Learning neural network-based boundary conditions for kinetic plasma sheath dynamics

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1 Normalized Vlasov-Fokker-Planck equation

The governing kinetic equation for our study is the 1D1V Vlasov-Dougherty-Fokker-Planck equation in the "flexible plasma normalization" [2]:

$$\partial_t f_s + v\partial_x f_s + (\omega_c \tau)\frac{Z_s}{A_s} E \partial_v f_s = (\nu_p \tau) \sum_s \nu_{ss'} \partial_v \left(\frac{T_{ss'}}{m_s} \partial_v f_s + (v - u_{ss'}) f_s \right). \tag{1}$$

The normalization constants appearing in this equation are as follows:

- Z_s and A_s are the normalized charge and mass of species s, expressed in units of the proton charge and mass.
- $\omega_c \tau$ is the normalized reference proton cyclotron frequency in a field B_0 which would give unit plasma beta: $|B_0|^2/2\mu_0 = n_0 T_0$.
- $\nu_p \tau$ is the normalized proton collision frequency.

For now we can take these normalization constants as given; we will need to concern ourselves with their definitions when we move on to translating a specific physical problem into our equation setup.

Equation (1) is coupled to the normalized Gauss's law,

$$\partial_x E = \frac{(\omega_p \tau)^2}{\omega_c \tau} \rho_c, \tag{2}$$

where

$$\rho_c = \sum_s Z_s \int f_s \, \mathrm{d}v \tag{3}$$

is the charge density. We will use the elliptic form of Gauss's law,

$$\partial_x^2 \phi = -\frac{(\omega_p \tau)^2}{\omega_c \tau} \rho_c, \tag{4}$$

with $E = -\partial_x \phi$.

1.1 Collision operator

Suggest we use the collision parameters derived in [1]

2 Domain and boundary conditions

We'll use a physical domain of length L_x, which by convention extends from $-L_x/2$ to $L_x/2$.

2.1 Absorbing wall

The simplest boundary condition that will produce a Langmuir sheath is the absorbing wall boundary condition. At a spatial boundary x_b with outward normal vector $\mathbf{n}(x_b)$, we have

$$f_s^b(x_b, v) = \mathbf{1}_{\mathbf{n}(x_b)\cdot v<0} f_s(x_b, v), \tag{5}$$

where $\mathbf{1}$ is the indicator function.

3 Straightforward DLR approximation

Each species distribution function f_s receives a separate low-rank decomposition:

$$f_s \approx \sum_{ij} X_{si}(x,t) S_{sij}(t) V_{sj}(v,t). \tag{6}$$

In what follows we will omit species subscripts.

Given that we need to apply boundary conditions in x, which should be applied to $K_j(x,t)$ rather than $X_i(x,t)$, we keep all spatial derivatives applied to K.

K step:

$$\begin{aligned}\partial_t K_j(x,t) + \sum_l \langle V_j, v V_l\rangle_v \partial_x K_l(x,t) + (\omega_c\tau)\frac{Z_s}{A_s}\sum_l \langle V_j, \partial_v V_l\rangle_v E K_l(x,t) \\ = \nu_p\tau \sum_{s'} \nu_{ss'} \sum_l \left[\frac{T_{ss'}}{m_s}\left\langle V_j \partial_v^2 V_l\right\rangle_v + \langle V_j, \partial_v(v V_l)\rangle_v - u_{ss'}\langle V_j, \partial_v V_l\rangle_v\right] K_l(x,t)\end{aligned}$$

S step:

$$\begin{aligned}\partial_t S_{ij}(x,t) + \sum_l \langle V_j, v V_l\rangle_v \langle X_i, \partial_x K_l\rangle_x + (\omega_c\tau)\frac{Z_s}{A_s}\sum_{kl} \langle V_j, \partial_v V_l\rangle_v \langle X_i, E X_k\rangle_x S_{kl}(t) \\ = \nu_p\tau \sum_{s'} \nu_{ss'} \sum_{kl} \left[\frac{1}{m_s}\langle X_i, T_{ss'} X_k\rangle_x \left\langle V_j, \partial_v^2 V_l\right\rangle_v + \delta_{ik}\langle V_j, \partial_v(v V_l)\rangle_v - \langle X_i, u_{ss'} X_k\rangle_x \langle V_j, \partial_v V_l\rangle_v\right] S_{kl}(x,t)\end{aligned}$$

L step:

$$\begin{aligned}\partial_t L_i(v,t) + \sum_k \langle X_i, \partial_x K_l\rangle_x v V_l(v,t) + (\omega_c\tau)\frac{Z_s}{A_s}\sum_k \langle X_i, E X_k\rangle_x \partial_v L_k(v,t) \\ = \nu_p\tau \sum_{s'} \nu_{ss'} \sum_k \left[\frac{1}{m_s}\langle X_i, T_{ss'} X_k\rangle_x \partial_v^2 L_k(v,t) + \delta_{ik}\partial_v(v L_k(v,t)) - \langle X_i, u_{ss'} X_k\rangle_x \partial_v L_k(v,t)\right]\end{aligned}$$

The boundary conditions on K_j are the projection of the full distribution function BCs:

$$\begin{aligned}K_j^b(x_b) &= \langle V_j, \mathbf{1}_{\mathbf{n}(x_b)\cdot v<0} f_s(x_b, v)\rangle_v & (7)\\ &= \left\langle V_j, \mathbf{1}_{\mathbf{n}(x_b)\cdot v<0}\left(\sum_{kl} X_k(x_b) S_{kl} V_l(v)\right)\right\rangle_v & (8)\\ &= \sum_l K_l(x_b)\left\langle V_j, \mathbf{1}_{\mathbf{n}(x_b)\cdot v<0} V_l(v)\right\rangle_v & (9)\end{aligned}$$

3.1 Spatial discretization

- I suggest using a finite volume scheme with MUSCL-like slope-limited reconstruction. Such an approach is extremely robust if using a TVD limiter, efficient, and easy to implement.
- For the spatial numerical flux, I suggest we use the kinetic flux vector splitting method applied to the DLR of the Boltzmann equation in [**huAdaptiveDynamicalLow2022**]
- A similar numerical flux can be applied for the $E\partial_v f$ terms.

4 Circuit model

Series RLC circuit equations for a discharging capacitor. Let Q be the charge in the capacitor, C the capacitance, R the resistance, and L the inductance. The circuit current is $I = \dot{Q}$, the rate of change of the charge. The voltages of each component are related by

$$V_R + V_L + V_C + V_p = 0, \tag{10}$$

where V_p is the voltage drop across the plasma, and V_R, V_L, V_C are the voltage drops across the resistor, inductor, and capacitor respectively. From the definition of capacitance we have $V_C = Q/C$. From Ohm's law we have $V_R = IR = \dot{Q}R$, and $V_L = \dot{I}L = \ddot{Q}L$.

References

[1] Evan Habbershaw et al. *A Nonlinear, Conservative, Entropic Fokker-Planck Model for Multi-Species Collisions*. Apr. 2024. arXiv: `2404.11775 [math-ph]`. (Visited on 05/22/2024).

[2] S. T. Miller and U. Shumlak. “A Multi-Species 13-Moment Model for Moderately Collisional Plasmas”. In: *Physics of Plasmas* 23.8 (Aug. 2016), p. 082303. ISSN: 1070-664X, 1089-7674. DOI: `10.1063/1.4960041`. (Visited on 02/11/2022).