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Conformal transformations and conformal invariance in gravitation

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Conformal transformations are frequently used tools in order to study relations between various theories of gravity and Einstein's general relativity theory. In this paper we discuss the rules of these transformations for geometric quantities as well as for the matter energy-momentum tensor. We show the subtlety of the matter energy-momentum conservation law which refers to the fact that the conformal transformation "creates" an extra matter term composed of the conformal factor which enters the conservation law. In an extreme case of the flat original spacetime the matter is "created" due to work done by the conformal transformation to bend the spacetime which was originally flat. We discuss how to construct the conformally invariant gravity theories and also find the conformal transformation rules for the curvature invariants R^2, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$ and the Gauss-Bonnet invariant in a spacetime of an arbitrary dimension. Finally, we present the conformal transformation rules in the fashion of the duality transformations of the superstring theory. In such a case the transitions between conformal frames reduce to a simple change of the sign of a redefined conformal factor.

1 Introduction

Conformal transformations of the metric tensor [1] are interesting characteristics of the scalar-tensor theories of gravity [2–5], including their conformally invariant versions [6–9]. The point is that these theories can be represented in the two conformally related frames: the Jordan frame in which the scalar field is non-minimally coupled to the metric tensor, and in the Einstein frame in which it is minimally coupled to the metric tensor. Besides, it is striking that the scalar-tensor theory of gravity is the low-energy limit of superstring theory [10–13]. It has been shown that some physical processes such as the inflation of the universe and density perturbations look different in conformally related frames [12, 13], and so it motivates discussions of these transformations within the framework of various theories of gravity.

Many studies have been devoted to the problem of a change of the geometrical and physical quantities under conformal transformations (see e.g. [3] for a current update, and [14] for a historical review). However, the transformation rules were not always presented in the most user-friendly way and also some simple mathematical properties of these transformations have not been explored in a detailed way. This is why we would like to collect all of these rules in one paper from the beginning to the end in order to have a compendium about the conformal transformations. In particular, we would like to explore the problem

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of the conformal transformation properties of the higher-order curvature invariants as well as conformal transformations as applied to superstring theory.

Our paper is organized as follows. In Sect. 2 we give a basic review of the idea of the conformal transformations of the metric tensor and discuss the transformation properties of the geometric quantities such as connection coefficients, Riemann tensor, Ricci tensor and Ricci scalar. In Sect. 3 we discuss the rules of conformal transformations of the curvature invariants R^2, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$ which may emerge as higher-order corrections to standard gravity. In Sect. 4 we discuss the conformal transformations of the matter energy-momentum tensor and emphasize that conformal transformations "create" matter. In Sect. 5 we discuss the construction of the conformally invariant theory of gravity. In Sect. 6 we show the rules of the conformal transformations in the fashion of duality transformations in superstring theory. In Sect. 7 we give our conclusions.

2 Conformal transformations in Einstein's theory of gravitation

Consider a spacetime $(\mathcal{M}, g_{ab})$, where $\mathcal{M}$ is a smooth n-dimensional manifold and g_{ab} is a Lorentzian metric on M. The following conformal transformation

$$\tilde{g}_{ab}(x) = \Omega^2(x) g_{ab}(x) \ , \tag{2.1}$$

where Ω is a smooth, non-vanishing function of the spacetime point is a point-dependent rescaling of the metric and is called a conformal factor. It must lie in the range $0 < \Omega < \infty$ $(a, b, k, l = 0, 1, 2, \ldots D)$. The conformal transformations shrink or stretch the distances between the two points described by the same coordinate system x^a on the manifold $\mathcal{M}$, but they preserve the angles between vectors (in particular null vectors which define light cones) which leads to a conservation of the (global) causal structure of the manifold [1]. If we take $\Omega =$ const. we deal with the so-called scale transformations [3]. In fact, conformal transformations are localized scale transformations $\Omega = \Omega(x)$.

On the other hand, the coordinate transformations $x^a \to \tilde{x}^a$ only change coordinates and do not change geometry so that they are entirely different from conformal transformations [1]. This is crucial since conformal transformations lead to a different physics [3]. Since this is usually related to a different coupling of a physical field to gravity, we will be talking about different frames in which the physics is studied (see also [7, 8] for a slightly different view).

In D spacetime dimensions the determinant of the metric $g = \det{[g_{ab}]}$ transforms as

$$\sqrt{-\tilde{g}} = \Omega^D \sqrt{-g} \ . \tag{2.2}$$

It is obvious from (2.1) that the following relations for the inverse metrics and the spacetime intervals hold

$$\tilde{g}^{ab} = \Omega^{-2} g^{ab} \ , \tag{2.3}$$

$$\mathrm{d}\tilde{\mathrm{s}}^2 = \Omega^2 \mathrm{ds}^2 \ . \tag{2.4}$$

Finally, the notion of conformal flatness means that

$$\tilde{g}_{ab} \Omega^{-2}(x) = \eta_{ab} \ , \tag{2.5}$$

where η_{ab} is the flat Minkowski metric.
The application of (2.1) to the Christoffel connection coefficients gives (compare [1])

$$\tilde{\Gamma}^c_{ab} = \Gamma^c_{ab} + \frac{1}{\Omega}\left(\delta^c_a \Omega_{,b} + \delta^c_b \Omega_{,a} - g_{ab} g^{cd} \Omega_{,d}\right) \ , \quad \tilde{\Gamma}^b_{ab} = \Gamma^b_{ab} + D\frac{\Omega_{,a}}{\Omega} \ , \tag{2.6}$$

$$\Gamma^c_{ab} = \tilde{\Gamma}^c_{ab} - \frac{1}{\Omega}\left(\tilde{\delta}^c_a \Omega_{,b} + \tilde{\delta}^c_b \Omega_{,a} - \tilde{g}_{ab} \tilde{g}^{cd} \Omega_{,d}\right) \ , \quad \Gamma^b_{ab} = \tilde{\Gamma}^b_{ab} - D\frac{\Omega_{,a}}{\Omega} . \tag{2.7}$$

The Riemann tensors, Ricci tensors, and Ricci scalars in the two related frames g_{ab} and $\tilde{g}_{ab}$ transform as (we use the signature (-+...++), the Riemann tensor convention $R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{cb}$, and the Ricci tensor definition $R_{bd} = R^a{}_{bad}$)

$$\tilde{R}^a{}_{bcd} = R^a{}_{bcd} + \frac{1}{\Omega}\left[\delta^a_d\Omega_{;bc} - \delta^a_c\Omega_{;bd} + g_{bc}\Omega^{;a}{}_{;d} - g_{bd}\Omega^{;a}{}_{;c}\right] \tag{2.8}$$
$$+ \frac{2}{\Omega^2}\left[\delta^a_c\Omega_{,b}\Omega_{,d} - \delta^a_d\Omega_{,b}\Omega_{,c} + g_{bd}\Omega^{,a}\Omega_{,c} - g_{bc}\Omega^{,a}\Omega_{,d}\right] + \frac{1}{\Omega^2}\left[\delta^a_d g_{bc} - \delta^a_c g_{bd}\right] g_{ef}\Omega^{,e}\Omega^{,f} ,$$

$$R^a{}_{bcd} = \tilde{R}^a{}_{bcd} - \frac{1}{\Omega}\left[\delta^a_d\Omega_{\tilde{;}bc} - \delta^a_c\Omega_{\tilde{;}bd} + \tilde{g}_{bc}\Omega^{\tilde{;}a}{}_{\tilde{;}d} - \tilde{g}_{bd}\Omega^{\tilde{;}a}{}_{\tilde{;}c}\right] \tag{2.9}$$
$$+ \frac{1}{\Omega^2}\left[\delta^a_d\tilde{g}_{bc} - \delta^a_c\tilde{g}_{bd}\right]\tilde{g}_{ef}\Omega^{,e}\Omega^{,f} ,$$

$$\tilde{R}_{ab} = R_{ab} + \frac{1}{\Omega^2}\left[2(D-2)\Omega_{,a}\Omega_{,b} - (D-3)\Omega_{,c}\Omega^{,c}g_{ab}\right] - \frac{1}{\Omega}\left[(D-2)\Omega_{;ab} + g_{ab}\Box\Omega\right] , \tag{2.10}$$

$$R_{ab} = \tilde{R}_{ab} - \frac{1}{\Omega^2}(D-1)\tilde{g}_{ab}\Omega_{,c}\Omega^{,c} + \frac{1}{\Omega}\left[(D-2)\Omega_{\tilde{;}ab} + \tilde{g}_{ab}\,\tilde{\Box}\,\Omega\right] , \tag{2.11}$$

$$\tilde{R} = \Omega^{-2}\left[R - 2(D-1)\frac{\Box\Omega}{\Omega} - (D-1)(D-4)g^{ab}\frac{\Omega_{,a}\Omega_{,b}}{\Omega^2}\right] , \tag{2.12}$$

$$R = \Omega^2\left[\tilde{R} + 2(D-1)\frac{\tilde{\Box}\,\Omega}{\Omega} - D(D-1)\tilde{g}^{ab}\frac{\Omega_{,a}\Omega_{,b}}{\Omega^2}\right] , \tag{2.13}$$

and the appropriate d'Alembertian operators change under (2.1) as

$$\tilde{\Box}\,\phi = \Omega^{-2}\left(\Box\phi + (D-2)g^{ab}\frac{\Omega_{,a}}{\Omega}\phi_{,b}\right) , \tag{2.14}$$

$$\Box\phi = \Omega^2\left(\tilde{\Box}\,\phi - (D-2)\tilde{g}^{ab}\frac{\Omega_{,a}}{\Omega}\phi_{,b}\right) . \tag{2.15}$$

In these formulas the d'Alembertian $\tilde{\Box}$ taken with respect to the metric $\tilde{g}_{ab}$ is different from $\Box$ which is taken with respect to a conformally rescaled metric g_{ab}. Same refers to the covariant derivatives $\tilde{;}$ and ; in (2.8)–(2.11). Note that some of these quantities are given in [1] in a different form. Also, notice that in $D = 4$ the rule (2.12) composes of the two terms only (and it is often presented in standard textbooks [24] that way), while the inverse rule (2.13) composes of the three terms. This reflects the fact that the rules of the simple and inverse transformations are not symmetric (see Sect. 6 for a discussion of a symmetric, duality-like, representation).

