LA homework Dec.17 name: 王柯皓 id: 2021533025

§ 6.3 (Page 655) :

6. Which of the following sets of matrices are orthonormal with respect to the inner product on M_{22} discussed in Example 6 of Section 6.1 ?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}, \begin{bmatrix} 0 & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$

(a)

10. Verify that the vectors

$$\mathbf{v}_1 = (1, -1, 2, -1), \quad \mathbf{v}_2 = (-2, 2, 3, 2),$$
$$\mathbf{v}_3 = (1, 2, 0, -1), \quad \mathbf{v}_4 = (1, 0, 0, 1)$$

form an orthogonal basis for R^4 with the Euclidean inner product; then use Theorem 6.3.2*a* to express each of the following as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

(a) $(1, 1, 1, 1)$

(b) $(\sqrt{2}, -3\sqrt{2}, 5\sqrt{2}, -\sqrt{2})$

Pf:

$\langle V_1, V_2\rangle = 1\times(-2) + (-1)\times2 + 2\times3 + (-1)\times2 = 0$

$\langle V_1, V_3\rangle = 1\times1 + (-1)\times2 + 2\times0 + (-1)\times(-1) = 0$

$\langle V_1, V_4\rangle = 1\times1 + (-1)\times0 + 2\times0 + (-1)\times1 = 0$

$\langle V_2, V_3\rangle = (-2)\times1 + 2\times2 + 3\times0 + 2\times(-1) = 0$

$\langle V_2, V_4\rangle = (-2)\times1 + 2\times0 + 3\times0 + 2\times1 = 0$

$\langle V_3, V_4\rangle = 1\times1 + 2\times0 + 0\times0 + (-1)\times1 = 0$

(b)

$$\begin{bmatrix} 1 & -1 & 2 & -1 & \sqrt{2} \\ -2 & 2 & 3 & 2 & -3\sqrt{2} \\ 1 & 2 & 0 & -1 & 5\sqrt{2} \\ 1 & 0 & 0 & 1 & -\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & -1 & \sqrt{2} \\ 0 & 0 & 7 & 0 & -\sqrt{2} \\ 0 & 3 & -2 & 0 & 4\sqrt{2} \\ 0 & 1 & -2 & 2 & -2\sqrt{2} \end{bmatrix}$$

$c = -\frac{\sqrt{2}}{7}$ $\quad b = \frac{26\sqrt{2}}{21}$ $\quad d = -\frac{37\sqrt{2}}{21}$ $\quad a = \frac{16\sqrt{2}}{21}$

$\frac{16\sqrt{2}}{21} V_1 + \frac{26\sqrt{2}}{21} V_2 - \frac{\sqrt{2}}{7} V_3 - \frac{37\sqrt{2}}{21} V_4$

In Exercises 14–15, the given vectors are orthogonal with respect to the Euclidean inner product. Find $\mathrm{proj}_W \mathbf{x}$, where $\mathbf{x} = (1, 2, 0, -2)$ and W is the subspace of R^4 spanned by the vectors.

14. (a) $\mathbf{v}_1 = (1, 1, 1, 1)$, $\mathbf{v}_2 = (1, 1, -1, -1)$

(b) $\mathbf{v}_1 = (0, 1, -4, -1)$, $\mathbf{v}_2 = (3, 5, 1, 1)$

$\langle X, V_1\rangle = 1\times0 + 2\times1 + 0\times(-4) + (-2)\times(-1) = 4$

$\langle X, V_2\rangle = 1\times3 + 2\times5 + 0\times1 + (-2)\times1 = 11$

$\|V_1\|^2 = 1 + 16 + 1 = 18$ $\quad \|V_2\|^2 = 9 + 25 + 1 + 1 = 36$

$\mathrm{proj}_W X = \frac{4}{18}(0, 1, -4, -1) + \frac{11}{36}(3, 5, 1, 1)$

$= (0, \frac{2}{9}, -\frac{8}{9}, -\frac{2}{9}) + (\frac{11}{12}, \frac{55}{36}, \frac{11}{36}, \frac{11}{36})$

$= (\frac{11}{12}, \frac{7}{4}, -\frac{7}{12}, \frac{1}{12})$

In Exercises 16–17, the given vectors are orthonormal with respect to the Euclidean inner product. Use Theorem 6.3.4*b* to find $\mathrm{proj}_W \mathbf{x}$, where $\mathbf{x} = (1, 2, 0, -1)$ and W is the subspace of R^4 spanned by the vectors.

