Signals and Systems

Lecturer: Dr. Lin Xu, Dr. Xiran Cai

Email: xulin1@shanghaitech.edu.cn

caixr@shanghaitech.edu.cn

Office: 3-428(Xu), 3-438(Cai) SIST

Tel: 20684449(Xu), 20684431(Cai)

ShanghaiTech University

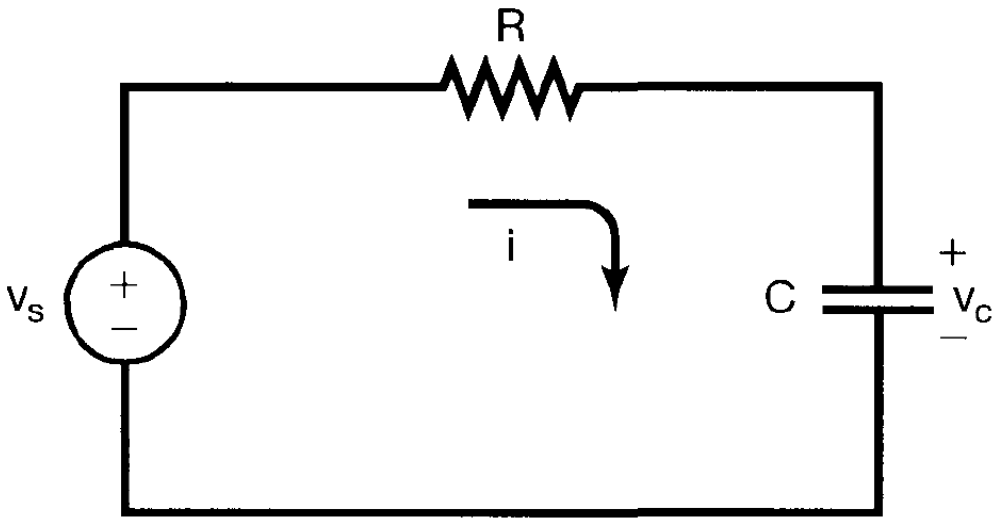
上海科技大学
ShanghaiTech University

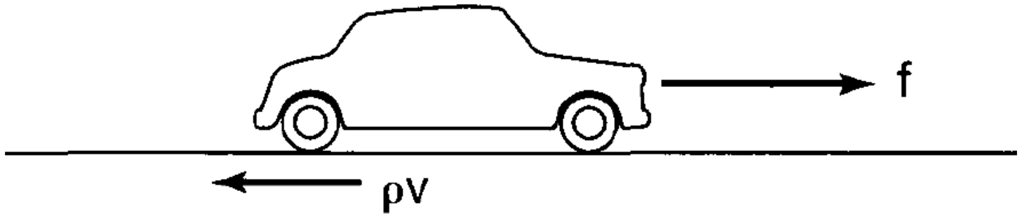
Chapter 1: An overview

- Continuous-Time and Discrete-Time Signals
- Transformations of the Independent Variable
- Exponential and Sinusoidal Signals
- The Unit Impulse and Unit Step Functions
- Continuous-Time and Discrete-Time Systems
- Basic System Properties

Continuous-Time and Discrete-Time Signals

- ***Signals*** describe a wide variety of physical phenomena





The voltage v_s and v_c are examples of signals.

The force f and velocity v are signals.

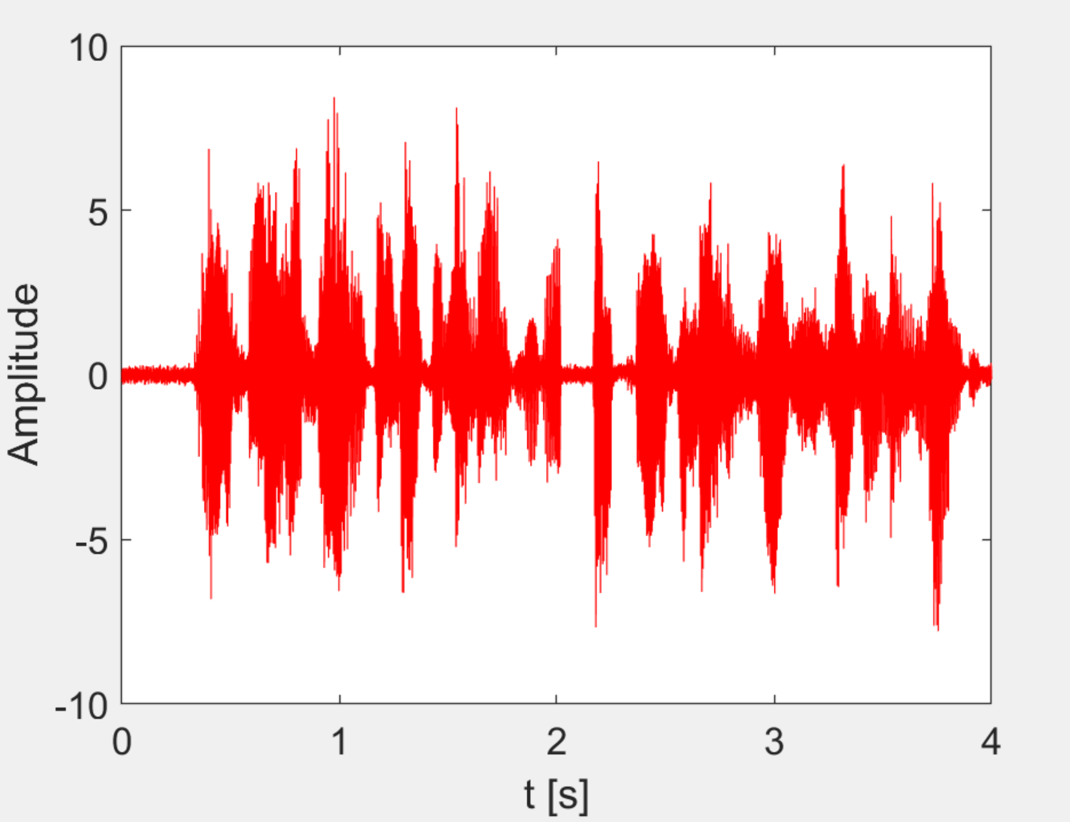
Continuous-Time and Discrete-Time Signals

- ***Mathematically, signals*** are represented as functions of one or more independent variables.

- Example of typical signals
  - Sound
  - Image
  - Video

Continuous-Time and Discrete-Time Signals

- ***Sound:*** represents acoustic pressure as a function of time



$f(t)$

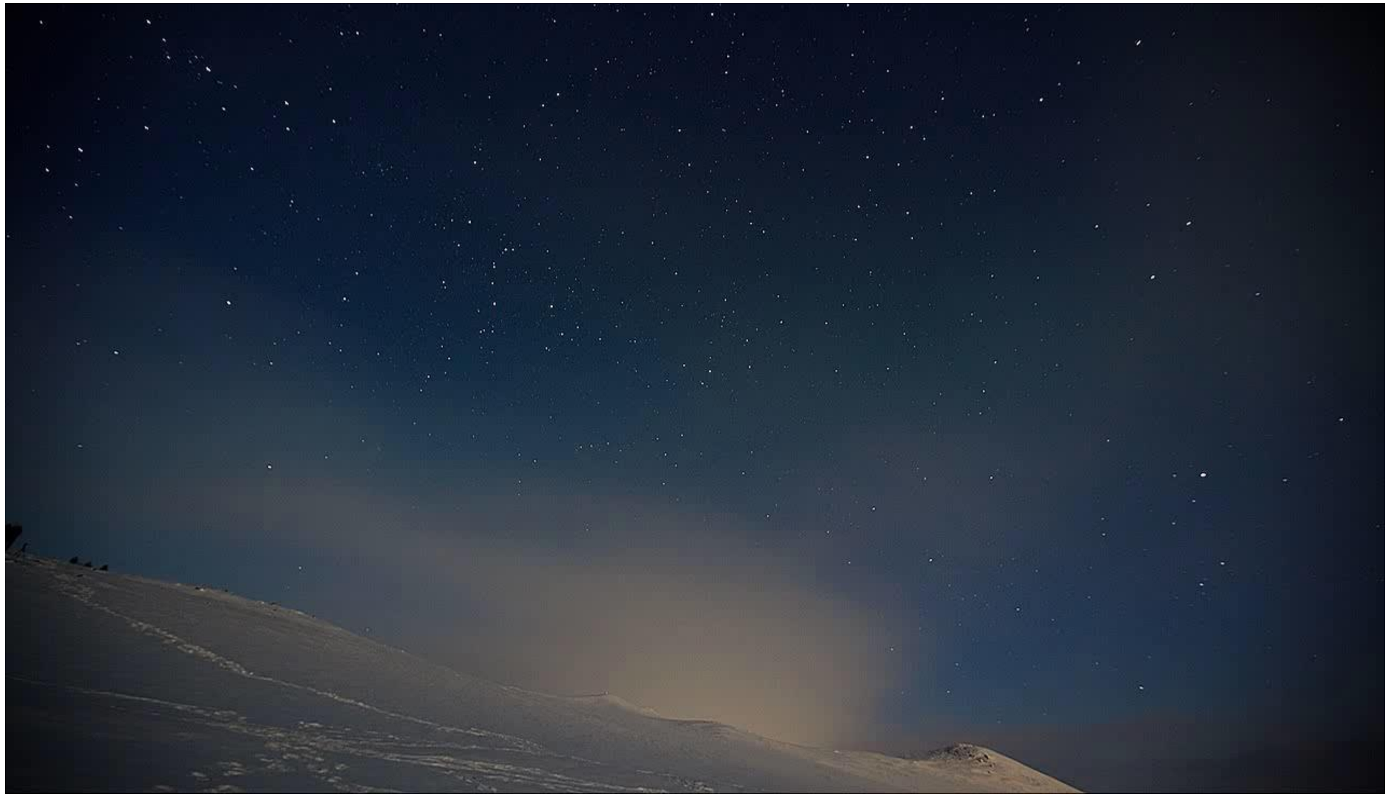
Continuous-Time and Discrete-Time Signals

- ***Picture:*** represents brightness as a function of two spatial variables

$f(x, y)$

Continuous-Time and Discrete-Time Signals

- ***Video:*** consists of a sequence of images, called frames, and is a function of 3 variables: 2 spatial coordinates and time.



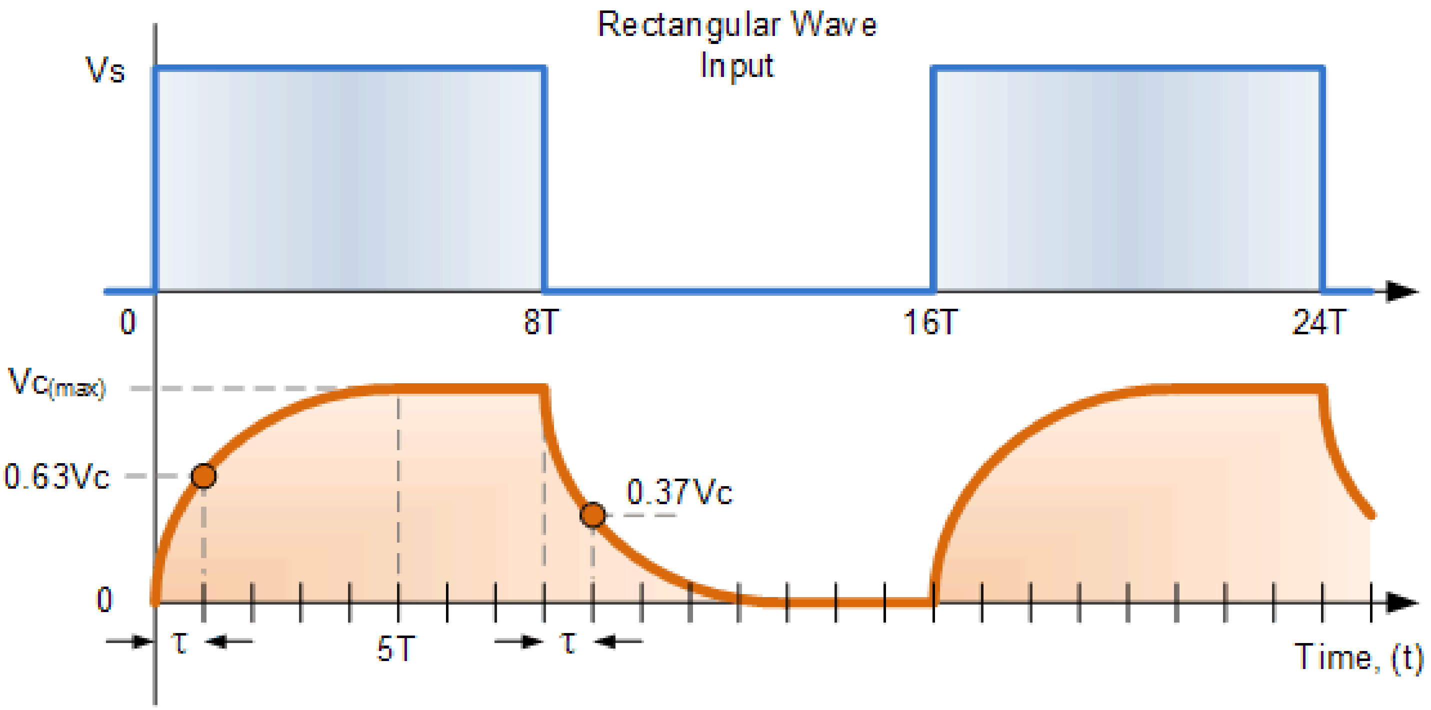
$f(x, y, t)$

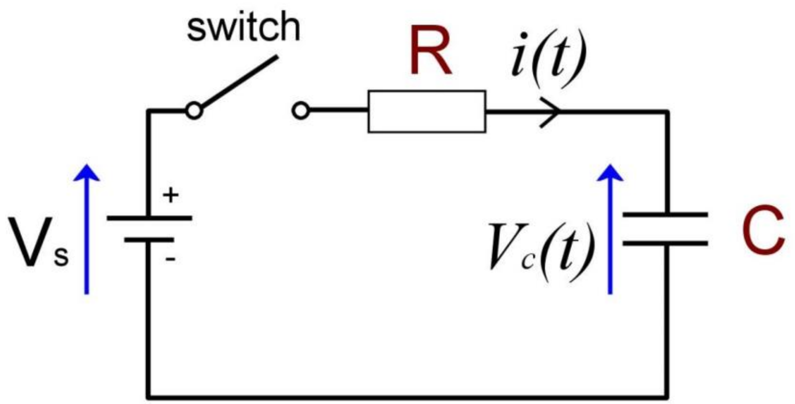
Continuous-Time and Discrete-Time Signals

- Independent variables can be one or more
- Focus on signals involving a single independent variable
- Generally refer to the independent variable as time, although it may not in fact represent time in specific applications
- Continuous-time and discrete-time signal

Continuous-Time and Discrete-Time Signals

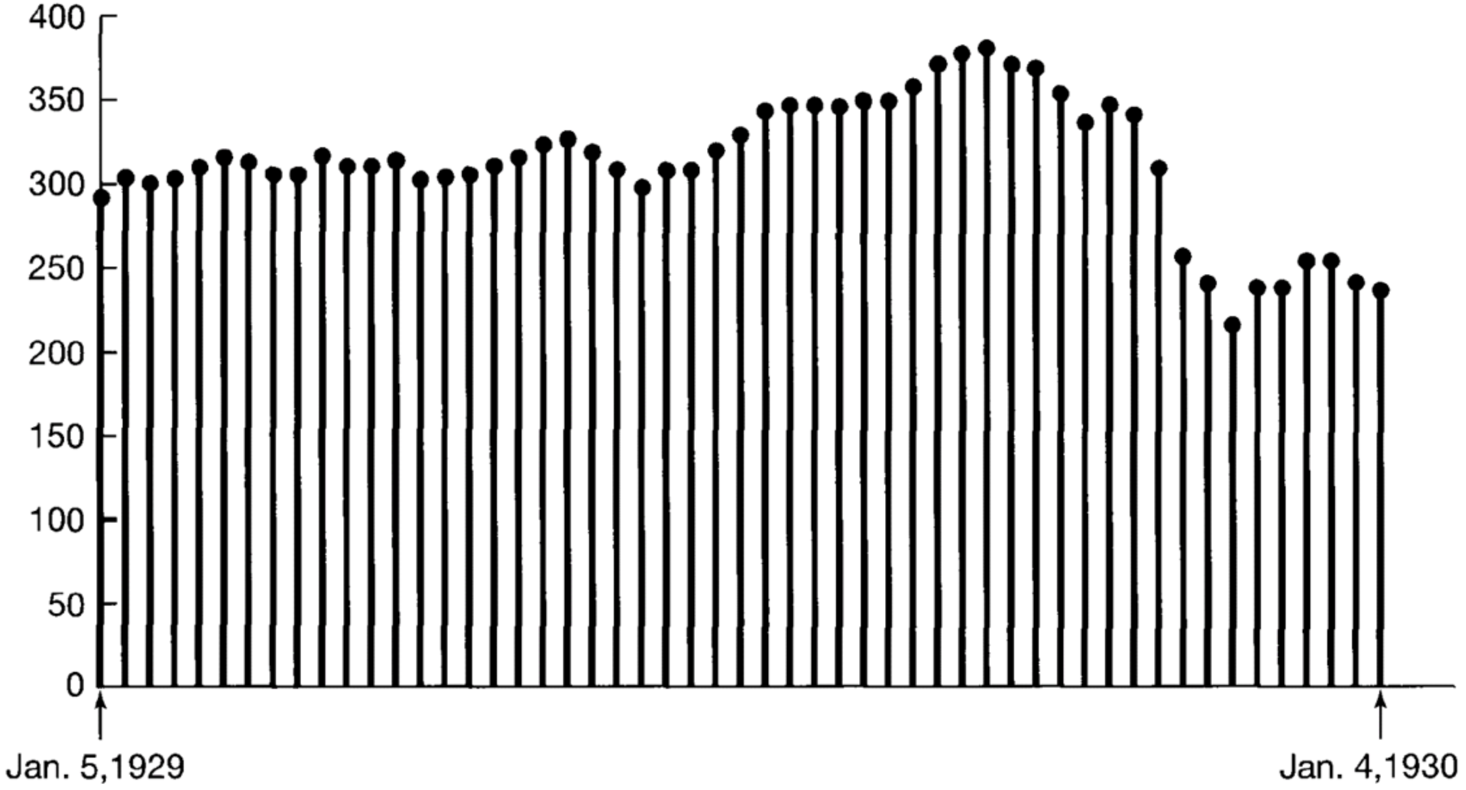
- ***Continuous-time signals:*** the independent variable is continuous, and signals are defined for a continuum of values