For the Einstein tensor we have

$$\tilde{G}_{ab} = G_{ab} + \frac{D-2}{2\Omega^2}\left[4\Omega_{,a}\Omega_{,b} + (D-5)\Omega_{,c}\Omega^{,c}g_{ab}\right] - \frac{D-2}{\Omega}\left[\Omega_{;ab} - g_{ab}\Box\Omega\right] , \tag{2.16}$$

$$G_{ab} = \tilde{G}_{ab} + \frac{D-2}{2\Omega^2}(D-1)\Omega_{,e}\Omega^{,e}\tilde{g}_{ab} + \frac{D-2}{\Omega}\left[\Omega_{\tilde{;}ab} - \tilde{g}_{ab}\,\tilde{\Box}\,\Omega\right] . \tag{2.17}$$

An important feature of the conformal transformations is that they preserve Weyl conformal curvature tensor $(D \geq 3)$

$$C_{abcd} = R_{abcd} + \frac{2}{D-2}\left(g_{a[d}R_{c]b} + g_{b[c}R_{d]a}\right) + \frac{2}{(D-1)(D-2)}Rg_{a[c}g_{d]b} , \tag{2.18}$$

which means that we have (note that one index is raised)

$$\tilde{C}^a{}_{bcd} = C^a{}_{bcd} \tag{2.19}$$

under (2.1). Using this property (2.19) and the rules (2.1)–(2.3) one can easily conclude that the Weyl Lagrangian [15]

$$\tilde{L}_w = -\alpha\sqrt{-\tilde{g}}\tilde{C}^{abcd}\tilde{C}_{abcd} = -\alpha\sqrt{-g}C^{abcd}C_{abcd} = L_w \tag{2.20}$$

is an invariant of the conformal transformation (2.1).

3 Conformal transformations in higher-order gravitation

In the physical theories which enter the dense quantum phase of the evolution of the universe one often applies quantum corrections to general relativity [16] which are composed of the curvature invariants R^2, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$ and the Gauss-Bonnet invariant R_{GB} [17] as well as their functions such as $f(R)$ [18], $f(R_{GB})$ [19] as well as $f(R^2, R_{ab}R^{ab}, R_{abcd}R^{abcd})$ [20,21]. In fact, the additional terms which come from the inclusion of these curvature invariants play the role of the corrections to the standard gravity action [22]. This is why it is useful to know the rules of the conformal transformations for all these quantities.

These rules are given as follows

$$\begin{aligned}\tilde{R}^2 = \Omega^{-4}\Big[&R^2 + 4(D-1)^2\Omega^{-2}\left(\Box\Omega\right)^2 + (D-1)^2(D-4)^2\Omega^{-4}g^{ab}\Omega_{,a}\Omega_{,b}g^{cd}\Omega_{,c}\Omega_{,d} \\ &- 4(D-1)R\Omega^{-1}\Box\Omega - 2R(D-1)(D-4)\Omega^{-2}g^{ab}\Omega_{,a}\Omega_{,b} \\ &+ 4(D-1)^2(D-4)\Omega^{-3}\Box\Omega g^{ab}\Omega_{,a}\Omega_{,b}\Big] \ ,\end{aligned} \tag{3.1}$$

$$\begin{aligned}\tilde{R}_{ab}\tilde{R}^{ab} = \Omega^{-4}\Big\{&R_{ab}R^{ab} - 2\Omega^{-1}\left[(D-2)R_{ab}\Omega^{;ab} + R\Box\Omega\right] \\ &+ \Omega^{-2}\Big[4(D-2)R_{ab}\Omega^{,a}\Omega^{,b} - 2(D-3)R\Omega_{,e}\Omega^{,e} + (D-2)^2\Omega_{;ab}\Omega^{;ab} \\ &+ (3D-4)\left(\Box\Omega\right)^2\Big] + \Omega^{-4}(D-1)(D^2-5D+8)\left(\Omega_{,a}\Omega^{,a}\right)^2 \\ &- \Omega^{-3}\left[(D-2)^2\Omega_{;ab}\Omega^{,a}\Omega^{,b} - (D^2-5D+5)\Box\Omega\Omega_{,e}\Omega^{,e}\right]\Big\} \ ,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\tilde{R}_{abcd}\tilde{R}^{abcd} = \Omega^{-4}\Big\{&R_{abcd}R^{abcd} - 8\Omega^{-1}R_{bc}\Omega^{;bc} + 4\Omega^{-2}\Big[\left(\Box\Omega\right)^2 + (D-2)\Omega_{;bc}\Omega^{;bc} \\ &- R\Omega_{,b}\Omega^{,b} + 4R_{bc}\Omega^{,b}\Omega^{,c}\Big] + 8\Omega^{-3}\left[(D-3)\Box\Omega\Omega_{,c}\Omega^{,c} - 2(D-2)\Omega_{;bc}\Omega^{,b}\Omega^{,c}\right] \\ &+ 2\Omega^{-4}D(D-1)\left(\Omega_{,a}\Omega^{,a}\right)^2\Big\} \ .\end{aligned} \tag{3.3}$$

In fact, out of these curvature invariants one forms the well-known Gauss-Bonnet term which is one of the Euler (or Lovelock) densities [17, 19]. Its conformal transformation (2.1) reads as

$$\begin{aligned}\tilde{R}_{GB} \equiv \tilde{R}_{abcd}\tilde{R}^{abcd} - 4\tilde{R}_{ab}\tilde{R}^{ab} + \tilde{R}^2 = \Omega^{-4}\Big\{&R_{GB} + 4(D-3)\Omega^{-1}\left[2R_{ab}\Omega^{;ab} - R\Box\Omega\right] \\ &+ 2(D-3)\Omega^{-2}\left[2(D-2)\left(\left(\Box\Omega\right)^2 - \Omega_{;ab}\Omega^{;ab}\right) - 8R_{ab}\Omega^{,a}\Omega^{,b} - (D-6)R\Omega_{,a}\Omega^{,a}\right] \\ &+ 4(D-2)(D-3)\Omega^{-3}\left[(D-5)\Box\Omega\Omega_{,a}\Omega^{,a} + 4\Omega_{;ab}\Omega^{,a}\Omega^{,b}\right] \\ &+ (D-1)(D-2)(D-3)(D-8)\Omega^{-4}\left(\Omega_{,a}\Omega^{,a}\right)^2\Big\} \ .\end{aligned} \tag{3.4}$$

The inverse transformation is given by

$$R_{GB} \equiv R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 = \Omega^4 \Big\{ \tilde{R}_{GB} - 4(D-3)\Omega^{-1} \left[2\tilde{R}_{ab}\Omega^{\tilde{;}ab} - \tilde{R}\, \tilde{\Box}\, \Omega \right] + 2(D-2)(D-3)\Omega^{-2} \left[2\left(\tilde{\Box}\,\Omega\right)^2 - 2\Omega_{\tilde{;}ab}\Omega^{\tilde{;}ab} - \tilde{R}\Omega_{\tilde{;}a}\Omega^{\tilde{;}a} \right] - (D-1)(D-2)(D-3)\Omega^{-3} \left[4\left(\tilde{\Box}\,\Omega\right)\Omega_{\tilde{;}a}\Omega^{\tilde{;}a} - D\Omega^{-1}\left(\Omega_{\tilde{;}a}\Omega^{\tilde{;}a}\right)^2 \right] \Big\} \, . \tag{3.5}$$

4 Conformal transformations of the matter energy-momentum tensor

So far we have considered only geometrical part. For the matter part we usually consider the matter action in the form

$$\tilde{S}_{\rm m} = \int \sqrt{-\tilde{g}} d^D x \tilde{\mathcal{L}}_{\rm m} = \int \sqrt{-g} d^D x \mathcal{L}_{\rm m} = S_{\rm m} \, , \tag{4.1}$$

where the Lagrangians in the conformally related frames transform as

$$\tilde{\mathcal{L}}_{\rm m} = \Omega^{-D}\mathcal{L}_{\rm m} \tag{4.2}$$

under the conformal transformation (2.1) [11]. Then, the energy-momentum tensor of matter in one conformal frame reads as

$$\tilde{T}^{ab}_{\rm m} = \frac{2}{\sqrt{-\tilde{g}}}\frac{\delta}{\delta\tilde{g}_{ab}}\left(\sqrt{-\tilde{g}}\tilde{\mathcal{L}}_{\rm m}\right) = \Omega^{-D}\frac{2}{\sqrt{-g}}\frac{\partial g_{cd}}{\partial\tilde{g}_{ab}}\frac{\delta}{\delta g_{cd}}\left(\sqrt{-g}\mathcal{L}_{\rm m}\right) \, , \tag{4.3}$$

which under (2.1) transforms as

$$\tilde{T}^{ab}_{\rm m} = \Omega^{-D-2}T^{ab}_{\rm m} \, , \qquad \tilde{T}^{a}_{{\rm m}b} = \Omega^{-D}T^{a}_{{\rm m}b}, \qquad \tilde{T}^{\rm m}_{ab} = \Omega^{-D+2}T^{\rm m}_{ab} \, , \qquad \tilde{T}_{\rm m} = \Omega^{-D}T_{\rm m} \, . \tag{4.4}$$

