

Application Problem 1a:

Compute $\int_0^3 x^2 dx$ using the Rectangle Rule with 1, 2, and 3 subintervals Compare your result with the exact solution. Compute the relative error, comment on your solution.

Exact Solution

$$y = \int_0^3 x^2 dx = \left[\frac{1}{3} x^3 \right]_0^3 = 9$$

Rectangle Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x) dx \approx \sum_{i=1}^n h f\left(\frac{x_{i-1} + x_i}{2}\right)$$

Where:

n = Number of Subintervals

$$h = \frac{x_{\text{final}} - x_{\text{initial}}}{n} \quad \text{Length of a Single Subinterval}$$

1 Subinterval :

$$h = \frac{3-0}{1} = 3 \quad \text{therefore} \quad y = 3 \left(\frac{0+3}{2} \right)^2 = 6.75$$

2 Subintervals :

$$h = \frac{3-0}{2} = 1.5 \quad \text{therefore} \quad y = 1.5 \left(\frac{0+1.5}{2} \right)^2 + 1.5 \left(\frac{1.5+3}{2} \right)^2 = 8.4375$$

3 Subintervals :

$$h = \frac{3-0}{3} = 1 \quad \text{therefore} \quad y = 1 \left(\frac{0+1}{2} \right)^2 + 1 \left(\frac{1+2}{2} \right)^2 + 1 \left(\frac{2+3}{2} \right)^2 = 8.75$$

Application Problem 1b:

Compute $\int_0^3 x^2 dx$ using the Trapezoidal Rule with 1, 2, and 3 subintervals Compare your result with the exact solution. Compute the relative error, comment on your solution.

Exact Solution

$$y = \int_0^3 x^2 dx = \left[\frac{1}{3} x^3 \right]_0^3 = 9$$

Trapezoidal Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x) dx \approx \sum_{i=1}^n \frac{h}{2} \left(f(x_{i-1}) + f(x_i) \right)$$

Where:

n = Number of Intervals

$$h = \frac{x_{\text{final}} - x_{\text{initial}}}{n} \quad \text{Length of a Single Subinterval}$$

1 Subinterval:

$$h = \frac{3-0}{1} = 3 \quad \text{therefore} \quad y = \frac{3}{2}(0^2 + 3^2) = 13.5$$

2 Subintervals:

$$h = \frac{3-0}{2} = 1.5 \quad \text{therefore} \quad y = \frac{1.5}{2}(0^2 + 1.5^2) + \frac{1.5}{2}(1.5^2 + 3^2) = 10.125$$

3 Subintervals:

$$h = \frac{3-0}{3} = 1 \quad \text{therefore} \quad y = \frac{1}{2}(0^2 + 1^2) + \frac{1}{2}(1^2 + 2^2) + \frac{1}{2}(2^2 + 3^2) = 9.5$$

Application Problem 1c:

Compute $\int_0^3 x^2 dx$ using the Simpson Rule with 2 and 4 subintervals Compare your result with the exact solution. Compute the relative error, comment on your solution.

Exact Solution

$$y = \int_0^3 x^2 dx = \left[\frac{1}{3} x^3 \right]_0^3 = 9$$

Simpson Rule for Integration

$$\int_{x_{i-1}}^{x_i} f(x) dx \approx \frac{2}{3} I_{\text{rectangular}} + \frac{1}{3} I_{\text{trapezoidal}}$$

Where:

$$I_{\text{rectangular}} = \text{Rectangle Rule Estimate}$$

$$I_{\text{trapezoidal}} = \text{Trapezoidal Rule Estimate}$$

2 Subintervals:

$$h = \frac{3-0}{2} = 1.5 \quad \text{therefore...}$$

$$I_{\text{rectangular}} = 1.5 \left(\frac{0+1.5}{2} \right)^2 + 1.5 \left(\frac{1.5+3}{2} \right)^2 = 8.4375$$

$$I_{\text{trapezoidal}} = \frac{3-0}{2} = 1.5 = \frac{1.5}{2} (0^2 + 1.5^2) + \frac{1.5}{2} (1.5^2 + 3^2) = 10.125$$

As a result

$$y = \frac{2}{3} I_{\text{rectangular}} + \frac{1}{3} I_{\text{trapezoidal}} = \frac{2}{3} (8.4375) + \frac{1}{3} (10.125) = 9$$

This is our exact solution!

