

Section 8: Electromagnetic Propulsion

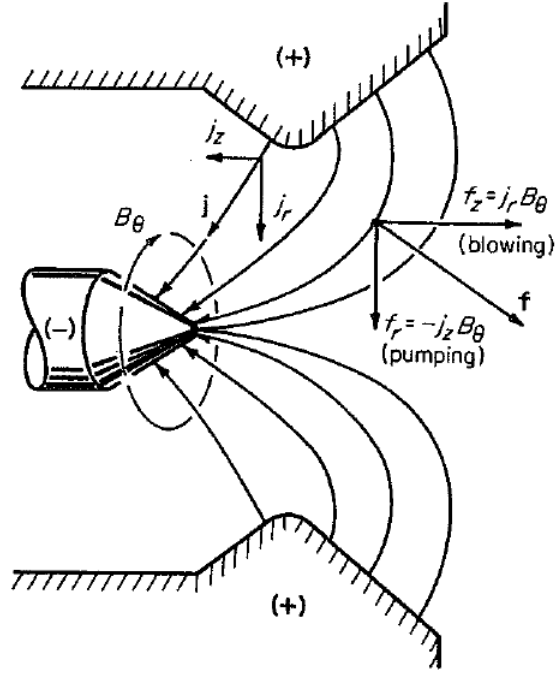
AE435
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1 Coaxial Self-Field Thrusters

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Schematic of a Magneto Plasma Dynamic (MPD) Accelerator:



Coaxial meaning the anode is surrounding the center-mounted cathode. We have a strong electric field between the anode and cathode which in turn forms an arc between the anode and cathode. The arc is passing through the plasma itself and that current creates a strong magnetic field.

In other words, a "self-field" thruster is one where the magnetic field is due to the plasma current. An "applied-field" thruster has a magnetic field that is externally applied (and will also have self-field too).

Can separate out two thrust components in a coaxial self-field thruster:

- **Blowing Forces**, which directly accelerate the gas in the axial direction
- **Pumping Forces**, which provide a radial pressure gradient (and, thus, pressure thrust)

For axisymmetric current (no-theta component)

$$\nabla \times \vec{B} = \mu \vec{j} \quad (8.8)$$

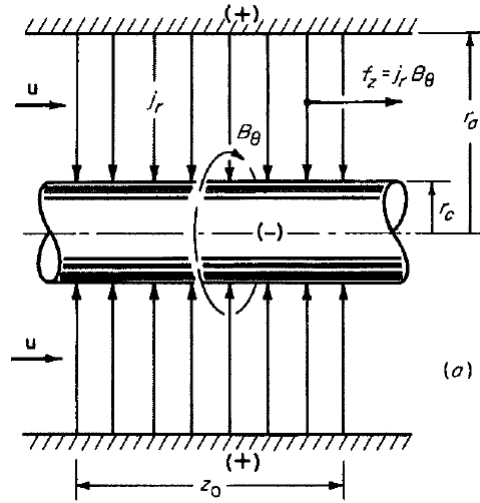
meaning the magnetic field is purely azimuthal.

$$\vec{B} = B(r, z) \hat{\theta} \quad \text{Purely Azimuthal}$$

1.1 Blowing Contribution

1.1.1 Cylindrical Cathode Tip

To evaluate the blowing contribution, we can consider a simplified model:



If we assume a radial uniform current density j_r at the cathode (as shown in the figure above), then

$$B_\theta(r, z) = \frac{\mu J}{2 \pi r} \left(1 - \frac{z}{z_o} \right) \quad (8.9)$$

where z_o is the length of the cathode and where the current into the cathode is:

$$J = \int_S \vec{j} \cdot d\vec{A} = 2 \pi r z_o j_r \quad (8.10)$$

which increases linearly with z_o .

