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False Vacuum Decay with Gravity

——Negative Mode Problem——

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Motivated by work of Lavrelashvili, Rubakov and Tinyakov (LRT), who claim that the WKB approximation would break down even at energy scale far below the Planck one, the decay of false vacuum under the presence of gravity is carefully investigated. First we approach the issue from a different point of view, namely, by solving the Wheeler-DeWitt equation under the boundary condition appropriate for the false vacuum decay. The resulting wave function shows no sign of the breakdown of the WKB approximation. Then, based on the canonical Hamiltonian form of the Euclidean path integral, we develop a method to analyze the number of negative modes around the Euclidean bounce solution describing the tunneling process and evaluate it in some limiting cases. Again, we find a strong evidence that LRT's claim is superficial.

§ 1. Introduction

The quantum tunneling in the early universe has been a matter of great interest in recent years. It has been discussed mainly in two different contexts; the creation of the universe in quantum cosmology, and the false vacuum decay and bubble nucleation in the inflationary universe. The former was pioneered by Hartle, and Hawking[1)] and Vilenkin,[2)] and has been developed by many others.[3)] The latter was originally discussed in the context of the old inflation scenario, but because of the so-called graceful exit problem, it was replaced by the new or chaotic inflation scenario unrelated to the quantum tunneling phenomena.[4)] But recently the extended inflation scenario was proposed by La and Steinhardt.[5)] In this scenario the graceful exit problem is solved by replacing the Einstein gravity theory with the Jordan-Brans-Dicke theory, yielding an effectively time-dependent bubble nucleation rate.

Among others, the most intriguing issues in the quantum tunneling in the early universe will be

(A) How the universe is created. Is it really created from "nothing"? And, if so, what is the probability distribution of various universes realized through the tunneling?

(B) If the early universe was in a metastable state (false vacuum), as in the inflation scenario, at what rate and to what sort of quantum state does the false vacuum make a transition?

With respect to (A), there exists the so-called cosmological constant problem.[6)] As for the cosmological constant, Λ, of our universe, the observed flatness and matter density indicate $\Lambda G \lesssim 10^{-122}$, according to the Einstein equations. This value seems too small to be explained as an expectation value of some quantum field. If the cosmological constant is identically zero, we need to know the reason. If the cosmological constant is tiny but not zero, which several observational facts seem to indicate, we need to know why such a small value was chosen.

An interesting point of view is that the cosmological constant is dynamically determined as an integral of motion. From this standpoint, Hawking proposed that the probability $P(\Lambda)$ that the universe was created quantum mechanically with a cosmological constant Λ will be given by the Euclidean path integral,[7]

$$P(\lambda)\propto\int[dg]\exp(-I[g,\Lambda])\,,$$

where $I[g,\Lambda]$ is the Euclidean action. Restricting the Euclidean configurations to the maximal symmetric space-time, S^4, the summation was evaluated by a saddle point; a solution of the Einstein equations. This estimation results that $P(\Lambda)$ diverges unnormalizably at $\Lambda=+0$ and so it was concluded that the effective cosmological constant is zero. Coleman extended this study by including the configurations with worm holes.[8] This requires $\Lambda=+0$ more strictly. These results are satisfactory if the cosmological constant is identically zero. However, if not, they need to be modified or a new mechanism should be considered.

With respect to (B), there seems to be two major unsolved problems: One is the effect of environmental degrees of freedom on the tunneling degree of freedom, or particle creation during the tunneling process and its back reaction. The other is the effect of gravity on the tunneling rate.

Concerning the former, Rubakov[9] has developed a method to calculate the particle production rate by means of a non-unitary Bogoliubov transformation; an extension of the Bogoliubov transformation method used to discuss the particle creation in curved space-time. Recently this line of thought was further developed by Vachaspati and Vilenkin.[10] However, to our knowledge, the backreaction of it to the tunneling process has not been studied so far.

As for the effect of gravity, the quantum tunneling with gravity was first discussed by Coleman and De Luccia.[11] Their approach is a straightforward extension of the instanton technique for the Euclidean path integral (EPI) which successfully describes the tunneling in the case of field theory on flat space-time.[12)~15)] But the extension to the case with gravity is highly non-trivial and has many problems to be solved. For example, when the flat Minkowski space-time is euclideanized, the Euclidean time t_E can take any real number just as the Lorentzian time t_L. On the other hand, when the gravity is switched on, the false vacuum state has the de Sitter metric in general; $ds^2=-dt_L{}^2+\cosh^2 t_L\, d\Omega^2$, while when it is euclideanized the metric becomes $ds^2=dt_E{}^2+\cos^2 t_E\, d\Omega^2$. Thus, although t_L can take any real number, t_E runs only between $-\pi/2$ and $+\pi/2$. This implies, in particular, that the conventional dilute gas (i.e., WKB) approximation used in EPI may easily become inappropriate. Further the periodicity in the Euclidean time may imply that the rate of the false vacuum decay evaluated on the basis of EPI is inevitably associated with finite temperature. If so, how should we evaluate the zero temperature transition amplitude? Another much more controversial problem concerning the gravity was pointed out by Rubakov,[17] who claimed that drastic particle production may occur during the tunneling of a universe with a positive cosmological constant from the Friedmann regime to the de Sitter regime. Moreover, based on the Lagrangian form of the EPI, Lavrelashvili, Rubakov and Tinyakov[18] (LRT) pointed out that the false vacuum decay with gravity can show a pathological feature depending on the potential shape of the tunneling field.[18] The existence of one and only one negative mode around the Euclidean classical solution, called the bounce, was essential to identify the decay rate with the EPI of the action in the WKB approximation[13),14] and it was proved for a wide class of potentials in the case of flat space-time.[16] However LRT argued that an infinite number of negative modes may appear around the bounce and consequently the WKB approximation may break down even in the case of a potential whose energy scale is far below the Planck one.

If the above claims are true, inclusion of the backreaction or of non-perturbative quantum corrections is essential to evaluate the process of creation of the universe or of the false vacuum decay. Then the actual tunneling probability would be absolutely different from that qualitatively expected so far.

We need to find the answers to all these problems to obtain a reliable physical picture of the early universe. However, it is of course a formidable task. Therefore, as a first step, we concentrate on the issue raised by LRT. We take two different approaches to it; one is based on the description of tunneling by the wave function and the other on the canonical Hamiltonian formulation of EPI.

The paper is organized as follows. In § 2, we review the EPI approach to quantum tunneling. In § 3, based on the Wheeler-DeWitt equation, we consider the WKB wave function describing a tunneling process. We find no sign of the breakdown of the WKB approximation. In § 4, we develop a method to systematically investigate the number of negative modes around a Euclidean classical solution with gravity, based on the canonical Hamiltonian formalism. Again, we find a strong evidence that the LRT pathology is an artifact due to their inadequate choice of gauge, inevitably implied by the Lagrangian formalism. Finally, we summarize our results in § 5.

Throughout this paper, the metric signature is $(-,+,+,+)$ for Lorentzian and $(+,+,+,+)$ for Euclidean space-time. We use the units $c=\hbar=1$. The Riemann tensor, Ricci tensor and Ricci scalar curvature are defined as

$$R^{\alpha}{}_{\beta\mu\nu}=\Gamma^{\alpha}{}_{\beta\nu,\mu}-\Gamma^{\alpha}{}_{\beta\mu,\nu}+\Gamma^{\alpha}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\beta\nu}-\Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\beta\mu}\,,$$

$$R_{\mu\nu}=R^{\alpha}{}_{\mu\alpha\nu}\,,$$

$$R=R^{\mu}{}_{\mu}\,.$$

§ 2. Euclidean path integral approach

Quantum tunneling of a simple one-dimensional quantum mechanical system has been well studied. In particular, when the WKB approximation is valid, the Euclidean path integral (EPI) method is known to work perfectly. The advantage of this method is that it can be easily extended to the case of field theory. However, as soon as gravity is turned on, there appear many conceptual as well as technical problems in the EPI method. In order to clearly understand the origin of such problems and to find a way to resolve them, in this section, we review the EPI approach to tunneling in quantum mechanics and field theory with and without gravity.[14)]

2.1. *Quantum tunneling in Mechanics*

Let us consider a one-particle system with a potential $V(q)$. We start with the canonical expression of Euclidean path integral,

$$\langle q_f|e^{-HT}|q_i\rangle=\int_{\substack{q(0)=q_i\\ q(T)=q_f}}\left[\frac{dp(\cdot)dq(\cdot)}{2\pi}\right]\exp\left\{\int_0^T\left(ip\frac{dq}{d\tau}-H\right)d\tau\right\}, \qquad (2\cdot1)$$

where H is the Hamiltonian of the system and the Euclideanization is defined by $t=-i\tau$.

Here two comments are in order: First, only the time is rotated to the imaginary axis but not the momentum p, hence the integral measure is still given in terms of the Lorentzian variables. Second, the momentum p is not defined at the end points $\tau=0$, T, hence its degrees of freedom is one less than those of the coordinate q (including the boundary values $q(0)$ and $q(T)$), out of infinite degrees of freedom associated with the functional measure.

For the Hamiltonian of the form,

$$H=\frac{1}{2}p^2+V(q), \qquad (2\cdot2)$$

the p-integration of Eq. (2·1) can be readily done to yield

$$\langle q_f|e^{-HT}|q_i\rangle=\int_{\substack{q(0)=q_i\\ q(T)=q_f}}\left[\frac{dq(\cdot)}{\sqrt{2\pi}}\right]\exp\{-S_E\}, \qquad (2\cdot3)$$

where

$$S_E=\int_0^T d\tau\left(\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2+V(q)\right), \qquad (2\cdot4)$$

which is the Euclidean action of the system related to the original Lorentzian action S_L as $S_L=iS_E$ with $t=-i\tau$.

