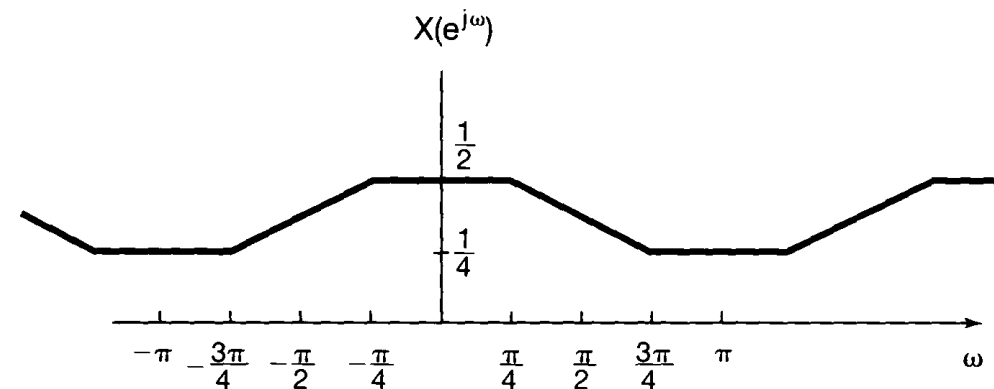
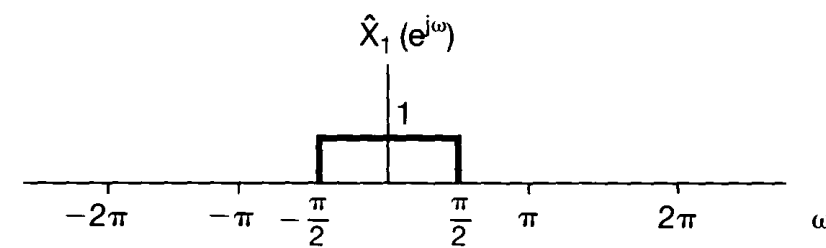
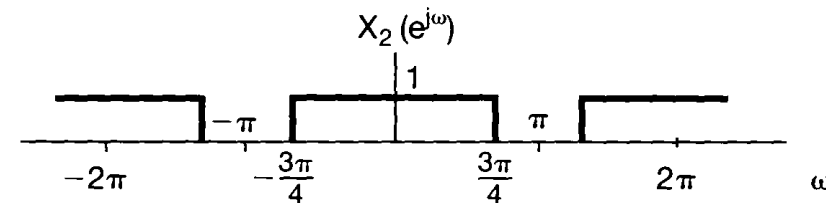
Solution

$$X(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\omega}) X_2(e^{j(\omega-\theta)}) d\theta$$

□ Convert to ordinary convolution

$$\text{Define } \hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}), & -\pi < \omega < \pi \\ 0, & \text{otherwise} \end{cases}$$

$$X(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$$



The Discrete-Time Fourier Transform (ch.5)

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- ❑ **Duality**
- ❑ Systems characterized by difference equations

Duality in the discrete FS

Consider $f[m] = \frac{1}{N} \sum_{r=\langle N \rangle} g[r] e^{-jr(2\pi/N)m}$ $f[m]$ and $g[r]$ are periodic

$m = k, r = n$ \Rightarrow $m = n, r = -k$

$$f[k] = \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk(2\pi/N)n}$$

$$f[n] = \frac{1}{N} \sum_{k=\langle N \rangle} g[-k] e^{jk(2\pi/N)n}$$

$g[n] \xrightarrow{\mathcal{FS}} f[k]$

$f[n] \xrightarrow{\mathcal{FS}} \frac{1}{N} g[-k]$

Every property of the discrete FS has a dual.