The body force density in the axial direction is then:

$$f_z = j_r B_\theta \quad (8.11)$$

Combined with Equation 8.9

$$f_z = \frac{\mu J^2}{4 \pi^2 r^2 z_o^2} (z_o - z) \quad (8.12)$$

To get the total force over the entire system we take the integral of the body force density over the gas volume:

$$F_z = \int_0^{z_o} \int_0^{2\pi} \int_{r_c}^{r_a} f_z r dr d\theta dz \quad (8.13)$$

Evaluating the thrust for this model:

$$F_z = \frac{\mu J^2}{4\pi^2 z_o^2} \int_0^{z_o} \int_0^{2\pi} \int_{r_c}^{r_a} \frac{(z_o - z)}{r^2} dr d\theta dz \quad (8.14)$$

yields

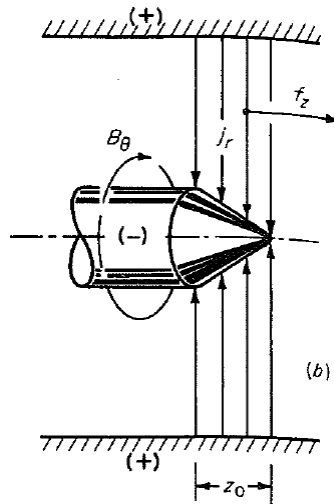
Cylindrical-Tipped Cathode Thrust:

$$F_z = \frac{\mu J^2}{4\pi} \ln \left(\frac{r_a}{r_c} \right) \quad (8.15)$$

This is the total blowing (axial) force when we have a cylindrical cathode. There is no pinching or pumping force for a cylindrical cathode. Notice it is dependent on J^2 which would make sense. If we were to increase the current, we would expect to get a larger force. It is also dependent on the geometry, the cathode and anode radius.

1.1.2 Conical Cathode Tip

You can model a conical tip cathode the same way:



If we assume a uniform current density j_a at the anode, then the cathode current density varies inversely with cathode radius (as the current increases, the radius must decrease). The new volume integral

$$F_z = \int_0^{z_o} \int_0^{2\pi} \int_{r_c(1-\frac{z}{z_o})}^{r_a} f_z r \, dr \, d\theta \, dz$$

yielding...

Conical-Tipped Cathode Thrust for Uniform Anode Current Density:

$$F_z = \frac{\mu J^2}{4\pi} \ln \left(\frac{r_a}{r_c} + \frac{1}{2} \right) \quad (8.16)$$

If we assume a uniform current density j_c at the cathode, then we get:

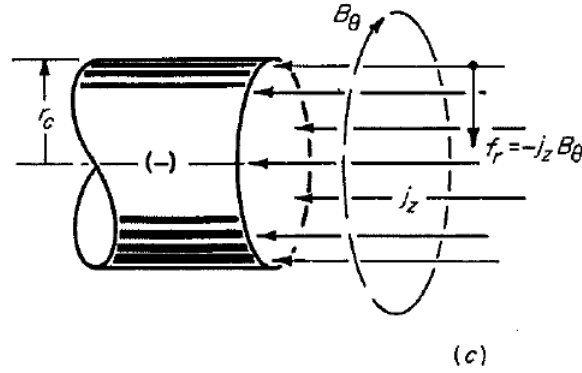
Conical-Tipped Cathode Thrust for Uniform Cathode Current Density:

$$F_z = \frac{\mu J^2}{4\pi} \ln \left(\frac{r_a}{r_c} + \frac{1}{4} \right) \quad (8.17)$$

Either way, current flow to a conical cathode tip increases the blowing force compared with a cylindrical cathode (Equation. 8.15). In coaxial thrusters with long solid-rod cathodes, the tip current fraction is $\sim 10\%$.

1.2 Pumping Contribution

To evaluate the pumping contribution, consider another simple model:



Here, all the current enters the face of the cathode, and the magnetic field becomes (as we saw within the arc column, Equation 7.76):

$$B_\theta(r, z) = \frac{\mu J^2 r}{2 \pi r_c^2} \quad (8.18)$$

The radial force density

$$f_r = j_z B_\theta = \frac{\mu J^2 r}{2 \pi^2 r_c^4} = -\frac{\partial p}{\partial r} \quad (8.19)$$

is balanced by the radial pressure gradient, $\frac{\partial p}{\partial r}$.

