Section 9: Electrostatic Propulsion

AE435

Spring 2018

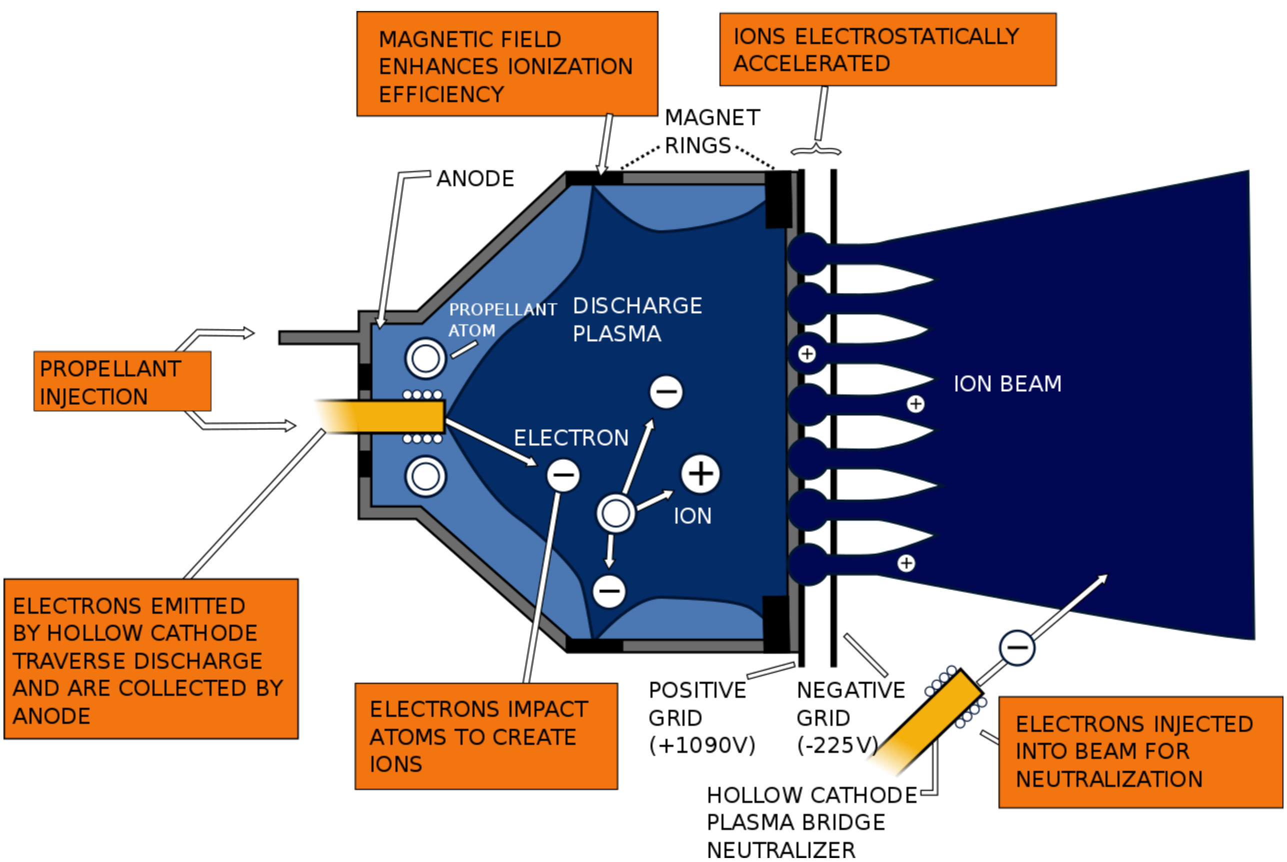
1 Gridded Ion Engines

Contents

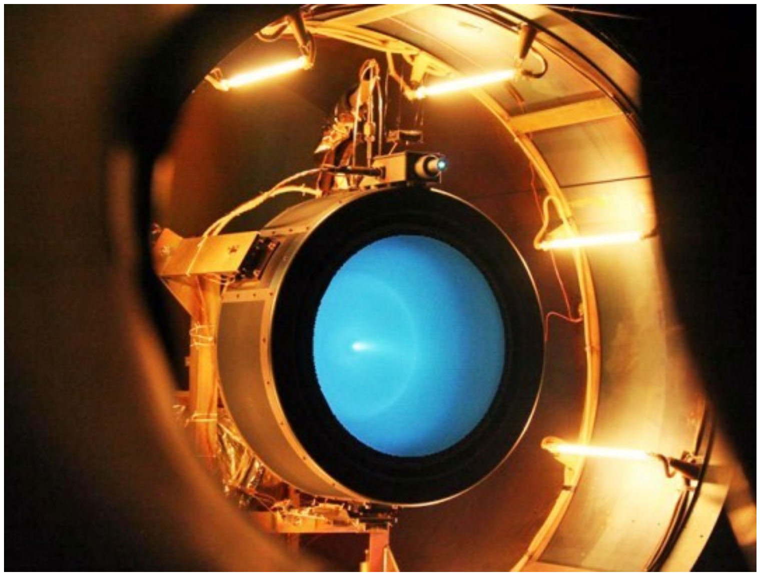
Gridded ion engines, a.k.a. ion thrusters, are characterized by the electrostatic acceleration of ions extracted from a plasma generator. Ion thruster geometries are best described in terms of three basic components:

- Ion accelerator
- Plasma generator
- Electron neutralizer

- Ion accelerator typically uses electrically biased multi-aperture grids to produce the ion beam.
- Neutralizer cathode is positioned outside the thruster body to provide electrons to neutralize the ion beam and maintain the potential of the thruster and spacecraft relative to the space plasma potential.
- Plasma generator can be direct current (DC) electron discharges, radio frequency (RF) discharges, or microwave discharges to produce the plasma.



