Section 2: Electromagnetics

AE435

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In this section, we will review the basics of charge, electricity, magnetism, and Maxwell equations.

5 Electromagnetic Induction

Contents

5.1 Faraday's Induction Law

From electrostatics, we have the identical statements (Equation 14)

$$\nabla \times \vec{E} = 0$$

Or in integral form

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = 0$$

When we have time-varying B-fields, though, we find experimentally that

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = \varepsilon_{mf} \tag{90}$$

This EMF, "electromotive force", is the potential difference that gives rise to a current and is due to a time-varying magnetic flux.

Consider a conductor element $\mathrm{d}\vec{l}$ moving at velocity $\vec{v}$ in a B-field, oriented OUT of the page. Electrons in $\mathrm{d}\vec{l}$ feel the Lorentz force which we first seen in Equation 64

$$\vec{F}_m = -q_e \, \vec{v} \times \vec{B} \tag{91}$$

Thus $\vec{v} \times \vec{B}$ acts just like an applied electric field E. If the circuit were closed, then the current would flow as if a battery were supplying a voltage:

$$\varepsilon_{mf} = \oint_c (\vec{v} \times \vec{B}) \cdot \mathrm{d}\vec{l} \tag{92}$$

Use vector identity to rearrange terms

$$\varepsilon_{mf} = -\oint_c \vec{B} \cdot (\vec{v} \times \mathrm{d}\vec{l}) = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot \hat{n} \, \mathrm{d}A$$

And note $(\vec{v} \times \mathrm{d}\vec{l}) = \hat{n} \, \frac{\mathrm{d}A}{\mathrm{d}t}$ that is the rate of change of the projected area enclosed by the circuit. Thus, using the definition of magnetic flux (Equation 88):

Faraday's Induction Law

$$\varepsilon_{mf} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot \hat{n} \, \mathrm{d}A = -\frac{\partial \Phi}{\partial t} \tag{93}$$

Example: Let's pull a rectangular loop of width l out of a uniform magnetic field $\vec{B}$ with a velocity $\vec{v}$.

Figure 22

For this simple case, the EMF

$$\varepsilon_{mf} = \oint_c (\vec{v} \times \vec{B}) \cdot \mathrm{d}\,\vec{l}$$

Which becomes

$$\varepsilon_{mf} = v\,B\,l$$

Since the vertical component is the only non-zero part. Given a loop resistance, R, the loop current is: (V = IR)

$$J = \frac{v\,B\,l}{R}$$

(we will assume J is small such that the self-field is negligible).
Another way to consider this is as a rate of change of magnetic flux:

$$\frac{\partial \Phi}{\partial t} = -B\frac{\mathrm{d}A}{\mathrm{d}t} = -Bvl$$

So again, $\varepsilon_{mf} = v\,B\,l$ In this case, the restoring force: $\vec{F} = q\,\vec{v} \times \vec{B} = J\,\vec{l} \times \vec{B}$ Has a magnitude

$$\vec{F} = J\,v\,B = \frac{B^2 v l^2}{R} \qquad \text{Force to Pull Loop}$$

Thus the power required to pull the loop out is

$$P = F \cdot v = \frac{B^2 v^2 l^2}{R}$$

EP application: Tether Propulsion

5.2 Faraday's Law

Consider a time-varying $\vec{B}$

Figure 23

The induced EMF in a closed loop (Equation 90)

$$\varepsilon_{mf} = \oint \vec{E} \cdot \mathrm{d}\,\vec{l}$$

is, via Stoke's theorem (Equation 73)

$$\varepsilon_{mf} = \oint \vec{E} \cdot \mathrm{d}\,\vec{l} = \int_S \nabla \times \vec{E} \cdot \hat{n}\,\mathrm{d}A$$

Applying Faraday's induction law (Equation 93):

$$\varepsilon_{mf} = -\frac{\partial}{\partial t}\int_S \vec{B} \cdot \hat{n}\,\mathrm{d}A = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\,\mathrm{d}A$$

We see that:

$$-\frac{\partial}{\partial t}\int_S \vec{B} \cdot \hat{n}\,\mathrm{d}A = \int_S \nabla \times \vec{E} \cdot \hat{n}\,\mathrm{d}A$$

Equal terms within the integral for arbitrary integration surfaces means that:

Generalized Form of Faraday's Law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{94}$$

5.3 Lenz's Law

Lenz's Law: The direction of the induced current is such that its magnetic field opposes the change in flux.

- If $\frac{\mathrm{d}B}{\mathrm{d}t}$ is negative (i.e., B decreasing with time)
- The resulting positive Emf induces a positive current
- $\nabla \times \vec{B} = \mu_o \vec{j}$ generates a positive B, opposing the change in flux

$$\dot{B} = \frac{\partial B}{\partial t}$$

We use this in Bdot probes for magnetic field measurement.

Figure 24

B-Dot Probe Theory A B-dot probe is used to measure the a time varying magnetic field produced by an electromagnetic waves propagating through a plasma. The term ?B-dot? comes from the mathematical notation $\dot{B} = \frac{\partial B}{\partial t}$. These waves can be measured in situ by inserting the probe in a glass tube in the plasma chamber since glass is not electrically conductive.

Faraday?s law can be used to show that a time varying magnetic field can produce a voltage in a loop of wire. We can see this directly from the theory: The induced voltage is directly related to the flux through the loop:

$$V = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \qquad \text{(A)}$$

Where the flux is defined as $\Phi = a\,N\,B$. Here a is the area of the loop and N is the number of loops in the wire. In our case, there is only one loop in the probe, thus $N = 1$. Substituting the equation for the flux into Equation A, the result is an equation that directly relates the voltage to the change in magnetic field in the plasma.

$$V = -\frac{\mathrm{d}\,(aB)}{\mathrm{d}t} \qquad \text{(B)}$$

If the area of the loop is constant, then a can be taken out of the time derivative from Equation B. Thus the time varying magnetic field is proportional to the amplitude of the induced voltage.

$$V = -a\,\frac{\mathrm{d}B}{\mathrm{d}t} \qquad \text{(C)}$$