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Probability of False Vacuum Decay in a Scalar Field

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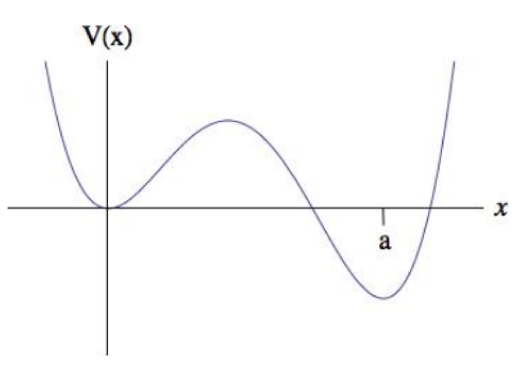
1 Introduction

Classically, a field (or a particle) in a potential with two distinct minima, like the one in 1, is stable in any of them. On the other hand, in a quantum field theory, the minima at 0 is unstable due to quantum fluctuations and would decay to the state with energy corresponding to the absolute minimum through quantum tunneling, making the other a false vacuum .

Figure 1: Potential with a false minima



The probability of tunneling through the potential barrier can be obtained by using the semiclassical approximation and has an exponential form [1]

$$\Gamma = Ae^{-B/\hbar}(1+O(\hbar)) \tag{1}$$

Using the Euclidean time formalism and the saddle point approximation for path integrals, A and B are calculated, first in quantum mechanics and then, by analogy, for the scalar field.

2 Barrier Penetration in Quantum Mechanics

In one dimensional quantum mechanics, the coefficient B in (1) is given by

$$B = 2\int_0^{q_0}\sqrt{2V(q)}dq. \tag{2}$$

For convenience, the false minima of the potential is located at 0 where $V(0) = 0$ and q_0 is point near the absolute minimum where $V(0) = 0$ again.

Going into imaginary time $t \to -i\tau$ (also known as a Wick rotation), the Euclidean time action S_E is obtained

$$S_E = \int\left[\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q)\right]dt \tag{3}$$

The corresponding equation of motion is

$$\frac{d^2q}{d\tau^2} = \frac{dV}{dq} = -\frac{d(-V)}{dq}$$

which can be interpreted as a particle moving in real time in a potential $-V(q)$, and the Euclidean energy of the particle is

$$\mathcal{E} = \frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 - V(q)$$

The solution with an Euclidean action that corresponds to (2) is the one where the particle that starts in the origin at $\tau = -\infty$, gets to q_0 and returns to the origin at $\tau = +\infty$, with $\mathcal{E} = 0$. Because it comes and goes, it is called the "bounce" solution q_B. Due to the time translation invariance we can choose any moment in time as the one when the particle reaches q_0. For convenience it is chosen to be $\tau = 0$. The velocity of the particle is zero at this time.

Using the condition $\mathcal{E} = 0$ in (3), taking into account that the motion of the particle for positive τ is the time reversal of the one for negative τ, and making a change of variable from τ to q_B, the Euclidean action for the bounce S_B can be written as

$$S_B = 2\int_0^{q_0}\sqrt{2V(q_B)}dq_B$$

showing that the semiclassical exponent of the tunneling probability B is equal to the Euclidean time action of the bounce.

$$B = S_B.$$

3 False Vacuum Decay in Field Theory

Consider a scalar field in a potential of the form like in 1. The false vacuum corresponds to the state ϕ_+ and the true vacuum to ϕ_-. In a similar way to the quantum mechanical case, the false vacuum is unstable, and the field would decay through tunneling to the true vacuum. Quantum fluctuations would make bubbles of true vacuum appear in certain regions of spacetime. If one of this bubbles is large enough so that its energy is classically favorable, it will start to grow converting the field from false to true vacuum.

