Homework 2

AE402 - Fall 2018
Emilio R. Gordon

Textbook Problem 2.3: A satellite is in an elliptic orbit with $e = 0.5$. Using a starting value of $E_0 = M$, determine the value of E to an accuracy of 10^-4 radians at the time the satellite is one-quarter period past periapses passage. List all iterations including the value of $F(E)$ by:

Constants Defined

$$\mu_{\text{Earth}} = 3.9860044188 \times 10^5 \, [km^3 s^{-2}]$$
$$e = 0.5$$
$$M = n\,(t - t_p) \text{ and we know } (t - t_p) = \frac{T}{4} \text{ and } n = \frac{2\,\pi}{T}$$
$$M = \frac{\pi}{2}\,[rad]$$

1. using the Newton algorithm

Iterations	Eccentric Anomaly Assumption
0	1.5707963267949
1	2.0707963267949
2	2.02142301987214
3	2.02097997437267

```
function [counter, values] = KeplersEqnNewtonMethod(M,e,E_0,tol)
x = E_0; i=0;
counter = [i];
values = [x];
    while abs(fn(M,x,e))>tol
        dx = -fn(M,x,e)/fnp(M,x,e);
        i=i+1;
        x = x+dx;
        counter = [counter; i];
        values = [values;x];
    end
end

%% Background Functions
function f = fn(M,En,e)
% Evaluate the given function.
 f = M-(En-e*sin(En));
end

function fp = fnp(M,En,e)
% Evaluate the derivative of the given function.
 fp = -1 + e*cos(En);
end
```

2. using the Laguerre algorithm (2.43)

Iterations	Eccentric Anomaly Assumption
0	1.5707963267949
1	2.0707963267949
2	2.02165684296529
3	2.02098006372809

```
function [counter, values] = KeplersEqnLaguerreMethod(M,e,E_0,tol)
x = E_0;
i=0;
n = 4;
counter = [i];
values = [x];
    while abs(fn(M,x,e))>tol
        if fnp(M,x,e)>0
            dx = -(n*fn(M,x,e))/(fnp(M,x,e)+sqrt((n-1)^2*(fnp(M,x,e))^2 - n*(n-1)*fn(M,x,e)*fnpp(M,x,e)));
            i=i+1;
            x = x+dx;
        elseif fnp(M,x,e)<0
            dx = -(n*fn(M,x,e))/(fnp(M,x,e)-sqrt((n-1)^2*(fnp(M,x,e))^2 - n*(n-1)*fn(M,x,e)*fnpp(M,x,e)));
            i=i+1;
            x = x+dx;
        else
            dx = 0;
            i=i+1;
            x = x+dx;
        end
        counter = [counter; i];
        values = [values;x];
    end
end

%% Background Functions

function f = fn(M,En,e)
% Evaluate the given function.
 f = M-(En-e*sin(En));
end

function fp = fnp(M,En,e)
% Evaluate the derivative of the given function.
 fp = -1 + e*cos(En);
end

function fpp = fnpp(M,En,e)
% Evaluate the derivative of the given function.
 fpp = -e*sin(En);
end
```

3. Repeat parts a) and b) using the starting value of Eq. (2.16)

$$E_o = \frac{M(1 - \sin u) + u \sin M}{1 + \sin M - \sin u}$$

Where $u \equiv M + e$. This give us $E_o = 3.06935014444381$.

Newton

Iterations	Eccentric Anomaly Assumption
0	3.06935014444381
1	2.09352572934448
2	2.02190453767231
3	2.02098009603149

Laguerre

Iterations	Eccentric Anomaly Assumption
0	3.06935014444381
1	2.21917526854539
2	2.03139126721541
3	2.02100963487613

4. Calculate the value of f, for the solution.

From before, we found that $E = 2.021$ [rad]. Using the following equation we can compute f using E.

