

Introduction to Machine Learning, Fall 2023

Homework 1

(Due Thursday, Oct. 26 at 11:59pm (CST))

October 11, 2023

1. [10 points] [Math review] Suppose $\{\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n\}$ form a random sample from a multivariate distribution:
 - (a) Prove that the covariance of \mathbf{X}_i is a semi positive definite matrix. [3 points]
 - (b) Assuming $\mathbf{X}_i \sim \mathcal{N}(\mu, \Sigma)$ which is a multivariate normal distribution, and samples X_i , derive the the log-likelihood $l(\mu, \Sigma)$ and MLE of μ [4 points]
 - (c) Suppose $\hat{\theta}$ is an unbiased estimator of θ and $\mathbf{Var}(\hat{\theta}) > 0$. Prove that $(\hat{\theta})^2$ is not an unbiased estimator of θ^2 . [3 points]

2. [10 points] Consider real-valued variables X and Y , in which Y is generated conditional on X according to

$$Y = aX + b + \epsilon, \text{ where } \epsilon \sim \mathcal{N}(0, \sigma^2).$$

Here ϵ is an independent variable, called a noise term, which is drawn from a Gaussian distribution with mean 0, and variance σ^2 . This is a single variable linear regression model, where a is the only weight parameter and b denotes the intercept. The conditional probability of Y has a distribution $p(Y|X, a, b) \sim \mathcal{N}(aX + b, \sigma^2)$, so it can be written as:

$$p(Y|X, a, b) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(Y - aX - b)^2\right).$$

- (a) Assume we have a training dataset of n i.i.d. pairs (x_i, y_i) , $i = 1, 2, \dots, n$, and the likelihood function is defined by $L(a, b) = \prod_{i=1}^n p(y_i|x_i, a, b)$. Please write the Maximum Likelihood Estimation (MLE) problem for estimating a and b . [3 points]
- (b) Estimate the optimal solution of a and b by solving the MLE problem in (a). [4 points]
- (c) Based on the result in (b), argue that the learned linear model $f(X) = aX + b$, always passes through the point (\bar{x}, \bar{y}) , where $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ and $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$ denote the sample means. [3 points]

3. [10 points] [Regression and Classification]

- (a) When we talk about linear regression, what does ‘linear’ regard to? [2 points]
- (b) Assume that there are n given training examples $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$, where each input data point x_i has m real valued features. When $m > n$, the linear regression model is equivalent to solving an under-determined system of linear equations $\mathbf{y} = \mathbf{X}\beta$. One popular way to estimate β is to consider the so-called ridge regression:

$$\underset{\beta}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda \|\beta\|_2^2$$

for some $\lambda > 0$. This is also known as Tikhonov regularization.

Show that the optimal solution β_* to the above optimization problem is given by

$$\beta_* = (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})^{-1} \mathbf{X}^T \mathbf{y}$$

Hint: You need to prove that given $\lambda > 0$, $\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I}$ is invertible. [5 points]

- (c) Is the given data set linear separable? If yes, construct a linear hypothesis function to separate the given data set. If no, explain the reason. [3 points]

Data	(1,3)	(4,4)	(3,-6)	(-2,1)	(-3,5)	(-6,-4)
Label	+1	-1	-1	+1	-1	-1