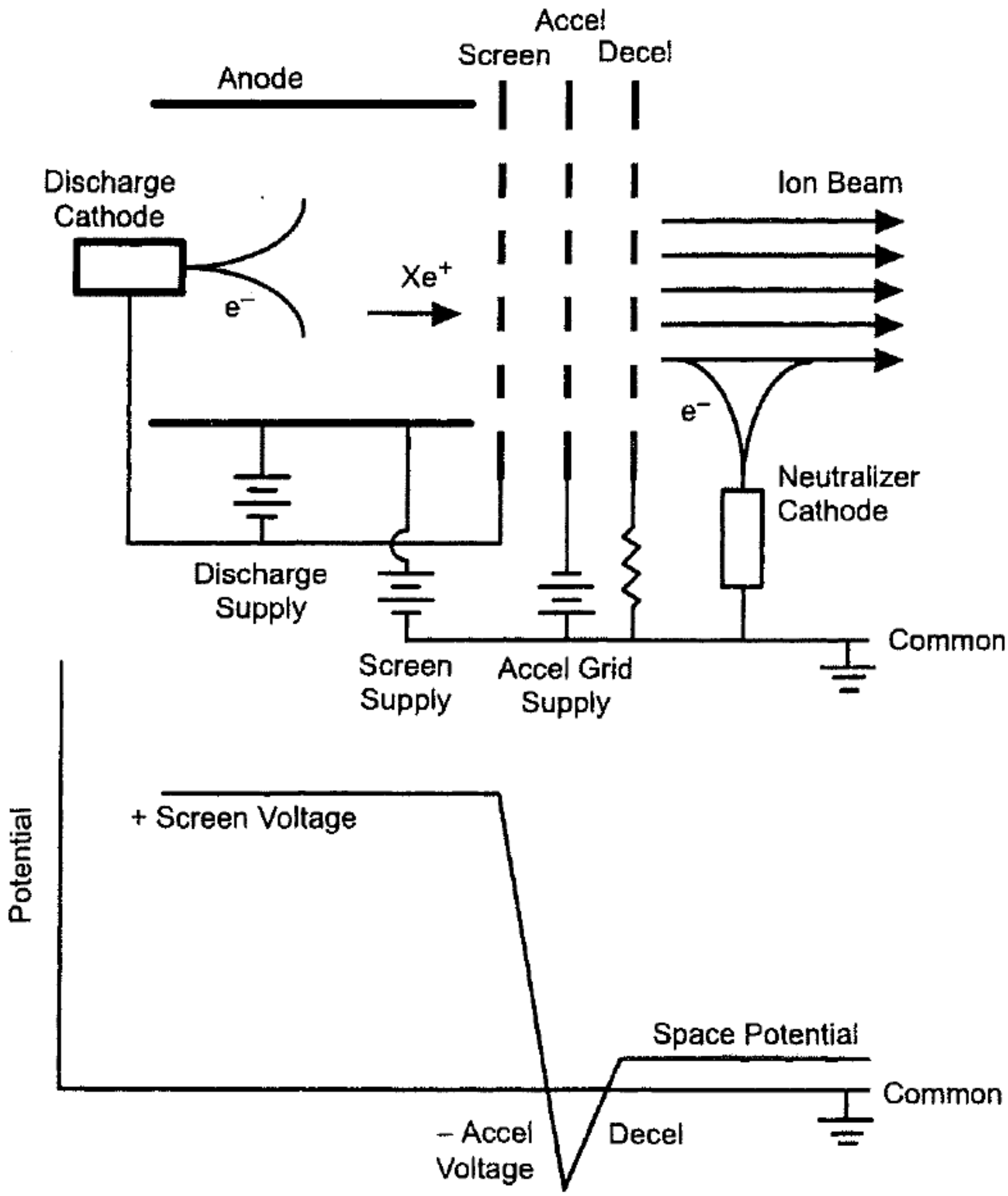
NEXIS 57-cm diameter ion engine (circa 2005), 20kW, 7000 sec, 100khr, JIMO, never flown in space



NEXT, 40-cm diameter ion engine, 7kW, 4200sec, never flown in space

NSTAR, 30-cm diameter ion engine, 2.3kW, 3000sec, flow on Deep-Space One in 1998 (a single unit) and DAWN in 2007 (3 units)

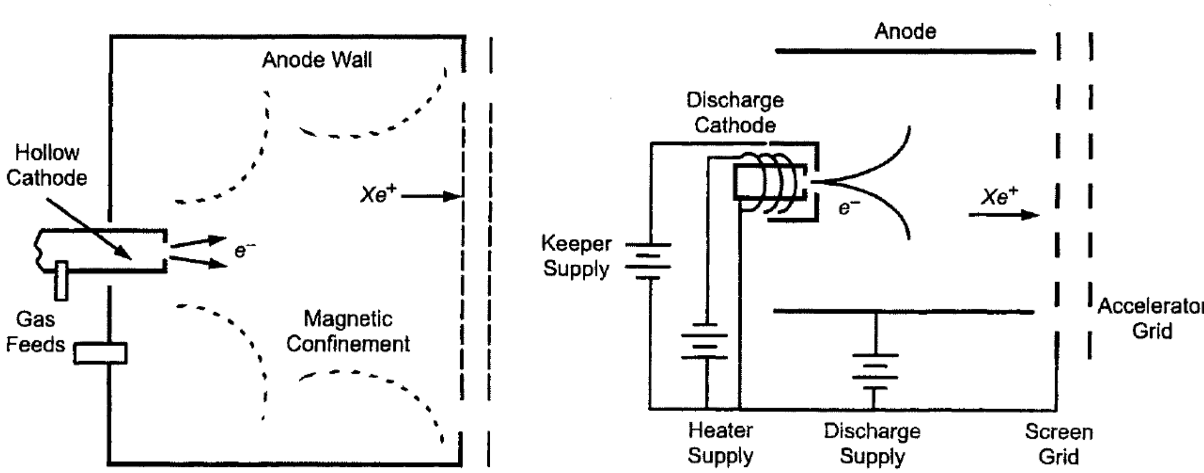
1.0.1 Electrical Schematic:



Decceleration grid to protect acceleration grid from CEX ions that cause pit-ngroove erosion of acceleration grid.

1.1 Plasma Generator - Discharge Chamber

The plasma generator we will focus on here is the DC electron discharge plasma generator, typically just called the discharge chamber. The purpose of the discharge chamber is to create a large volume of uniform plasma from which ions can be extracted and accelerated by the ion accelerator.



The goal is to inject neutral gas atoms, ionize them, and then supply them to the ion accelerator. Therefore there are two main important performance metrics:

1. How efficiently ions are created and supplied to the accelerator. This is called the ion production cost (or discharge loss), which has units of eV/ion (or W/A).

$$\eta_d = \frac{P_{in}}{I_b} = \frac{\text{Power In}}{\text{Beam Current}} \tag{9.1}$$

2. How efficiently the injected neutral atoms are converted into ions. Specifically the fraction of injected neutrals that become supplied to the accelerator. This is called the mass utilization efficiency.