$$\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$$

$$f = 2.4466\,[rad]$$

Homework Problem 2 Code:

```
%{
An object is detected by AF Space Command in Colorado Springs. Its position
and velocity are determined below to be:
%}
        % Given
                r = [-4743;4743;0];
                v = [-5.879;-4.223;0];
                mu_E = 3.9860044188e5;

%% Part A: What is the semi-major axis of the object's orbit?
                a = 1/((-((norm(v))^2/mu_E))+(2/norm(r)))

%% Part B: What is the eccentricity of the orbit?
                e = sqrt(1-((norm(cross(r,v)))^2/(mu_E*a)))

%% Part C.A: What is the instantaneous true anomaly, f?
                f = acosd((((a*(1-e^2))/norm(r))-1)/e)
                r_dot = dot(r,v)/norm(r) %Positive therefore f is positive

%% Part C.B: What is the Instantaneous Mean Anomaly, M?
                E = acosd((1-(norm(r)/a))/e)
                M = E - e*sind(E)

%% Part D: How much time (in minutes) will pass between detection of the object and its impact on the earths surface
                r_impact = 6378
                n = sqrt(mu_E/a^3)

                f_impact = acos((((a*(1-e^2))/r_impact)-1)/e)
                E_impact = 2*atan(sqrt((1-e)/(1+e))*tan(f_impact/2))
                M_impact = E_impact - e*sin(E_impact)

                t = M_impact/n
                t_min = (t/60)
%% Part E: What will the speed of the object be at impact?
                v_impact = sqrt(mu_E*((2/r_impact)-(1/a)))
```

Textbook Problem 2.19: A spacecraft is in an earth orbit having a semi-major axis of 7200 km and eccentricity of 0.06.

Constants Defined

$$\mu_{\text{Earth}} = 3.9860044188 \times 10^5 \, [km^3 s^{-2}]$$
$$e = 0.06$$
$$a = 7200 \, [km]$$

1. What is the period of the orbit (in minutes)?

$$T = 2\pi \sqrt{\frac{a^3}{\mu_{\text{Earth}}}} = 6080.1 \, [\text{sec}] = 101.3 \, [\text{min}]$$

2. Find the true anomaly of the spacecraft 60 minutes after it passes perigee. Using the results from (a) what quadrants could the spacecraft be in.

   (a) Calculate Mean Anomaly

$$M = \sqrt{\frac{\mu_{\text{Earth}}}{a^3}} \, t \cdot 60 \, \text{sec} = 3.72[\text{rad}]$$

   (b) Calculate Eccentric Anomaly using Newtons Method. Initial guess, $E_0 = M$ and a tolerance of 10^{-14}

   (c) Calculate True Anomaly

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan\frac{E}{2} \quad \rightarrow \quad f = -2.625 \, [\text{rad}]$$

   This result makes since given that a quarter period is around 25 minutes. This means that after 60 minutes, it should be somewhere in the third quadrant.

3. Using the true anomaly found in (b), find the position vector of the spacecraft 60minutes after it passes perigee. Express the results on the basis vectors shown in Fig. 2.1.

   (a) Calculate the Semi-Minor Axis

$$b = a\sqrt{1-e^2} = 7187.03 \, [\text{km}]$$

   (b) Calculate Position Vector

$$r = \begin{bmatrix} a \cos(E) - e \\ \frac{b}{a} (a \sin(E) \\ 0 \end{bmatrix} = \begin{bmatrix} -6579.82 \\ -3740.82 \\ 0 \end{bmatrix} [\text{km}]$$

4. We have the exact result. Now use the f and g series (2.35) to find (an approximation to) the position vector $\mathbf{r}$ of the spacecraft 60minutes after it passes perigee.

(a) F and G Series

$$t = 60\,[\text{min}] = 3600\,[\text{sec}]$$

$$r_o = a\,(1-e)\,\hat{\imath} = 6768\,\hat{\imath}\,[\text{km}]$$

$$v_o = \sqrt{\mu_{\text{Earth}}\left(\frac{2}{r_o} - \frac{1}{a}\right)}\,\hat{\jmath} = 7.90\,\hat{\jmath}\,[\text{km/s}]$$

$$H_o = \frac{\mu_{\text{Earth}}}{r_o^3} = 1.29 \times 10^{-6}\left[\frac{1}{s^2}\right]$$

$$P_o = 0 \qquad \text{Because } r_o \text{ and } v_o \text{ are perpendicular}$$

(b) Calculate Position Vector

$$\mathbf{r} = r_o\left(1 - \left(\frac{t^2}{2}\right)H_o\right) + v_o\left(t - \left(\frac{t^3}{6}\right)H_o\right) = \begin{bmatrix} -49620.7 \\ -50551.7 \\ 0 \end{bmatrix}\ [\text{km}]$$

5. How well do the two estimates of $\mathbf{r}$ compare?

   The two estimates of r do not show any similarities or proximities to each other. The problem with this approach is that we are expanding our solution around a specific point far from the point we currently know. With this method, it is a known fact that as $t - t_o$ gets larger, the accuracy of our solution will decrease. A solution to this would be to include higher order terms to our series to recover some accuracy however this will come at the expense of computation time and complexities.

