- Support Vector Machines (SVMs).

Maria-Florina Balcan

03/25/2015

Support Vector Machines (SVMs).

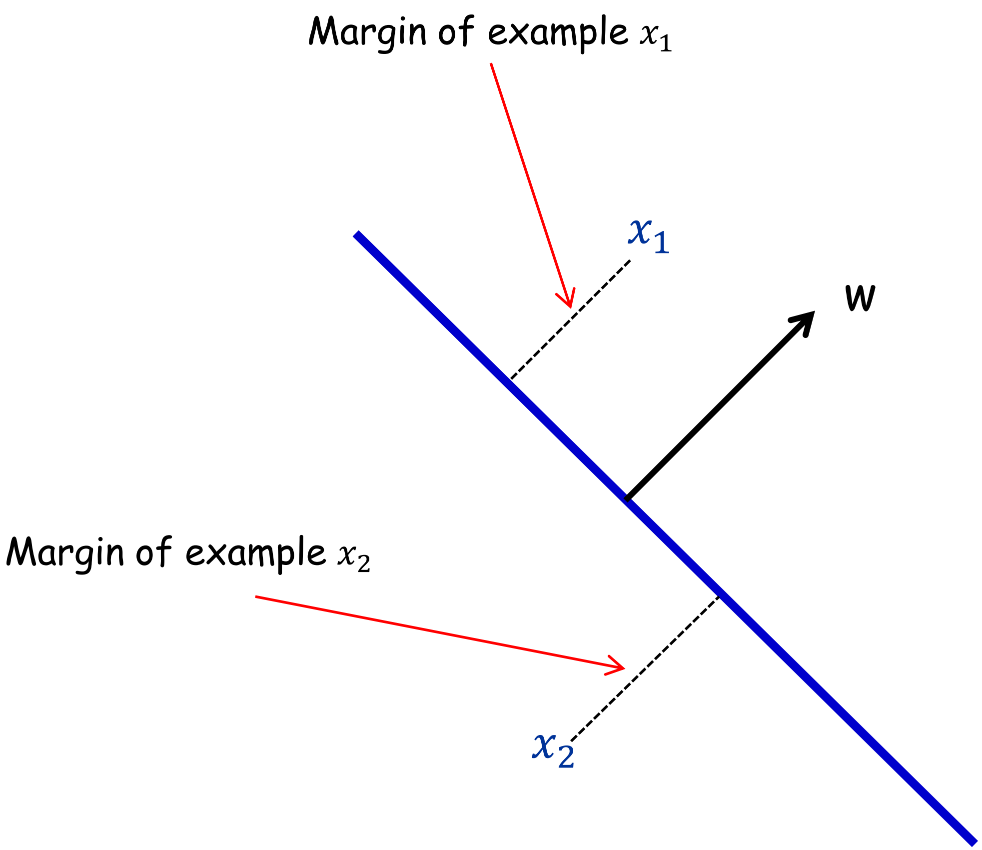
One of the most theoretically well motivated and practically most effective classification algorithms in machine learning.

Directly motivated by Margins and Kernels!

Geometric Margin

WLOG homogeneous linear separators $[w_0 = 0]$.

Definition: The margin of example x w.r.t. a linear sep. w is the distance from x to the plane $w \cdot x = 0$.