The Euclidean action for a scalar field theory is

$$S_E = \int d\tau d^3x\left[\frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{2}(\nabla\phi)^2 + U(\phi)\right].$$

The bounce solution starts and ends at the false vacuum at $\tau = \pm\infty$ and has a turning point at $\tau = 0$ where its (imaginary) temporal derivative is zero. Its Euclidean energy is again zero which means that the Euclidean action of the false vacuum $S_E(\phi_+)$ is also zero since $U(\phi_+) = 0$. The bounce action must be finite and the integrand is only zero at ϕ_+, this means that at large distances the field must still be in the false vacuum. This matches with the image that a bubble of true vacuum appears somewhere, but far from it the field is still in the false vacuum.

The boundary conditions of the bounce solutions can be satisfied by considering spherically symmetric fields $\phi(\rho)$, invariant under 4-dimensional rotations where ρ is the distance in Euclidean spacetime

$$\rho^2 = \tau^2 + \mathbf{x}^2$$

The boundary conditions then becomes

$$\lim_{\rho\to\pm\infty}\phi(\rho) = \rho_+$$

and the equation of motion is

$$\frac{d^2\rho}{d\rho^2} + \frac{3}{\rho}\frac{d\rho}{d\rho} = U'(\phi)$$

It can be shown that an $O(4)$-invariant bounce solution always exists by analyzing the classical motion of a particle in a potential $-U(\phi)$ in the presence of a friction force [1].

Finally, using this new coordinate, the B coefficient, which is equal to the Euclidean action of the bounce, is

$$B = S_E = 2\pi^2\int_0^\infty \rho^3 d\rho\left[\frac{1}{2}\left(\frac{d\phi_B}{d\rho}\right)^2 + U(\phi_B)\right]$$

4 False Vacuum Decay in the Path Integral Formalism

Since the false vacuum is unstable, its energy $E(\phi_+)$ would have an imaginary part, related to the decay probability by [2]

$$\mathrm{Im}(E(\phi_+)) = -\frac{1}{2}\Gamma(\phi_+).$$

The Euclidean time path integral is given by

$$\langle\phi_f, T/2|\, e^{-HT/\hbar}\, |\phi_i, -T/2\rangle = \int D[\phi(x)]e^{-S_E/\hbar}.$$

The energy of the lowest energy state $E(\phi_+)$ can be obtained in the limit of $T\to\infty$ as this term dominates the path integral

$$e^{-E(\phi_+)T/\hbar} = \int D[\phi(x)]e^{-S_E/\hbar}.$$

4.1 Saddle Point Approximation

It is well known that the action S is a minimum for the classical trajectory of a particle x_{cl}. The same holds for the Euclidean action with the classical trajectory in imaginary time

$$\frac{\delta S_E[x_{\mathrm{cl}}]}{\delta x(\tau)} = 0.$$

It is possible then to expand any trajectory around the classical one

$$x(\tau) = x_{cl}(\tau) + \eta(\tau)$$

Doing the same for the action

$$\begin{aligned} S_E[x] &= S_E[x_{cl} + \eta] \\ &= S_E[x_{cl}] + \frac{1}{2}\iint d\tau_1 d\tau_2 \frac{\delta^2 S_E[x_{cl}]}{\delta x(\tau_1)\delta x(\tau_2)}\eta(\tau_1) + O(\eta^3) \\ &\approx S_E^{\mathrm{cl}} + S_E^{(2)}[\eta(\tau)] \end{aligned}$$

The second order term $S_E^{(2)}[\eta(\tau)]$ can be obtained by calculating the second functional derivative, reducing this term to

$$S_E^{(2)} = \frac{1}{2}\int \eta(\tau)\left(-\frac{d^2}{d\tau^2} + V''(x_{\mathrm{cl}})\right)\eta(\tau) \tag{4}$$

Introducing a complete set of orthonormalized eigenvalues of the operator in 4 and ignoring higher order terms, the path integral reduces to

$$I = Ne^{-S_E^{\mathrm{cl}}/\ \hbar}\int D[\eta(\tau)]e^{-S_E^{(2)}/\hbar}. \tag{5}$$

(N is a normalization constant). Choosing the measure of integration (the factor of $\sqrt{2\pi\hbar}$ is for convenience)

$$D[\eta(\tau)] = \prod\frac{da_\lambda}{\sqrt{2\pi\hbar}},$$

eq. (5) becomes a product of gaussian integrals that give the determinant of the operator. Finally the path integral in the saddle point approximation is given by

$$I = Ne^{-S_E^{\mathrm{cl}}/\hbar}\left[\det\left(-\frac{d^2}{d\tau^2} + V''(x_{cl})\right)\right]^{-1/2}. \tag{6}$$

As can be clearly seen in the expression above, the saddle point method is only valid if the determinant is different from zero, which usually doesn't happens when the system has a symmetry.

