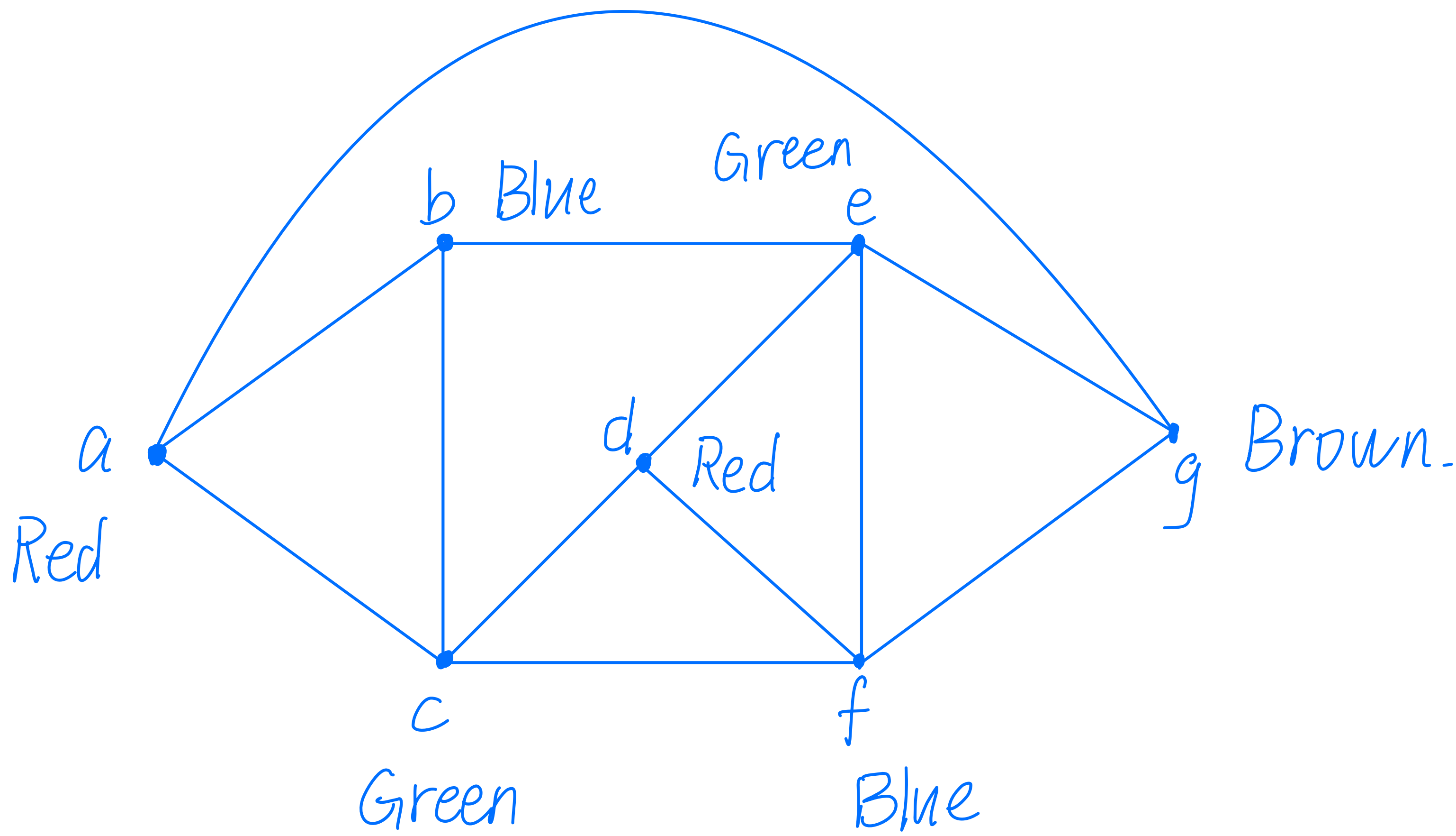
1. (10 points) Find a planar graph G with $\chi(G) = 4$.