Continuous-Time and Discrete-Time Signals

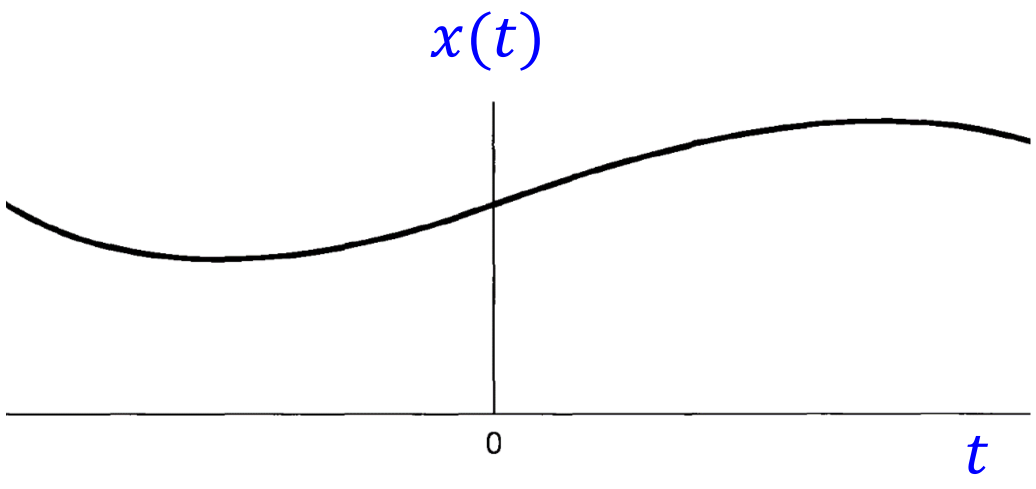
- ***Discrete-time signals:*** defined only at discrete times, and the independent variable takes on only a discrete set of values

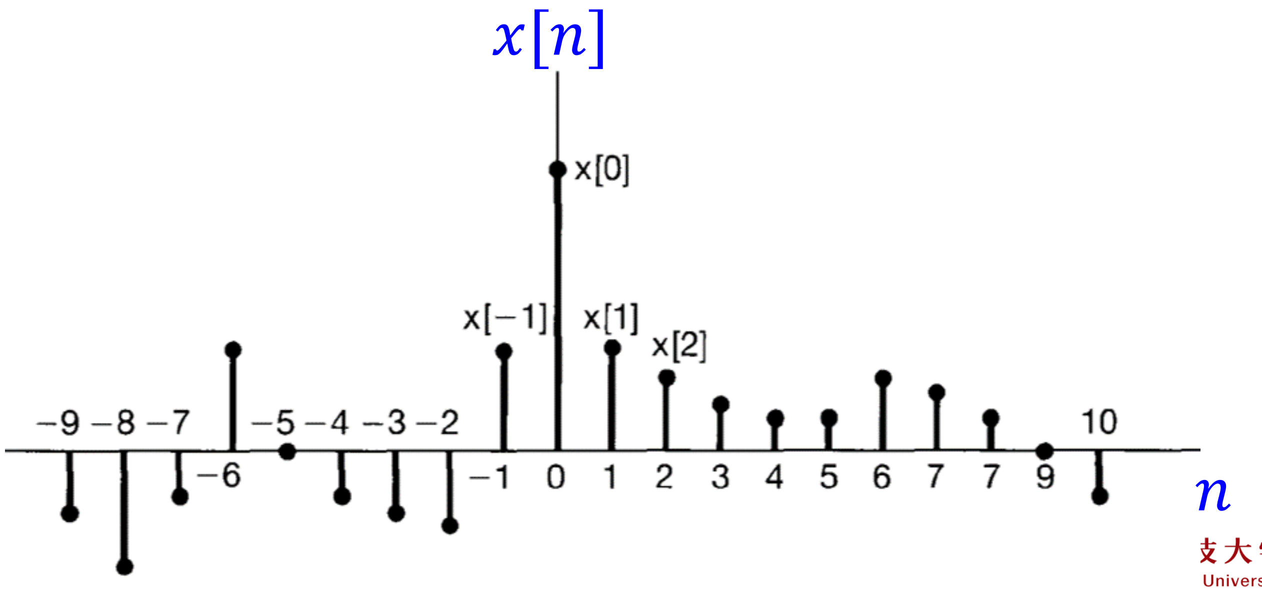


An example of a discrete-time signal: The weekly Dow-Jones stock market index from January 5, 1929, to January 4, 1930.

Continuous-Time and Discrete-Time Signals

- ***Continuous-time signals:*** t denote the independent variable, enclosed in $(\cdot)$
- ***Discrete-time signals:*** n denote the independent variable, enclosed in $[\cdot]$
- $x[n]$
  - discrete in nature; or sampling of continuous-time signal
  - defined only for integer values of n



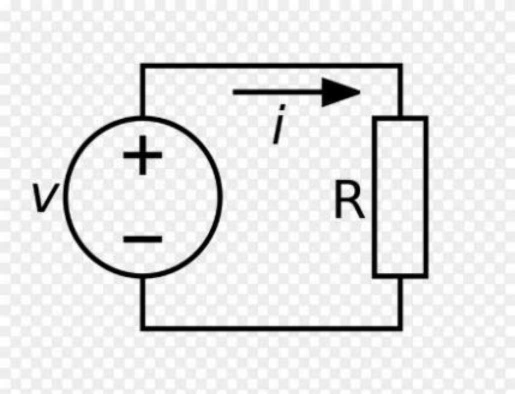


Continuous-Time and Discrete-Time Signals

Energy and power

- $v(t)$ and $i(t)$ are voltage and current across a resistor R, the instantaneous power is

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$



- The total energy over the time interval $t_1 \le t \le t_2$ is

$$E_R = \int_{t_1}^{t_2} p(t)\,dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt$$

- The average power over the time interval $t_1 \le t \le t_2$ is

$$P_R = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)\,dt$$

Continuous-Time and Discrete-Time Signals

Signal energy and power

- Similarly, for any signal $x(t)$ or $x[n]$, the total energy is defined as

$$E = \int_{t_1}^{t_2} |x(t)|^2\, dt \qquad t_1 \le t \le t_2 \qquad \text{Continuous-time signal}$$

$$E = \sum_{n=n_1}^{n_2} |x[n]|^2 \qquad n_1 \le n \le n_2 \qquad \text{Discrete-time signal}$$

- The average power is defined as

$$P = \frac{E}{t_2 - t_1} \quad \text{Continuous} \qquad\qquad P = \frac{E}{n_2 - n_1 + 1} \quad \text{Discrete}$$

Continuous-Time and Discrete-Time Signals

Signal energy and power

- Over infinite time interval $-\infty \leq t \leq \infty$ or $-\infty \leq n \leq \infty$

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt \qquad \text{Continuous}$$

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad \text{Discrete}$$

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

Continuous

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Discrete

Continuous-Time and Discrete-Time Signals

Signal energy and power

- Finite-energy signal: $E_\infty < \infty$

$$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$$

$$P_\infty = \lim_{N\to\infty} \frac{E_\infty}{2N+1} = 0$$

- Finite-power signal: $P_\infty < \infty, E_\infty = \infty$

- Infinite energy & power signal $P_\infty \to \infty, E_\infty \to \infty$

Continuous-Time and Discrete-Time Signals

Signal energy and power

- Examples:

$$(1)\ x(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases} \qquad E_\infty < \infty, P_\infty = 0$$

$$(2)\ x[n] = 4 \qquad P_\infty < \infty, E_\infty = \infty$$

$$(3)\ x(t) = t \qquad P_\infty \to \infty, E_\infty \to \infty$$

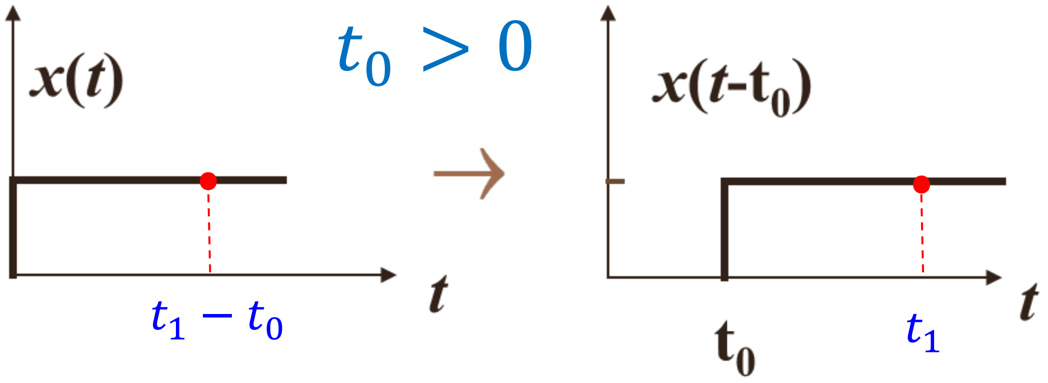
Chapter 1: An overview

- Continuous-Time and Discrete-Time Signals
- Transformations of the Independent Variable
- Exponential and Sinusoidal Signals
- The Unit Impulse and Unit Step Functions
- Continuous-Time and Discrete-Time Systems
- Basic System Properties

Transformation of the independent variable

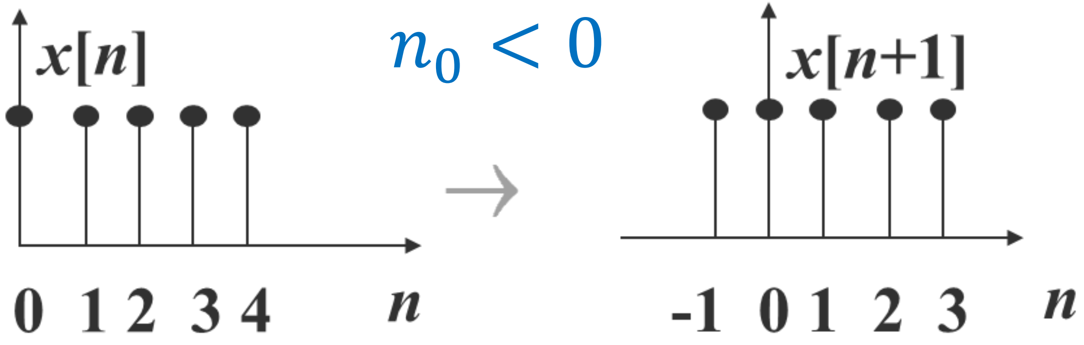
Time shift

$x(t) \longrightarrow x(t-t_0) = y(t)$



$$y(t)\Big|_{t=t_1} = x(t-t_0)\Big|_{t=t_1} = x(t_1-t_0) = x(t)\Big|_{t=t_1-t_0}$$

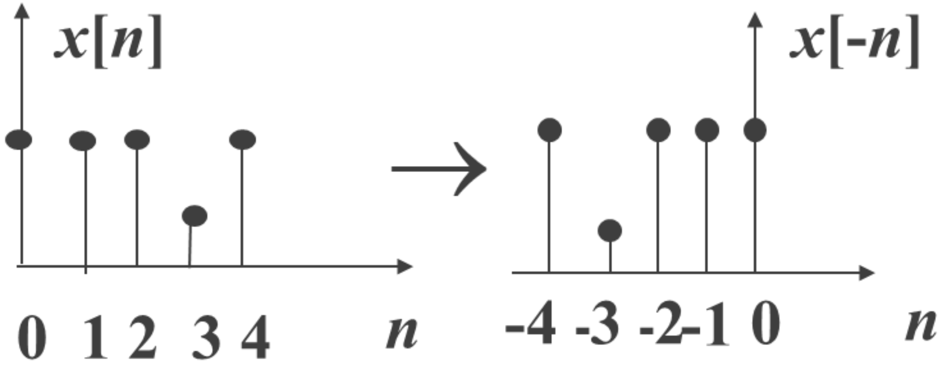
$x[n] \longrightarrow x[n-n_0]$



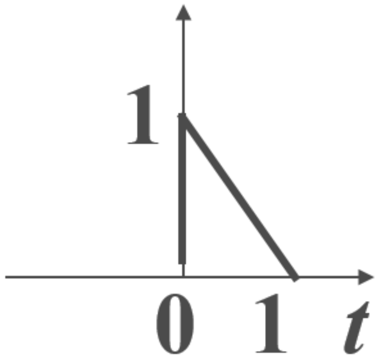
Transformation of the independent variable

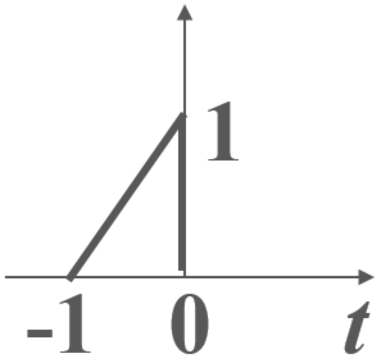
Time reversal

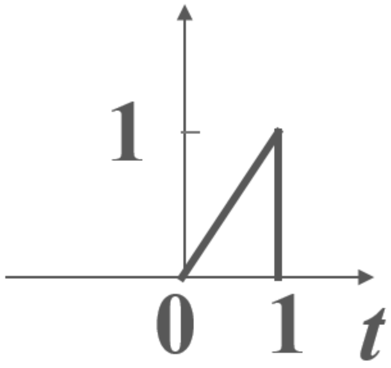
$$x[n] \longrightarrow x[-n]$$



$$x(t) \longrightarrow x(-t) \longrightarrow x(1-t)$$







Example

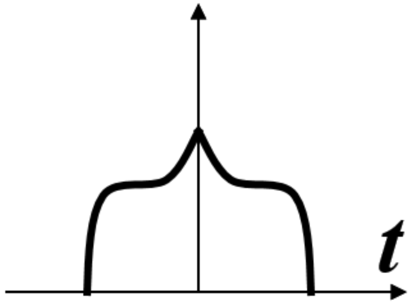
Transformation of the independent variable

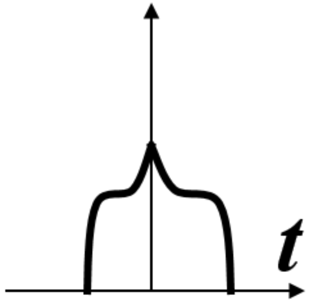
Time scaling

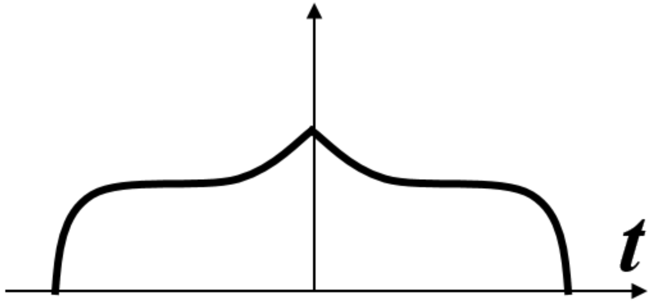
$x(t) \longrightarrow x(2t)$ Compressed

$x(t) \longrightarrow x(t/2)$ Stretched

$x(t) \longrightarrow x(2t) \longrightarrow x(t/2)$







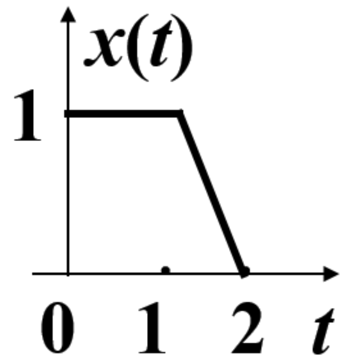
Transformation of the independent variable

General: Let $x(t) \rightarrow x(\alpha t + \beta)$

- if $|\alpha| > 1$, compressed
- if $|\alpha| < 1$, stretched
- if $\alpha < 0$, reversed
- if $\beta \neq 0$, shifted

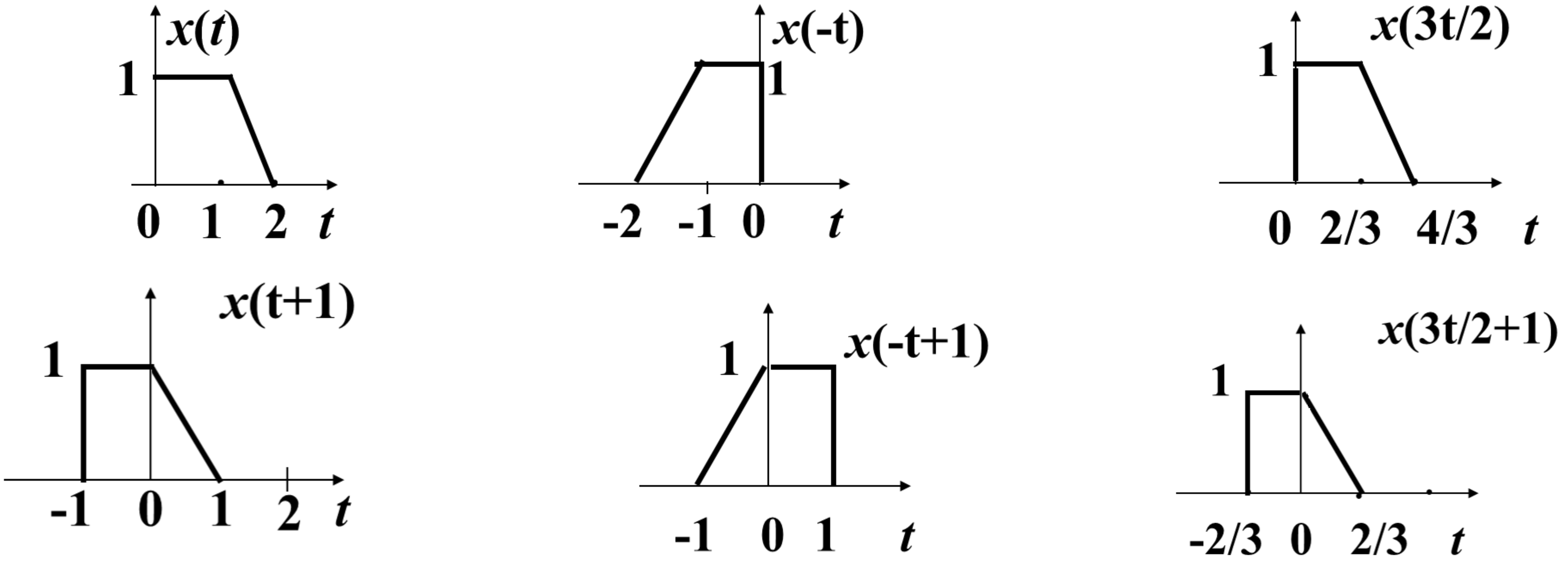
Example1: Given the signal $x(t)$, to illustrate