Usually Eq. (2·3) is taken as the starting point of the EPI method. However, for a more complicated system (e.g., with the Hamiltonian not in the form of Eq. (2·2)), it may well happen that the right-hand side of Eq. (2·3) is ill-defined or ambiguous. Hence in this paper we take a standpoint that Eq. (2·1) is fundamental and go back to it whenever necessary.

Now, the left-hand side of Eq. (2·3) is the transition amplitude from the state $q=q_i$ to $q=q_f$ after an interval of $t=-iT$ in the direction of imaginary time, and only the lowest energy state contributes to it in the limit $T\to\infty$,

$$\langle q_f|e^{-HT}|q_i\rangle \xrightarrow[T\to\infty]{} e^{-E_0T}\langle q_f|0\rangle\langle 0|q_i\rangle. \qquad (2\cdot5)$$

Therefore, if the asymptotic form of the right-hand side of Eq. (2·3) at $T\to\infty$ is known for $q_i = q_f = q_1$, we can read off the ground state energy eigenvalue E_0 and the overlapping amplitude of the state $|q_1\rangle$ with the ground state; $|\langle 0|q_1\rangle|$.

In the WKB approximation, the path integral in Eq. (2·3) is evaluated in terms of the Euclidean classical solution with the boundary condition $q(0)=q(T)=q_1$ and the perturbation around it. Since the particle motion in Euclidean time is described by that with the inverted potential $V_E(q)=-V(q)$, the only solution would be $q(\tau)\equiv q_1$ if $q=q_1$ were the absolute minimum of $V(q)$. However, if $q=q_1$ is a local minimum of $V(q)$, there exist a non-trivial solution $\bar{q}(\tau)$ called the bounce, which stays near $q=q_1$ for most of the time but rolls down the inverted potential, and reflected back to $q=q_1$ within some finite time-scale, say $\delta\tau$, determined from the shape of the potential. Then one can construct an infinite number of approximate solutions that have an arbitrary number of bounces which are sufficiently separated from each other during the large time interval T. Evaluating the path integral by summing up all these solutions is called the dilute gas approximation and it is known to be equivalent to the WKB approximation.

Given the bounce solution $\bar{q}(\tau)$, fluctuations around it can be path-integrated by decomposing them into the eigenmode functions around the bounce. The result is[14]

$$\int_{q(0)=q(T)=q_1}\left[\frac{dq(\cdot)}{\sqrt{2\pi}}\right]\exp(-S_E[q(\tau)])=N'e^{-B}\prod_n \lambda_n^{-\frac{1}{2}}, \tag{2·6}$$

where N' is a normalization constant, λ_n are the eigenvalues, and B is the classical action integral of the bounce,

$$B=S_E[\bar{q}(\tau)]=2\int_{q_1}^{q_2}dq\sqrt{2V(q)} \tag{2·7}$$

with the understanding that the potential is adjusted by adding a constant to $V(q_1)=0$.

In the above, it is tacitly assumed that all the eigenvalues λ_n are positive definite. However, in fact, there generally exist a zero mode and a negative mode in the decay of a metastable state. We do not go into details here, but only mention that the zero mode corresponds to the freedom of position of a bounce within the interval T and the negative mode describes the instability of the initial state, yielding a factor of the imaginary unit i to the path integral by a suitable analytic continuation.[14]

Now, summing up all possible configurations of the bounces, one obtains

$$\begin{aligned}\langle q_1|e^{-HT}|q_1\rangle &= \sum_n \frac{T^n}{n!}e^{-nB}K^n\sqrt{\frac{\omega}{\pi}}e^{-\omega T/2}\\ &=\sqrt{\frac{\omega}{\pi}}\exp\{-(\omega/2-Ke^{-B})T\},\end{aligned} \tag{2·8}$$

where K is given by

$$K=\sqrt{\frac{B}{2\pi}}\frac{\prod_{n\neq 1}\lambda_n^{-\frac{1}{2}}}{\sqrt{\omega/\pi}\,e^{-\omega T/2}}=\sqrt{\frac{B}{2\pi}}\left|\frac{\det'[-\partial_\tau^2+V''(\bar{q})]}{\det[-\partial_\tau^2+\omega^2]}\right|^{-1/2}, \tag{2·9}$$

and det′ means the exclusion of the zero mode from the evaluation. Then the metastable ground state energy is read as

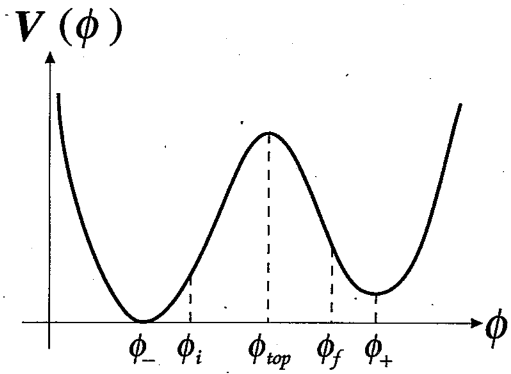

Fig. 1. Illustration of a potential with false vacuum. For the Coleman-De Luccia solution, the scalar field varies from ϕ_i to ϕ_f, while for the Hawking-Moss solution, it stays at the top of the barrier $\phi=\phi_{top}$.

$$E_0=\frac{\omega}{2}-Ke^{-B}. \tag{2·10}$$

As mentioned, the negative mode gives rise to the factor i in K. Omitting the detail of how the analytic continuation is done, the decay rate of the metastable state is given by

$$\Gamma=|K|e^{-B}. \tag{2·11}$$

2.2. *False vacuum decay in field theory*

Extension of the results in the previous subsection to quantum field theory is straightforward in the case of a single real scalar field.[12] For the Euclidean action of the form,

$$S_E=\int d^4x\left[\frac{1}{2}\left\{\left(\frac{d\phi}{d\tau}\right)^2+(\nabla\phi(x))^2\right\}+V(\phi(x))\right] \tag{2·12}$$

with the potential $V(\phi)$ having a false vacuum at $\phi=\phi_+$ as illustrated in Fig. 1, the bounce solution is $O(4)$-symmetric, i.e., $\phi(\tau, x)=\phi(\rho)$ where $\rho^2=\tau^2+x^2$. Then the equation of motion is

$$\frac{d^2\phi}{d\rho^2}+\frac{3}{\rho}\frac{d\phi}{d\rho}=V'(\phi) \tag{2·13}$$

with the boundary condition at infinity,

$$\phi(\infty)=\phi_+, \tag{2·14}$$

and the regularity condition at the origin,

$$\frac{d\phi}{d\rho}(0)=0. \tag{2·15}$$

Only changes from the case of quantum mechanics are that (i) there exists effectively a friction term in the equation of motion for the bounce due to spatial degrees of freedom, and (ii) there appear four zero modes, instead of one, because of the four-dimensional nature. Taking into account these changes, the decay rate of the false vacuum per unit volume is given by

$$\Gamma/V=|K|e^{-B}; \quad B=S_E[\bar{\phi}(\rho)]-S_E[\phi_+], \tag{2·16}$$

where

$$K=\left(\frac{B}{2\pi}\right)^2\frac{|\det'[-\nabla^2+V''(\bar{\phi})]|^{-1/2}}{|\det[-\nabla^2+V''(\phi_+)]|^{-1/2}}, \tag{2·17}$$

and $\bar{\phi}(\rho)$ denotes the bounce solution. The above formula holds provided there exists one and only one negative mode. The uniqueness of the negative mode was proved by Coleman[16] rather generally, for a wide class of systems.

Now, we take the effect of gravity into account. Quantum tunneling with gravity was first discussed by Coleman and De Luccia.[11] Since the Einstein theory is expected to be valid for energy scale far below the Planck energy, it is appropriate to begin with the following Euclidean action,

$$S_E=\int d^4x\sqrt{g}\Big[-\frac{1}{2\kappa}R+\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+V(\phi)\Big], \tag{2·18}$$

where $\kappa=8\pi G$ is the gravitational constant, and it will be assumed that the potential has a local minimum at $\phi=\phi_+$ as before and the absolute minimum at $\phi=\phi_-$ with $V(\phi_-)=0$ (see Fig. 1).

As soon as gravity is turned on, there appear various difficulties, both conceptual and technical ones. Among others, the most troublesome is that the Euclidean action (2·18) is not bounded below. It is related to the existence of the so-called conformal mode of gravity and there is a proposal to overcome this difficulty,[19),20)] but this issue is beyond the scope of this paper. Here we note that, even in the case of the false vacuum decay without gravity, the path integral is dominated by the bounce which is not the minimum but the saddle point of the action, and the result is insensitive to the global feature of the potential. Hence it is reasonable to assume this is also true in the case with gravity, i.e., we assume the dominant contribution to the path integral comes from an $O(4)$-symmetric classical solution of the Euclidean action (2·18).

To find an $O(4)$-symmetric solution, we consider the metric of the form,

$$ds^2=N^2(\xi)d\xi^2+a^2(\xi)d\Omega_{(3)}^2\,, \tag{2·19}$$

where $d\Omega_{(3)}^2$ is the metric on the unit 3-sphere, and the scalar field which depends only on the variable ξ; $\phi(x^\mu)=\phi(\xi)$. Then, after a suitable integration by parts the action becomes

$$S_E=2\pi^2\int d\xi\Big[\frac{1}{N}\Big(-\frac{3}{\kappa}a\dot{a}^2+\frac{1}{2}a^3\dot{\phi}^2\Big)-\frac{3}{\kappa}NaQ(a,\phi)\Big], \tag{2·20}$$

where

$$Q(a,\phi):=1-\frac{1}{3}\kappa a^2V(\phi)\,. \tag{2·21}$$

The variation of S_E with respect to ϕ gives the field equation,

$$\ddot{\phi}+\Big(3\frac{\dot{a}}{a}-\frac{\dot{N}}{N}\Big)\dot{\phi}=N^2\frac{dV}{d\phi}\,, \tag{2·22}$$

and with respect to N a constraint equation (the Hamiltonian constraint),

$$\dot{a}^2-\frac{\kappa}{6}a^2\dot{\phi}^2=N^2Q(a,\phi)\,. \tag{2·23}$$

As well known, this constraint equation is a consequence of the time reparametrization invariance of the system and the variation with respect to the scale factor a does not give any new equation. To solve this system, let us impose the gauge condition,

$N\equiv 1$, here.

