Determining PDE Directions

$$a\,\frac{\partial^2 T}{\partial x^2} + b\,\frac{\partial^2 T}{\partial x\,\partial y} + c\,\frac{\partial^2 T}{\partial y^2} + d\,\frac{\partial T}{\partial x} + e\,\frac{\partial T}{\partial y} + g\,T + h = 0$$

Such that the slope (dx/dy) is controlled by the sign of $(b^2 - 4ac)$. In other words, If...

$(b^2 - 4ac) < 0 \rightarrow$ the slope is imaginary (all directions)

$\rightarrow$ **Elliptic PDE**

$(b^2 - 4ac) = 0 \rightarrow$ There is only one slope (information uniformly in one direction)

$\rightarrow$ **Parabolic PDE**

$(b^2 - 4ac) > 0 \rightarrow$ There are two slopes (information in two paths)

$\rightarrow$ **Hyperbolic PDE**

First Order Forward Difference Scheme

$$\left.\frac{\partial f}{\partial x}\right|_i = \frac{f_{+1} - f_i}{\Delta x} + O(\Delta x)$$

First order because of the error. Forward Difference because of the index.

First Order Backward Difference Scheme

$$\left.\frac{\partial f}{\partial x}\right|_i = \frac{f_i - f_{i-1}}{\Delta x} + O(\Delta x)$$

Second Order Backward Difference Scheme

$$\left.\frac{\partial f}{\partial x}\right|_i = \frac{3\,f_i - 4\,f_{i-1} + f_{i-2}}{2\,\Delta x} + O(\Delta x^2)$$

Second Order Central Difference Scheme for the First Derivative

$$\left.\frac{\partial f}{\partial x}\right|_i = \frac{f_{i+1} - f_{i-1}}{2\,\Delta x} + O(\Delta x^2)$$

Second Order Central Difference Scheme for the Second Derivative

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_i = \frac{f_{i+1} - 2\,f_i + f_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

Types of Boundary Conditions

Dirichlet (fixed): $f = C_1$

Neumann (derivative): $\frac{\partial f}{\partial x} = C_2$

Cauchy (mixed): $f + C_1\,\frac{\partial f}{\partial x} = C_2$

Derivative Boundary Conditions with Ghost Cell Approach

Recall Neumann BC:

$$\frac{\partial f}{\partial x} = C_1$$

Recall Second Order Central Difference:

$$\left.\frac{\partial f}{\partial x}\right|_i = \frac{f_{i+1} - f_{i-1}}{2\,\Delta x} + O(\Delta x)$$

Equating these and equating for the ghost cell (f_{n+1}) we get:

$$[f_{i+1} - f_{i-1}] = 2\,C_1\,\Delta x \quad \rightarrow \quad f_{i+1} = 2\,C_1\,\Delta x + f_{i-1}$$

Substitute the ghost term into the GDE (thereby eliminating the ghost cell (f_{n+1})). The resulting equation is the boundary condition equation ready for implementation.