- $x(t + 1)$
- $x(-t + 1)$
- $x(3t/2)$
- $x(\frac{3t}{2} + 1)$



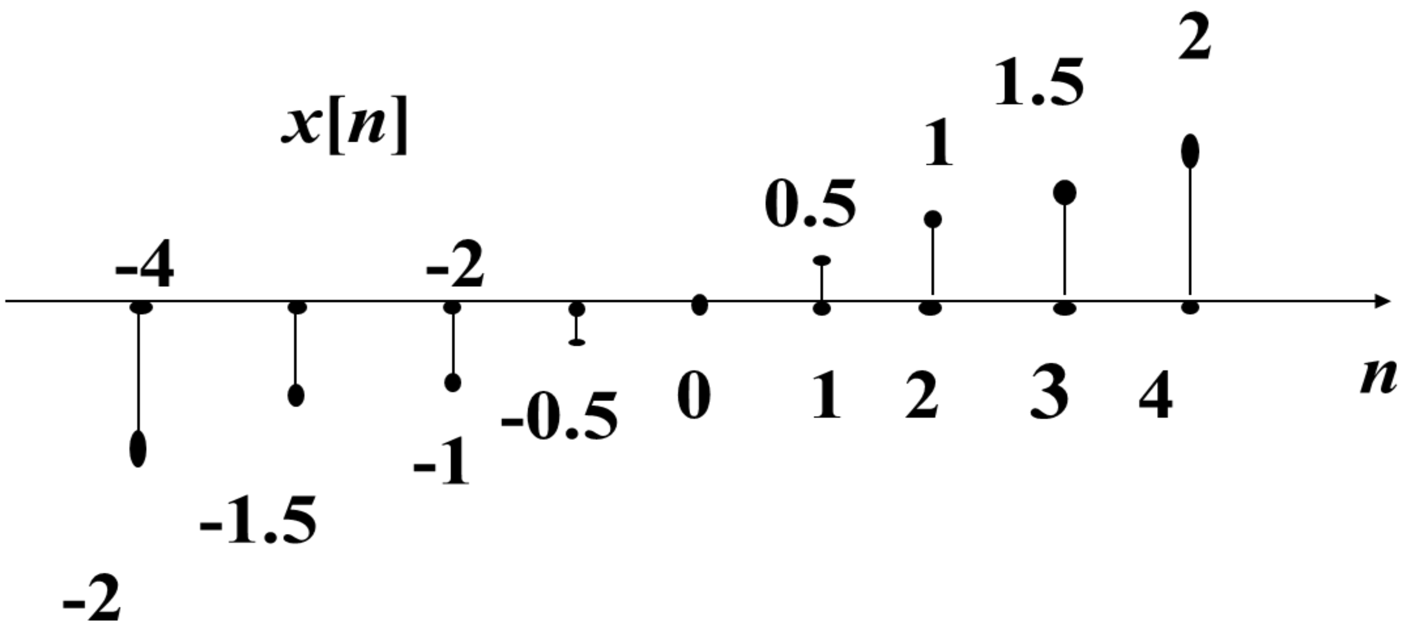
Transformation of the independent variable

- $x(t+1)$ $x(-t+1)$ $x(3t/2)$ $x(\frac{3t}{2}+1)$

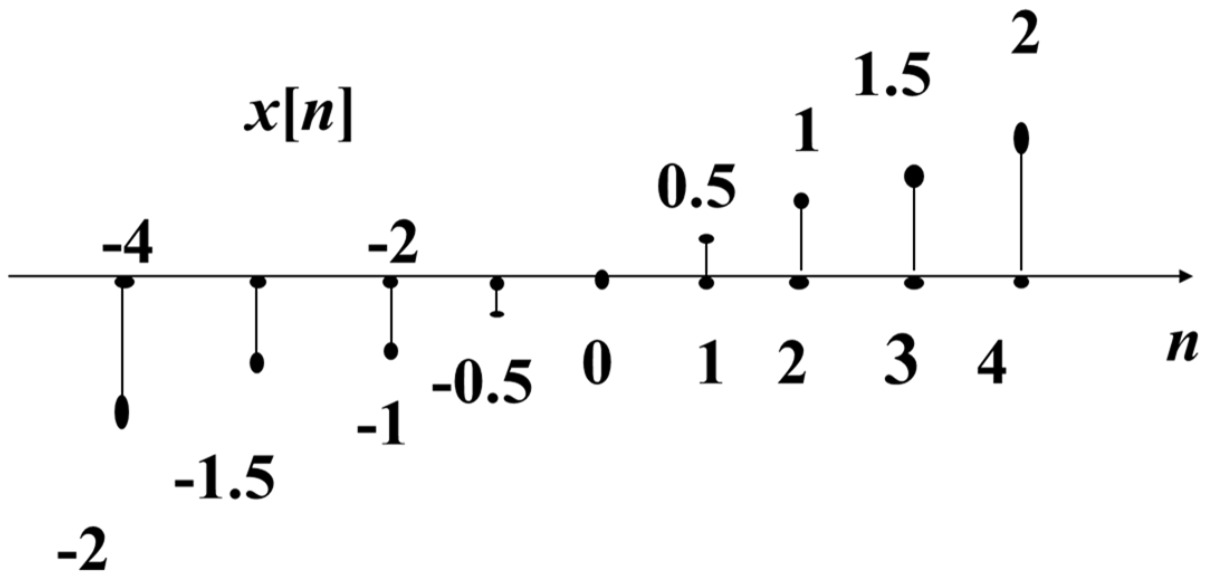


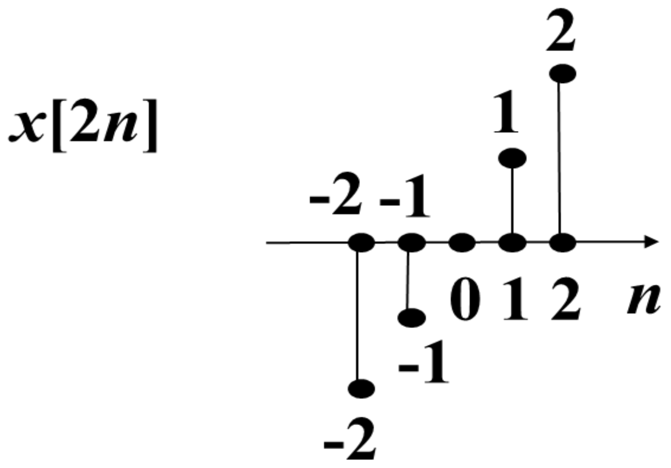
Transformation of the independent variable

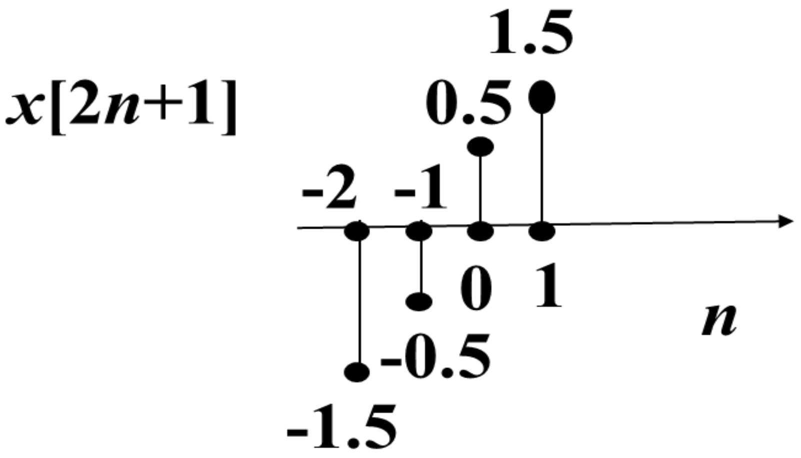
- ***Example2:*** A discrete signal $x[n]$ is shown below, sketch and label following signals:
  - $x[2n]$
  - $x[2n+1]$



Transformation of the independent variable

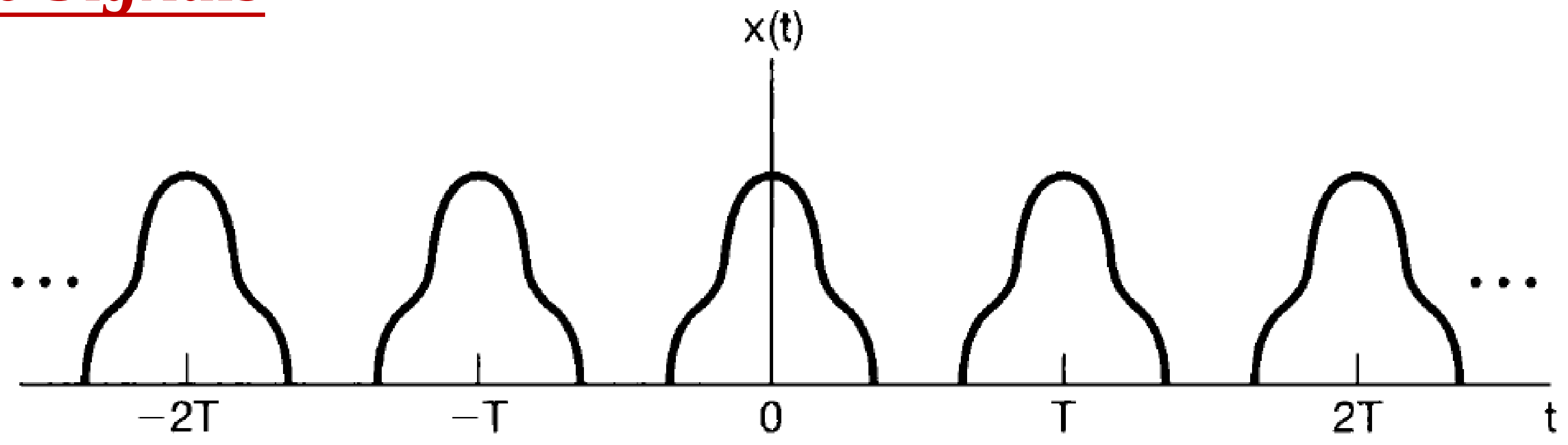






Transformation of the independent variable

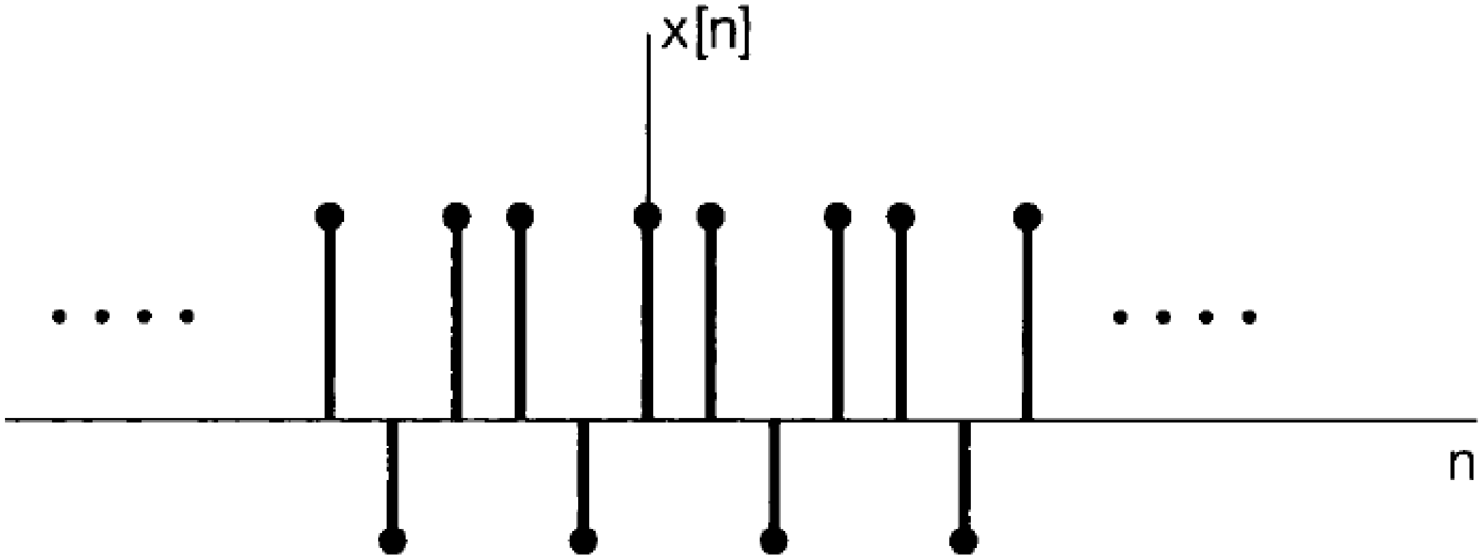
Periodic Signals



- Continuous-time: $x(t) = x(t + T)$ for all t
- Fundamental period
  - The smallest positive value of T for which $x(t) = x(t + T)$ holds

Transformation of the independent variable

Periodic Signals



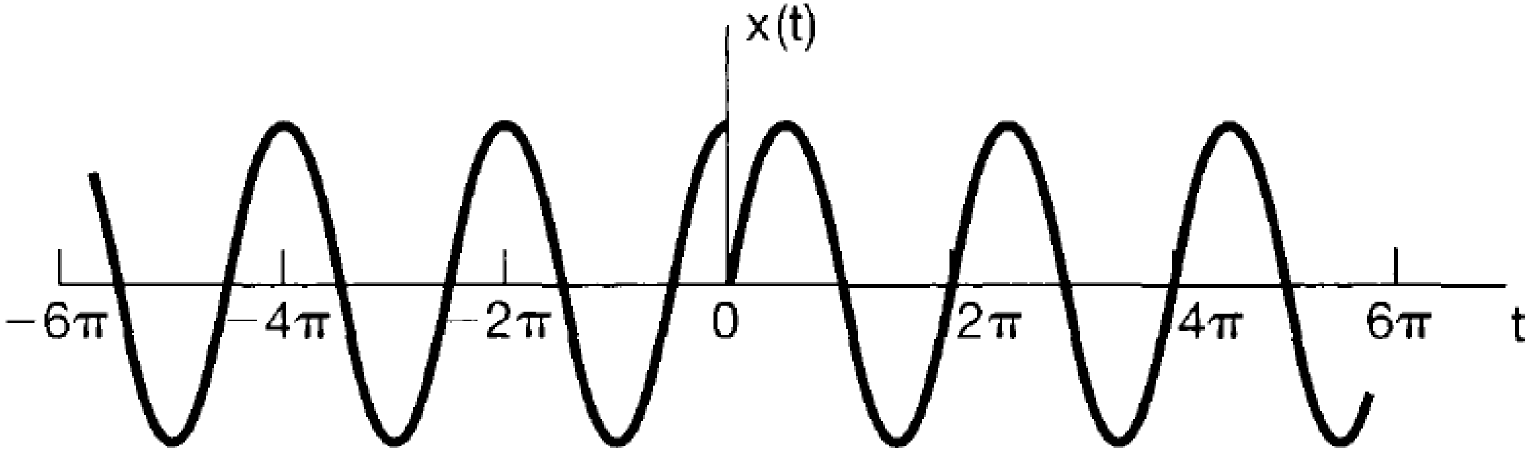
- Discrete-time: $x[n] = x[n+N]$ for all n
- Fundamental period
  - The smallest positive value of N for which $x[n] = x[n+N]$ holds

Transformation of the independent variable

Periodic Signals?