For the present case, even the trivial solution, $\phi(x^\mu)\equiv\phi_+$, is not so trivial because Eq. (2·23) must be solved for the scale factor a. The solution is $a(\xi)=H^{-1}\sin H\xi$ where $H=\sqrt{\kappa V(\phi_+)/3}$. This space-time is a 4-sphere of radius H^{-1} and is called the Euclidean de Sitter space. An important difference from the case without gravity is that this solution is closed in the ξ-direction as well and there is no asymptotic infinity. This implies that when one looks for a non-trivial solution corresponding to the bounce solution in flat space-time, one cannot set the boundary condition at $\xi=\infty$ as in Eq. (2·14). Instead, one can only require the regularity of the solution similar to Eq. (2·15) at some finite value of ξ, say $\xi=\xi_f$, where a vanishes again. Thus the boundary conditions are

$$a(0)=\dot{\phi}(0)=0 \quad \text{and} \quad a(\xi_f)=\dot{\phi}(\xi_f)=0\,. \tag{2·24}$$

Two solutions satisfying the boundary conditions are known; the Coleman-De Luccia solution (CD)[11] and the Hawking-Moss solution (HM).[21] The relation between these two solutions was studied by Jensen and Steinhardt[22] and recently by Samuel and Hiscock.[23] The definition of the CD solution is that $\phi(0)=\phi_i$ is on the true vacuum side and $\phi(\xi_f)=\phi_f$ on the false vacuum side as illustrated in Fig. 1, and $\phi(\xi)$ varies monotonically. While, the definition of the HM solution is that ϕ always stays at the top of the potential barrier; $\phi(\xi)\equiv\phi_{\rm top}$.

It is almost trivial that an HM type solution always exists. But the existence of the CD solution is not so trivial. Jensen and Steinhardt[22] showed that the CD solution exists when $-V''(\phi_{\rm top})>4H^2$. As the potential is deformed gradually to the limit $-V''(\phi_{\rm top})\to 4H^2+0$, we have found an approximate CD solution which approaches the HM solution (see § 4). If this is the unique CD solution, which seems reasonable, we can conclude that the CD solution converges asymptotically to the HM solution, $\phi(\xi)\to\phi_{\rm top}$ in the limit $-V''(\phi_{\rm top})\to 4H^2+0$.

Knowing the behavior of the classical solution, one can extrapolate the previous results to the case with gravity and evaluate the rate of the false vacuum decay. However, it is not clear if this extrapolation is meaningful, because of several reasons: First of all, since the Hamiltonian of a compact space vanishes by the constraint equation, the reasoning used in the case without gravity to relate the vacuum persistent amplitude and the path integral does not apply. Further, even if one ignores this point, since the Euclidean time interval T is at most of $O(H^{-1})$, only a finite number of the bounces can be placed far apart from each other. Then the dilute gas approximation may become invalid easily, which means the breakdown of the WKB approximation. Nevertheless, the expression for the decay rate (2·16) can be extended naturally:

$$\Gamma/V=|K|e^{-B}\,,\quad B=S_E[\bar{\phi}(\xi),\bar{a}(\xi)]-S_E[\phi\equiv\phi_+]\,, \tag{2·25}$$

where $\bar{\phi}(\xi)$ and $\bar{a}(\xi)$ are the classical solution having the lowest action among $O(4)$-symmetric ones. In the next section, we shall see that this is consistent with the wave function approach based on the Wheeler-DeWitt equation, at least when the thin-wall approximation is valid. Thus, as a working hypothesis, we assume the above extension is correct. Then, one expects the perturbation around the classical solution to have one and only one negative mode. We focus on this issue of negative mode in § 4.

§ 3. Wave function approach

The purpose of this section is to examine the validity of the WKB approximation by considering the wave function which describes the false vacuum decay with gravity. Before doing so, let us first briefly review a pathological feature in the false vacuum decay observed by Lavrelashvili, Rubakov and Tinyakov[18] (LRT) when gravity is included.

Let a non-trivial $O(4)$-symmetric Euclidean classical solution with the boundary conditions (2·24) be

$$N=1\,, \quad a=\bar{a}\,, \quad \phi=\bar{\phi}\,. \tag{3·1}$$

LRT considered $O(4)$-symmetric perturbations around this solution. Then N, a and ϕ are still the only variables of the system. Hence the perturbation may be described as

$$N=1+\nu\,, \quad a=\bar{a}+\alpha\,, \quad \phi=\bar{\phi}+\varphi\,. \tag{3·2}$$

Since the system has the Hamiltonian constraint, one must impose a gauge condition to eliminate unphysical modes from the action. The gauge condition chosen by LRT is to set $a=\bar{a}$, i.e., $\alpha=0$. Then with the help of the Hamiltonian constraint, we can eliminate ν from the second order terms of the action to obtain

$$S_{\mathrm{E}}{}^{(2)}=\pi^2\int d\xi\frac{\bar{a}^3}{\bar{Q}}\left[\dot{\bar{a}}^2\dot{\varphi}^2-\frac{\kappa\bar{a}^2V'\dot{\bar{\phi}}}{3}\varphi\dot{\varphi}+\left(\frac{\kappa\bar{a}^2V'^2}{6}+V''\bar{Q}\right)\varphi^2\right], \tag{3·3}$$

where $\bar{Q}$ is the background value of Q:

$$\bar{Q}=1-\frac{1}{3}\kappa\bar{a}^2V(\bar{\phi})=\dot{\bar{a}}^2-\frac{1}{6}\kappa\bar{a}^2\dot{\bar{\phi}}^2\,. \tag{3·4}$$

Here one finds the coefficient of the kinetic term is proportional to $\bar{a}^3\dot{\bar{a}}^2/\bar{Q}$. As a result the action can be made unboundedly large and negative if the background solution has a region $\bar{Q}<0$, by letting φ rapidly oscillate in this region. This implies the existence of infinitely many negative modes.

LRT showed that there exists a model in which the thin-wall approximation is valid and the energy scale is far below the Planck one, but the classical solution has the region $\bar{Q}<0$ at the wall. Hence they concluded that the WKB approximation would break down even at sufficiently low energy scale. LRT proposed an interpretation of this phenomenon as follows. If these negative modes are stabilized at some finite amplitude of the perturbation, a violently varying field configuration will be realized. This violently varying field configuration means the creation of a great many ϕ particles in the $\bar{Q}<0$ region during the tunneling and the tunneling rate will be drastically changed.

We now examine if the WKB approximation is really invalidated in the $\bar{Q}<0$

region, based on the wave function approach to the false vacuum decay. We let $\hbar$ appear explicitly in the rest of this section to make the order of the WKB approximation clear. The basic framework of our discussion owes to formulations developed in Refs. 24)~28).

As well known, a system with gravity is described by the Wheeler-DeWitt equation which is the quantum version of the Hamiltonian constraint. For our purpose, we may restrict our attention to $O(4)$-symmetric field configurations. Then the Wheeler-DeWitt equation takes the form,

$$H^0|\Psi\rangle=\left[\frac{1}{2\pi^2}\frac{1}{2a^3}P_\phi{}^2-\frac{1}{2\pi^2}\frac{\kappa}{12a}P_a{}^2-2\pi^2\frac{3}{\kappa}aQ(a,\ \phi)\right]|\Psi\rangle=0\,, \tag{3·5}$$

where P_a and P_ϕ are the momentum operators conjugate to a and ϕ, respectively. In the WKB approximation, we expand the wave function as $|\Psi\rangle=\exp[-W_0(a,\ \phi)/\hbar - W_1(a,\ \phi)+\cdots]$. Then in the lowest order of $\hbar$ we have

$$-\frac{1}{2\pi^2}\frac{1}{2a^3}\left(\frac{\partial W_0}{\partial\phi}\right)^2+\frac{1}{2\pi^2}\frac{\kappa}{12a}\left(\frac{\partial W_0}{\partial a}\right)^2-2\pi^2\frac{3}{\kappa}aQ=0\,. \tag{3·6}$$

If we define $\phi(\xi)$ and $a(\xi)$ by

$$\frac{\partial W_0}{\partial\phi}=2\pi^2a^3\frac{d\phi}{d\xi}\,,\quad \frac{\partial W_0}{\partial a}=-2\pi^2\frac{6a}{\kappa}\frac{da}{d\xi}\,, \tag{3·7}$$

it can be readily verified that the so determined variables satisfy the Euclidean equations of motion, (2·22) and (2·23), with the gauge condition $N=1$. Thus we obtain W_0 along the 1-parameter family of configurations as

$$W_0{}^{O(4)}(\xi)-W_0{}^{O(4)}(0)=-2\pi^2\frac{6}{\kappa}\int_0^\xi d\xi'\,\bar{a}(\xi')\bar{Q}(\xi')\,, \tag{3·8}$$

where the bar ($^-$) denotes the solution of the classical equation of motion, as before, and the superscript $O(4)$ on W_0 means it describes only the $O(4)$ symmetric configurations. Note that because of the Hamiltonian constraint, the above is equal to the classical action integral (see Eq. (2·20)) of the $O(4)$-symmetric solution ($\bar{a}(\xi)$, $\bar{\phi}(\xi)$).