Integrating over r , you get a parabolic pressure profile:

$$p(r) = p_o + \frac{\mu J^2}{4 \pi^2 r_c^2} \left[1 - \left(\frac{r}{r_c} \right)^2 \right] \quad (8.20)$$

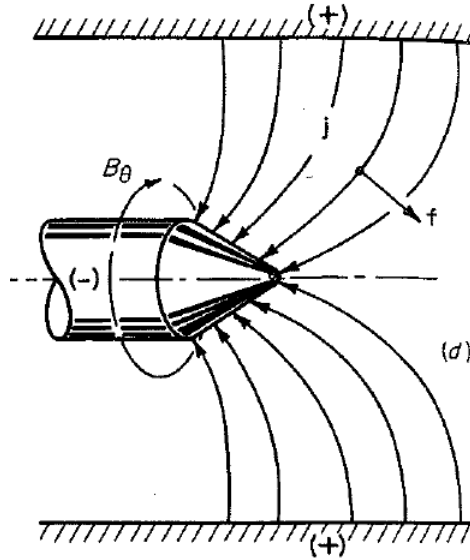
which can be integrated over the cathode tip to give a pressure force.

$$F_c = 2 \pi \int_0^{r_c} (p - p_o) r \, dr = \frac{\mu J^2}{8 \pi} \quad (8.21)$$

p is the pressure in the arc while p_o is the ambient pressure. To get the total force, we add up the blowing and pumping contributions.

1.3 Maecker Model

A more general model, including all of the above:



Adding together all the blowing and pumping contributions, gives:

Maecker Formula

$$F = \frac{\mu J^2}{4\pi} \left(\ln \left(\frac{r_a}{r_c} \right) + \frac{3}{4} \right) \quad (8.22)$$

This is the Maecker formula, derived in 1955.

- Thrust is independent of flow rate and current flow details. The only thing that matters is the current, J , and the geometry.
- The specific impulse varies as, $\frac{J^2}{\dot{m}}$

Proof:

$$I_{SP} = \frac{F}{g_o \dot{m}} = \frac{\mu J^2}{4\pi g_o \dot{m}} \left(\ln \left(\frac{r_a}{r_c} \right) + \frac{3}{4} \right)$$

How well does Maecker compare to real thruster data?

First we define a thrust coefficient:

$$C_T = \frac{4\pi}{\mu_o} \frac{F}{J^2} \quad (8.23)$$

so that thrust

$$F = C_T \frac{\mu_o}{4\pi} J^2$$

Notice that for the Maecker formula, C_T is a constant for a particular geometry:

What this model is essentially saying then is that once I have built my thruster (such that all geometries are fixed), when I run my thruster and change the current input, there is no change in the thrust coefficient. My thrust coefficient does not change. In the figure below, the Maecker formula is constant and stays constant. We see that it models the thrust coefficient almost close for high current states but fails when operating at lower currents.

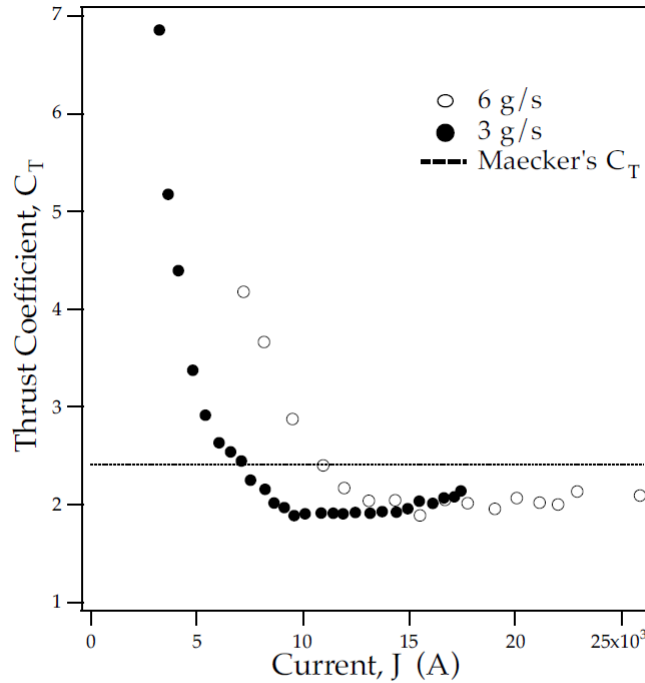


Figure 1: Comparison of the Maecker thrust coefficient with that measured[3] for the Princeton Benchmark Thruster with two different argon mass flow rates. ($r_a/r_c = 5.26$)

The actual data (from Choueiri, "On the Thrust of Self-field MPD Thrusters", IEPC-97-121, 1997) comparison with the Princeton Benchmark Thruster (PBT):

- Is generally much lower than the Maecker C_T at high J
- Much higher than the Maecker C_T at low J
- Shifts with mass flow rate

Why are we seeing this?

