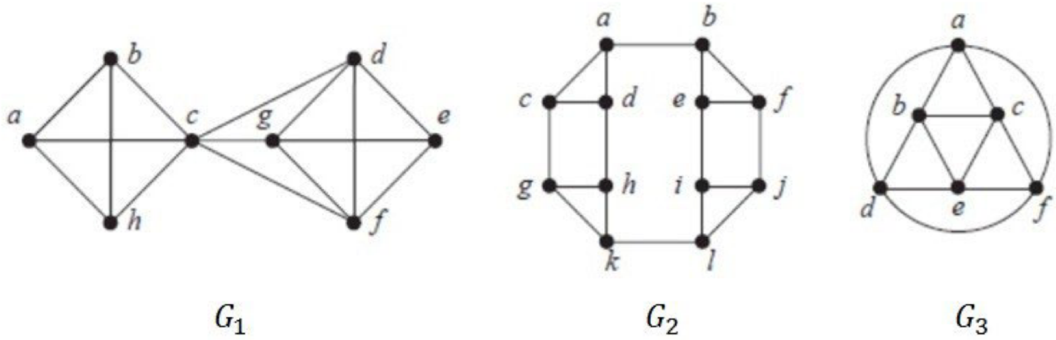
1. (10 points) Determine $\kappa(G_i)$, $\lambda(G_i)$ and $\delta(G_i) = \min_{v \in V} \deg(v)$ for each of the following graphs and verify that $\kappa(G_i) \leq \lambda(G_i) \leq \delta(G_i)$, where $i = 1, 2, 3$.



G_1: $K(G_1) = 1$ $\lambda(G_1) = 3$ $\delta(G_1) = 3$

$K(G_1) \leq \lambda(G_1) \leq \delta(G_1)$

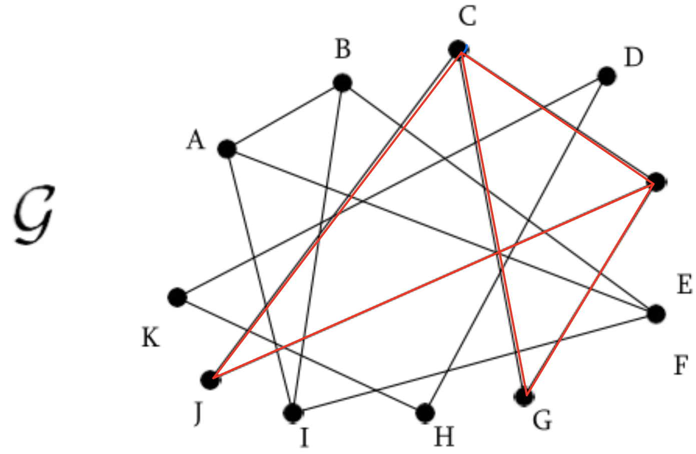
G_2: $K(G_2) = 2$ $\lambda(G_2) = 2$ $\delta(G_3) = 3$

