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Small weak epsilon-nets in families of rectangles

Bachelor's Thesis
Computer Science - IT Analyst

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August 2020

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Abstract

Let P be a set of n points in $\mathbb{R}^2$, $\varepsilon > 0$. A set of points Q is called a *weak ε-net* for P with respect to a family $\mathcal{S}$ of objects (e.g. axis-parallel rectangles or convex sets) if every set from $\mathcal{S}$ containing more than εn points of P contains a point from Q.

Let $\mathcal{R}$ be the family of all axis-parallel rectangles in $\mathbb{R}^2$ and $\varepsilon_k^{\mathcal{R}}$ be the smallest real number such that for any P there exists a weak $\varepsilon_k^{\mathcal{R}}$-net of size k. The work by Aronov et al. suggests that the inequality $\varepsilon_k^{\mathcal{R}} \leq \frac{2}{k+3}$ may hold. In this work we present the complete proofs of this inequality for $k = 1, \ldots, 5$ and prove that this bound is tight for $k = 1, 2, 3$.

Besides, it is not clear how to construct optimal nets. Langerman conjectured that k-point $\frac{2}{k+3}$-nets can be chosen from some specific set of points which are the intersections of grid lines, where the grid is of size $k \times k$. We give counterexamples to this conjecture for nets of size 3 through 6.

Contents

Introduction

Let P be a set of n points in $\mathbb{R}^2$. A point q (not necessarily in P) is called a *centerpoint* of P if each closed half-plane containing q contains at least $\lceil \frac{n}{3} \rceil$ points of P, or, equivalently, any convex set that contains more than $\frac{2}{3}n$ points of P must also contain q.

Each finite point set in $\mathbb{R}^2$ has at least one centerpoint. Besides, it is known that the constant $\frac{2}{3}$ is the minimum possible — see, e.g., [1]. This constant can be improved if we use some other small number of points — clearly, the more points we use, the smaller this constant is. However, for most families of objects (e.g. convex sets, axis-parallel rectangles) it is still not clear what the optimal constant is and even the optimal order of magnitude is not known. Let us consider the following definitions:

Definition 1. Let P be an n-point set in $\mathbb{R}^2$ and $\varepsilon > 0$ be fixed. Consider a family $\mathcal{S}$ of sets in $\mathbb{R}^2$. A set of points $Q \subset \mathbb{R}^2$ is called a weak ε-net for P with respect to $\mathcal{S}$, if for any $S \in \mathcal{S}$ with $|S \cap P| > \varepsilon n$, we have $S \cap Q \neq \varnothing$. Further, if $Q \subseteq P$, then Q is called a *strong* ε-net for P with respect to $\mathcal{S}$.

It is common to use weak inequality when defining ε-nets, but a strong one is more suitable for our problem.

The notion of ε-net first appeared in [2]. Note that if we fix $|Q| = 1$, $\varepsilon = \frac{2}{3}$, $\mathcal{S}$ — the family of all convex sets, then in Definition 1 the set Q consists of the centerpoint of P. Since in this work we do not consider strong epsilon-nets, we will simply write "net" instead of "weak net".

One common question about ε-nets is to find the smallest net for any set of points P of size n. That is, to provide the best possible upper bound on the size of an ε-net using function of ε. Aronov et al. [3] proposed the following definition, which reverses this dependency: fix the size of a net, and try to find the smallest epsilon which can be achieved with a net of this size:

Definition 2. Let $i \in \mathbb{N}$ and $\mathcal{S}$ be a family of sets in $\mathbb{R}^2$. Let $\varepsilon_i^{\mathcal{S}} \in [0, 1]$ denote the smallest real number such that for any finite point set $P \subset \mathbb{R}^2$ there exists an i-point $\varepsilon_i^{\mathcal{S}}$-net for P with respect to $\mathcal{S}$.

For example, $\varepsilon_1^{\mathcal{C}} = \frac{2}{3}$ where $\mathcal{C}$ is the family of all convex sets in the plane.

Now we briefly mention some of the known bounds on the size of an ε-net for any P and $\mathcal{S}$ which use only ε and other parameters depending on $\mathcal{S}$. For most natural families of sets, such as convex sets, axis-parallel rectangles, etc. the best known lower bound is the trivial $\Omega(\frac{1}{\varepsilon})$. The best known general upper bound for any set system $\mathcal{S}$ is $O(\frac{d}{\varepsilon} \ln \frac{1}{\varepsilon})$, provided that the VC-dimension[1] of the range space $(\mathbb{R}^2, \mathcal{S})$ is d [2, 4].