- Example:

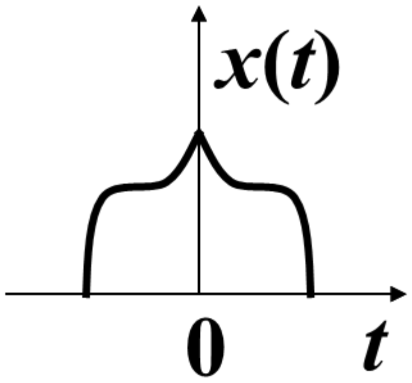
$$x(t) = \begin{cases} \cos(t) & \text{if } t < 0 \\ \sin(t) & \text{if } t \geq 0 \end{cases}$$



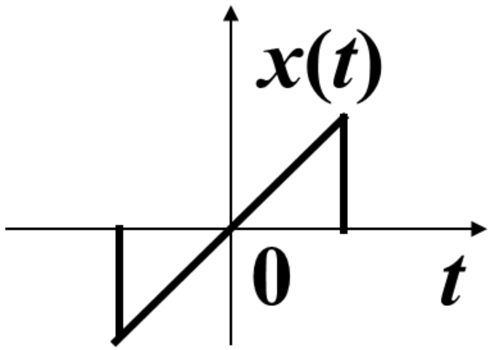
Transformation of the independent variable

Even and Odd Signals

- Even signal
  - $x(t) = x(-t) \quad x[n] = x[-n]$



- Odd signal
  - $x(t) = -x(-t) \quad x[n] = -x[-n]$



Transformation of the independent variable

Even and Odd Signals

- Any signal can be broken into a sum of two signals
  - One even and one odd

$$x(t) = x_e(t) + x_o(t)$$

$$x_e(t) = E_v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

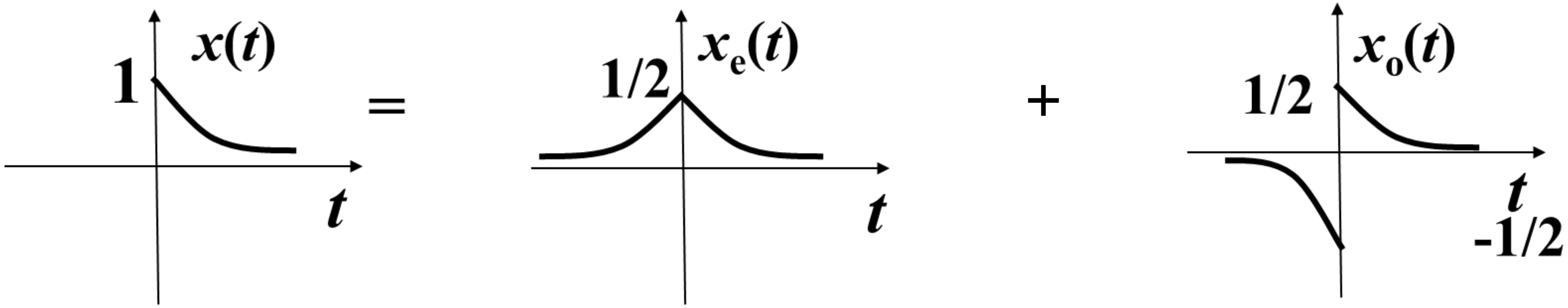
$$x_o(t) = O_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

Transformation of the independent variable

Even and Odd Signals

$$x_e(t) = E_v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

$$x_o(t) = O_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$



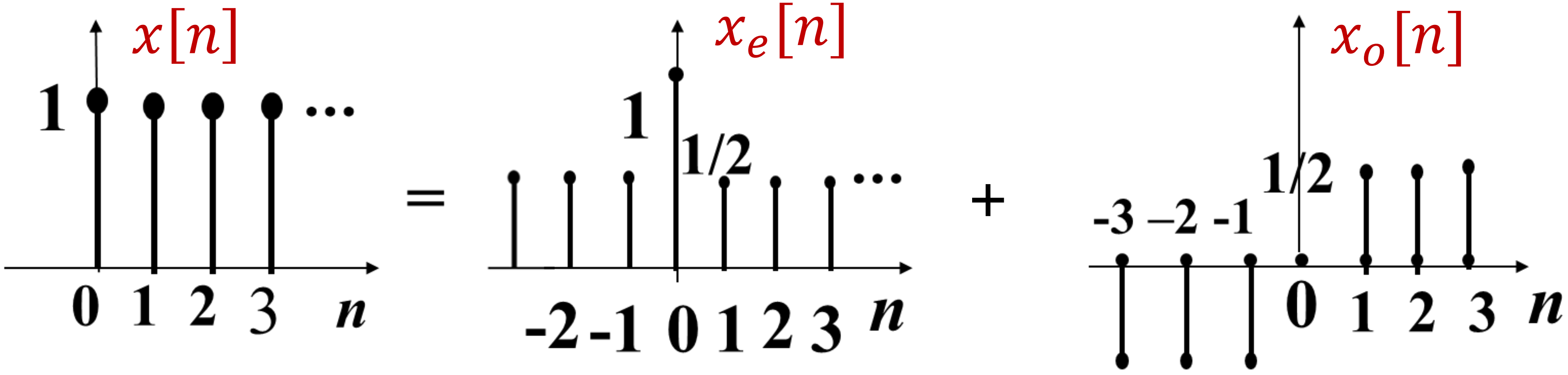
Transformation of the independent variable

Even and Odd Signals

$$x[n] = x_e[n] + x_o[n]$$

$$x_e[n] = (x[n] + x[-n])/2$$

$$x_o[n] = (x[n] - x[-n])/2$$



Chapter 1: An overview

- Continuous-Time and Discrete-Time Signals
- Transformations of the Independent Variable
- Exponential and Sinusoidal Signals
- The Unit Impulse and Unit Step Functions
- Continuous-Time and Discrete-Time Systems
- Basic System Properties

Exponential and Sinusoidal Signals

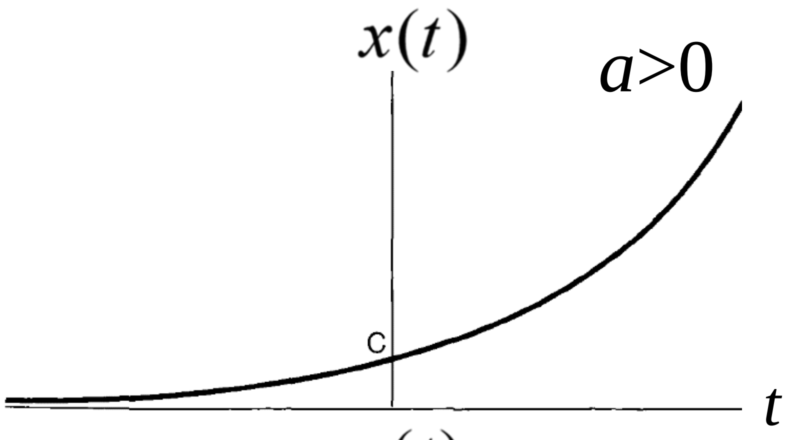
Continuous-Time Complex Exponential and Sinusoidal Signals

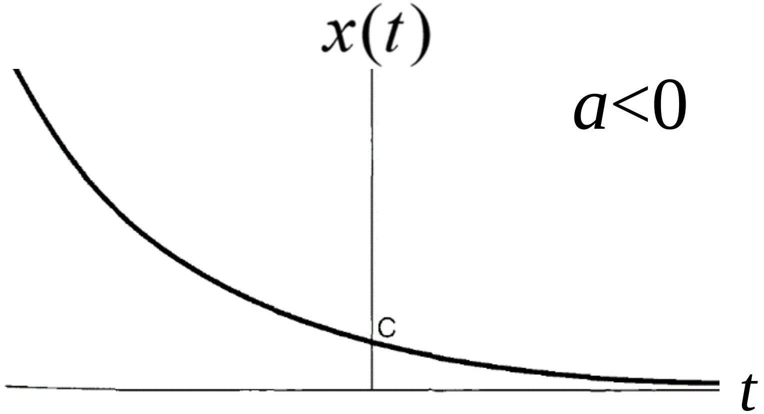
- General case

$$x(t) = ce^{at}$$

C and a are complex number

- Real exponential signal
  - C and a are real
  - $a>0$, as $t \uparrow$, $x(t) \uparrow$
  - $a<0$, as $t \uparrow$, $x(t) \downarrow$
  - $a=0$, $x(t)$ is constant





Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

- Periodic exponential signals
  - c is real, specifically 1
  - a is purely imaginary

$$x(t) = e^{j\omega_0 t}$$

  - Fundamental period T_0?

$$x(t) = e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T} \Rightarrow e^{j\omega_0 T} = 1$$

$$\Rightarrow \omega_0 T = 2k\pi,\ k = \pm 1, \pm 2, \ldots \Rightarrow T = \frac{2k\pi}{\omega_0} \Rightarrow T_o = \frac{2\pi}{|\omega_0|}$$

  - T_0 is undefined for $\omega_0 = 0$

Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

- Sinusoidal Signals $x(t) = A\cos(\omega_0 t + \emptyset)$

- Closely related to complex exponential signals

$$e^{j(\omega_0 t+\emptyset)} = \cos(\omega_0 t + \emptyset) + j\sin(\omega_0 t + \emptyset)$$

$$A\cos(\omega_0 t + \emptyset) = A \cdot Re\{e^{j(\omega_0 t+\emptyset)}\}$$

$$A\sin(\omega_0 t + \emptyset) = A \cdot Im\{e^{j(\omega_0 t+\emptyset)}\}$$

- Fundamental frequency ω_0

$$A\cos(\omega_0 t + \emptyset) = \frac{A}{2}e^{j\emptyset}e^{j\omega_0 t} + \frac{A}{2}e^{-j\emptyset}e^{-j\omega_0 t}$$

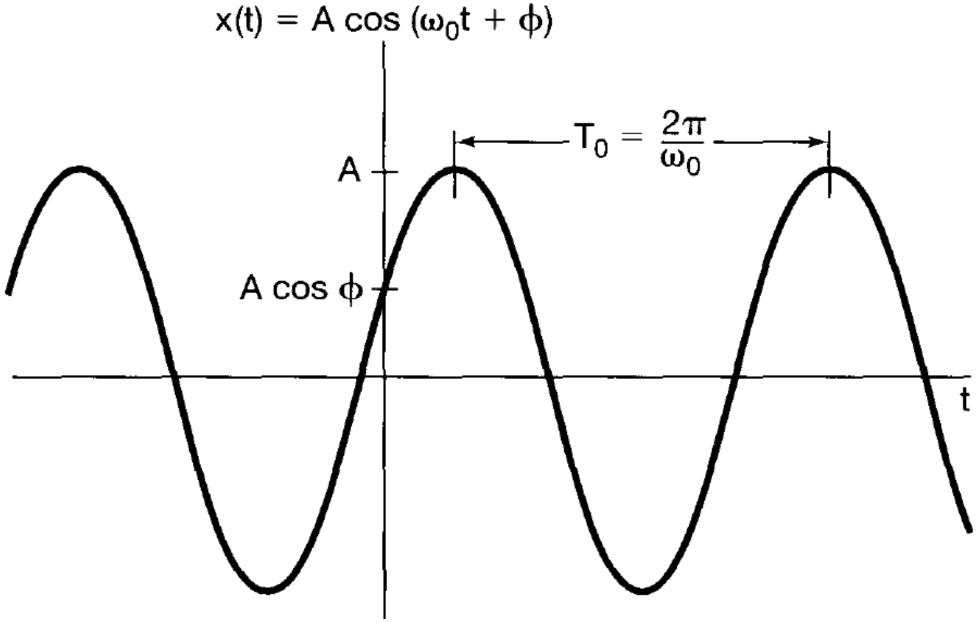
Exponential and Sinusoidal Signals

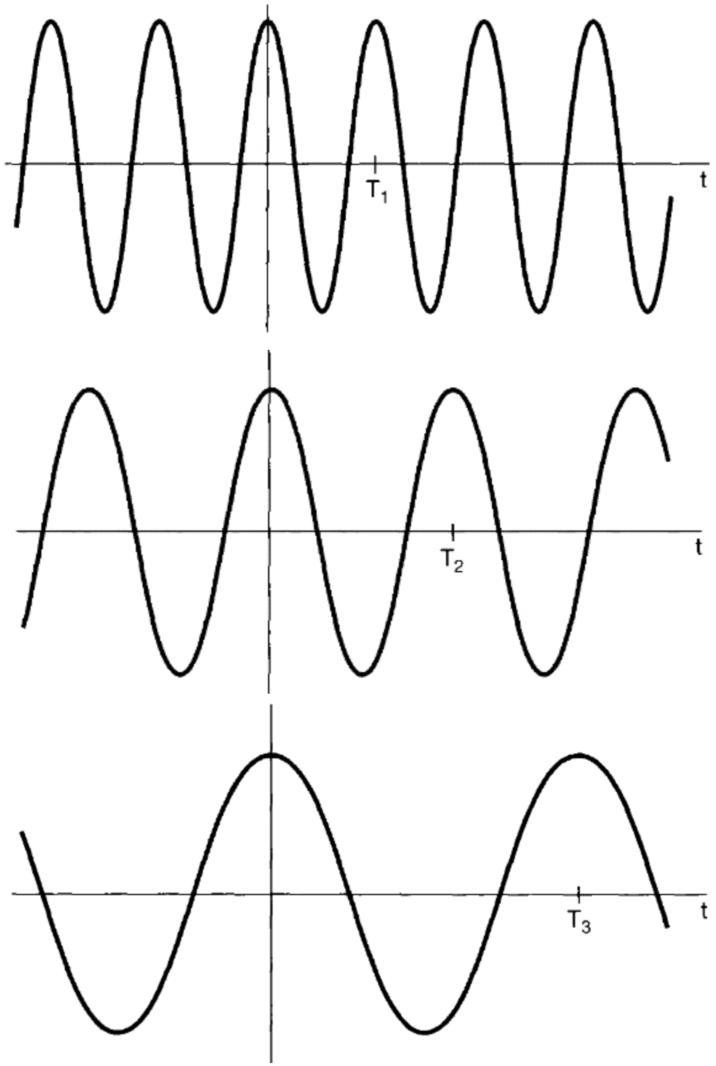
Continuous-Time Complex Exponential and Sinusoidal Signals

- Sinusoidal Signals

$$x(t) = A\cos(\omega_0 t + \emptyset)$$

- Fundamental frequency ω_0





$\omega_3 < \omega_2 < \omega_1$

$T_3 > T_2 > T_1$

Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

- $e^{j\omega_0 t}$ and $A\cos(\omega_0 t + \emptyset)$： infinite total energy but finite average power

$$E_{period} = \int_0^{T_0} \left| e^{j\omega_0 t} \right|^2 dt = \int_0^{T_0} 1dt = T_0$$

$$p_{period} = \frac{1}{T_0} E_{period} = 1$$

- Total energy: infinite
- Average power: finite

Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

- Harmonically related complex exponentials

- Sets of periodic exponentials (with different frequencies), all of which are periodic with a common period T_0

$$e^{j\omega t} = e^{j\omega(t+T_0)} = e^{j\omega t}e^{j\omega T_0}$$

$$\omega T_0 = 2k\pi,\ k = 0, \pm 1, \pm 2, \dots.$$

$$\omega = 2k\pi/T_0 = k\omega_0, \text{ with } \omega_0 = 2\pi/T_0$$

- $\phi_k(t) = e^{jk\omega_0 t}$, $k = 0, \pm 1, \pm 2, \dots$ is a harmonically related set.
- For any $k \neq 0$, fundamental frequency $|k|\omega_0$; fundamental period

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$$

Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

- Examples – Periodic or not?

$$(1)\ x_1(t) = je^{j10t} \qquad \omega_0 = 10,\ T_0 = \frac{2\pi}{10} = \frac{\pi}{5}$$

$$(2)\ x_2(t) = e^{(-1+j)t}$$

Aperiodic

$$(3)\ x_3(t) = 2\cos(3t + \frac{\pi}{4}) \qquad \omega_0 = 3,\ T_0 = \frac{2\pi}{3}$$

$$(4)\ x(t) = 2\cos(3t + \frac{\pi}{4}) + 3\cos(2t - \frac{\pi}{6})$$

$$T_{01} = \frac{2\pi}{3}, \quad T_{02} = \pi \qquad T_0 = SCM(T_{01}, T_{02}) = 2\pi$$

Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

- General case

$$x(t) = Ce^{at}$$

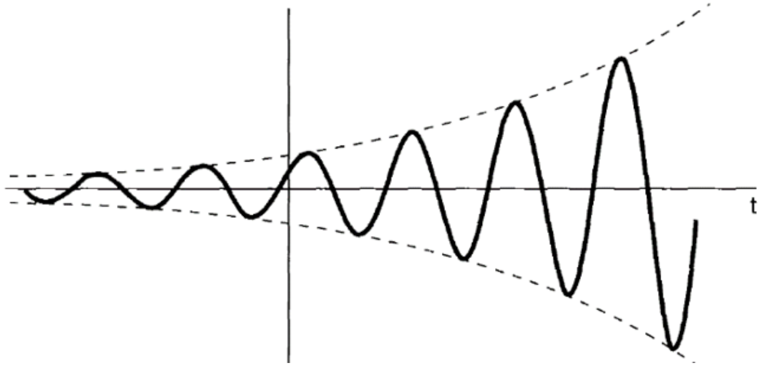
C and a are complex numbers

$$C = |C|e^{j\theta}, a = r + j\omega_0$$

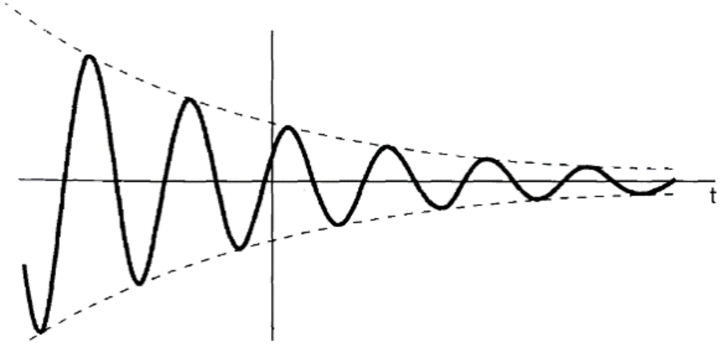
$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$

