Section 2: Electromagnetics

AE435

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In this section, we will review the basics of charge, electricity, magnetism, and Maxwell equations.

1 Electrostatics

Contents

1.1 Coulomb's Law

Coulomb's Law is the measure of force between charges.

Case 1: Two Particles

Consider two charges q_2 and q_2 located at $\vec{r_1}$ and $\vec{r_2}$.

Figure 1

The force on q_1 due to q_2 is

Coulomb Force

$$\overrightarrow{F_{12}} = c(\frac{q_1 q_2}{r_{12}^2})\frac{\overrightarrow{r_{12}}}{r_{12}} \quad \sim \quad \frac{1}{r^2} \tag{1}$$

Where:

$$\begin{aligned}
\overrightarrow{r_{12}} = \vec{r_1} - \vec{r_2} & \qquad \text{The Sum of Position Vectors} \\
r_{12} = |\overrightarrow{r_{12}}| & \qquad \text{The Magnitude of } \overrightarrow{r_{12}} \\
\\
c = & \qquad \text{Coulomb's Constant} \\
& \qquad = \quad 8.9875x10^9 = \frac{1}{4\pi\epsilon_0}[\frac{Nm^2}{c^2}] \\
\\
\epsilon_0 = & \qquad \text{Permittivity of Free Space} \\
& \qquad = \quad 8.854x10^{-12}[\frac{c^2}{Nm^2}]
\end{aligned}$$

Coulomb force scales with the square of the distance as shown by $\sim \frac{1}{r^2}$ in Equation 1.

Case 2: Many Particles - Coulomb Law

Definition 1. Principle of superposition:
Attraction between any pair can be calculated with Equation 1, regardless of the number of particles in the ensemble.

So, let $\vec{r_i}$ be location of test particle q_i. If we have N charged particles, the force on q_i is the linear superposition of the individual forces,

Multiple Particle Coulomb Force

$$\overrightarrow{F_{ij}} = \frac{q_i}{4\pi\epsilon_0}\sum_{j\neq i}^{N} q_i\left(\frac{\overrightarrow{r_{ij}}}{r_{ij}^3}\right) \tag{2}$$

Where:

$$\overrightarrow{r_{ij}} = \vec{r_i} - \vec{r_j} \qquad \text{The vector from test particle } q_i \text{ to field particle } q_j$$

Case 3: Continuum Generalizations

Let us define

- **Volume Charge Density** $[\frac{C}{m^3}]$
$$\rho_e = \lim_{\Delta \cancel{V} \to 0} \frac{\Delta q}{\Delta \cancel{V}} \tag{3}$$
- **Surface Charge Density** $[\frac{C}{m^2}]$
$$\sigma_e = \lim_{\Delta \mathrm{S} \to 0} \frac{\Delta q}{\Delta S} \tag{4}$$

With these, the force acting on charge q_o due to distributed charge sources are:

Continuum Charge Coulomb Force

$$\overrightarrow{F_{q_o}} = \frac{q_i}{4\pi\epsilon_0}\left[\int_{\cancel{V}} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}\rho_e(\vec{r}')\mathrm{d}\vec{\cancel{V}} + \int_S \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}\sigma_e(\vec{r}')\mathrm{d}\vec{S}\right] \tag{5}$$

Where:

$$\begin{aligned} \vec{r}' &= \text{The location within } \cancel{V} \text{ or location on S} \\ \vec{r} &= \text{The location of } q_o \\ \sigma_e \text{ and } \rho_e &= \text{Functions of position } \vec{r}' \end{aligned}$$

1.2 Electric Field

$$\vec{E} = \lim_{q_0 \to 0} \frac{F_{q_o}}{q_o} \tag{6}$$

The force acting on a specific charge q_o from a collection of other charges per unit charge, as the specific charge tends to zero.

We set $q_0 \to 0$ so its presence does not influence the ambient charge.

