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MATH 55: Discrete Mathematics

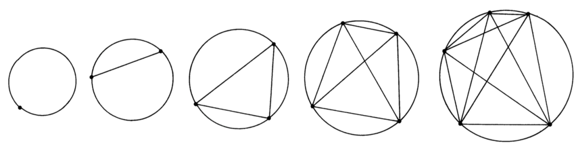
June 18 — Summer 2018

Lecture 0: Entr'acte

1 Proof by examples doesn't always work

Let's begin with a few problems.

Problem 1. *How many regions is each circle split into? See if you can find a pattern.*



Problem 2. *For $n = 2, 3, 4, \dots,$ check: is n prime? Is $2^n - 1$ prime? Find a pattern.*

Problem 3. *Can every even number greater than 2 be written as the sum of two primes?*

For Problem 1, we can see that the first five circles have 1, 2, 4, 8, and 16 regions. For Problem 2, we get the following table:

n	2^{n-1}	Is n prime?	Is $2^n - 1$ prime?
2	3	Yes	Yes
3	7	Yes	Yes
4	15	No	$15 = 3 \cdot 5$
5	31	Yes	Yes
6	63	No	$63 = 7 \cdot 9$
7	127	Yes	Yes
8	255	No	$255 = 5 \cdot 51$
9	511	No	$511 = 7 \cdot 73$
10	1023	No	$1023 = 3 \cdot 341$

As for Problem 3, we can write out a few problems easily enough.

$$
\begin{aligned}
4 &= 2 + 2, \\
6 &= 3 + 3, \\
8 &= 3 + 5, \\
10 &= 5 + 5, \\
12 &= 5 + 7, \\
14 &= 3 + 11, \\
16 &= 5 + 11.
\end{aligned}
$$

There seem to be some patterns at play here, but will they carry on forever? Maybe, maybe not. We can at least pose a few *conjectures*:

Conjecture 1. *The n-th circle in Problem 1 will be split into 2^{n-1} regions.*

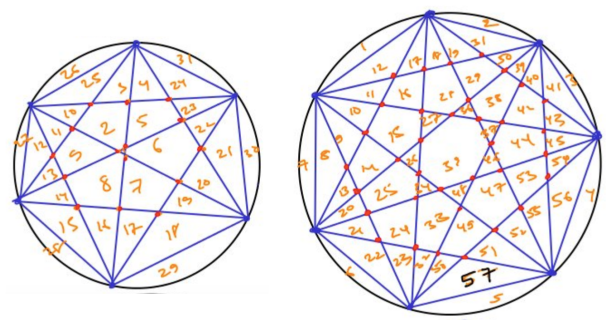
Conjecture 2. *If n is prime, then $2^n - 1$ is prime.*

Conjecture 3. *If n is not prime, then $2^n - 1$ is not prime.*

Conjecture 4 (Goldbach's Conjecture)**.** *Every even number can be written as the sum of two primes.*

Let's look at these problems a little more closely to see if these patterns really do hold.

If we go a bit farther with the circles in Problem 1, the pattern suddenly breaks down: the next two circles have only 31 and 57 regions, respectively, and the next few circles have just 99, 163, and 256 regions.



The pattern in Problem 2 breaks down for $n = 11$, since $2^{1}1 - 1 = 2047 = 23 \cdot 89$, so Conjecture 2 is false. It turns out that Conjecture 3 is true, but by now you should be a little wary.

What about the last conjecture? The problem was posed in a series of letters in 1742 between Christian Goldbach and Leonhard Euler, and Euler considered it to be a fully certain theorem despite being unable to prove it. To this day nobody knows for certain if the conjecture is true, although it has been verified for all numbers up to 4,000,000,000,000,000,000.

Remark. In the lecture it was mistakenly claimed that Goldbach's conjecture was one of the seven Millennium Problems. This is incorrect, although it is true that the conjecture is one of the most famous unsolved problems in mathematics.

So what do we learn from all of this?

Lesson 1. *Checking a small number of examples does not prove that a general pattern will hold!*

2 Proof by authority doesn't always work

Fine, then. How else do we know that something is true? How about the Pythagorean Theorem?

Theorem 1 (Pythagorean Theorem)**.** *If a, b, and c are the legs and hypotenuse (respectively) of a right triangle, then $a^2 + b^2 = c^2$.*

How do we know this is true? Well, you may well have seen a proof of it in a previous math class, but the first time you learned it was probably a proof by authority: someone told you it was true, and you believed them. We get quite a lot of information this way, but it's far from perfect.

One example, coming from an attempt to extend the Pythagorean Theorem:

Theorem 2 (Fermat's Last Theorem)**.** *For all $n \geq 3$, there are no positive integer solutions to $a^n + b^n = c^n$.*

Posing this claim in 1637, Pierre de Fermat wrote in the margins of another book "I have discovered a truly remarkable proof of this theorem which this margin is too small to contain." It is somewhat doubtful that his proof was correct, if it existed at all, since the problem stood for over three hundred and fifty years. A full proof was finally published by Andrew Wiles in 1995, but it used modern techniques that Fermat almost certainly did not know about.