Application Problem 2a:

Compute $\int_0^3 x^2 dx$ using the 1-point, 2-point Gauss Quadrature Rule. Compare your result with the exact solution (ie. compute the relative error) and comment on your solution

- **Exact Solution**

$$y = \int_0^3 x^2 dx = \left[\frac{1}{3} x^3 \right]_0^3$$

- **1 Point Gauss Quadrature Rule**

Such that $w_1 = 2$ and $\xi_1 = 0$:

$$\begin{aligned} I &= \int_0^3 x^2 dx = \frac{3}{2} \int_{-1}^1 f\left(\frac{3\xi + 3}{2}\right) d\xi \\ &= \frac{3}{2} \int_{-1}^1 \left(\frac{3\xi + 3}{2}\right)^2 d\xi \\ &= 2 \left(\frac{3(0) + 3}{2}\right)^2 = \frac{18}{4} = 4.5 \end{aligned}$$

- **2 Point Gauss Quadrature Rule**

Such that $w_1 = w_2 = 1$, $\xi_1 = -\frac{\sqrt{3}}{3}$ and $\xi_2 = \frac{\sqrt{3}}{3}$:

$$\begin{aligned} I &= w_1 f(\xi_1) + w_2 f(\xi_2) \\ &= \frac{3}{2} \left(\frac{3(-\frac{\sqrt{3}}{3}) + 3}{2}\right)^2 + \frac{3}{2} \left(\frac{3(\frac{\sqrt{3}}{3}) + 3}{2}\right)^2 \\ &= 9 \end{aligned}$$

Application Problem 2b:

Compute $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx$ using the 1-point, 2-point Gauss Quadrature Rule. Compare your result with the exact solution (ie. compute the relative error) and comment on your solution

- **Exact Solution**

$$y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx = \left[\sin(x) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2$$

First we must do a Coordinate Transformation to get this in the form we want.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) dx = \frac{\pi}{2} \int_{-1}^1 \cos\left(\frac{\pi}{2} \xi\right) d\xi$$

- **1 Point Gauss Quadrature Rule** Such that $w_1 = 2$ and $\xi_1 = 0$:

$$I = \frac{\pi}{2} \left[\cos\left(\frac{\pi}{2} \xi_1\right) w_1 \right]$$

$$= \frac{\pi}{2} \left[\cos\left(\frac{\pi}{2} (0)\right) 2 \right]$$

$$= 3.14$$

• **2 Point Gauss Quadrature Rule**

Such that $w_1 = w_2 = 1$, $\xi_1 = -\frac{\sqrt{3}}{3}$ and $\xi_2 = \frac{\sqrt{3}}{3}$:

$$I = \frac{\pi}{2} \left[w_1 \cos\left(\frac{\pi}{2} \xi_1\right) + w_2 \cos\left(\frac{\pi}{2} \xi_2\right) \right]$$

$$= \frac{\pi}{2} \left[1 \cos\left(\frac{\pi}{2} \left(-\frac{\sqrt{3}}{3}\right)\right) + 1 \cos\left(\frac{\pi}{2} \left(\frac{\sqrt{3}}{3}\right)\right) \right]$$

$$= 1.9352 \quad \approx 3\% \text{ off}$$

The solution alternates between over estimate and under estimate.

Application Problem 3:

Compute $\int_{-1}^1 \int_{-1}^1 \exp(2x) \cdot \ln(3+y) dy dx$ using the 1 * 1, 2 * 2, 3 * 3 Gauss Quadrature rule. Compare your result with the exact solution ($I_{ex} = 7.829967$). Compute the relative error and comment on your solution.

