538 Riddler Challenge

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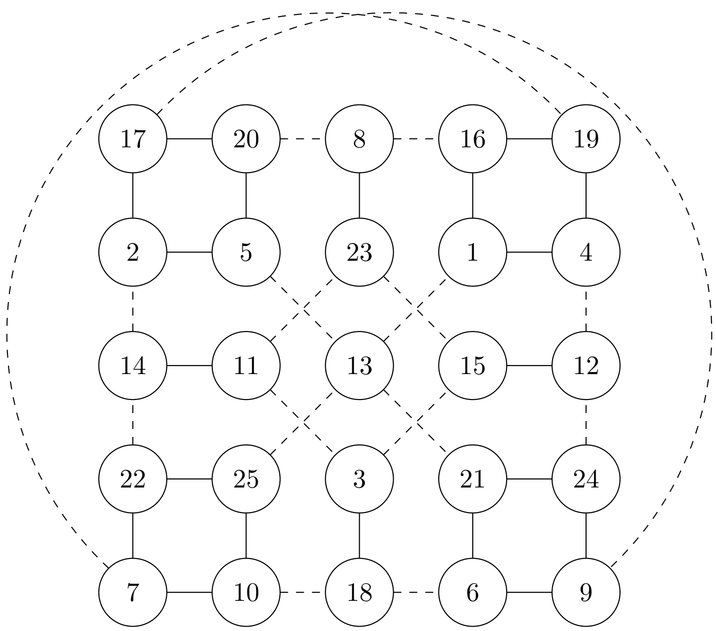
1 The Problem

The challenge is to visit every square in a 5×5 grid exactly once, where we are allowed to jump three squares horizontally or vertically or two squares diagonally.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Let's represent the grid by a graph where the vertices are the squares of the grid, and where two vertices are connected if we can jump from one to the other. In the graph below, the dashed lines represent diagonal moves and the solid lines represent horizontal moves.



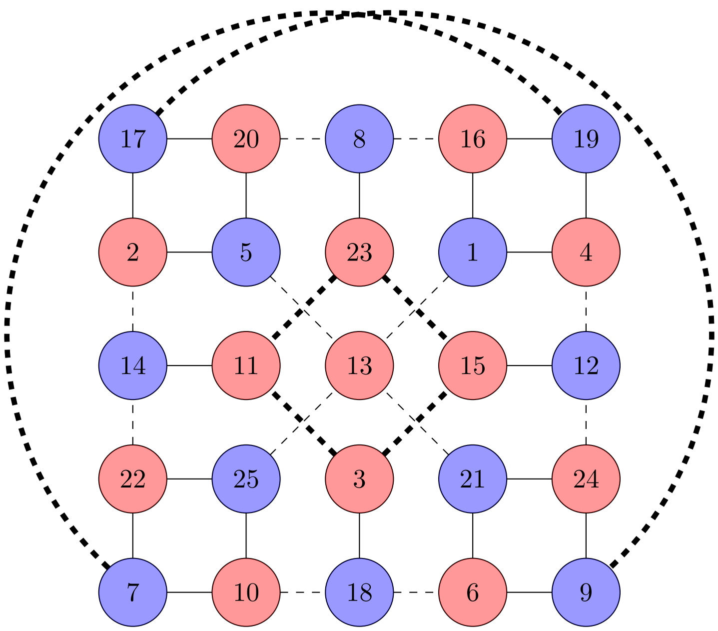
2 Solution

From here we could try to find a solution by trial and error, but let's try to be a bit more systematic, and solve a more challenging problem while we're at it: can we visit every square on the board and, with one extra move, return to our original position?

If we could only move *one* square at a time horizontally or vertically and were not allowed to move diagonally, then the answer would be no. To see this, color the grid like a checkerboard. If we want to visit each square once and return to our starting spot, then we will need 25 moves in all. But since with each move our color alternates red/blue/red/... and since 25 is odd, after 25 moves the color of our current square must be different from the color of the square we started on.

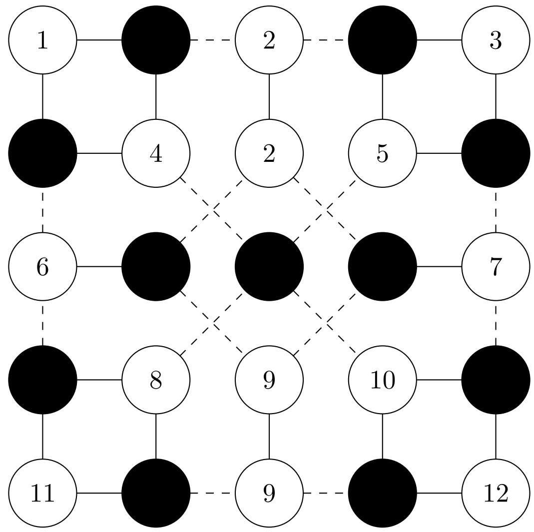
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

With the graph for the Riddler problem, things are a little different. We can color the board mostly as before, but this time there are six possible moves (shown in bold) that do not lead to a change in color. If we want to return to our starting position after 25 moves, we must use an odd number of these bold moves.