- **Traditional Explanation:** Electrothermal component of thrust. We are heating up the gas! In other words, we don't have just electromagnetic force ($J \times B$), we also have some electrothermal acceleration. If we look at the geometry, the anode has a converging diverging shape so we will have some type of electrothermal component to it. But it is hard to see why thrust would decrease with increasing J ? A better explanation...
- **Modern:** Pumping forces on the rear of the chamber. Not much experimental verification yet, but Choueiri's model (which includes all surfaces) does pretty well.

1.4 Tikhonov Model

Tikhonov (1976) came up with an improvement on the Maecker model. Assumptions:

- Quasi 1-D flow
- Single-fluid single-temperature MHD plasma
- Isothermal fluid
- High Magnetic Reynolds Number (plasma carries magnetic field with it as it convects)

Model:

$$C_T = \frac{\gamma + 1}{2} + \frac{\alpha_o^{-2}}{2} \quad (8.24)$$

The dimensionless parameter α_o is given by:

$$\alpha_o = \frac{\gamma \mu_o}{8 \pi a_o} \frac{J^2}{\dot{m}}$$

where

γ = Adiabatic Index

$$a_o = \sqrt{\gamma R T} = \text{Speed of Sound}$$

Notice how that the results are independent of geometry, $\frac{r_a}{r_c}$. There is also no model for ionization potential ε_i . So if we were to switch gases, there is no accounting for how the ionization potential changes for the type of gas being used.

However, it does a better job than the Maecker Formula of following the general trend of the data:

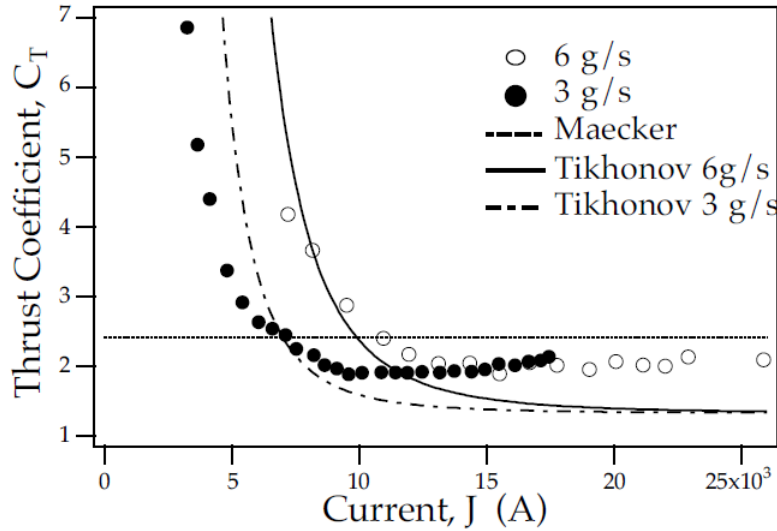


Figure 2: Comparison of the Tikhonov thrust coefficient with the measured thrust coefficient of the PBT plotted versus the measured current for two different argon mass flow rates. Also shown is the constant Maecker C_T . (The ion acoustic speed was calculated for a temperature of 1 eV)

Unfortunately, when $\frac{J^2}{m}$ is high, the Tikhonov formula predicts that:

$$C_T \cong \frac{\gamma + 1}{2} \quad (8.25)$$

meaning that for $\gamma \gg 1$ this model seriously under-predicts the high-thrust, $\frac{J^2}{m}$.

1.5 Choueiri Model

Choueiri (1997) came up with the best model yet where we are basically calculating the force on all of the surfaces of the control volume.

