Section 1

AE435

Spring 2018

Definition 1. Electric Propulsion - The use of electric power to accelerate propellant for propulsion.

Key Concepts:

Note 1. Overcome limitations of the Rocket Equation which expresses mission capability in terms of "characteristic velocity" or "Delta-V" ΔV.

The Rocket Equation

$$\Delta V = c \ln(\frac{M_o}{M_f})$$

Where:

$$\begin{aligned} c &= \text{Effective Exhaust Velocity} \\ M_o &= \text{Initial Mass} \\ M_f &= \text{Final Mass} \end{aligned}$$

For a **Chemical Thruster**, the effective exhaust velocity, c, is limited by the enthalpy of combustion. In other words, it is limited by the chemical bond energy of the propellant.

Effective Exhaust Velocity for a Chemical Thruster

$$c = U_e + \frac{A_e}{\dot{m}}(P_e - P_\infty)$$

Where:

$$\begin{aligned} U_e &= \text{Exhaust Velocity} \\ A_e &= \text{Exit Area} \\ \dot{m} &= \text{Mass Flow Rate} \\ P_e &= \text{Exit Pressure} \\ P &= \text{Ambient Pressure (0 in Vacuum)} \end{aligned}$$

For **Electric Propulsion**, the effective exhaust velocity, c, is limited by the power supplied.

Note 2. Electric Propulsion is "Power Limited"

It is difficult to generate power in space (International Space Station $\tilde{1}$20kW). Power limited means the power of the exhaust beam is limited. In other words

$$P_{\text{jet}} = \frac{1}{2}\dot{m}c^2$$

is limited. Where P_{jet} is the power in the exhaust beam, the amount of kinetic energy being expelled per unit time.

- **Corollary 1:** Power Limited & High $c \Rightarrow$ Low $\dot{m}$ & Low Thrust
- **Corollary 2:** Low Thrust $\Rightarrow$ Low Acceleration (10^{-3} - 10^{-6} g's)
- **Corollary 3:** Low Acceleration $\Rightarrow$ Long Trip Times $\Rightarrow$ No Manned Missions

Note 3. For each mission, (ΔV & Time) there is an optimum c.

It is not the case to say "Lets just do a high ΔV", NO! For each optimum c there is also an optimum thruster. The specific impulse level needed for a mission determines the thruster we choose to use (Ion, Hall, etc.).

1. Find the ΔV for mission
2. Find the optimum c
3. Pick the "best" thruster.

1 Introduction to Electric Propulsion

1.1 Space Propulsion

See handwritten notes .

1.2 Comparison of Chemical Propulsion and Electric Propulsion - An Example

Consider a small 5lb chemical thruster capable of 22N of thrust. From Equation 2, in a vacuum,

$$\text{Thrust} = \frac{d}{dt}(\text{momentum}) = \dot{m}u_e + A_e(P_e - 0) = \dot{m}c$$

For a hydrazine monopropellant, $I_{SP} = 220\text{sec}$.

Specific Impulse

$$I_{SP} = \frac{\text{impulse}}{\text{propellant weight}} = \frac{\int T dt}{gM_{prop}} \tag{1}$$

Where, g is always $9.81\frac{m}{s^2}$

For a constant thrust, T, for time t_t and

$$\dot{m} = \frac{M_{prop}}{t_t} \tag{2}$$

then

$$I_{SP} = \frac{Tt_t}{gM_{prop}} = \frac{\dot{m}c}{g\dot{m}} = \frac{c}{g} \tag{3}$$

So, $I_{SP} = 220\text{sec} \rightarrow c = 2156\frac{\text{m}}{\text{s}}$

$\dot{m}$ for 5lb thruster is $\dot{m} = \frac{T}{c} = 0.1\frac{\text{kg}}{\text{s}}$

Exhaust Beam Power

$$P_{\text{jet}} = \frac{1}{2}\dot{m}c^2 = \frac{1}{2}Tc \tag{4}$$

Which for our problem equates to $= 24kW$ which is HUGE! (Space Shuttle produces power in the GW range). The smallest electric propulsion is <1W

1.3 Thruster Efficiency

Recall that thrusters are essentially just energy conversion device. For Electric Propulsion:

Figure 1

Thruster Efficiency:

$$\eta_t = \frac{\text{Beam Power as Thrust}}{\text{Power to Thruster}} = \frac{\frac{1}{2}Tc}{P_{in}} \tag{5}$$

Power Processing Unit Efficiency:

$$\eta_{PPU} = \frac{\text{Power to Thruster}}{\text{Power from Source}} = \frac{P_{in}}{P} \tag{6}$$

Overall System Efficiency:

$$\eta_t = \frac{\text{Beam Power as Thrust}}{\text{Power from Source}} = \frac{\frac{1}{2}Tc}{P} = \eta_t \eta_{PPU} \tag{7}$$

The PPU takes the power and puts it in a form the thruster can use. For Electric Propulsion, thruster effieciency (η_t) ranges from 5% to 90% depending on the thruster.

- PPT: 0%-2%
- Hall: 50% - 60%
- Ion: 80%

1.4 Thrust Discussion

What is and isn't a force?

Fundamental Forces	Caused by...
Gravity	Gravitation Field (mass)
Pressure	Particle Momentum/Energy
Shear	Particle Collisions (Viscosity)
Electric Force	E-field (charge)
Magnetic Foce	B-field (charge motion)

There is no force called "Thrust." Thrust is caused by shear and pressure when considering the exhaust of a chemical thruster. For electric propulsion however, we now have thrust caused by all of the fundamental forces listed. Thrust is the resulting effect when other forces push-on/propels a space craft, rocket, gas turbine engine etc.

