Application Problem 1:

Application : Compute $\int_0^3 x^2 dx$ using

- Rectangle rule with 1, 2 and 3 subintervals
- Trapezoidal rule with 1, 2 and 3 subintervals
- Simpson rule with 2 and 4 subintervals

Compare your result with the exact solution (i.e., compute the relative error) and comment on your solution.

- **Exact Solution**

$$y = \int_0^3 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^3 = 9$$

- **Rectangle Rule Solution**

Rectangle Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} h_i\, f(\frac{x_{i-1}+x_i}{2})$$

Where:

$$\begin{aligned} n &= \text{Number of Subintervals} \\ h_i &= (x_i - x_{i-1}) \\ h &= \frac{x_{\text{final}} - x_{\text{initial}}}{n} \\ &= \text{Length of a Single Subinterval} \end{aligned}$$

1 Subinterval:

$$h = \frac{3-0}{1} = 3 \quad \text{therefore} \quad y = 3\left(\frac{0+3}{2}\right)^2 = 6.75$$

2 Subintervals:

$$h = \frac{3-0}{2} = 1.5 \quad \text{therefore} \quad y = 1.5\left(\frac{0+1.5}{2}\right)^2 + 1.5\left(\frac{1.5+3}{2}\right)^2 = 8.4375$$

3 Subintervals:

$$h = \frac{3-0}{3} = 1 \quad \text{therefore} \quad y = 1\left(\frac{0+1}{2}\right)^2 + 1\left(\frac{1+2}{2}\right)^2 + 1\left(\frac{2+3}{2}\right)^2 = 8.75$$

- **Trapezoidal Rule Solution**

Trapezoidal Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} \frac{h_i}{2}\left(f(x_{i-1}) + f(x_i)\right)$$

Where:

$$\begin{aligned} n &= \text{Number of Intervals} \\ h_i &= (x_i - x_{i-1}) \\ h &= \frac{x_{\text{final}} - x_{\text{initial}}}{n} \\ &= \text{Length of a Single Subinterval} \end{aligned}$$

1 Subinterval:

$$h = \frac{3-0}{1} = 3 \quad \text{therefore} \quad y = \frac{3}{2}\big(0^2 + 3^2\big) = 13.5$$

2 Subintervals:

$$h = \frac{3-0}{2} = 1.5 \quad \text{therefore} \quad y = \frac{1.5}{2}\big(0^2 + 1.5^2\big) + \frac{1.5}{2}\big(1.5^2 + 3^2\big) = 10.125$$

3 Subintervals:

$$h = \frac{3-0}{3} = 1 \quad \text{therefore} \quad y = \frac{1}{2}\big(0^2 + 1^2\big) + frac12\big(1^2 + 2^2\big) + frac12\big(2^2 + 3^2\big) = 9.5$$

- **Simpson Rule Solution**

Simpson Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \frac{2}{3}\, I_{\text{rectangular}} + \frac{1}{3}\, I_{\text{trapezoidal}}$$

Where:

$$I_{\text{rectangular}} = \text{Rectangle Rule Estimate}$$
$$I_{\text{trapezoidal}} = \text{Trapezoidal Rule Estimate}$$

2 Subintervals:

$$h = \frac{3-0}{2} = 1.5$$

therefore

$$I_{\text{rectangular}} = 1.5\left(\frac{0+1.5}{2}\right)^2 + 1.5\left(\frac{1.5+3}{2}\right)^2 = 8.4375$$

$$I_{\text{trapezoidal}} = \frac{3-0}{2} = 1.5 = \frac{1.5}{2}\left(0^2 + 1.5^2\right) + \frac{1.5}{2}\left(1.5^2 + 3^2\right) = 10.125$$

As a result

$$y = \frac{2}{3}\, I_{\text{rectangular}} + \frac{1}{3}\, I_{\text{trapezoidal}}$$

$$= \frac{2}{3}\,(8.4375) + \frac{1}{3}\,(10.125)$$

$$= 9 \qquad \leftarrow \text{This is our exact solution!}$$

1 Problem 2

2 Basic Templates

Note 1. This is how you make numbered notes

Exercise 1. This is how you make numbered exercises

Definition 1. This is how you make numbered definitions

Rule 1. This is how you make numbered rules

Simpson Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \frac{2}{3} I_{\text{rectangular}} + \frac{1}{3} I_{\text{trapezoidal}}$$

Where:

$$I_{\text{Rectangular}} = \text{Rectangle Rule Estimate}$$
$$I_{\text{Trapezoidal}} = \text{Trapezoidal Rule Estimate}$$

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Compare your result with the exact solution (i.e., compute the relative error) and comment on your solution.