2. (20 points) Let G be a planar graph and d(v) = 3 for any vertex v. Show there is a face with at most 5 edges.

Suppose G has n vertices, m edges and r faces.

Since $d(v) = 3$, then $n \geq 4$

if all faces has more than 5 edges.

then $2m > 5r$

according to Euler's Formula.

$2m > 5r = 5(m-n+2) = 5m - 5n + 10$ (1)

Since $2m = d(v)n = 3n$ (2)

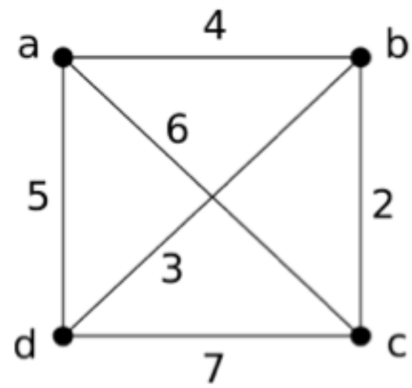
from (1) (2) we can get $n > 20$

but $\{n \in \mathbb{Z}, n > 20\} \supseteq \{n \in \mathbb{Z}, n \geq 4\}$.

So it contradicts.

thus there is a face with at most 5 edges.

3. (10 points) Solve the traveling salesperson problem for this graph.

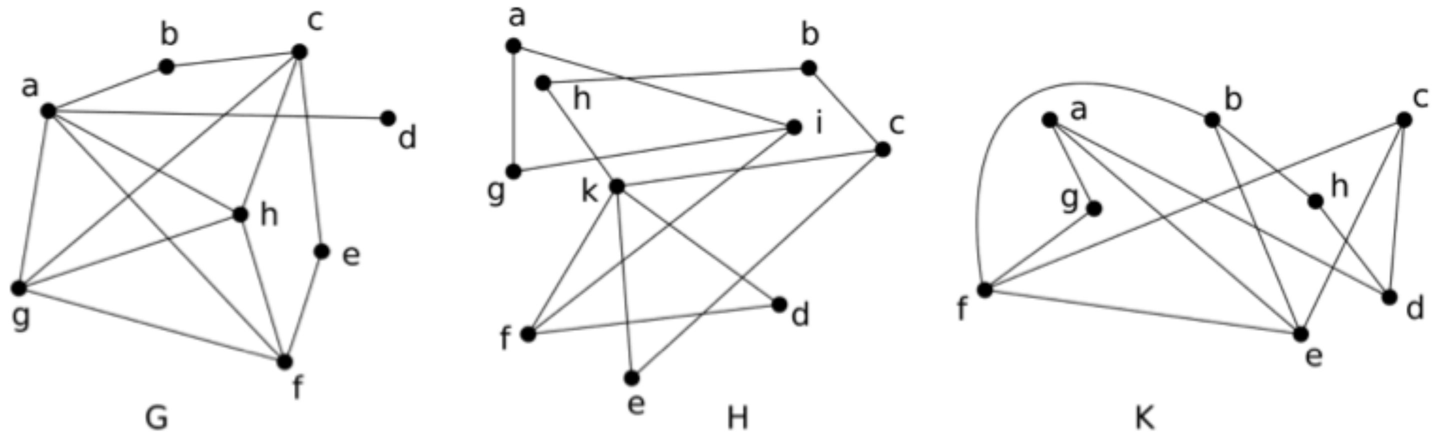


Route	Tot. dist.
a, b, c, d, a	18
a, b, d, c, a	20
a, c, b, d, a	16
a, c, d, b, a	20
a, d, b, c, a	16
a, d, c, b, a	18

So the solution is $a \to c \to b \to d \to a$

or $a \to d \to b \to c \to a$.

4. (10 points) Are the graphs G, H, K below planar?



G: NO

H: YES.