Consider a Control Volume containing a rocket and all exhaust. Apply Newton's Law:

Figure 2

Theory:
The change in momentum of mass in the C.V. equals the sum of forces on the C.V. (gravity). If $g = 0$, the momentum in the C.V. is constant because it contains all the exhaust.

Proof
Change in $\vec{P}$ (momentum) in time Δt as we accelerate to the effective exhaust velocity, c:

$$\begin{aligned} \text{(before):} \quad & \vec{P}(t) = (m + \Delta m)\vec{v} \\ \text{(after):} \quad & \vec{P}(t + \Delta t) = m(\vec{v} + \Delta\vec{v}) + \Delta m(\vec{v} + \Delta\vec{v} - \vec{c}) \\ \text{(difference):} \quad & \Delta P = \vec{P}(t + \Delta t) - \vec{P}(t) = m\Delta\vec{v} - \Delta m\vec{c} + \Delta m\Delta\vec{v} \end{aligned}$$

Take the limit, $\Delta t \to dt$:

$$\frac{\Delta P}{dt} = \frac{d\vec{P}}{dt} = m\frac{d\vec{v}}{dt} - \frac{dm}{dt}\vec{c}$$

Equating $\frac{d\vec{P}}{dt}$ with the gravitational force $\vec{F_g}$, we get:

$$m\frac{d\vec{v}}{dt} - \frac{dm}{dt}\vec{c} = F_g \qquad \text{or} \qquad m\dot{\vec{v}} = F_g + \dot{m}\vec{c}$$

Where:

$$\begin{aligned} m\dot{\vec{v}} &= \text{Change in Rocket Momentum} \\ F_g &= \text{Gravity Causes Rocket Momentum Change} \\ \dot{m}\vec{c} &= \text{Rate of Increase of Exhaust Momentum} \end{aligned}$$

Note that the third term in the equation $m\dot{\vec{v}} = F_g + \dot{m}\vec{c}$ has units of force but is not actual a force. This we call Thrust! c, comes from the fundamental forces initially discussed. We will develop models to predict what these forces will be on a certain Electric Thruster and use those to predict the thrust on an Electric Propulsion device.

Some points to be made on the "after" equation $\vec{P}(t+\Delta t) = m(\vec{v}+\Delta\vec{v}) + \Delta m(\vec{v}+\Delta\vec{v}-\vec{c})$ are:

$$\begin{aligned} \vec{v} + \Delta\vec{v} &= \text{Some change because we expelled some small amount of mass in exhaust.} \\ -\vec{c} &= \text{This is negative because the rocket and the exhaust are going in opposite directions.} \end{aligned}$$

To emphasize its importance, we highlight the following equation explained above.

$$\boxed{m\dot{\vec{v}} = F_g + \dot{m}\vec{c}} \tag{8}$$

1.5 Thrust-to-Power $\frac{T}{P}$ $[\frac{N}{W}]$

The Thrust-to-Power ratio is an important space craft design parameter.

$$\frac{T}{P} = \frac{2\eta}{c} \propto \frac{1}{c} \qquad \text{and} \qquad \frac{T}{P} \propto \frac{1}{I_{SP}}$$

With the overall efficiency equation from Section 1.3 and with Equation 3 from Section 1.2 shown above we can achieve an equation for the Thrust-to-Power ratio.

Exercise 1.
For Electric Propulsion with $\eta = 0.5$ and $I_{SP} = 1000$sec, find the Thrust-to-Power ratio.

$$\frac{T}{P} = 10^{-4}\frac{\text{N}}{\text{W}} \qquad \text{or} \qquad 0.1\frac{\text{N}}{\text{kW}}$$

Thrust Time

$$t_t = \frac{\text{velocity change}}{\text{acceleration}} = \frac{v_2 - v_1}{\frac{T}{m}} \tag{9}$$

Exercise 2.
Given a 100kg spacecraft with P=1kW and a velocity change, $V_2 - V_1 = 1000\frac{\mathbf{m}}{\mathbf{s}}$, find the thrust time, t_t.

$$t_t = \frac{1000}{\frac{0.1}{100}} = 10^6\text{sec} = 11.6\text{days}$$

1.6 Design of Space Missions using Electric Propulsion

Considering Electric Propulsion for a space mission, we need to know:

1. Characteristic velocity or "delta-V", ΔV
2. Time available

Of these, "delta-V" is easier to define so start with that.

Free Body Diagram:

Figure 3

Equation of Motion:

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{T} - \vec{D} - m\vec{g} \tag{10}$$

Writing thrust as $T = \frac{\mathrm{d}m}{\mathrm{d}t}c$ in the direction of the flight velocity, $\vec{v}$ we get.

$$\mathrm{d}v = c\frac{\mathrm{d}m}{m} - \frac{D}{m}\mathrm{d}t - g\sin\gamma\mathrm{d}t$$

or...

