Section 2: Electromagnetics

AE435

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In this section, we will review the basics of charge, electricity, magnetism, and Maxwell equations.

3 Magnetostatics

"Charge in motion creates a magnetic field"

Contents

3.1 Electric Current

Current

$$J \equiv \frac{\partial Q}{\partial t} \qquad \left[\frac{c}{s}\right] = [A] = \text{Amps} \tag{50}$$

We'll use J, although sometimes we see I used for current. Currents can flow in a range of media: metals, semiconductors, fluids, gases and plasmas.

METALS
In metals, there are fixed ionic cores with bound inner ions:

Figure 12

Outer valence electrons get freely traded from ion to ion in response to electric fields. In other words, say we put 10 electrons in one end of a wire and we get 10 electrons out the other end. Those won't be the same 10 electrons we put in though.

GASES AND PLASMAS
In gases and plasmas, both electrons AND ions move:

Figure 13

- Most of the conduction is by electrons, because they're much lighter.
- In thermal motion, both ions and electrons are as likely to cross plane in one direction as another, so no net current.
- Under electric field, drift velocity of species (ions toward cathode, electrons toward anode) gives rise to current.

3.2 Current Density

Consider flux of particles through surface element dS:

Figure 14

Since particle with charge q crossing dS carries incremental current $q\,\vec{v}\cdot\hat{n}$. For N particles per unit volume, current crossing dS is then:

$$\mathrm{d}J = N\,q\,\vec{v}\cdot\hat{n}\,\mathrm{d}S$$

For multiple species, sum over all:

$$\mathrm{d}J = \sum_i N_i\,q_i\,\vec{v}_i\cdot\hat{n}\,\mathrm{d}S$$

Define current density as a vector current per unit area:

Current Density

$$\vec{j} = \sum_i N_i\,q_i\,\vec{v}_i \qquad \left[\frac{A}{m^2}\right] \tag{51}$$

So, the total current across a surface S is:

Total Current Across a Surface

$$J = \int_S \vec{j}\cdot\hat{n}\,\mathrm{d}S \tag{52}$$

3.3 Continuity

Consider volume $\cancel{V}$ enclosed by surface S, within a current density field

Figure 15

Current crossing in or out of $\cancel{V}$ through S is:

$$J = -\int_S \vec{j} \cdot \hat{n}\, \mathrm{d}S = -\int_{\cancel{V}} \nabla \cdot \vec{j}\, \mathrm{d}\cancel{V} \tag{53}$$

Note that positive current refers to what is going into the volume.

We also know from the definition of current (Equation 50), and from definition of volume charge density, and for a steady control volume:

$$J = \frac{\partial Q}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\cancel{V}} \rho_e\, \mathrm{d}\cancel{V} = \int_{\cancel{V}} \frac{\partial \rho_e}{\partial t}\, \mathrm{d}\cancel{V} \tag{54}$$

Combining Equation 54 and Equation 53, we get:

$$\int_{\cancel{V}} \left(\frac{\partial \rho_e}{\partial t} + \nabla \cdot \vec{j} \right) \mathrm{d}\cancel{V} = 0$$

This must be true for any arbitrary volume, therefore the integrand must vanish at every point, giving:

Continuity: Conservation of Charge

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \vec{j} = 0 \tag{55}$$

Where:

$$\rho_e = \text{Charge density}$$

$$\frac{\partial \rho_e}{\partial t} = \text{Time rate of change of some quantity within the volume}$$

$$\nabla \cdot \vec{j} = \text{The cause for that quantity to change with time}$$

The right hand side being equal to zero tells us that nothing else can change the continuity. If the right hand side were to not equal zero, then we would be describing a case where something is changing. For the case of charge, that could be a chemical reaction that changes how continuity behaves.

Sometimes, it may be beneficial to recall how continuity was described in terms of fluid dynamics.