First-principles model of the Princeton Benchmark Thruster:

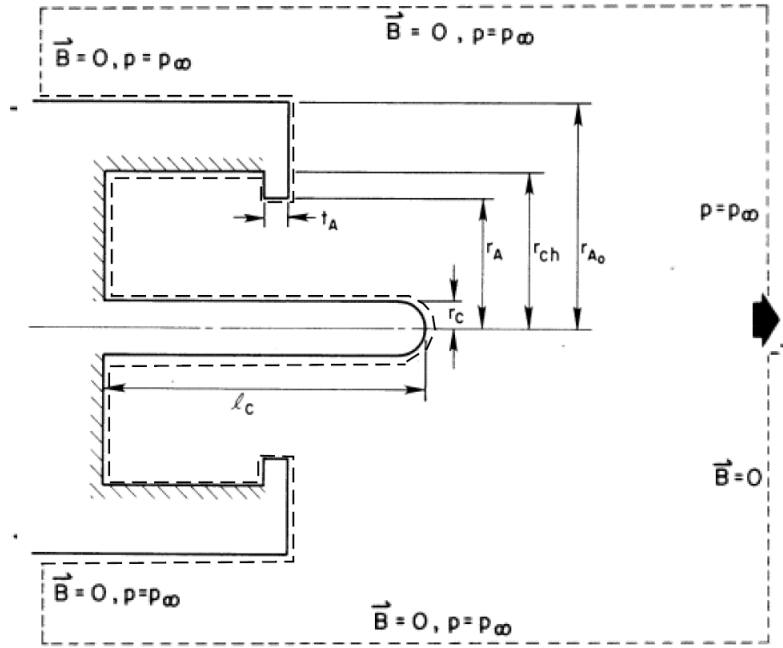


Figure 3: Control volume used in deriving a thrust equation for the Princeton benchmark thruster[6]. The dashed line represents the boundary of the control volume used in the thrust analysis. For the particular thruster used in the thrust experiments we have $r_c = .95$ cm, $r_a = 5.1$ cm, $r_{ao} = 9.3$ cm, $r_{ch} = 6.4$ cm, $t_a = .95$ cm and $l_c = 10$ cm.

Choueiri considers the Blowing (b) and Pumping/Pinching (p) contributions from the:

- Backplate (BP)
- Anode inner face (AIF)
- Anode outer face (AOF)
- Cathode tip (CT)

Each of these curves in the Figure 6 below represents the individual contributions from each of the surfaces listed above. If we add all of the pinching and blowing contributions on all of the surfaces, we arrive at a sum or total thrust coefficient, C_T , that closely matched experimental data.

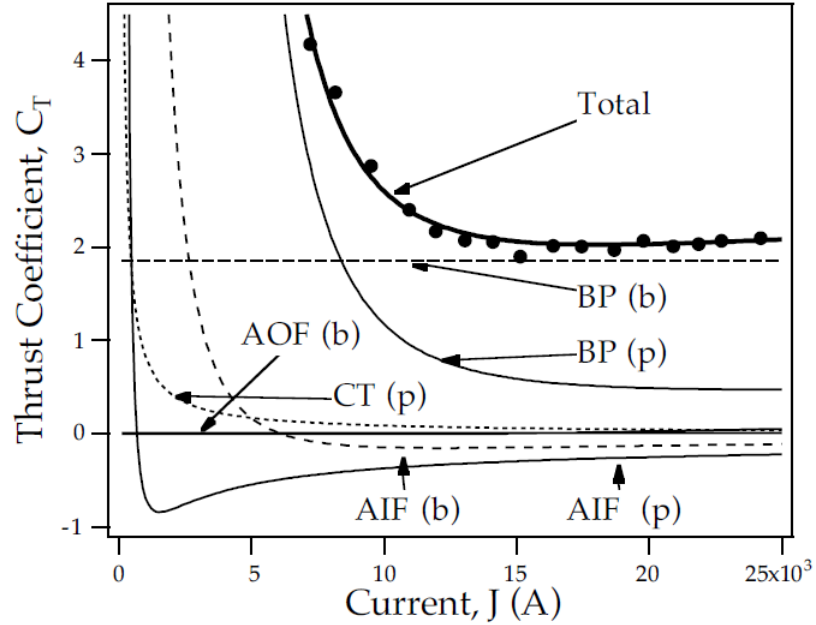


Figure 6: Contributions of the various blowing and pinching components to C_T along with their total sum for the PBT operated with 6 g/s of argon. Also shown are the measured data.