"Delta-V"

$$\int c\frac{\mathrm{d}m}{m} = \int \mathrm{d}v + \int \frac{D}{m}\mathrm{d}t + \int g\sin\gamma\mathrm{d}t \tag{11}$$

Where:

$$\int \mathrm{d}v = \text{Velocity Change (can be positive, negative or zero)}$$
$$\int \frac{D}{m}\mathrm{d}t = \text{Drag Loss (zero in a vacuum or for impulsive thrust)}$$
$$\int g\sin\gamma\mathrm{d}t = \text{Gravity Loss (zero when g=0, } \gamma = 0 \text{ or impulsive)}$$

Equation 11 tells us the mass change required to achieve a velocity change in the presence of drag and gravity. The LHS of the equation is called "Delta-V", ΔV, and is sometimes referred to as the "characteristic velocity", c. Not to be confused with c^* for chemical propulsion.

Note 4.

For the Space Shuttle:

$$\text{Total delta-V} = 9347\frac{\text{m}}{\text{s}}$$

$$\text{Actual Orbit Velocity is only} = 7790\frac{\text{m}}{\text{s}}$$

$$\text{Gravity Loss} = 1080\frac{\text{m}}{\text{s}}$$

$$\text{Drag Loss} = 120\frac{\text{m}}{\text{s}}$$

$$\text{Vertical-Horizonal} = 360\frac{\text{m}}{\text{s}}$$

There is a $1557\frac{\text{m}}{\text{s}}$ difference between the actual orbit velocity and the total delta-v for the space shuttle. In the case of no gravity or drag, we see that Δv is just a matter of velocity change.

1.6.1 Impulsive Maneuvers

Consider the case where c=constant and we perform an impulsive maneuver. In addition, assume that we are in space therefore excluding gravity and drag. Then Equation 11 becomes:

$$\Delta v = c\ln(\frac{M_1}{M_2}) = v_2 - v_1 + \int \frac{D}{m}dt + \int g\sin\gamma dt$$

$$\text{No Drag, } D = 0$$
$$\text{No Gravity or Impulse, d}t = 0$$

Gravity Free Rocket Equation

$$\Delta v = c\ln(\frac{M_1}{M_2}) = v_2 - v_1 \tag{12}$$

Where:

$$M_1 = \text{Initial Mass}$$
$$M_2 = \text{Final Mass}$$

Exercise 3. We want to <u>accelerate</u> to v_2, <u>reverse</u>, and <u>decelerate</u> to <u>zero</u> velocity. Start by...

$$\Delta V_A = c\ln(\frac{M_1}{M_2}) = V_2 - 0$$
$$\Delta V_B = c\ln(\frac{M_2}{M_3}) = 0 - V_2$$
$$\Delta V_{total} = |\Delta V_A| + |\Delta V_B| = c\ln(\frac{M_1}{M_2}) + c\ln(\frac{M_2}{M_3}) = c\ln(\frac{M_1}{M_3}) = 2V_2$$

While $\Delta v_{total} \neq 0$, $\Delta v_{total} = 2v_2$
The Net Change in velocity however will be $v_2 - v_2 = 0$
"delta-V" is the mass required to do that maneuver.

In general, **How do we calculate "delta-V"?**

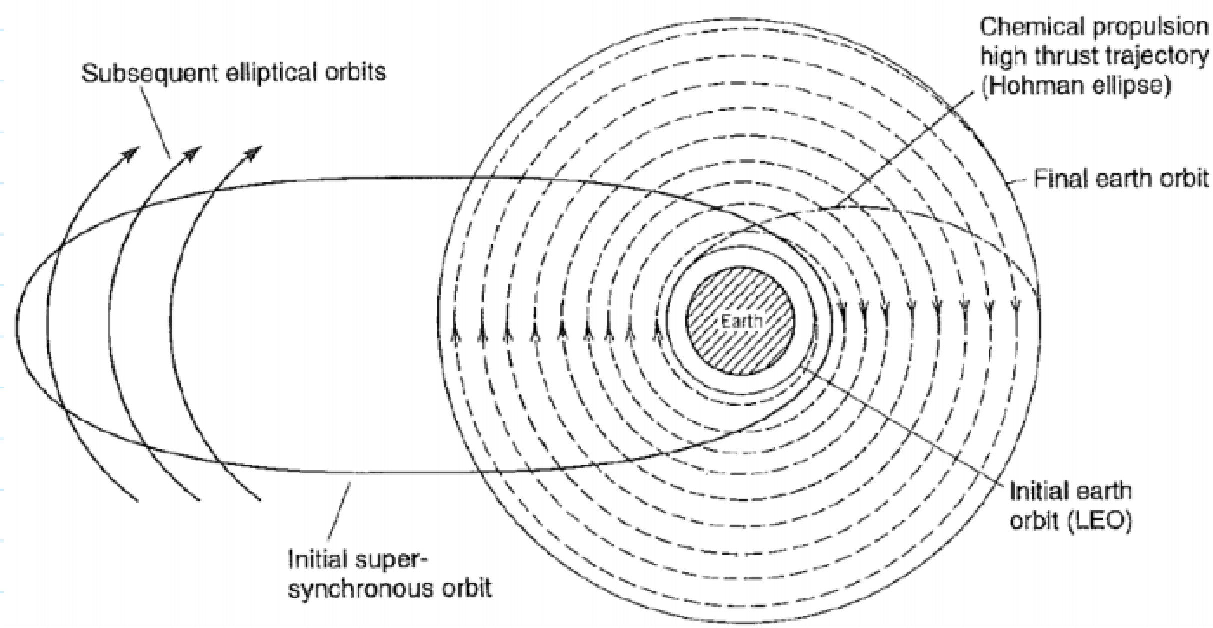
1. Select a trajectory
2. Use LHS of Equation 11 to find the mass required to do maneuver.

