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§ 7.1 (Page 708) :

6. Let a rectangular $x'y'$-coordinate system be obtained by rotating a rectangular xy-coordinate system counterclockwise through the angle $\theta = 3\pi/4$.

(a) Find the $x'y'$-coordinates of the point whose xy-coordinates are $(-2, 6)$.

(b) Find the xy-coordinates of the point whose $x'y'$-coordinates are $(5, 2)$.

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$ $(4\sqrt{2}, -2\sqrt{2})$

(b) $\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{7\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} \end{bmatrix}$ $\left(-\frac{7\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$

§ 7.2 (Page 722) :

In Exercises 2–9, find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$.

7. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{vmatrix}$

$= \lambda^3 - 6\lambda^2 + 9\lambda = 0$

$\lambda_1 = 0 \quad \lambda_2 = 3$

$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

$x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad v_2 = \frac{x_2}{\|x_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$

$x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad v_3 = \frac{x_3}{\|x_3\|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$.

$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

11. Prove that if A is any $m \times n$ matrix, then A^TA has an orthonormal set of n eigenvectors.

if A is any $m \times n$ matrix

then A^TA is a $n \times n$ matrix

So there is an orthogonal matrix P such that

$P^{-1}(A^TA)P$ is diagonal.

the n column vectors of P are eigenvectors of A

since P is orthogonal. these column vectors are orthonormal

so A^TA has n orthonormal eigenvectors.

12. (a) Show that if $\mathbf{v}$ is any $n \times 1$ matrix and I is the $n \times n$ identity matrix, then $I - \mathbf{v}\mathbf{v}^T$ is orthogonally diagonalizable.

(b) Find a matrix P that orthogonally diagonalizes $I - \mathbf{v}\mathbf{v}^T$ if

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{matrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{matrix}$$

(a). $I - vv^T$ is a $n \times n$ matrix

so it is orthogonally diagonalizable.

(b) let $A = I - vv^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$

$\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 1 \\ 0 & \lambda - 1 & 0 \\ 1 & 0 & \lambda \end{vmatrix}$

$= (\lambda - 1)(\lambda^2 - 1) = 0.$

$\lambda_1 = 1 \quad x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -1 \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$

$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$

17. Show that if A is a symmetric orthogonal matrix, then 1 and -1 are the only possible eigenvalues.

17. If A is an orthogonal matrix then its eigenvalues have absolute value 1, but may be complex. Since the eigenvalues of a symmetric matrix must be real numbers, the only possible eigenvalues for an orthogonal symmetric matrix are 1 and -1.