To scale this to other thrusters, Choueiri looked at how ionization sucks energy away from useful work (plasma acceleration). When the jet power equals the ionization energy loss rate:

$$\frac{1}{2} F u_e = \dot{m} \frac{\varepsilon_i}{M} \quad (8.26)$$

where

ε_i = Ionization Energy

M = Heavy-Particle (Ion or Neutral) Mass

We can solve for a critical ionization speed

Critical Ionization Speed

$$u_{ci} = \left(\frac{2 \varepsilon_i}{M} \right)^{\frac{1}{2}} \quad (8.27)$$

Where we can use values for typical propellants experimentally collected or known:

Propellant	T_i [°C]	T_g [°C]	\mathcal{E}_i [ev]	M [amu]	u_{ci} [m/s]
Argon (Ar)	-189	-186	15.76	39.9	8731
Bismuth (Bi)	271	1559	7.29	209.0	2594
Krypton (Kr)	-157	-153	14.00	83.8	5678
Lithium (Li)	181	1342	5.39	6.94	12245
Xenon (Xe)	N/A	N/A	12.10	131.3	4217

We can nondimensionalize J as:

$$\xi = \frac{J}{J_{ci}} \quad (8.28)$$

Where a critical current is defined as,

$$J_{ci} = \left(4\pi \frac{\dot{m} u_{ci}}{\mu_o C_T} \right)^{\frac{1}{2}} \quad (8.29)$$

which relates u_{ci} to C_T and J . Using the Maecker approach, the dimensionless current can be expressed as:

Dimensionless Current

$$\xi = \frac{J \left(\frac{\mu_o}{4\pi} \ln \left(\frac{r_a}{r_c} \right) \right)^{\frac{1}{2}}}{\dot{m}^{\frac{1}{2}} \left(\frac{2\mathcal{E}_i}{M} \right)^{\frac{1}{4}}} \quad (8.30)$$

This formulation has been shown to relate thruster performance pretty well to propellant mass and ionization energy. As such, Choueiri derived the following scaling laws for C_T :

Choueiri Scaling Laws for C_T

$$\begin{aligned} C_T &\sim \xi^{-4} && \text{for } \xi < 1 \\ C_T &\sim \ln \left(\frac{r_a}{r_c} + \xi^2 \right) && \text{for } \xi > 1 \end{aligned}$$

which can be combined into a single formula as:

Choueiri Scaling Law for C_T

$$C_t = \frac{\nu}{\xi^4} + \ln \left(\frac{r_a}{r_c} + \xi^2 \right) \quad (8.31)$$

where

$$\nu = \frac{\dot{m}}{\dot{m}^*}$$

From 6 g/s argon data, the scaling flow rate is derived to be:

Scaling Flow Rate

$$\dot{m}^* = 66 \quad [g/s]$$

This scaling flow rate is not a theoretical model. It is a scaling model based on data from Argon but we take it to be 66 [g/s] for all gases that we work with.

Figure 7 in Choueiri shows how this scaling method (not quite a model) applies to argon and xenon data.