First, it is not possible to use only a single move out of the middle four if we hope to complete a full cycle. If we traveled along the route 18-3-15-12, for example, then we would also be forced to use the route 11-23 because otherwise squares 11 and 23 would be stranded. If we use two moves, they must appear opposite each other and not adjacent, since (for example) if we traveled the route 14-11-3-15-12 then square 23 would be stranded. We also cannot use all four of these moves, since that would create a closed loop on its own.

Second, if we hope to find a cycle then we must use at least one of the two long arcs 17-9 or 7-19. If we were to remove the arcs from this graph, resolving to do without them, then we could split the graph into 12 separate pieces by removing only 11 vertices. But if the graph did have a cycle, then cutting that cycle in 11 places would split it into at most 11 pieces. Thus the graph with the long arcs removed has no cycle.



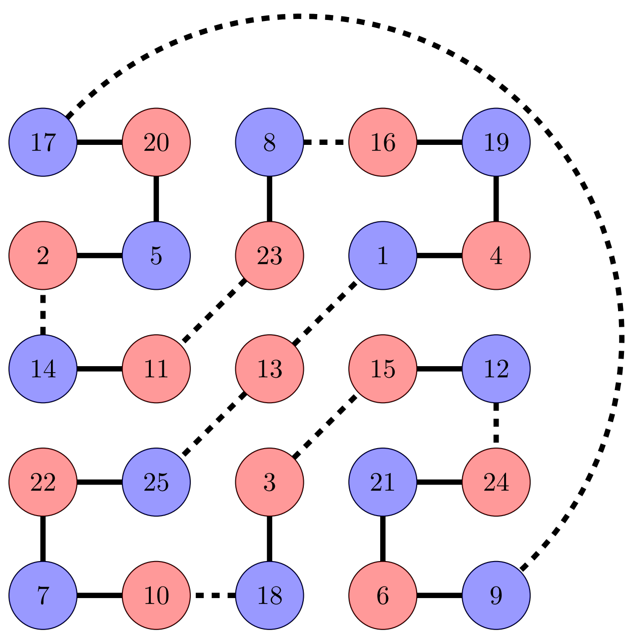
Therefore, any tour around the graph must involve one of the following:

1. One of the two long arcs and two non-adjacent segments from the middle square, or
2. Both outer arcs and three segments from the middle square.

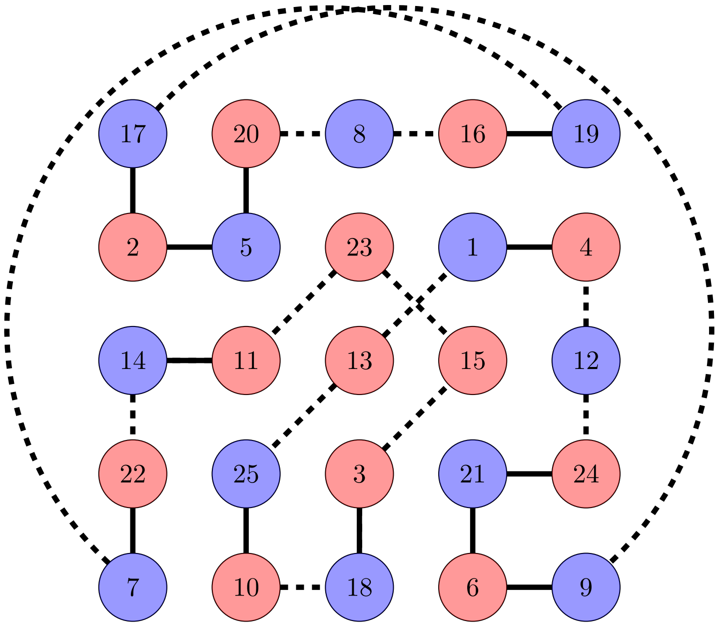
There are several solutions of both types, and we present one of each.

2.1 Solution 1

This solution has a nice 180-degree rotational symmetry.



2.2 Solution 2



3 Non-Planarity

As an aside, it's interesting to note that we cannot draw the graph with no edge crossings, no matter how we might try. As proof, we can pick out two sets of three nodes (colored red and blue) and show that each red node can be connected to each blue node along non-intersecting paths. Therefore, as with the better-known Three Utilities puzzle, there is no way to draw this graph so that no two edges cross.