QUIZ 2022/5/19

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1. (15 points) Find the potential function (势函数) φ of $\vec{F}$, where the minimum of φ is -1, and

$$\vec{F} = \frac{1}{x^2+y^2+z^2+2xy+1}(x+y, x+y, z).$$

$\varphi(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} \vec{F}\,dr + C$

$= \frac{1}{2}\ln\left((x+y)^2+z^2+1\right) + C$

$\because \min \varphi(x,y,z) = 1$

$\therefore \lim_{(x,y,z)\to(0,0,0)} \varphi(x,y,z) = \frac{1}{2}\ln 1 + C = -1$ $\therefore C = -1$

$\therefore \varphi(x,y,z) = \frac{1}{2}\ln\left((x+y)^2+z^2+1\right) - 1$

2. (15 points) Calculate

$$\oint_L \overset{P}{y^2}\,\mathrm{d}x + \overset{Q}{z^2}\,\mathrm{d}y + \overset{R}{x^2}\,\mathrm{d}z,$$

where the curve L is given by: $\begin{cases} x^2+y^2+z^2=1 \\ x+y+z=1 \end{cases}$, counterclockwise (逆时针方向) looking from the point $O(0,0,0)$.

根据Stokes公式 $\oint_L y^2dx + z^2dy + x^2dz = \iint_S 2\,zdydz + 2\,xdzdx + 2\,ydxdy$

$= 2\iint_S zdydz + xdzdx + ydxdy$

令 $x = \sin\theta\cos\varphi$ $y = \sin\theta\sin\varphi$ $z = \cos\theta$.

原式 =

3. (15 points) Calculate

$$\iiint_V (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where V : spatial (空间的) domain defined by $x^2+y^2+(z-2)^2 \geq 4$, $x^2+y^2+(z-1)^2 \leq 9$ and $z \geq 0$.

记 $x^2+y^2+(z-2)^2=4$ 围成区域 V_1, $x^2+y^2+(z-1)^2 \leq 9$ 围成区域 V_2

则 $\iiint_V (x^2+y^2)\,dxdydz = \iiint_{V_2} (x^2+y^2)\,dV - \iiint_{V_1} (x^2+y^2)\,dV$

(1) 令 $x = r\sin\varphi\cos\theta \quad y = r\sin\varphi\sin\theta \quad z = 2 + r\cos\varphi$

$\iiint_{V_1} (x^2+y^2)\,dV = \int_0^2 dr \int_0^{\pi} d\varphi \int_0^{2\pi} r^4 \sin^3\varphi\, d\theta = \frac{256}{15}\pi$

(2) 令 $x = r\sin\varphi\cos\theta \quad y = r\sin\varphi\sin\theta \quad z = 1 + r\cos\varphi$

$\iiint_{V_2} (x^2+y^2)\,dV = \int_0^3 dr \int_0^{\pi} d\varphi \int_0^{2\pi} r^4 \sin^3\varphi\, d\theta = \frac{648\pi}{5}$

$\therefore \iiint_V (x^2+y^2)\,dV = \frac{648\pi}{5} - \frac{256\pi}{15} = \frac{1688\pi}{15}$

4. (15points) Suppose $f(x,y,z) \geq 0$ has continuous partial derivative (偏导数) on Ω : $x^2+y^2+z^2 \leq R^2$, and $f|_{\partial\Omega} = 0$.
Prove that:

$$\iiint_\Omega f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq \frac{\pi R^4}{3} \max_{(x,y,z)\in\Omega} \|\nabla f\|$$

$\because f(x,y,z)$ 在 Ω 上连续.

$\therefore \exists (l,m,n) \in \Omega$ 使得 $\iiint_\Omega f(x,y,z)\,dxdydz = f(l,m,n)\cdot D$

其中 D 为区域 Ω 的体积 $\therefore D = \frac{4\pi R^3}{3}$