Electric Propulsion is <u>**NOT**</u> good for impulsive maneuvers (maybe PPT thrusters). Electric Propulsion is usually good at low thrust maneuvers.

1.6.2 Low Thrust Maneuvers

For zero drag and zero gravity there is no difference and we use Equation 12 for Low Thrust Maneuvers as well. **But** we must include gravity within our orbit, so there will be some difference.

The image below is an example of a low thrust trajectory. Here we have a series of quasi-circular spirals, burn all the way ($\gamma \neq 0$ in Equation 11) and definitely NOT impulsive. Now we have a gravity term (loss) in Equation 11, so we expect delta-V (mass requirement) will be higher than for an impulsive maneuver. We have to carry some of propellant to higher altitude, which takes energy which requires more propellant consumption.



Side-Note: Supersynchronous orbit: Spacecraft launched into very eccentric orbit with CP, supersynchronous elliptical orbit; from there electric propulsion can continuously and effectively be fired to attain a GEO orbit. Apogee > GEO, EP used to circularize and reduce Apogee to GEO

In 1961, Edelbaum solved this quasi-circular spiral trajectory problem, assuming quasi-circular spirals, apply a small inclination change per orbit.

Figure 4

He also assumed optimum change in thrust angle, as the spacecraft spirals out, the thrust angle changes optimally.

Edelbaum Equation

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos(\frac{\pi}{2}\theta)} \tag{13}$$

Where:

$$\begin{aligned} \theta &= \text{Inclination Angle (in degrees)} \\ V_{\#} &= \text{Circular Orbit Velocity} \end{aligned}$$

Before moving forward with the exercise, lets recall the definition and equations for a circular orbit velocity

Circular Orbit Velocity

$$v_{\text{circular}} = \sqrt{\frac{\mu}{r}} \tag{14}$$

Where:

$$\begin{aligned} \mu &= \text{Standard Gravitational Parameter} \\ &= G(M+m) \\ \mu_{\text{earth}} &= 3.986\text{x}10^{14}\frac{m^3}{s^2} \\ &= 3.986\text{x}10^{5}\frac{km^3}{s^2} \\ r &= \text{Radius of Orbit} \end{aligned}$$

and an escape velocity.

Escape Velocity

$$v_{\text{escape}} = \sqrt{2}v_{\text{circular}} \tag{15}$$

Figure 5

Exercise 4.
Find the escape delta-V from LEO. Assume $v_1 = 7800\frac{m}{s}$ orbital velocity

- Chemical Propulsion: $\Delta V_{\text{chemical}}$

$$v_{\text{escape}} = \sqrt{2}7800 = 11030\frac{m}{s}$$
$$\Delta V_{\text{chemical}} = v_{\text{escape}} - v_1 = 3230\frac{m}{s}$$

- Electrical Propulsion: $\Delta V_{\text{electric}}$

  We will be using the Edelbaum equation for the case of Electric Propulsion. One thing to consider, since we are moving from LEO to an escaped path, our final orbit radius is $r = \infty$.

Figure 6

As a result

$$v_{\text{circular}} = 0 \quad \text{when r} = 0$$
$$v_1 = 7800\frac{m}{s}$$
$$v_2 = 0\frac{m}{s}$$
$$\theta = 0^\circ\text{, no inclination change}$$

Edelbaum Equation:

$$\Delta V_{\text{electrical}} = \sqrt{7800^2 + 0^2 - 0} = 7800\frac{m}{s}$$

Note how $\Delta V_{\text{electric}}$ is higher! Why would we want to use it then? The reason is because it could have less propellant mass then chemical propulsion.

Consider the case where a chemical thruster and electric thruster use the same propellant mass. Then,

$$\frac{(I_{SP})_{\text{electric}}}{(I_{SP})_{\text{chemical}}} = \frac{\Delta V_{\text{electric}}}{\Delta V_{\text{chemical}}} = \frac{7800}{3230} = 2.4$$

Therefore, we have to have 2.4 times more I_{SP} for electric propulsion than chemical. If a typical chemical thruster has an $I_{SP} \approx 300$sec then an electrical thruster must have $I_{SP} \approx 750$sec. Luckily, electric thrusters tend to range around $I_{SP} \approx 1000+$sec meaning we most likely are saving on propellant mass.

1.6.3 Power Supply Penalty

The so-called "power supply penalty" places a premium on maximizing the specific power (in W/kg) of power supplies used for electric propulsion systems. Electric Propulsion has "massive" power generating system (PPU and Power Supply) and is low thrust, so its flight regime is much different than for chemical rockets.

The performance of the electric rocket can be analyzed in terms of the power and relevant masses.

Initial Mass Breakdown

$$m_o = m_p + m_{pl} + m_{pp} \tag{16}$$

Where:

$$\begin{aligned} m_o &= \text{Initial Mass} \\ m_p &= \text{Propellant Mass} \\ m_{pl} &= \text{Payload Mass} \\ m_{pp} &= \text{Power Plant Mass} \\ &= m_{\text{thruster}} + m_{\text{tank}} + m_{\text{feed system}} + m_{\text{PPU}} + m_{\text{power source}} \end{aligned}$$

Let α be the mass specific power of the power plant, that is:

Specific Power

$$\alpha = \frac{P}{m_{pp}} \quad \frac{\text{kW}}{\text{kg}} \tag{17}$$

Where:

$$\begin{aligned} P &= \text{Electrical Power Output of the Power Plant with mass } m_{pp} \\ m_{pp} &= \text{Power Plant Mass} \end{aligned}$$

Also (confusingly) $\alpha = \frac{m_{pp}}{P} \frac{\text{kg}}{\text{kW}}$ as seen in some cases like below. In class, we will most often use α as seen in Equation 17.