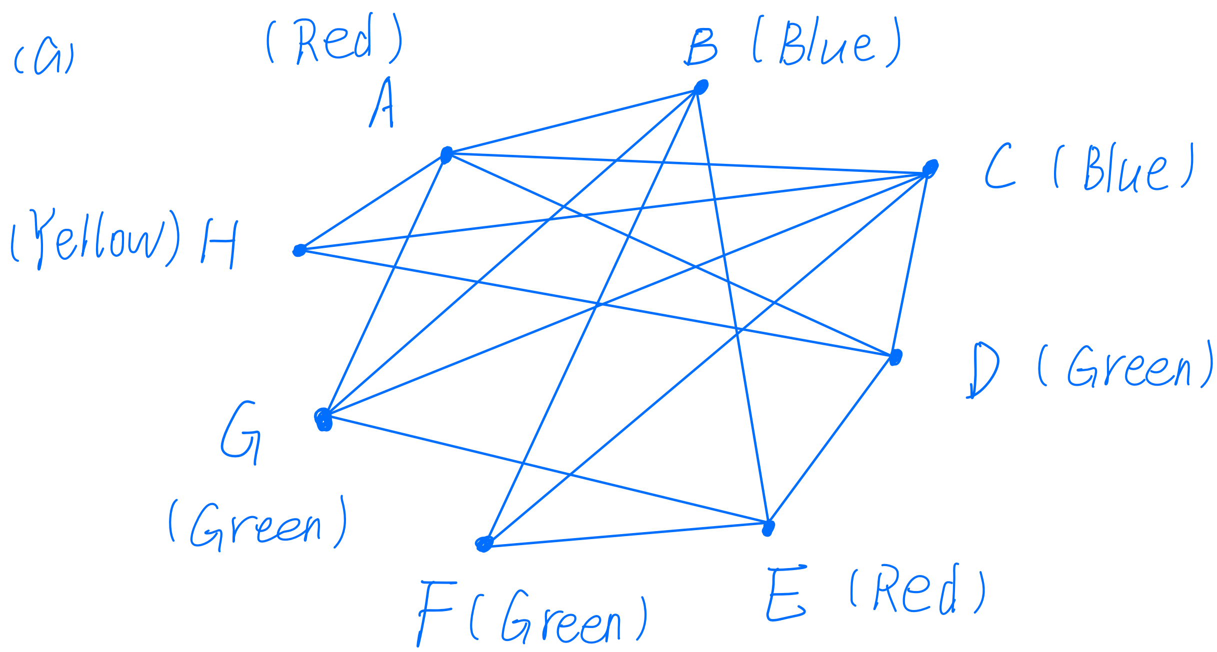
K: NO.

5. (10 points) The letters A, B, C, D, E, F, G and H denote 8 fishes. In the table below, a circle means that the fishes can cohabit in the same aquarium, a cross means that they cannot.

	A	B	C	D	E	F	G	H
A	∘	×	×	×	∘	∘	×	×
B	×	∘	∘	∘	×	×	×	∘
C	×	∘	∘	×	∘	×	×	×
D	×	∘	×	∘	×	∘	∘	×
E	∘	×	∘	×	∘	×	×	∘
F	∘	×	×	∘	×	∘	∘	∘
G	×	×	×	∘	×	∘	∘	∘
H	×	∘	×	×	∘	∘	∘	∘

(a) Model this problem by a graph.
(b) Find the chromatic number of the graph.
(c) Deduce the minimal number of aquarium needed for the fishes.

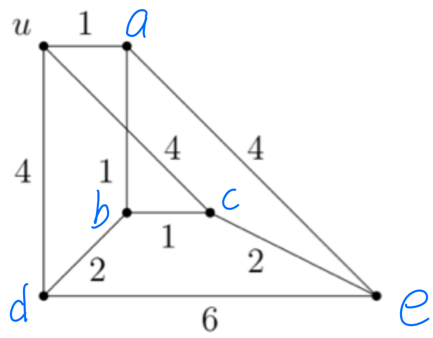
(a)



(b) 4

(c) 4.

6. (30 points) For the weighted graph shown in the figure use Dijkstra's algorithm to compute the distance $d(u, v)$ for every $v \in V$. For each step k of the algorithm write down explicitly the set S_k and the labels $L_k(v)$ for every $v \in V$.