6. The series (2.35) becomes less accurate as $(t - t_o)$ increases, because it comes from a Taylor series expansion. In fact, as $(t - t_o)$ increases it will eventually reach a point such that the Taylor series will not converge. To improve the result we will employe a clever trick. **Use apogee passage to defined the epoch time**. You will need to redefine and determine the epoch position and velocity vectors and the new time elapsed since epoch. Find a new estimate for $\mathbf{r}$.

(a) F and G Series

$$t_{\text{apogee}} = \frac{\text{Periode}}{2} = 3040.04\,[\text{sec}]$$

$$t_o = t - t_{\text{apogee}} = 559.96\,[\text{sec}]$$

$$r_o = a\,(1-e)\,\hat{\imath} = -7632\,\hat{\imath}\,[\text{km}]$$

$$v_o = \sqrt{\mu_{\text{Earth}}\left(\frac{2}{r_o} - \frac{1}{a}\right)}\,\hat{\jmath} = -7.01\,\hat{\jmath}\,[\text{km/s}]$$

$$H_o = \frac{\mu_{\text{Earth}}}{r_o^3} = 8.97 \times 10^{-7}\left[\frac{1}{s^2}\right]$$

$$P_o = 0 \qquad \text{Because } r_o \text{ and } v_o \text{ are perpendicular}$$

(b) Calculate Position Vector

$$\mathbf{r} = r_o\left(1 - \left(\frac{t^2}{2}\right)H_o\right) + v_o\left(t - \left(\frac{t^3}{6}\right)H_o\right) = \begin{bmatrix} -6559.15 \\ -3739.61 \\ 0 \end{bmatrix}\,[\text{km}]$$

7. Does this improve the accuracy of the result.

This estimate gives a much better result when comparing to our results of the position vector from part C. When expanding the taylor series around the apogee we get a much better result since the point we want to find is closer to our known point (position at apogee). In addition, this is likely due to the fact that the quality $t - t_o$ is much smaller than if we had set our t to be at periapsis. Apoapsis and periapsis are good choices for expanding around due to the fact that the velocity and position vectors are orthogonal, thus simplifying our expression.

```matlab
function HW2P3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Part A
                a = 7200; % Semi-Major Axis [km]
                e = 0.06; % Eccentricity
                mu_earth = 3.9860044188e5; % Standard Gravitational Parameter [km^3/s^2]
        % Period of an Orbit
                T_sec = 2*pi*sqrt(a^3/mu_earth); %[sec]
                T_min = (T_sec/60); % [min]
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Part B
                t = 60; % [min]
                t = t*60; % [sec]
        % Calculate Mean Anomaly
                M = sqrt(mu_earth/a^3)*t;
        % Calculate Eccentric Anomaly using Newtons Method
                E_0 = M;
                tol = 10^-14;
                [iterations, values] = KeplersEqnNewtonMethod(M,e,E_0,tol);
                E = values(end);
        % Caclulate True Anomaly
                f = 2*atan2(tan(E/2),sqrt((1-e)/(1+e)));
        % Verify Logics
                T_min/4;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Part C:
        % Calculate the Semi-Minor Axis
                b = a*sqrt(1-e^2);
        % Calculate Position Vector
                r = [a*(cos(E)-e); (b/a)*(a*sin(E)); 0];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Part D:
                t = 60; % [min]
                t = t*60; % [sec]
                r_o = [a*(1-e); 0; 0];
                v_o = [0; sqrt(mu_earth*((2/norm(r_o))-(1/a))); 0];
                H_o = mu_earth/norm(r_o)^3;
                P_o = 0;
                r = r_o*(1-(t^2/2)*H_o) + v_o*(t - (t^3/6)*H_o);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Part F:
        % Define a new elapsed time past epoch.
                t_apogee = T_sec/2 %Time at apogee
                t_o = t - t_apogee
                r_o = [-a*(1+e); 0; 0]
                v_o = [0; -sqrt(mu_earth*((2/norm(r_o))-(1/a))); 0]
                H_o = mu_earth/norm(r_o)^3
                P_o = 0
                r = r_o*(1-(t_o^2/2)*H_o) + v_o*(t_o - (t_o^3/6)*H_o)
end
```