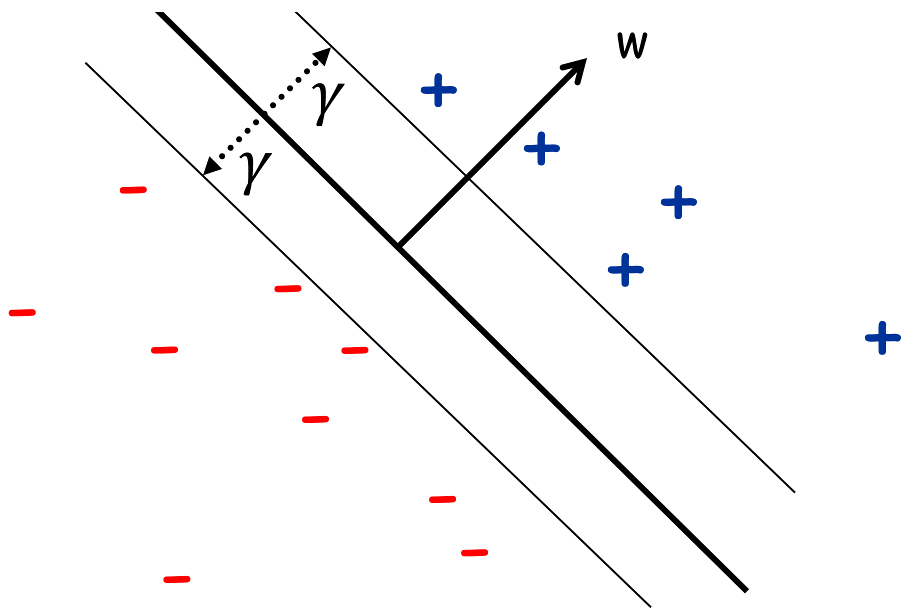
If $||w|| = 1$, margin of x w.r.t. w is $|x \cdot w|$.

Geometric Margin

Definition: The margin of example x w.r.t. a linear sep. w is the distance from x to the plane $w \cdot x = 0$.

Definition: The margin γ_w of a set of examples S wrt a linear separator w is the smallest margin over points $x \in S$.

Definition: The margin γ of a set of examples S is the maximum γ_w over all linear separators w.

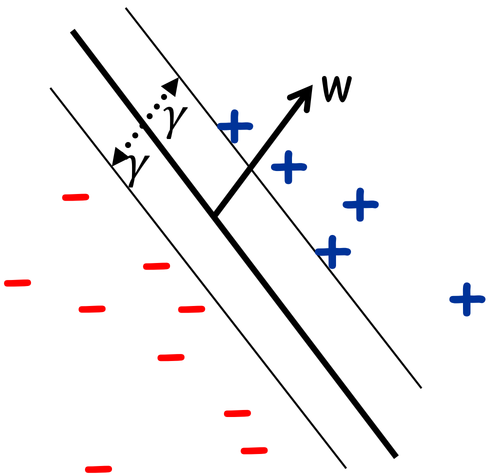


Margin Important Theme in ML

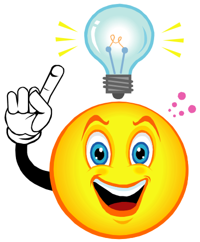
Both sample complexity and algorithmic implications.

Sample/Mistake Bound complexity:

- If large margin, # mistakes Peceptron makes is small (independent on the dim of the space)!
- If large margin γ and if alg. produces a large margin classifier, then amount of data needed depends only on R/γ [Bartlett & Shawe-Taylor '99].



Algorithmic Implications



Suggests searching for a large margin classifier… SVMs

Support Vector Machines (SVMs)

Directly optimize for the maximum margin separator: SVMs

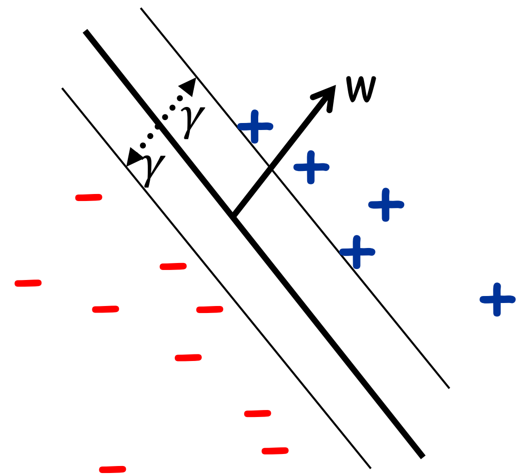
First, assume we know a lower bound on the margin γ