In order to calculate the decay rate of false vacuum, we need to know what the initial and final states are. For simplicity, consider the situation in which the thin-wall approximation is valid as illustrated in Fig. 2. That is,

$$\bar{\phi}(\xi)\approx\begin{cases}\phi_-; & 0\le\xi<\xi_{\rm in}\,,\\ \phi_+; & \xi_{\rm out}<\xi\le\xi_f\,,\end{cases}$$

$$\bar{a}(\xi)\approx\begin{cases}\xi; & 0\le\xi<\xi_{\rm in}\,,\\ H^{-1}\cos H(\xi-\xi_f); & \xi_{\rm out}<\xi\le\xi_f\,,\end{cases} \tag{3·9}$$

where $H=\sqrt{\kappa V(\phi_+)/3}$, and we have assumed $V(\phi_-)=0$ and $H\Delta\xi=H(\xi_{\rm out}-\xi_{\rm in})\ll1$. In this case, the state expected to come out after the tunneling should contain the true vacuum bubble and which can be analytically continued to the Lorentzian expanding bubble solution. The only possibility for such a state is the 3-hypersurface Σ_f shown

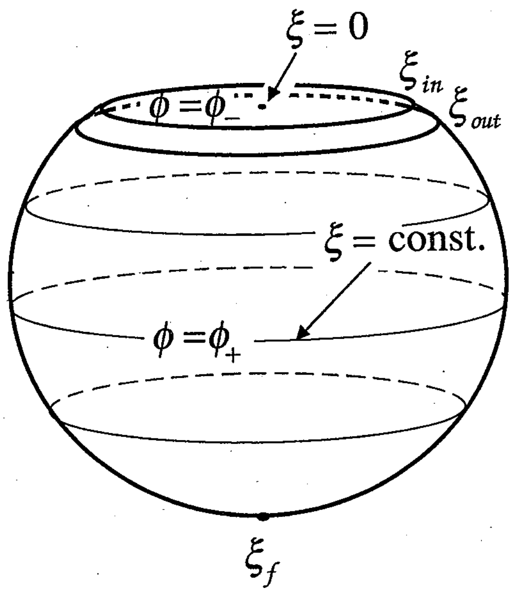

Fig. 2. Geometry of the $O(4)$-symmetric Euclidean solution when the thin-wall approximation is valid. The global topology is S^4.

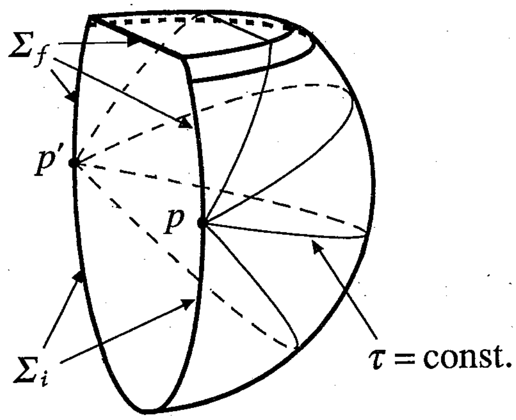

Fig. 3. A half of the Euclidean manifold and a foliation of it by the hypersurfaces parametrized by the time τ, which are relevant to the wave function describing the false vacuum decay. See text for details.

in Fig. 3. Then, as a natural extension of the flat space-time case, the 3-hypersurface Σ_i in Fig. 3 may be regarded as the initial state.

Now, since these states do not respect the $O(4)$-symmetry, the wave function described by $W_0^{O(4)}$ has no direct meaning to the false vacuum decay. To justify the identification of the hypersurfaces Σ_i and Σ_f as the initial and final states, respectively, and to understand the role of $W_0^{O(4)}$, we tentatively fix the geometry to the false vacuum de Sitter space-time and consider the bubble nucleation in this background.[29] The metric of the de Sitter space-time can be represented in several ways,

$$ds^2=d\xi^2+H^{-2}\sin^2 H\xi[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta d\varphi^2)]\,, \tag{3·10}$$

$$ds^2=(1-H^2r^2)d\tau^2+\frac{dr^2}{1-H^2r^2}+r^2(d\theta^2+\sin^2\theta d\varphi^2)\,. \tag{3·11}$$

These coordinates are related as

$$\cos H\xi=\sqrt{1-H^2r^2}\cos H\tau\,,$$

$$\sin\chi=Hr\sqrt{1-(1-H^2r^2)\cos^2(H\tau/\arcsin Hr)}\,. \tag{3·12}$$

If one were to focus only on $O(4)$-symmetric states, the coordinates (3·10) would be the most convenient. However, when considering the bubble nucleation in the false vacuum background, the static de Sitter coordinates (3·11) are more relevant, since the time coordinate τ naturally parametrizes the states connecting the initial false vacuum state and the final state with a true vacuum bubble. The important feature of the coordinates (3·11) is that the τ=const. hypersurface covers the region only inside the horizon; $r<H^{-1}$. Thus the hypersurfaces Σ_i and Σ_f have open topologies. This is not a defect since it is reasonable to exclude the possibility that the tunneling occurs coherently over a region greater than the horizon scale.

We now recover the coupling with gravity and consider the transition amplitude from the initial state; Σ_i, to the final state; Σ_f, connected by a family of the field configurations distinguished by a parameter τ, which is analogous to the static de Sitter time coordinate. Since these field configurations can be obtained simply by a certain coordinate transformation; $(\xi, \chi)\to(\tau, r)$ as

$$(N=1, N_i=0, \bar{a}(\xi), \bar{\phi}(\xi))\to(N(r;\tau), N_i(r;\tau), \bar{h}_{ij}(r;\tau), \bar{\phi}(r;\tau)), \qquad (3\cdot13)$$

they are also the solution of the Euclidean equation of motion. We note that $N_i(H^{-1};\tau)=0$ holds. Then from a generalized version of Eq. (3·8), W_0 along the 1-parameter family of configurations $(\bar{h}_{ij}(r;\tau), \bar{\phi}(r;\tau))$ is given by

$$W_0[\bar{h}_{ij}(r;\tau_f), \bar{\phi}(r;\tau_f)]-(\tau_f\to\tau_i)=4\pi\int_M d\tau L_E{}^{(3+1)}$$

$$=\int_M L_E{}^{(4)}-(\text{surface term}), \qquad (3\cdot14)$$

where M refers to *a half* of the Euclidean manifold as shown in Fig. 3, $L_E{}^{(4)}$ is the original Euclidean scalar-Einstein-Hilbert Lagrangian and $L_E{}^{(3+1)}$ is the Euclidean Lagrangian in the (3+1)-formalism, i.e., $L_E{}^{(3+1)}=-ip^i dq_i/d\tau+H(p^i, q_i)$ for general canonical variables (p^i, q_i). These are different by the surface term. Note that, by construction, W_0 is given by the action integral of the Lagrangian in the (3+1)-formalism.

It should be clear now that the volume integral term in the last line of Eq. (3·14) is given by a half of the action of the $O(4)$-symmetric solution. That is

$$\int_M L_E{}^{(4)}=\frac{1}{2}(W_0{}^{O(4)}(\xi_f)-W_0{}^{O(4)}(0))$$

$$=\frac{1}{2}S_E[\bar{\phi}(\xi), \bar{a}(\xi)]. \qquad (3\cdot15)$$

On the other hand, we need a little more consideration to evaluate the surface term in Eq. (3·14). The explicit form of the surface term is

$$(\text{surface term})=\frac{1}{\kappa}\left[\int d\sigma_\mu(n^\mu\nabla_\nu n^\nu)-\int d\sigma_\nu(n^\mu\nabla_\mu n^\nu)\right], \qquad (3\cdot16)$$

where n_μ is the vector unit normal to the τ=const. hypersurface and $d\sigma_\mu$ is the surface element on the boundary of M. In the present case, the boundary of the region M consists of the three parts; the initial hypersurface Σ_i, the final hypersurface Σ_f, and the hypersurface Σ_h which corresponds to the horizon, $r=H^{-1}$. It is easily seen that only Σ_h gives a non-vanishing surface term for an $O(4)$-symmetric solution. To estimate its contribution, consider the boundary surface of a τ=const. hypersurface, indicated by p and p' in Fig. 3. Though p and p'seem to be discrete from each other in the figure, they are, in fact, one connected 2-dimensional surface and topologically S^2. Thus Σ_h has the topology $D^1\times S^2$. When the thin-wall approximation is valid, the configuration near Σ_h is given by $\phi=\phi_+$ and the false vacuum de Sitter space-time with the static metric (3·11). Since the vector n^μ is expressed in terms of $(\tau, r, \theta, \varphi)$ components as

$$n^\mu=\left(\frac{1}{\sqrt{1-H^2r^2}}, 0, 0, 0\right), \tag{3·17}$$

Eq. (3·16) is evaluated as

$$\begin{aligned}(\text{surface term})&=-\frac{1}{\kappa}\int_{\Sigma_h} d\sigma_r(n^\tau\nabla_\tau n^r)\\ &=-\frac{4\pi}{\kappa}\int_0^{\pi/H} d\tau H^2r^3|_{r=H^{-1}}=-\frac{4\pi^2}{\kappa H^2}.\end{aligned} \tag{3·18}$$

This is just a half of the background action $S_E[\phi\equiv\phi_+]$.

Combining the results in Eqs. (3·15) and (3·18), and noting that the probability amplitude is given by the ratio of the square of the initial and final wave functions, we finally find the decay rate as

$$\Gamma/V\sim\exp(-S_E[\bar{\phi}(\xi), \bar{a}(\xi)]+S_E[\phi\equiv\phi_+]), \tag{3·19}$$

which is completely in agreement with the expression obtained by the EPI method, Eq. (2·25), in the lowest WKB order.