3.4 Ohm's Law

Experiments show that

Ohm's Law

$$\vec{j} = \sigma(\vec{E})\,\vec{E} \tag{56}$$

Where:

$$\begin{aligned} \vec{j} &= \text{Current} \\ \sigma(\vec{E}) &= \text{Conductivity} \\ \vec{E} &= \text{Description} \end{aligned}$$

Where the conductivity, σ, has units of Siemens/m. Note that σ is **NOT** the surface charge density! In common conductors (such as metals, electrolytes and unmagnetized plasmas) $\sigma(\vec{E}) = \sigma$ is a constant. These are called **linear media** or **ohmic media**.

If you take the reciprocal of the conductivity, you get the resistivity,

Resistivity

$$\sigma = \frac{1}{\rho} \qquad \Big[\text{Ohm-m}\Big] \tag{57}$$

Where:

$$\begin{aligned} \rho &= \text{Resistivity} \\ &\rho \text{ is NOT volume change density} \end{aligned}$$

For steady currents, $\frac{\partial \rho_e}{\partial t} = 0$. By continuity then:

$$\nabla \cdot \vec{j} = 0 \tag{58}$$

Now substituting in conductivity:

$$\nabla \cdot \sigma\,\vec{E} = 0 \tag{59}$$

For a homogeneous media, meaning it has constant σ over space:

$$\nabla \cdot \vec{E} = 0 \tag{60}$$

which for electrostatic fields is just Laplace's Equation $\left(\vec{E} = -\nabla\phi\right)$

$$\nabla^2\phi = 0 \tag{61}$$

Given boundary conditions of ϕ or $\vec{j}$ at surfaces of a medium, we can solve for $\vec{j}$ in the medium.

For Copper, if we begin heating it (increasing its temperature), we see that

- Resistance **Increases**
- Power required to heat **Increases**
- Temperature **Increases**

And the cycle continues until it melts.

For Plasma, if we begin heating it (increasing its temperature), we see that

- Resistance **Decreases**
- Power required to heat **Increases**
- Temperature **Increases**

The resistance of plasma goes down as we heat it. This means that if we want to continue heating it, we must continue supplying more and more power.

3.5 Magnetic Field

Recall the description of force on a charge, Coulomb's Law, from section II.1.
Once again, we will look at a **Two-Particle Model**:

The force on q due to q_1 is Equation 1, $\vec{F}_e = \frac{1}{4\pi\epsilon_0}\left(\frac{q\,q_1}{r^2}\right)\frac{\vec{r}}{r}$.

Figure 16

But what if the charges are moving? Let q move at a velocity $\vec{v}$ and q_1 move at a velocity $\vec{v}_1$. In this case, there's now another force on q due to q_1. This force is the **Magnetic Force.**

$$\vec{F}_m = \frac{\mu_o}{4\pi}\left(\frac{q\,q_1}{r^2}\right)\vec{v}\times\left(\vec{v}_1\times\frac{\vec{r}}{r}\right) \tag{62}$$

Where μ_0 is the **Permeability of Free Space** and is proportional to:

$$\frac{\mu_o}{4\pi} = 10^{-7} \qquad \left[\frac{N\cdot s^2}{c^2}\right] \tag{63}$$

By grouping the terms relating to the field charge q_1 , we can define a vector field that represents the force on the test charge q moving at velocity $\vec{v}_1$ such that:

$$\vec{F}_m = \frac{\mu_o}{4\pi}\left(\frac{q\,q_1}{r^2}\right)\vec{v}\times\left(\vec{v}_1\times\frac{\vec{r}}{r}\right)$$

Magnetic Force

$$\vec{F}_m = q\,\vec{v} \times \vec{B} \tag{64}$$

Where

$$\vec{B} = \frac{\mu_o}{4\,\pi}\,\frac{q_1}{r^2}\left(\vec{v}_1 \times \frac{\vec{r}}{r}\right) \qquad \left[\frac{N\,s}{c\,m}\right] = [T] \tag{65}$$

This vector field, $\vec{B}$, is the magnetic induction which is the magnetic field intensity, and has SI units Tesla. Note: 1 Tesla $\approx$ 10,000 Gauss.

Therefore, the total force acting on test charge q is the combination of both the Electric AND Magnetic forces, which is called the Lorentz Force.