$Re\{x(t)\} = |C|e^{rt}\cos(\omega_0 t + \theta), r > 0$



$Re\{x(t)\} = |C|e^{rt}\cos(\omega_0 t + \theta), r < 0$



Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

- General case

$$x[n] = C\alpha^n$$

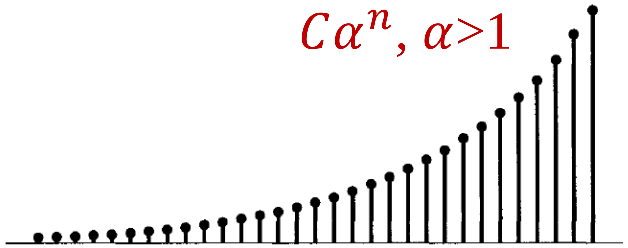
c and α are complex numbers

$$x[n] = Ce^{\beta n} \qquad \alpha = e^{\beta}$$

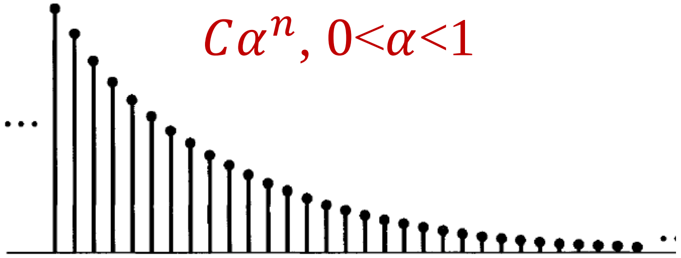
- Real Exponential Signals

C and α are real numbers

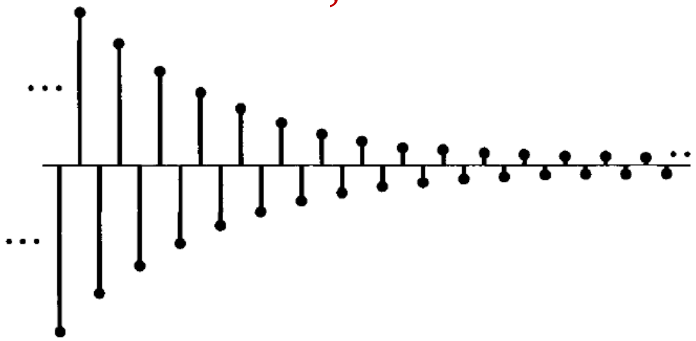
$C\alpha^n$, α>1



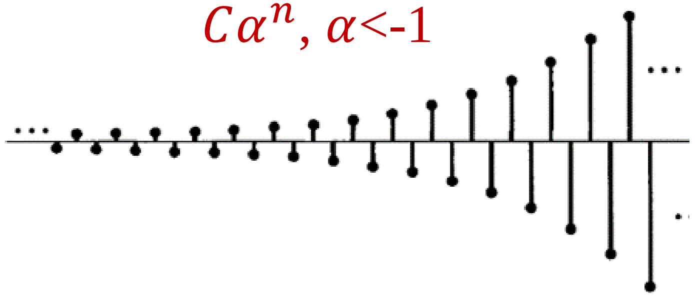
$C\alpha^n$, 0<α<1



$C\alpha^n$, -1<α<0



$C\alpha^n$, α<-1



Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

- Sinusoidal signals
  - c is real, specifically 1; β is purely imaginary

$$x[n] = e^{j\omega_0 n}$$

Closely related $\quad A\cos(\omega_0 n + \emptyset)$

$$e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$$

$$A\cos(\omega_0 n + \emptyset) = A/2 \cdot e^{j\emptyset}e^{j\omega_0 n} + A/2 \cdot e^{-j\emptyset}e^{-j\omega_0 n}$$

  - Infinite total energy but finite average power

$$|e^{j\omega_0 n}|^2 = 1$$

上海科技大学 ShanghaiTech University

Exponential and Sinusoidal Signals

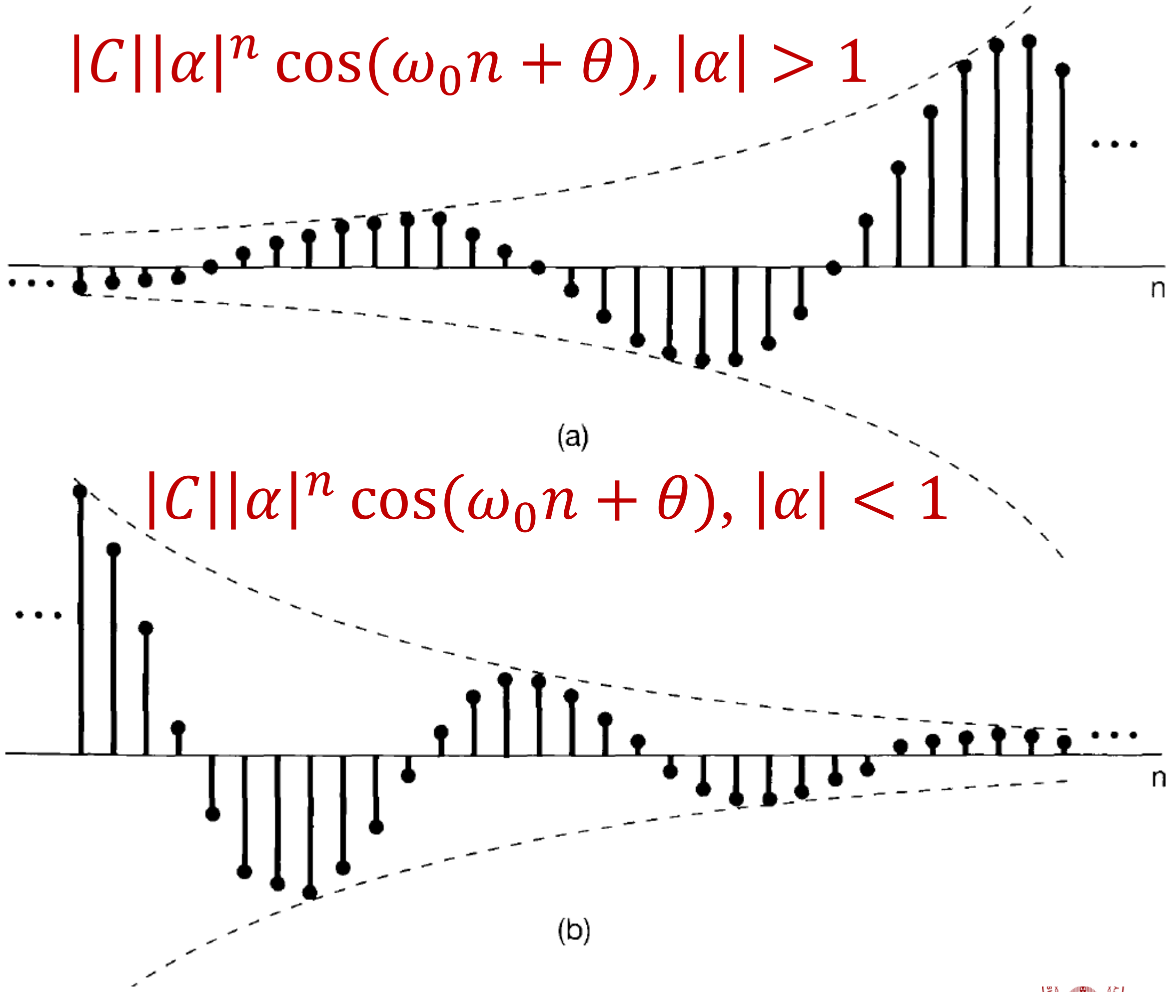
Discrete-Time Complex Exponential and Sinusoidal Signals

- General Signals

$$x[n] = C\alpha^n$$

$$C = |C|e^{j\theta}, \alpha = |\alpha|e^{j\omega_0}$$

$$x[n] = |C||\alpha|^n \cos(\omega_0 n + \theta)$$
$$+j\,|C||\alpha|^n \sin \omega_0 n + \theta$$



Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

- Periodicity properties $x[n] = e^{j\omega_0 n}$ **Focusing on ω_0**
  - $e^{j\omega_0 n}$: same value at ω_0 and $\omega_0 + 2k\pi$

$$e^{j(\omega_0+2k\pi)n} = e^{j2k\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$$

  - Only consider interval $0 \leq \omega_0 \leq 2\pi$ or $-\pi \leq \omega_0 \leq \pi$
- From 0 to π: $\omega_0 \uparrow$, oscillation rate of $e^{j\omega_0 n} \uparrow$
- From π to 2π: $\omega_0 \uparrow$, oscillation rate of $e^{j\omega_0 n} \downarrow$
- Maximum oscillation rate at $\omega_0 = \pi$

$$e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

Exponential and Sinusoidal Signals

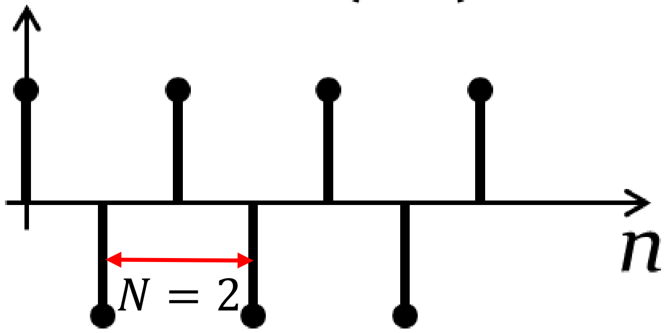
Discrete-Time Complex Exponential and Sinusoidal Signals

- Periodicity properties

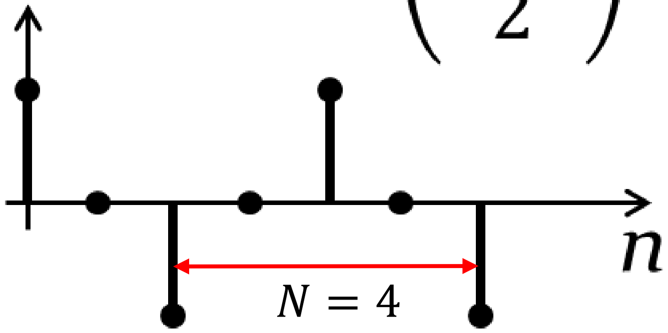
  - Q: Which one is a higher frequency signal?

$\omega_0 = \pi \qquad \omega_0 = 3\pi/2$

$\cos(\pi n)$



$\cos\left(\frac{3\pi n}{2}\right) = \cos\left(\frac{\pi n}{2}\right)$

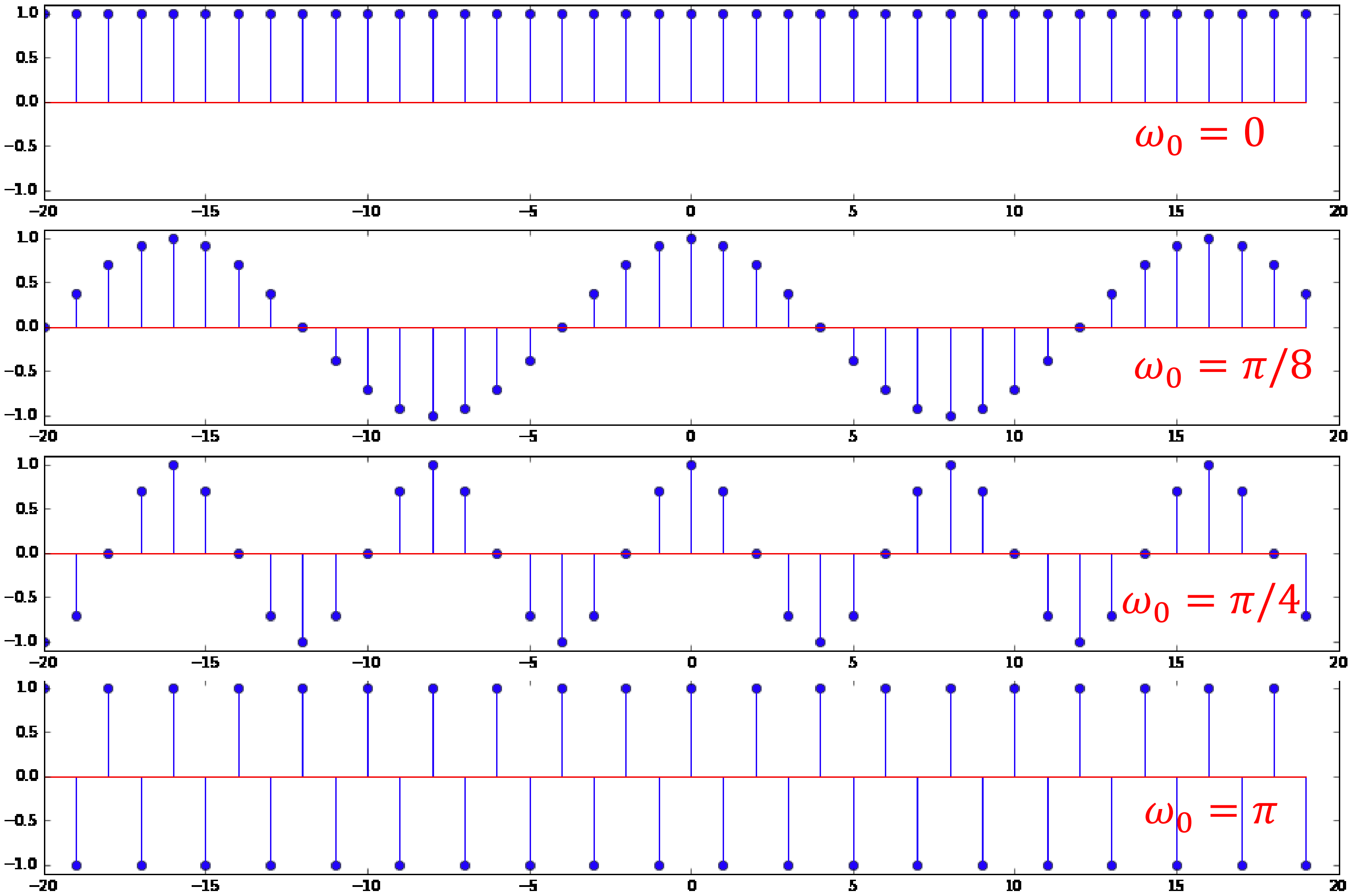


Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

- Periodicity properties

$$\cos(\omega_0 n)$$

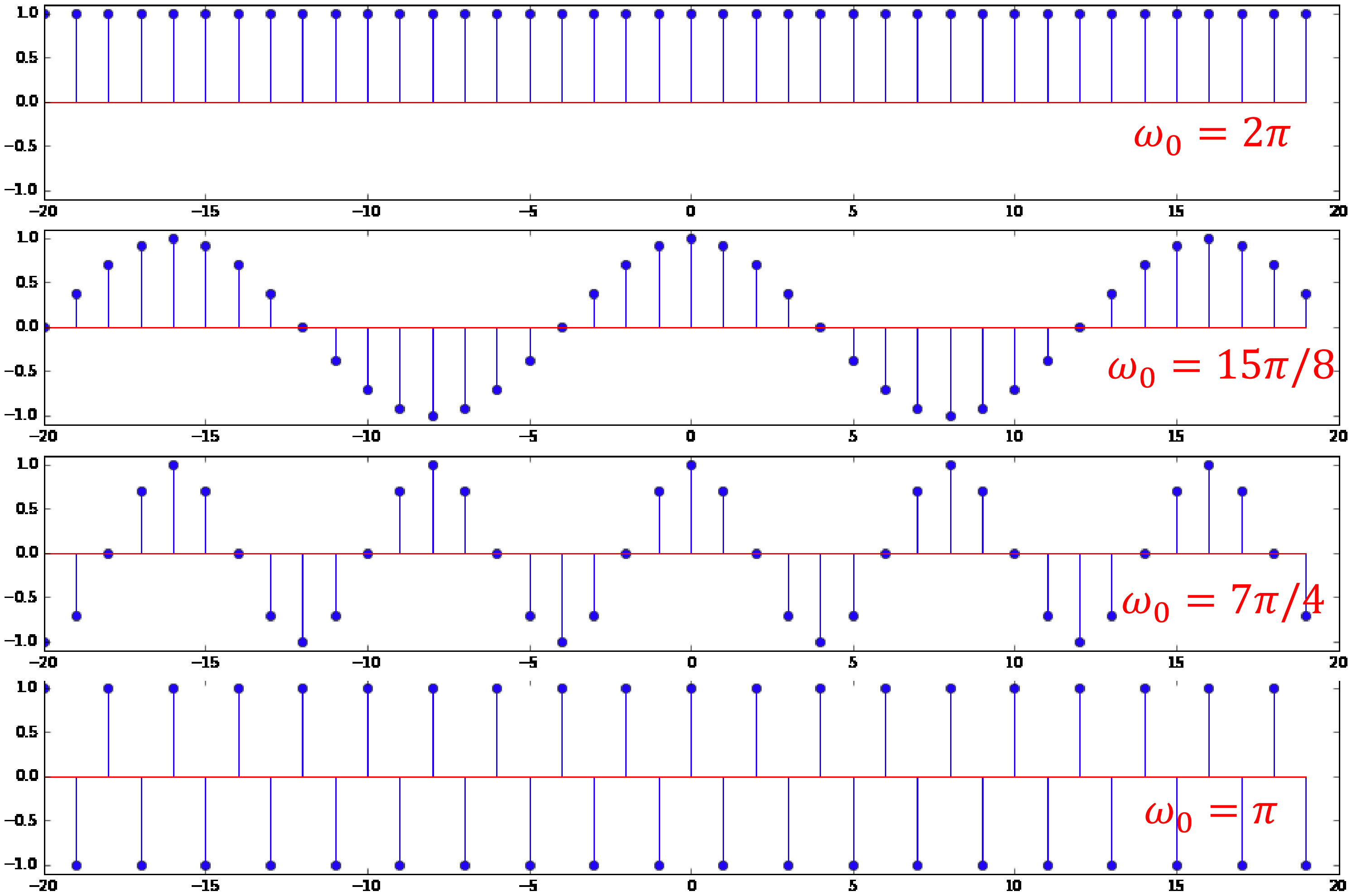


Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

- Periodicity properties

$$\cos(\omega_0 n)$$



Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

- Periodicity properties $x[n] = e^{j\omega_0 n}$ **Focusing on *n***
  - In order for $e^{j\omega_0 n}$ to be periodic with N>0, must

$$e^{j\omega_0(n+N)} = e^{j\omega_0 N} e^{j\omega_0 n} = e^{j\omega_0 n}$$

$$\omega_0 N = 2\pi m,\ m \text{ integer number}$$

$$\frac{\omega_0}{2\pi} = \frac{m}{N}$$

  - $\omega_0/2\pi$: rational number
  - Fundamental frequency: $2\pi/N = \omega_0/m$
  - Fundamental period: $N = m(2\pi/\omega_0)$

