Rectangle Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} h\, f\left(\frac{x_{i-1}+x_i}{2}\right)$$

Where:

$$\begin{aligned} n &= \text{Number of Subintervals} \\ h &= \frac{x_{\text{final}} - x_{\text{initial}}}{n} \\ &= \text{Length of a Single Subinterval} \end{aligned}$$

Trapezoidal Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \sum_{i=1}^{n} \frac{h}{2}\left(f(x_{i-1}) + f(x_i)\right)$$

Where:

$$\begin{aligned} n &= \text{Number of Intervals} \\ h &= \frac{x_{\text{final}} - x_{\text{initial}}}{n} \\ &= \text{Length of a Single Subinterval} \end{aligned}$$

Simpson Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \approx \frac{2}{3} I_{\text{rectangular}} + \frac{1}{3} I_{\text{trapezoidal}}$$

Where:

$$\begin{aligned} I_{\text{rectangular}} &= \text{Rectangle Rule Estimate} \\ I_{\text{trapezoidal}} &= \text{Trapezoidal Rule Estimate} \end{aligned}$$

Gauss Quadrature Rule for Integration

$$\int_{-1}^{1} f_n(\xi)\,\mathrm{d}\xi = \sum_{k=1}^{q} f_n(\xi_k)\,w_k$$

Basic Idea: We will try to minimize the number of sampling points needed to integrate exactly a polynomial of a chosen degree n on the domain [-1, 1].

Where:

w_k = Weighting Function

- **To integrate exactly (n=1)** $f_1(\xi) = \alpha_0 + \alpha_1\,\xi$

$$\begin{cases} w_1 = 2 \\ \xi_1 = 0 \end{cases}$$

- **To integrate exactly (n=3)** $f_3(\xi) = \alpha_0 + \alpha_1\,\xi + \alpha_2\,\xi^2 + \alpha_3\,\xi^3$

  We need 2 sampling points (and 2 weights)

$$\begin{cases} w_1 = 1 & w_2 = 1 \\ \xi_1 = -\frac{\sqrt{3}}{3} & \xi_2 = \frac{\sqrt{3}}{3} \end{cases}$$

- **To integrate exactly (n=5)** $f_5(\xi) = \alpha_0 + \alpha_1\,\xi + \alpha_2\,\xi^2 + \alpha_3\,\xi^3 + \alpha_4\,\xi^4 + \alpha_5\,\xi^5$

  We need 3 sampling points (and 3 weights):

$$\begin{cases} w_1 = \frac{8}{9} & w_2 = \frac{5}{9} & w_3 = \frac{5}{9} \\ \xi_1 = 0 & \xi_2 = \sqrt{\frac{3}{5}} & \xi_3 = -\sqrt{\frac{3}{5}} \end{cases}$$

Coordinate Transformation

$$\xi = \frac{2x}{b-a} + \frac{a+b}{a-b} = \frac{2x-(a+b)}{b-a}$$

such that

$$\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{(b-a)\xi+(a+b)}{2}\right)\mathrm{d}\xi$$

Forward Difference Scheme

$$f'(x) = \frac{f(x+h)-f(x)}{h} - \frac{h}{2}f''(\xi) \qquad \text{with } \xi \in (x, x+h)$$

Central Difference Scheme

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h} - \frac{h^2}{6}f'''(\xi) \qquad \text{with } \xi \in (x, x+h)$$

Richardson Extrapolation

$$a_o = F(h) + \frac{F(h)-F(qh)}{q^p-1} + O(h^r) \qquad \text{or} \qquad a_o = \frac{F(qh)-q^p\,F(h)}{1-q^p} + O(h^r)$$

Where:

$p = 1$ For Forward Difference Scheme since error is linear.

$p = 2$ For Central Difference Scheme since error is squared.