Lorentz Force

$$\vec{F}_{\text{total}} = \vec{F}_{\text{E}} + \vec{F}_{\text{M}} = q\,\vec{E} + q\,\vec{v} \times \vec{B} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{66}$$

Now lets consider, **What is the work done by a stationary magnetic field?**

$$\dot{W} = \vec{F}_m \cdot \vec{v} = q\,\vec{v} \times \vec{B} \cdot \vec{v} = 0$$

Essentially, a stationary magnetic field cannot do work/accelerate a fluid/particle/charge.

3.6 Forces on Conductors

Consider a conducting circuit immersed in a B-field. Charges with a number density N move at a velocity $\vec{v}$ along a wire section, $\mathrm{d}\,\vec{l}$, with a cross-sectional area A.

Figure 17

The total charge in this section $\mathrm{d}\,\vec{l}$ is:

$$\mathrm{d}q = N\,q\,A\,|\mathrm{d}\,\vec{l}\,|$$

And so the total magnetic force on this section is:

$$\mathrm{d}\vec{F}_m = \mathrm{d}q\,\vec{v}\times\vec{B} = N\,q\,A\,|\mathrm{d}\,\vec{l}\,|\,\vec{v}\times\vec{B}$$

Since the current runs along the wire, $\vec{v} \parallel \mathrm{d}\,\vec{l}$, then we can swap locations in the vector equation:

$$\mathrm{d}\vec{F}_m = N\,q\,A\,|\vec{v}|\mathrm{d}\,\vec{l}\times\vec{B}$$

Recall that the current density magnitude, (Equation 51: $j = N\,q\,|\vec{v}|$), and that total current (Equation 52: $J = j\,A$), then our equation becomes:

$$\mathrm{d}\vec{F}_m = J\,\mathrm{d}\,\vec{l}\times\vec{B}$$

Integrating over the entire circuit, the **Total Magnetic Force on the Circuit** is:

$$\vec{F}_m = \oint J\,\mathrm{d}\,\vec{l}\times\vec{B} \tag{67}$$

3.7 Biot-Savart Law

Now look at 2 circuits

- A source circuit that produces a B-field
- A test circuit that the B-field exerts force upon

Figure 18

The source circuit element $\mathrm{d}\vec{l}_1$, is at $\vec{r}_1$ and the test circuit element $\mathrm{d}\vec{l}$ is at $\vec{r}_2$

We define the relative position to be: $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$

The force on $\mathrm{d}\vec{l}$ is, from (Equation 62)

$$\mathrm{d}\vec{F}_m(\vec{r}_2) = \frac{\mu_o}{4\pi}\,\frac{N\,q\,|\mathrm{d}\vec{l}\,|\,A}{|\vec{r}_{12}|^2}\,\big(N_1\,q_1\,|\mathrm{d}\vec{l}_1|\,A_1\big)\,\vec{v}\times\left(\vec{v}_1\times\frac{\vec{r}_{12}}{|\vec{r}_{12}|}\right)$$

Where

- $N\,q\,|\mathrm{d}\vec{l}\,|\,A$ is the test circuit charge reacting to the B-field.
- $N_1\,q_1\,|\mathrm{d}\vec{l}_1|\,A_1$ is the source circuit charge creating the B-field.

As before $\vec{v}\parallel \mathrm{d}\vec{l}$ and $\vec{v}_1\parallel \mathrm{d}\vec{l}_1$ so we can swap their places

$$\mathrm{d}\vec{F}_m(\vec{r}_2) = \frac{\mu_o}{4\pi}\,\frac{N\,q\,|\vec{v}|\,A}{|\vec{r}_{12}|^2}\,(N_1\,q_1\,|\vec{v}_1|\,A_1)\mathrm{d}\vec{l}\times\left(\mathrm{d}\vec{l}_1\times\frac{\vec{r}_{12}}{|\vec{r}_{12}|}\right) \tag{68}$$

- Test circuit current $J = N\,q\,|\vec{v}|\,A$
- Source circuit current, $J_1 = N_1\,q_1\,|\vec{v}_1|\,A_1$

Again, integrate over both circuits to get the **Total Magnetic Force on the Test Circuit due to the Source Circuit Current:**

$$\vec{F}_m = \frac{\mu_o}{4\pi} J\, J_1 \oint_1 \oint_2 \frac{\mathrm{d}\vec{l} \times (\mathrm{d}\vec{l}_1 \times \vec{r}_{12})}{|\vec{r}_{12}|^3} \tag{69}$$

Now take out everything in $\mathrm{d}\vec{F}_m$ that is not associated with the test circuit.