Input: γ, S={(x_1, y_1), …,(x_m, y_m)};

Find: some w where:

- $||w||^2 = 1$
- For all i, $y_i w \cdot x_i \geq \gamma$

Output: w, a separator of margin γ over S



Realizable case, where the data is linearly separable by margin γ

Support Vector Machines (SVMs)

Directly optimize for the maximum margin separator: SVMs

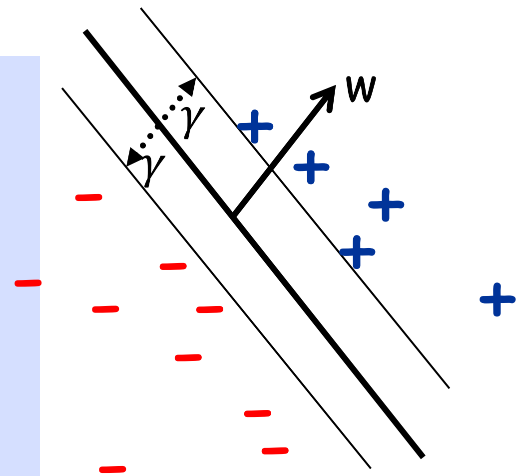
E.g., search for the best possible γ

<u>Input</u>: S={(x_1, y_1), …,$(\mathrm{x}_\mathrm{m}, y_\mathrm{m})$};

<u>Find</u>: some w and maximum γ where:

- $||\mathrm{w}||^2 = 1$
- For all i, $y_i w \cdot x_i \geq \gamma$

<u>Output</u>: maximum margin separator over S



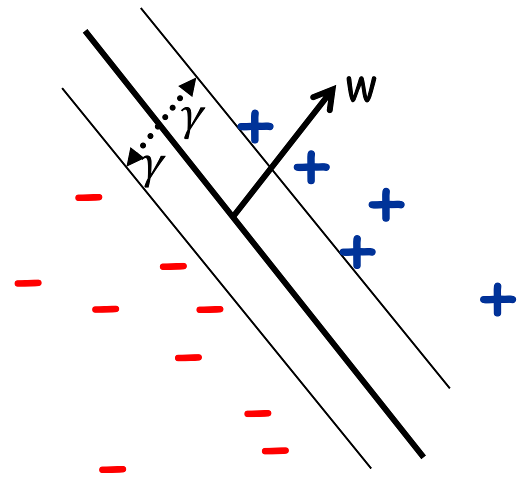
Support Vector Machines (SVMs)

Directly optimize for the maximum margin separator: SVMs

Input: S={(x_1, y_1), …,(x_m, y_m)};

Maximize γ under the constraint:

- $||w||^2 = 1$
- For all i, $y_i w \cdot x_i \geq \gamma$



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objective function

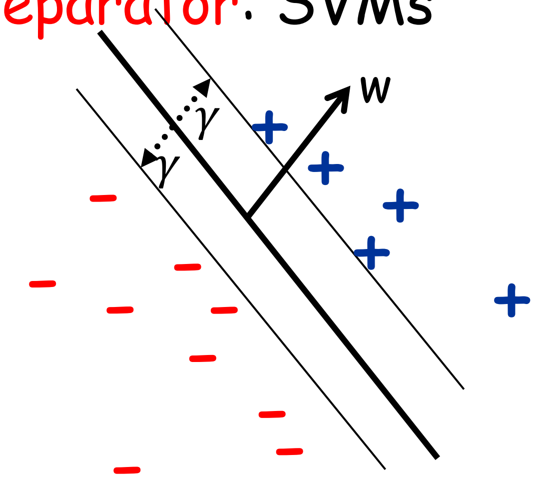
constraints

This is a **constrained optimization** problem.