□ Examples

$\left\{ \begin{array}{l} x[n - n_0] \xleftrightarrow{\mathcal{FS}} a_k e^{-jk(2\pi/N)n_0} \\ e^{jm(2\pi/N)n} x[n] \xleftrightarrow{\mathcal{FS}} a_{k-m} \end{array} \right.$

$\left\{ \begin{array}{l} \sum_{r=\langle N \rangle} x[r] y[n - r] \xleftrightarrow{\mathcal{FS}} N a_k b_k \\ x[n] y[n] \xleftrightarrow{\mathcal{FS}} \sum_{l=\langle N \rangle} a_l b_{k-l} \end{array} \right.$

Examples

$$x[n] = \begin{cases} \frac{1}{9} \frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ 5/9, & n = \text{multiple of } 9 \end{cases}$$

$$a_k = ?$$

① Dual in the frequency domain: $n \rightarrow k$

② Time domain signal

$$b_k = \begin{cases} \frac{1}{9} \frac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \neq \text{multiple of } 9 \\ 5/9, & k = \text{multiple of } 9 \end{cases}$$

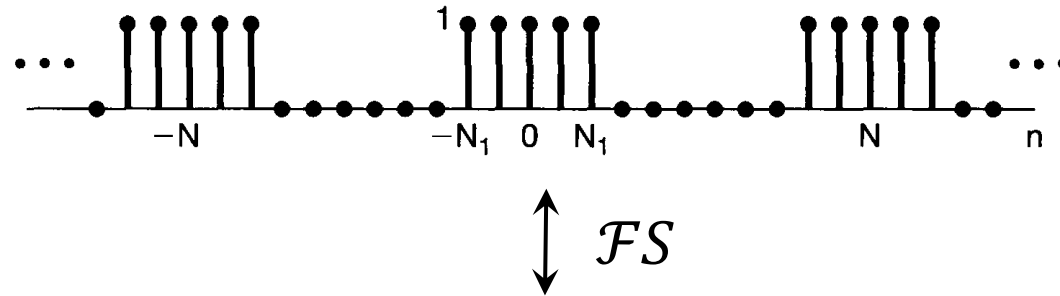
$$g[n] \xleftrightarrow{\mathcal{FS}} b_k$$

Duality



Examples

$$x[n] = \begin{cases} \frac{1}{9} \frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ 5/9, & n = \text{multiple of } 9 \end{cases} \quad a_k = ?$$



$$\begin{cases} \frac{1}{N} \frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}, & k \neq 0, \pm N, \pm 2N, \dots \\ (2N_1 + 1)/N, & k = 0, \pm N, \pm 2N, \dots \end{cases}$$

Duality



Examples

$$x[n] = \begin{cases} \frac{1}{9} \frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ 5/9, & n = \text{multiple of } 9 \end{cases}$$

$$a_k = \frac{1}{N} g[-k] = \begin{cases} 1/9, & |k| \leq 2 \\ 0, & 2 < |k| \leq 4 \end{cases}$$

③ Duality

① Dual in the frequency domain: $n \rightarrow k$

$$g[n] = \begin{cases} 1, & |n| \leq 2 \\ 0, & 2 < |n| \leq 4 \end{cases}$$

② Time domain signal

$$g[n] \xleftrightarrow{\mathcal{FS}} b_k$$

$$b_k = \begin{cases} \frac{1}{9} \frac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \neq \text{multiple of } 9 \\ 5/9, & k = \text{multiple of } 9 \end{cases}$$

$g[n]$ is periodic ($N = 9$)

Duality between discrete FT and continuous FS

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \text{Discrete FT} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad \text{Continuous FS} \quad a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

$n \rightarrow k\omega_0, \omega \rightarrow -t$

Properties of discrete-time Fourier Transform

Examples

$$x[n] = \frac{\sin(\pi n/2)}{\pi n}$$

① DT → CT FS:

$n \rightarrow k\omega_0$

$$X(e^{j\omega}) = ?$$

$$a_k = \frac{\sin(k\omega_0\pi/2)}{k\omega_0\pi}$$

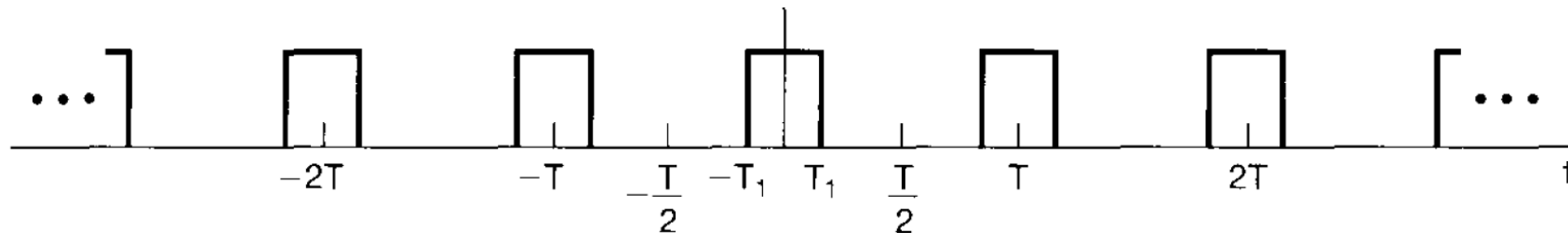


$$\omega_0 = 1$$

$$g(t) = \begin{cases} 1, & |t| \leq T_1 \quad (T_1 = \pi/2) \\ 0, & T_1 < |t| \leq T \end{cases}$$

② CT signal

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi}, k \neq 0$$





Properties of discrete-time Fourier Transform

Examples

$$x[n] = \frac{\sin(\pi n/2)}{\pi n}$$

$$X(e^{j\omega}) = ?$$

$$g(t) = \begin{cases} 1, & |t| \leq T_1 \text{ (} T_1 = \pi/2 \text{)} \\ 0, & T_1 < |t| \leq T \end{cases}$$

② CT signal

① DT \rightarrow CT FS: $n \rightarrow k\omega_0$

$$a_k = \frac{\sin(k\pi/2)}{k\pi}$$

$$\omega_0 = 1$$

$$\frac{\sin(\pi k/2)}{\pi k} = \frac{1}{T} \int_T g(t) e^{-jkt} dt = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} g(t) e^{-jkt} dt$$

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} g(\omega) e^{-j\omega n} d\omega \xrightarrow{n = k\omega_0, t = \omega} \frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} g(\omega) e^{j\omega n} d\omega$$

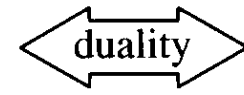
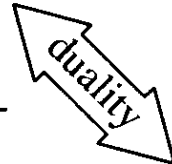
$$\therefore X(e^{j\omega}) = g(\omega) = \begin{cases} 1, & |\omega| \leq \pi/2 \\ 0, & \pi/2 < |\omega| \leq \pi \end{cases}$$

Duality



Summary FS and FT expressions

	Continuous time		Discrete time	
	Time domain	Frequency domain	Time domain	Frequency domain
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ <p>continuous time periodic in time</p>	$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$ <p>discrete frequency aperiodic in frequency</p>	$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$ <p>discrete time periodic in time</p>	$a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$ <p>discrete frequency periodic in frequency</p>
Fourier Transform	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ <p>continuous time aperiodic in time</p>	$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ <p>continuous frequency aperiodic in frequency</p>	$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ <p>discrete time aperiodic in time</p>	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$ <p>continuous frequency periodic in frequency</p>



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- ☒ Systems characterized by difference equations

System characterized by difference equations



$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

$$\sum_{k=0}^N a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^M b_k e^{-jk\omega} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^M b_k e^{-jk\omega}}{\sum_{k=0}^N a_k e^{-jk\omega}}$$

System characterized by difference equations



Examples Causal LTI system

$$y[n] - ay[n-1] = x[n], |a| < 1 \quad h[n] = ?$$

Solution

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$h[n] = a^n u[n]$$

System characterized by difference equations



Examples Causal LTI system

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n] \quad h[n] = ?$$

Solution

$$\begin{aligned} H(e^{j\omega}) &= \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}} = \frac{2}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)} \\ &= \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}} \end{aligned}$$

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

System characterized by difference equations



Examples

Causal LTI system

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n] \quad x[n] = \left(\frac{1}{4}\right)^n u[n] \quad y[n] = ?$$

Solution

$$\begin{aligned} Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) = \left[\frac{2}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)} \right] \left[\frac{1}{1 - \frac{1}{4}e^{-j\omega}} \right] \\ &= \frac{2}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)^2} = -\frac{4}{1 - \frac{1}{4}e^{-j\omega}} - \frac{2}{\left(1 - \frac{1}{4}e^{-j\omega}\right)^2} + \frac{8}{1 - \frac{1}{2}e^{-j\omega}} \end{aligned}$$

$$y[n] = \left\{ -4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n \right\} u[n]$$