Typical α values:

- Very High Performance: $\alpha = 1\frac{\text{kg}}{\text{kw}}$
- High Performance: $\alpha = 10\frac{\text{kg}}{\text{kw}}$
- Moderate Performance: $\alpha = 30\frac{\text{kg}}{\text{kw}}$
- Low Performance: $\alpha = 100\frac{\text{kg}}{\text{kw}}$

For Solar Electric Propulsion:

- Solar Panel: $150\frac{\mathrm{W}}{\mathrm{kg}} \rightarrow \alpha_{\mathrm{electric}} = 6.7\frac{\mathrm{kg}}{\mathrm{kw}}$
- PPU: $\alpha_{\mathrm{PPU}} = 2.5\frac{\mathrm{kg}}{\mathrm{kw}}$
- Thruster, Mount, FeedSystem, Tank, etc: $\alpha_{\mathrm{t}} = 0.8\frac{\mathrm{kg}}{\mathrm{kw}}$
- Therefore... $\alpha_{\mathrm{overall}} = 10\frac{\mathrm{kg}}{\mathrm{kw}}$

Research is focused on achieving higher α values.

The efficiency η is

$$\boxed{\eta = \frac{\frac{1}{2}Tc}{P} = \frac{\frac{1}{2}\dot{m}c^2}{P}} \tag{18}$$

Where c is the effective exhaust velocity and $\dot{m}$ is the mass flow rate. Then

$$m_{pp} = \frac{m_p c^2}{2\alpha t_t \eta} \qquad \dot{m} = \frac{m_p}{t_t}$$

Where t_t is the burn or thrusting/propulsive time.

Using Equation 12, the "Gravity-Free Rocket Equation", (no drag and no gravity so ΔV is simply the change in the velocity of the craft), where m_f is the final mass

$$\frac{m_o}{m_f} = \exp(\frac{\Delta V}{c})$$

Where

$$m_o = m_p + m_f \qquad \text{and} \qquad m_f = m_{pl} + m_{pp}$$

We will now begin to derive the payload mass fraction.

Step 1: Invert $\frac{m_o}{m_f}$

$$\frac{m_o}{m_f} = \exp(\frac{\Delta V}{c}) \qquad \rightarrow \qquad \frac{m_f}{m_o} = \exp(\frac{-\Delta V}{c})$$

Step 2: Substitute the equations for m_f and m_o

$$m_o = m_p + m_f \qquad \rightarrow \qquad m_f = m_o - m_p$$

$$\frac{m_f}{m_o} = \frac{m_o - m_p}{m_o} = 1 - \frac{m_p}{m_o}$$

$$1 - \frac{m_p}{m_o} = \exp(\frac{-\Delta V}{c}) \qquad \rightarrow \qquad \frac{m_p}{m_o} = 1 - \exp(\frac{-\Delta V}{c})$$

Step 3: Do the same as in Step 2: this time for $\frac{m_{pl}}{m_o}$

$$\frac{m_f}{m_o} = \frac{m_{pl} + m_{pp}}{m_o} = \frac{m_{pl}}{m_o} + \frac{m_{pp}}{m_o}$$

$$\frac{m_{pl}}{m_o} + \frac{m_{pp}}{m_o} = \exp(\frac{-\Delta V}{c}) \qquad \rightarrow \qquad \frac{m_{pl}}{m_o} = \exp(\frac{-\Delta V}{c}) - \frac{m_{pp}}{m_o}$$

Step 4: Substitute m_{pp} after making the correct simplifications.

$$m_{pp} = \frac{m_p c^2}{2\alpha t_t \eta} \qquad \rightarrow \qquad \frac{m_{pp}}{m_p} = \frac{c^2}{2\alpha t_t \eta}$$

$$\frac{m_{pp}}{m_p}\frac{m_p}{m_o} = \frac{m_{pp}}{m_o} = \frac{c^2}{2\alpha t_t \eta}(1 - \exp(\frac{-\Delta V}{c}))$$

$\frac{m_{pp}}{m_p}$ and $\frac{m_p}{m_o}$ are known, therefore

$$\frac{m_{pl}}{m_o} = \exp\frac{-\Delta V}{c} - \frac{m_{pp}}{m_o}$$

The resulting equation is known as the payload fraction.

Payload Fraction

$$\frac{m_{pl}}{m_o} = \exp\frac{-\Delta V}{c} - \frac{c^2}{2\alpha t_t \eta}(1 - \exp(\frac{-\Delta V}{c}))$$

We can define a "characteristic speed" or "characteristic velocity" as Stuhlinger [1964] and Irving [1959] did.