- Famous example of constrained optimization: **linear programming**, where objective fn is linear, constraints are linear (in)equalities

Support Vector Machines (SVMs)

Directly optimize for the maximum margin separator: SVMs



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Maximize γ under the constraint:

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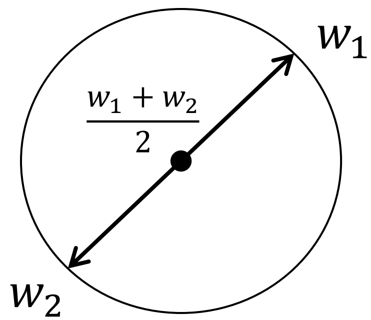
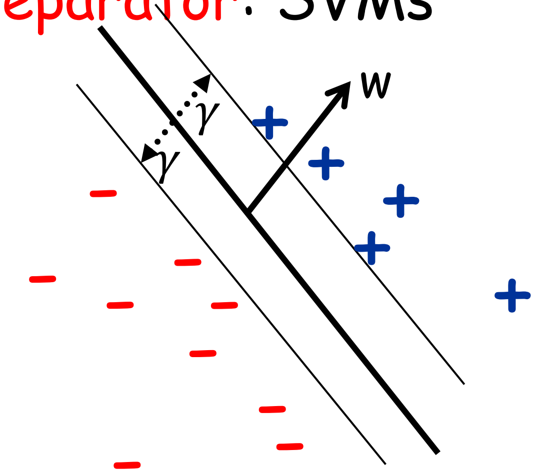
This constraint is non-linear.

In fact, it's even non-convex



Support Vector Machines (SVMs)

Directly optimize for the maximum margin separator: SVMs

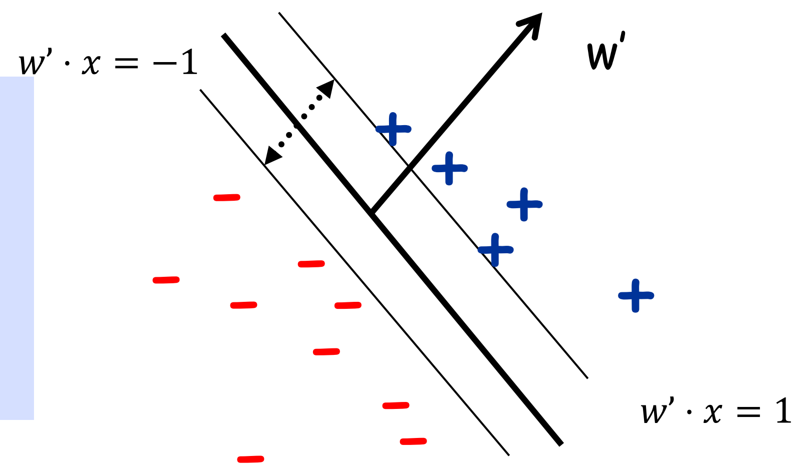


Input: S={(x_1, y_1), …,(x_m, y_m)};

Maximize γ under the constraint:

- $||w||^2 = 1$
- For all i, $y_i w \cdot x_i \geq \gamma$

$w' = w/\gamma$, then max γ is equiv. to minimizing $||w'||^2$ (since $||w'||^2 = 1/\gamma^2$).
So, dividing both sides by γ and writing in terms of w′ we get:



Input: S={(x_1, y_1), …,(x_m, y_m)};

Minimize $||w'||^2$ under the constraint:

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Support Vector Machines (SVMs)

Directly optimize for the maximum margin separator: SVMs

Input: S={(x_1, y_1), …,$(\mathrm{x}_\mathrm{m}, y_\mathrm{m})$};

$\mathrm{argmin}_w \ ||w||^2$ s.t.:

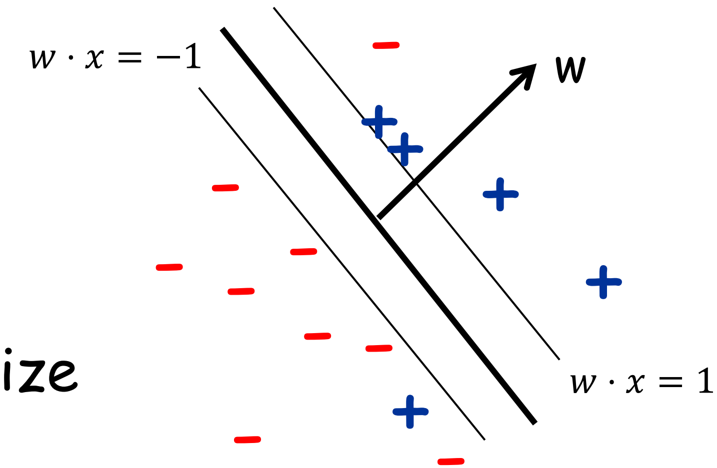
- For all i, $y_i w \cdot x_i \geq 1$

This is a **constrained optimization** problem.

- The objective is convex (quadratic)
- All constraints are linear
- Can solve efficiently (in poly time) using standard **quadratic programing** (QP) software

Support Vector Machines (SVMs)

Question: what if data isn't perfectly linearly separable?



Issue 1: now have two objectives

- maximize margin
- minimize # of misclassifications.

Ans 1: Let's optimize their sum: minimize

$||w||^2 + C(\text{\# misclassifications})$

where C is some tradeoff constant.

Issue 2: This is computationally hard (NP-hard).

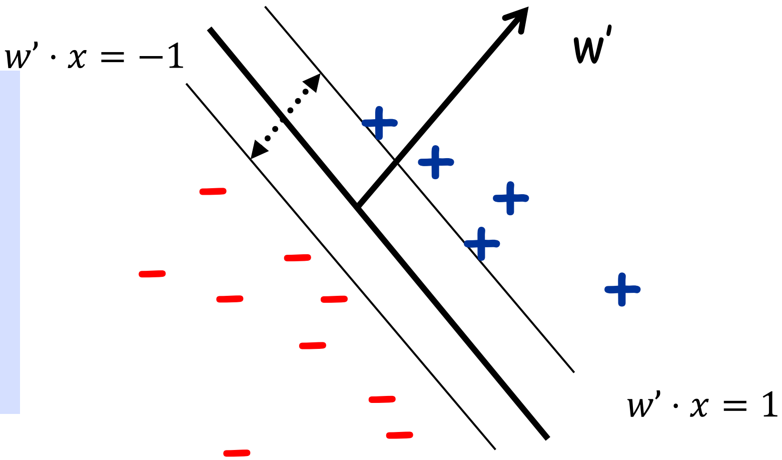
[even if didn't care about margin and minimized # mistakes]

NP-hard [Guruswami-Raghavendra'06]

Support Vector Machines (SVMs)

Question: what if data isn't perfectly linearly separable?

Replace "# mistakes" with upper bound called "hinge loss"



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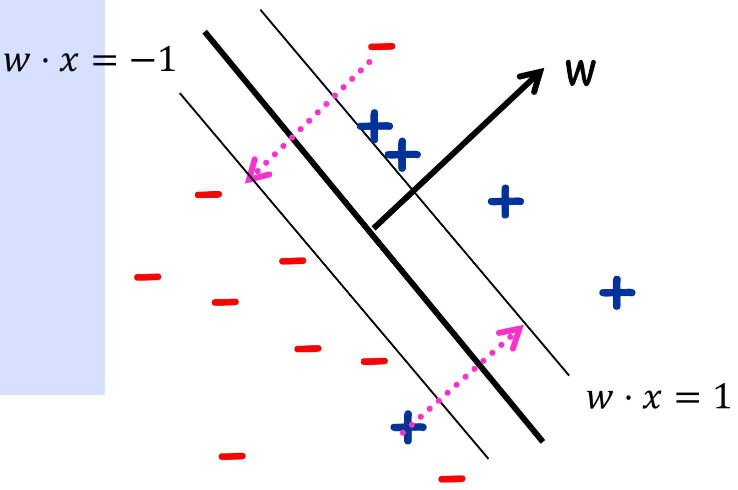