$k=0$ (initialization): $S_0 = \phi$, $L_0(u)=0$

$L_0(a)=L_0(b)=L_0(c)=L_0(d)=L_0(e)=\infty$

$k=1$: $v:=u \rightsquigarrow S_1=\{u\}$

$L_0(u)+d(u,a)=1 < L_0(a) \rightsquigarrow L_1(a)=1$

$L_0(u)+d(u,c)=4 < L_0(c) \rightsquigarrow L_1(c)=4$

$L_0(u)+d(u,d)=4 < L_0(d) \rightsquigarrow L_1(d)=4$

$k=2$: $v:=a \rightsquigarrow S_1=\{u,a\}$

$L_1(a)+d(a,b)=2 < L_1(b) \rightsquigarrow L_2(b)=2$

$L_1(a)+d(a,e)=5 < L_1(e) \rightsquigarrow L_2(e)=5$

$k=3$: $v:=d \rightsquigarrow S_1=\{u,a,d\}$

$L_2(d)+d(d,b)=6 > L_2(b)$

$L_2(d)+d(d,e)=10 > L_2(e)$

$k=4$: $v:=b \rightsquigarrow S_1=\{u,a,d,b\}$

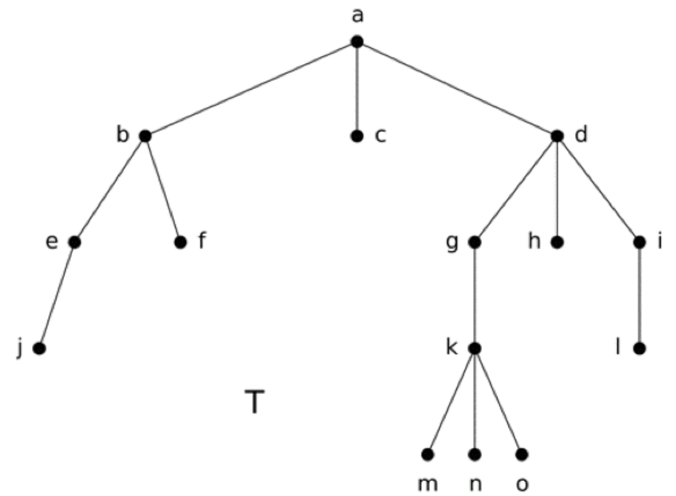
$L_3(b)+d(b,c)=3 < L_3(c) \rightsquigarrow L_4(c)=3$.

$k=5$: $v:=c \rightsquigarrow S_1=\{u,a,d,b,c\}$

$L_4(c)+d(c,e)=5=L_4(e)$

$k=6$: $v:=e \rightsquigarrow S_1=\{u,a,d,b,c,e\}$

7. (10 points)



(a) Given the rooted tree T, answer the following questions:

i. Is T a m-ary tree for some positive integer m? If not, what is the minimal number of edges to add to T to make it a m-ary tree?

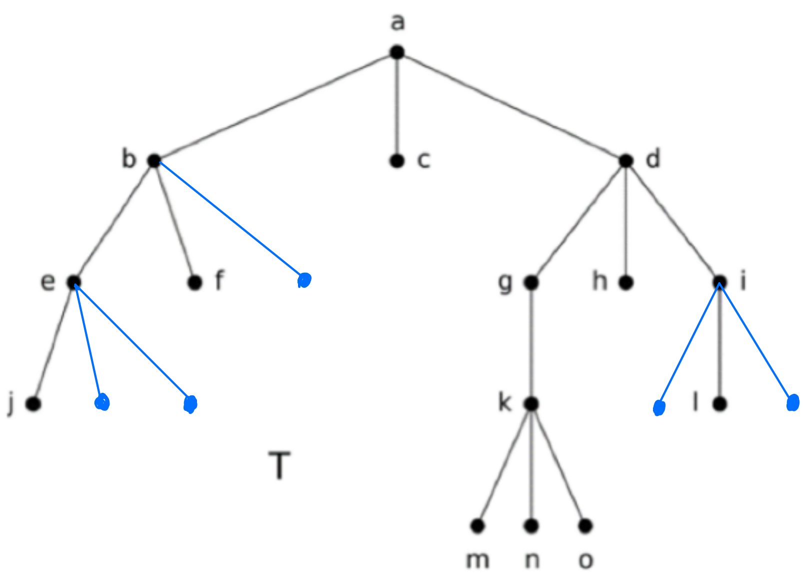
ii. Is T a full m-ary tree for some positive integer m? If not, what is the minimal number of edges to add to T to make it a full m-ary tree? Draw the corresponding m-ary tree.

iii. Is T balanced? If not, what is the minimal number of edges to add to T to make it balanced? Draw the corresponding balanced tree.

