Section 4.5 Character of Second Order PDE: Slide 37 Sample Problem

For the vairous Aerospace related PDE's; What is $(B^2 - 4AC)$? What type of PDE is it?

Determining PDE Directions

$$a\,\frac{\partial^2 T}{\partial x^2} + b\,\frac{\partial^2 T}{\partial x\,\partial y} + c\,\frac{\partial^2 T}{\partial y^2} + d\,\frac{\partial T}{\partial x} + e\,\frac{\partial T}{\partial y} + g\,T + h = 0$$

Such that the slope (dx/dy) is controlled by the sign of $(b^2 - 4ac)$. In other words, If...

$(b^2 - 4ac) < 0 \rightarrow$ the slope is imaginary (all directions)

$\rightarrow$ **Elliptic PDE**

$(b^2 - 4ac) = 0 \rightarrow$ There is only one slope (information uniformly in one direction)

$\rightarrow$ **Parabolic PDE**

$(b^2 - 4ac) > 0 \rightarrow$ There are two slopes (information in two paths)

$\rightarrow$ **Hyperbolic PDE**

The Inviscid Stream-Function Equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \left| \quad \begin{array}{l} A = 1 \\ B = 0 \\ C = 1 \end{array} \right. \qquad (B^2 - 4AC) = -4$$

The PDE is Ellipitic

The Inviscid Linearized 2-D Compressible Flow Equation:

$$(M_o - 1)\,\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \left| \quad \begin{array}{l} A = 1 \\ B = 0 \\ C = 1 \end{array} \right. \qquad (B^2 - 4AC) > 0$$

The PDE is Hyperbolic

The Beam Torsion Equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2\,\mu\,\Theta \quad \left| \quad \begin{aligned} & A = (M_o - 1) \\ & B = 0 \\ & C = 1 \\ & H = 2\,\mu\,\Theta \end{aligned} \right. \qquad (B^2 - 4AC) - 4 \qquad \text{The PDE is Elliptic}$$

The Beam Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\,\frac{\partial^2 u}{\partial x^2} \quad \left| \quad \begin{aligned} & A = \frac{E}{\rho} \\ & B = 0 \\ & C = -1 \end{aligned} \right. \qquad (B^2 - 4AC) > 0 \qquad \text{The PDE is Hyperbolic}$$

The Thermal 1-D Heat Diffusion Equation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{\rho\,c}{k}\,\frac{\partial T}{\partial t} \quad \left| \quad \begin{aligned} & A = 1 \\ & B = 0 \\ & C = 0 \end{aligned} \right. \qquad (B^2 - 4AC) = 0 \qquad \text{The PDE is Parabolic}$$

The Laminar Boundary Layer Equation:

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = v\,\frac{\partial^2 u}{\partial y^2} \quad \left| \quad \begin{aligned} & A = 0 \\ & B = 0 \\ & C = v \end{aligned} \right. \qquad (B^2 - 4AC) = 0 \qquad \text{The PDE is Parabolic}$$

Sample Problem

How do we discretize a partial function of two variables?

$$\left.\frac{\partial f}{\partial x \partial y}\right|_{i,j}$$

Let's Expand this a bit

$$\left.\frac{\partial f}{\partial x \partial y}\right|_{i,j} = \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}\right) \tag{1}$$

What is the discretized form of $\frac{\partial f}{\partial y}$? Well...

$$\left.\frac{\partial f}{\partial y}\right|_{i,j} = \frac{f_{i,j+1} - f_{i,j-1}}{2\,\Delta y} \tag{2}$$

Substituting this Equation 2 into Equation 1, we get....

$$\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x}\left(\frac{f_{i,j+1} - f_{i,j-1}}{2\,\Delta y}\right)$$

$$= \frac{\left.\frac{\partial f}{\partial y}\right|_{i+1,j} - \left.\frac{\partial f}{\partial y}\right|_{i-1,j}}{2\,\Delta y}$$

...

$$= \frac{f_{i+1,j+1} - f_{i+1,j-1} - f_{i-1,j+1} + f_{i-1,j-1}}{4\,\Delta x\,\Delta y}$$