Input: S={(x_1, y_1), …,(x_m, y_m)};

Find $\text{argmin}_{w,\xi_1,\ldots,\xi_m} ||w||^2 + C \sum_i \xi_i$ s.t.:

- For all i, $y_i w \cdot x_i \geq 1 - \xi_i$

  $\xi_i \geq 0$

ξ_i are "slack variables"

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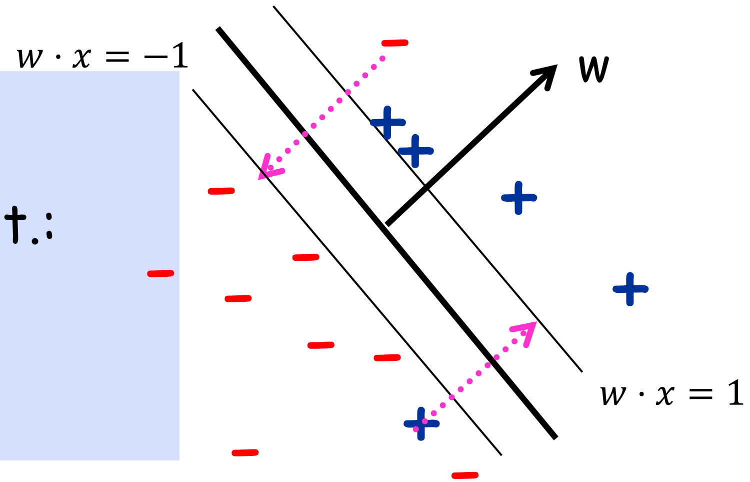

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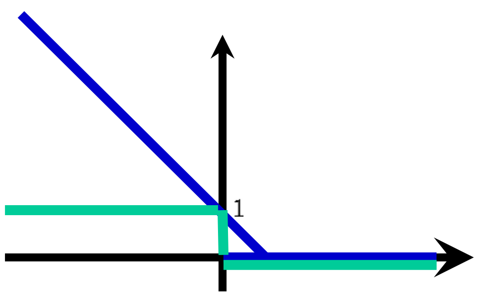
C controls the relative weighting between the twin goals of making the $||w||^2$ small (margin is large) and ensuring that most examples have functional margin ≥ 1.



$l(w, x, y) = \max(0, 1 - y\, w \cdot x)$

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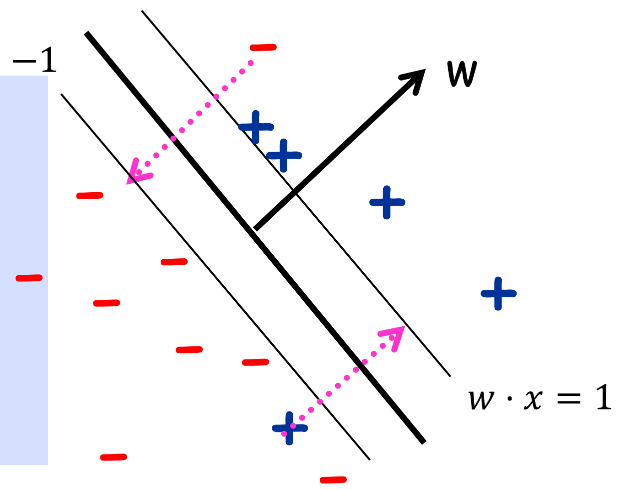

