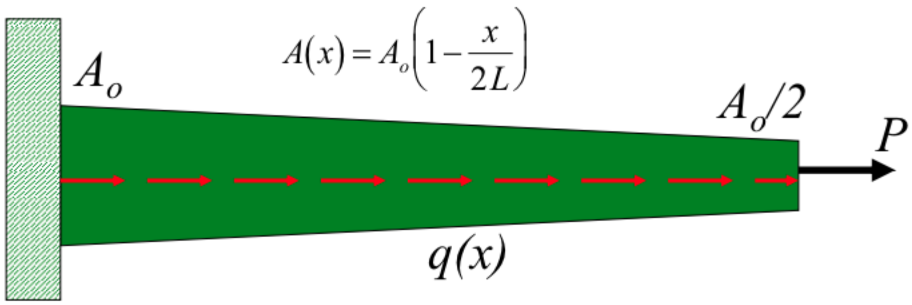
Application Problem 1: Beam Extension



$$\text{DEQ}: \quad \frac{d}{dx}\left(EA(x)\frac{du}{dx}\right) = -q(x) \quad \text{on } 0 \le x \le L$$

$$\text{BC}: \quad \begin{cases} u(0) = 0 \\ EAu'\big|_{x=L} = P \end{cases}$$

- **Discretize the DEQ with the second-order central finite difference scheme**
- **Write all the terms entering the resulting linear system, assuming n equally spaced grid points**
- **Use two approaches for the BC at x=L**
  - **Backward difference scheme**
  - **Ghost cell approach**

Discretize the DEQ with the second-order central finite difference scheme

Further computing the DEQ given provides us with:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(EA_o\left(1-\frac{x}{2L}\right)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = EA_o\left(1-\frac{x}{2L}\right)\frac{\partial^2 u}{\partial x^2} - EA_o\left(\frac{1}{2L}\right)\frac{\partial u}{\partial x}$$

The **Second Order Central Difference Scheme** tells us...

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

$$\left.\frac{\partial u}{\partial x}\right|_i = \frac{u_{i+1} - u_{i-1}}{2\,\Delta x} + O(\Delta x^2)$$

Substituting the discretized equations given by the second order central difference scheme, we get

$$EA_o\left(1-\frac{x}{2L}\right)\left[\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\right] - \frac{EA_o}{2L}\left[\frac{u_{i+1} - u_{i-1}}{2\,\Delta x}\right] = -q(x_i)$$

Expanding these terms, simplifying the equation and combining like terms, we get an equation in the form

$$a_i\,u_{i-1} + b_i\,u_i + c_i\,u_{i+1} = d_i$$

Where

$$a_i = EA_o\left(1-\frac{x_i}{2L}\right)\left(\frac{1}{\Delta x^2}\right) + \frac{EA_o}{4L\Delta x}$$

$$b_i = -EA_o\left(\frac{2}{\Delta x^2}\right)\left(1-\frac{x_i}{2L}\right)$$

$$c_i = \left(\frac{EA_o}{\Delta x^2}\right)\left(1-\frac{x_i}{2L}\right) - \left(\frac{EA_o}{4L\Delta x}\right)$$

$$d_i = -q(x_i)$$

The resulting equation is the final discretized form of the GDE.

Write all the terms entering the resulting linear system, assuming n equally spaced grid points

To begin we know our problem will have the following form...

$$\begin{bmatrix} n \times n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} n \times 1 \end{bmatrix}$$

We will start with the interior by applying the GDE

$$\begin{bmatrix} - & - & - & - & \cdots & - \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ - & \cdots & - & - & - & - \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} - \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ - \end{bmatrix}$$

Now lets put in the Boundary Conditions.

We will start with the displacement boundary conditions $\mathbf{u(0) = 0}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ - & \cdots & - & - & - & - \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} 0 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ - \end{bmatrix}$$

Implementing our final Boundary Condition is a bit more tricky....

Implement the boundary condition $\mathbf{EAu'|_{x=L} = P}$ using a Backward Difference Scheme

Lets consider our GDE and BC at $x = L$. At $x = L$, $i = n$ the final row in our matrix. The backward Difference scheme tells us...

Second Order Backward Difference Scheme

$$\left.\frac{\partial f}{\partial x}\right|_i = \frac{3\,f_i - 4\,f_{i-1} + f_{i-2}}{2\,\Delta x} + O(\Delta x^2)$$

which in our case results in

$$\left.\frac{\partial u}{\partial x}\right|_{i=n} = \frac{3\,u_n - 4\,u_{n-1} + u_{n-2}}{2\,\Delta x} = \frac{P}{EA}$$

The resulting equation is the boundary condition equation ready for implementation.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & \cdots & 0 & \frac{1}{2\Delta x} & -\frac{2}{\Delta x} & \frac{3}{2\Delta x} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} 0 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ \frac{P}{EA} \end{bmatrix}$$

Implement the boundary condition $\mathbf{EAu'|_{x=L} = P}$ using the Ghost Cell Approach

Lets take a moment to consider our GDE and BC at $x = L$. At $x = L$, $i = n$ the final row in our matrix. The Ghost Cell Approach tells us to...

Consider our Neumann Boundary Condition

$$\left.\frac{\partial u}{\partial x}\right|_n = \frac{P}{EA}$$

Recall the Second Order Central Difference Scheme:

$$\left.\frac{\partial f}{\partial x}\right|_i = \frac{u_{i+1} - u_{i-1}}{2\,\Delta x} + O(\Delta x)$$

Equating these and equating for the ghost cell (u_{n+1}) we get:

$$u_{n+1} = \frac{2\Delta x\, P}{EA} + u_{n-1}$$

Substitute the ghost term into the GDE (thereby eliminating the ghost cell (u_{n+1})).

$$a_n\, u_{n-1} + b_n\, u_n + c_n\, u_{n+1} = d_n \quad \rightarrow \quad a_i n\, u_{n-1} + b_n\, u_n + c_n \left(\frac{2\Delta x\, P}{EA} + u_{n-1}\right) = d_n$$

Which resolves to be...

$$u_{n-1}\,(a_n + c_n) + b_n\, u_n = d_n - c_n\,\frac{2\Delta x\, P}{EA}$$

The resulting equation is the boundary condition equation ready for implementation.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & \cdots & 0 & 0 & a_n + c_n & b_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} 0 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ d_n - c_n\,\frac{2\,\Delta x\, P}{EA} \end{bmatrix}$$