Characteristic Speed

$$v_c = \sqrt{2\alpha\eta t_t} \tag{19}$$

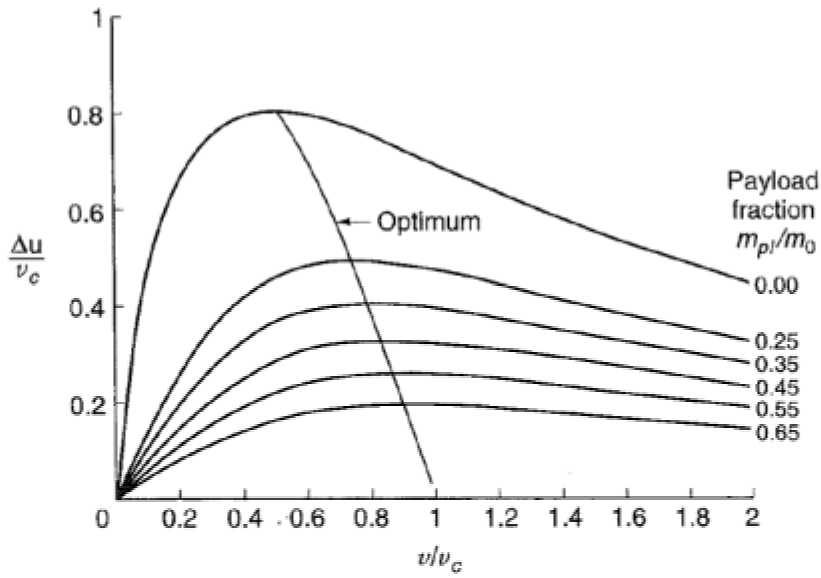
It has no physical meaning but represents the speed that the power plant mass m_{pp} would have if its full power output were converted to kinetic energy of its own mass. With the characteristic speed, we can take the inverse and rewrite our Payload Fraction as

Payload Fraction (Inverted; In terms of Characteristic Speed)

$$\frac{m_o}{m_{pl}} = \frac{\exp\frac{\Delta V}{c}}{1-(\exp(\frac{\Delta V}{c})-1)\frac{c^2}{v_c^2}} \tag{20}$$

Here we see the payload ration in terms of the power-plant properties! We typically want the $\frac{m_{pl}}{m_o}$ value to be high because we want to transport more cargo.

A plot of $\frac{m_{pl}}{m_o}$ for $\frac{c}{v_c}$ is shown below.



- For a given payload fraction there is an optimum value of c (I_{SP}) corresponding to the most delta-V.
- Peaks exists cause m_{pp} increases with I_{SP} (commonly called the "power supply penalty") but m_p decreases with I_{SP}.
- For a given mission (delta-V) there exists a theoretical optimum range of I_{SP} and thus optimum propulsion system.
- Peaks are for

$$\frac{\Delta V}{V_c} \leq 0.805 \qquad \text{and} \qquad 0.805 < \frac{c}{V_c} < 1.0$$

  - $\Delta v \propto v_c \rightarrow \Delta v^2 \propto t_t$: Optimum operating time is proportional to Δv^2. As a result t_t is long for large Δv.
  - $\Delta v \propto v_c$ with $c \propto v_c$, so $\Delta v \propto c$ and $c \propto I_{SP}$. Optimum $I_{SP} \propto \Delta v$, therefore large Δv requires large I_{SP}.

The peak of the curves can be found by differentiating Equation 20. The following equation gives the optimum c for a given delta-V and characteristic speed v_c.

Payload Fraction Peaks: Optimum Line Equation

$$\frac{c}{\Delta V}(\exp(\frac{\Delta V}{c}) - 1) - \frac{1}{2}(\frac{v_c}{c})^2 - \frac{1}{2} = 0 \tag{21}$$

Another way to look at this tradeoff between power supply mass and propellant mass is....

$$m_p = \dot{m} t_t \qquad \text{(Equation 2)}$$

$$T = \dot{m} c$$

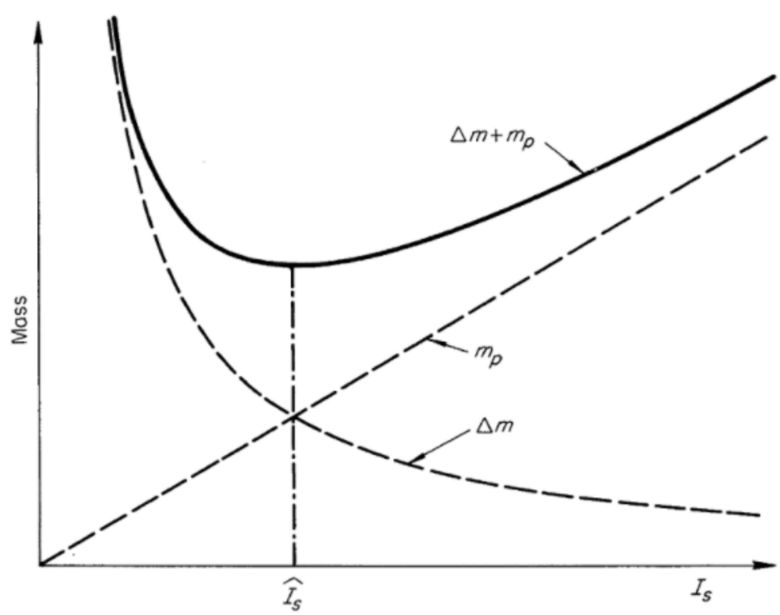
$$\rightarrow m_p = \frac{T t_t}{c} \qquad \text{Substitute with Equation 3}$$

$$\rightarrow \qquad m_p = \frac{T t_t}{g I_{SP}} \propto \frac{1}{I_{SP}} \tag{22}$$