Mass Utilization Efficiency

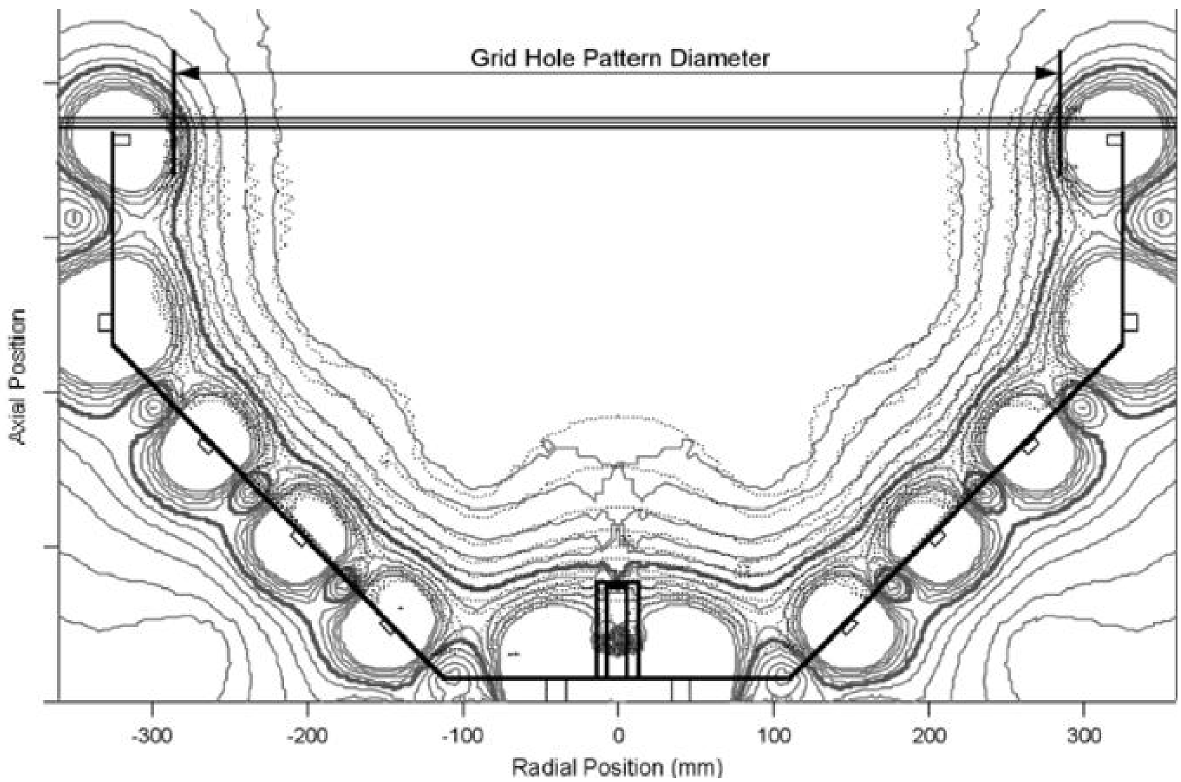
$$\eta_{md} = \frac{I_b}{\frac{\dot{m}_o}{M}\, e} \tag{9.2}$$

Where

$$\begin{aligned} \dot{m}_o &= \text{Neutral Particle Input Flow Rate} \\ e &= \text{Electron Charge} \\ &= 1.60217662 \times 10^{-19}\, C \\ M &= \text{Ion Mass} \end{aligned}$$

Clearly we want low ion production cost (will never be less than ionization energy! 12.13 eV/xenon-ion), and high mass utilization efficiency.

An example of modern state-of-the-art discharge chamber is shown here, the NEXIS ion engine.



Electrons created and emitted by the cathode (center cylinder) are called primary electrons, and are accelerated toward the anode body (surrounding metallic shell). These primary electrons have ionizing collisions with neutral propellant (typically xenon, neutral pressure of a few mTorr) injected into the chamber.

To reduce ion propulsion cost and increase mass utilization efficiency, the pathlength of the electrons (distance they must travel to get to the anode), and thereby their probability of having an ionizing collision, is increased with a constant magnetic field. A cusped field configuration has been shown to provide best performance in terms of providing a flat beam profile with low ion production cost and high utilization efficiency.

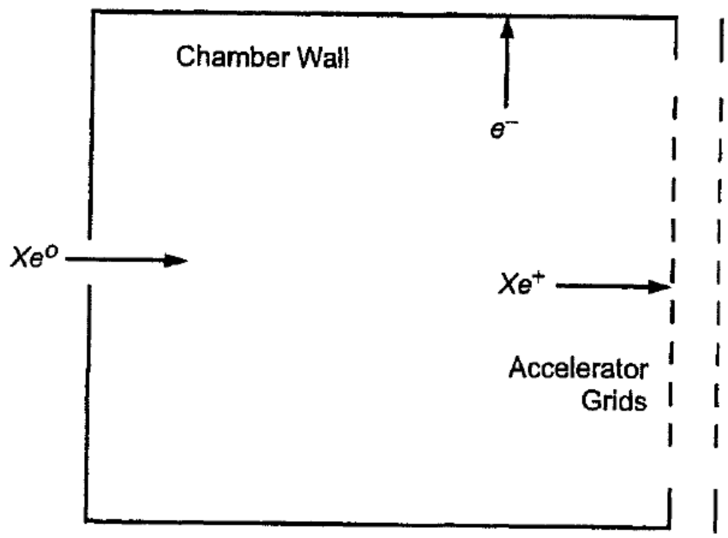
Cusps are created using alternating polarity of permanent (typically samarium cobalt) magnets along the shell of the discharge chamber. A good rule of thumb for high efficiency is to keep the 50-60 G magnetic field contour from intersecting the anode wall.

Models have been developed to predict the effect of cusp magnetic field on electron collection at the anode, and these effects have been incorporated into discharge chamber models. But here, we develop a simple idealized model that neglects the magnetic field entirely. While it is simple, it is still a good approximation to current experimental data.

1.1.1 Idealized Model for Predicting Discharge Chamber Performance

Power is injected by arbitrary means into a volume filled with neutral gas to produce ionization and neutral gas excitation.

- All the ions go to the grids
- Equal number of plasma electrons go to the wall to conserve charge.
- Discharge chamber has a volume, V.



Ions only flow to grids, perfect confinement everywhere else. Current of ions to the grids is the Bohm current (due to the plasma sheath that forms along the surface of the accelerator grids). The Bohm current is:

Bohm Current

$$I_{\text{Bohm}} = I_i = \frac{1}{2} n_i \, e \, v_a \, A \tag{9.3}$$

Where

$$\begin{aligned} v_a = v_{\text{Bohm}} &= \sqrt{\frac{k \, T_e}{M}} \\ &= \text{Ion Acoustic/Bohm Velocity} \\ A &= \text{Area of Grid} \end{aligned}$$

The fraction of ions that arrive at the grids and get accelerated into the beam is T_g, the "effective grid transparency", so the beam current is:

Beam Current

$$I_b = \frac{1}{2}\, n_i\, e\, v_a\, A\, T_g \tag{9.4}$$

Ions in the discharge chamber are produced at a rate given by the ionization rate coefficient:

Ionization Rate Coefficient:

$$I_p = n_o\, n_e\, e\, < \sigma_i\, v_e > \not V \tag{9.5}$$

Where

$$n_o = \text{Neutral Density}$$
$$< \sigma_i\, v_e > = \text{Ionization Rate Coefficient; Average over Velocity Distribution Function}$$

Power is conserved in the system, so the power put into the plasma is equal to the power that comes out in the form of charged particles and radiation. To first order, the power put in goes into ionization and excitation of neutral gas, heating of the electrons, and power that is carried to the walls and the grids by the ions and electrons. Then:

Plasma Power Input

$$P_{in} = I_p\, U^+ + I^*\, U^* + I_i\, \varepsilon_i + \frac{n_e \not V}{\tau}\, \varepsilon_e \tag{9.6}$$

where

$$\begin{aligned} U^+ &= \text{Ionization potential} \\ U^* &= \text{Excitation potential} \\ \varepsilon_i &= \text{Energy carried by ions to grids} \\ \varepsilon_e &= \text{Energy carried by electrons to anode/leaving plasma} \\ I^* &= \text{Production rate of excited species} \\ \tau &= \text{Mean confinement time of electrons} \end{aligned}$$

The production rate of excited species is given by (similar to (9.5)):

$$I^* = \sum_j n_o\, n_e\, e\, < \sigma^*\, v_e >_j \not V \tag{9.7}$$

Where σ^* is the cross-section for excitation to level j and we sum over all possible excitation levels.

