Application Problem 1:

Application : Compute $\int_0^3 x^2 dx$ using

- Rectangle rule with 1, 2 and 3 subintervals
- Trapezoidal rule with 1, 2 and 3 subintervals
- Simpson rule with 2 and 4 subintervals

Compare your result with the exact solution (i.e., compute the relative error) and comment on your solution.

- **Exact Solution**

$$y = \int_0^3 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^3 = 9$$

- **Rectangle Rule Solution**

Rectangle Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} h_i\, f(\frac{x_{i-1}+x_i}{2})$$

Where:

$$\begin{aligned} n &= \text{Number of Subintervals} \\ h_i &= (x_i - x_{i-1}) \\ h &= \frac{x_{\text{final}} - x_{\text{initial}}}{n} \\ &= \text{Length of a Single Subinterval} \end{aligned}$$

1 Subinterval:

$$h = \frac{3-0}{1} = 3 \qquad \text{therefore} \qquad y = 3\left(\frac{0+3}{2}\right)^2 = 6.75$$

2 Subintervals:

$$h = \frac{3-0}{2} = 1.5 \quad \text{therefore} \qquad y = 1.5\left(\frac{0+1.5}{2}\right)^2 + 1.5\left(\frac{1.5+3}{2}\right)^2 = 8.4375$$

3 Subintervals:

$$h = \frac{3-0}{3} = 1 \qquad \text{therefore} \qquad y = 1\left(\frac{0+1}{2}\right)^2 + 1\left(\frac{1+2}{2}\right)^2 + 1\left(\frac{2+3}{2}\right)^2 = 8.75$$

- **Trapezoidal Rule Solution**

Trapezoidal Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} \frac{h_i}{2}\left(f(x_{i-1}) + f(x_i)\right)$$

Where:

$$\begin{aligned} n &= \text{Number of Intervals} \\ h_i &= (x_i - x_{i-1}) \\ h &= \frac{x_{\text{final}} - x_{\text{initial}}}{n} \\ &= \text{Length of a Single Subinterval} \end{aligned}$$

1 Subinterval:

$$h = \frac{3-0}{1} = 3 \qquad \text{therefore} \qquad y = \frac{3}{2}\big(0^2 + 3^2\big) = 13.5$$

2 Subintervals:

$$h = \frac{3-0}{2} = 1.5 \quad \text{therefore} \qquad y = \frac{1.5}{2}\big(0^2 + 1.5^2\big) + \frac{1.5}{2}\big(1.5^2 + 3^2\big) = 10.125$$

3 Subintervals:

$$h = \frac{3-0}{3} = 1 \qquad \text{therefore} \qquad y = \frac{1}{2}\big(0^2 + 1^2\big) + frac12\big(1^2 + 2^2\big) + frac12\big(2^2 + 3^2\big) = 9.5$$

- **Simpson Rule Solution**

Simpson Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \frac{2}{3}\, I_{\text{rectangular}} + \frac{1}{3}\, I_{\text{trapezoidal}}$$

Where:

$$I_{\text{rectangular}} = \text{Rectangle Rule Estimate}$$
$$I_{\text{trapezoidal}} = \text{Trapezoidal Rule Estimate}$$

2 Subintervals:

$$h = \frac{3-0}{2} = 1.5$$

therefore

$$I_{\text{rectangular}} = 1.5\left(\frac{0+1.5}{2}\right)^2 + 1.5\left(\frac{1.5+3}{2}\right)^2 = 8.4375$$

$$I_{\text{trapezoidal}} = \frac{3-0}{2} = 1.5 = \frac{1.5}{2}\left(0^2 + 1.5^2\right) + \frac{1.5}{2}\left(1.5^2 + 3^2\right) = 10.125$$

As a result

$$y = \frac{2}{3}\, I_{\text{rectangular}} + \frac{1}{3}\, I_{\text{trapezoidal}}$$

$$= \frac{2}{3}\,(8.4375) + \frac{1}{3}\,(10.125)$$

$$= 9 \qquad \leftarrow \text{This is our exact solution!}$$

Application Problem 2:

Application : Compute $\int_{0}^{3} x^2 dx$ and $\int_{-\pi/2}^{\pi/2} \cos(x) dx$ using the 1 - and 2 - point GQ rule

Compare your result with the exact solution (i.e., compute the relative error) and comment on your solution.