To calculate the path integral all field configurations with any number of bounces needs to be considered.

4.2 Zero modes

The path integral eq. (6) for the trivial solution $\phi(x) = \phi_+$ is

$$I_0 = \left[\det\left(-\partial^2 + \omega^2\right)\right]^{-1/2}$$

since its action is zero and its eigenvalues positive, with $U''(\phi_+) = \omega^2$. It gives a positive energy so it doesn't contribute to the decay probability.

Naively, we could write the contribution of the one bounce configuration as

$$I_B = e^{-S_B/\hbar}\left[\det\left(\partial^2 + U''(\phi_{cl})\right)\right]^{-1/2} \tag{7}$$

but it would diverge, since the operator for the bounce has zero and even a negative eigenvalue. The zero modes are related to the invariance under spatiotemporal translations since it is not important where the center of the bounce is located.

The zero modes need to be integrated independently giving a non-exponential factor

$$\prod_i\int(2\pi\hbar)^{-1/2}da_i^{(0)} = \left(\frac{B}{2\pi\hbar}\right)^2 TV.$$

4.3 Negative mode

For the bounce in 1D quantum mechanics, the zero mode is proportional to the velocity which becomes zero at $\tau = 0$, in other words, it has a node and cannot be the lowest energy state. Since its energy is already zero, the energy of the lowest energy eigenvector must be negative.

The integral of this eigenvalue diverges, but it can be calculated by distorting the path of integration to the complex plane and integrating over half the imaginary line [3]. This would contribute an i and a $1/2$ to eq. (7).

Taking together all of this details, the contribution of the bounce to the path integral is

$$I_B = \frac{i}{2}\left(\frac{B}{2\pi\hbar}\right)^2 TVe^{-S_B/\hbar}\left[\det'\left(\partial^2 + U''(\phi_{cl})\right)\right]^{-1/2} \tag{8}$$

where det′ denotes the determinant without taking into account the zero modes.

4.4 Multibounce contribution

Defining

$$K = \frac{i}{2}\left(\frac{B}{2\pi\hbar}\right)^2\left[\frac{\det'\left(\partial^2 + U''(\phi_{cl})\right)}{\det\left(-\partial^2 + \omega^2\right)}\right]^{-1/2}$$

it can be seen that eq. (8) is proportional to I_0 by $iVTKe^{-B}$. The same is true for an arbitrary number of bounces, then

$$I_n = \frac{1}{n!}(iVTKe^{-B})^n I_0$$

where the $1/n!$ factor appears because any two bubbles are the same.

Summing over all bounce contributions

$$I = \sum_n I_n = I_0 \exp iVTKe^{-B}.$$

The decay probability per unit volume of the false vacuum in a scalar field is then

$$\frac{\Gamma}{V} = \left(\frac{B}{2\pi\hbar}\right)^2\left[\frac{\det'\left(\partial^2 + U''(\phi_{cl})\right)}{\det\left(-\partial^2 + \omega^2\right)}\right]^{-1/2} e^{-S_B/\hbar}$$

5 Conclusions

The saddle point approximation allowed us to calculate the Euclidean time path integral, getting an explicit expression for A and recovering the result for B obtained only using the semiclassical approximation. While this tools are very powerful in making this kind of computations, it leaves some questions open about the physics of the problem. In the past few years, a different approach to imaginary time has been proposed, known as Picard−Lefschetz theory, where the trajectory and fields are complexified instead of time. This work could be expanded by exploring this new method and comparing to the one used in order to gain a better insight into the process of false vacuum decay.

References

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