Now we want to use the least amount of m_p so we want a large I_{SP}. BUT power supply penalty, using Equation 17, $\frac{\text{Equation 7}}{\text{Equation 18}}$ and Equation 3 we arrive at:

$$m_{pp} = \frac{P}{\alpha} = \frac{Tc}{2\eta\alpha} = \frac{Tg}{2\eta\alpha} I_{SP} \propto I_{SP} \tag{23}$$

If we have large I_{SP}, then we must have a large power plant to carry along. There must be an optimum! **Optimum = least propellant AND power supply mass.**



$$\frac{\partial(m_p + m_{pp})}{\partial(gI_{SP})} = 0 = \frac{-Tt_t}{(gI_{SP})^2} + \frac{T}{2\eta\alpha}$$

Which gives us

Optimum c for Minimum Mass

$$(gI_{SP})_{\text{optimum}} = \sqrt{2\eta\alpha t_t} = v_c = c \tag{24}$$

Equation 21 gives us the optimum c for a max delta-V because there is no linear relation between mass and delta-V (ln relation). Equation 24 gives us the optimum c for minimum mass.

1.6.4 Spacecraft Design Procedure

Given the payload mass, m_pl, and vehicle velocity increment, (delta-V), a spacecraft design procedure could proceed as:

1. Pick an arbitrary payload mass fraction $(\frac{m_{pl}}{m_o}) \quad \rightarrow \quad$ which would yield an optimum $\frac{\Delta v}{v_c}$ either from the figure or Equation 21
2. From the given Δv determine v_c from Equation 21
3. From optimum $\frac{c}{v_c}$ at given payload mass fraction (Equation 21), determine corresponding c and I_{SP}.
4. Select an engine that can deliver this optimum I_{SP} (and from the thruster's α and η) find the thrusting time t_t alongside Equation 19, with v_c
5. Calculate the propellant mass, m_p, using the rocket equation (Equation 12) and your picked payload ratio (your guess/pick in step 1)
6. Check that vehicle electrical power, volume, mass, and mission time are not exceeded. In other words, reconcile the results from Steps 1-5 with the other spacecraft design parameters (e.g., available power, mass, volume, mission time). Will it fit within the spacecraft and mission constraints? If...
   - **Yes**, Awesome!
   - **No**, go back to step 1with lower payload ratio.

As noted in step 1, there is no unique criterion for selecting payload mass fraction. We simply pick and hope the resulting mass, volume, power requirements can be met.

More sophisticated approaches use a "dual optimum" technique where they optimize by seeking shortest burn time with the highest payload mass fraction for the vehicle. How doe we optimize time and payload ration? By using transfer rates!

1.6.5 Transfer Rate

What about t_t? We said at beginning of Section I.6. that for a space mission design, we need to know Δv and t_t. Only then can we calculate $\frac{m_{pl}}{m_o}$.

Side Note:
For some missions, electric propulsion may get to GEO sooner and cost less, when one considers the entire spacecraft and launch vehicle timeline

Figure 6

But back to the spacecraft mission in space. How can we determine t_t? (How do we minimize the time required?)

One way to resolve t_t is to treat it as a variable. We did not do this previously. Previously we found an optimum c (or I_{SP}) to either

- minimize mass $m_p + m_{pp}$, given η , α , t_t. This was Equation 25.
- maximize payload ratio, given η , α , t_t. This was Equation 21.

Now, we treat t_t as a variable, and search for a "dual" optimum, we want to transfer the most payload in the shortest time. **Maximize m_{pl} while Minimizing t_t.**

$$m_{pl} = m_o - m_p - m_{pp} \qquad \text{Equation 16}$$

$$\frac{m_p}{m_o} = 1 - \exp[\frac{-\Delta V}{c}]$$

$$\frac{m_{pp}}{m_o} = \frac{Tc}{2\eta\alpha m_o} \qquad \text{Equation 23}$$

$$\boxed{\frac{m_{pl}}{m_o} = \exp[\frac{-\Delta V}{c}] - \frac{Tc}{2\eta\alpha m_o} \qquad (25)}$$