Exponential and Sinusoidal Signals

Discrete-Time Complex Exponential and Sinusoidal Signals

- Periodicity properties

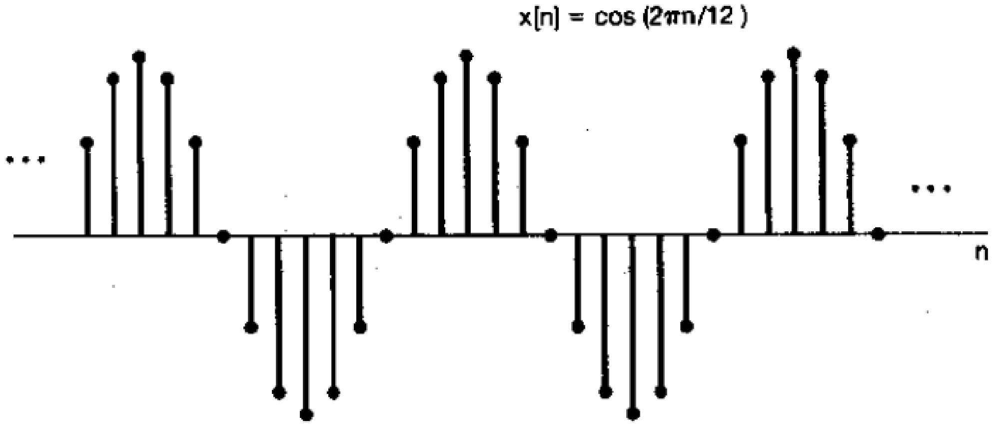
$$x[n] = \cos(2\pi n/12)$$ periodic N=12

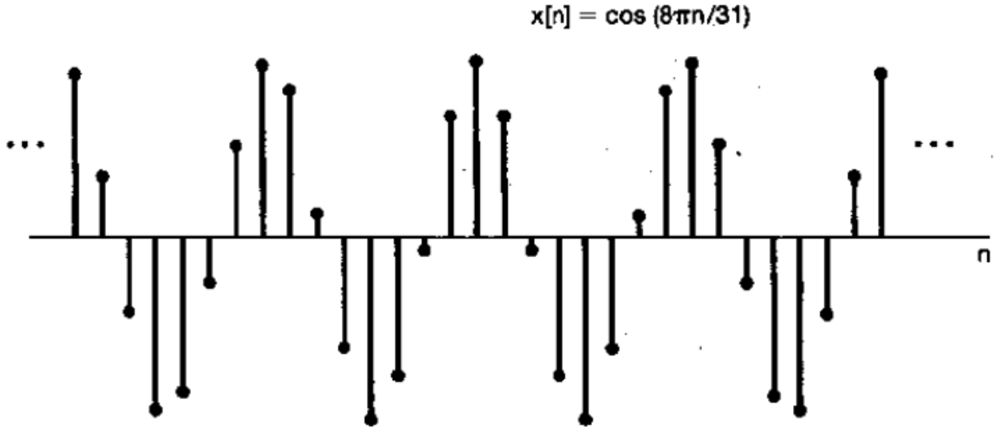
$$x[n] = \cos(8\pi n/31)$$ periodic N=31

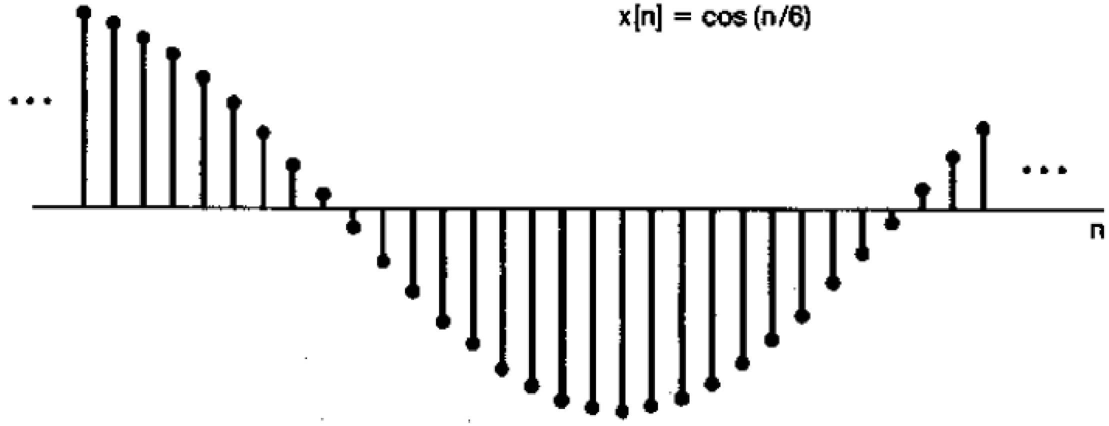
$$x[n] = \cos(n/6)$$ aperiodic

$$x[n] = e^{j\left(\frac{2\pi n}{3}\right)} + e^{j\left(\frac{3\pi n}{4}\right)}$$ periodic, N=24

Exponential and Sinusoidal Signals







Exponential and Sinusoidal Signals

Periodicity properties: discrete-time vs. continuous-time

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct ω_0	**Identical signals for values of ω_0 separated by multiples of 2π**
Periodic for any ω_0	**Only if $\omega_0=2\pi m/N$ for some integers $N>0$ and m**
Fundamental frequency ω_0	**ω_0/m**
Fundamental period $2\pi/\omega_0$	**$N=m(2\pi/\omega_0)$**

Chapter 1: An overview

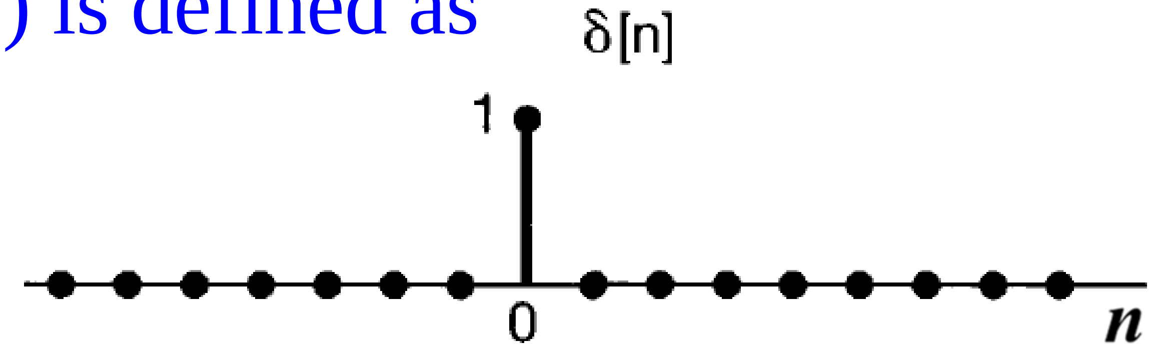
- Continuous-Time and Discrete-Time Signals
- Transformations of the Independent Variable
- Exponential and Sinusoidal Signals
- The Unit Impulse and Unit Step Functions
- Continuous-Time and Discrete-Time Systems
- Basic System Properties

The Unit Impulse and Unit Step Functions

Discrete-time unit impulse and unit step sequences

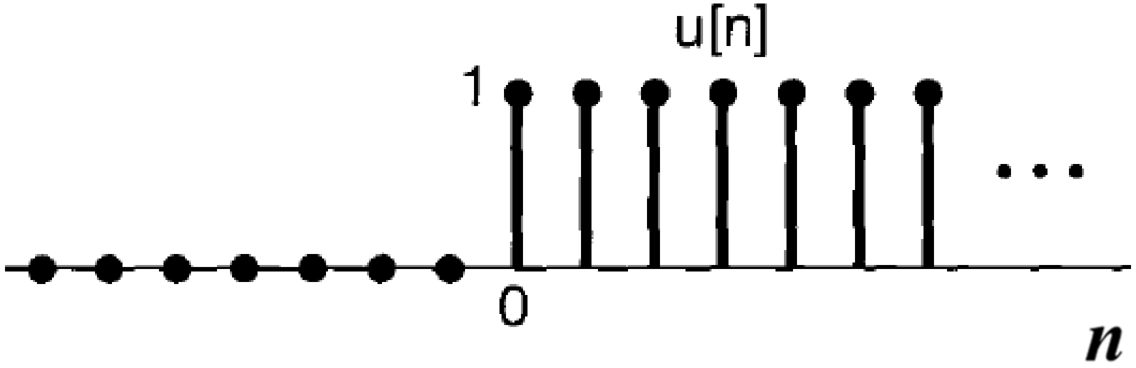
- Unit impulse (unit sample) is defined as

$$\delta[n]=\begin{cases}0, n\neq 0\\ 1, n=0\end{cases}$$



- Unit step is defined as

$$u[n]=\begin{cases}0, n<0\\ 1, n\geq 0\end{cases}$$



The Unit Impulse and Unit Step Functions

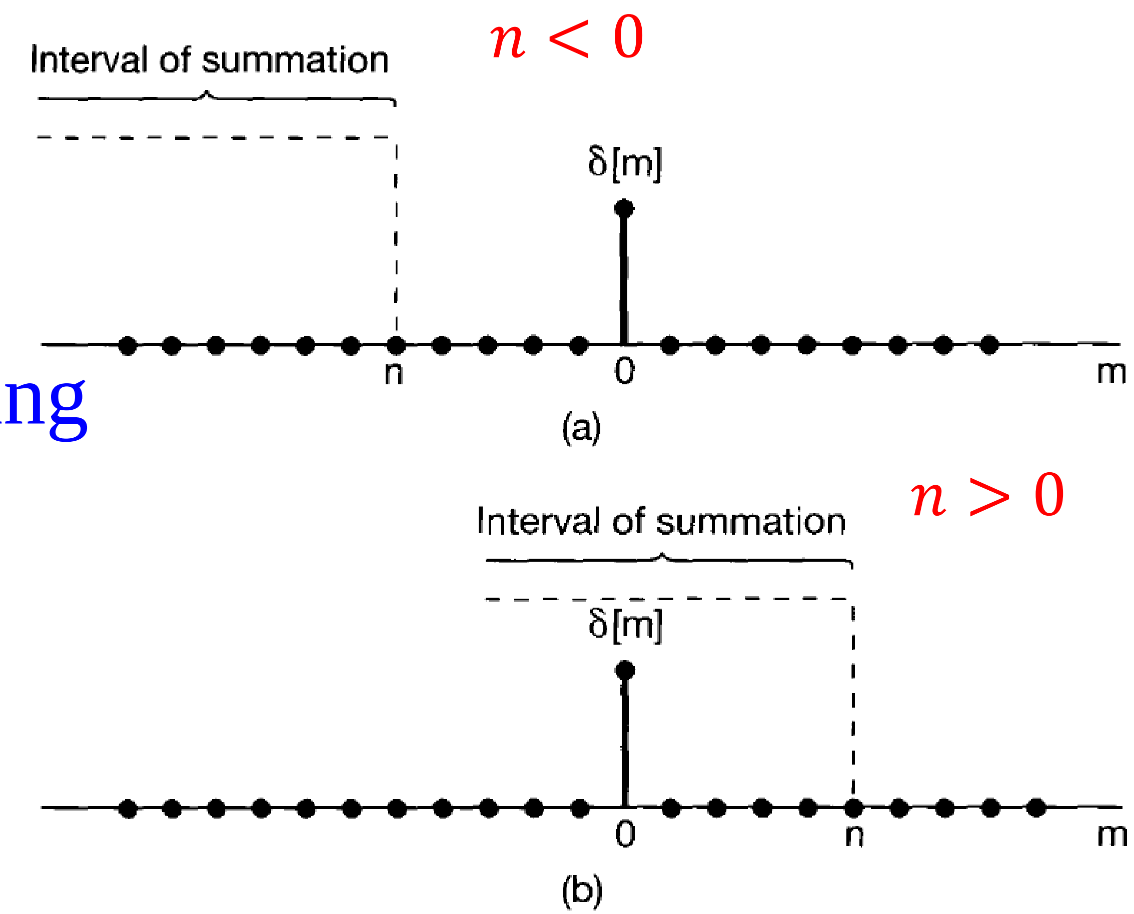
Discrete-time unit impulse and unit step sequences

- The impulse is the first difference of the step

$$\delta[n] = u[n] - u[n-1]$$

- Conversely, the step is the running sum of unit sample

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$



The Unit Impulse and Unit Step Functions

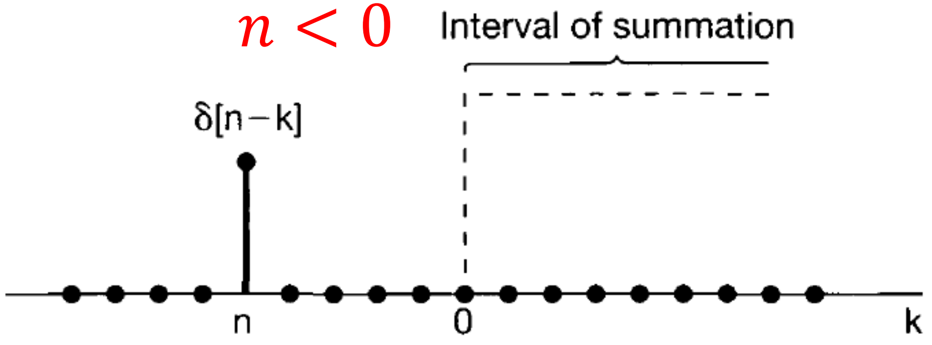
Discrete-time unit impulse and unit step sequences

- Let $m = n - k$,

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k]$$

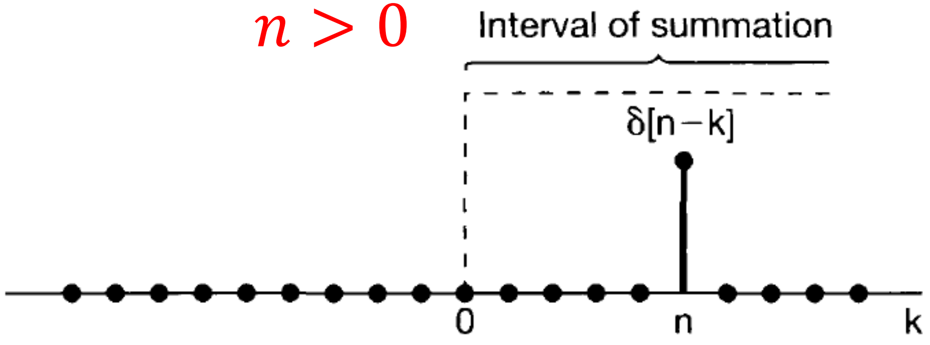
or $$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

$n < 0$



(a)

$n > 0$



(b)

The Unit Impulse and Unit Step Functions

Discrete-time unit impulse and unit step sequences

- Sampling property

$$x[n]\delta[n] = x[0]\delta[n]$$

- More generally

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

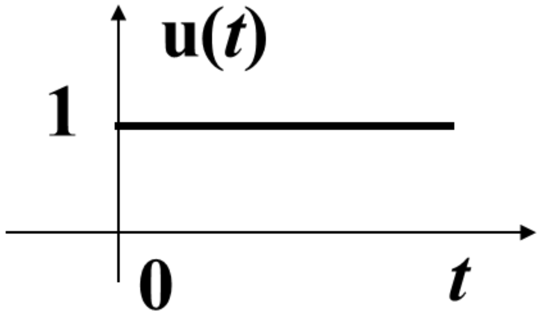
The Unit Impulse and Unit Step Functions

Continuous-time unit impulse and unit step sequences

- Unit step

$$u(t) = \begin{cases} 0, & t < o \\ 1, & t > 0 \end{cases}$$

Discontinuous at t=0



- The continuous unit step $u(t)$ is the running integral of unit impulse $\delta(t)$

$$u(t) = \int_{-\infty}^{t} \delta(\tau) \mathrm{d}\tau$$

- $\delta(t)$ the first derivative of $u(t)$

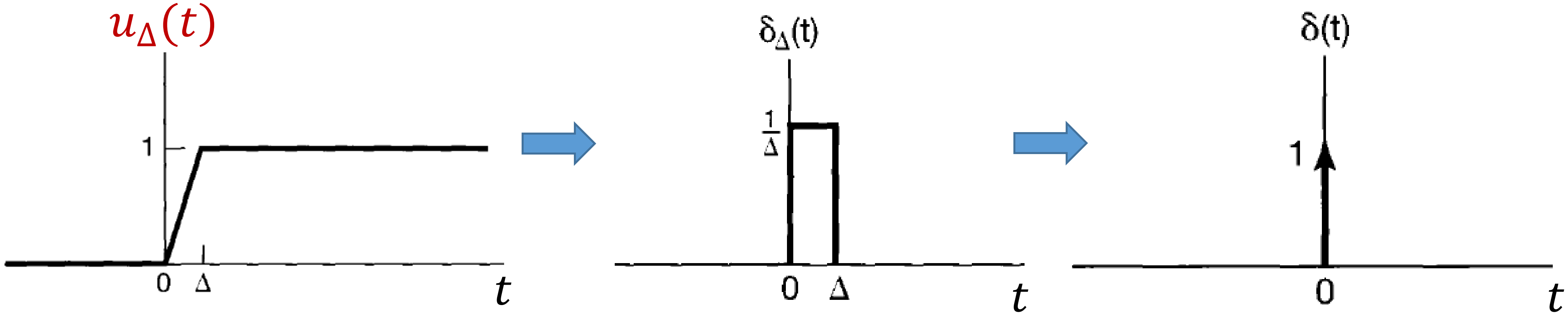
$$\delta(t) = \frac{du(t)}{dt}$$

上海科技大学
ShanghaiTech University

The Unit Impulse and Unit Step Functions

Continuous-time unit impulse and unit step sequences

- $u(t)$ is discontinuous at $t = 0$, How we get $\delta(t)$?
  - *Consider* $u_\Delta(t)$



$$u(t) = \lim_{\Delta \to 0} u_\Delta(t) \qquad \delta_\Delta(t) = \frac{d\, u_\Delta(t)}{dt} \qquad \delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

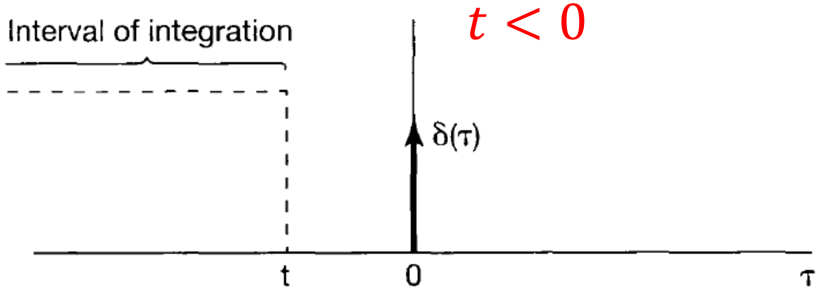
- arrow at $t = 0$: area of the pulse is concentrated at $t = 0$
- arrow height and "1“: *area* of the impulse