Adding Equation 2 and Equation 5 then dividing thru by q results in:

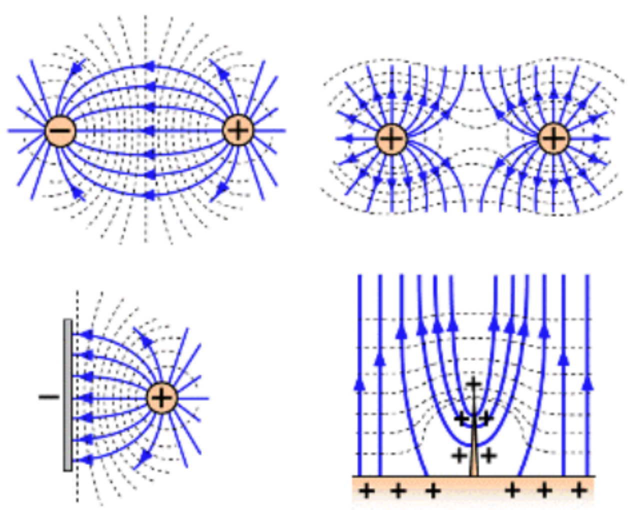
Electric Field Equation

$$\vec{E} = \frac{q_i}{4\pi\epsilon_0}\left[\sum_{i=1}^{N} q\frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3} + \int_{V} \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\rho_e(\vec{r}')\mathrm{d}\vec{V} + \int_{S} \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\sigma_e(\vec{r}')\mathrm{d}\vec{S}\right] \tag{7}$$

Where:

$$\vec{E} = \vec{E}(\vec{r}) \quad \text{Electric Field is a function of position } \vec{r}$$

We can use integral approach to solve problems, but this can get complex. We can also visualize the Electric Field via field lines, curves that are everywhere tangent to the field.



1.3 Conductors and Insulators

Definition 2. Conductor: Free charges, respond to external Electric field with charge motion.

Definition 3. Insulator: Bound charges, no motion. Also often called a ”dielectric”

1.4 Gauss’s Law

Definition 4. Gauss’s Law: Relates the electric field at a surface to the charge enclosed within that surface. The total flux that passes through any closed surface is proportional to the electric charge enclosed by a surface. This surface is the Gaussian Surface.

Case 1: Single Charge

The electric field for a single charge is

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}} \tag{8}$$

If we take the surface integral around an arbitrary volume surrounding the charge we get:

Figure 2

Gauss’ Law for Electric Fields

$$\oint_S \vec{E} \cdot \hat{n}\,\mathrm{d}A = \frac{q}{4\pi\epsilon_0}\oint_S \frac{\vec{r}\cdot\hat{n}}{r^3}\,\mathrm{d}A \tag{9}$$

Where:

$$\left(\frac{\vec{r}}{r}\right)\cdot\hat{n}\,\mathrm{d}A = \text{The project of dA on a plane perpendicular to } \vec{r}$$

If we divide the projected area, $\left(\frac{\vec{r}}{r}\right) \cdot \hat{n}\, \mathrm{d}A$, by r^2, we arrive at the solid angle $\mathrm{d}\Omega$.

Figure 3

The total solid angle subtended by a surface totally enclosing the charge is 4π, so

$$\oint_S \frac{\vec{r} \cdot \hat{n}}{r^3}\, \mathrm{d}A = 4\pi$$

As a result, Equation 9 becomes:

$$\boxed{\oint_S \vec{E} \cdot \hat{n}\, \mathrm{d}A = \frac{q}{\epsilon_0}} \tag{10}$$

Case 2: Many Point Charges

For the case of many point charges, we take the sum:

$$\oint_S \vec{E} \cdot \hat{n}\, \mathrm{d}A = \frac{1}{\epsilon_0} \sum_{i=1}^{N} qi$$

Case 3: Distributed Charge

For a distributed charge, ρ_e, we take the integral over the enclosed volume:

Integral Form of Gauss's Law

$$\oint_S \vec{E} \cdot \hat{n}\, \mathrm{d}A = \frac{1}{\epsilon_0} \int_{\not{V}} \rho_e\, \mathrm{d}\not{V} \tag{11}$$

Also recall Divergence Theorem (also known as Gauss's Theorem):

Divergence Theroem

$$\oint_S \vec{F} \cdot \hat{n} \, \mathrm{d}A = \int_{\forall} \nabla \cdot \vec{F} \, \mathrm{d}\forall \tag{12}$$

which applies to any vector field $\vec{F}$, so we arrive that...

$$\oint_S \vec{E} \cdot \hat{n} \, \mathrm{d}A = \int_{\forall} \nabla \cdot \vec{E} \mathrm{d}\forall = \frac{1}{\epsilon_0} \int_{\forall} \rho_e \, \mathrm{d}\forall$$

Differential Form of Gauss's Law

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho_e \tag{13}$$

1.5 Electrostatic Potential and Energy

One can show that the curl of the electric field is zero for an electrostatic field.
In general $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0$ as we will see later.

$$\nabla \times \vec{E} = 0 \tag{14}$$

We also use the vector identity:

$$\nabla \times \nabla \phi = 0 \tag{15}$$

From Equation 14 and Equation 15, we can see the relation $\vec{E} = \nabla \phi$, such that the vector field $\vec{E}$ is related to the gradient in some scalar field.