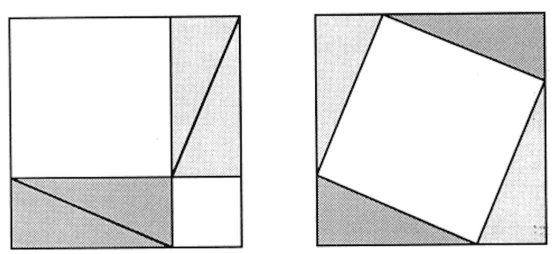
What's more, even if you did have a reliable authority to tell you that a pattern existed, the answer wouldn't necessarily be satisfying or illuminating. For example:

Theorem 3. *The n-th circle from Problem 1 is divided into $\frac{n^4-6n^3+23n^2-18n+24}{24}$ regions.*

The answer is correct, but it doesn't help you understand *why* it's correct, and it doesn't give you any indication of how to solve similar problems. We'll come back to this later when we discuss proof techniques and problem solving.

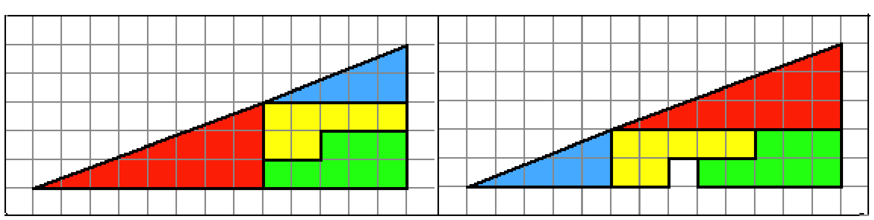
3 Proof by pictures doesn't always work

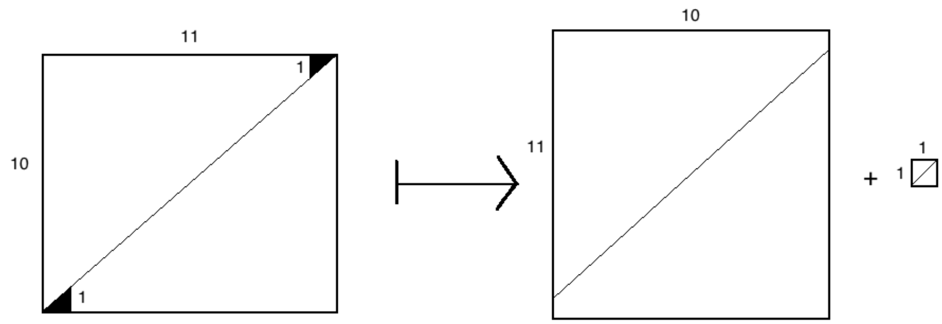
Well, how about picture proofs? The Pythagorean Theorem has a nice one:



Exercise 1. *Explain why this picture proves the Pythagorean Theorem.*

But we can also come up with pictures that appear to prove false things. The following two pictures appear to prove that $32.5 = 31.5$ and that $110 = 111$! The latter picture was proposed by Martin Gardner as a method for generating infinite gold: if you have a 10×11 gold slab, you can slice it according to the diagram, and after rearranging you will have an 11×10 slab and a 1×1 piece left over!





Well, those last two claims seem intuitively wrong, so we know that something has to be amiss. But that just raises the question: if the last two pictures were "cheating" somehow, how do we know that the first picture wasn't? Maybe we just believed it because we already thought that the Pythagorean was true in the first place!

- Birthday paradox.
- HHH or THH first?
- 10 flips: probability of getting 3 heads or 3 tails in a row?

4 Intuition definitely doesn't always work

How good is intuition, anyway? As it turns out, pretty terrible. Consider the following problems:

Problem 4. *There are about 40 people in the room and about 365 days in a year. What are the odds that two people in the room share a birthday?*

a) Less than 1/9 *b) About 1/9* *c) About 1/3* *d) About 1/2* *e) More than 1/2*

Problem 5. *Flip a coin 10 times in a row. What are the odds that you get at least three heads in a row or at least three tails in a row?*

a) About 1/10 *b) About 1/4* *c) About 1/2* *d) About 2/3* *e) About 4/5*

Problem 6. *Flip a coin until either of the sequences HHH or THH comes up. Which is more likely to come up first, or are they equally likely?*

For Problem 4 the answer turns out to be about 9/10, and for Problem 5 it's a little more than 8/10–both much higher then most people would guess. And testing Problem 6 during the lecture had 4 people flipping HHH first and 24 flipping THH first, even though the general consensus was that they would be equally likely!

5 Total precision

So if examples, pictures, authority, and intuition are all potentially unreliable, then how do we learn any mathematical truths at all? By using deduction, that's how! One of the goals of this course will be to start with premises we believe beyond a shadow of a doubt to be true, then proceed using pure logical reasoning to ironclad conclusions.

But one thing we need to do first is get our definitions and language sorted out. First consider the statement

Claim 4. *"We are good at math."*

Is this statement true? Well, we can't say whether it is unless we know what "we" means. It could be the royal "we", meaning just the speaker. It could mean the speaker and the listener, but it could also mean the speaker and *not* the listener: if A tells B "we went to the movies yesterday", A could be referring to A and B, but could also mean A and C but not B. Or "we" could be a polite way of referring to the audience only, as when a professor tells the class that "we need to turn in our homework on time"!

So if we want to say for certain whether something is true, we first need exact definitions for everything, or at least ones that everyone can agree on. This has the effect of sacrificing all of the nuance that comes with natural language in favor of precision.

For example, consider the following problem.

Problem 7. *At what point does a heap of sand, removing one grain at a time, cease to be a heap?*

If you are a linguist or a philosopher, you could spend quite a bit of time pondering the concept vagueness in human language. If you are a mathematician, this is an obstacle to be removed, perhaps in the following manner.

Definition 1. A **heap** is a pile of sand with at least one million grains.

Problem solved! Kind of. At any rate, in our quest for total rigor, we should keep in mind this lesson:

Lesson 2. *Mathematical language is not the same as natural language!*