The Unit Impulse and Unit Step Functions

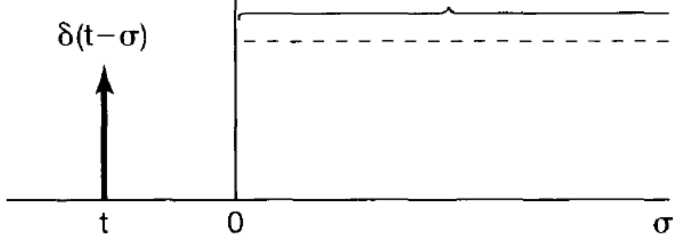
Continuous-time unit impulse and unit step sequences

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad \text{Let } \sigma = t - \tau \qquad u(t) = \int_{0}^{\infty} \delta(t-\sigma)d\sigma$$

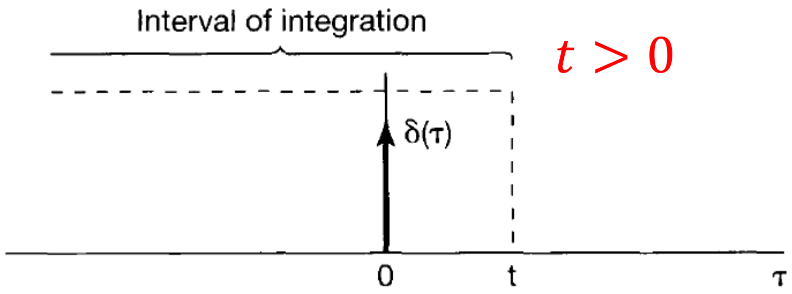
$t < 0$



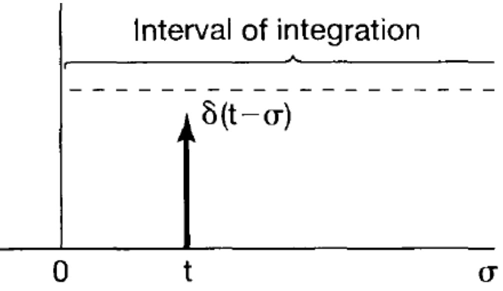
$t < 0$



$t > 0$



$t > 0$



上海科技大学
ShanghaiTech University

The Unit Impulse and Unit Step Functions

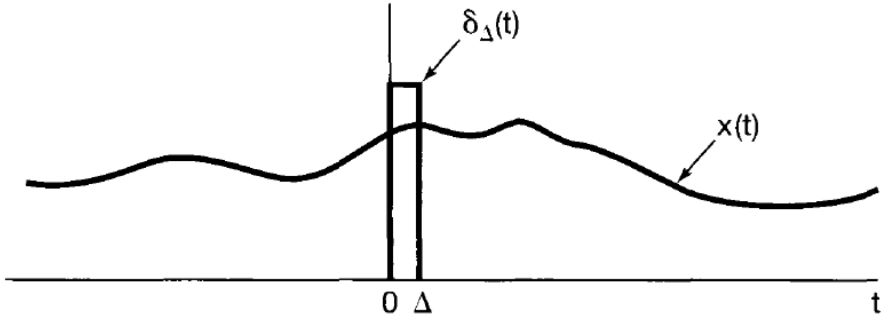
Continuous-time unit impulse and unit step sequences

- Sampling property

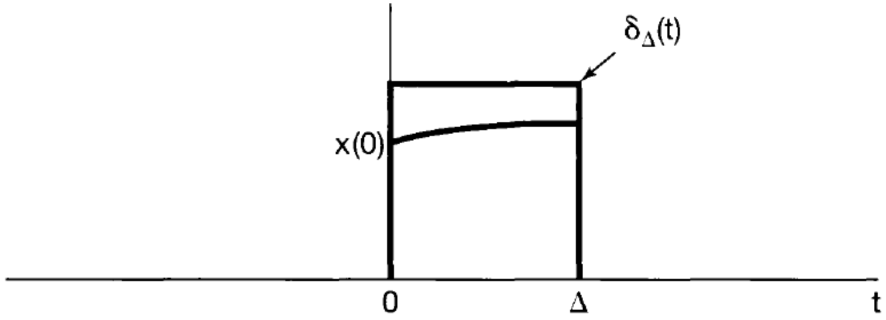
$$x_1(t) = x(t)\delta_\Delta(t)$$

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$$

$$x(t)\delta(t) = \lim_{\Delta\to 0} x(t)\delta_\Delta(t) = x(0)\lim_{\Delta\to 0}\delta_\Delta(t) = x(0)\delta(t)$$



(a)



(b)

- More generally

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

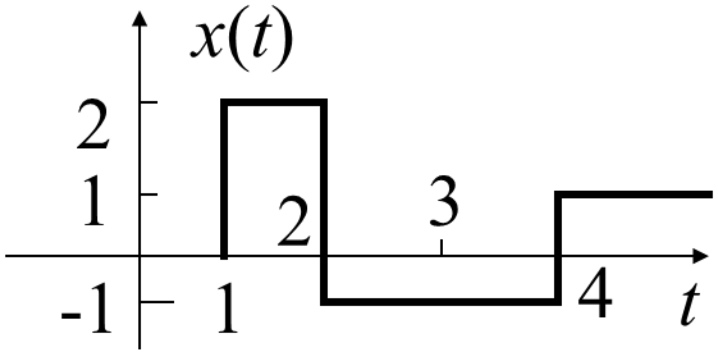
The Unit Impulse and Unit Step Functions

Continuous-time unit impulse and unit step sequences

- Example:

  (1) Calculate and sketch the $x'(t)$;

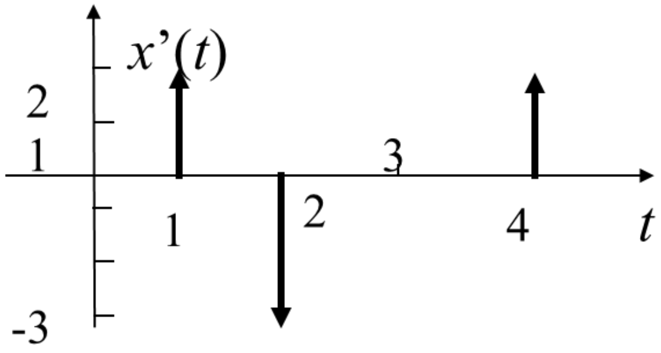
  (2) Recover $x(t)$ from $x'(t)$.



- Solutions:

  (1) $x(t) = 2u(t-1) - 3u(t-2) + 2u(t-4)$

  $$\therefore \quad x'(t) = 2\delta(t-1) - 3\delta(t-2) + 2\delta(t-4)$$



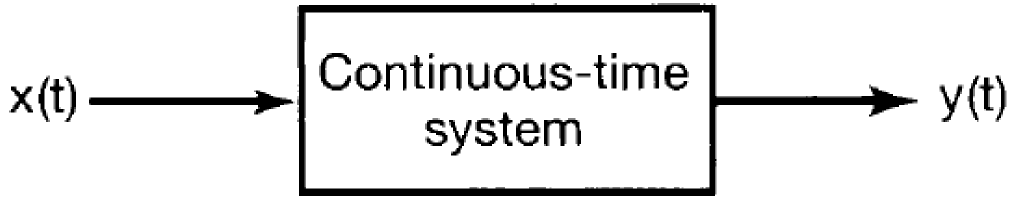
  (2) $x(t) = \int_0^{\infty} x'(t)dt$

Chapter 1: An overview

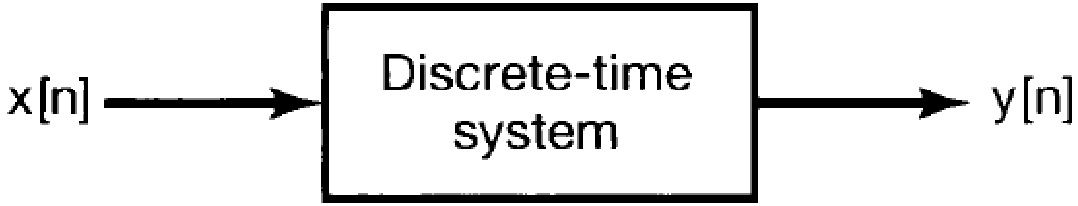
- Continuous-Time and Discrete-Time Signals
- Transformations of the Independent Variable
- Exponential and Sinusoidal Signals
- The Unit Impulse and Unit Step Functions
- Continuous-Time and Discrete-Time Systems
- Basic System Properties

Continuous-Time and Discrete-Time Systems

- Continuous-Time Systems: Input and output are continuous



- Discrete-Time Systems: Input and output are discrete



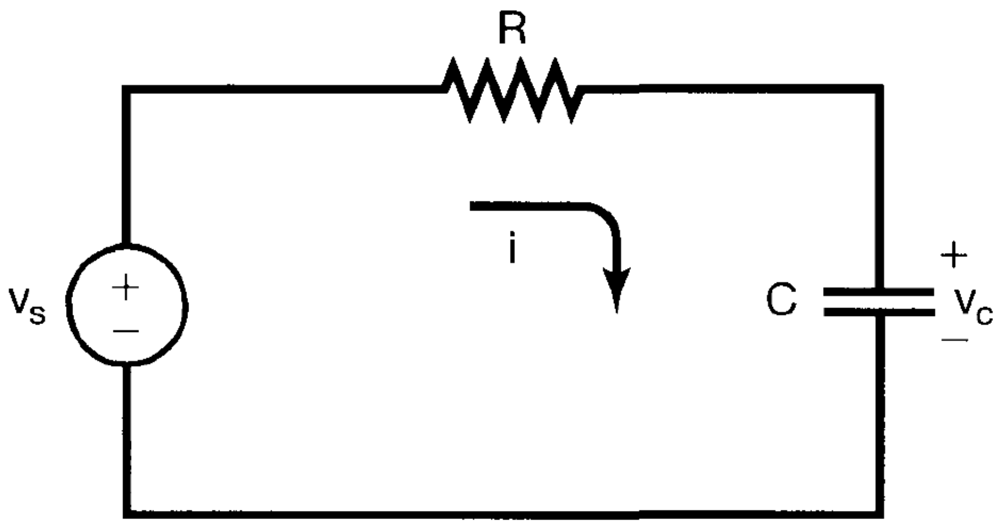
Continuous-Time and Discrete-Time Systems

Examples of systems

- RC circuit

$$i(t) = \frac{v_s(t) - v_c(t)}{R}$$

$$i(t) = C\frac{dv_c(t)}{dt}$$



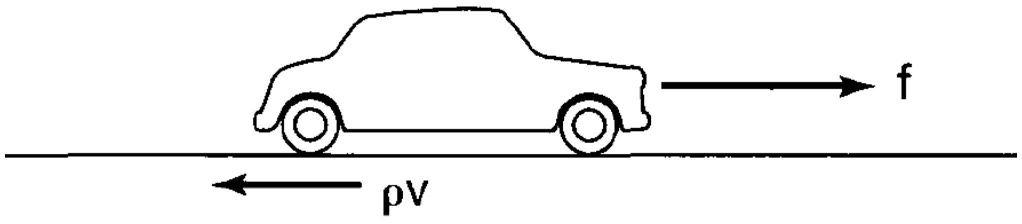
$$\Longrightarrow \quad \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

Continuous-Time and Discrete-Time Systems

Examples of systems

- Moving car

$$\frac{dv(t)}{dt} = \frac{1}{m}(f(t) - \rho v(t))$$



$$\Rightarrow \quad \frac{dv(t)}{dt} + \frac{\rho}{m} v(t) = \frac{1}{m} f(t)$$

In general: $$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

Continuous-Time and Discrete-Time Systems

Examples of systems

- Balance in a bank account:

$$y[n] = 1.01y[n-1] + x[n]$$

$y[n]$: balance at the end of the nth month; $x[n]$: net deposit; Interest rate: 1%

$$y[n] - 1.01y[n-1] = x[n]$$

Continuous-Time and Discrete-Time Systems

Examples of systems

- Digital simulation of a differential equation $$\frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

  - Approximate $dv(t)/dt$ at $t = n\Delta$ by $\frac{v(n\Delta)-v((n-1)\Delta)}{\Delta}$

$$\frac{v(n\Delta) - v((n-1)\Delta)}{\Delta} + \frac{\rho}{m}v(n\Delta) = \frac{1}{m}f(n\Delta)$$

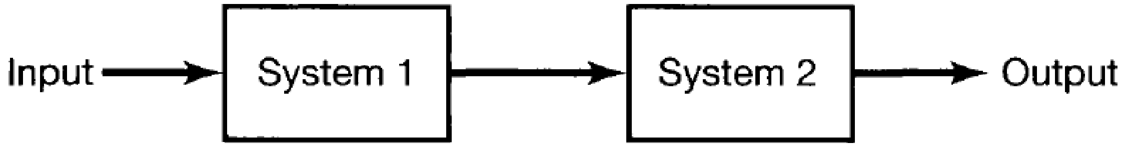
  - Let $v[n] = v(n\Delta)$ $$v[n] - \frac{m}{m+\rho\Delta}v[n-1] = \frac{\Delta}{m+\rho\Delta}f[n]$$

  - In general $$y[n] + ay[n-1] = bx[n]$$

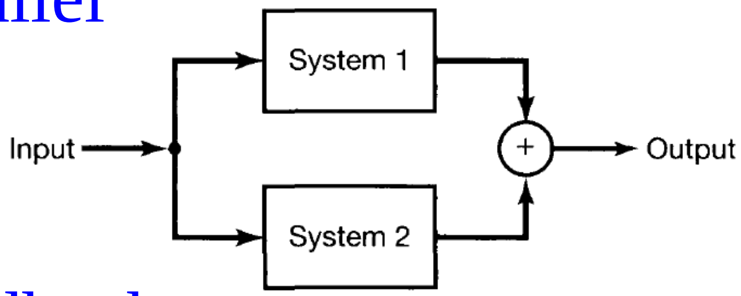
Continuous-Time and Discrete-Time Systems

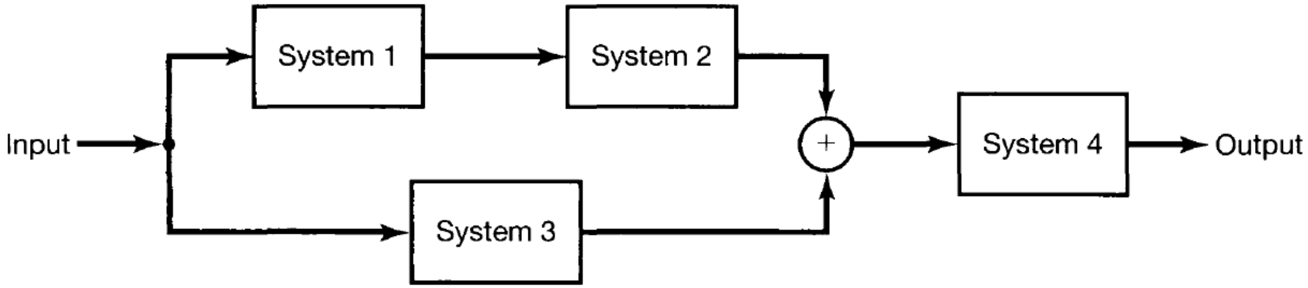
Interconnections of systems

- Series (or cascade)

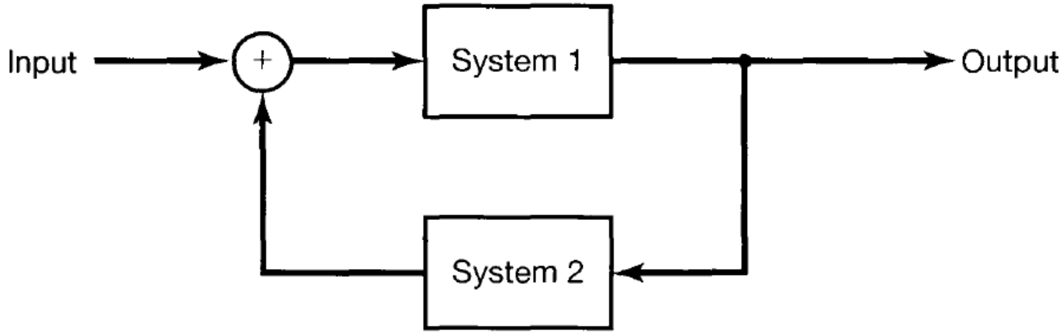


- Parallel





- Feedback



Chapter 1: An overview

- Continuous-Time and Discrete-Time Signals
- Transformations of the Independent Variable
- Exponential and Sinusoidal Signals
- The Unit Impulse and Unit Step Functions
- Continuous-Time and Discrete-Time Systems
- Basic System Properties

Basic System Properties

System with and without memory

- System without memory:
  - Output is dependent only on the current input
  - Examples:

$$y[n] = (2x[n] - x^2[n])^2$$

$$y(t) = Rx(t)$$

$$y(t) = x(t)$$

$$y[n] = x[n]$$

Basic System Properties

System with and without memory

- System with memory:
  - Output is dependent on the current and previous inputs
  - Examples:

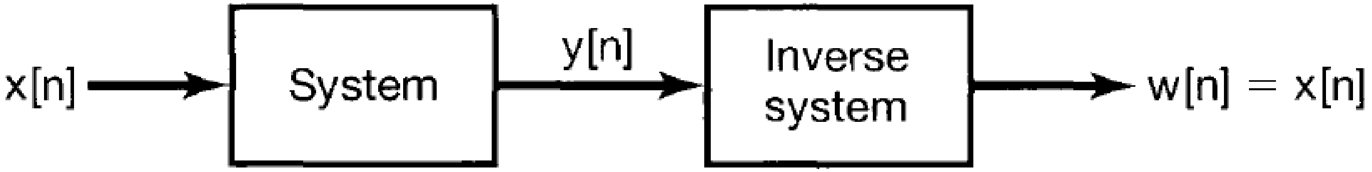
$$y[n] = \sum_{k=-\infty}^{n} x[k] \qquad y[n] = x[n-1] \qquad y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

  - Memory: retaining or storing information about input values at times
  - Physical systems, memory is associated with the storage of energy

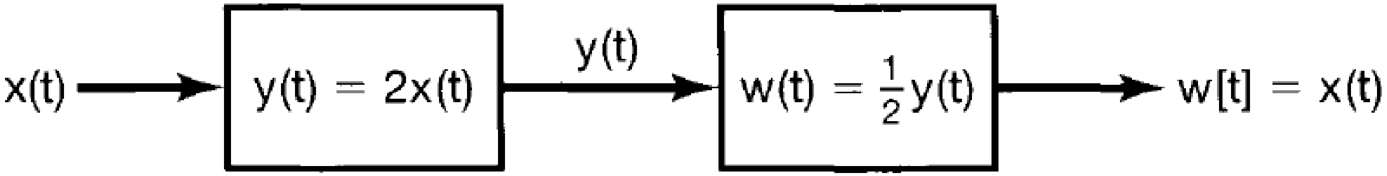
Basic System Properties

<u>*Invertibility and inverse system*</u>

- Invertible
  - Distinct inputs lead to distinct outputs.