For the matter in the form of the perfect fluid with the four-velocity v^a ($v_a v^b = -1$), the energy density ϱ and the pressure $\rm p$

$$T^{ab}_{\rm m} = (\varrho + {\rm p})v^a v^b + {\rm p}g^{ab} \, , \tag{4.5}$$

the conformal transformation gives

$$\tilde{T}^{ab}_{\rm m} = (\tilde{\varrho} + \tilde{\rm p})\tilde{v}^a \tilde{v}^b + \tilde{\rm p}\tilde{g}^{ab} \, , \tag{4.6}$$

where

$$T^{ab}_{\rm m} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{ab}}\left(\sqrt{-g}\mathcal{L}_{\rm m}\right) \, , \tag{4.7}$$

and

$$\tilde{v}^a = \frac{\rm dx^a}{\rm d\tilde{s}} = \frac{1}{\Omega}\frac{\rm dx^a}{\rm ds} = \Omega^{-1}v^a \, . \tag{4.8}$$

Therefore, the relation between the pressure and the energy density in the conformally related frames reads as

$$\tilde{\varrho} = \Omega^{-D}\varrho \, , \tag{4.9}$$

$$\tilde{\rm p} = \Omega^{-D}{\rm p} \, . \tag{4.10}$$

It is easy to note that the imposition of the conservation law in the first frame

$$T^{ab}_{\mathrm{m};b} = 0 \; , \tag{4.11}$$

gives in the conformally related frame

$$\tilde{T}^{ab}_{\mathrm{m}\tilde{;}b} = -\frac{\Omega^{,a}}{\Omega}\tilde{T}_{\mathrm{m}} \; . \tag{4.12}$$

From (4.12) it appears obvious that the conformally transformed energy-momentum tensor is conserved only, if the trace of it vanishes ($\tilde{T}_{\mathrm{m}} = 0$) [3, 6, 9, 28]. For example, in the case of barotropic fluid with

$$\mathrm{p} = (\gamma - 1)\varrho \qquad \gamma = \mathrm{const.}, \tag{4.13}$$

it is conserved only for the radiation-type fluid $\mathrm{p} = [1/(D-1)]\varrho$.

Similar considerations are also true if we first impose the conservation law in the second frame

$$\tilde{T}^{ab}_{\mathrm{m}\tilde{;}b} = 0 \; , \tag{4.14}$$

which gives in the conformally related frame (no tildes)

$$T^{ab}_{\mathrm{m};b} = \frac{\Omega^{,a}}{\Omega}T_{\mathrm{m}} \; . \tag{4.15}$$

Finally, it follows from (4.4) that that vanishing of the trace of the energy-momentum tensor in one frame necessarily requires its vanishing in the second frame, i.e., if $T_{\mathrm{m}} = 0$ in one frame, then $\tilde{T}_{\mathrm{m}} = 0$ in the second frame and vice versa. This means only the traceless type of matter fulfills the requirement of energy conservation.

We now use the formulas (2.16)–(2.17) to discuss the formulation of the Einstein field equations for the case of the conformal transformations. Let us assume then the validity of the Einstein field equations in one of the conformal frames (no tildes) as follows

$$G_{ab} = \kappa^2 T^{\mathrm{m}}_{ab}, \qquad G^{ab} = \kappa^2 T^{ab}_{\mathrm{m}} \; , \tag{4.16}$$

so that the imposition of the Bianchi identity gives

$$G^{ab}_{\ \ ;b} = 0 \Rightarrow T^{ab}_{\mathrm{m}\ ;b} = 0 \; , \tag{4.17}$$

However, due to (2.17) in a conformally related frame (with tildes) one has

$$G_{ab} = \tilde{G}_{ab} + \tilde{T}^{\Omega}_{ab}, \qquad G^{ab} = \Omega^4\left(\tilde{G}^{ab} + \tilde{T}^{ab}_{\Omega}\right) \; , \tag{4.18}$$

where

$$\tilde{T}^{\Omega}_{ab} = -\frac{D-2}{2\Omega^2}(D-1)\Omega_{,e}\Omega^{,e}\tilde{g}_{ab} + \frac{D-2}{\Omega}\left[\Omega_{\tilde{;}ab} - \tilde{g}_{ab}\,\tilde{\Box}\,\Omega\right] \; . \tag{4.19}$$

Using these one can write down the Einstein equations (4.16) as

$$\tilde{G}_{ab} + \tilde{T}^{\Omega}_{ab} = \kappa^2\Omega^{D-2}\tilde{T}^{m}_{ab} \; , \tag{4.20}$$

or, alternatively as

$$\tilde{G}^{ab} = \kappa^2\Omega^{D+2}\tilde{T}^{ab}_{m} - \tilde{T}^{ab}_{\Omega} \; . \tag{4.21}$$

Left-hand side of (4.21) is geometrical, so that the imposition of the Bianchi identity gives

$$\tilde{G}^{ab}_{\ \ \tilde{;}b} = 0 = \kappa^2(\Omega^{D+2}\tilde{T}^{ab}_{m})_{\tilde{;}b} - \tilde{T}^{ab}_{\Omega\tilde{;}b} \; . \tag{4.22}$$

Using (4.4), (4.11) and (4.12) after some manipulations one gets

$$\kappa^2 \left[(D+2)\frac{\Omega_{,b}}{\Omega} T_m^{ab} - \Omega^{,a}\Omega T_m \right] = \tilde{T}^{ab}_{\Omega \tilde{;} b} \, . \tag{4.23}$$

The right-hand side of (4.23) can be obtained after differentiating of (4.19). The conclusion is that we need to take into account an extra matter term in (4.21) which comes from conformal factor so that one may read (4.21) in the form

$$\tilde{G}^{ab} = \kappa^2 \tilde{T}^{ab}_{total} \, , \tag{4.24}$$

and

$$\tilde{T}^{ab}_{total} = \hat{T}^{ab}_m - \frac{1}{\kappa^2}\tilde{T}^{ab}_{\Omega} \, , \tag{4.25}$$

where

$$\hat{T}^{ab}_m = \Omega^{D+2}\tilde{T}^{ab}_m \, . \tag{4.26}$$

Now, let us make the transformation in an inverse way. Assume

$$\tilde{G}_{ab} = \kappa^2 \tilde{T}^{\mathrm{m}}_{ab}, \qquad \tilde{G}^{ab} = \kappa^2 \tilde{T}^{ab}_{\mathrm{m}} \, , \tag{4.27}$$

so that the imposition of the Bianchi identity gives

$$\tilde{G}^{ab}_{\ \tilde{;}\, b} = 0 \Rightarrow \tilde{T}^{ab}_{\mathrm{m}\, \tilde{;}\, b} = 0 \, , \tag{4.28}$$

In a conformally related frame (without tildes) one has

$$\tilde{G}_{ab} = G_{ab} + T^{\Omega}_{ab}, \qquad \tilde{G}^{ab} = \Omega^{-4}\left(\tilde{G}^{ab} + \tilde{T}^{ab}_{\Omega}\right) \, , \tag{4.29}$$

where

$$T^{\Omega}_{ab} = -\frac{D-2}{2\Omega^2}\left[4\Omega_{,a}\Omega_{,b} + (D-5)\Omega_{,c}\Omega^{,c}g_{ab}\right] - \frac{D-2}{\Omega}\left[\Omega_{;ab} - g_{ab}\Box\Omega\right] \, . \tag{4.30}$$

Now, the field equations (4.27) give

$$G_{ab} + T^{\Omega}_{ab} = \kappa^2 \Omega^{-D-2} T^{\mathrm{m}}_{ab} \, , \tag{4.31}$$

or, alternatively

$$G^{ab} = \kappa^2 \Omega^{-D+2} T^{ab}_{\mathrm{m}} - T^{ab}_{\Omega} \, , \tag{4.32}$$

and the Bianchi identity gives

$$G^{ab}_{\ ;b} = 0 = \kappa^2 \left(\Omega^{-D+2} T^{ab}_{\mathrm{m}}\right)_{;b} - T^{ab}_{\Omega;b} \, . \tag{4.33}$$

In order to be consistent with the previous derivation we assume (4.11) which finally gives

$$\kappa^2(-D+2)\Omega^{-D+2}\frac{\Omega_{,b}}{\Omega} T^{ab}_{\mathrm{m}} = T^{ab}_{\Omega;b} \, . \tag{4.34}$$

As an amazing example of the subtlety of the conformal transformation applied to matter energy-momentum tensor let us consider the “creation” of the Friedmann universes out of the flat Minkowski spacetime. In order to do that we start with the flat Minkowski spacetime with a flat Minkowski metric $g_{ab} = \eta_{ab}$ and $\tilde{g}_{ab} = \Omega^2 \eta_{ab}$. Assuming no matter energy-momentum tensor we have from (4.4) that

$T^{ab}_{\rm m} = 0$ which implies $\tilde{T}^{ab}_{\rm m} = 0$. Then, from (4.21) we have that despite $G_{ab} = 0$ one has the matter "created", i.e.,

$$\tilde{G}^{ab} = -\tilde{T}^{ab}_{\Omega} \neq 0 \, . \tag{4.35}$$

For the flat Friedmann universe being "created" out of the flat Minkowski universe one has ($a, b = 0, 1, 2, 3$)

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2) = \Omega^2(\eta)\eta_{ab}dx^a dx^b \, , \tag{4.36}$$

and the conformal factor $\Omega = a(\eta)$ is equal to the scale factor while η is the conformal time. In this simple case one has

$$\tilde{T}^{0}_{0\,\Omega} = -3\frac{\dot{a}}{a^4}, \quad \tilde{T}^{\mu}_{\mu\,\Omega} = \frac{1}{a^4}\left[2a\ddot{a} - \dot{a}^2\right] \, , \tag{4.37}$$

where $\mu = 1, 2, 3$ and the dot represents the derivative with respect to conformal time η. The other components of the tensor $\tilde{T}^b_a$ are zero.

