Application Problem 1a:

Compute $\int_0^3 x^2\,\mathrm{d}x$ using the Rectangle Rule with 1, 2, and 3 subintervals Compare your result with the exact solution. Compute the relative error, comment on your solution.

Exact Solution

$$y = \int_0^3 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^3 = 9$$

Rectangle Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} h\, f(\frac{x_{i-1}+x_i}{2})$$

Where:

$$n = \text{Number of Subintervals}$$

$$h = \frac{x_{\text{final}} - x_{\text{initial}}}{n} \qquad \text{Length of a Single Subinterval}$$

1 Subinterval :

$$h = \frac{3-0}{1} = 3 \qquad \text{therefore} \qquad y = 3\left(\frac{0+3}{2}\right)^2 = 6.75$$

2 Subintervals :

$$h = \frac{3-0}{2} = 1.5 \quad \text{therefore} \qquad y = 1.5\left(\frac{0+1.5}{2}\right)^2 + 1.5\left(\frac{1.5+3}{2}\right)^2 = 8.4375$$

3 Subintervals :

$$h = \frac{3-0}{3} = 1 \qquad \text{therefore} \qquad y = 1\left(\frac{0+1}{2}\right)^2 + 1\left(\frac{1+2}{2}\right)^2 + 1\left(\frac{2+3}{2}\right)^2 = 8.75$$

Application Problem 1b:

Compute $\int_0^3 x^2\,\mathrm{d}x$ using the Trapezoidal Rule with 1, 2, and 3 subintervals Compare your result with the exact solution. Compute the relative error, comment on your solution.

Exact Solution

$$y = \int_0^3 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^3 = 9$$

Trapezoidal Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} \frac{h}{2}\Big(f(x_{i-1}) + f(x_i)\Big)$$

Where:

$$n = \text{Number of Intervals}$$

$$h = \frac{x_{\text{final}} - x_{\text{initial}}}{n} \qquad \text{Length of a Single Subinterval}$$

1 Subinterval:

$$h = \frac{3-0}{1} = 3 \qquad \text{therefore} \qquad y = \frac{3}{2}\big(0^2 + 3^2\big) = 13.5$$

2 Subintervals:

$$h = \frac{3-0}{2} = 1.5 \quad \text{therefore} \qquad y = \frac{1.5}{2}\big(0^2 + 1.5^2\big) + \frac{1.5}{2}\big(1.5^2 + 3^2\big) = 10.125$$

3 Subintervals:

$$h = \frac{3-0}{3} = 1 \qquad \text{therefore} \qquad y = \frac{1}{2}\big(0^2 + 1^2\big) + \frac{1}{2}\big(1^2 + 2^2\big) + \frac{1}{2}\big(2^2 + 3^2\big) = 9.5$$

Application Problem 1c:

Compute $\int_0^3 x^2\,\mathrm{d}x$ using the Simpson Rule with 2 and 4 subintervals Compare your result with the exact solution. Compute the relative error, comment on your solution.

Exact Solution

$$y = \int_0^3 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^3 = 9$$

Simpson Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \frac{2}{3}\,I_{\text{rectangular}} + \frac{1}{3}\,I_{\text{trapezoidal}}$$

Where:

$$I_{\text{rectangular}} = \text{Rectangle Rule Estimate}$$

$$I_{\text{trapezoidal}} = \text{Trapezoidal Rule Estimate}$$

2 Subintervals:

$$h = \frac{3-0}{2} = 1.5 \qquad \text{therefore...}$$

$$I_{\text{rectangular}} = 1.5\left(\frac{0+1.5}{2}\right)^2 + 1.5\left(\frac{1.5+3}{2}\right)^2 = 8.4375$$

$$I_{\text{trapezoidal}} = \frac{3-0}{2} = 1.5 = \frac{1.5}{2}\left(0^2+1.5^2\right) + \frac{1.5}{2}\left(1.5^2+3^2\right) = 10.125$$

As a result

$$y = \frac{2}{3}\,I_{\text{rectangular}} + \frac{1}{3}\,I_{\text{trapezoidal}} = \frac{2}{3}\,(8.4375) + \frac{1}{3}\,(10.125) = 9$$

This is our exact solution!