Textbook Problem 2.11: An earth-orbiting satellite has a period of 15.743 hr and a perigee radius of 2 earth radii

1. Determine the semi-major axis of the orbit.
   We know...
$$\frac{2\,\pi}{T} = \sqrt{\frac{\mu_{\text{Earth}}}{a^3}} \quad \rightarrow \quad a = \left(\frac{\mu_{\text{Earth}}}{\left(\frac{2\,\pi}{T}\right)^2}\right)^{\frac{1}{3}} = 31889.86\,[km]$$

2. Determine the position and velocity vectors at perigee
$$r_o = \begin{bmatrix} r_{\text{perigee}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 12742 \\ 0 \\ 0 \end{bmatrix} \text{ [km]} \qquad v_o = \begin{bmatrix} 0 \\ \sqrt{\mu_{\text{Earth}}\left(\frac{2}{r_{\text{perigee}}} - \frac{1}{a}\right)} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 7.076 \\ 0 \end{bmatrix} \text{ [km/s]}$$

3. using the f and g functions, eqns. (2.26) and (2.28), find the position vector r letting perigee passage be the epoch time t_o

   (a) Calculate Mean Anomaly
$$M = \sqrt{\frac{\mu_{\text{Earth}}}{a^3}}\, t \cdot 60 \cdot 60\,\text{sec} = 3.99\,\text{[rad]}$$

   (b) Calculate the Eccentricity
$$e = 1 - \frac{r_{\text{perigee}}}{a} = 0.60$$

   (c) Calculate the Semi-Minor Axis
$$b = a\,\sqrt{1 - e^2} = 25501.43\,\text{[km]}$$

   (d) Calculate Eccentric Anomaly using Newtons Method. Initial guess, $E_0 = M$ and a tolerance of 10^{-14}. After 4 iterations, the eccentric anomaly is found to be:
$$E = 3.68\,\text{[rad]}$$

   (e) Calculate the f and g functions.
$$f = 1 - \frac{a}{r_o}\,(1 - \cos E) = -3.65$$
$$g = t - \sqrt{\frac{a^3}{\mu_{\text{Earth}}}}\,(E - \sin E) = -1854.62$$

$$r = f \cdot r_o + g \cdot v_o = \begin{bmatrix} -46491.43 \\ -13122.73 \\ 0 \end{bmatrix} \text{ [km]}$$

```matlab
function HW2P4

T = 15.743; % Period [hr]
mu_earth = 3.9860044188e5; % Standard Gravitational Parameter km^3/s^2
R_earth = 6371; % Radius of Earth [km]

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Part A: Determine the semimajor axis of the orbit
                T = T*60*60; % Period [sec]
                r_perigee = 2*R_earth; % Radius at perigee [km]
        % Semi-Major Axis
                a = (mu_earth/((2*pi)/(T))^2)^(1/3);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Part A': Determine the position and velocity vectors at perigee
                r_o = [r_perigee; 0; 0]
                v_o = [0; sqrt(mu_earth*((2/r_perigee)-(1/a))); 0]
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Part B: Determine the position and velocity vectors after perigee
                t = 10;% Perigee Passage [hr]
                t = t*60*60;% Perigee Passage [sec]
        % Calculate Mean Anomaly
                M = sqrt(mu_earth/a^3)*t;
        % Calculate Eccentricity
                e = 1-(r_perigee/a);
        % Calculate the Semi-Minor Axis
                b = a*sqrt(1-e^2);
        % Calculate Eccentric Anomaly using Newtons Method
                E_0 = M;
                tol = 10^-14;
                [iterations, values] = KeplersEqnNewtonMethod(M,e,E_0,tol);
                E = values(end);
        % Calculate f and g functions (E_o = 0 b/c at perigee)
                f = 1 - (a/norm(r_o))*(1-cos(E));
                g = t-sqrt(a^3/mu_earth)*(E-sin(E));
                r = f*r_o + g*v_o
end
```