We call ϕ the **electric potential**. And actually use

$$\vec{E} = -\nabla \phi(\vec{r}) \tag{16}$$

For a single point charge q_i at $\vec{r_i}$, by integrating Equation 16, we arrive at:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{|\vec{r} - \vec{r_1}|} + \text{constant}$$

On any curve linking point P_1 to point P_2 ,

$$\phi_{12}(\vec{r}) = \int_{P_i}^{P_2} \nabla \phi(\vec{r}) \cdot \mathrm{d}\,\vec{l} = -\int_{P_i}^{P_2} \vec{E}(\vec{r}) \cdot \mathrm{d}\,\vec{l}$$

Which tells us the work per unit charge to move from P_1 to P_2.
More generally,

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r_i}|} + \int_V \frac{\rho_e(\vec{r})}{|\vec{r} - \vec{r}'|} \,\mathrm{d}\vec{V} + \int_S \frac{\sigma_e(\vec{r})}{|\vec{r} - \vec{r}'|} \,\mathrm{d}A' \right] + \text{constant}$$

Taking line integral from some reference location to $\vec{r}$,

$$\int_{\text{ref}}^{\vec{r}} \vec{E}(\vec{r}) \cdot \mathrm{d}\vec{r} = -\int_{\text{ref}}^{\vec{r}} \nabla \phi(\vec{r}) \cdot \mathrm{d}\vec{r} = -\int_{\text{ref}}^{\vec{r}} \mathrm{d}\phi = -\phi(\vec{r}) - \phi_{ref}$$

Now if we set $\phi_{ref} = 0$ at $r \to \infty$. Then

$$\phi(\vec{r}) = -\int_{\text{ref}}^{\vec{r}} \vec{E}(\vec{r}) \cdot \mathrm{d}\vec{r} \tag{17}$$

The Potential Energy, U, associated with a force $\vec{F}$ is

$$U(\vec{r}) = -\int_{\text{ref}}^{\vec{r}} \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r}$$

Since $\vec{F} = q\vec{E}$ for electrostatic force as seen in Equation 16, the electrostatic potential is

Electrostatic Potential

$$\phi(\vec{r}) = \frac{U(\vec{r})}{q} \tag{18}$$

The electrostatic potential defines the potential energy per unit charge.

If we define $\phi(\infty) = 0$ then $U(\vec{r})$ is the energy required to bring a test charge from $\vec{r} = \infty$ to $\vec{r}$.

1.6 Poisson and Laplace Equations

For a charge distribution $q(\vec{r})$:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\partial q}{|\vec{r} - \vec{r}'|} \tag{19}$$

Such that

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{r} - \vec{r}')\, \mathrm{d}q}{|\vec{r} - \vec{r}'|^3} \tag{20}$$

Since $\vec{E} = -\nabla\phi$ from Equation 16. We can solve for $\phi(\vec{r})$ and $\vec{E}(\vec{r})$ if we know $q(\vec{r})$ charge distribution. Alternatively, we can start with the differential form of Gauss's law (Equation 13) and Equation 16,

$$\nabla \cdot \vec{E} = \frac{\rho_e}{\epsilon_o} \qquad \text{and} \qquad \vec{E} = -\nabla\phi$$

$$\nabla \cdot (-\nabla\phi) = \frac{\rho_e}{\epsilon_o}$$

$$\text{Use laplacian: } \nabla^2 = \nabla \cdot \nabla \rightarrow$$

Poisson Equation

$$\nabla^2\phi = -\frac{\rho_e}{\epsilon_o} \tag{21}$$

Where:

$$\begin{aligned} \rho_e = \rho_e(\vec{r}) &\qquad \text{charge density distribution} \\ \phi = \phi(\vec{r}) &\qquad \text{electric potential distribution} \end{aligned}$$

The Poisson Equation relates charge density distribution $\rho_e(\vec{r})$ to electric potential $\phi(\vec{r})$ distribution.

Alternatively,

Laplace Equation

$$\nabla^2\phi = 0 \tag{22}$$

For region with no free (space) charge $\rho_e = 0$