Application Problem 2a:

Compute $\int_0^3 x^2 \mathrm{d}x$ using the 1-point, 2-point Gauss Quadrature Rule. Compare your result with the exact solution (ie. compute the relative error) and comment on your solution

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- **Exact Solution**

$$y = \int_0^3 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^3$$

- **1 Point Gauss Quadrature Rule**

Such that $w_1 = 2$ and $\xi_1 = 0$:

$$I = \int_0^3 x^2 \mathrm{d}x = \frac{3}{2}\int_{-1}^{1} f\left(\frac{3\xi + 3}{2}\right)\mathrm{d}\xi$$

$$= \frac{3}{2}\int_{-1}^{1} \left(\frac{3\xi + 3}{2}\right)^2 \mathrm{d}\xi$$

$$= 2\left(\frac{3\,(0) + 3}{2}\right)^2 = \frac{18}{4} = 4.5$$

- **2 Point Gauss Quadrature Rule**

Such that $w_1 = w_2 = 1$, $\xi_1 = -\frac{\sqrt{3}}{3}$ and $\xi_2 = \frac{\sqrt{3}}{3}$:

$$I = w_1\, f(\xi_1) + w_2\, f(\xi_2)$$

$$= \frac{3}{2}\left(\frac{3\,(-\frac{\sqrt{3}}{3}) + 3}{2}\right)^2 + \frac{3}{2}\left(\frac{3\,(\frac{\sqrt{3}}{3}) + 3}{2}\right)^2$$

$$= 9$$

Application Problem 2b:

Compute $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\mathrm{d}x$ using the 1-point, 2-point Gauss Quadrature Rule. Compare your result with the exact solution (ie. compute the relative error) and comment on your solution

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- **Exact Solution**

$$y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\mathrm{d}x = \Big[\sin(x)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2$$

First we must do a Coordinate Transformation to get this in the form we want.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\mathrm{d}x = \frac{\pi}{2}\int_{-1}^{1} \cos\left(\frac{\pi}{2}\,\xi\right)\mathrm{d}\xi$$

- **1 Point Gauss Quadrature Rule** Such that $w_1 = 2$ and $\xi_1 = 0$:

$$I = \frac{\pi}{2}\Big[\cos(\frac{\pi}{2}\,\xi_1)\,w_1\Big]$$

$$= \frac{\pi}{2}\Big[\cos\big(\frac{\pi}{2}\,(0)\big)\,2\Big]$$

$$= 3.14$$

- **2 Point Gauss Quadrature Rule**

  Such that $w_1 = w_2 = 1$, $\xi_1 = -\dfrac{\sqrt{3}}{3}$ and $\xi_2 = \dfrac{\sqrt{3}}{3}$:

$$I = \frac{\pi}{2}\left[w_1\ \cos(\frac{\pi}{2}\,\xi_1) + w_2\ \cos(\frac{\pi}{2}\,\xi_2)\right]$$

$$= \frac{\pi}{2}\left[1\ \cos\left(\frac{\pi}{2}\,(-\frac{\sqrt{3}}{3})\right) + 1\ \cos\left(\frac{\pi}{2}\,(\frac{\sqrt{3}}{3})\right)\right]$$

$$= 1.9352 \quad \approx 3\%\ \text{off}$$

The solution alternates between over estimate and under estimate.

Application Problem 3:

Compute $\int_{-1}^{1}\int_{-1}^{1} \exp(2x) * \ln(3+y)\mathrm{d}y\mathrm{d}x$ using the $1*1,\ 2*2,\ 3*3$ Gauss Quadrature rule. Compare your result with the exact solution ($I_{ex} = 7.829967$). Compute the relative error and comment on your solution.

- **Exact Solution**

$$I_{\text{exact}} = \int_{-1}^{1}\int_{-1}^{1} e^{2\,x} \cdot \ln(3+y)\,\mathrm{d}y\,\mathrm{d}x = 7.829967$$

- **1*1 Point Gauss Quadrature Rule** Such that $w = w_1 \times w_1 = 4$ and $\xi_1 = \eta_1 = 0$:

$$I = w\,f(\xi_1, \eta_1)$$

$$= 4\,e^{0} \cdot \ln(3) = 4.39449$$

- **2*2 Point Gauss Quadrature Rule**

Such that $w_k = 1$ and $(\xi_k, \eta_k) = \big(\pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3}\big)$:

$$I = \sum w_k\,f(\xi_k, \eta_k)$$

$$= 1\,\exp\left[2\left(-\frac{\sqrt{3}}{3}\right)\right]\,\ln\left(3+\left(-\frac{\sqrt{3}}{3}\right)\right) \quad + \quad 1\,\exp\left[2\left(-\frac{\sqrt{3}}{3}\right)\right]\,\ln\left(3+\left(\frac{\sqrt{3}}{3}\right)\right)$$

$$+\,1\,\exp\left[2\left(\frac{\sqrt{3}}{3}\right)\right]\,\ln\left(3+\left(\frac{\sqrt{3}}{3}\right)\right) \quad + \quad 1\,\exp\left[2\left(\frac{\sqrt{3}}{3}\right)\right]\,\ln\left(3+\left(-\frac{\sqrt{3}}{3}\right)\right)$$

$$= 7.532767$$

Application Problem 4:

Derive the second-order central difference approximations of the first and second derivatives

Application Problem 5:

Let $f(x) = \sin(x)$. Compute $f'(1)$ using the Forward Difference Scheme with $h = 0.25$ and $h = 0.5$. Then Improve your solution by using the Richardson's Extrapolation Scheme. Compare your three approximations with the exact solution.

Forward Difference Scheme

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(\xi) \qquad \text{with } \xi \in (x, x+h)$$

- Exact

$$f'(1)_{\text{exact}} = 0.5403$$

- h=0.25

$$f'(x) = \frac{\sin(x + 0.25) - \sin(x)}{0.25} \quad \rightarrow \quad f'(x) = 0.43055$$

- h=0.50

$$f'(x) = \frac{\sin(x + 0.5) - \sin(x)}{0.5} \quad \rightarrow \quad f'(x) = 0.312048$$

Richardson Extrapolation

$$a_o = F(h) + \frac{F(h) - F(qh)}{q^p - 1} + O(h^r) \qquad \text{or} \qquad a_o = \frac{F(qh) - q^p\, F(h)}{1 - q^p} + O(h^2)$$

For this problem

$$h = 0.25 \qquad f'(1) = 0.430551 = F(h)$$

$$h = 0.50 \qquad f'(1) = 0.312048 = F(qh)$$

Where $q = 2$ and $p = 1$ since we are using the Forward Difference Scheme and error (h) is linear.

$$a_o = \frac{F(qh) - q\, F(h)}{1 - q} = \frac{0.312048 - 2\,(0.430551)}{1 - 2} = 0.54806$$

Application Problem 6:

Let $f(x) = \exp(1 + 3\,x)$. Compute $f'(2)$ using the Central Difference Scheme with $h = 0.04$ and $h = 0.08$. Then Improve your solution by using the Richardson's Extrapolation scheme. Compare your three approximations with the exact solution.

Central Difference Scheme

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(\xi) \qquad \text{with } \xi \in (x, x+h)$$

- Exact

$$f'(2)_{\text{exact}} = 3289.899$$

- h=0.04

$$f'(2) = F(h) = \frac{\exp(1 + 3\,(2 + 0.04)) - \exp(1 + 3\,(2 - 0.04))}{2\,(0.04)} = 3297.801$$

- h=0.08

$$f'(2) = F(qh) = \frac{\exp(1 + 3\,(2 + 0.08)) - \exp(1 + 3\,(2 - 0.08))}{2\,(0.08)} = 3321.574$$

Richardson Extrapolation

$$a_o = F(h) + \frac{F(h) - F(qh)}{q^p - 1} + O(h^r) \qquad \text{or} \qquad a_o = \frac{F(qh) - q^p\,F(h)}{1 - q^p} + O(h^2)$$

For this problem

$$h = 0.04 \qquad f'(2) = 3297.801 = F(h)$$

$$h = 0.08 \qquad f'(2) = 3321.574 = F(qh)$$

Where $q = 2$ and $p = 2$ since we are using the Central Difference Scheme and error (h) is squared

$$a_o = \frac{F(qh) - q^2\,F(h)}{1 - q^2} = \frac{3321.574 - 4\,(3297.801)}{1 - 4} = 3289.877$$

Application Problem 7:

Transform the mass/spring/dashpot equation

$$\mathbf{m\,x'' = m\,g - k\,x - c\,x'}$$

into a system of 1^{st} **-order ODE.**

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = mg - kx - c\frac{\mathrm{d}x}{\mathrm{d}t}$$

Divide by m in order to isolate the second order term.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = g - \frac{k}{m}\,x - \frac{c}{m}\frac{\mathrm{d}x}{\mathrm{d}t}$$

Lets say $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ Therefore...

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g - \frac{k}{m}\,x - \frac{c}{m}\,y$$

Where

$$y = \frac{\mathrm{d}x}{\mathrm{d}t}$$

We have transformed the second order ODE into two coupled first order ODE.

Application Problem 8:

Use Euler Scheme to solve

$$\mathbf{y' = y \qquad on \qquad 0 \leq t \leq 4 \qquad with \qquad y(0) = 1}$$

Use h=1 and compute the relative error at t=4