One can give the following physical interpretation of the effect under study. The energy-momentum tensor in a Friedmann universe is "created" out of the flat Minkowski space due to the work done by the conformal transformation to bend flat space and become curved.

5 Conformally invariant gravitation

Let us start with the vacuum Einstein-Hilbert action of general relativity in D spacetime dimensions which read as (in $D = 4$ dimensions $\xi = 1/6$ and $\kappa^2 = 8\pi G \equiv 6$)

$$\mathrm{S}_{EH} = \frac{\xi}{2}\int \mathrm{d^D x}\sqrt{-\tilde{\mathrm{g}}}\tilde{\mathrm{R}} \, , \tag{5.1}$$

where

$$\xi = \frac{1}{4}\frac{D-2}{D-1} \, . \tag{5.2}$$

This is the so-called Einstein frame action. The application of the formula (2.12) to (5.1) gives a new action

$$\mathrm{S}_{EH} = \frac{1}{2}\int \mathrm{d^D x}\sqrt{-\mathrm{g}}\xi\Omega^{\mathrm{D}-2}\left[\mathrm{R} - 2(\mathrm{D}-1)\frac{\Box\Omega}{\Omega} - \frac{(\mathrm{D}-1)(\mathrm{D}-4)}{\Omega^2}\mathrm{g}^{\mu\nu}\Omega_{,\mu}\Omega_{,\nu}\right] \, , \tag{5.3}$$

which is not conformally invariant, apart from the case of the global transformations of the trivial type $\tilde{g}_{\mu\nu} = \mathrm{const.} \times g_{\mu\nu}$. However, if we start with the action (Jordan frame action)

$$\tilde{\mathrm{S}}_C = \frac{1}{2}\int \mathrm{d^D x}\sqrt{-\tilde{\mathrm{g}}}\xi\tilde{\mathrm{R}}\tilde{\Phi}^2 \, , \tag{5.4}$$

then the result of the conformal transformation will be as follows

$$\tilde{\mathrm{S}}_C = \frac{1}{2}\int \mathrm{d^D x}\sqrt{-\mathrm{g}}\Omega^{\mathrm{D}-2}\tilde{\Phi}^2\left[\xi\mathrm{R} - \frac{(\mathrm{D}-2)}{2}\frac{\Box\Omega}{\Omega} - \frac{(\mathrm{D}-2)(\mathrm{D}-4)}{4\Omega^2}\mathrm{g}^{\mu\nu}\Omega_{,\mu}\Omega_{,\nu}\right] \, , \tag{5.5}$$

so that we can redefine the scalar field as

$$\tilde{\Phi} = \Omega^{\frac{2-D}{2}}\Phi \tag{5.6}$$

to get

$$\tilde{\mathrm{S}}_C = \frac{1}{2}\int \mathrm{d^D x}\sqrt{-\mathrm{g}}\Phi^2\left[\xi\mathrm{R} - \frac{(\mathrm{D}-2)}{2}\frac{\Box\Omega}{\Omega} - \frac{1}{4}(\mathrm{D}-2)(\mathrm{D}-4)\mathrm{g}^{\mu\nu}\frac{\Omega_{,\mu}\Omega_{,\nu}}{\Omega^2}\right] . \tag{5.7}$$

Now we add

$$\mathrm{S}_{\tilde{\Phi}} = -\frac{1}{2}\int \mathrm{d^D x}\sqrt{-\tilde{\mathrm{g}}}\tilde{\Phi}\ \tilde{\Box}\ \tilde{\Phi} . \tag{5.8}$$

First we use (2.14) with $\phi = \tilde{\Phi}$ to get

$$\tilde{\Box}\ \tilde{\Phi} = \Omega^{-2}\left[\Box\tilde{\Phi} + (D-2)g^{\mu\nu}\frac{\Omega_{,\mu}}{\Omega}\tilde{\Phi}_{,\nu}\right] . \tag{5.9}$$

Then, from (5.6) we have

$$\begin{aligned}\Box\tilde{\Phi} = & \frac{D(D-2)}{4}\Omega^{-\frac{D+2}{2}}g^{\mu\nu}\Omega_{,\mu}\Omega_{,\nu}\Phi + \frac{2-D}{2}\Omega^{-\frac{D}{2}}\Phi\Box\Omega \\ & -(D-2)\Omega^{-\frac{D}{2}}\Omega_{,\mu}\Phi_{,\nu}g^{\mu\nu} + \Omega^{\frac{2-D}{2}}\Box\Phi .\end{aligned} \tag{5.10}$$

The derivatives of $\tilde{\Phi}$ give

$$\tilde{\Phi}_{,\nu} = \frac{2-D}{2}\Omega^{-\frac{D}{2}}\Omega_{,\nu}\Phi + \Omega^{\frac{2-D}{2}}\Phi_{,\nu} , \tag{5.11}$$

$$\tilde{\Phi}^{,\mu} = \tilde{g}^{\mu\nu}\tilde{\Phi}_{,\nu} = \frac{1}{\Omega^2}\left[\frac{2-D}{2}\Omega^{-\frac{D}{2}}\Omega^{,\mu}\Phi + \Omega^{\frac{2-D}{2}}\Phi^{,\mu}\right] . \tag{5.12}$$

On the other hand, from (5.9) and (5.10) we get

$$\tilde{\Box}\ \tilde{\Phi} = \Omega^{-2}\left[\frac{2-D}{2}\Omega^{-\frac{D}{2}}\Phi\Box\Omega + \Omega^{\frac{2-D}{2}}\Box\Phi - \frac{(D-2)(D-4)}{4}\Omega^{-\frac{D+2}{2}}g^{\mu\nu}\Omega_{,\mu}\Omega_{,\nu}\Phi\right] , \tag{5.13}$$

which after the substitution into (5.8) gives

$$\mathrm{S}_{\tilde{\Phi}} = -\frac{1}{2}\int d^D x\sqrt{-g}\left[\frac{2-D}{2}\Phi^2\frac{\Box\Omega}{\Omega} + \Phi\Box\Phi - \frac{1}{4}(D-2)(D-4)g^{\mu\nu}\frac{\Omega_{,\mu}\Omega_{,\nu}}{\Omega^2}\Phi^2\right] . \tag{5.14}$$

Now we notice that the total action in an original frame

$$\tilde{\mathrm{S}} = \mathrm{S}_{\tilde{\Phi}} + \tilde{\mathrm{S}}_C = \frac{1}{2}\int \mathrm{d^D x}\sqrt{-\tilde{\mathrm{g}}}\tilde{\Phi}\left(\frac{1}{4}\frac{\mathrm{D}-2}{\mathrm{D}-1}\tilde{\mathrm{R}}\tilde{\Phi} - \tilde{\Box}\ \tilde{\Phi}\right) , \tag{5.15}$$

is, in fact, conformally invariant, since

$$\mathrm{S} = \mathrm{S}_\Phi + \mathrm{S_C} = \frac{1}{2}\int \mathrm{d^D x}\sqrt{-\mathrm{g}}\Phi\left(\frac{1}{4}\frac{\mathrm{D}-2}{\mathrm{D}-1}\mathrm{R}\Phi - \Box\Phi\right) , \tag{5.16}$$

with S_Φ and S_C defined for the quantities without tildes. The conformally invariant actions (5.15) and (5.16) are the basis to derive the equations of motion via the variational principle. The scalar field equations of motion are

$$\left(\tilde{\Box} - \frac{1}{4}\frac{D-2}{D-1}\tilde{R}\right)\tilde{\Phi} = \Omega^{-\frac{D+2}{2}}\left(\Box - \frac{1}{4}\frac{D-2}{D-1}R\right)\Phi = 0 , \tag{5.17}$$

and they are also conformally invariant having the structure of the Klein-Gordon equation with the mass term replaced by the curvature term [25].