With Equation 9.5, 9.7, Equation 9.6 becomes:

$$P_{in} = n_o\, n_e\, < \sigma_i\, v_e > \cancel{V} \left[U^+ + \frac{< \sigma^*\, v_e >_j}{< \sigma_i\, v_e >}\, U^* \right] + I_i\, \varepsilon_i + \frac{n_e\, \cancel{V}}{\tau}\, \varepsilon_e \tag{9.8}$$

where we assume there is one "average" excited state, j, with corresponding energy U^*. ($U^* \sim 10$ eV for Xe)

Assuming the plasma is quasi-neutral ,$n_e \approx n_i$, and that ions and electrons leave at the same rate, such that the confinement time of ions and electrons is the same, then:

$$I_i = \frac{1}{2}\, n_i\, e\, v_a\, A = \frac{n_i\, e\, \cancel{V}}{\tau} \tag{9.9}$$

such that:

$$\tau = \frac{2\, \cancel{V}}{v_a\, A} \tag{9.10}$$

We see that larger volume to surface increases confinement time. Therefore we expect discharge chambers with larger characteristic size (e.g., radius) will have higher confinement time of charged particles (good so electrons can have ionization collisions). Larger discharge chambers tend to have better performance (but there's a mass and size penalty for the spacecraft!).

Now to determine how much energy an ion and electron takes with it when it leaves the discharge chamber, ε_i and ε_e . We must consider the potential profile at the anode wall. It is an electron repelling sheath, since the plasma potential is higher than the anode potential.

Figure: Potential Profile at the Anode Wall

Assuming Maxwellian electrons in the bulk plasma, these electrons are decelerated and repelled by the sheath potential. The electron current density reaching the wall is given by moments (integral) of the distribution function (see 4.26):

$$\begin{aligned} j_e &= e\,n \int_{-\infty}^{\infty} \mathrm{d}v_x \int_{-\infty}^{\infty} \mathrm{d}v_y \int_{\sqrt{\frac{2\,e\,\phi}{m}}}^{\infty} v_z \left(\frac{m}{2\,\pi\,k\,T_e}\right)^{\frac{3}{2}} \exp\left(\frac{-mv^2}{2\,k\,T_e}\right) \mathrm{d}v_z \\ &= \frac{1}{4}\,e\,n \sqrt{\frac{8\,k\,T_e}{\pi\,m}} \exp\left(\frac{-e\,\phi}{k\,T_e}\right) \end{aligned} \tag{9.11}$$

The flux of kinetic energy (or kinetic power) reaching the wall is:

$$\begin{aligned} P_e &= n \int_{-\infty}^{\infty} \mathrm{d}v_x \int_{-\infty}^{\infty} \mathrm{d}v_y \int_{\sqrt{\frac{2\,e\,\phi}{m}}}^{\infty} v_z \left(\frac{1}{2}\,m\,v^2\right) \left(\frac{m}{2\,\pi\,k\,T_e}\right)^{\frac{3}{2}} \exp\left(\frac{-mv^2}{2\,k\,T_e}\right) \mathrm{d}v_z \\ &= \frac{1}{4}\,n\,e \sqrt{\frac{8\,k\,T_e}{\pi\,m}} \left(2\,\frac{k\,T_e}{e} + \phi\right) \exp\left(\frac{-e\,\phi}{k\,T_e}\right) \end{aligned} \tag{9.12}$$

Therefore the average energy transported by an electron from the plasma is the ratio of the power per electron to the flux of electrons:

$$\varepsilon_e = \frac{P_e}{j_e} = 2\frac{k\,T_e}{e} + \phi = 2\,T_{e\,[eV]} + \phi \tag{9.13}$$

Energy removed by ions when they leave the plasma. Ions first fall through the pre-sheath potential, which is approximately Te/2 [eV] to produce the Bohm velocity, and then through the sheath potential, ϕ . Each ion then removes from the plasma a total energy per ion of:

$$\varepsilon_i = \frac{1}{2}\,\frac{k\,T_e}{e} + \phi = \frac{1}{2}\,T_{e\,[eV]} + \phi \tag{9.14}$$

The plasma potential, ϕ, is found from electron current leaving the plasma, which for a stable sheath is:

$$I_{\text{anode}} = \frac{1}{4} \sqrt{\frac{8\,k\,T_e}{\pi\,m}}\,e\,n_e\,A_a \exp\left(\frac{-e\,\phi}{k\,T_e}\right) \tag{9.15}$$

Since we have assumed ambipolar ion and electron flow to the wall (ion and electron loss rates are same), we can equate (9.15) and (9.3):

Floating Potential

$$\phi = \frac{k\,T_e}{e}\,\ln\left[\frac{A_a}{A}\,\sqrt{\frac{2\,M}{m\,\pi}}\right] \tag{9.16}$$

Where

$$A = \text{Grid Area}$$

$$A_a = \text{Anode Area}$$

This is normally called the "floating potential" since it's the potential such that the ion and electron current are equal (we set (9.15) equal to (9.3)). But in this model there are no applied potentials to draw a net current.

This potential, ϕ , is the potential setup to balance the electron and ion loss rate. Since electrons are less massive, they leave the plasma first, leaving behind a slight positive charge (or potential, ϕ). This potential retards the loss of electrons to keep the plasma quasi-neutral.

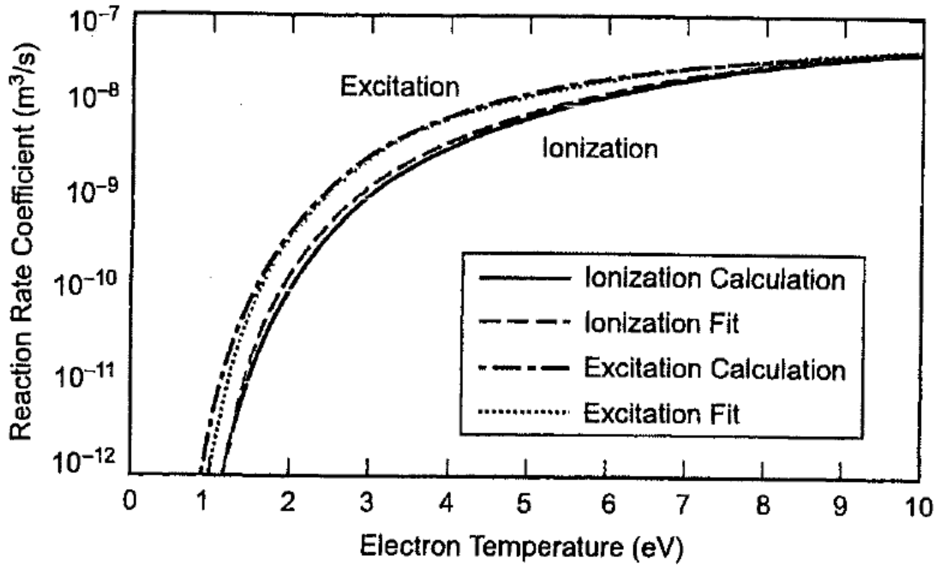
Need Electron Temperature for this model. Find electron temperature by equating ion production and loss rates, (9.3) equal to (9.5):

$$\frac{\sqrt{\frac{k\,T_e}{M}}}{<\sigma_i\,v_e>} = \frac{2\,n_o\,\cancel{V}}{A} \tag{9.17}$$