- Exact Solution

$$I_{\text{exact}} = \int_{-1}^1 \int_{-1}^1 e^{2x} \cdot \ln(3+y) dy dx = 7.829967$$

- 1*1 Point Gauss Quadrature Rule Such that $w = w_1 \times w_1 = 4$ and $\xi_1 = \eta_1 = 0$:

$$I = w f(\xi_1, \eta_1)$$

$$= 4 e^0 \cdot \ln(3) = 4.39449$$

- 2*2 Point Gauss Quadrature Rule

Such that $w_k = 1$ and $(\xi_k, \eta_k) = (\pm \frac{\sqrt{3}}{3}, \pm \frac{\sqrt{3}}{3})$:

$$I = \sum w_k f(\xi_k, \eta_k)$$

$$= 1 \exp \left[2 \left(-\frac{\sqrt{3}}{3} \right) \right] \ln \left(3 + \left(-\frac{\sqrt{3}}{3} \right) \right) + 1 \exp \left[2 \left(-\frac{\sqrt{3}}{3} \right) \right] \ln \left(3 + \left(\frac{\sqrt{3}}{3} \right) \right)$$

$$+ 1 \exp \left[2 \left(\frac{\sqrt{3}}{3} \right) \right] \ln \left(3 + \left(\frac{\sqrt{3}}{3} \right) \right) + 1 \exp \left[2 \left(\frac{\sqrt{3}}{3} \right) \right] \ln \left(3 + \left(-\frac{\sqrt{3}}{3} \right) \right)$$

$$= 7.532767$$

Application Problem 4:

Derive the second-order central difference approximations of the first and second derivatives

Application Problem 5:

Let $f(x) = \sin(x)$. Compute $f'(1)$ using the Forward Difference Scheme with $h = 0.25$ and $h = 0.5$. Then Improve your solution by using the Richardson's Extrapolation Scheme. Compare your three approximations with the exact solution.

Forward Difference Scheme

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(\xi) \quad \text{with } \xi \in (x, x+h)$$

- Exact

$$f'(1)_{\text{exact}} = 0.5403$$

- $h=0.25$

$$f'(x) = \frac{\sin(x+0.25) - \sin(x)}{0.25} \rightarrow f'(x) = 0.43055$$

- $h=0.50$

$$f'(x) = \frac{\sin(x+0.5) - \sin(x)}{0.5} \rightarrow f'(x) = 0.312048$$

Richardson Extrapolation

$$a_o = F(h) + \frac{F(h) - F(qh)}{q^p - 1} + O(h^r) \quad \text{or} \quad a_o = \frac{F(qh) - q^p F(h)}{1 - q^p} + O(h^2)$$

For this problem

$$h = 0.25 \quad f'(1) = 0.430551 = F(h)$$

$$h = 0.50 \quad f'(1) = 0.312048 = F(qh)$$

Where $q = 2$ and $p = 1$ since we are using the Forward Difference Scheme and error (h) is linear.

$$a_o = \frac{F(qh) - q F(h)}{1 - q} = \frac{0.312048 - 2(0.430551)}{1 - 2} = 0.54806$$

Application Problem 6:

Let $f(x) = \exp(1 + 3x)$. Compute $f'(2)$ using the Central Difference Scheme with $h = 0.04$ and $h = 0.08$. Then Improve your solution by using the Richardson's Extrapolation scheme. Compare your three approximations with the exact solution.

Central Difference Scheme

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(\xi) \quad \text{with } \xi \in (x, x+h)$$

- Exact

$$f'(2)_{\text{exact}} = 3289.899$$

- h=0.04

$$f'(2) = F(h) = \frac{\exp(1 + 3(2 + 0.04)) - \exp(1 + 3(2 - 0.04))}{2(0.04)} = 3297.801$$

- h=0.08

$$f'(2) = F(qh) = \frac{\exp(1 + 3(2 + 0.08)) - \exp(1 + 3(2 - 0.08))}{2(0.08)} = 3321.574$$

Richardson Extrapolation

$$a_o = F(h) + \frac{F(h) - F(qh)}{q^p - 1} + O(h^r) \quad \text{or} \quad a_o = \frac{F(qh) - q^p F(h)}{1 - q^p} + O(h^2)$$

For this problem

$$h = 0.04 \quad f'(2) = 3297.801 = F(h)$$

$$h = 0.08 \quad f'(2) = 3321.574 = F(qh)$$

Where $q = 2$ and $p = 2$ since we are using the Central Difference Scheme and error (h) is squared

$$a_o = \frac{F(qh) - q^2 F(h)}{1 - q^2} = \frac{3321.574 - 4(3297.801)}{1 - 4} = 3289.877$$