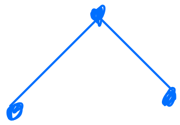
(b) Let n be a power of 2. How many steps are needed to add n numbers using a tree-connected network of $n - 1$ processors? Explain your answer.

(a) i. $m = 3$

ii. NO. The minimal number is 5

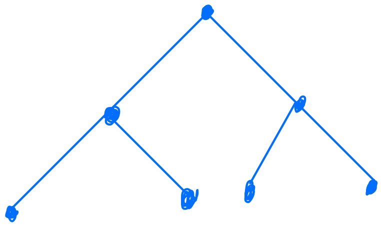


iii. YES.



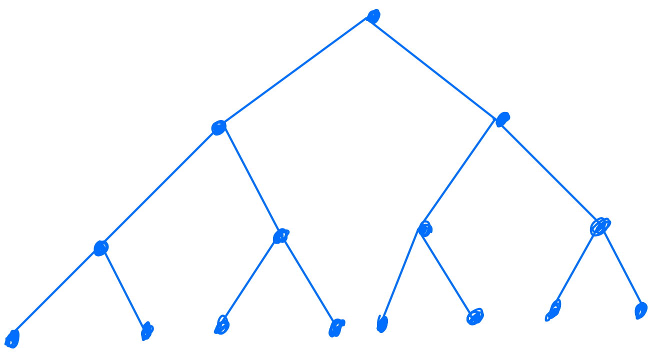
$n=2$

steps $=1$



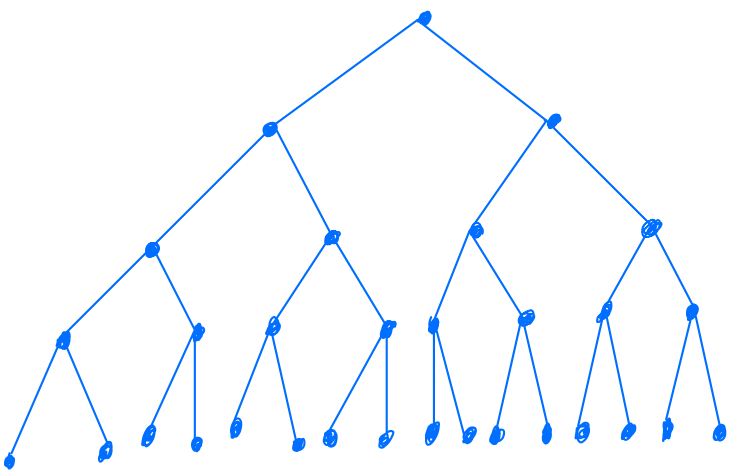
$n=4$

steps $=2$.



$n=8$

steps $=3$



$n=16$

steps $=4$.

- - - -

$n= \cdots$

steps $= \cdots$

(b). steps $=\log_2 n$. the number of steps are equal to the height of tree.

8. (20 points) Show that a connected simple graph G is a tree $\Longleftrightarrow$ every edge e of G is a bridge (i.e. $G \setminus e$ is not connected).

(1) $\Rightarrow$ Since connected simple graph G is a tree.

for every edge e of G

it is connected with two vertices a, b

if we remove the edge e.

then there is no edge between a and b

Since G is a connected simple graph

then G\e is not connected.

(2) $\Leftarrow$ Since every edge e of G is a bridge

then there is a unique simple path between any two of its vertices.

namely the graph G doesn't have simple circuits.

So a connected simple graph G is a tree.