Now, we must examine if there is any point in the above procedure where the WKB approximation could be invalidated. It would appear as a singularity in the next order term W_1 in the wave function, corresponding to the factor K in the EPI approach. If one were to evaluate the wave function along the 1-parameter family of the $O(4)$-symmetric configurations $(\bar{a}(\xi), \bar{\phi}(\xi))$, one might expect to encounter a singularity in W_1 at $\bar{Q}=0$, since $dW_0/d\xi=0$ there which is analogous to the classical turning point in a one-dimensional quantum mechanical system.*) However, in the above we have arrived at the same expression for the decay rate as in the EPI approach by evaluating the wave function along the family of configurations $(\bar{h}_{ij}(r;\tau), \bar{\phi}(r;\tau))$ parametrized by τ. This family has nothing to do with the configuration which satisfies $\bar{Q}=0$, where the breakdown of the WKB approximation was claimed by LRT. Therefore it is most probable that the WKB approximation is valid everywhere along this family. The precise proof of it is a remaining issue, but we conclude that, at least for the wave function considered here, no physical significance can be attributed to the $\bar{Q}<0$ region.

§ 4. Hamiltonian formalism of EPI with gravity

In the previous section, we have seen that the $\bar{Q}<0$ region of the $O(4)$-symmetric solution does not seem to invalidate the WKB approximation. In order to strengthen

*) One can show by a careful consideration that this expectation is generally incorrect for a multi-dimensional system. We plan to come back to this issue in future publication.

this conclusion, in this section, we develop a method to investigate the number of negative modes associated with such a Euclidean solution and show that one can choose a gauge in which no pathological behavior of the mode functions appears in the $\bar{Q}<0$ region.

The existence of the $\bar{Q}<0$ region is due to the fact that the kinetic term of the Einstein-Hilbert action has an indefinite metric, which is closely related to the existence of the conformal mode. Although this gives rise to various difficulties when considering quantum gravity, it should not do any harm to quantum tunneling phenomena at sufficiently low energy scale. As a method to take care of the conformal mode, Gibbons, Hawking and Perry[19] proposed the conformal rotation technique in which the integration contour for the conformal mode is taken to be parallel to the imaginary axis, and it was justified in the case of linearized gravity.[20] In what follows, we shall see that a similar situation arises in our case and as a result the pathology associated with the $\bar{Q}<0$ region disappears.

Before presenting our formulation, we mention that the $\bar{Q}<0$ region appears in almost all the systems when gravity is included. This is because any $O(4)$-symmetric solution has the 3-surface at which $\dot{\bar{a}}=0$ where the spatial volume is maximum, which implies $\bar{Q}<0$ unless $\dot{\bar{\phi}}$ exactly vanishes there (see Eq. (3·4)). Thus except for the HM solution, for which gravity is effectively decoupled, it is practically impossible to analyze the number of negative modes in the action (3·3). We also note that in the gauge $a=\bar{a}$ chosen by LRT, the Hamiltonian constraint (2·23) implies

$$N^2=\frac{\dot{\bar{a}}^2-\dfrac{1}{6}\kappa\bar{a}^2\dot{\phi}^2}{\bar{Q}-\dfrac{1}{3}\kappa\bar{a}^2\varDelta V(\phi)}, \tag{4·1}$$

where $\varDelta V(\phi):=V(\phi)-V(\bar{\phi})$. Thus, given an arbitrary configuration of ϕ, the lapse function diverges in general near the point $\bar{Q}=0$. The origin of this trouble is that the maximum 3-volume of the geometry, which is a physical gauge-invariant quantity, is automatically fixed at the background value in the gauge $a=\bar{a}$. In fact, a similar trouble always appears for any gauge-fixing at the Lagrangian level when only the configuration variables are involved.

In principle, these difficulties associated with LRT's gauge can be circumvented and the normalizable mode functions can be constructed by adding a small imaginary part to the time coordinate; $\xi\to\xi\pm i\epsilon$ and requiring a suitable analyticity of the perturbation φ off the real axis of the complex ξ-plane. However, if the regularity of the perturbation can be guaranteed by setting another appropriate gauge condition, it is not necessary to complexify the time coordinate to determine the mode functions. We shall see below that such a gauge condition exists. For this purpose, we extend the results from Makino and Sasaki,[30] who gave a canonical formulation of the scalar field perturbation coupled with gravity in a spatially flat universe, to the spatially closed Euclidean background.

We begin with the Euclidean action written in the canonical (3+1)-form,

$$S_E=\int d^4x\Big[-i\pi^{ij}\dot{h}_{ij}-i\pi_\phi\dot{\phi}+Nh^{1/2}\Big\{2\kappa h^{-1}\pi^{ij}\pi_{ij}-\kappa h^{-1}\pi^2-\frac{1}{2\pi}{}^{(3)}R+\frac{1}{2}h^{-1}\pi_\phi{}^2$$

$$+\frac{1}{2}h^{ij}\nabla_i\phi\nabla_j\phi+V(\phi)\Big\}+N^j\{\pi_\phi\nabla_j\phi-2\nabla^i\pi_{ij}\}\Big], \tag{4·2}$$

where the lapse function N and the shift vector N^i are regarded as Lagrange multipliers. On the $O(4)$-symmetric background, the perturbations can be expanded in terms of scalar, vector and tensor harmonics on the 3-sphere. It is known that these three types of perturbations are completely decoupled from each other in the action. As we are interested in the mode coupled to the scalar field, we focus only on the scalar-type perturbations. As for the $O(4)$-symmetric background, we choose the gauge $\bar{N}=1$ as before and denote the time coordinate by ξ. Then the conjugate momenta to the background variables, $\bar{a}$ and $\bar{\phi}$, are given by

$$\bar{P}_a=-i\frac{6}{\kappa}\bar{a}\dot{\bar{a}}\,,\quad \bar{P}_\phi=i\bar{a}^3\dot{\bar{\phi}}\,. \tag{4·3}$$

Note that the conjugate momenta take pure imaginary values on the Euclidean background.

The scalar-type perturbation of the configuration variables can be described as

$$h_{ij}=(\bar{a}^2+2h_L)\gamma_{ij}+2\Big(-\Delta^{-1}D_iD_j+\frac{1}{3}\gamma_{ij}\Big)h_T\,,$$

$$\phi=\bar{\phi}+\varphi\,, \tag{4·4}$$

and the lapse and shift as

$$N=1+\bar{a}^{-2}A\,,\quad N^i=\bar{a}^{-2}D^iB\,, \tag{4·5}$$

where γ_{ij}, D_i and Δ are the metric, the covariant derivative and the Laplacian, respectively, on the unit 3-sphere. The variables h_L and h_T represent the trace and traceless parts, respectively, of the metric perturbation. Then the conjugate momentum variables are written as

$$\pi^{ij}=\Big(\frac{\bar{P}_a}{6\bar{a}}+\frac{1}{6}P_L\Big)\gamma^{ij}+\frac{3}{4}\Big(-\Delta^{-1}D^iD^j+\frac{1}{3}\gamma^{ij}\Big)P_T\,,$$

$$\pi_\phi=\bar{P}_\phi+P_\varphi\,, \tag{4·6}$$

where P_L, P_T and P_φ are so defined that they are canonically conjugate to h_L, h_T and φ, respectively.

We now expand all the perturbation variables by the harmonics on the unit 3-sphere,[31]

$$q(\xi,x)=\sum_{klm}q_{klm}(\xi)Y_{klm}(x)\,, \tag{4·7}$$

where q represents a perturbation variable and

$$(\Delta+\boldsymbol{k}^2)Y_{klm}=0;\ \boldsymbol{k}^2:=k(k+2)\,,\quad k=0,1,2,\cdots. \tag{4·8}$$

We assume the harmonics Y_{klm} are orthonormalized. After this expansion, say, the traceless part of the metric perturbation is expressed as

$$\left(-\Delta^{-1}D^iD^j+\frac{1}{3}\gamma^{ij}\right)h_T=\sum_{k,l,m}h_{T,klm}Y_{klm,ij}\,, \tag{4·9}$$

where $Y_{ij}:=(\boldsymbol{k}^2)^{-1}D_iD_jY+(1/3)\gamma_{ij}Y$ is the traceless tensor constructed from the scalar harmonic function. We remark that Y_{ij} does not exist for the $k=0$ and $k=1$ modes.

Using the orthonormality of the harmonics, the quadratic part of the action (4·2) reduces to

$$S_E{}^{(2)}=\sum_{k,l,m}\int d\xi(-iP_{L,klm}\dot{h}_{L,klm}-iP_{T,klm}\dot{h}_{T,klm}-iP_{\varphi,klm}\dot{\varphi}_{klm}+H_{klm})\,, \tag{4·10}$$

where H_{klm} is the Hamiltonian for the (k, l, m)-mode perturbation. Since each (k, l, m)-mode can be treated separately, henceforth we omit the subscript klm for simplicity. After a bit tedious calculation, the explicit form of the Hamiltonian H is obtained as

$$H=H_0+(A+3h_L)\lambda_1+\boldsymbol{k}^2B\lambda_2\,, \tag{4·11}$$

where

$$\begin{aligned}H_0=&-\frac{\kappa\bar{a}}{12}P_L{}^2+\frac{\kappa\bar{P}_a}{3\bar{a}^2}P_Lh_L-\frac{1}{\bar{a}^3}\left\{-\frac{12\bar{P}_\phi{}^2}{\bar{a}^4}+\frac{2\kappa\bar{P}_a{}^2}{3\bar{a}^2}-\frac{5}{\kappa}\boldsymbol{k}^2+\frac{12}{\kappa}\right\}h_L{}^2\\&+\frac{3\kappa\bar{a}}{4}\frac{\boldsymbol{k}^2}{\boldsymbol{k}^2-3}P_T{}^2+\frac{\kappa\bar{P}_a}{3\bar{a}^2}P_Th_T\\&-\frac{1}{\bar{a}^3}\frac{\boldsymbol{k}^2-3}{\boldsymbol{k}^2}\left\{-\frac{2\bar{P}_\phi{}^2}{3\bar{a}^4}-\frac{\kappa\bar{P}_a{}^2}{27\bar{a}^2}+\frac{\boldsymbol{k}^2+12}{9\kappa}\right\}h_T{}^2\\&+\frac{4}{3\kappa\bar{a}^3}(\boldsymbol{k}^2-3)h_Th_L+\frac{1}{2\bar{a}^3}P_\varphi{}^2+\frac{\bar{a}}{2}(\boldsymbol{k}^2+\bar{a}^2V'')\varphi^2-\frac{6\bar{P}_\phi}{\bar{a}^5}P_\varphi h_L\end{aligned} \tag{4·12}$$

and

$$\begin{aligned}\lambda_1=&-\frac{\kappa\bar{P}_a}{6\bar{a}^2}P_L+\frac{\bar{P}_\phi}{\bar{a}^5}P_\varphi-\frac{1}{\bar{a}^3}\left\{\frac{3\bar{P}_\phi{}^2}{\bar{a}^4}-\frac{\kappa\bar{P}_a{}^2}{6\bar{a}^2}+\frac{2}{\kappa}(\boldsymbol{k}^2-3)\right\}h_L\\&-\frac{2}{3\kappa\bar{a}^3}(\boldsymbol{k}^2-3)h_T+\bar{a}V'\varphi\,,\\\lambda_2=&-\frac{1}{3}P_L+P_T+\frac{\bar{P}_a}{3\bar{a}^3}h_L+\frac{8\bar{P}_a}{9\bar{a}^3}\frac{\boldsymbol{k}^2-3}{\boldsymbol{k}^2}h_T+\frac{\bar{P}_\phi}{\bar{a}^2}\varphi\,.\end{aligned} \tag{4·13}$$