Where

$$A = \text{Ion Loss/ Grid Area}$$
$$n_o = \text{Neutral Densities}$$

This equation can be solved for T_e, but not analytically. The reaction rate coefficient has been calculated using empirically determined collision cross-sections:



Electron Energy (eV)	Ionization (m^3/s)	Excitation (m^3/s)
0.5	4.51×10^{-25}	1.99×10^{-22}
0.6	3.02×10^{-23}	4.01×10^{-21}
0.7	6.20×10^{-22}	3.61×10^{-20}
0.8	6.04×10^{-21}	1.95×10^{-19}
0.9	3.58×10^{-20}	7.44×10^{-19}
1.0	1.50×10^{-19}	2.21×10^{-18}
1.5	1.16×10^{-17}	6.64×10^{-17}
2.0	1.08×10^{-16}	4.02×10^{-16}
2.5	4.24×10^{-16}	1.23×10^{-15}
3.0	1.08×10^{-15}	2.66×10^{-15}
3.5	2.13×10^{-15}	4.66×10^{-15}
4.0	3.59×10^{-15}	7.12×10^{-15}
4.5	5.43×10^{-15}	9.93×10^{-15}
5.0	7.61×10^{-15}	1.30×10^{-14}
5.5	1.01×10^{-14}	1.61×10^{-14}
6.0	1.28×10^{-14}	1.94×10^{-14}
6.5	1.57×10^{-14}	2.26×10^{-14}
7.0	1.88×10^{-14}	2.57×10^{-14}
7.5	2.20×10^{-14}	2.87×10^{-14}
8.0	2.53×10^{-14}	3.14×10^{-14}
8.5	2.86×10^{-14}	3.34×10^{-14}
9.0	3.20×10^{-14}	3.41×10^{-14}
9.5	3.55×10^{-14}	3.21×10^{-14}
10.0	3.90×10^{-14}	2.48×10^{-14}

Now to calculate the discharge loss (a.k.a. Ion Production Cost):

Discharge Loss

$$\eta_d = \frac{P_{in}}{I_b} = \frac{2\, n_o \, < \sigma_i \, v_e > \cancel{V}}{v_a \, A \, T_g} \left[U^+ + \frac{< \sigma^* \, v_e >}{< \sigma_i \, v_e >} \, U^* \right]$$

$$+ \frac{1}{T_g \, e} \left[2.5 \, k \, T_e + 2 \, k \, T_e \, \ln \left(\frac{A_a}{A} \sqrt{\frac{2\, M}{m \, \pi}} \right) \right] \tag{9.18}$$

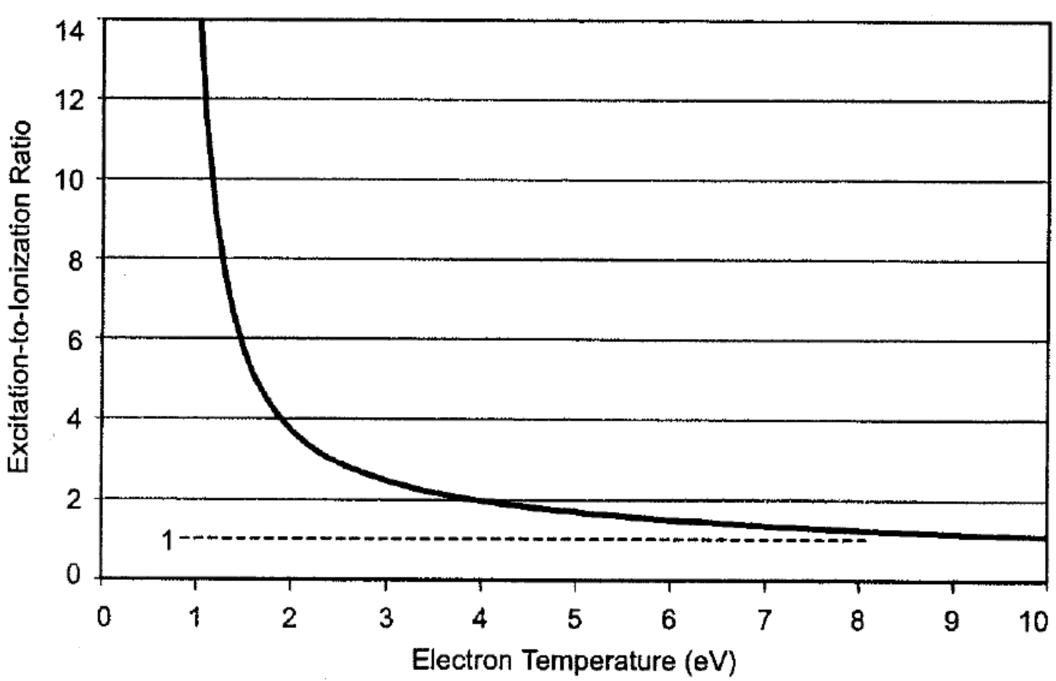
Where

$$v_a = v_{\text{Bohm}} = \sqrt{\frac{k \, T_e}{M}}$$

$$M = \text{Ion Mass}$$

- Grid transparency, T_g, directly affects the discharge loss
- Input power is distributed between:
  1. 1st term: producing ions and excited neutrals
  2. 2nd term: heating the electrons that are lost to the walls

Need the ratio of the excitation to ionization reaction rates:



At temperatures below about 8eV, the excitation rate exceeds the ionization rate. This higher excitation rate results in more of the power being radiated to the walls than producing ions. This explains at least part of the inefficiency inherent in XENON plasma generators, but similar results are found for other inert gas propellants.

Also need T_g ($\sim$ 80%), xenon ionization potential (12.13 eV), xenon excitation potential ($\sim$10 eV), diameter of grids (A), electron loss area (A_a $\sim$chamber wall and screen grid).

The other performance parameter is the mass utilization efficiency.

