Application Problem 1a:

Compute $\int_0^3 x^2\,\mathrm{d}x$ using the <u>Rectangle Rule with 1, 2, and 3 subintervals</u> Compare your result with the exact solution. Compute the relative error, comment on your solution.

Exact Solution

$$y = \int_0^3 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^3 = 9$$

Rectangle Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} h\, f(\frac{x_{i-1}+x_i}{2})$$

Where:

$$n = \text{Number of Subintervals}$$

$$h = \frac{x_{\text{final}} - x_{\text{initial}}}{n} \qquad \text{Length of a Single Subinterval}$$

1 Subinterval :

$$h = \frac{3-0}{1} = 3 \qquad \text{therefore} \qquad y = 3\left(\frac{0+3}{2}\right)^2 = 6.75$$

2 Subintervals :

$$h = \frac{3-0}{2} = 1.5 \quad \text{therefore} \qquad y = 1.5\left(\frac{0+1.5}{2}\right)^2 + 1.5\left(\frac{1.5+3}{2}\right)^2 = 8.4375$$

3 Subintervals :

$$h = \frac{3-0}{3} = 1 \qquad \text{therefore} \qquad y = 1\left(\frac{0+1}{2}\right)^2 + 1\left(\frac{1+2}{2}\right)^2 + 1\left(\frac{2+3}{2}\right)^2 = 8.75$$

Application Problem 1b:

Compute $\int_0^3 x^2\,\mathrm{d}x$ using the <u>Trapezoidal Rule with 1, 2, and 3 subintervals</u> Compare your result with the exact solution. Compute the relative error, comment on your solution.

Exact Solution

$$y = \int_0^3 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^3 = 9$$

Trapezoidal Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} \frac{h}{2}\Big(f(x_{i-1}) + f(x_i)\Big)$$

Where:

$$n = \text{Number of Intervals}$$

$$h = \frac{x_{\text{final}} - x_{\text{initial}}}{n} \qquad \text{Length of a Single Subinterval}$$

1 Subinterval:

$$h = \frac{3-0}{1} = 3 \qquad \text{therefore} \qquad y = \frac{3}{2}\big(0^2 + 3^2\big) = 13.5$$

2 Subintervals:

$$h = \frac{3-0}{2} = 1.5 \quad \text{therefore} \qquad y = \frac{1.5}{2}\big(0^2 + 1.5^2\big) + \frac{1.5}{2}\big(1.5^2 + 3^2\big) = 10.125$$

3 Subintervals:

$$h = \frac{3-0}{3} = 1 \qquad \text{therefore} \qquad y = \frac{1}{2}\big(0^2 + 1^2\big) + \frac{1}{2}\big(1^2 + 2^2\big) + \frac{1}{2}\big(2^2 + 3^2\big) = 9.5$$

Application Problem 1c:

Compute $\int_0^3 x^2\,\mathrm{d}x$ using the Simpson Rule with 2 and 4 subintervals Compare your result with the exact solution. Compute the relative error, comment on your solution.

Exact Solution

$$y = \int_0^3 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^3 = 9$$

Simpson Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \frac{2}{3}\,I_{\text{rectangular}} + \frac{1}{3}\,I_{\text{trapezoidal}}$$

Where:

$$I_{\text{rectangular}} = \text{Rectangle Rule Estimate}$$

$$I_{\text{trapezoidal}} = \text{Trapezoidal Rule Estimate}$$

2 Subintervals:

$$h = \frac{3-0}{2} = 1.5 \qquad \text{therefore...}$$

$$I_{\text{rectangular}} = 1.5\left(\frac{0+1.5}{2}\right)^2 + 1.5\left(\frac{1.5+3}{2}\right)^2 = 8.4375$$

$$I_{\text{trapezoidal}} = \frac{3-0}{2} = 1.5 = \frac{1.5}{2}\left(0^2+1.5^2\right) + \frac{1.5}{2}\left(1.5^2+3^2\right) = 10.125$$

As a result

$$y = \frac{2}{3}\,I_{\text{rectangular}} + \frac{1}{3}\,I_{\text{trapezoidal}} = \frac{2}{3}\,(8.4375) + \frac{1}{3}\,(10.125) = 9$$

This is our exact solution!

Application Problem 2a:

Compute $\int_0^3 x^2 \mathrm{d}x$ using the 1-point, 2-point Gauss Quadrature Rule. Compare your result with the exact solution (ie. compute the relative error) and comment on your solution