As the action does not contain the time derivatives of A and B, which are the lapse and shift perturbations, respectively, there appear the following two constraint equations:

$$\lambda_1\approx 0\,,\quad \lambda_2\approx 0\,, \tag{4·14}$$

which correspond to the Hamiltonian and momentum constraints, respectively. Here and in the rest of this section, the approximate equality ($\approx$) means weak equality. It is then easily verified that these constraints are of the first class.

To handle the present first class constrained system, we resort to the gauge-fixing method, in which one requires the dynamical variables to satisfy necessary and sufficient number of extra constraints by hand to kill all the gauge degrees of freedom. With this procedure, all the constraints become of the second class. In this context LRT's treatment is equivalent to take $h_L \approx 0$ as the gauge condition. Here, we choose the gauge condition in such a way that the 3-metric perturbation satisfies the longitudinal gauge condition, i.e., $A=-h_L$ and $B=h_T=0$. To do so, we take

$$\chi_1=\bar{a}^2 P_L-3\bar{P}_\phi\varphi-\frac{\bar{P}_a}{\bar{a}}h_L\approx 0\,,\quad \chi_2=\frac{1}{\bar{a}^2}h_T\approx 0\,, \tag{4·15}$$

as our gauge condition. Then the consistency condition requires

$$\chi_3=A+h_L+\frac{1}{3}h_T\approx 0\,,\quad \chi_4=B\approx 0\,. \tag{4·16}$$

Certainly these conditions are in accord with the longitudinal gauge. We can easily verify that all the constraints are now of the second class. This allows us to replace the weak equality by the strong one. Hence we have

$$h_T=0\,,\quad P_T=0\,,\quad P_L-\frac{\bar{P}_a}{\bar{a}^3}h_L-\frac{3\bar{P}_\phi}{\bar{a}^2}\varphi=0\,,$$

$$\frac{\bar{P}_\phi}{\bar{a}^4}P_\varphi-\frac{1}{\bar{a}^2}\left(\frac{2}{\kappa}(\boldsymbol{k}^2-3)+3\frac{\bar{P}_\phi{}^2}{\bar{a}^4}\right)h_L-\left(\frac{\kappa\bar{P}_a\bar{P}_\phi}{2\bar{a}^3}-\bar{a}^2V'\right)\varphi=0\,. \tag{4·17}$$

Since there are four relations among six dynamical variables for each (k, l, m)-mode in general (for $k=0$ and $k=1$, this is not so; we will come back to this point later), there are only two physical degrees of freedom among the variables. For convenience, we choose the pair (h_L, P_L) as these variables. Note that a different choice of a pair will be related to our choice by a certain canonical transformation. Then, substituting relations (4·17) into the action (4·10), and integrating by parts appropriately, we obtain a rather simple expression,

$$S^{(2)}=(\boldsymbol{k}^2-3)\int d\xi\Big[-iP\dot{h}_L+\frac{\kappa^2\bar{P}_\phi{}^2}{8\bar{a}^3}P^2$$
$$+\frac{2\bar{a}}{\kappa^2\bar{P}_\phi{}^2}\left\{\boldsymbol{k}^2-4-\frac{\kappa\bar{a}^3\bar{P}_a}{3}\left(\frac{\kappa\bar{P}_a}{3\bar{a}^5}-\frac{V'}{\bar{P}_\phi}\right)\right\}h_L{}^2\Big]\,, \tag{4·18}$$

where we have introduced the variable P defined by

$$P:=-\frac{2\bar{a}^4}{3\kappa\bar{P}_\phi{}^2}P_L\,, \tag{4·19}$$

in order that it be canonically conjugate to h_L.

Although one could use the action (4·18) for the analysis of the mode functions, it turns out that the action can be transformed in a much simpler form by the following canonical transformation:

$$P_h=\frac{\kappa\bar{P}_\phi}{2}P+\frac{2\bar{a}^6V'}{\kappa\bar{P}_\phi{}^2}h_L\,,$$

$$h=\frac{2}{\kappa\bar{P}_\phi}h_L\,. \tag{4·20}$$

The resulting action takes the form,

$$S^{(2)}=(\boldsymbol{k}^2-3)\int d\xi\Big[-iP_h\dot{h}+\frac{1}{2\bar{a}^3}P_h{}^2+\frac{\bar{a}}{2}\Big\{\boldsymbol{k}^2-6+\bar{a}^2\Big(\mu^2(\bar{\phi})+\frac{1}{6}\bar{R}\Big)\Big\}h^2\Big], \tag{4·21}$$

where $\bar{R}$ is the background 4-scalar curvature,

$$\bar{R}=6\Big[-\frac{\ddot{\bar{a}}}{\bar{a}}-\Big(\frac{\dot{\bar{a}}}{\bar{a}}\Big)^2+\frac{1}{\bar{a}^2}\Big], \tag{4·22}$$

and μ^2 is given by

$$\mu^2(\bar{\phi})=-V''+2\frac{\ddot{\bar{\phi}}\Big(\ddot{\bar{\phi}}+2\dfrac{\dot{\bar{a}}}{\bar{a}}\dot{\bar{\phi}}\Big)}{\dot{\bar{\phi}}^2}-\frac{\kappa}{2}\dot{\bar{\phi}}^2\,. \tag{4·23}$$

We should mention that the canonical transformation (4·20) contains imaginary coefficients because $\bar{P}_\phi$ is pure imaginary in the Euclidean background. However, this is in fact a correct choice in order to preserve the Hermiticity of canonical variables; the Hermiticity is guaranteed if all the transformation coefficients become real when analytically continued back to the Lorentzian space-time.

Here we comment on the physical degrees of freedom of $k=0$ and $k=1$ modes. For $k=0$, the variables h_T and P_T are absent from the beginning. However, at the same time, the shift vector perturbation B is also absent. As a result the number of physical degrees of freedom turns out to be the same as that for $k\geq 2$ modes. Thus the action $S^{(2)}$ given in Eq. (4·18) is perfectly relevant for the $k=0$ mode as well. As for $k=1$, on the other hand, h_T and P_T are still absent from the beginning, while B comes into play. Hence there remains no physical degree of freedom. This can be also seen from Eq. (4·18); the action $S^{(2)}$ vanishes identically for $k=1$ since $\boldsymbol{k}^2=k(k+2)=3$. Thus only allowable $k=1$ mode perturbations are global coordinate transformation which are the algebraic solution of the constraint equations and which are the counterpart of the zero eigenmodes of the bounce in flat space-time. In this respect, it is interesting that the inclusion of gravity naturally guarantees the exclusion of zero modes from the path integral.

Now, given the final form of the action (4·21), we consider the path integral,

$$Z^{(2)}:=\int[dP_h dh]\exp[-S^{(2)}]\,. \tag{4·24}$$

In order to obtain the eigenvalue equation for the mode functions, one needs to path-integrate over the momentum P_h first. It can be readily done for the $k\geq 2$ modes. However, for the $k=0$ mode, we encounter a problem; the overall sign of the action is *negative,* hence the integration over P_h cannot be done along the real axis. Thus we have to analytically continue the path integral somehow to guarantee its convergence. We assume this is effectively done by taking the integration contour of P_h along the imaginary axis. To keep the integration measure invariant, it should be at least necessary to associate the rotation of h; $h\to -ih$ with that of P_h; $P_h\to iP_h$. In addition, we argue that this analytic continuation will give rise to a factor proportional to the imaginary unit i.

Let us explain the reason for the above claim. The path integral is ordinarily defined as the continuum limit of integrations with respect to canonical variables at discrete points. As discussed in the beginning of § 2, in this discretized path integral, the number of momentum degrees of freedom to be integrated is different from that of coordinate degrees of freedom by one (whether it is one more or one less depends on whether the boundary is fixed or to be integrated). Therefore,if we define the path integral over P_h by rotating its contour to the imaginary axis, it gives rise to a factor i to the integral measure of each coordinate degree of freedom, save one imaginary unit due to the difference in the number of momentum and coordinate degrees of freedom. If one takes the continuum limit after this assignment of the imaginary factor to each momentum and coordinate degree of freedom, the factor i will arise in the resultant normalization factor. In addition, this also implies that the number of negative modes may depend on the choice of physical canonical variables, since if one can find a canonical transformation by which the kinetic term of the transformed variables is positive definite, the factor i should not be contained in the normalization factor but should arise from the negative mode in the path integral.

Although our argument may sound rather naive and we do not have a precise proof for the conjecture, we are confident that it is true. As a supporting evidence for it, a system in which this conjecture holds perfectly well is discussed in Appendix B.