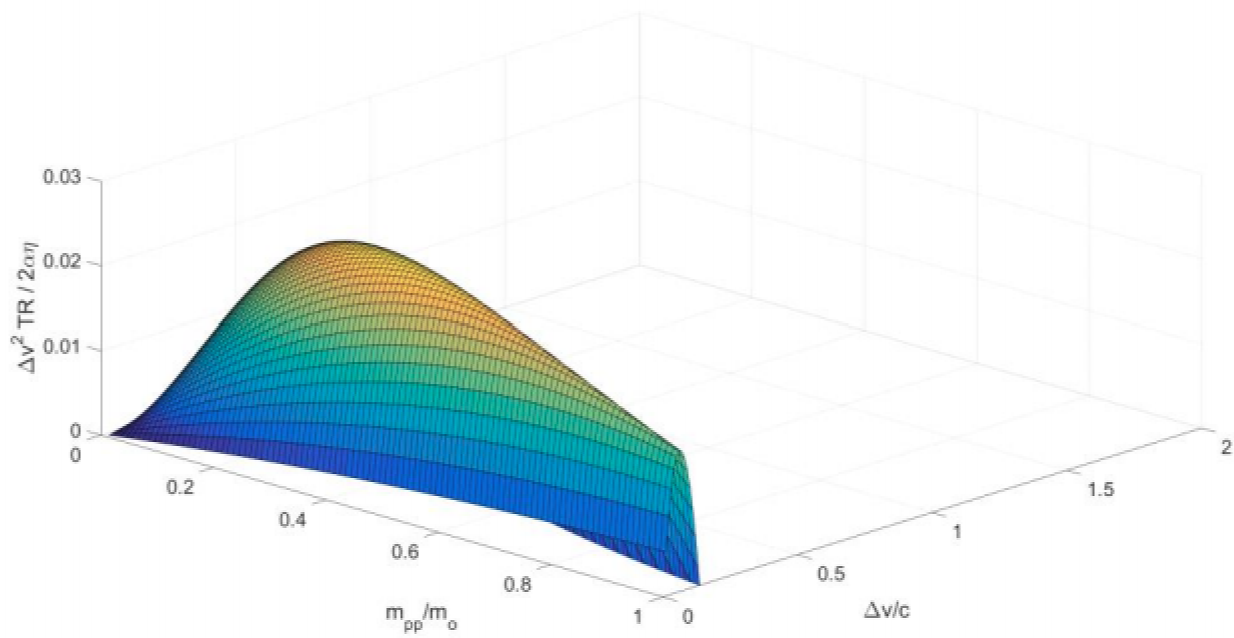
Transfer Rate

$$\text{TR} = \frac{\frac{m_{pl}}{m_o}}{t_t} \tag{26}$$

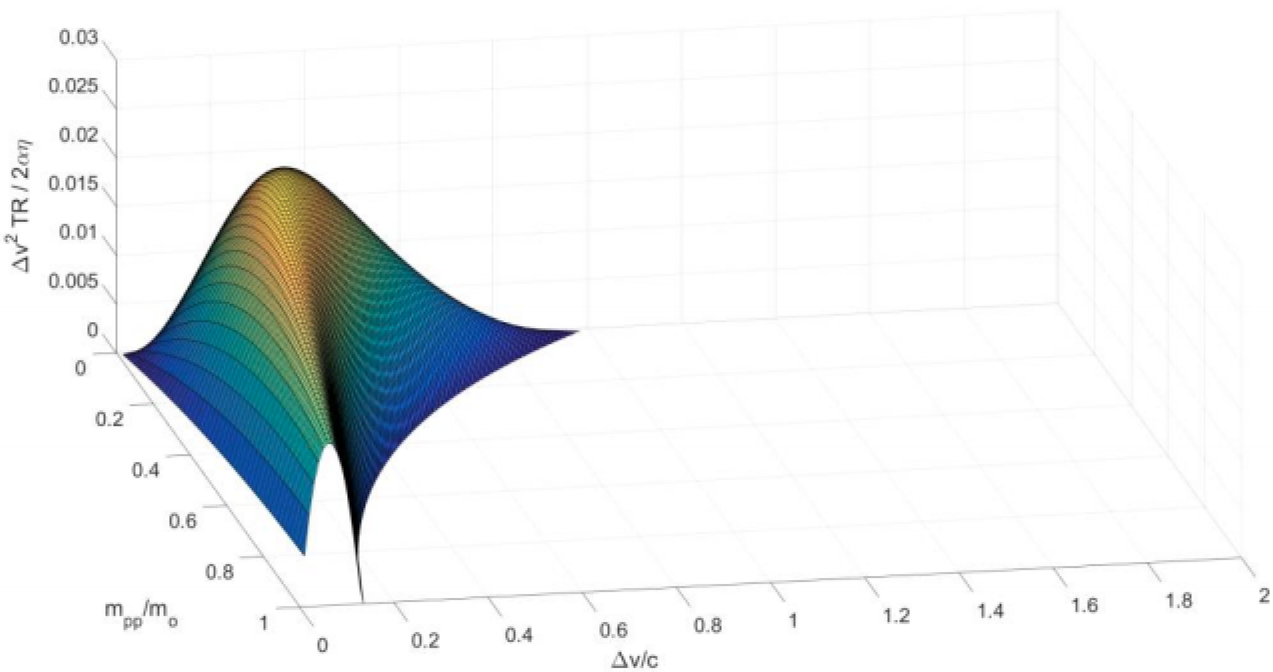
From Pukniel and Burton, Journals Spacecraft and Rockets, Vol 45, No1, 2008. For a LEO→ GEO → LEO returning SpaceTug. For fixed η, α, P, T:

$$\frac{(\Delta v)^2 \text{TR}}{2\alpha\eta} = (\frac{m_{pp}}{m_o})(\frac{\Delta v}{c})^2 \frac{\exp[\frac{-\Delta v}{c}] - (\frac{m_{pp}}{m_o})\exp[\frac{\Delta v}{c}]}{1 - \exp[\frac{-\Delta v}{c}] + (\frac{m_{pp}}{m_o})(\exp[\frac{\Delta v}{c}] - 1)} \tag{27}$$

A plot of Equation 27 shows...



a maximum.



Optimum is at..

$$\frac{m_{pp}}{m_o} = 18.5\% \qquad \frac{m_p}{m_o} = 45\% \qquad \frac{m_{pl}}{m_o} = 36.5\% \qquad \frac{\Delta v}{c} = 0.43 \qquad \frac{(\Delta v)^2 \text{TR}}{2\alpha\eta} = 0.0279$$

Summary: There are many ways to optimize an electric spacecraft

Optimization Gives: I_{SP}, $\frac{m_p}{m_o}$, $\frac{m_{pp}}{m_o}$, t_t

Now we know the parameters for which to select the "best" thruster for a particular mission. Moving forward we need to know about the different types of electric propulsion thrusters, but first we need to review Electromagnetics.