The conformally invariant field equations in D dimensions are

$$\left(\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}\right)\frac{1}{4}\frac{D-2}{D-1}\tilde{\Phi}^2 \quad + \tilde{\Phi}_{,\mu}\tilde{\Phi}_{,\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\Phi}_{,\alpha}\tilde{\Phi}^{,\alpha}$$
$$+ \frac{1}{4}\frac{D-2}{D-1}\left[\tilde{g}_{\mu\nu}\tilde{\Phi}\,\tilde{\Box}\,(\tilde{\Phi}^2) - \tilde{\Phi}(\tilde{\Phi}^2)_{\tilde{;}\mu\nu}\right] = 0\,. \tag{5.18}$$

Since

$$\tilde{\Box}\left(\tilde{\Phi}^2\right) = 2\tilde{\Phi}_{,\alpha}\tilde{\Phi}^{,\alpha} + 2\tilde{\Phi}\,\tilde{\Box}\,\tilde{\Phi}\,, \tag{5.19}$$

$$\left(\tilde{\Phi}^2\right)_{\tilde{;}\mu\nu} = 2\tilde{\Phi}_{,\mu}\tilde{\Phi}_{,\nu} + 2\tilde{\Phi}\tilde{\Phi}_{\tilde{;}\mu\nu}\,, \tag{5.20}$$

then by using (5.19)–(5.20), Eqs. (5.18) can be cast to

$$\left(\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}\right)\frac{1}{4}\frac{D-2}{D-1}\tilde{\Phi}^2 \quad + \frac{1}{2(D-1)}\left[D\tilde{\Phi}_{,\mu}\tilde{\Phi}_{,\nu} - \tilde{g}_{\mu\nu}\tilde{\Phi}_{,\alpha}\tilde{\Phi}^{,\alpha}\right]$$
$$+ \frac{1}{2}\frac{D-2}{D-1}\left[\tilde{g}_{\mu\nu}\tilde{\Phi}\,\tilde{\Box}\,\tilde{\Phi} - \tilde{\Phi}\tilde{\Phi}_{\tilde{;}\mu\nu}\right] = 0\,. \tag{5.21}$$

Notice that the scalar field equations of motion (5.17) can be obtained by the contraction of (5.18) or (5.21). In order to prove the conformal invariance of the field equations (5.21) it is necessary to know the rule of the conformal transformations for the twice covariant derivative of a scalar field which reads as

$$\tilde{\Phi}_{\tilde{;}\mu\nu} = -\frac{1}{2}(D-2)\Omega^{-\frac{D}{2}}\Phi\Omega_{;\mu\nu} + \Omega^{\frac{2-D}{2}}\Phi_{;\mu\nu} + \frac{1}{4}(D-2)(D+4)\Omega^{-\frac{D+2}{2}}\Phi\Omega_{,\mu}\Omega_{,\nu} \tag{5.22}$$
$$- \frac{D}{2}\Omega^{-\frac{D}{2}}\left(\Phi_{,\mu}\Omega_{,\nu} + \Omega_{,\mu}\Phi_{,\nu}\right) - \frac{1}{2}(D-2)\Omega^{-\frac{D+2}{2}}\Phi g_{\mu\nu}\Omega_{,\rho}\Omega^{,\rho} + \Omega^{-\frac{D}{2}}g_{\mu\nu}\Phi_{,\rho}\Omega^{,\rho}\,.$$

Inserting (2.10), (2.12), (5.11), (5.13) and (5.23) into (5.21) one is able to prove the conformal invariance of the gravitational equations of motion which now have the same form in a conformally related frame, i.e.,

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\frac{1}{4}\frac{D-2}{D-1}\Phi^2 \quad + \frac{1}{2(D-1)}\left[D\Phi_{,\mu}\Phi_{,\nu} - g_{\mu\nu}\Phi_{,\alpha}\Phi^{,\alpha}\right]$$
$$+ \frac{1}{2}\frac{D-2}{D-1}\left[g_{\mu\nu}\Phi\Box\Phi - \Phi\Phi_{;\mu\nu}\right] = 0\,. \tag{5.23}$$

The field equations (5.21) and (5.23) generalize those of Hoyle-Narlikar theory onto an arbitrary number of spacetime dimensions. For $D = 4$ these are exactly the same field equations as in the Hoyle-Narlikar theory [23, 24]. Note that the scalar field equations of motion (5.17) can be obtained by the appropriate contraction of Eqs. (5.21) and (5.23) so that they are not independent and do not supply any additional information [29].

The actions (5.15) and (5.16) are usually represented in a different form by the application of the expression for a covariant d'Alembertian for a scalar field in general relativity

$$\tilde{\Box}\,\tilde{\Phi} = \frac{1}{\sqrt{-\tilde{g}}}\,\tilde{\partial}_a\left(\sqrt{-\tilde{g}}\,\tilde{\partial}^a\,\tilde{\Phi}\right)\,, \tag{5.24}$$

which after integrating out the boundary term, gives [3]

$$\tilde{S} = \frac{1}{2}\int d^4x\sqrt{-\tilde{g}}\left[\xi\tilde{R}\tilde{\Phi}^2 + \tilde{\partial}_a\,\tilde{\Phi}\,\tilde{\partial}^a\,\tilde{\Phi}\right]\,, \tag{5.25}$$

and the second term is just a kinetic term for the scalar field (cf. [1, 16]). Equations (5.25) are also conformally invariant since the application of the formulas (2.2), (2.12) and (5.6) together with the appropriate integration of the boundary term gives the same form of the equations

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[\xi R\Phi^2 + \partial_a\Phi\partial^a\Phi\right] . \tag{5.26}$$

Because of the type of non-minimal coupling of gravity to a scalar field $\tilde{\Phi}$ or Φ in (5.25) or (5.26) appropriately and the relation to Brans-Dicke theory we say that these equations are presented in the Jordan frame [3, 6]. It is interesting that the action (5.26) for $D = 4$ is just the Brans-Dicke action [2] with the Brans-Dicke parameter $\omega = -3/2$ [9].

Note that in $D = 4$ dimensions the action (5.15) reads as

$$\tilde{\mathrm{S}} = \frac{1}{2}\int \mathrm{d}^4\mathrm{x}\sqrt{-\tilde{\mathrm{g}}}\tilde{\Phi}\left(\frac{1}{6}\tilde{\mathrm{R}}\tilde{\Phi} - \tilde{\Box}\,\tilde{\Phi}\right) , \tag{5.27}$$

together with (5.6) as

$$\tilde{\Phi} = \Omega^{-1}\Phi \tag{5.28}$$

which, in fact, is conformally invariant, since the conformally transformed action has the same form, i.e.,

$$\mathrm{S} = \frac{1}{2}\int \mathrm{d}^4\mathrm{x}\sqrt{-\mathrm{g}}\Phi\left(\frac{1}{6}\mathrm{R}\Phi - \Box\Phi\right) . \tag{5.29}$$

Now, one can see that the original form of the Einstein-Hilbert action (5.1) can be recovered from (5.27) (or, alternatively (5.29)) provided we assume that

$$\kappa^2 = \frac{6}{\tilde{\Phi}^2} = \mathrm{const.} \tag{5.30}$$

Let us now notice that we can formally attach an energy-momentum tensor of the scalar field in both frames writing (one gets these expressions by putting $\Omega = \tilde{\Phi}$ in (4.19) and $\Omega = \Phi$ in (4.30))

$$^{\tilde{\Phi}}\tilde{T}_{ab} \equiv \frac{1}{6}\left[\tilde{g}_{ab}\tilde{\Phi}_{,c}\tilde{\Phi}^{,c} - 4\tilde{\Phi}_{,a}\tilde{\Phi}_{,b}\right] + \frac{1}{3}\tilde{\Phi}\left[\tilde{\Phi}_{\tilde{;}ab} - \tilde{g}_{ab}\,\tilde{\Box}\,\tilde{\Phi}\right] , \tag{5.31}$$

and

$$^{\Phi}T_{ab} \equiv \frac{1}{6}\left[g_{ab}\Phi_{,c}\Phi^{,c} - 4\Phi_{,a}\Phi_{,b}\right] + \frac{1}{3}\Phi\left[\Phi_{;ab} - g_{ab}\Box\Phi\right] , \tag{5.32}$$

which allow to brief Eqs. (5.21) and (5.23) to the familiar Einstein form

$$\tilde{G}_{ab} = \tilde{R}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{R} = \frac{6}{\tilde{\Phi}^2}\left(^{\tilde{\Phi}}\tilde{T}_{ab}\right) \equiv^{con\tilde{\Phi}} \tilde{T}_{ab} = \tilde{T}^{\Omega}_{ab} , \tag{5.33}$$

and

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = \frac{6}{\Phi^2}\left(^{\Phi}T_{ab}\right) \equiv^{con\Phi} T_{ab} = T^{\Omega}_{ab} , \tag{5.34}$$

and we have added the abbreviation "con" to mark the fact that these tensors are conserved due to the Bianchi identity (compare with (4.19) and (4.30)). Notice that the contraction of (5.32) gives

$$\left(^{\Phi}T\right) = -\Phi\Box\Phi , \tag{5.35}$$

which in view of (5.17) gives the condition

$$R = -\frac{6}{\Phi^2}\left(^{\Phi}T\right) = 6\frac{\Box\Phi}{\Phi} , \tag{5.36}$$

which is just a contraction of (5.34).

If so, by the application of Bianchi identity to the left-hand sides of (5.33) and (5.34) we formally get the conservation laws for these conserved energy-momentum tensors, i.e.,

$$ {}^{con\tilde{\Phi}}\tilde{T}^{a}_{\ b\tilde{;}a} = 0 \, , \tag{5.37} $$

and

$$ {}^{con\Phi}T^{a}_{\ b;a} = 0 \, . \tag{5.38} $$

Writing down the conservation law as if it referred to tensors ${}^{\tilde{\Phi}}\tilde{T}^{ab}$ and ${}^{\Phi}T^{ab}$ gives

$$ {}^{\tilde{\Phi}}\tilde{T}^{a}_{\ b\tilde{;}a} = 2\frac{\tilde{\Phi}_{,a}}{\tilde{\Phi}}\ {}^{\tilde{\Phi}}\tilde{T}^{a}_{\ b} = \frac{1}{3}\tilde{\Phi}\tilde{\Phi}_{,a}\left(\tilde{R}^{a}_{b} - \frac{1}{2}\delta^{a}_{b}\tilde{R}\right) \, , \tag{5.39} $$

and

$$ {}^{\Phi}T^{a}_{\ b;a} = 2\frac{\Phi_{,a}}{\Phi}\,{}^{\Phi}T^{a}_{b} = \frac{1}{3}\Phi\Phi_{,a}\left(R^{a}_{b} - \frac{1}{2}\delta^{a}_{b}R\right) \, , \tag{5.40} $$

and they look like the matter was created.