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- **Exact Solution**

$$y = \int_0^3 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^3$$

- **1 Point Gauss Quadrature Rule**

  Such that $w_1 = 2$ and $\xi_1 = 0$:

$$I = \int_0^3 x^2 \mathrm{d}x = \frac{3}{2}\int_{-1}^{1} f\left(\frac{3\xi + 3}{2}\right)\mathrm{d}\xi$$

$$= \frac{3}{2}\int_{-1}^{1} \left(\frac{3\xi + 3}{2}\right)^2 \mathrm{d}\xi$$

$$= 2\left(\frac{3\,(0) + 3}{2}\right)^2 = \frac{18}{4} = 4.5$$

- **2 Point Gauss Quadrature Rule**

  Such that $w_1 = w_2 = 1$, $\xi_1 = -\frac{\sqrt{3}}{3}$ and $\xi_2 = \frac{\sqrt{3}}{3}$:

$$I = w_1\, f(\xi_1) + w_2\, f(\xi_2)$$

$$= \frac{3}{2}\left(\frac{3\,(-\frac{\sqrt{3}}{3}) + 3}{2}\right)^2 + \frac{3}{2}\left(\frac{3\,(\frac{\sqrt{3}}{3}) + 3}{2}\right)^2$$

$$= 9$$

Application Problem 2b:

Compute $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\mathrm{d}x$ using the 1-point, 2-point Gauss Quadrature Rule. Compare your result with the exact solution (ie. compute the relative error) and comment on your solution

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- **Exact Solution**

$$y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\mathrm{d}x = \Big[\sin(x)\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2$$

First we must do a Coordinate Transformation to get this in the form we want.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\mathrm{d}x = \frac{\pi}{2}\int_{-1}^{1} \cos\left(\frac{\pi}{2}\,\xi\right)\mathrm{d}\xi$$

- **1 Point Gauss Quadrature Rule** Such that $w_1 = 2$ and $\xi_1 = 0$:

$$I = \frac{\pi}{2}\Big[\cos(\frac{\pi}{2}\,\xi_1)\,w_1\Big]$$

$$= \frac{\pi}{2}\Big[\cos\big(\frac{\pi}{2}\,(0)\big)\,2\Big]$$

$$= 3.14$$

- **2 Point Gauss Quadrature Rule**

  Such that $w_1 = w_2 = 1$, $\xi_1 = -\dfrac{\sqrt{3}}{3}$ and $\xi_2 = \dfrac{\sqrt{3}}{3}$:

$$I = \frac{\pi}{2}\left[w_1 \cos(\frac{\pi}{2}\,\xi_1) + w_2 \cos(\frac{\pi}{2}\,\xi_2)\right]$$

$$= \frac{\pi}{2}\left[1 \cos\left(\frac{\pi}{2}\,(-\frac{\sqrt{3}}{3})\right) + 1 \cos\left(\frac{\pi}{2}\,(\frac{\sqrt{3}}{3})\right)\right]$$

$$= 1.9352 \quad \approx 3\% \text{ off}$$

The solution alternates between over estimate and under estimate.

Application Problem 3:

Compute $\int_{-1}^{1}\int_{-1}^{1} \exp(2x) * \ln(3+y) \mathrm{d}y\mathrm{d}x$ **using the** $1*1$**,** $2*2$**,** $3*3$ **Gauss Quadrature rule. Compare your result with the exact solution** $(I_{ex} = 7.829967)$**. Compute the relative error and comment on your solution.**

- **Exact Solution**

$$I_{\text{exact}} = \int_{-1}^{1}\int_{-1}^{1} e^{2\,x} \cdot \ln(3+y)\,\mathrm{d}y\,\mathrm{d}x = 7.829967$$

- **1*1 Point Gauss Quadrature Rule** Such that $w = w_1 \times w_1 = 4$ and $\xi_1 = \eta_1 = 0$:

$$I = w\, f(\xi_1, \eta_1)$$

$$= 4\, e^0 \cdot \ln(3) = 4.39449$$

- **2*2 Point Gauss Quadrature Rule**

Such that $w_k = 1$ and $(\xi_k, \eta_k) = \big(\pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3}\big)$:

$$I = \sum w_k\, f(\xi_k, \eta_k)$$

$$= 1\ \exp\left[2\left(-\frac{\sqrt{3}}{3}\right)\right] \ln\left(3 + \left(-\frac{\sqrt{3}}{3}\right)\right) \quad + \quad 1\ \exp\left[2\left(-\frac{\sqrt{3}}{3}\right)\right] \ln\left(3 + \left(\frac{\sqrt{3}}{3}\right)\right)$$

$$+ 1\ \exp\left[2\left(\frac{\sqrt{3}}{3}\right)\right] \ln\left(3 + \left(\frac{\sqrt{3}}{3}\right)\right) \quad + \quad 1\ \exp\left[2\left(\frac{\sqrt{3}}{3}\right)\right] \ln\left(3 + \left(-\frac{\sqrt{3}}{3}\right)\right)$$

$$= 7.532767$$

Application Problem 4:

Derive the second-order central difference approximations of the first and second derivatives

Application Problem 5:

Let $f(x) = \sin(x)$. Compute $f'(1)$ using the Forward Difference Scheme with $h = 0.25$ and $h = 0.5$. Then Improve your solution by using the Richardson's Extrapolation Scheme. Compare your three approximations with the exact solution.

Application Problem 6:

Let $f(x) = \exp(1 + 3\,x)$. Compute $f'(2)$ using the Central Difference Scheme with $h = 0.04$ and $h = 0.08$. Then Improve your solution by using the Richardson's Extrapolation scheme. Compare your three approximations with the exact solution.