$K(G_2) \leq \lambda(G_2) \leq \delta(G_2)$

G_3: $K(G_3) = 4$ $\lambda(G_3) = 4$ $\delta(G_3) = 4$

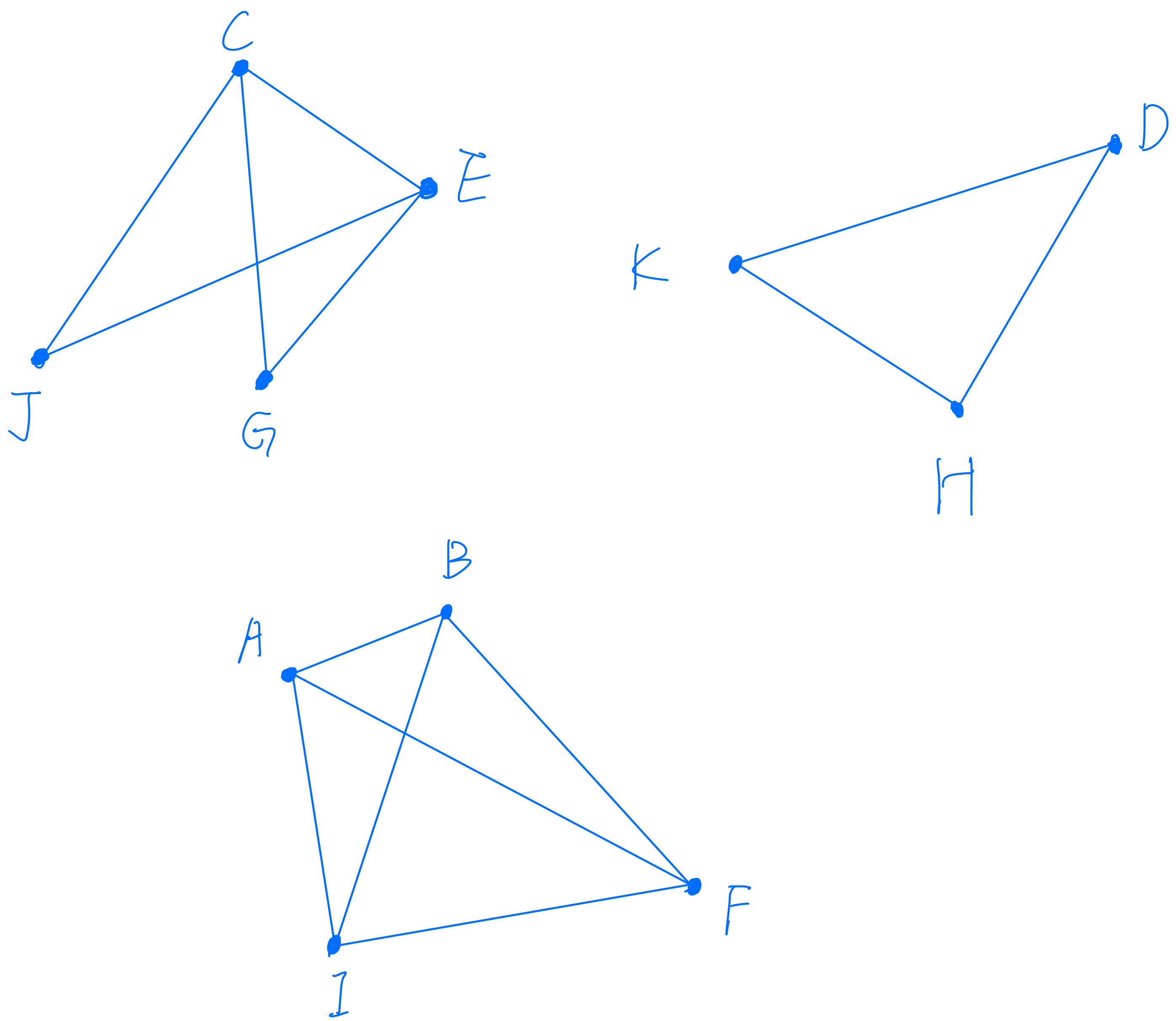
$K(G_4) \leq \lambda(G_3) \leq \delta(G_3)$

2. (10 points) Is the graph G below connected? Give the connected components of G.



G is not connected.

the connected components.



3. (20 points) Let G be connected graph. If e is an edge, such that removing E gives a non-connected graph, then e is called a bridge.

- Prove that if G contains no vertices of odd degree then G is bridgeless.

if G is a connected graph which contains vertices of odd degree, G can be divided into several connected components that are connected by one edge from those vertices whose degrees are odd.

Then these edges are all bridges.

So if G contains no vertices. G can't be divided into connected components. So G is bridgeless.

4. (10 points) Determine for which values the complete bipartite graph $K_{m,n}$ has a.) an Euler circuit. b.) an Euler path.

a). m, n are all even.

(their degrees are all even).

b. for m and n, one equals to 2

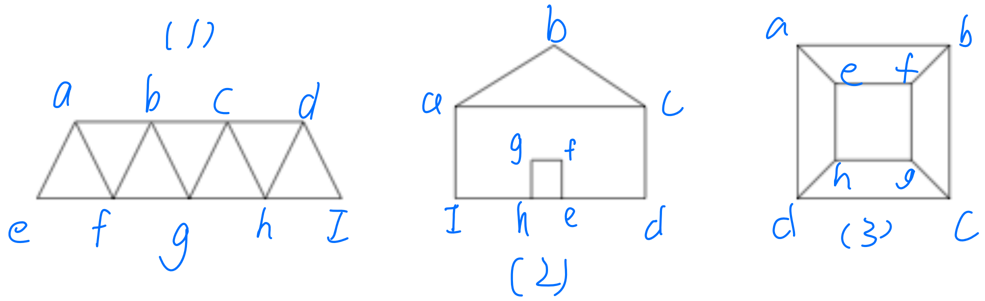
and another is odd.