Define B created by the source circuit:

$$\mathrm{d}\vec{B}(\vec{r}_2) = \frac{\mu_o}{4\pi} \frac{J_1}{|\vec{r}_{12}|^3} \mathrm{d}\vec{l}_1 \times \vec{r}_{12}$$

So that we get a Lorentz force

$$\mathrm{d}\vec{F}_m(\vec{r}_2) = J\, \mathrm{d}\vec{l} \times \mathrm{d}\vec{B}(\vec{r}_2)$$

We can also write J_1 as a distributed

$$J_1 = \int_S \vec{j}_1(\vec{r}_1) \cdot \mathrm{d}\vec{S}_1$$

Such that

$$\vec{B}(\vec{r}_2) = \frac{\mu_o}{4\,\pi} \int_S \oint \frac{\vec{j}_1 \times \vec{r}_{12}}{|\vec{r}_{12}|^3}\, \mathrm{d}\vec{l}_1 \mathrm{d}\vec{S}_1$$

Or since $\mathrm{d}\vec{l}_1 \mathrm{d}\vec{S}_1 = \mathrm{d}V\!\!\!/$

Biot-Savart Law

$$\vec{B}(\vec{r}_2) = \frac{\mu_o}{4\,\pi} \int_{V\!\!\!/} \frac{\vec{j}_1 \times \vec{r}_{12}}{|\vec{r}_{12}|^3}\, \mathrm{d}V\!\!\!/ \tag{70}$$

If we want to find the B-field anywhere, we have to integrate the current everywhere throughout the entire volume.

Results of Biot-Savart law can show that...

- The monopole law follows from Biot-Savart such that

$$\nabla \cdot \vec{B} = 0 \tag{71}$$

- For steady-state currents where $\nabla \cdot \vec{j} = 0$

$$\nabla \times \vec{B} = \mu_o \vec{j} \tag{72}$$

This is **Ampere's Law** (in a static sense), which shows that

- $\vec{B}$ curls around $\vec{j}$
- Current is the source of $\vec{B}$

Equation 72 shows that the B-Field along a loop is due to the current of the loop enclosed. Compare this with Gauss's Law ($\nabla \cdot \vec{E} = \frac{\rho_e}{\epsilon_o}$) which shows that charge is the source of $\vec{E}$.

We can integrate Ampere's law over a surface

$$\int_S \nabla \times \vec{B} \cdot \hat{n}\, \mathrm{d}A = \mu_o \int_S \vec{j} \cdot \hat{n} \mathrm{d}A$$

And apply Stokes's theorem on a closed curve C around surface S

$$\oint_c \vec{A} \cdot \mathrm{d}\,\vec{l} = \int_S \nabla \times \vec{A} \cdot \hat{n}\, \mathrm{d}A \tag{73}$$

To get the integral form

$$\oint_c \vec{B} \cdot \mathrm{d}\,\vec{l} = \mu_o \int_S \vec{j} \cdot \hat{n}\, \mathrm{d}A = \mu_o\, J \tag{74}$$

Or integrating current density through the surface:

Example: Calculate the magnetic field, $\vec{B}$, from a straight wire with current, J, from radius, $\vec{r}$. Utilize the right hand rule such that the thumb points in the direction of current.

$\vec{B}$ is radially symmetric

$$\vec{B} = B_\theta(r)\hat{\theta}$$

On circle centered at radius r from wire

$$\oint \vec{B} \cdot \mathrm{d}\,\vec{l} = \mu_o J = 2\pi r B(r)$$

$$\vec{B}(r) = \frac{\mu_o J}{2\pi r}\,\hat{\theta}$$

Figure 19