Another point is that Eqs. (5.21) or (5.23) apparently could give directly the vacuum Einstein field equations for $\tilde{\Phi} = \sqrt{6}/\kappa = \sqrt{6/8\pi G} =$ const. (cf. Eq. (5.30)). The same is obviously true for the field equations (5.23) with the same value of $\Phi = \sqrt{6}/\kappa = \sqrt{6/8\pi G}$ = const. However, this limit is restricted to the case of vanishing Ricci curvature $R = 0$ or $\tilde{R} = 0$ (so only flat Minkowski space limit is allowed) which can be seen from the scalar field equations of motion (5.17).

The admission of the matter part (4.1) into the action (5.29) allows to generalize the field equations (5.23) to

$$ \left(R_{ab} - \frac{1}{2}g_{ab}R\right)\frac{1}{6}\Phi^2 + \frac{1}{6}\left[4\Phi_{,a}\Phi_{,b} - g_{ab}\Phi_{,c}\Phi^{,c}\right] + \frac{1}{3}\left[g_{ab}\Phi\Box\Phi - \Phi\Phi_{;ab}\right] = T^{\mathrm{m}}_{ab} \, , \tag{5.41} $$

or

$$ G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = \frac{6}{\Phi^2}\left[{}^{\Phi}T_{ab} + T^{\mathrm{m}}_{ab}\right] \, , \tag{5.42} $$

and this last formula (5.42) is equivalent to (4.31). These equations (5.42), after contraction, give modified field equations (5.17)

$$ \left(\Box - \frac{1}{6}R\right)\Phi = \frac{T_{\mathrm{m}}}{\Phi} \, , \tag{5.43} $$

where we have used a generalized relation (5.36)

$$ R = -\frac{6}{\Phi^2}\left({}^{\Phi}T + T_{\mathrm{m}}\right) \, . \tag{5.44} $$

However, the variation of the action with matter term which does not depend on the scalar field Φ, i.e.,

$$ \mathcal{L}_{\mathrm{m}} \neq \mathcal{L}_{\mathrm{m}}(\Phi) \tag{5.45} $$

leaves the scalar field equation of motion (5.17) intact. In view of (5.43) this necessarily requires that the trace of the energy-momentum tensor of matter must vanish, i.e.,

$$ T_{\mathrm{m}} = 0. \tag{5.46} $$

It means that only the traceless type of matter can be made consistent with the matter energy-momentum tensor independent of the scalar field Φ. This fact is often expressed as a statement that "photons weigh, but the Sun does not" [27].

In a conformally related frame we apply the matter term (4.1) and (4.3) to get

$$\left(\tilde{R}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{R}\right)\frac{1}{6}\tilde{\Phi}^2 + \frac{1}{6}\left[4\tilde{\Phi}_{,a}\tilde{\Phi}_{,b} - \tilde{g}_{ab}\tilde{\Phi}_{,c}\tilde{\Phi}^{,c}\right] + \frac{1}{3}\left[\tilde{g}_{ab}\tilde{\Phi}\Box\tilde{\Phi} - \tilde{\Phi}\tilde{\Phi}_{;ab}\right] = \tilde{T}^m_{ab} , \quad (5.47)$$

or (cf. (4.20))

$$\tilde{G}_{ab} = \tilde{R}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{R} = \frac{6}{\tilde{\Phi}^2}\left[{}^{\Phi}\tilde{T}_{ab} + \tilde{T}^m_{ab}\right] , \quad (5.48)$$

which after contraction give a modified equation (5.17)

$$\left(\tilde{\Box} - \frac{1}{6}\tilde{R}\right)\tilde{\Phi} = \frac{\tilde{T}_m}{\tilde{\Phi}} . \quad (5.49)$$

From (4.12) it appears transparent that the conformally transformed energy-momentum tensor is conserved only if the trace of it vanishes ($\tilde{T} = 0$) [6, 28]. For example, in the case of barotropic fluid (4.13) it vanishes only for radiation $\mathrm{p} = (1/3)\varrho$. This means that only the photons may obey the equivalence principle and this is not the case for other types of matter since with non-vanishing trace in (4.12) we deal with creation of matter process (compare with Self Creation Cosmology of [27] which has the same field equations (5.47) and (4.12), but Eq. (5.49) is the same only for a vanishing curvature scalar $\tilde{R}$).

The tracelessness of the energy-momentum tensor as a result of the imposition of the conservation law saves the equivalence principle - conformally invariant matter follows geodesic trajectories. An interesting solution is to allow both traceless and traceful matter in which radiation fulfills the equivalence principle and follow geodesics while ordinary matter does not [27]. We will come to this problem later. Of course, in that case ordinary matter is "created" during the evolution and the conservation law is not fulfilled. This will also happen in our case which is alike in Hoyle-Narlikar theory [23].

Let us notice that the application of the Bianchi identity to (5.42) gives

$${}^{\Phi}T^a_{b;a} = 2\frac{\Phi_{,a}}{\Phi}\left({}^{\Phi}T^a_b + T^a_{\mathrm{m}b}\right) - T^a_{\mathrm{m}b;a} = \frac{1}{3}\Phi\Phi_{,a}\left(R^a_b - \frac{1}{2}\delta^a_b R\right) - T^a_{\mathrm{m}b;a} . \quad (5.50)$$

The assumption of the energy-momentum conservation for matter part

$$T^a_{\mathrm{m}b;a} = 0 , \quad (5.51)$$

would give the condition for the equivalence principle to be fulfilled – the assumption which was made in Brans-Dicke theory (see e.g. [28]). However, one can make an assumption that the matter is created in self-creation fashion [27], i.e., that

$$T^a_{\mathrm{m}b;a} = f(\Phi)\Box\Phi = f(\Phi)T_{\mathrm{m}} , \quad (5.52)$$

which would make an agreement with a general requirement of conformal invariance given by (4.15).

Because of the restriction to allow only traceless type of matter, for consistency of the theory, we now admit a potential term into the action in order to get more degrees of freedom while trying to keep conformal invariance. Let us first introduce the potential for the scalar field $\tilde{\Phi}$ (or Φ) itself. First, we try the common mass term

$$\tilde{V}(\tilde{\Phi}) = \frac{1}{2}\tilde{m}^2\tilde{\Phi}^2 . \quad (5.53)$$

The action now looks as follows

$$\tilde{\mathrm{S}} = \frac{1}{2}\int \mathrm{d}^4\mathrm{x}\,\sqrt{-\tilde{\mathrm{g}}}\left[\frac{1}{6}\tilde{\mathrm{R}}\tilde{\Phi}^2 + \tilde{\partial}_{\mathrm{a}}\,\tilde{\Phi}\,\tilde{\partial}^{\mathrm{a}}\,\tilde{\Phi} + \tilde{\mathrm{m}}^2\tilde{\Phi}^2\right] + \tilde{\mathrm{S}}_{\mathrm{m}} , \quad (5.54)$$

which after using (2.1) and (5.28) transforms into

$$\mathrm{S} = \frac{1}{16\pi}\frac{1}{2}\int \mathrm{d}^4\mathrm{x}\,\sqrt{-\mathrm{g}}\left[\frac{1}{6}\mathrm{R}\Phi^2 + \partial_\mathrm{a}\Phi\partial^\mathrm{a}\Phi + \mathrm{m}^2\Phi^2\right] + \mathrm{S_m}\ . \tag{5.55}$$

Thus, it is conformally invariant, if the mass scales as

$$\tilde{m} = \Omega^{-1} m\ , \tag{5.56}$$

and the redefined mass term reads as (cf. Eq. (5.28))

$$V(\Phi) = \frac{1}{2} m^2 \Phi^2\ . \tag{5.57}$$

Formally, the mass term contributes to the energy-momentum tensor as follows

$${}^V T_{ab} = \frac{1}{2} g_{ab} m^2 \Phi^2\ , \tag{5.58}$$

$${}^V T = 2m^2\Phi^2\ . \tag{5.59}$$

The appropriate field equations which result from (5.54) are

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R = \frac{6}{\Phi^2}\left[{}^\Phi T_{ab} + T^\mathrm{m}_{ab}\right] + \frac{1}{4} m^2 g_{ab}\ , \tag{5.60}$$

from which we can immediately realize that the mass term plays the role of the cosmological constant. The contraction of (5.60) gives

$$\left(\Box - m^2\Phi - \frac{1}{6}R\right) = 0\ , \tag{5.61}$$

while varying (5.55) with respect to Φ we obtain

$$\left(\Box - m^2\Phi - \frac{1}{6}R\right) = \frac{1}{\Phi} T_m\ , \tag{5.62}$$

which again is consistent only for traceless matter

$$T_m = -\varrho + 3p = 0\ . \tag{5.63}$$

On the other hand, the application of Bianchi identity to (5.60) gives

$${}^\Phi T^a_{b;a} = 2\frac{\Phi_{,a}}{\Phi}\left[{}^\Phi T^a_b + T^a_{\mathrm{m}b}\right] - T^a_{\mathrm{m}b;a}\ , \tag{5.64}$$

and the conservation law for ordinary matter

$$T^a_{\mathrm{m}b;a} = 0 \tag{5.65}$$

may or may not be imposed. It is interesting to note that the contribution to (5.64) from the mass term (5.57) has been cancelled.