(the graph has exactly 2 vertices of

odd degree)

5. (15 points)

(a) Can you draw the following pictures without lifting the pen? Explain.



(1) Yes. (1) has exactly 2 vertices (a, d) of odd degree.

circuit: a, e, f, a, b, f, g, b, c, g, h, c, d, h, I

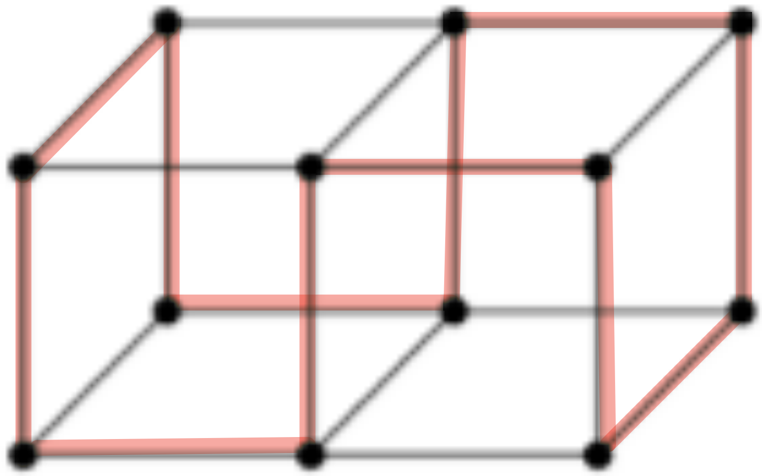
(2) NO.

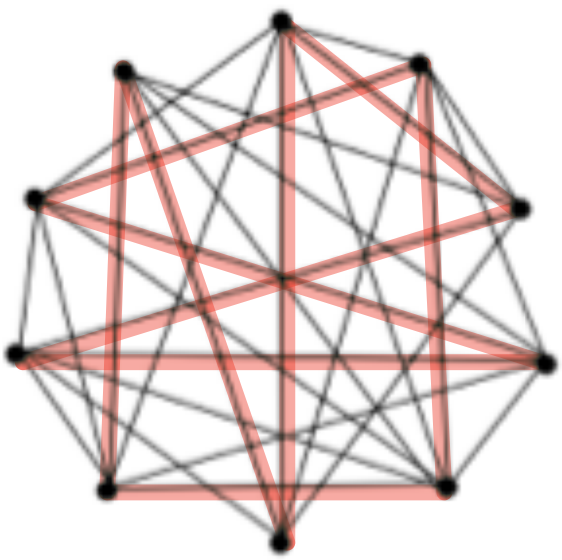
the graph doesn't has more than 2 vertices of odd degree

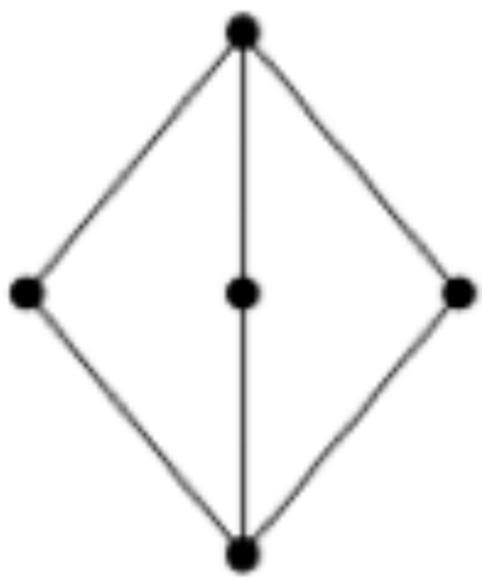
(3) NO.

the graph doesn't has more than 2 vertices of odd degree

(b) Do the following graphs admit any Hamilton circuit? If yes, draw one, otherwise, explain why there is no Hamilton circuit.





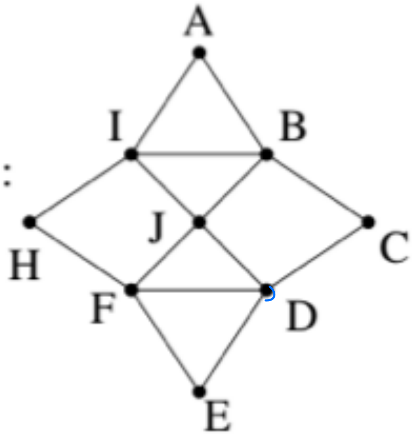


NO.

