1. (20 points) Let $A = (a, \alpha), B = (b, \beta) \in \mathbb{Z}^2$. Show that if A, B satisfy the T-condition, then there is a T-route from A to B. (**T-condition**: (1) $b > a$; (2) $b - a \geq |\beta - \alpha|$; (3) $b - a + \beta - \alpha$ is even.)

let $A = P_0, P_1, \cdots, P_k = B$ be a T-route from A to B

where $P_i = (x_i, y_i)$ $\quad x_0 = a,\ y_0 = \alpha,\ x_k = b,\ y_k = \beta$

$x_i - x_{i-1} = 1;\ y_i - y_{i-1} \in \{\pm 1\}$ for every $i = 1, 2, \cdots, k$

(1) $b - a = x_k - x_0 = (x_k - x_{k-1}) + (x_{k-1} - x_{k-2}) + \cdots + (x_1 - x_0) = k > 0$

so $b > a$

(2) $|\beta - \alpha| = |y_k - y_0| = |(y_k - y_{k-1}) + (y_{k-1} - y_{k-2}) + \cdots + (y_1 - y_0)|$

$\leq |y_k - y_{k-1}| + |y_{k-1} - y_{k-2}| + \cdots + |y_1 - y_0|$

$= k = b - a$

so $b - a \geq |\beta - \alpha|$

(3) $b - a + \beta - \alpha = \sum_{i=1}^{k} (y_i - y_{i-1} + x_i - x_{i-1})$

Since $y_i - y_{i-1} + x_i - x_{i-1} \in \{0, 2\}$

then $2 \mid (b - a + \beta - \alpha)$, namely $b - a + \beta - \alpha$ is even.

Conclusion: if A, B satisfy the T-condition, then there is a T-route from A to B

2. (20 points) At the end of a basketball match between team A and team B, the result is 80:81. What is the number of possibilities that A's score is always less than B's score during the entire match? A possibility can be described with the sequence of intermediate results during the entire match. For example, $0:1, 0:2, \ldots, 0:81, 1:81, 2:81, \ldots, 80:81$ describes one of the possibilities that A's score is always less than B's score during the entire match. (**Hint:** Use the idea of counting T-routes.)

for $A_n = n$, the number of posibilities that A's score is less than B's score is $B_n = 81 - n$ $(0 \le n \le 80, n \in \mathbb{Z})$

So the number of all posibibilities is

$$N = 81 + 80 + \cdots + 1$$

$$= \frac{(81+1) \times 81}{2}$$

$$= 3321$$

3. (20 points) Let n, r be positive integers such that $r \geq n$. Determine the number of vectors $(x_1, x_2, \ldots, x_n)$ such that $x_1 + x_2 + \cdots + x_n = r$ and $x_1, x_2, \ldots, x_n \in \mathbb{Z}^+$.

let $X = \{ (x_1, \cdots, x_n) ; x_1, \cdots, x_n \in \mathbb{N}$ and $x_1 + \cdots + x_n = r \}$.

Y : the set of all r-combinations of $[n]$ with repetition

$f : X \to Y \quad (x_1, \cdots, x_n) \mapsto \{ x_1 \cdot 1, x_2 \cdot 2, \cdots, x_n \cdot n \}$

Since f is bijective

Hence, $|X| = |Y| = \binom{n+r-1}{r}$

4. (20 points) Let $\{a_n\}_{n\geq s}, \{b_n\}_{n\geq s}$ be two sequences such that $a_n = \sum_{k=s}^{n}(-1)^{n-k}\binom{n}{k}b_k$ for all $n \geq s$. Show that $b_n = \sum_{k=s}^{n}\binom{n}{k}a_k$ for all $n \geq s$.

$$\sum_{k=s}^{n}\binom{n}{k}a_k = \sum_{k=s}^{n}\binom{n}{k}\sum_{i=s}^{k}(-1)^{k-i}\binom{k}{i}b_i$$

$$= \sum_{i=s}^{n}\sum_{k=i}^{n}(-1)^{k-i}\binom{n}{k}\binom{k}{i}b_i$$

$$= b_n$$

5. (20 points) Suppose that $n+1 \geq k \geq 2$. Provide a combinatorial proof of $S_2(n+1,k) = S_2(n,k-1) + k \cdot S_2(n,k)$. (**Hint**: Interpret both sides of the equation as the number of elements in a set X)

$$S_2(n+1,k) = \frac{1}{k!}\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}(k-i)^{n+1}$$

$$S_2(n,k-1) + k \cdot S_2(n,k) = \frac{1}{(k-1)!}\sum_{i=0}^{k-2}(-1)^i\binom{k-1}{i}(k-1-i)^n$$

$$+ k \cdot \frac{1}{k!}\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}(k-i)^n$$

$$= \frac{1}{(k-1)!}\left[\sum_{i=0}^{k-2}(-1)^i\binom{k-1}{i}(k-1-i)^n + \sum_{i=0}^{k-1}(-1)^i\binom{k}{i}(k-i)^n\right]$$

$$= \frac{1}{(k-1)!}\left[\sum_{i=0}^{k-2}(-1)^i\frac{(k-1)!}{i!(k-1-i)!}(k-1-i)^n + \sum_{i=0}^{k-1}(-1)^i\frac{k!}{i!(k-i)!}(k-i)^n\right]$$

$$= \frac{1}{k!}\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}(k-i)^{n+1}$$

$$= S_2(n+1,k)$$

Conclusion. $S_2(n+1,k) = S_2(n,k-1) + k \cdot S_2(n,k)$