Section 3: Electrical Conductivity

AE435

Spring 2018

In this unit, we will explore electrical conductivity within an ionized gas. That is, a gas that consists of charged particles that can freely move under the influence of their own inertia, and whose motion will be influenced by body forces (Electric and Magnetic forces, ignoring gravity) and momentum-transfer collisions (with other particles, with boundaries).

In the presence of both electric and magnetic fields, the constitutive equation for conductivity is:

$$\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \tag{1}$$

While the definition of current density is:

$$\vec{j} = \sum_i n_i \, q_i \, \vec{v_i} \tag{2}$$

Where i denotes one of the species in the plasma, and $\vec{v_i}$ is the **drift velocity**, or mean velocity for species i .

Conductivity is really determined by the equation of motion for charge carriers, including:

- Inertia term
- Body forces
- Momentum-transfer collisions between species

We must understand how charged particles (plasma particles, ions, electrons) move (their equation of motion) in the presence of electric and magnetic fields, and in the presence of other particles (collisions). We start by ignoring collisions (just analyze motion in E and B fields), then add collisions.

1 Charged Particle Motions

In this section, we will look at single-particle motion. Motion of a single particle in electric and magnetic fields.

Contents

1.1 Uniform, Steady B-Field, No E-Field

Start by ignoring collisions (reasonable for low density and low temperature).

Let $\vec{B} = B\,\hat{z}$

Then Lorentz force, $\vec{F} = q\,(\vec{E} + \vec{v} \times \vec{B})$, with $\vec{E} = 0$ is:

$$\vec{F} = q\,\vec{v} \times \vec{B} = m\,\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \tag{3}$$

Looking at the $\vec{v} \times \vec{B}$ term,

$$\begin{aligned} \vec{v} \times \vec{B} &= (\vec{v}_y\vec{B}_z - \vec{v}_z\vec{B}_y)\hat{x} - (\vec{v}_x\vec{B}_z - \vec{v}_z\vec{B}_x)\hat{y} + (\vec{v}_x\vec{B}_y - \vec{v}_y\vec{B}_x)\hat{z} \\ &= \vec{v}_y B\hat{x} - \vec{v}_x B\hat{y} \end{aligned} \tag{4}$$

As a result, Equation 3 becomes

$$\begin{aligned} &\text{x-comp:} \quad m\,\frac{\mathrm{d}\vec{v}_x}{\mathrm{d}t} = q\,B\,v_y = m\dot{v}_x \\ &\text{y-comp:} \quad m\,\frac{\mathrm{d}\vec{v}_y}{\mathrm{d}t} = -q\,B\,v_x = m\dot{v}_y \\ &\text{z-comp:} \quad m\,\frac{\mathrm{d}\vec{v}_z}{\mathrm{d}t} = 0 = m\dot{v}_z \end{aligned} \tag{5}$$

So, there is NO force on the particle acting in the z-direction (along B, of course not, B can't do work on particle). Any motion Along B remains same.

What about motion perpendicular to B?

How about x and y motion?

Take derivative of (3.5a) and (3.5b):

$$\begin{aligned} m\,\ddot{v}_x &= q\,B\,\dot{v}_y = -\frac{(q\,B)^2}{m}v_x \\ m\,\ddot{v}_y &= -q\,B\,\dot{v}_x = -\frac{(q\,B)^2}{m}v_y \end{aligned} \tag{6}$$

So

$$\ddot{v}_x = -(\frac{q\,B}{m})^2 v_x$$
$$\ddot{v}_y = -(\frac{q\,B}{m})^2 v_y \tag{7}$$

Equations 7 takes the form of a Simple Harmonic Oscillator ($m\ddot{x} + kx = 0$) with Frequency

Cyclotron/Gyro Frequency

$$\omega_B = \frac{|q|B}{m} \qquad\qquad B[T] \quad m[kg] \quad q[c] \tag{8}$$

The solution of Equation 7 is

$$v_x = C\,\cos(\omega_B t - \alpha)$$

Where

$$C = \text{Constant}$$
$$\alpha = \text{Phase Angle, choose} = 0$$

We will introduce $v_\perp = \sqrt{v_x^2 + v_y^2}$ which is the amplitude of the velocity perpendicular to $\vec{B}$.