$\deg(u) \geq \frac{n}{2}$ (n is order of graph) doesn't hold for every $u \in V$.

(c) Do the graphs G_1, G_2, G_3 and G_4 below admit any Euler path or Euler circuit? If yes, draw one, otherwise, explain why there is no Euler path nor Euler circuit.

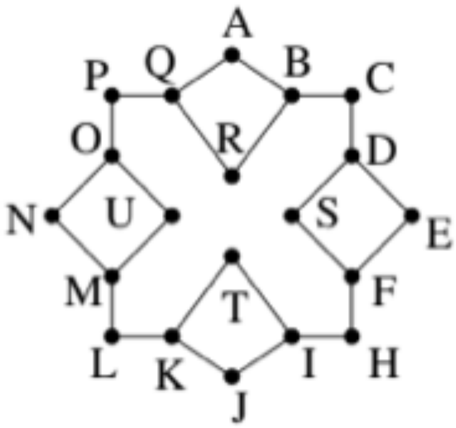
G_1:



Euler circuit

A, I, H, F, J, I, B, J, D, F, E, D, C, B, A

G_2:

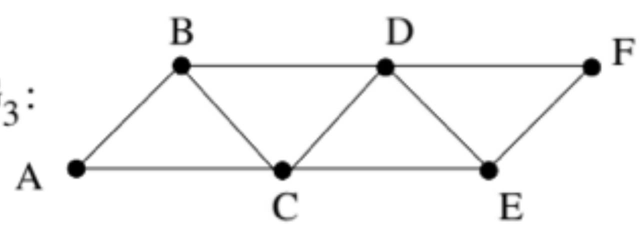


NO.

(1) $2 \mid \deg(x)$ doesn't hold for every $x \in V$

(2) G doesn't have exactly 2 vertices of odd degree.

G_3:



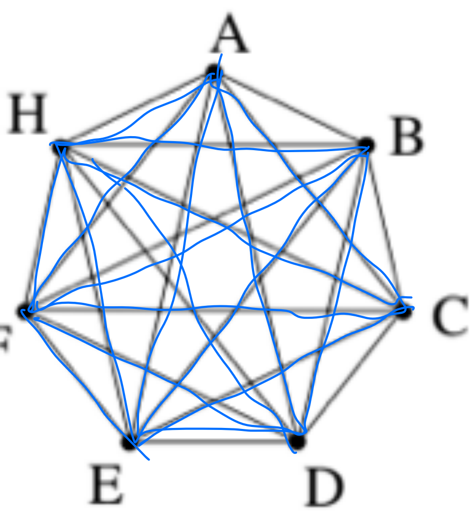
Euler Path

B, A, C, B, D, C, E, D, F, E

no Euler circuit

$2 \mid \deg(x)$ doesn't hold for every $x \in V$

G_4:



Euler circuit

A, F, C, H, E, D, A, H, E, B, H, F, D, B, F, E, C, H, D, C, B, A

6. (10+10 points) A simple graph with $2p$ vertices is such that each of its vertices is of degree at least p. Show that this graph is connected. Do we have the same result for a simple graph with n vertices such that each vertex v has $deg(v) \geq (n-1)/2$?

(1) if $P=1$, the graph has only 2 vertices and degree are all 1. so the graph is connected.

if $P \geq 2$, then $2P \geq 3$

Since $\deg(u) \geq P$ for every $u \in V$. then G has a Hamilton circuit

So the graph is connected.

(2). if $n=1$ then $\deg(V) \geq 0$ for each vertex V. ($\deg(V)=0$)

so the graph is connected.

if $n=2$ then $\deg(V) \geq \frac{1}{2}$ for each vertex V. ($\deg(V)=1$)