16. (a) $\mathbf{v}_1 = \left(0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}}\right), \mathbf{v}_2 = \left(\frac{1}{2}, \frac{5}{6}, \frac{1}{6}, \frac{1}{6}\right)$

(b) $\mathbf{v}_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \mathbf{v}_2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

$\mathrm{proj}_W X = \langle X, V_1\rangle V_1 + \langle X, V_2\rangle V_2$

$= \left(\frac{1}{2}+1-\frac{1}{2}\right)\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \left(\frac{1}{2}+1+\frac{1}{2}\right)\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

$= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + (1, 1, -1, -1)$

$= \left(\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

20. Find the vectors $\mathbf{w}_1$ in W and $\mathbf{w}_2$ in $W^{\perp}$ such that $\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{x}$ and W are as given in

(a) Exercise 16(a).

$W_1 = \mathrm{proj}_W X = \left(\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

$W_2 = X - \mathrm{proj}_W X = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

$\frac{\sqrt{6}}{3}$

$\frac{3}{2}$ $\quad \frac{\sqrt{3}}{\sqrt{2}}$ $\quad \frac{\sqrt{6}}{2}$

25. Let R^3 have the inner product

$$\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3$$

Use the Gram–Schmidt process to transform $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (1, 1, 0)$, $\mathbf{u}_3 = (1, 0, 0)$ into an orthonormal basis.

$V_1 = U_1 = (1, 1, 1)$

$V_2 = U_2 - \frac{\langle U_2, V_1\rangle}{\|V_1\|^2} V_1 = (1, 1, 0) - \frac{1+2}{6}(1, 1, 1) = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

$V_3 = U_3 - \frac{\langle U_3, V_1\rangle}{\|V_1\|^2} V_1 - \frac{\langle U_3, V_2\rangle}{\|V_2\|^2} V_2 = (1, 0, 0) - \frac{1}{6}(1, 1, 1) - \frac{1}{3}\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) = \left(\frac{2}{3}, -\frac{1}{3}, 0\right)$

so the orthonormal basis is

$W_1 = \frac{V_1}{\|V_1\|} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$W_2 = \frac{V_2}{\|V_2\|} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$

$W_3 = \frac{V_3}{\|V_3\|} = \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0\right)$

29. Find the QR-decomposition of the matrix, where possible.

(a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ let $u_1=(1,0,1)$ $u_2=(2,1,4)$

(b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$ $v_1=u_1=((1,0,1)$ $v_2=u_2-\frac{\langle u_2,v_1\rangle}{\|v_1\|^2}v_1=(2,1,4)-\frac{6}{2}(1,0,1)=(-1,1,1)$

so $q_1=\frac{v_1}{\|v_1\|}=(\frac{1}{\sqrt2},0,\frac{1}{\sqrt2})$ $q_2=\frac{v_2}{\|v_2\|}=(-\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3})$

$\langle u_1,q_1\rangle=\frac{2}{\sqrt2}=\sqrt2$ $\langle u_2,q_1\rangle=3\sqrt2$ $\langle u_2,q_2\rangle=\sqrt3$

$\therefore \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}=\begin{bmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt3} \\ 0 & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt3} \end{bmatrix}\begin{bmatrix} \sqrt2 & 3\sqrt2 \\ 0 & \sqrt3 \end{bmatrix}$

33. ***Calculus required*** Let P_2 have the inner product

$$\langle \mathbf{p},\mathbf{q}\rangle=\int_0^1 p(x)q(x)\,dx$$

Apply the Gram–Schmidt process to transform the standard basis $S=\{1,x,x^2\}$ into an orthonormal basis.

let $u_1=1$ $u_2=x$ $u_3=x^2$

$v_1=u_1=1$ $\|v_1\|^2=1$

$v_2=u_2-\frac{\langle u_2,v_1\rangle}{\|v_1\|^2}v_1=x-\frac{\frac12}{1}x1=x-\frac12$ $\|v_2\|^2=\int_0^1(x^2-x+\frac14)\,dx$

$=\frac13x^3-\frac12x^2+\frac14x\Big|_0^1$

$=\frac{1}{12}$

$v_3=u_3-\frac{\langle u_3,v_1\rangle}{\|v_1\|^2}v_1-\frac{\langle u_3,v_2\rangle}{\|v_2\|^2}v_2$

$=x^2-\frac13-\frac{\frac{1}{12}}{\frac{1}{12}}(x-\frac12)=x^2-x+\frac16$ $\|v_2\|=\frac{1}{2\sqrt3}$

$\|v_3\|^2=\int_0^1(x^4-2x^3+\frac43x^2-\frac13x+\frac1{36})\,dx$

$=(\frac15x^5-\frac12x^4+\frac49x^3-\frac16x^2+\frac1{36}x)\Big|_0^1=\frac1{180}$ $\|v_3\|=\frac{1}{6\sqrt5}$

so the orthonormal basis

is $w_1=\frac{v_1}{\|v_1\|}=1$

$w_2=\frac{v_2}{\|v_2\|}=2\sqrt3x-\sqrt3$

$w_3=6\sqrt5x^2-6\sqrt5x+\sqrt5$