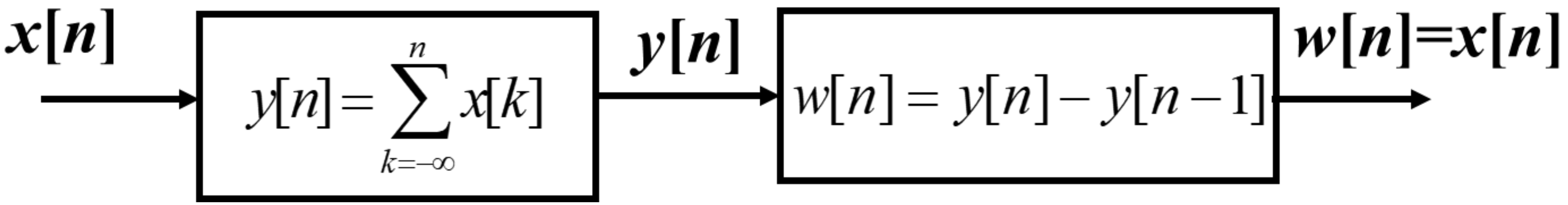
$$y(t) = 2x(t) \qquad\qquad w(t) = \frac{1}{2}y(t)$$



Basic System Properties

Invertibility and inverse system

- Invertible
  - Examples: Accumulator $$y[n] = \sum_{k=-\infty}^{n} x[k]$$
  - The difference between two successive outputs is precisely the inputs $$y[n] - y[n-1] = x[n]$$



Basic System Properties

Invertibility and inverse system

- Noninvertible

$y[n] = 0$ All $x[n]$ leads to the same $y[n]$

$y(t) = x^2(t)$ Cannot determine the sign of the inputs

Basic System Properties

Causality

- Causal: the output at any time depends only on the inputs at the present time and in the past

$y(t) = Rx(t)$	Causal
$y[n] = \sum_{k=-\infty}^{n} x[k]$	Causal
$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$	Causal
$y[n] = x[n] - x[n+1]$	Non-causal
$y(t) \ = x(n+1)$	Non-causal

Basic System Properties

Causality

- Examples

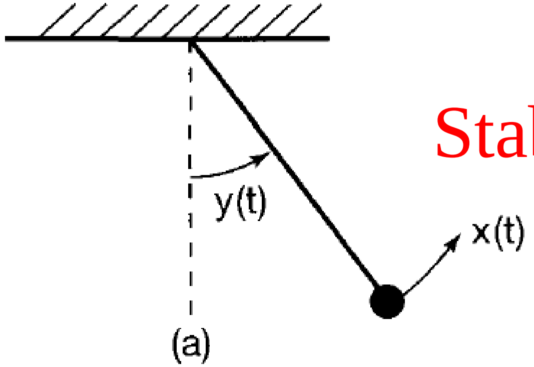
$$y[n] = x[-n]$$ Non-causal

$$y(t) = x(t)\cos(t+1)$$ Causal

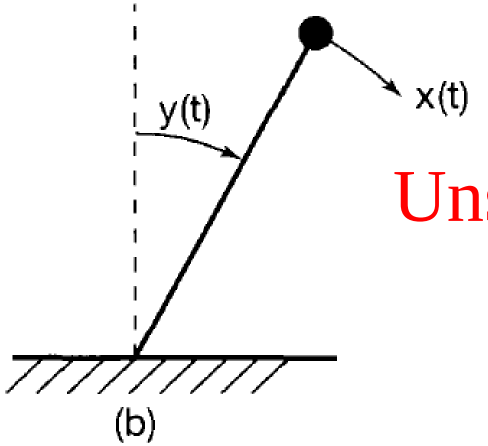
Basic System Properties

Stability

- Informally: small inputs lead to responses that do not diverge.



Stable



Unstable

A bank account balance

$$y[n] = x[n] + (1+\alpha) \times y[n-1]$$

Unstable

Basic System Properties

Stability

- Formally: bounded input leads to bounded output

  - Bounded: $|y(t)| < B$

$$y[n] = \frac{1}{2M+1}\sum_{k=-M}^{+M} x[n-k]$$ Stable

$$y[n] = \sum_{k=-\infty}^{n} u[k] = (n+1)u[n]$$ Unstable

Basic System Properties

Stability

- Examples

$S_1 : y(t) = tx(t)$ Unstable

$S_2 : y(t) = e^{x(t)}$ Stable

$$|x(t)| < B \quad \rightarrow \quad -B < x(t) < B \quad \rightarrow \quad e^{-B} < y(t) < e^{B}$$

Basic System Properties

Time Invariance

- Time invariant: a time shift in the input signal results in an identical time shift in the output signal

If $x[n] \rightarrow y[n]$

Then $x[n-n_0] \rightarrow \mathrm{y}[n-n_0]$

If $x(t) \rightarrow y(t)$

Then $x(t-t_0) \rightarrow y(t-t_0)$

$$x_1(t) \rightarrow \boxed{\text{System}} \rightarrow y_1(t)$$

$$x_2(t) \rightarrow \boxed{\text{System}} \rightarrow y_2(t)$$

$$y_2(t) = f\{x_2(t)\}$$

$$y_2'(t) = y_1(t-t_0)$$

$$y_2(t) = y_2'(t)\ ?$$

Basic System Properties

Time Invariance

- Examples: $y(t) = \sin[x(t)]$

$$x_1(t) \rightarrow \boxed{\sin[x(t)]} \rightarrow y_1(t)$$

$$x_2(t) \rightarrow \boxed{\sin[x(t)]} \rightarrow y_2(t)$$

If $x_2(t) = x_1(t - t_0)$

$$y_2(t) = f\{x_2(t)\}$$

$$f\{\cdot\} = \sin\{\cdot\}$$

$$y_2(t) = \sin[x_1(t - t_0)]$$

$$y_2'(t) = y_1(t - t_0)$$

$$y_1(t) = \sin[x_1(t)]$$

$$y_2'(t) = \sin[x_1(t - t_0)]$$

$$\therefore y_2(t) = y_2'(t)$$

Basic System Properties

Time Invariance

- Examples: $y[n] = nx[n]$

$$x_1[n] \longrightarrow \boxed{nx[n]} \longrightarrow y_1[n]$$

$$x_2[n] \longrightarrow \boxed{nx[n]} \longrightarrow y_2[n]$$

If $x_2[n] = x_1[n - n_0]$

$$y_2[n] = f\{x_2[n]\}$$
$$= n \cdot x_1[n - n_0]$$

$$y_2'[n] = y_1[n - n_0]$$
$$y_1[n] = n \cdot x_1[n]$$

$$\therefore y_2[n] \neq y_2'[n]$$

Basic System Properties

Time Invariance

- Examples: $y(t) = x(2t)$

$$x_1(t) \longrightarrow \boxed{x(2t)} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{x(2t)} \longrightarrow y_2(t)$$

$$If\ x_2(t) = x_1(t - t_0)$$

$$y_2(t) = f\{x_2(t)\}$$

$$= x_1(2t - t_0)$$

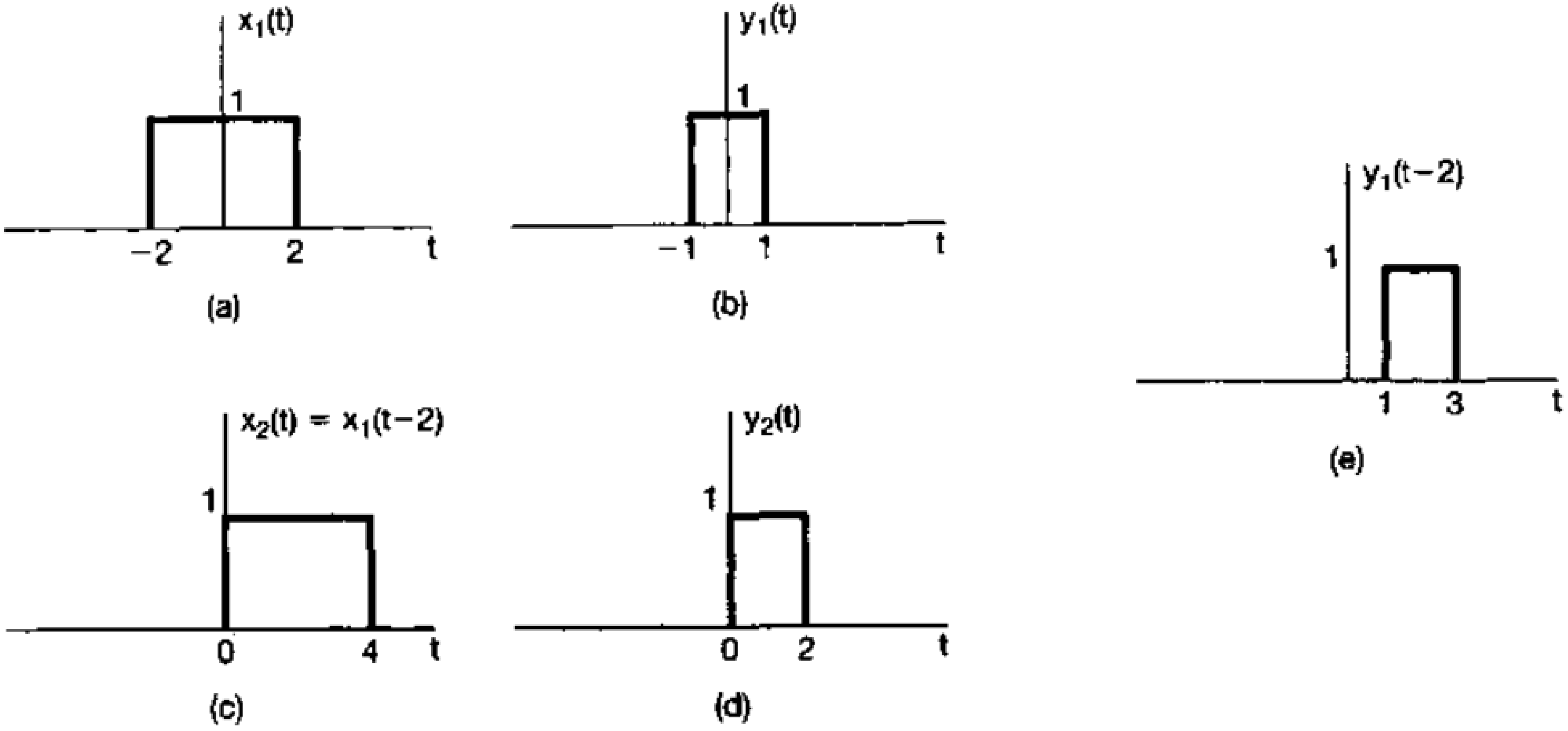
$$y_2'(t) = y_1(t - t_0)$$

$$y_1(t) = x_1(2t)$$

$$y_2'(t) = x_1[2(t - t_0)]$$

$$\therefore y_2(t) \neq y_2'(t)$$

Basic System Properties



Basic System Properties

Linearity

- **Linear** $x_1(t) \rightarrow y_1(t), \qquad x_2(t) \rightarrow y_2(t)$

$$a x_1(t) + b x_2(t) \rightarrow a y_1(t) + b y_2(t)$$

Superposition property
(additivity and homogeneity)

$$x_1(t) \rightarrow \boxed{\text{System}} \rightarrow y_1(t)$$

$$x_2(t) \rightarrow \boxed{\text{System}} \rightarrow y_2(t)$$

$$x_3(t) \rightarrow \boxed{\text{System}} \rightarrow y_3(t)$$

If $x_3(t) = a x_1(t) + b x_2(t)$

$$y_3(t) = f\{x_3(t)\}$$

$$y_3'(t) = a y_1(t) + b y_2(t)$$

$$y_3(t) = y_3'(t) \text{ ?}$$

Basic System Properties

Linearity

- Examples $y(t) = tx(t)$

$$x_1(t) \rightarrow \boxed{tx(t)} \rightarrow y_1(t) = tx_1(t)$$

$$x_2(t) \rightarrow \boxed{tx(t)} \rightarrow y_2(t) = tx_2(t)$$

$$x_3(t) \rightarrow \boxed{tx(t)} \rightarrow y_3(t) = tx_3(t)$$

If $x_3(t) = ax_1(t) + bx_2(t)$

$$y_3(t) = f\{x_3(t)\}$$
$$= t[ax_1(t) + bx_2(t)]$$

$$y_3'(t) = ay_1(t) + by_2(t)$$

$$y_3(t) = y_3'(t)$$

Basic System Properties

Linearity

- Examples $y(t) = x^2(t)$

$$x_1(t) \rightarrow \boxed{x^2(t)} \rightarrow y_1(t) = x_1^2(t)$$

$$x_2(t) \rightarrow \boxed{x^2(t)} \rightarrow y_2(t) = x_2^2(t)$$

$$x_3(t) \rightarrow \boxed{x^2(t)} \rightarrow y_3(t) = x_3^2(t)$$

If $x_3(t) = ax_1(t) + bx_2(t)$

$$y_3(t) = f\{x_3(t)\}$$

$$= [ax_1(t) + bx_2(t)]^2$$

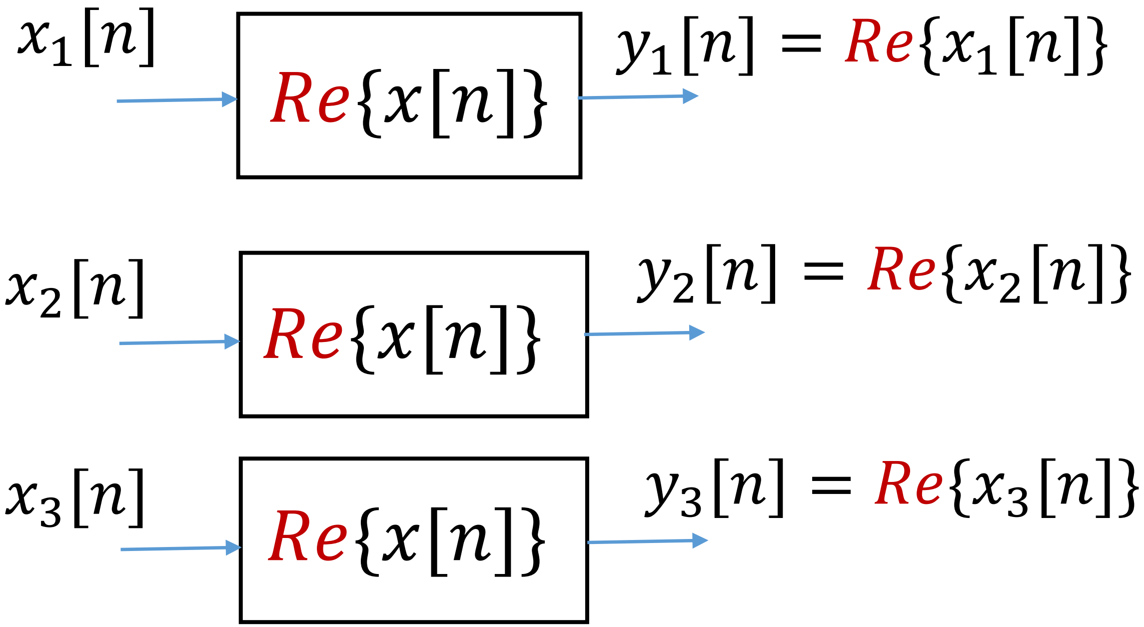
$$y_3'(t) = ay_1(t) + by_2(t)$$

$$y_3(t) \neq y_3'(t)$$

Basic System Properties

Linearity

- Examples $y[n] = Re\{x[n]\}$



If $x_3[n] = ax_1[n] + bx_2[n]$

$$y_3[n] = f\{x_3[n]\}$$
$$= Re\{ax_1[n] + bx_2[n]\}$$

$$y_3'[n] = ay_1[n] + by_2[n]$$
$$= aRe\{x_1[n]\} + bRe\{x_2[n]\}$$

If a and b are complex numbers

$$y_3[n] \neq y_3'[n]$$

Basic System Properties

Linearity

- Examples $y[n] = 2x[n] + 3$

$x_1[n] \rightarrow \boxed{2x[n] + 3} \rightarrow y_1[n] = 2x_1[n] + 3$

$x_2[n] \rightarrow \boxed{2x[n] + 3} \rightarrow y_2[n] = 2x_2[n] + 3$

$x_3[n] \rightarrow \boxed{2x[n] + 3} \rightarrow y_3[n] = 2x_3[n] + 3$

If $x_3[n] = ax_1[n] + bx_2[n]$

$$y_3[n] = f\{x_3[n]\}$$

$$= 2(ax_1[n] + bx_2[n]) + 3$$

$$y_3'[n] = ay_1[n] + by_2[n]$$

$$= a(2x_1[n] + 3) + b(2x_1[n] + 3)$$

$$y_3[n] \neq y_3'[n]$$