# Section 5: Collisions

AE435

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## 1 Cross Sections

The collision "cross-section" is like the target size a particle "sees" as it moves through a collection of particles.

Some cross-sections tell us how (to what angle) the incident particle will be scattered due to a particle type of collision, these are "differential cross-sections".

If we integrate the differential cross-section over all possible scatter angles you get the "total" or "effective" cross-section for that particular type of collision.

Finally, if you sum the effective (or total) cross-section for all possible collision types, you get the "total" collision cross-section.

## Contents

## 1.1 Effective Cross Sections

Consider

- A particle of type j encountering
- A particle of type k
- In collision of type $\beta$

The probability of this over a path length d$x$ is:

**Probability of Encounter**

$$\mathrm{d}P_{jk}^{(\beta)} = n_k \, Q_{jk}^{(\beta)} \, \mathrm{d}x \tag{2}$$

Where

$$\begin{aligned} Q_{jk}^{(\beta)} &= \text{Corresponding Collision Cross-Section} \\ &= [m^{-3} \, m^2 \, m^1] = [1] \quad \text{Unitless} \end{aligned}$$

For a flux ($\Gamma$) of test particles, j, entering a cloud of field particles, k, the flux leaving the cloud is:

$$\Gamma + \mathrm{d}\Gamma = \Gamma(1 - \mathrm{d}P_{jk}^{(\beta)}) = \Gamma(1 - n_k \, Q_{jk}^{(\beta)} \, \mathrm{d}x) \tag{3}$$

**Figure 3**

Cancelling like terms yields:

$$d\Gamma = \Gamma\, n_k\, Q_{jk}^{(\beta)}\, dx) \tag{4}$$

Which integrates to:

$$\Gamma = \Gamma_o \exp\left(-n_k\, Q_{jk}^{(\beta)}\, x\right) = \Gamma_o \exp\left(-\frac{x}{\lambda_{\text{mfp}}}\right) \tag{5}$$

where $\Gamma_o$ is the initial flux. The mean free path now becomes

$$\lambda_{\text{mfp}} = \frac{1}{n_k\, Q_{jk}^{(\beta)}} \tag{6}$$

A good first estimate for $Q_{jk}^{(\beta)}$ is the... atomic cross section

**Atomic Cross Section**

$$a_o = \frac{\epsilon_o\, h^2}{\pi\, m_e\, q_e^2} \tag{7}$$

Where the Bohr radius $\pi a_o^2$ describes the orbit for an electron with the lowest possible non-zero momentum.

Returning to the collisional cross-section, note that

$$Q_{jk}^{(\beta)} = Q_{kj}^{(\beta)} \tag{8}$$

We can define a...

**Total ("Effective") Cross-Section**

$$Q_{jk} = \sum_{\beta} Q_{jk}^{(\beta)} \tag{9}$$

which is the sum of all the different types of collisions (elastic, excitation, ionization, etc.).

### 1.2 Differential Cross-Sections

In addition to Effective Cross Sections, we also have differential cross-sections $q(\theta)$ defined by the scattering angle,

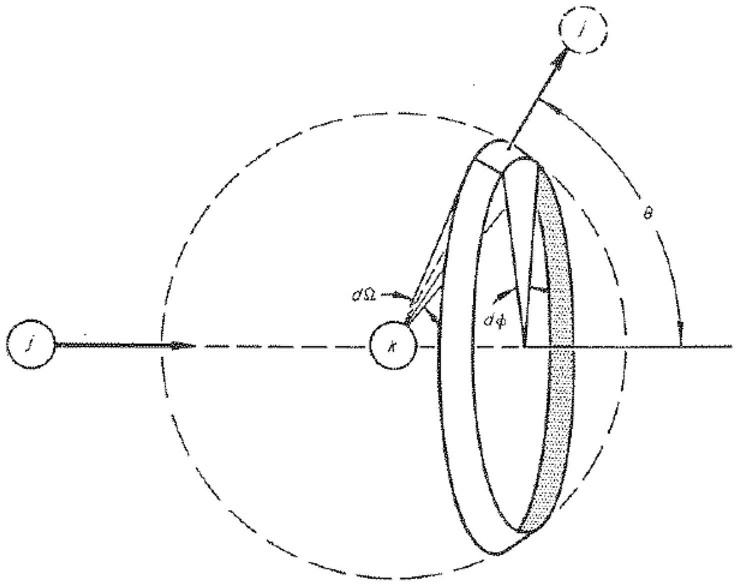


Fig. 4-2 Nomenclature for definition of differential cross section.

The probability of scattered particle emerging into solid angle:

$$\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{10}$$

is

$$\mathrm{d}P(\theta) = q(\theta)\, \mathrm{d}\Omega = q(\theta)\ \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{11}$$

Integrating over the full $4\pi$ of the solid angle around the scattering center gives us the:

**Total Collisional Cross-Section**

$$Q = \int_0^{2\pi} \int_0^{\pi} q(\theta)\ \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{12}$$

For a type $-\beta$ collision, the mean free path is

$$\lambda_{\mathrm{mfp}}^{(\beta)} = \frac{1}{n_k\, Q_{jk}^{(\beta)}} \tag{13}$$

as previously defined in Equation 6 and Equation 4.15

We can now define...

**Collision Frequency**

$$\nu_{jk}^{(\beta)} = n_k \, Q_{jk}^{(\beta)} \, \overline{v_{jk}} \tag{14}$$

Where

$$\overline{v_{jk}} = \text{Collision Speed} \quad = \text{Relative Speed Between j and k}$$

The collision speed $\overline{v_{jk}}$ is well-defined in beam experiments, but less so for thermal plasmas.

**General Equation for Collision Speed in Thermal Plasmas:**

$$\overline{v_{jk}} \cong v_{th} = \left( \frac{8\,k\,T}{\pi\,m} \right)^{\frac{1}{2}} \qquad \text{Equation 4.32} \tag{15}$$

This relation applies for the faster particle species in mixed thermal plasmas.

**Example:** Consider a xenon plasma with equal electron and ion temperature ($T_e = T_i = 3\text{eV}$).

As a result...

$$\begin{cases} V_e = 750 \, \frac{km}{s} \\ V_i = 1.5 \, \frac{km}{s} \end{cases}$$

Meaning that...

$$\begin{aligned} \nu &= n \; < Q_{jk} \, v_{jk} > \\ &= n \, \overline{Q_{jk} \, v_{jk}} \end{aligned}$$

Such that

$$\overline{Q\,V} = \int_0^\infty Q \, v \, f(v) \, \mathrm{d}v$$

Note: This mean collision rate is only applicable for test particles with velocities much much greater than field particles.

Finally, we can define a:

**Mean Collision Rate for a Swarm of Particles**

$$n_j \, \nu_{jk} = \int \int n_j(\vec{v}_j) \, n_k(\vec{v}_k) \, Q_{jk} \left(|\vec{v}_j - \vec{v}_k|\right) |\vec{v}_j - \vec{v}_k| \, \mathrm{d}\vec{v}_j \, \mathrm{d}\vec{v}_k \tag{16}$$

Where

- We can approximate $n_j(\vec{v}_j)$ and $n_k(\vec{v}_k)$ as Maxwellian Distribution Functions
- $Q_{jk} \left(|\vec{v}_j - \vec{v}_k|\right)$ shows the dependence of $Q$ on relative velocity, $|\vec{v}_j - \vec{v}_k|$.

Opposed to the mean collision rate prior to Equation 16, this applies for test particles and field particles with approximately the same velocity. It considers a way for accounting for the fact that $j$ and $k$ particles have some sort of distribution associated with them.