Choose at $t = 0$ and $v_y = 0$ such that...

$$v_x = v_\perp \cos(\omega_B t) = \dot{x} \tag{9a}$$

from Equation 4a in Section 2

$$v_y = \mp\frac{1}{\omega_B}\,\dot{v_x} = \mp\frac{1}{\omega_B}\omega_B\,v_\perp\,\sin(\omega_B t) = \mp\,v_\perp\,\sin(\omega_B t) = \dot{y} \tag{9b}$$

Where the $\pm$ or $\mp$ is the signage for an ion (top) or electron (bottom). To clarify, here, (-) is for an ion and (+) is for an electron.

Moving on, we integrate Equation 9 to get

$$x - x_o = \frac{v_\perp}{\omega_B}\sin(\omega_B t) \qquad \text{and} \qquad y - y_o = \pm\frac{v_\perp}{\omega_B}\sin(\omega_B t) \tag{10}$$

Recall our earlier notation such that, for this case, (+) is for the ion and (-) for the electron.

From this, we can define

Larmor/Gyro Radius

$$r_L = \frac{v_\perp}{\omega_B} = \frac{m\, v_\perp}{q\, B} \tag{11}$$

Plotting this motion:

Figure 1

We see that the particle moves in circular motion with radius r_L at a frequency of ω_B.

Time	Location	(Equation 10)	Velocity	(Equation 9)
$t = 0$	$x = 0$	$y = r_L$	$V_x = V_\perp$	$V_y = 0$
$t = \frac{\pi}{2\,\omega_B}$	$x = r_L$	$y = 0$	$V_x = 0$	$V_y = -V_\perp$
$t = \frac{\pi}{\omega_B}$	$x = 0$	$y = -r_L$	$V_x = -V_\perp$	$V_y = 0$
$t = \frac{3\,\pi}{2\,\omega_B}$	$x = -r_L$	$y = 0$	$V_x = 0$	$V_y = V_\perp$

To summarize: charged particles execute cyclotron motion (gyro-motion) in magnetic fields. Frequency of this motion is ω_B, the cyclotron frequency. Radius of this circular motion is r_L, the Larmor radius.

Note that ions and electrons rotate in different directions. They move so that the Bfield due to particle motion cancels/opposes the applied magnetic field (right hand rule).

Plasma particles tend to reduce the applied Bfield, and thus are diamagnetic.

Also remember there is a V_z along B. In other words, $V_{||}$ remains constant or is unchanged.

1.2 Uniform, Steady B-Field and E-Field

Now add a constant Electric field. In this case the particle motion is composed of two parts:

1. Gyro-motion (Section 3, 1.1)
2. Guiding center drift

We can choose our axes so that $\vec{B} = B\,\hat{z}$ and $\vec{E}$ lies in the x-z plane,

Figure 2

The lorentz force equation is our equation of motion, $\vec{F} = q\,(\vec{E} + \vec{v} \times \vec{B}) = m\frac{dv}{dt}$.

The z-component or the component along/parallel to the B-field direction is:

$$\begin{aligned} \frac{\mathrm{d}v_z}{\mathrm{d}t} &= \frac{q}{m}E_z \\ v_z &= \frac{q}{m}\,E_z\,t + v_{z,0} \end{aligned} \tag{12}$$

Acceleration of the particle is along $\vec{B}$. $v_z = v_{||}$ changes with time meaning it is no longer constant as before. This is due to the E-field component along $\vec{B}$ which accelerates the particle along $\vec{B}$.