To calculate it we recognize that neutrals entering the discharge chamber can leave the discharge chamber as neutrals through the grid, or become ionization and leave as ions through the grids.

$$\dot{m}_{o,out} = \dot{m}_{o,in} - \frac{I_b\,M}{e} \tag{9.19}$$

The mass flow rate of neutrals out of the discharge chamber can be written as:

Mass Flow Rate of Neutrals Out of Discharge Chamber

$$\dot{m}_{o,out} = \frac{1}{4}\,M\,n_o\,v_o\,A_g\,T_a\,\eta_c \tag{9.20}$$

Where

$$\begin{aligned}
V_o &= \text{Neutral velocity} = \sqrt{\frac{8\,k\,T_o}{\pi\,M}} \\
&\quad T_o - \text{Neutral Temp} \\
&\quad T_o \sim 200 - 300^\circ C \\
n_o &= \text{Neutral density} \\
A_g &= \text{Grid area} \\
T_a &= \text{Grid transparency to neutrals} \\
\eta_c &= \text{Clausing factor} \\
\frac{1}{4}\,M\,n_o\,v_o\,A_g &= \text{Random flux of neutrals to grid}
\end{aligned}$$

The **Clausing Factor** accounts for the fact that the grids are finite thickness (not infinitely thin), related to gas flow restriction in short tubes. Typical grids have small thickness-to-length ratios, so Clausing factor must be calculated with numerical kinetic theory/Monte Carlo techniques, but a value of $\sim$ 0.5 is typical for ion engines.

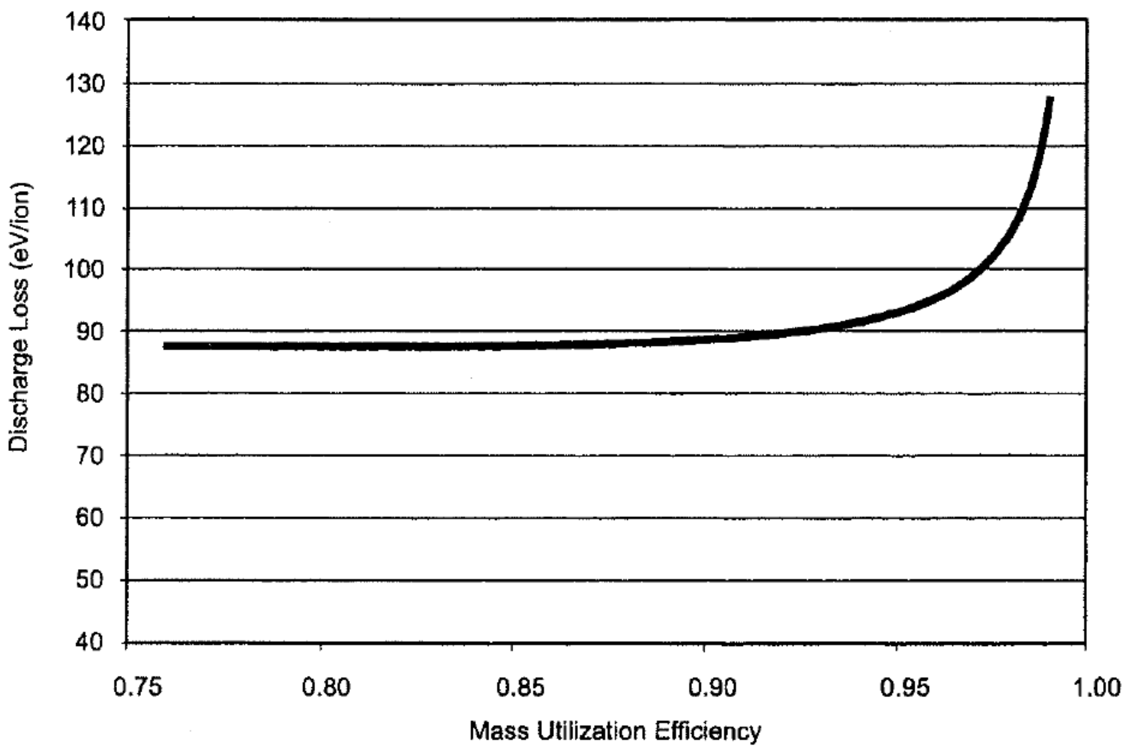
The mass utilization efficiency was given as (9.2), which will us (9.4) for the beam current.

$$\eta_{md} = \frac{I_b}{\frac{\dot{m}_{o,in}\,e}{M}}$$

Using (9.19) and (9.20), we can solve for the neutral density in the discharge chamber:

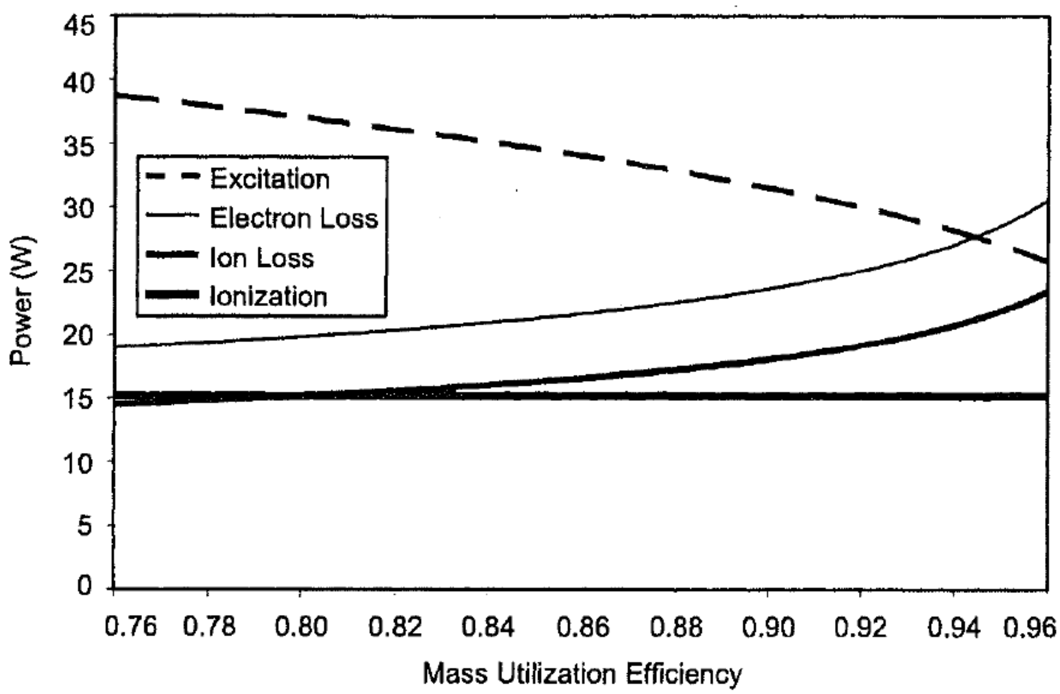
$$n_o = \frac{\frac{4\,\dot{m}_{o,in}\,e}{M}\,(1-\eta_{md})}{v_o\,A_g\,T_a\,\eta_c} = \frac{4\,I_b}{v_o\,e\,A_g\,T_a\,\eta_c}\,\frac{(1-\eta_{md})}{\eta_{md}} \tag{9.21}$$

Typical efficiencies for a discharge chamber using Equation 9.2 and Equation 9.18: 20-cm diameter, 30-cm long.