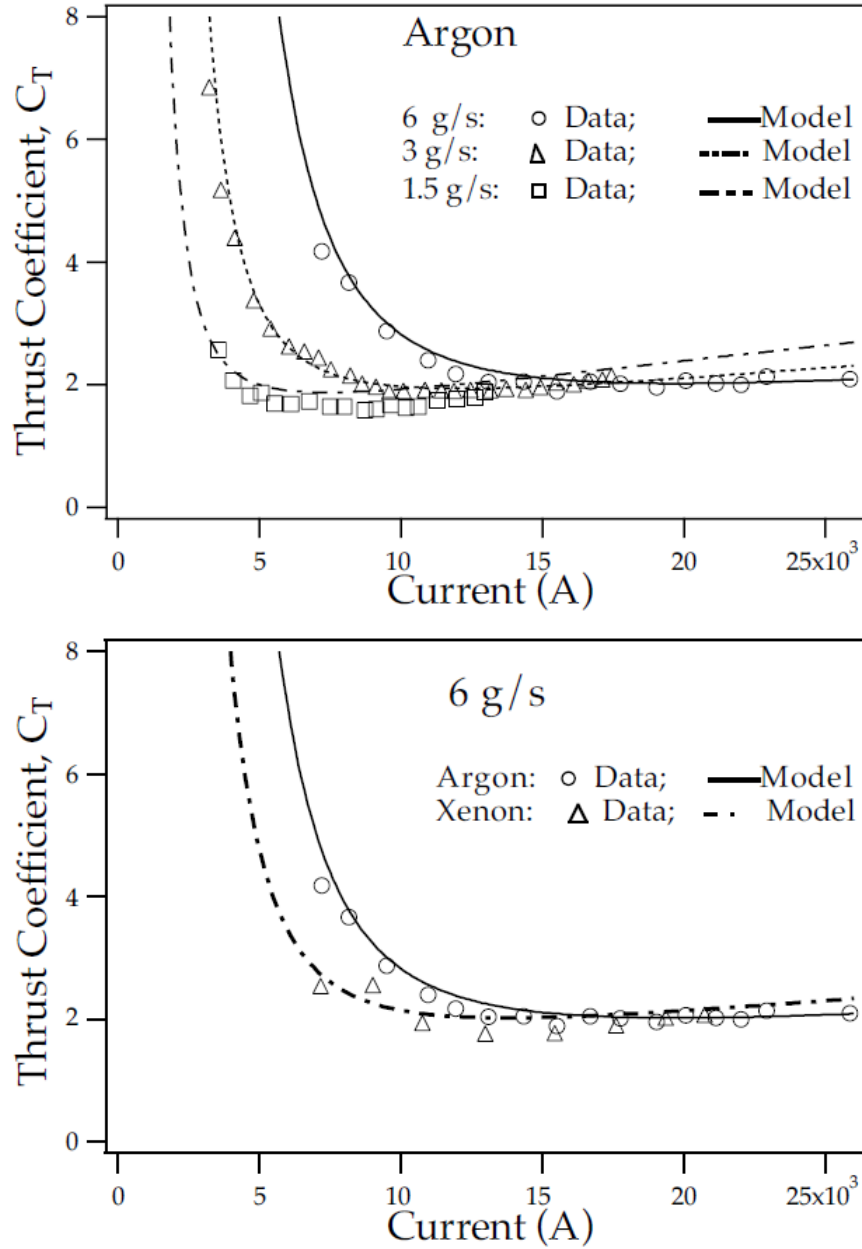


Figure 7: Comparison of the scaling formula in Eq. (39) with argon and xenon thrust measurements for various mass flow rates.

1.6 Power Analysis

$$P_e = \frac{1}{2} F u_e = J^2 Z \quad (8.32)$$

Use impedance Z instead of ohmic resistance R because propulsion systems can have reactive elements (capacitance, inductance) in addition to resistance.

Thus, for $T=F$ of Maecker Formula (Equation 8.22)...

$$Z = \frac{F u_e}{2 J^2} = \frac{1}{2} \frac{\mu_o}{4 \pi} \left[\ln \left(\frac{r_a}{r_c} \right) + \frac{3}{4} \right] u_e = \frac{1}{2} k u_e \quad (8.33)$$

Example: Consider a system with an exit velocity of 40,000 m/s, an anode to cathode ratio of 5 and a current of 20 kA. Find the Thrust, Exit Jet Power, Impedance and Voltage for this system.

We have:

$$u_e = 40000 \quad [m/s] \quad , \quad \frac{r_a}{r_c} \sim 5 \quad , \quad J = 20 \quad [kA]$$

We can find the thrust using Equation 8.22.

$$F = \frac{\mu J^2}{4 \pi} \left(\ln \left(\frac{r_a}{r_c} \right) + \frac{3}{4} \right) = 94.4 [N]$$

Leading us to now find...

$$P_e = \frac{1}{2} F u_e = 1.9 \times 10^6 [W]$$

$$Z = 0.00472 [\omega]$$

$$V = J Z = 94.4 [Volts]$$

This is a fairly low voltage! The actual voltage of the system will be higher since ohmic resistance (due to ω_o) and voltage sheaths not included.