The x and y components (perpendicular to B-field direction) for

$$\begin{aligned} \dot{v}_x &= \frac{q}{m}\,E_x \pm \omega_B\,v_y && \text{(Now has } E_x\text{)} \\ \dot{v}_y &= 0 \pm \omega_B\,v_x && \text{(Same as Equation 5)} \end{aligned} \tag{13}$$

Differentiating with constant $\vec{E}$

$$
\begin{aligned}
\ddot{v}_x &= -\omega_B^2\, v_x && \text{(Same as Equation 7)} \\
\ddot{v}_y &= -\omega_B^2 \left(v_y + \frac{E_x}{B} \right) && \text{(Now has } E_x\text{)}
\end{aligned}
\tag{14}
$$

The solution of these equations is:

$$
\begin{aligned}
v_x &= v_\perp \cos(\omega_B t) \\
v_y &= \mp v_\perp \sin(\omega_B t) - \frac{E_x}{B}
\end{aligned}
\tag{15}
$$

- Note that compared with Equation 9 from before, there is now a drift in the -y direction (for $E_x > 0$).

$$\text{Drift} = -\frac{E_x}{B}$$

- Even though E was only in x and z directions, it causes drift in y direction.
- This is called guiding center drift since the point about which the particle gyrates is moving.

We will now begin looking at sketches of charged particle motion.

Case 1: Gyromotion that drifts in $\mathbf{E} \times \mathbf{B}$ (-y) direction. Gyromotion that stays in the x-y plane.

$$\vec{B} = B\hat{z} \qquad \vec{E} = E_x\hat{x} \qquad v_{||} = v_z = 0$$

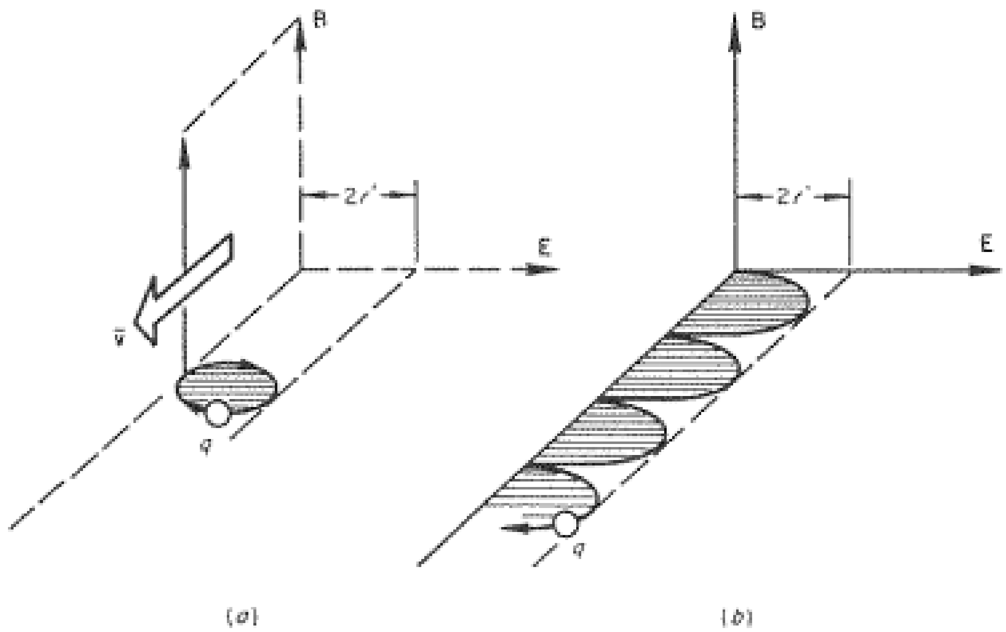


Fig. 5-2 Charged-particle drift from rest in crossed electric and magnetic fields. *(a)* Transformed system; *(b)* laboratory system.

In this case, a charge is starting from rest in uniform perpendicular $\vec{E}$ and $\vec{B}$ fields. Here the total force is normal to v and to B

$$\vec{F} = q\,(\vec{E} + \vec{v} \times \vec{B}) = m\frac{dv}{dt}$$

and the initial velocity, v_o, is simply the negative of the drift velocity, v_{drift}.

$$v_o^1 = v_o - v_{\text{drift}} = -v_{\text{drift}} = -\frac{\vec{E} \times \vec{B}}{B^2}$$