Amount of power to produce 1A of beam current (with 80% transparency Tg) is about 90W. While it only takes 12.13eV to ionize a xenon atom, even in an idealized thruster it takes 7.5 times this energy to produce and deliver an ion into the beam due to other losses.

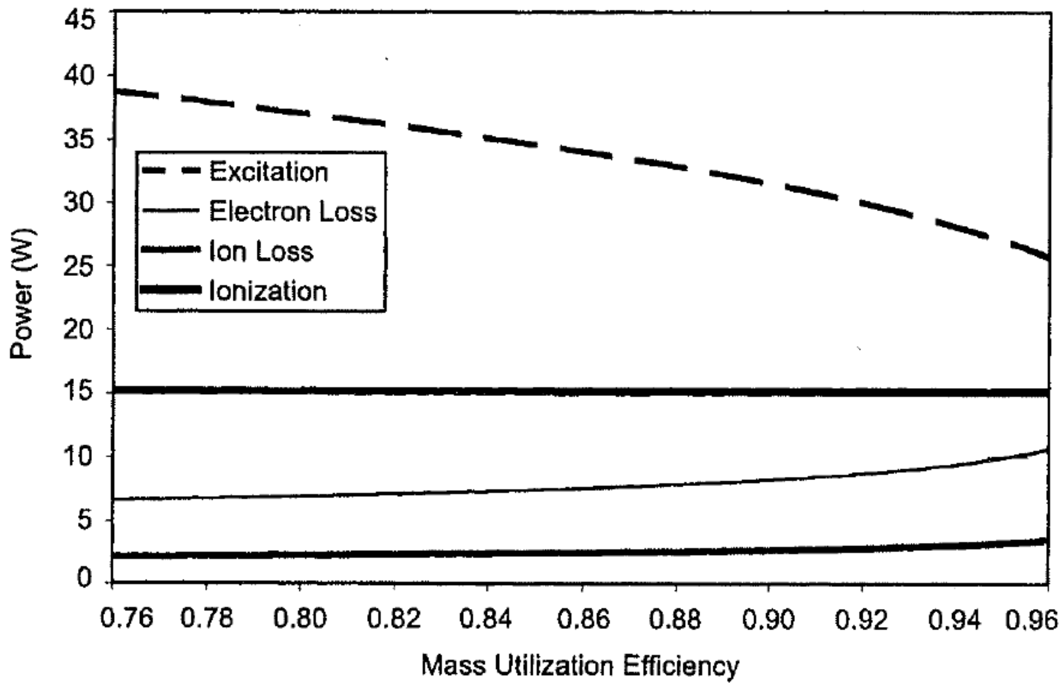
What are the losses??



This plot assumes the ionization power is constant, with 1A beam current ($1/0.8 * 12.13eV = 15.1W$). Electrons lost to screen grid and anode (20cm diam, 30cm long, $A_a = 2200cm^2$).

Major power loss is excitation at low mass utilization where electron temp is low. Electron and ion convection losses to the wall increase with mass utilization because there are less neutrals, so have higher electron temp, higher plasma potential, and thereby increased energy lost per electron and ion.

State-of-the-art discharge chambers use magnetic confinement to reduce the effective anode area (A_a), the Bfield reduces the area of the anode the electrons can actually get to. This effect is illustrated below:



Effective anode area reduced to $1cm^2$. Reduced anode area does not affect electron loss rate, since ion and electron loss rates are same in this model (electron loss is same as ion loss to grid). Reduced area changes plasma potential relative to loss area potential in order to maintain charge balance as seen in (9.16). As seen in figure, ionization and excitation power is not changed. But the energy loss due to electron and ion convection out of the system is significantly reduced (from 15-30W to 2-10W).

1.2 Grid Ion Optics

To accelerate ions a potential difference must be established between the plasma produced inside the thruster plasma generator (discharge chamber), and the ambient space plasma.

A grid is used to establish this potential difference. The grid must have holes smaller than the Debye length of the plasma, otherwise the plasma would simply leak out through the holes.

Figure 2: Grid Optics

At very high grid potential ($V_g \gg T_e$) the thin sheath boundary is in the Child-Langmuir sheath regime, and the current flow into the sheath is limited.

Child-Langmuir Law (Current Limit in a Plane Diode):

$$j_i = \frac{4}{9}\,\varepsilon_o \left(\frac{2\,q}{M}\right)^{\frac{1}{2}} \frac{V^{\frac{3}{2}}}{d^2} \tag{9.22}$$

Where

$$V = \text{Voltage across gap/grids}$$
$$d = \text{Distance across gap/between grids}$$

The amount of current that an ion accelerator can extract and focus into a beam is limited by the Child-Langmuir law, and is called the perveance:

Perveance

$$P \equiv \frac{I_B}{V^{\frac{3}{2}}}$$

The maximum perveance would be the perveance with the maximum current, which is the Child-Langmuir current, so:

Maximum Perveance

$$P_{\max} = \frac{4}{9}\,\varepsilon_o\sqrt{\frac{2\,q}{M}} \qquad \left[\frac{\text{Amps}}{\text{Volts}^{3/2}}\right] \tag{9.23}$$

Where $P_{\max} = 4.8 \times 10^{-9}$ for a singly-ionized xenon.

With a beamlet area of $\frac{\pi}{4}\,D^2$, the max perveance is:

Maximum Perveance with Beamlet Area $\frac{\pi}{4}\,\mathbf{D^2}$

$$P_{\max} = \frac{\pi\,\varepsilon_o}{9}\sqrt{\frac{2\,q}{M}}\left(\frac{D^2}{d^2}\right) \qquad \left[\frac{\text{Amps}}{\text{Volts}^{3/2}}\right] \tag{9.24}$$