On the other hand, the self-interacting scalar field potential

$$\tilde{U}(\tilde{\Phi}) = \frac{\tilde{\lambda}}{4}\tilde{\Phi}^4\ , \tag{5.66}$$

with the coupling constant $\tilde{\lambda}$ is self-conformally-invariant only in $D = 4$ spacetime dimensions. In order to see this, we start with the action with self-interaction potential which under conformal transformation changes as

$$\begin{aligned} \mathrm{S} &= \frac{1}{2}\int \mathrm{d^D x}\,\sqrt{-\tilde{\mathrm{g}}}\left[\frac{1}{6}\tilde{\mathrm{R}}\tilde{\Phi}^2 + \tilde{\partial}_{\mathrm{a}}\,\tilde{\Phi}\,\tilde{\partial}^{\mathrm{a}}\,\tilde{\Phi} + \frac{\tilde{\lambda}}{4}\tilde{\Phi}^4\right] \\ &= \frac{1}{2}\int \mathrm{d^D x}\,\sqrt{-\mathrm{g}}\left[\frac{1}{6}\mathrm{R}\Phi^2 + \partial_{\mathrm{a}}\Phi\partial^{\mathrm{a}}\Phi + \frac{\tilde{\lambda}}{4}\Omega^{\mathrm{D}-4}\Phi^4\right] . \end{aligned} \tag{5.67}$$

From this, we can immediately see that in order to get conformal invariance one has to rescale the coupling constant as

$$\tilde{\lambda} = \Omega^{4-D}\lambda\,, \tag{5.68}$$

which in $D = 4$ dimensions gives a self-invariance. Now the self-interaction potential reads as

$$U(\Phi) = \frac{\lambda}{4}\Phi^4\,. \tag{5.69}$$

For a more general framework see e.g. [5]. This can also be seen using similar arguments as in Eq. (5.60).

It emerges that it is possible to maintain the conformal invariance of a fermion field $\Psi(x)$ described by the Dirac equation, which applies to spin-$1/2$ particles like quarks, electrons or protons.
The Dirac equation can be derived in the framework of the classical Lagrange field theory by varying the action

$$S_D = \int \mathrm{d^D x}\sqrt{-\mathrm{g}}\bar{\Psi}\left[\imath\partial\!\!\!/ - \mathrm{m}\right]\Psi\,, \tag{5.70}$$

where $\partial\!\!\!/ = \gamma^a\partial/\partial x^a$ and γ^a are Dirac matrices, obeying the anti-commutation relation

$$\left\{\gamma^a,\gamma^b\right\}_+ = 2g^{ab}\,, \tag{5.71}$$

and

$$\gamma^{a\dagger} = \gamma^0\gamma^a\gamma^0\,. \tag{5.72}$$

The convention we use is

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,\text{and } \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{5.73}$$

where $\vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ are the Pauli matrices. The adjoint field $\bar{\Psi}$ is defined by

$$\bar{\Psi} = \Psi^\dagger(x)\gamma^0 = \frac{\delta\mathcal{L}_D}{\delta\Psi_{,0}}\,. \tag{5.74}$$

The action (5.70) transforms under (2.1) as

$$\tilde{S}_D = \int \mathrm{d^D x}\sqrt{-\tilde{\mathrm{g}}}\bar{\tilde{\Psi}}\left(\imath\tilde{\partial\!\!\!/} - \tilde{\mathrm{m}}\right)\tilde{\Psi}\,. \tag{5.75}$$

and is conformally invariant provided that the fermion field transforms as [16]

$$\bar{\tilde{\Psi}} = \Omega^{\frac{1-D}{2}}\Psi \tag{5.76}$$

and the masses scale as in (5.56).

For the vector boson (spin-1) field V_μ with strength $F_{ab} = \partial_a V_b - \partial_a V_b$ action

$$\tilde{S}_V = \int \mathrm{d^D x}\sqrt{-\tilde{g}}\left[-\frac{1}{4}\tilde{\mathrm{F}}_{\mathrm{ab}}\tilde{\mathrm{F}}^{\mathrm{ab}} + \frac{1}{2}\tilde{\mathrm{m}}^2\tilde{\mathrm{V}}_{\mathrm{a}}\tilde{\mathrm{V}}^{\mathrm{a}}\right] , \tag{5.77}$$

the application of (2.1) gives

$$S_V = \int \mathrm{d^D x}\sqrt{-g}\left[-\frac{1}{4}\mathrm{F}_{\mathrm{ab}}\mathrm{F}^{\mathrm{ab}} + \frac{1}{2}\Omega^{\mathrm{D}-2}\tilde{\mathrm{m}}^2\mathrm{V}_{\mathrm{a}}\mathrm{V}^{\mathrm{a}}\right] , \tag{5.78}$$

which means that it can be conformally invariant [26], provided the masses scale according to (5.56), i.e.,

$$\tilde{\mathrm{m}} \;\rightarrow\; \Omega^{\frac{2-D}{2}}\mathrm{m} .$$

It means that scaling of mass is an important possibility to maintain the conformal invariance of the theory.

However, despite the idea is formally correct, it is a bit striking in its physical context, for (5.56) makes the mass term in (5.54) coordinate-dependent. This is due to the fact that Ω is coordinate-dependent. In other words, the mass m in (5.56) can be invariant and coordinate-independent, while after the transformation the mass $\tilde{m}$, as being equal to $\Omega^{-1}m$, is coordinate-dependent. Then, in this case, $\tilde{m}$ is not exactly what we mean by an invariant mass, but it is rather a certain scalar field in the universe. This approach is acceptable if we believe that the mass can really be a cosmic field (cf. [23,26,27,29]). In the simple Friedmann cosmology framework, it means that the mass is only the time-coordinate dependent and it scales with the expansion of the universe becoming effectively a cosmic scalar field [26].

In a more traditional approach [36], one tries to keep the covariant meaning of mass after conformal transformation. This can be achieved, if one introduces a new scalar field χ which transforms conformally in a way similar to (5.28), i.e., by

$$\tilde{\chi} = \Omega^{-1}\chi. \tag{5.79}$$

In this case the mass of the scalar field Φ changes into

$$\tilde{m}^2 = \frac{m^2}{M^2}\tilde{\chi}^2 , \tag{5.80}$$

with M being a dimensional parameter. In such an approach, it is the field $\tilde{\chi}$ (sometimes called the cosmion [37]), which takes a coordinate-dependence rather than $\tilde{m}$. This saves the problem of a mass non-invariance which is faced, if one assumes (5.56). This argument may also be applied to a fermion and a vector mass terms given in (5.75) and in (5.78), as well as to a coupling constant rescaling in (5.68).

Let us finally mention that very general conformally invariant actions based on gauged Wess-Zumino-Witten terms were studied in [30]. Also, multidimensional $f(R)$ theory models which explored conformal transformations were studied in [31].

6 Conformal transformations as duality transformations in superstring theory

It emerges that the conformal transformations under some special conditions may behave like duality transformations in superstring theory [33]. This can be shown easily by defining a conformal factor as

$$\Omega(x) = e^{\omega(x)} , \tag{6.1}$$

where $\omega(x)$ is a new scalar so that

$$\tilde{g}_{ab}(x) = e^{2\omega(x)}g_{ab}(x) , \tag{6.2}$$

and one can get much simpler rules of the transformation for the geometric quantities of the Sect. 2. The Einstein frame (with tildes) and Jordan frame (without tildes) quantities can then be obtained by the simple duality transformation of the form

$$\Omega \leftrightarrow \Omega^{-1} , \tag{6.3}$$

which is equivalent to

$$\omega \leftrightarrow -\omega , \tag{6.4}$$

and corresponds to weak-strong coupling regime duality in superstring theories if ω is a dilaton field [10, 11]. Using (6.1) one has

$$\frac{\Omega_{,a}}{\Omega} = \omega_{,a} \tag{6.5}$$

$$\frac{\Omega_{;ab}}{\Omega} = \omega_{;ab} + \omega_{,a}\omega_{,b} , \tag{6.6}$$

$$\frac{\Box\Omega}{\Omega} = \Box\omega + \omega_{,a}\omega_{,b} , \tag{6.7}$$

etc. For the quantities calculated with respect to the Einstein frame metric one has to replace ; with $\tilde{;}$ and $\Box$ with $\tilde{\Box}$.

Below we give the rules of the conformal transformations of the geometric quantities of Sect. 2 with the conformal factor given by (6.1). This proves especially useful while making higher-order curvature calculations which involve complicated geometrical terms.