Keeping in mind the above procedure for the $k=0$ mode, the momentum path integrals are done to yield

$$Z^{(2)}=N'\int[dh]\exp\left[-\frac{|\boldsymbol{k}^2-3|}{2}\int d\xi\bar{a}^3\left\{\dot{h}^2+\left(\frac{\boldsymbol{k}^2-6}{\bar{a}^2}+\mu^2+\frac{1}{6}\bar{R}\right)h^2\right\}\right], \qquad (4\cdot25)$$

where N' is a normalization constant which contains one imaginary unit for $k=0$, while none for $k\geq 2$. The action (4·25), as expected, shows no special feature in the $\bar{Q}<0$ region. Furthermore, in Appendix A, it is shown that the regularity of h at the north and south poles of the background (i.e., at $\bar{a}=0$) guarantees that of the perturbed geometry. Hence we can evaluate the number of negative modes by analyzing the eigenvalue equation derived from this action. In particular, since the kinetic term is positive definite, the number of negative modes is at most finite, in contrast with LRT's argument.

From now on, we concentrate on the $O(4)$-symmetric (i.e., $k=0$) perturbations and study the number of negative modes around the HM solution and the CD solution near the HM limit, which can be discussed analytically.

In the case of the HM solution, the number of negative modes can be evaluated by the action (3·3), because the system becomes equivalent to the one of a free scalar field on the fixed de Sitter background. Substituting the HM solution,

$$\bar{a}(\xi)=\frac{1}{H}\sin H\xi\,,\quad \bar{\phi}(\xi)=\phi_{\rm top}\,;\quad H=\sqrt{\kappa V(\phi_{\rm top})/3}\,, \qquad (4\cdot26)$$

into Eq. (3·3), and noting that $\bar{Q}=\dot{\bar{a}}^2$, the eigenvalue equation for φ is derived to be

$$\left[-\frac{1}{\bar{a}(\xi)^3}\frac{d}{d\xi}\bar{a}(\xi)^3\frac{d}{d\xi}+V''(\phi_{\rm top})\right]\varphi=\lambda\varphi\,. \tag{4·27}$$

Using the normalizability condition of the mode functions, we find the eigenvalues as

$$\lambda_n=n(n+3)H^2+V''(\phi_{\rm top});\ n=0,1,2,\cdots. \tag{4·28}$$

Combining this result with the discussion given near the end of § 2, we find that there is one and only one negative mode when $-V''(\phi_{\rm top})\le 4H^2$, i.e., if the HM solution is the unique solution satisfying the boundary condition, though at $-V''(\phi_{\rm top})=4H^2$ a zero mode appears in addition. This result supports the validity of the decay rate formula (2·25). As we shall see soon, there exists a CD solution which approaches the HM solution asymptotically at $-V''(\phi_{\rm top})=4H^2+0$. The appearance of the zero mode at this limit implies that the HM solution splits into two different classical solutions smoothly. Thus from these facts, we conclude that the action of the CD solution is continuously connected to that of the HM solution at $-V''(\phi_{\rm top})=4H^2$, which was previously shown by Samuel and Hiscock[23] numerically. On the other hand, when $-V''(\phi_{\rm top})>4H^2$, more than two negative modes appear. However, it is not a counter example of the hypothesis that only one negative mode exists, since the CD solution will have smaller value of the action than the HM solution.[23]

Now we consider the CD solution near the HM limit. In this case, we investigate the eigenmodes of the action (4·25). In order to find a solution in this limit, we assume the potential around $\phi=\phi_{\rm top}$ to have the following form,

$$V(\phi)=V(\phi_{\rm top})-\frac{M^2}{2}(\phi-\phi_{\rm top})^2+\frac{m}{3}(\phi-\phi_{\rm top})^3+\frac{\nu}{4}(\phi-\phi_{\rm top})^4+\cdots, \tag{4·29}$$

where $M^2=4H^2(1+\chi^2)$ with $\chi\ll 1$. Then, after a bit tedious manipulation, the solution is found to be

$$\bar{a}(\xi)=\frac{1}{\tilde{H}}\sin\tilde{H}\xi\left[1+\varepsilon^2\frac{\kappa'}{8}\sin^2\tilde{H}\xi+O(\varepsilon^3)\right],$$

$$\bar{\phi}(\xi)=\phi_{\rm top}+\varepsilon H\left[-\cos\tilde{H}\xi+\varepsilon\frac{\mu'}{12}(1-2\cos^2\tilde{H}\xi)\right.$$

$$\left.+\varepsilon^2\left(\frac{3\kappa'-4\nu'}{56}\sin^2\tilde{H}\xi-\frac{\mu'^2}{36}\cos^2\tilde{H}\xi\right)\cos\tilde{H}\xi+O(\varepsilon^3)\right], \tag{4·30}$$

where $\tilde{H}:=H(1+\kappa'\varepsilon^2/24)$, $\varepsilon^2:=84\chi^2/(16\kappa'+9\nu')$, $\kappa':=\kappa H^2$, $\mu':=m/H$, $\nu':=\nu+\mu'^2/18$. As seen from the definition of ε, there is a condition on the form of the potential for the existence of the CD solution in this limit:

$$\nu'+\frac{16}{9}\kappa'=\nu+\frac{m^2}{18H^2}+\frac{16}{9}\kappa H^2>0\,, \tag{4·31}$$

that is, ν cannot be too large and negative. We do not know what happens to the CD solution if this condition is not satisfied. Here, we assume it is satisfied.

Substituting the above solution into the action (4·25), we obtain the eigenvalue equation for h,

$$\left[-\frac{1}{\bar{a}^3}\frac{d}{d\xi}\bar{a}^3\frac{d}{d\xi}+(c_0+c_1\cos\tilde{H}\xi+c_2\cos^2\tilde{H}\xi)H^2+O(\varepsilon^3)\right]h=\tilde{\lambda}h\,;$$

$$c_0:=\left(\frac{10}{7}\kappa'+\frac{1}{18}\mu'^2+\frac{3}{7}\nu'\right)\varepsilon^2\,,$$

$$c_1:=-\frac{2}{3}\mu'\varepsilon\,,$$

$$c_2:=\left(-\frac{18}{7}\kappa'-\frac{1}{6}\mu'^2+\frac{3}{7}\nu'\right)\varepsilon^2\,. \tag{4·32}$$

In the first non-vanishing order of χ, the lowest eigenvalue of this equation is found to be

$$\tilde{\lambda}_0=\frac{24}{5}\chi^2H^2\,. \tag{4·33}$$

This is positive definite. Consequently, the action (4·25) has no negative mode. Note in passing that, although Eq. (4·32) involves the other potential parameters (m and ν) as well, $\tilde{\lambda}_0$ depends only on χ.

Thus provided that N' in Eq. (4·25) contains the imaginary unit, the path integral over the perturbation around the CD solution is proportional to i. This again supports the formula (2·25) and the fact that the action of the CD solution is lower than that of the HM solution. In fact, if we take the $\chi\to 0$ limit, the HM solution (4·26) and the CD solution (4·30) merges smoothly and the zero mode appears in both of Eqs. (4·27) and (4·32), but the former has a negative mode while the latter has none. Thus in this limit, our conjecture is justified. In this respect, although the uniqueness of the negative mode does not hold in the false vacuum decay with gravity, the property that the path integral around the bounce should give one imaginary unit seems to be maintained.

§ 5. Conclusion

We have carefully investigated the decay of false vacuum under the presence of gravity, with particular emphasis on the issue of negative modes in the path integral over fluctuations around the $O(4)$-symmetric Euclidean classical solution. In a situation when the WKB approximation is valid, the path integral over fluctuations must yield an imaginary factor. In the case of field theory on flat space-time, this is guaranteed by the existence of a unique negative mode. In the case with gravity, however, Lavrelashvili, Rubakov and Tinyakov[18] (LRT) argued that infinite number of negative modes may appear even at low energy far below the Planck scale, hence renders the WKB approximation invalid.

In this paper, we have approached this issue of negative modes from two different points of view: (1) Analysis of the wave function which describes the tunneling process based on the Wheeler-DeWitt equation, and (2) Hamiltonian formalism of the path integral over fluctuations around the Euclidean background.

From (1), we have found that, although the validity of the WKB approximation could be a bit delicate issue if the wave function were to be evaluated along a 1-parameter family of the $O(4)$-symmetric configurations, the wave function along a family of configurations appropriate for the false vacuum decay shows no sign of the breakdown of the WKB approximation, and the resultant decay rate coincides precisely with that expected from the Euclidean path integral method.

From (2), we have argued that the origin of LRT's claim is due to their inadequate choice of gauge, inevitably implied by the Lagrangian formalism, in which it is impossible to take the variation of the maximum 3-volume of the $O(4)$-symmetric hypersurfaces. Then, based on the Hamiltonian formalism, we have derived the action for the perturbation around the Euclidean background in another choice of gauge, which is free from any pathological feature. With the help of this action, we have evaluated the number of negative modes in the Hawking-Moss solution and the Coleman-De Luccia solution in the Hawking-Moss limit.

We have found that the number of negative modes does depend on the choice of canonical variables. However, we have presented a conjecture that its difference is to be canceled by the difference in the path integral measure and have shown that this conjecture holds for the CD solution in the limit of HM solution. Consequently, we have found the resultant path integral have the same number of the imaginary factor, namely just one i, which is in agreement with the straightforward extension of the Euclidean path integral formula for the decay rate in flat space-time.

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Appendix A
——*Regularity of Mode Functions at $\bar{a}=0$*——

In this Appendix, we examine the behavior of the mode functions for h of Eq. (4·25) in the neighborhood of the north and south poles, i.e., at $\bar{a}=0$, and show that the regularity of the mode functions guarantees that of the geometry there.

