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§ 4.10 (Page 492)

4. Let $T_1(x_1, x_2, x_3) = (4x_1,\ -2x_1 + x_2,\ -x_1 - 3x_2)$ and $T_2(x_1, x_2, x_3) = (x_1 + 2x_2,\ -x_3, 4x_1 - x_3)$.

(a) Find the standard matrices for T_1 and T_2.

(b) Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

(c) Use the matrices obtained in part (b) to find formulas for $T_1(T_2(x_1, x_2, x_3))$ and $T_2(T_1(x_1, x_2, x_3))$.

(a) $\begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}$ (b) $[T_2 \circ T_1] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 3 & 0 \\ 17 & 3 & 0 \end{bmatrix}$

$[T_1 \circ T_2] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 0 \\ -2 & 4 & -1 \\ -1 & -2 & 3 \end{bmatrix}$

$T_2(T_1(X_1, X_2, X_3)) = (2X_2,\ X_1 + 3X_2,\ 17X_1 + 3X_2)$

(C) $T_1(T_2(X_1, X_2, X_3)) = (4X_1 + 8X_2,\ -2X_1 + 4X_2 - X_3,\ -X_1 - 2X_2 + 3X_3)$

6. Find the standard matrix for the stated composition in R^2.

(a) A rotation of 60°, followed by an orthogonal projection on the x-axis, followed by a reflection about the line $y = x$.

(b) A dilation with factor $k = 2$, followed by a rotation of 45°, followed by a reflection about the y-axis.

(c) A rotation of 15°, followed by a rotation of 105°, followed by a rotation of 60°.

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$

(C) $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -\sin 15^\circ & -\cos 15^\circ \\ \cos 15^\circ & -\sin 15^\circ \end{bmatrix}\begin{bmatrix} \cos 15^\circ & -\sin 15^\circ \\ \sin 15^\circ & \cos 15^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

9. Determine whether $T_1 \circ T_2 = T_2 \circ T_1$.

(a) $T_1 : R^2 \to R^2$ is the orthogonal projection on the x-axis, and $T_2 : R^2 \to R^2$ is the orthogonal projection on the y-axis.

(b) $T_1 : R^2 \to R^2$ is the rotation through an angle θ_1, and $T_2 : R^2 \to R^2$ is the rotation through an angle θ_2.

(c) $T_1 : R^2 \to R^2$ is the orthogonal projection on the x-axis, and $T_2 : R^2 \to R^2$ is the rotation through an angle θ.

(a) $T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $T_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T_1 \circ T_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $T_2 \circ T_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore T_1 \circ T_2 = T_2 \circ T_1$

(b) $T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$ $T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$

$T_1 \circ T_2 = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} = T_2 \circ T_1$

(C) $T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $T_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$T_1 \circ T_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{bmatrix}$

$T_2 \circ T_1 = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix}$

$\therefore T_1 \circ T_2 \neq T_2 \circ T_1$

14. Determine whether the matrix operator $T: R^3 \to R^3$ defined by the equations is one-to-one; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2, w_3)$.

(a) $w_1 = x_1 - 2x_2 + 2x_3$
$w_2 = 2x_1 + x_2 + x_3$
$w_3 = x_1 + x_2$

$[T] = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ one-to-one

$[T^{-1}] = [T]^{-1} = \begin{bmatrix} 1 & -2 & 4 \\ -1 & 2 & -3 \\ -1 & 3 & -5 \end{bmatrix}$

$T^{-1}(w_1, w_2, w_3) = (w_1 - 2w_2 + 4w_3, -w_1 + 2w_2 - 3w_3, -w_1 + 3w_2 - 5w_3)$

22. Find the standard matrix for the given matrix operator.

(a) $T: R^3 \to R^3$ reflects a vector about the xz-plane and then contracts that vector by a factor of $\frac{1}{5}$.

(b) $T: R^3 \to R^3$ projects a vector orthogonally onto the xz-plane and then projects that vector orthogonally onto the xy-plane.

(c) $T: R^3 \to R^3$ reflects a vector about the xy-plane, then reflects that vector about the xz-plane, and then reflects that vector about the yz-plane.

(a) $[T] = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$.

(c) $[T] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

29. Let A be an $n \times n$ matrix such that $\det(A) = 0$, and let $T: R^n \to R^n$ be multiplication by A.

(a) What can you say about the range of the matrix T? Give an example that illustrates your conclusion.

(b) What can you say about the number of vectors that T maps into $\mathbf{0}$?

(a) the range of T_A is R^m $(m < n)$.

(b) the number of vectors $\leq n$.

30. Prove: If the matrix transformation $T_A: R^n \to R^n$ is one-to-one, then A is invertible.

Since the matrix transformation $T_A: R^n \to R^n$ is one-to-one
the rank $(A) = n$.
so A is invertible.

§ 8.1 (Page 791)

In Exercises 1–8, determine whether the function is a linear transformation. Justify your answer.

1. $T: V \to R$, where V is an inner product space, and $T(\mathbf{u}) = \|\mathbf{u}\|$.

Not a linear transformation

because $T(\vec{0}) \neq \vec{0}$

2. $T: R^3 \to R^3$, where $\mathbf{v}_0$ is a fixed vector in R^3 and $T(\mathbf{u}) = \mathbf{u} \times \mathbf{v}_0$.

a linear transformation

6. $T: M_{22} \to R$, where

(a) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 3a - 4b + c - d$ linear

(b) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a^2 + b^2$ linear

7. $T: P_2 \to P_2$, where

(a) $T\left(a_0 + a_1x + a_2x^2\right) = a_0 + a_1\left(x+1\right) + a_2(x+1)^2$

(b) $T\left(a_0 + a_1x + a_2x^2\right) = \left(a_0 + 1\right) + \left(a_1 + 1\right)x + \left(a_2 + 1\right)x^2$

(a) linear

(b) nonlinear $T(\vec{0}) \neq \vec{0}$

8. $T: F(-\infty, \infty) \to F(-\infty, \infty)$, where

(a) $T(f(x)) = 1 + f(x)$ linear

(b) $T(f(x)) = f(x+1)$ linear.