Since the rule of the conformal transformation (6.4) now is dual-symmetric, one easily sees that the transition from the quantities in a "no-tilde" frame into a "tilde" frame can be just made by the simple replacement of the "no-tilde" quantities to a "tilde" quantities and the replacement of ω into $-\omega$, $\omega_{,a}$ into $-\omega_{,a}$, $\omega_{;ab}$ into $-\omega_{\tilde{;}ab}$, $\Box\omega$ into $-\tilde{\Box}\,\omega$ etc. Due to this, in order to avoid a simple repetition of the conformal transformation rules in the text, we present only "one-way" transformations of the geometric quantities starting from the "tilde" frame and terminating in the "no-tilde" frame as follows further.
The Christoffel connection coefficients transform as

$$\Gamma^c_{ab} = \tilde{\Gamma}^c_{ab} - \left(\tilde{g}^c_a\omega_{,b} + \tilde{g}^c_b\omega_{,a} - \tilde{g}_{ab}\tilde{g}^{cd}\omega_{,d}\right) , \qquad \Gamma^b_{ab} = \tilde{\Gamma}^b_{ab} - D\omega_{,a}. \tag{6.8}$$

The Riemann tensor transforms as

$$\begin{aligned} R^a{}_{bcd} = \tilde{R}^a{}_{bcd} &- \left[\delta^a_d\omega_{\tilde{;}bc} - \delta^a_c\omega_{\tilde{;}bd} + \tilde{g}_{bc}\omega^{\tilde{;}a}{}_{\tilde{;}d} - \tilde{g}_{bd}\omega^{\tilde{;}a}{}_{\tilde{;}c}\right] \\ &+ \left[\delta^a_c\omega_{,b}\omega_{,d} - \delta^a_d\omega_{,b}\omega_{,c} + \tilde{g}_{bd}\omega^{,a}\omega_{,c} - \tilde{g}_{bc}\omega^{,a}\omega_{,d}\right] + \left[\delta^a_d\tilde{g}_{bc} - \delta^a_c\tilde{g}_{bd}\right]\tilde{g}_{ef}\omega^{,e}\omega^{,f} , \end{aligned} \tag{6.9}$$

the Ricci tensor transforms as

$$R_{ab} = \tilde{R}_{ab} + (D-2)\left[\omega_{,a}\omega_{,b} - \tilde{g}^{cd}\omega_{,c}\omega_{,d}\tilde{g}_{ab}\right] + \left[(D-2)\omega_{\tilde{;}ab} + \tilde{g}_{ab}\,\tilde{\Box}\,\omega\right] , \tag{6.10}$$

the Ricci scalar transforms as

$$R = e^{2\omega}\left\{\tilde{R} - (D-1)\left[(D-2)\tilde{g}^{ab}\omega_{,a}\omega_{,b} - 2\,\tilde{\Box}\,\omega\right]\right\} , \tag{6.11}$$

and the d'Alembertian transforms as

$$\Box\phi = e^{2\omega}\left[\tilde{\Box}\,\phi - (D-2)\tilde{g}^{ab}\omega_{,a}\phi_{,b}\right] . \tag{6.12}$$

In fact, the obtained formulas (6.9), (6.10), and (6.11) agree with the formulas (25), (26), and (27) given in [32], provided one puts $\omega = -\sigma$ in their notation. The Einstein tensor transforms as

$$G_{ab} = \tilde{G}_{ab} + (D-2)\left[\omega_{,a}\omega_{,b} + \frac{1}{2}(D-3)\tilde{g}^{cd}\omega_{,c}\omega_{,d}\tilde{g}_{ab}\right] + (D-2)\left[\omega_{\tilde{;}ab} - \tilde{g}_{ab}\,\tilde{\Box}\,\omega\right]. \tag{6.13}$$

The curvature invariants transform as

$$\begin{aligned} R^2 = e^{4\omega}\Big\{&\tilde{R}^2 + 4(D-1)\,\tilde{\Box}\,\omega\left(\tilde{\Box}\,\omega + R\right) \\ &+ (D-1)(D-2)\tilde{g}^{ab}\omega_{,a}\omega_{,b}\left[(D-1)(D-2)\tilde{g}^{ef}\omega_{,e}\omega_{,f} - 4(D-1)\,\tilde{\Box}\,\omega - 2\tilde{R}\right]\Big\}\,, \end{aligned} \tag{6.14}$$

$$\begin{aligned} R_{ab}R^{ab} = e^{4\omega}\Big\{&\tilde{R}_{ab}\tilde{R}^{ab} + 2\left[(D-2)\tilde{R}_{ab}\omega^{\tilde{;}ab} + \tilde{R}\,\tilde{\Box}\,\omega\right] \\ &+ (D-2)^2\omega_{\tilde{;}ab}\omega^{\tilde{;}ab} + (3D-4)\left(\tilde{\Box}\,\omega\right)^2 + 2(D-2)\left[\tilde{R}_{ab}\omega^{,a}\omega^{,b} - \tilde{R}\tilde{g}^{ef}\omega_{,e}\omega_{,f}\right] \\ &- (D-2)^2\omega_{\tilde{;}ab}\omega^{,a}\omega^{,b} - (D^2+D-3)\,\tilde{\Box}\,\omega\tilde{g}^{ef}\omega_{,e}\omega_{,f} \\ &+ (D-1)(D^2-4D+7)\left(\tilde{g}^{ef}\omega_{,e}\omega_{,f}\right)^2\Big\}\,, \end{aligned} \tag{6.15}$$

$$\begin{aligned} R_{abcd}R^{abcd} = e^{4\omega}\Big\{&\tilde{R}_{abcd}\tilde{R}^{abcd} + 8\tilde{R}_{bc}\omega^{\tilde{;}bc} + 8\tilde{R}_{bc}\omega^{,b}\omega^{,c} - 4\tilde{R}\tilde{g}^{cd}\omega_{,c}\omega_{,d} + 4\left(\tilde{\Box}\,\omega\right)^2 \\ &+ 4(D-2)\omega_{\tilde{;}bc}\omega^{\tilde{;}bc} - 8(D-2)\left[\tilde{\Box}\,\omega\tilde{g}^{cd}\omega_{,c}\omega_{,d} - \omega_{\tilde{;}bc}\omega^{,b}\omega^{,c}\right] \\ &+ 2(D-1)(D-2)\left(\tilde{g}^{cd}\omega_{,c}\omega_{,d}\right)^2\Big\}\,. \end{aligned} \tag{6.16}$$

Again, the formulas (6.15), (6.15), and (6.16) agree with the formulas (31), (30), and (29) of [32]. Finally, the Gauss-Bonnet invariant transforms as (compare the formula (35) of [32])

$$\begin{aligned} R_{GB} \equiv R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 = e^{4\omega}\Big\{&\tilde{R}_{GB} - 4(D-3)\left(2\tilde{R}_{ab}\omega^{\tilde{;}ab} - \tilde{R}\,\tilde{\Box}\,\omega\right) \\ &+ 4(D-3)(D-2)\left[\left(\tilde{\Box}\,\omega\right)^2 - \omega_{\tilde{;}ab}\omega^{\tilde{;}ab}\right] - 2(D-3)(D-4)\tilde{R}\tilde{g}^{cd}\omega_{,c}\omega_{,d} \\ &- 8(D-3)\tilde{R}_{ab}\omega^{,a}\omega^{,b} - 4(D-2)(D-3)^2\,\tilde{\Box}\,\omega\tilde{g}^{cd}\omega_{,c}\omega_{,d} \\ &- 8(D-2)(D-3)\omega_{\tilde{;}ab}\omega^{,a}\omega^{,b} + (D-1)(D-2)(D-3)(D-4)\left(\tilde{g}^{cd}\omega_{,c}\omega_{,d}\right)^2\Big\}\,. \end{aligned} \tag{6.17}$$

An interesting example of the duality-like symmetry was considered in [34] for a metric-dilaton model of the form

$$S = \int d^4x\sqrt{-g}\left[A(\phi)g^{ab}\partial_a\phi\partial_b\phi + B(\phi)R + C(\phi)\right]\,, \tag{6.18}$$

where $A(\phi)$, $B(\phi)$, and $C(\phi)$ are the functions of the dilaton which obey some constraints. A generalized variant of this theory was studied in [35].

7 Conclusion

In this paper we discussed the rules of conformal transformations for geometric quantities in general relativity such as connection coefficients, Riemann tensor, Ricci tensor, Ricci scalar, Einstein tensor and the d'Alembertian operator in an arbitrary spacetime dimension D. Since the conformal transformations are also used to investigate higher-order gravity theories, we also found the conformal transformation rules for the curvature invariants R^2, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$ and, as a consequence, for the Gauss-Bonnet invariant in D spacetime dimensions.

We devoted some effort in order to discuss precisely the conformal transformations of the matter energy-momentum tensor and, in particular, the energy-momentum tensor composed of the conformal factor Ω. We showed that the conserved energy-momentum tensor is not the same as the energy-momentum tensor of the conformal factor and this is the reason why we may say that the conformal transformation "creates" an extra matter term composed of the conformal factor which enters the conservation law. In other words, an empty Minkowski space after conformal transformation may produce a non-zero energy-momentum tensor composed of the conformal factor Ω.

We also discussed how to construct the conformally invariant gravity. Its simplest version is a special case of the scalar-tensor Brans-Dicke theory–the one with the Brans-Dicke parameter $\omega = -3/2$. It can be made conformally invariant due to the admission a non-minimal coupling of the scalar field to gravity as well as due to the admission of the appropriate kinetic term for a scalar field. We have shown that the massive scalar field, self-interacting scalar field, the Dirac field and the vector field theories can also be made conformally invariant at the expense of the rescaling appropriate fields and, in particular, of the mass scaling.

Finally, we presented already obtained rules of the conformal transformations for the geometrical quantities in the fashion of the duality transformation as in superstring theory. In such a case the transitions between conformal frames can just be obtained by a simple change of the sign of the quantity $\omega = \ln \Omega$, where Ω is the conformal factor. We found these rules the easiest of all possibilities.

We are aware of the fact that many studies of the problem of the conformal transformations of the geometrical and physical quantities have been done so far. However, we decided to collect all of these rules in one paper in order to give the reader a fairly comprehensive collection of these transformations for future reference.

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