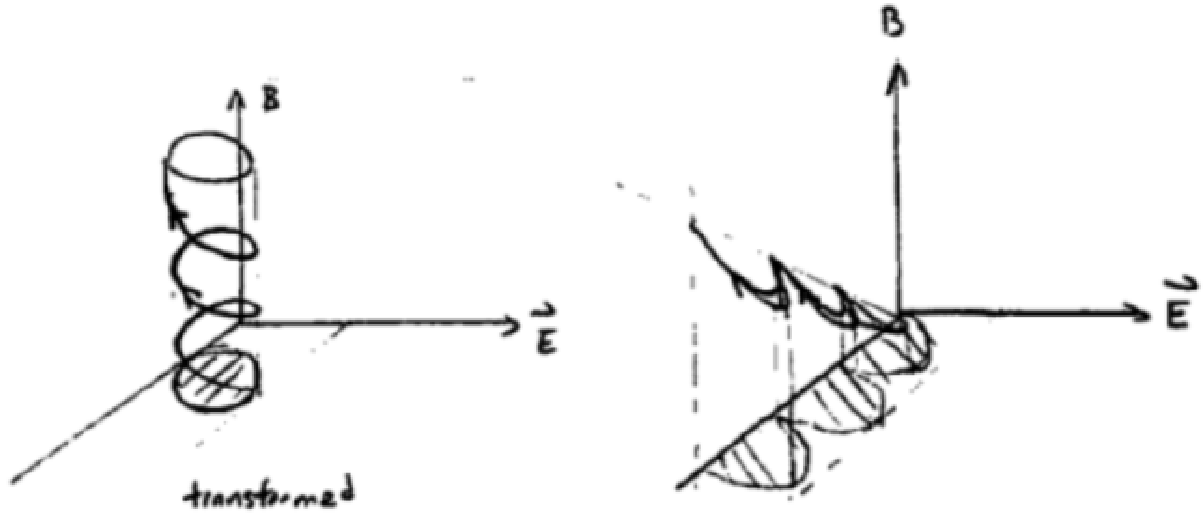
The motion in a coordinate system moving with the drift velocity relative to the laboratory frame is a circle of radius:

$$r^1 = \frac{m\,v_o^1}{q\,B} = \frac{m\,v_{\text{drift}}}{q\,B} = \frac{m\,E}{q\,B^2}$$

which is transcribed with constant tangential speed, $v_{\text{drift}} = \frac{E}{B}$, in the plane normal to $\vec{B}$. In the laboratory frame, this motion transforms into a cycloid in the same plane, advancing in the $\vec{E} \times \vec{B}$ direction with drift speed $\frac{E}{B}$ as seen in the figure above. Note that this drift is normal to $\vec{E}$ and independent of the charge sign.

Case 2: Gyromotion that drifts in the $\mathbf{E} \times \mathbf{B}$ (-y) direction, with an initial parallel velocity component. Helix with constant pitch, pushed over on its side.

$$\vec{B} = B\hat{z} \qquad \vec{E} = E_x\hat{x} \qquad v_{||} = v_z \neq 0$$



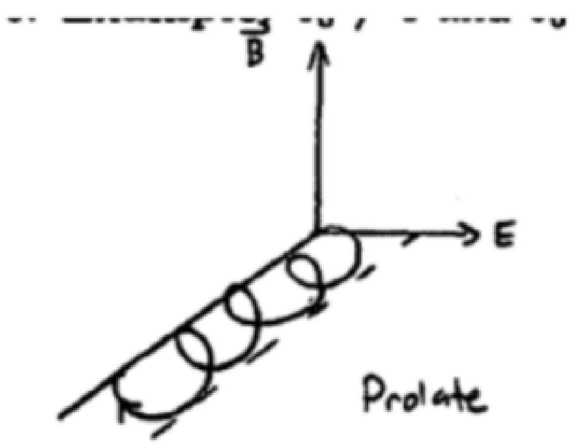
In this case, we now have a velocity in the z-direction. As a result it not only has a gyromotion but is also moving up at the same time. On top of that, it also is moving to the right because of our drift velocity. Note however that the velocity parallel to the z-component is constant!