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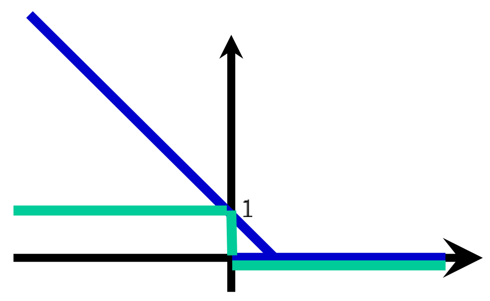
Replace the number of mistakes with the hinge loss

$$||w||^2 + C(\text{\# misclassifications})$$



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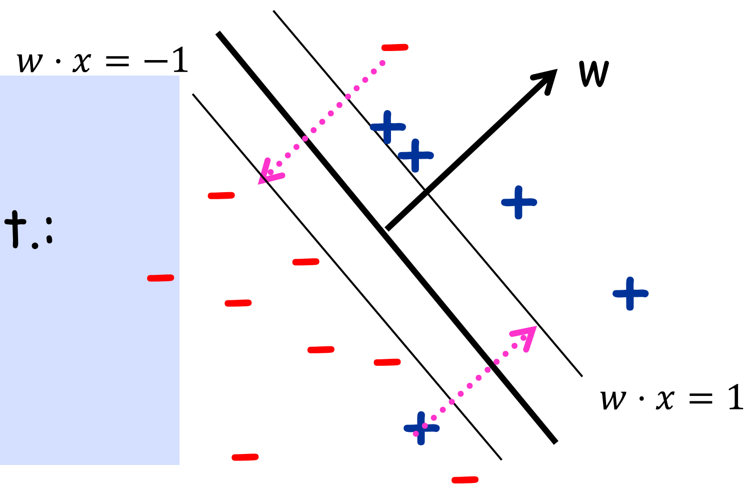

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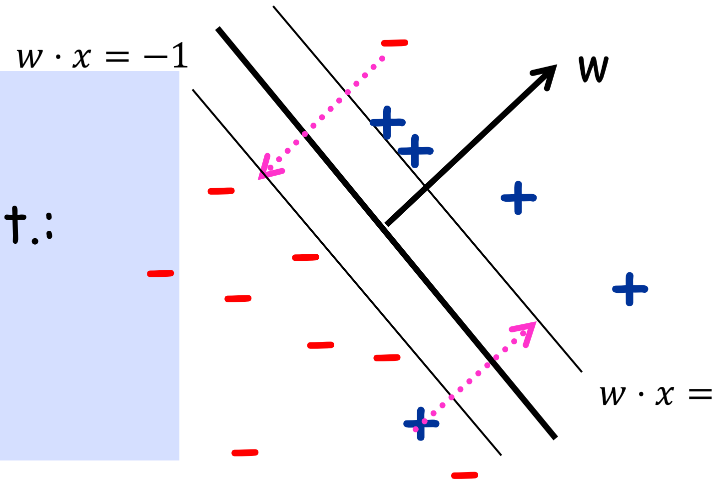

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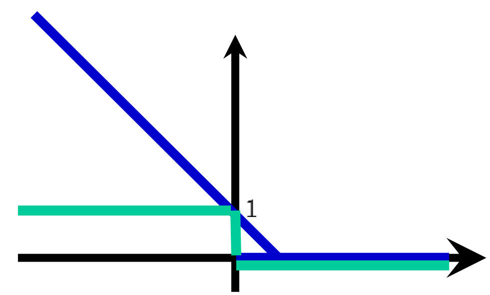
- For all i, $y_i w \cdot x_i \geq 1 - \xi_i$
  $\xi_i \geq 0$

Total amount have to move the points to get them on the correct side of the lines $w \cdot x = +1/-1$, where the distance between the lines $w \cdot x = 0$ and $w \cdot x = 1$ counts as "1 unit".



$l(w, x, y) = \max(0, 1 - y\, w \cdot x)$

What if the data is far from being linearly separable?

Example:  vs 

No good linear separator in pixel representation.