- **Exact Solution**

$$y = \int_0^3 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^3$$

- **1 Point Gauss Quadrature Rule**

  Such that $w_1 = 2$ and $\xi_1 = 0$:

$$I = \int_0^3 x^2 \mathrm{d}x = \frac{3}{2}\int_{-1}^{1} f\left(\frac{3\xi + 3}{2}\right)\mathrm{d}\xi$$

$$= \frac{3}{2}\int_{-1}^{1} \left(\frac{3\xi + 3}{2}\right)^2 \mathrm{d}\xi$$

$$= 2\left(\frac{3\,(0) + 3}{2}\right)^2 = \frac{18}{4} = 4.5$$

- **2 Point Gauss Quadrature Rule**

  Such that $w_1 = w_2 = 1$, $\xi_1 = -\frac{\sqrt{3}}{3}$ and $\xi_2 = \frac{\sqrt{3}}{3}$:

$$I = w_1\, f(\xi_1) + w_2\, f(\xi_2)$$

$$= \frac{3}{2}\left(\frac{3\,(-\frac{\sqrt{3}}{3}) + 3}{2}\right)^2 + \frac{3}{2}\left(\frac{3\,(\frac{\sqrt{3}}{3}) + 3}{2}\right)^2$$

$$= 9$$

- **Exact Solution**

$$y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\mathrm{d}x = \Big[\sin(x)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2$$

First we must do a Coordinate Transformation to get this in the form we want.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\mathrm{d}x = \frac{\pi}{2}\int_{-1}^{1} \cos\left(\frac{\pi}{2}\,\xi\right)\mathrm{d}\xi$$

- **1 Point Gauss Quadrature Rule**

Such that $w_1 = 2$ and $\xi_1 = 0$:

$$I = \frac{\pi}{2}\Big[\cos(\frac{\pi}{2}\,\xi_1)\,w_1\Big]$$

$$= \frac{\pi}{2}\Big[\cos\big(\frac{\pi}{2}\,(0)\big)\,2\Big]$$

$$= 3.14$$

- **2 Point Gauss Quadrature Rule**

Such that $w_1 = w_2 = 1$, $\xi_1 = -\frac{\sqrt{3}}{3}$ and $\xi_2 = \frac{\sqrt{3}}{3}$:

$$I = \frac{\pi}{2}\Big[w_1\,\cos(\frac{\pi}{2}\,\xi_1) + w_2\,\cos(\frac{\pi}{2}\,\xi_2)\Big]$$

$$= \frac{\pi}{2}\left[1\,\cos\left(\frac{\pi}{2}\,(-\frac{\sqrt{3}}{3})\right) + 1\,\cos\left(\frac{\pi}{2}\,(\frac{\sqrt{3}}{3})\right)\right]$$

$$= 1.9352 \qquad \approx 3\%\text{ off}$$

The solution alternates between over estimate and under estimate.

Application Problem 3:

Application : Compute $\int_{-1}^{1}\int_{-1}^{1} \exp(2x) * \ln(3+y)\,dy\,dx$ using the 1 * 1, 2 * 2 and 3 * 3 GQ rule

Compare your result with the exact solution (I_{ex} = 7.829967) (i.e., compute the relative error) and comment on your solution.

- **Exact Solution**

$$I_{\text{exact}} = \int_{-1}^{1}\int_{-1}^{1} e^{2\,x} \cdot \ln(3+y)\,\mathrm{d}y\,\mathrm{d}x = 7.829967$$

- **1*1 Point Gauss Quadrature Rule**

Such that $w = w_1 \times w_1 = 4$ and $\xi_1 = \eta_1 = 0$:

$$I = w\,f(\xi_1, \eta_1)$$

$$= 4\,e^0 \cdot \ln(3) = 4.39449$$

- **2*2 Point Gauss Quadrature Rule**

Such that $w_k = 1$ and $(\xi_k, \eta_k) = \big(\pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3}\big)$:

$$I = \sum w_k\,f(\xi_k, \eta_k)$$

$$= 1\,\exp\left[2\left(-\frac{\sqrt{3}}{3}\right)\right]\,\ln\left(3 + \left(-\frac{\sqrt{3}}{3}\right)\right) \quad + \quad 1\,\exp\left[2\left(-\frac{\sqrt{3}}{3}\right)\right]\,\ln\left(3 + \left(\frac{\sqrt{3}}{3}\right)\right)$$

$$+ 1\,\exp\left[2\left(\frac{\sqrt{3}}{3}\right)\right]\,\ln\left(3 + \left(\frac{\sqrt{3}}{3}\right)\right) \quad + \quad 1\,\exp\left[2\left(\frac{\sqrt{3}}{3}\right)\right]\,\ln\left(3 + \left(-\frac{\sqrt{3}}{3}\right)\right)$$

$$= 7.532767$$