Section 4: Kinetic Theory

AE435

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1 Pressure, Temperature, and Internal Energy

This section draws from Chapter 1 in the Vincenti and Kruger's Introduction Physical Gas Dynamics

In Kinetic Theory we model a gas as a collection of particles/molecules.

Contents

1.1 Pressure

In this section we will model a gas as a collection of particles in a cubical box with sides of length l. We will make the following assumptions:

1. Particles have no internal structure
2. Equilibrium
3. Mean free path ($\lambda >> l$) (no intermolecular collisions)

The assumption of equilibrium makes the other two assumptions reasonable - at equilibrium, the number of particles in a certain state must not change. Gas then acts as if the other two assumptions are true.

Figure 1

A particle in the box moves with a velocity:

$$\vec{c} = c_1\,\hat{x}_1 + c_2\,\hat{x}_2 + c_3\,\hat{x}_3 \tag{1}$$

Such that the speed C is given by

$$c^2 = c_1^2 + c_2^2 + c_3^2 \tag{2}$$

Our analysis begins with pressure so we will be looking at the particles momentum. Consider...

1. A particle of mass, m, moving in the $\hat{x}$-direction. The particle will have a momentum, $m\, c_1\, \hat{x}_1$.
2. If it bounces elastically from the wall at $x_1 = l$, the momentum change is, $2\, m\, c_1\, \hat{x}_1$.
3. The time between collisions is $\frac{2\, l}{c_1}$.
4. As a result the force (change in total momentum per unit time) on this wall from 1 particle is:

$$\vec{F}_l = \frac{\text{Momentum}}{\text{Time}} = \frac{2\, m\, c_1 \hat{x}_1}{\frac{2\, l}{c_1}} = \frac{m c_1^2}{l}\, \hat{x}_1$$

5. Pressure is the normal component of the force per unit area, so the pressure on this wall from 1 particle is:

$$P_l = \frac{\frac{m c_1^2}{l}}{l^2} = \frac{m c_1^2}{l^3} = \frac{m c_1^2}{\cancel{V}}$$

   Note that l^3 is the volume, $\cancel{V}$, of the box

6. Corresponding equations give the pressure on the other 2 sets of walls. Pressure on each wall is equal (by equilibrium) so the mean pressure is:

$$P = \frac{m\, (c_1^2 + c_2^2 + c_3^2)}{3\, \cancel{V}} = \frac{m\, c^2}{3\, \cancel{V}}$$

7. For many particles, the pressure becomes:

$$P = \frac{\sum_i\ m_i\, c_i^2}{3\, \cancel{V}} \qquad \left[\frac{N}{m^2} \cdot \frac{m}{m} = \frac{J}{m^3}\right] \tag{3}$$

   Note how the term , $m_i\, c_i^2$ relates back to kinetic energy.

1.2 Translational Energy

The energy of translation for the ensemble of gas particles is

$$E_{\text{translational}} = \frac{1}{2} \sum_i m_i \, c_i^2 \tag{4}$$

So from Equation 3

$$P \, \forall = \frac{2}{3} E_{\text{translational}} \tag{5}$$

Compare this with the empirically-derived equation of state (the "perfect gas law") from classical thermo:

$$P \, \forall = \mathcal{N} \hat{R} T \tag{6}$$

Where

$$\begin{aligned} \mathcal{N} &= \text{Number of moles} \\ \hat{R} &= \text{Gas constant per mole} \\ T &= \text{Temperature} \end{aligned}$$

Therefore the energy of translation for the particles in the volume, $\forall$,is:

$$E_{\text{translational}} = \frac{3}{2} \mathcal{N} \hat{R} T = \frac{1}{2} \sum_i m_i \, c_i^2 \tag{7}$$

This can be written in terms of average kinetic energy per particle:

$$\widetilde{e}_{\text{translational}} = \frac{E_{\text{translational}}}{N}$$

Where

$$N = \text{Number of Particles Within } \forall.$$

Now, with Equation 7, this becomes:

$$\widetilde{e}_{\text{translational}} = \frac{3}{2}\,\frac{\mathcal{N}}{N}\,\hat{R}\,T = \frac{3}{2}\,\hat{R}\hat{N}\,T \tag{8}$$

Where

$$\hat{N} = \frac{N}{\mathcal{N}} = 6.022140857 \times 10^{23} \qquad \left[\frac{\text{Particles}}{\text{Mole}}\right]$$
$$= \text{ Avogadro's Number}$$

Another way to express Equation 8 is:

$$\widetilde{e}_{\text{translational}} = \frac{3}{2}\,k\,T \tag{9}$$

Where

$$k = 1.3807 \times 10^{-23} \qquad \left[\frac{J}{K}\right]$$
$$= \text{ Boltzmann Constant}$$

Now we can rewrite the pressure in terms of the translational kinetic energy of the particles, using Equation 3, Equation 7 and Equation 9.

$$P = \frac{2}{3}\,\frac{\mathcal{N}\,\hat{N}\,\widetilde{e}_{tr}}{\cancel{V}} = \frac{2}{3}\,\frac{\mathcal{N}\,\hat{N}}{\cancel{V}}\left(\frac{3}{2}\,k\,T\right) = \frac{\mathcal{N}\,\hat{N}\,k\,T}{\cancel{V}} \tag{10}$$

If we multiply both sides by the total mass within the volume, M, and rearrange:

$$P = \frac{\mathcal{N}\,\hat{N}\,k\,T}{\cancel{V}}\left(\frac{M}{M}\right) = \frac{M}{\cancel{V}}\left(\frac{\mathcal{N}\,\hat{N}\,k}{M}\right)T \tag{11}$$

Where

$$\rho = \frac{M}{\cancel{V}} \qquad \text{Mass Density}$$

$$\frac{\hat{R}}{M} = R \qquad \text{Gas Constant Per Unit Mass}$$

Leading us to...

$$P = \rho\,R\,T \tag{12}$$