So the graph is connected

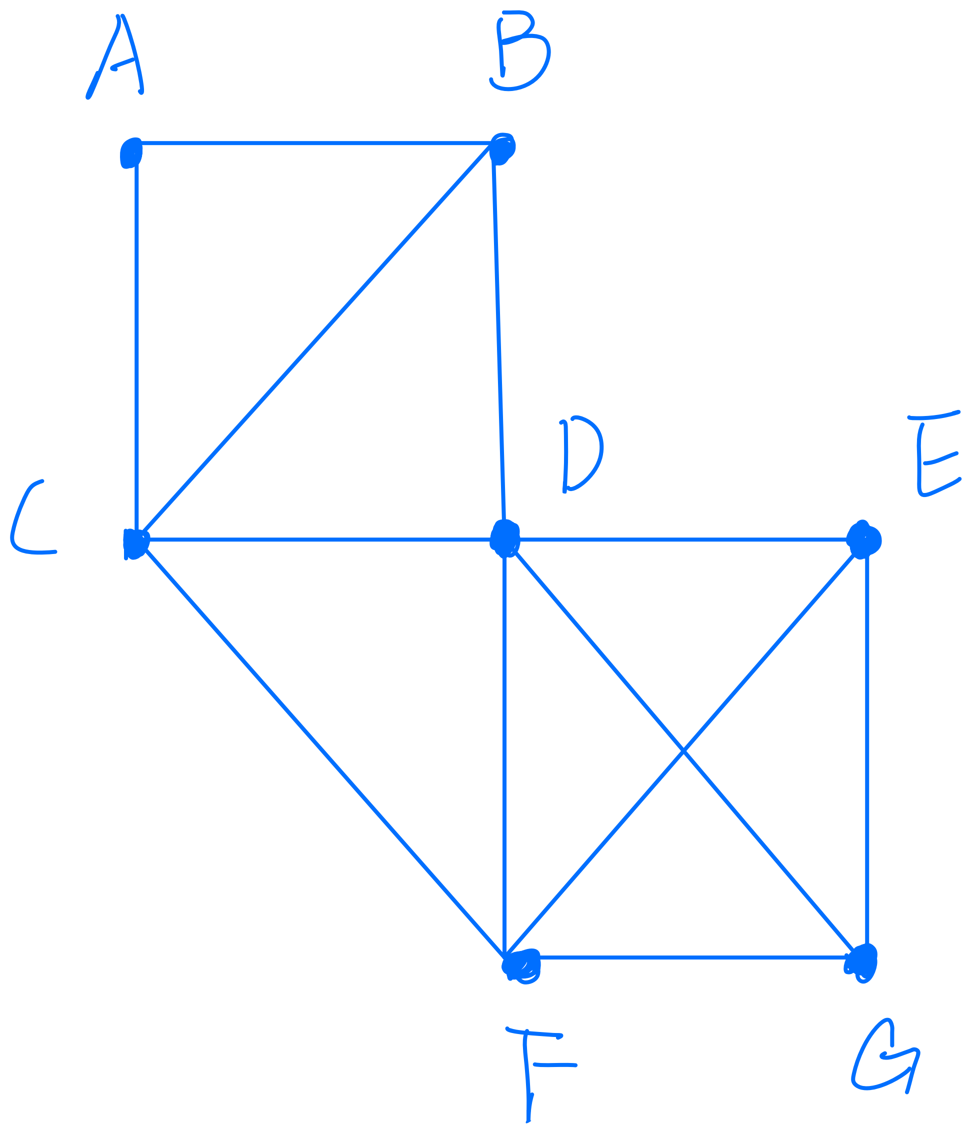
if $n \geq 3$ then $\deg(V) \geq \frac{n-1}{2}$, $\deg(u) \geq \frac{n-1}{2}$ for all $\{u,v\} \notin E$. that is $\deg(V)+\deg(u) \geq n-1$

so the graph doesn't has a Hamilton circuit.

thus the graph is not connected.

In conclusion, the simple graph with n vertices such that each vertex v has $\deg(V) \geq \frac{n-1}{2}$ is connected when $n=1$ or $n=2$

7. (10 points) Draw a simple connected graph with 7 vertices without any Euler path (nor Euler circuit!).



8. (5 points) Let G be a directed graph with n vertices, and let M be its adjacency matrix. Assume M^n is not the zero matrix (i.e. at least one coefficient is not zero). Show that G contains at least one circuit.

Since M^n is not the zero matrix

Suppose the non-zero coefficient is the (i, j) entry of the matrix M^n

Since G be a directed graph with adjacency matrix M with respect to the ordering of vertices $v_1, \cdots, v_n$

So the number of different paths of length $n \geq 1$ from v_i to v_j equals to that non-zero coefficient.

We know that the length of a circuit is n

So the number of circuits is at least one.

In conclusion, G contains at least one circuit