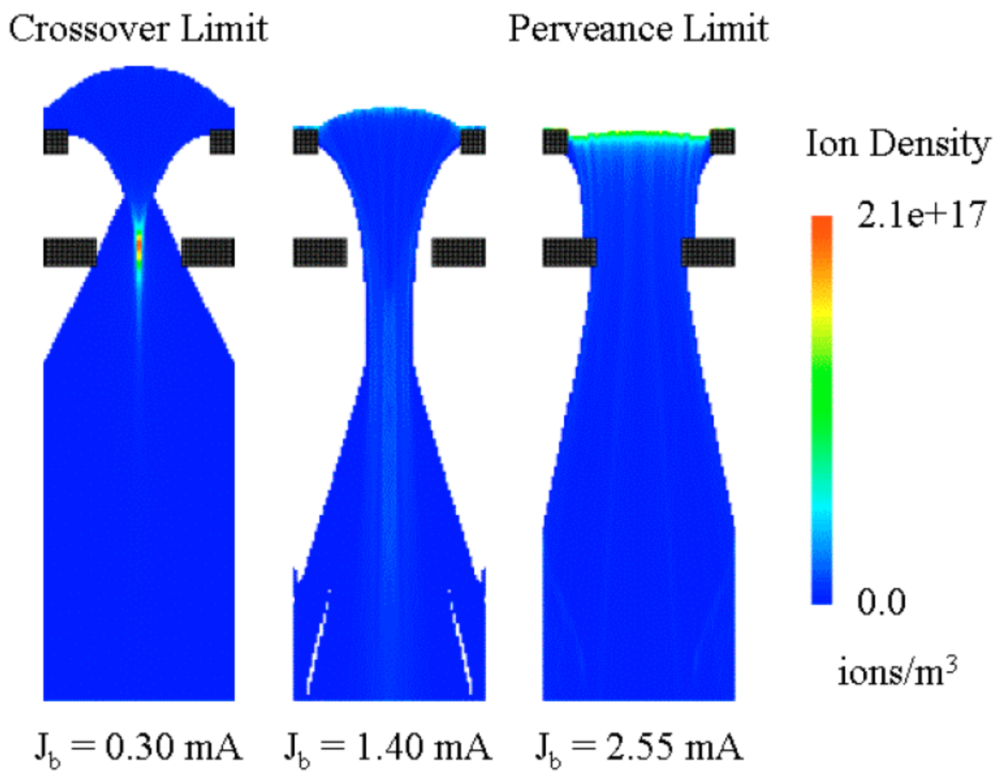
Where

$$d = \text{Effective grid gap}$$
$$D = \text{Beamlet diameter}$$

Maximum perveance requires the grid gap be smaller than the beamlet diameter.

1.2.1 Cross-over limit to Perveance limit

Ion trajectories that do not intersect the grids and have minimal divergence result from operating at or near the optimal ion current density and voltage for the grid geometry (center figure). Operating at significantly less than the optimal perveance, called "under-perveance" (corresponding to higher voltages or lower beamlet currents), increase the Child-Langmuir sheath length and pushes the sheath farther into the discharge plasma. This causes ions to be launch from the edge of the sheath and "cross-over" producing excessive erosion of the grids (far left figure). Similar, operating at higher than optimal perveance (higher beamlet current, lower voltage), reduces the CL sheath thickness and the plasma boundary pushes toward the screen grid (far right figure).



Farnell, AIAA-2004-3818

The thruster ion optics assembly serves three main purposes:

- Extract ions from the discharge chamber
- Accelerate ions to generate thrust
- Prevent electron backstreaming

Ideal grid assembly would extract and accelerate all ions that approach the grid, and block all neutral gas outflow. Also would want long lifetime, high current densities, and ion trajectories that are parallel with the thrust axis (no cosine-losses/divergence).

But grids have finite transparency, the screen grid transparency is:

$$T_s = \frac{I_b}{I_i} \tag{9.25}$$

It depends on the plasma parameters in the discharge chamber because the sheath edge is normally pushed slightly into the plasma by the applied voltage if the screen grid is relatively thin, creating a non-planar, hemispherical sheath boundary. This modified sheath thickness can be accounted for.

Consider a typical two grid ion optics setup:

Figure 3: Two Grid Ion Optics Setup

Maximum Current Density given by Child-Langmuir:

$$j_{\max} = j_{cl} = P_{\max} = \frac{4}{9}\,\varepsilon_o\,\sqrt{\frac{2\,e}{M}}\,\frac{V_T^{3/2}}{l_e^2} \tag{9.26}$$

where

V_t = Total Voltage Across Sheath Between The Two Grids

l_e = Sheath Thickness

Sheath Thickness is given by:

Sheath Thickness

$$l_e = \sqrt{(l_g + t_s)^2 + \frac{d_s^2}{4}} \tag{9.27}$$

This sheath thickness accounts for this non-planar condition and has been found to be useful in predicting the space-charge limited (Child-Langmuir) current in ion thruster grid configurations. Note, the value of the grid gap, l_g , is the "hot" grid gap. The grids expand during operation due to increased temperature!

We can now find the maximum thrust per unit area of an ion thruster. The thrust of an ion thruster is:

Thrust of an Ion Thruster

$$T = \frac{\mathrm{d}}{\mathrm{d}t}(m\,v) = \gamma\,\dot{m}_i\,v_i \tag{9.28}$$

where

$$\begin{aligned}\gamma &= \text{Correction due to divergence (cosine loss) and multiply-charged species}\\ \dot{m}_i &= \text{Ions mass flow rate}\\ v_i &= \text{Ions velocity}\end{aligned}$$

Assuming ions start from rest, then:

$$v_i = \sqrt{\frac{2\,e\,V_b}{M}} \tag{9.29}$$

where

$$V_b = \text{Net voltage ion accelerated through (beam voltage)}$$

and using mass flow rate (8.34):

$$\dot{m}_i = \frac{I_b\,M}{e} \tag{9.30}$$

The thrust is then:

$$T = \gamma\,I_b\,M\,\sqrt{\frac{2\,e\,V_b}{M}}$$

or

Thrust Density

$$\frac{T}{A_g} = \frac{j\,\gamma\,T_s\,M}{e}\,\sqrt{\frac{2\,e\,V_b}{M}} \tag{9.31}$$

where A_g is the active area of the grids (with extraction apertures, just the total grid area, e.g., area of a 30-cm-diam circle)

The effective electric field in the grid gap is:

Effective Electric Field in the Grid Gap:

$$E = \frac{V_T}{l_e} \tag{9.32}$$

where

$$V_T = \text{Total voltage across the gap, } V_T \text{ , that is:}$$

$$V_T = V_{\text{screen}} + |V_{\text{acceleration}}| = V_s + |V_a| = \frac{V_b}{R} \tag{9.33}$$

where we get the ratio of the net beam voltage to the total voltage (how much of the total voltage is used to accelerate the ion)

Using the maximum possible current density in (9.31), the Child-Langmuir space-charge limited current (9.26), and the electric field of (9.32), then:

$$\begin{aligned} \frac{T}{A_g} &= \frac{4}{9}\,\frac{\varepsilon_o\,\gamma\,T_s}{e}\,\sqrt{\frac{2\,e}{M}}\,\frac{V_T^{3/2}}{l_e^2}\,M\sqrt{\frac{2\,e\,V_b}{M}} \\ &= \frac{8}{9}\,\varepsilon_o\,\gamma\,T_s\,\sqrt{R}\,E^2 \end{aligned} \tag{9.34}$$

Max thrust density increases with screen grid transparency and the square of the electric field. So max thrust is possible if use thin high transparency grids operated near the perveance limit and at max possible electric field in the gap.

Also note, thrust density is independent of propellant mass.

Finally, note R describes the relative magnitude of the accel grid bias relative to the screen potential.

$$R = \frac{V_b}{V_T} = \frac{V_b}{V_s + |V_a|} \tag{9.35}$$

Operating with small values of R increase the total voltage between the screen and acceleration grids, which from (9.26) results in higher current density of ions accelerated from the thruster. But, smaller R also results in higher energy ion bombardment of acceleration grid. Magnitude of acceleration grid bias usually minimized to value required to just avoid electron back streaming and the value of R typically ranges from 0.8 to 0.9.