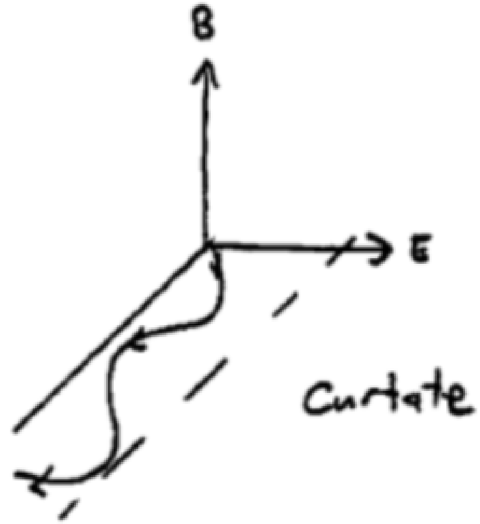
Case 3: Gyromotion that drifts in $\mathbf{E} \times \mathbf{B}$ (-y) direction, WITH an initial perpendicular velocity.

$$\vec{B} = B\hat{z} \qquad \vec{E} = E_x\hat{x} \qquad v_{||} = v_z = 0 \qquad v_{\perp,0} \neq 0$$

Prolate - initial velocity opposes $E \times B$ drift

Curtate - initial velocity same direction as $E \times B$ drift





If $\vec{v}_o \neq 0$ but has some component parallel to $\vec{B}$, the motion in the transformed system will be a helix instead of a circle, and in the laboratory frame, an inclined helix whose axis lies in the $\vec{B}$, $\vec{E} \times \vec{B}$ plane. If $\vec{v}_o$ also has a component normal to $\vec{B}$, this will contribute to the $\vec{v}^1 \times \vec{B}$ force, and hence, change the radius of gyration and angular velocity, without affecting the drift velocity. The orbit then becomes a prolate cycloid (i.e. has loops) or a curtate cycloid depending on $\vec{v}^1_{0,\perp} \lessgtr \frac{E}{B}$. The cycloid always progresses along the positive $\vec{E} \times \vec{B}$ axis, but its position relative to that axis depends on the direction of $\vec{v}^1_{0,\perp}$.

Note: the very special case of $\vec{v}^1_o = 0$, that is, $\vec{v}_o = v_{\text{drift}} = (\vec{E} \times \vec{B})/B^2$. A particle injected with this velocity moves in a straight line, undisturbed by the fields.

Case 4: Gyromotion with drift, with acceleration along B.

$$\vec{B} = B\hat{z} \qquad \vec{E} = E_x\hat{x} + E_z\hat{z}$$

Now, helix with increasing pitch (expanding helix) pushed over on its side in lab frame.