SVM philosophy: "Use a Kernel"

Support Vector Machines (SVMs)

Input: S={(x_1, y_1), …,(x_m, y_m)};

Find $\text{argmin}_{w,\xi_1,\dots,\xi_m} ||w||^2 + C\sum_i \xi_i$ s.t.:

- For all i, $y_i w \cdot x_i \geq 1 - \xi_i$

  $\xi_i \geq 0$

Primal form

Which is equivalent to:

Input: S={(x_1, y_1), …,(x_m, y_m)};

Find $\text{argmin}_{\alpha} \frac{1}{2}\sum_i \sum_j y_i y_j \alpha_i \alpha_j x_i \cdot x_j - \sum_i \alpha_i$ s.t.:

- For all i, $0 \leq \alpha_i \leq C_i$

  $y_i \alpha_i = 0$

  i

Lagrangian Dual

SVMs (Lagrangian Dual)

<u>Input</u>: $S=\{(x_1, y_1), \ldots, (x_m, y_m)\}$;

<u>Find</u> $\operatorname{argmin}_\alpha \frac{1}{2} \sum_i \sum_j y_i y_j\, \alpha_i \alpha_j x_i \cdot x_j - \sum_i \alpha_i$ s.t.:

- For all i, $0 \le \alpha_i \le C_i$

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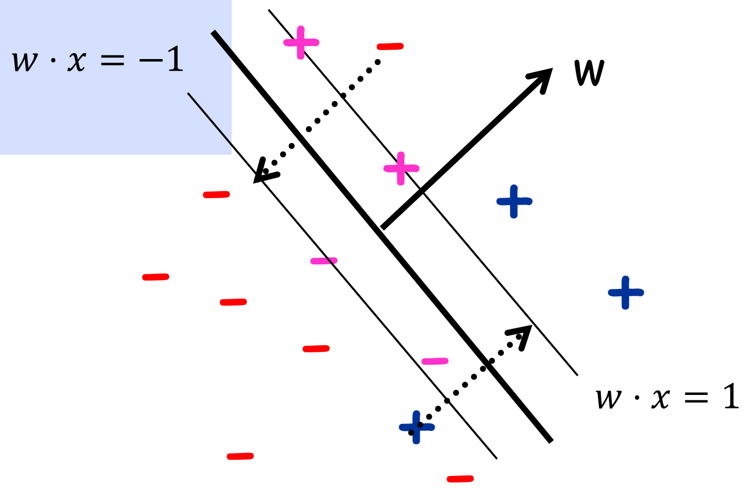

- Final classifier is: $w = \sum_i \alpha_i y_i x_i$

- The points x_i for which $\alpha_i \neq 0$ are called the "support vectors"

Kernelizing the Dual SVMs

<u>Input</u>: $S=\{(x_1, y_1), \ldots, (x_m, y_m)\}$;

<u>Find</u> $\text{argmin}_\alpha \frac{1}{2} \sum_i \sum_j y_i y_j \, \alpha_i \alpha_j x_i \cdot x_j - \sum_i \alpha_i$ s.t.:

- For all i, $0 \leq \alpha_i \leq C_i$

$$\sum_i y_i \alpha_i = 0$$

Replace $x_i \cdot x_j$ with $K(x_i, x_j)$.

- Final classifier is: $w = \sum_i \alpha_i y_i x_i$
- The points x_i for which $\alpha_i \neq 0$ are called the "support vectors"
- With a kernel, classify x using $\sum_i \alpha_i y_i K(x, x_i)$

Support Vector Machines (SVMs).

One of the most theoretically well motivated and practically most effective classification algorithms in machine learning.

Directly motivated by Margins and Kernels!

What you should know

- The importance of margins in machine learning.
- The primal form of the SVM optimization problem
- The dual form of the SVM optimization problem.
- Kernelizing SVM.
- Think about how it's related to Regularized Logistic Regression.