First, let us consider the regularity of the mode functions for h. Near the poles, the background solution takes the form,

$$\bar{a}=\xi+O(\xi^3),$$

$$\bar{\phi}=\phi_i+\frac{V'(\phi_i)}{8}\xi^2+O(\xi^4), \tag{A·1}$$

where we have set $\xi=0$ and $\bar{\phi}=\phi_i$ at $\bar{a}=0$. Substituting Eq. (A·1) into Eq. (4·25), the latter reduces to

$$\sim\int[dh]\exp\left[-\frac{|\boldsymbol{k}^2-3|}{2}\int d\xi\{\xi^3 h'^2+\xi(\boldsymbol{k}^2+O(\xi^2))h^2\}\right]. \tag{A·2}$$

Hence the eigenvalue equation for h has the following asymptotic form at $\xi=0$:

$$\left[-\frac{1}{\xi^3}\frac{d}{d\xi}\xi^3\frac{d}{d\xi}+\frac{\boldsymbol{k}^2}{\xi^2}+O(1)\right]h=\lambda h\,. \tag{A·3}$$

This implies the asymptotic behavior of the mode functions as

$$h\propto\xi^k\,,\quad\text{or}\quad h\propto\xi^{-k-2}\,, \tag{A·4}$$

and the former should be imposed for the regularity of h.

On the other hand, the regularity of the scalar curvature at $\bar{a}=0$ requires

$$\delta R\propto g^{ij}\ddot{h}_{ij}\propto\xi^k Y_k\,. \tag{A·5}$$

In terms of the metric perturbation, this is equivalent to

$$h_L\propto\xi^{k+4}\,. \tag{A·6}$$

From Eq. (4·20), one has $h_L\propto\xi^4 h$. Hence, Eq. (A·6) implies $h\propto\xi^k$, which is just the regularity condition of h we have required. Thus the regularity of the geometrical perturbations at $\bar{a}=0$ is guaranteed.

Appendix B
——*Canonical Transformation and the Number of Negative Modes*——

In § 4, we have seen that the normalization constant N' in Eq. (4·25) should have one imaginary factor i for the $O(4)$-symmetric (i.e., $k=0$) mode. We have argued that the analytic continuation of the path integral over P_h should give rise to this imaginary factor. We have then conjectured that the number of negative modes may change under the canonical transformation of variables, but the normalization factor of the transformed system contains a factor which cancels the effect of the change and the resultant value of the path integral remains invariant. In a sense, this is analogous to gauge field theory, in which the path integral over unphysical degrees of freedom is canceled by that over the ghost term and/or the Faddeev-Popov determinant. In this Appendix, we present an analysis of a system which retains some generality and which is perfectly in accordance with our conjecture.

We consider a system which has features in common with the $O(4)$-symmetric perturbation around the HM solution. Let us assume that the Euclidean background is given by the unit $(N+1)$-dimensional sphere,

$$ds_E{}^2=d\tau^2+a(\tau)^2 d\Omega_N{}^2;$$

$$a(\tau)=\cos\tau\,,\quad -\frac{\pi}{2}\le\tau\le\frac{\pi}{2}\,, \tag{B·1}$$

where $d\Omega_N{}^2$ is the metric on the unit N-sphere ($N=0, 1, 2, \cdots$). Assume also that the Euclidean action for the $O(N)$-symmetric perturbation around this background is given by

$$S_E=\int d\tau\left[-ip\dot{q}+\frac{1}{2a^N}p^2-\frac{a^N}{2}\omega^2q^2\right]. \tag{B·2}$$

In this system, the eigenvalue equation for q takes the form,

$$-\frac{1}{a^N}(a^N\dot{q}_n)^{\cdot}-\omega^2q_n=\lambda_nq_n\,, \tag{B·3}$$

and the eigenvalues are easily obtained as

$$\lambda_n=n(n+N)-\omega^2\,,\quad (n=0,1,2,3,\cdots)\,. \tag{B·4}$$

Thus there exists one and only one negative mode in the system if $0<\omega^2\leq N+1$, which we assume in the following. Then after a suitable analytic continuation, the path integral,

$$Z:=\int[d\pi dX]e^{-S_E} \tag{B·5}$$

will be proportional to i.

Now we consider a canonical transformation of the variables and examine if our conjecture on the factor i of the normalization constant holds or not. Let the canonical transformation be

$$\begin{aligned}p&=\alpha\Pi+\beta X\,,\\ q&=\gamma\Pi+\delta X\,,\end{aligned} \tag{B·6}$$

where $\alpha\delta-\beta\gamma=1$. The action is rewritten in terms of the new variables as

$$\begin{aligned}S_E&=\tilde{S}_E-i\left[\frac{1}{2}\alpha\gamma\Pi^2+\frac{1}{2}\beta\delta X^2+\beta\gamma\Pi X\right]_{\text{boundary}}\\ &=\tilde{S}_E-i\left[\frac{1}{2}\frac{\alpha}{\gamma}q^2+\frac{1}{2}\frac{\delta}{\gamma}X^2-\frac{1}{\gamma}qX\right]_{\text{boundary}},\end{aligned} \tag{B·7}$$

where

$$\tilde{S}_E=\int dt\left[-i\Pi\dot{X}+\frac{A}{2a^N}\Pi^2+\frac{1}{2}a^NBX^2+C\Pi X\right] \tag{B·8}$$

with

$$\begin{aligned}A&=\alpha(\alpha-ia^N\dot{\gamma})+a^N\gamma(i\dot{\alpha}-\omega^2a^N\gamma)\,,\\ B&=\frac{\beta}{a^N}\left(\frac{\beta}{a^N}-i\dot{\delta}\right)+\delta\left(\frac{i\dot{\beta}}{a^N}-\omega^2\delta\right),\\ C&=\alpha\left(\frac{\beta}{a^N}-i\dot{\delta}\right)-a^N\gamma\left(\frac{i\dot{\beta}}{a^N}-\omega^2\delta\right).\end{aligned} \tag{B·9}$$

It is easy to see that the boundary term in the last line of Eq. (B·7) corresponds to the matrix element between the two coordinate eigenstates; $\langle q|X\rangle$. At this point, as mentioned in § 4, it is important to note that the coefficients α, β, γ and δ should be chosen in a way as to retain the Hermiticity of the canonical variables when analytically continued back to the Lorentzian time. Provided this is satisfied, the boundary term will be harmless to the path integral and no imaginary factor will arise from it. Hence we ignore it in the rest of our discussion.

As a special class of canonical transformations, we take the coefficients as

$$\alpha=i\dot{a}\,,\quad \beta=C_N a^{1+N}\,,\quad \gamma=-\frac{a^{1-N}}{C_N}\,,\quad \delta=-i\dot{a}\,, \tag{B·10}$$

where C_N is a constant. Then

$$A=-\frac{1}{C_N}\left\{\left(\frac{\omega^2}{C_N}+1\right)a^2+(C_N-N+1)\dot{a}^2\right\},$$

$$B=C_N\left\{(C_N+1)a^2+\left(\frac{\omega^2}{C_N}+N+1\right)\dot{a}^2\right\},$$

$$C=\frac{C_N{}^2-NC_N-\omega^2}{C_N}i\dot{a}a\,. \tag{B·11}$$

We choose the constant C_N so as to satisfy $C=0$:

$$C_N=\frac{N\pm\sqrt{N^2+4\omega^2}}{2}\,, \tag{B·12}$$

which gives

$$A=-\frac{C_N-N+1}{C_N}\,,\quad B=C_N(C_N+1)\,. \tag{B·13}$$

In what follows, we examine the property of the canonically transformed system in detail.

(i) $C_N=\dfrac{N-\sqrt{N^2+4\omega^2}}{2}$:

(i·a) $N=0$:

In this case $C_N=-\omega<0$ and we have $A=(1-\omega)/\omega$ and $B=\omega(\omega-1)$. Since $0<\omega\leq 1$, these imply $A\geq 0$, $-1<B\leq 0$ and $-1<AB\leq 0$. It then follows that the system has the same property as the original one, i.e., the ordinary sign for the kinetic term and the existence of a unique negative mode, except for the case $\omega=1$. (When $\omega=1$, the dynamics disappears. We do not know its reason.)

(i·b) $N\geq 1$:

In this case, since $C_N-N+1<0$ and $C_N<0$, we find $A<0$.

(ii) $C_N=\dfrac{N+\sqrt{N^2+4\omega^2}}{2}$:

In this case, A is always negative for any N, since $C_N-N+1>0$ and $C_N>0$.

In the case of (i-b) or (ii), the kinetic term has the wrong sign; $A<0$. Hence, if we path-integrate over Π by rotating it to the imaginary axis and recall the argument given at the beginning of this Appendix, we have

$$\tilde{Z}:=\int[d\Pi dX]e^{-\tilde{S}_E}$$

$$=iK'\int[idX]\exp\left[-\int d\tau\left(\frac{a^N}{2A}\dot{X}^2+\frac{a^N}{2}BX^2\right)\right]$$

$$=iK'\int[d\tilde{X}]\exp\left[-\int d\tau\frac{a^N}{2(-A)}(\dot{\tilde{X}}^2-\tilde{\omega}^2\tilde{X}^2)\right], \qquad (B\cdot14)$$

where $\tilde{X}=iX$, K' is a real number, and

$$\tilde{\omega}^2:=-AB=\begin{cases}1+\omega^2-\sqrt{N^2+4\omega^2} & \text{for (i·b)},\\ 1+\omega^2+\sqrt{N^2+4\omega^2} & \text{for (ii)}.\end{cases} \qquad (B\cdot15)$$

Therefore, in the case of (i·b), we have $\tilde{\omega}^2\leq0$ and there is no negative mode in the system. On the other hand, in the case of (ii), we have $N+1<\tilde{\omega}^2\leq2(N+2)$ and there exist two negative modes. However, for the even number of negative modes, the path integral can be made real with a suitable choice of analytic continuation. Thus, in both of the cases, we conclude $\tilde{I}\propto i$, i.e., the property that the path integral should give one imaginary unit is retained under canonical transformations.

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