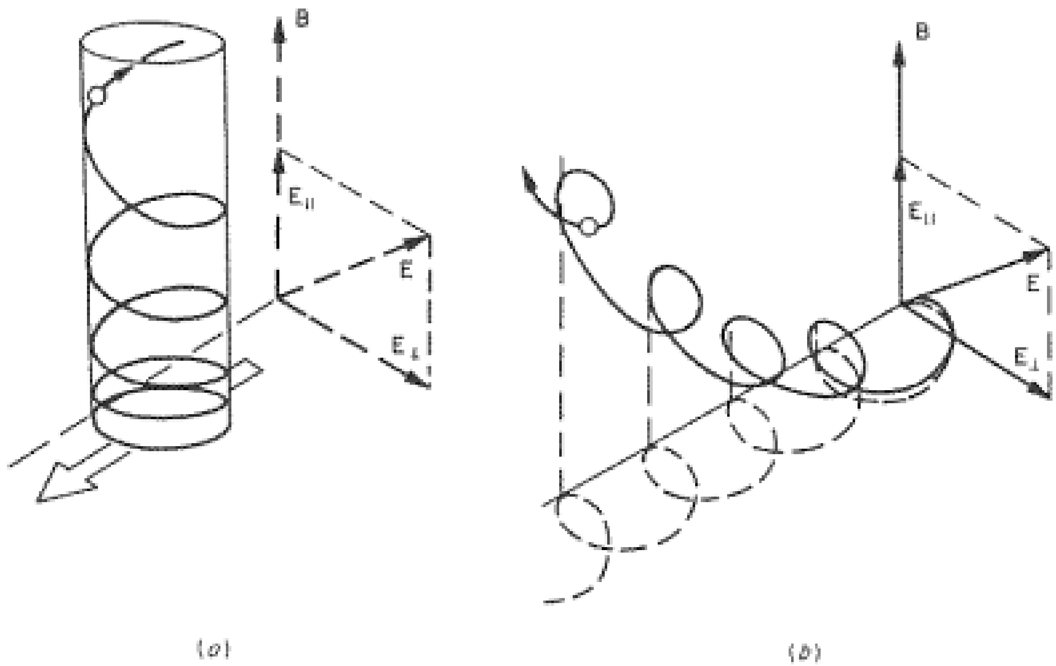


Fig. 5-3 Charged-particle drift in arbitrarily oriented electric and magnetic fields. *(a)* Transformed system; *(b)* laboratory system.

Finally, if we include a component of $\vec{E}$ parallel to $\vec{B}$, we find in the transformed frame, a helix of quadratically increasing pitch:

$$\vec{v}^{\,1}_{||} = \frac{q\,E_{||}}{m}\,t + \vec{v}^{\,1}_{o,||}$$

In the laboratory frame, this transforms into a parabolically displaced helix, whose axis lies in the $\vec{B}$, $\vec{E} \times \vec{B}$ plane.

We will now shift focus to the **Magnitude and Direction of an $\mathbf{E} \times \mathbf{B}$ Drift**

The magnitude and direction of the guiding center drift can be found from Lorentz Equation from Section 2 Equation 66.

To begin, we chose to ignore $m\,\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$ since it only gives gyromotion and not drift. Also recall from Equation 15 that the drift was independent of time.

$$\vec{E} + \vec{v} \times \vec{B} = 0 \quad \Rightarrow \qquad \vec{E} = -\vec{v} \times \vec{B}$$

$$\vec{E} \times \vec{B} = \vec{B} \times (\vec{v} \times \vec{B}) \quad = \qquad \vec{v}B^2 - \vec{B}(\vec{v} \cdot \vec{B})$$

However, we want the transverse component only. As a result, we get:

$$\vec{v}_{E\times B} = \frac{\vec{E} \times \vec{B}}{B^2} \qquad \text{Where B is the magnitude of the B-field} \tag{16}$$

The magnitude of which is...

$$|\vec{v}_{E\times B}| = \frac{E}{B} \tag{17}$$

PHYSICALLY, what gives rise to this motion?

1. Ion begins to accelerate in an E-field.
2. This causes $v_\perp$ to increase and the instantaneous Larmor radius to also increase.
3. This causes the magnetic force to increase (since force, $\vec{v} \times \vec{B}$, is perpendicular to the velocity), causing the ion to turn.
4. This causes the Ion to turn so much that its velocity is now opposing $\vec{E}$, thereby going against $\vec{E}$.
5. This now causes the ion to slow down, meaning it decelerates and its instantaneous Larmor radius decreases.
6. The Magnetic force still causes ion to turn, eventually it gets turned back around such that velocity is aligned with Efield, and the process repeats.

This interaction of the electric and magnetic forces on the charged particle cause motion that looks like:

Figure 3

Finally, note that VExB is independent of q, m, $\vec{v}_{\perp}$

1.3 Arbitrary Force Drift

The results of the previouse section are actually a special case. They are the special case for when an Electric force is present with a Magnetic force. In other words, when an electric field is present with a magnetic field.

More generally, a charged particle drift occurs when any other force is present with the magnetic force.

Generic Equation of Motion:

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = F + q\,\vec{v} \times \vec{B} \tag{18}$$

Where F is any other applied force and $q\,\vec{v} \times \vec{B}$ is the applied magnetic force. From this:

Arbitrary Force Drift

$$\vec{v}_F = \frac{1}{q}\frac{\vec{F} \times \vec{B}}{B^2} \tag{19}$$

if $\vec{F} = q\,\vec{E}$ then

$$\vec{v} = \frac{\vec{E} \times \vec{B}}{B^2} \qquad \text{Equation 16}$$

What if instead we used gravity, $\vec{F}_g = m\,\vec{g}$. Then:

$$\vec{v}_{F_g} = \frac{m}{q}\frac{\vec{g} \times \vec{B}}{B^2} \tag{20}$$

1.4 Non-Uniform Fields

Still more drifts resulting from nonuniform fields can be superimposed on these motions. For instance:

Grad-B drift

$$\vec{V}_{\nabla B} = \pm\frac{1}{2}\, v_{\perp}\, r_L \frac{\vec{B} \times \nabla B}{B^2} \tag{21}$$

Which is the case for a non-uniform magnetic field:

In addition we can have:

Curvature drift

$$\vec{V}_R = \frac{m v_{||}^2}{q\, B^2}\, \frac{\vec{R}_c \times \vec{B}}{R_c^2}$$

Where

$$R_c = \text{Magnitude of } \vec{R}_c$$

$$B = \text{Magnitude of } \vec{B}$$

$$\vec{R}_c = \text{Radius of Curvature of the B-field}$$

Which is the case for a non-uniform magnetic field in a vacuum where being in a vacuum results in curved field lines.

1.5 Conductivity in Collisionless Plasma

Considering a two-species plasma (electrons and ions) in simple crossed E and B fields. Both species move in the same direction with the same drift velocity ($|\vec{v}_{E\times B}| = \frac{E}{B}$).

Particles have different mass and perpendicular velocity, and thus different gyromotion, meaning they have different cyclotron frequencies and Larmor radii. Despite this however, they all move in the ExB direction with same speed.

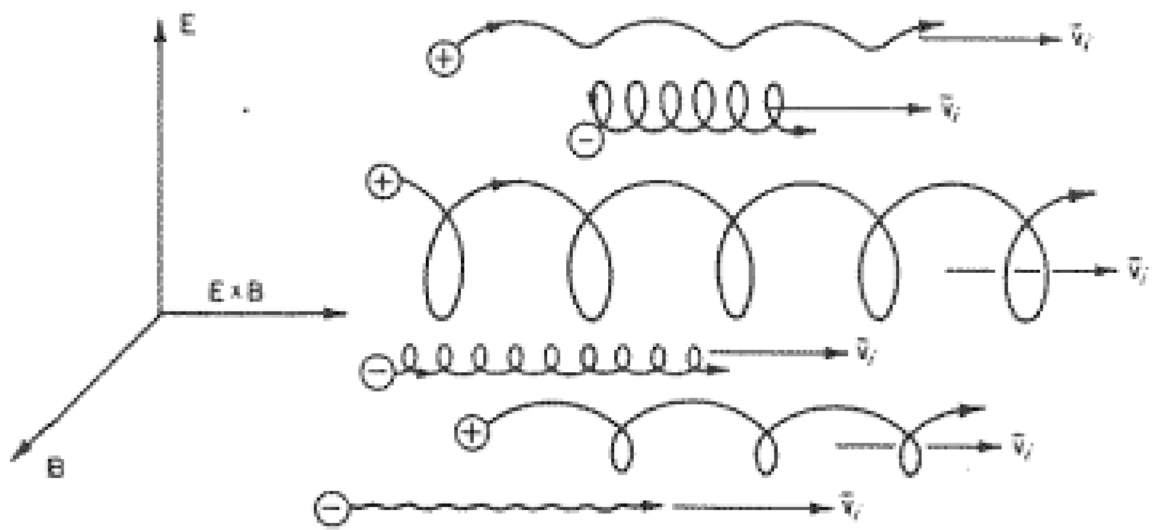


Fig. 5-4 Schematic representation of common migration velocities of thermal electrons and ions.

Thus, for quasineutrality (s̃ame number of positive particles as negative particles), the net current density in the crossed-field direction is:

$$\vec{j} = \sum_k n_k\, q_k\, \vec{v}_k = n_i\, q_i\, \vec{v}_i - n_e\, q_e\, \vec{v}_e = 0$$

But what if there is a parallel component to the Efield?

- $\vec{v}_i \neq \vec{v}_e$; $q\vec{E}$ pulls ions and electrons in opposite directions.
- Since we are considering the case with no collisions, we have unopposed acceleration (what is there to stop them? It's collisionless), which leads to $\vec{v} \to \infty$ as $t \to \infty$
- As a result current becomes ill-defined.

Conclusion: Collisions must play an important role in limiting the unopposed acceleration of the particles.