Let $\mathcal{R}$ denote the set of all axis-parallel rectangles in the plane. Aronov, Ezra and Sharir [5] proved that smallest ε-nets for $\mathcal{R}$ are of size $O(\frac{1}{\varepsilon} \log\log \frac{1}{\varepsilon})$ which remains the best known upper bound by today. This bound implies that $\varepsilon_k^{\mathcal{R}} = O(\frac{\log\log k}{k})$. In another work Aronov et al. [3] remark that $\varepsilon_k^{\mathcal{R}} \leq \frac{2}{k+3}$ for $1 \leq k \leq 5$, and it would be interesting to check whether this inequality holds for any k. Some general results regarding upper bounds on $\varepsilon_k^{\mathcal{R}}$ are also provided in [3], but they fall asymptotically far behind the $O(\frac{\log\log k}{k})$ bound, which in turn is weaker than the conjectured $\frac{2}{k+3}$ bound. If the latter bound is true, then for $\mathcal{R}$ the size of an optimal ε-net is $O(\frac{1}{\varepsilon})$.

Besides the bound itself, it is also interesting whether ε-nets have some structure, that is, maybe we can effectively choose a net from some relatively small set of possible positions, which are easily defined by the point set P itself. It was shown in [3] (for $k = 2$) and claimed in [6] (for $k = 4, 6$) that $\frac{2}{k+3}$-nets can be chosen from a set of grid points which will be defined in Section 2. Langerman [7] has even conjectured that this holds for any k — see Conjecture 2.1.

In this work we only consider nets with respect to the family $\mathcal{R}$ of all axis-parallel rectangles in the plane. The aim of this thesis is to present known results which match the above-mentioned $\frac{2}{k+3}$ bound for small ε-nets for $\mathcal{R}$, as also to disprove Conjecture 2.1.

In Section 1 we reduce the problem to considering only point sets in general position which satisfy certain divisibility constraints, which simplifies the problem. We then present bounds for ε-nets of size $1, \ldots, 5$. Tight bounds of $\frac{2}{k+3}$ are given for $\varepsilon_k^{\mathcal{R}}$ for $k = 1, 2, 3$, and upper bounds of $\frac{2}{k+3}$ are given for $k = 4, 5$.

In Section 2 we discuss and disprove the conjecture that $\frac{2}{k+3}$-nets can be chosen from the set of intersections of the grid lines. Counterexamples for nets of size $3, 4, 5, 6$ are provided. In particular, a result of [6] is shown to be wrong. Despite we do not provide general construction, the ones we give strongly suggest that analogous counterexamples exist for all $k \geq 3$.

[1] The VC-dimension is an indicator of combinatorial complexity of geometric set systems. See e.g. [1].

1 Weak epsilon-nets of size 1 to 5

In this section we prove that it holds $\varepsilon_k^{\mathcal{R}} \leq \frac{2}{k+3}$ for $1 \leq k \leq 5$, and $\varepsilon_k^{\mathcal{R}} \geq \frac{2}{k+3}$ for $1 \leq k \leq 3$. Proofs for $k = 1, 2$ and the lower bound for $k = 3$ follow the proofs from [3]. Our proof for $k = 4$ is modified from [6]. Upper bounds for $k = 3, 5$ are given in [3] and are stated there as corollaries of a more general statement, but we prefer proving them directly. We remark that the best known lower bounds for $k = 4, 5$ are $\varepsilon_4^{\mathcal{R}} \geq \frac{5}{16}$ and $\varepsilon_5^{\mathcal{R}} \geq \frac{1}{6}$ — see [3].

From now on, we say that a point set P is in *general position* if no two points of P have the same x- or y-coordinate. We remark that it somewhat differs from the standard meaning of the term "general position". Similarly, we write "rectangle" instead of "axis-parallel rectangle".

We begin by arguing that when proving any result on ε-nets of fixed size with respect to $\mathcal{R}$, we can assume that the given point set P is in general position, and $|P|$ is divisible by any fixed number we want.

1.1 On general position and divisibility

In this subsection we prove the following useful lemma:

Lemma 1.1. *Given any set of n points $P \subset \mathbb{R}^2$, $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists a set P' of nq points with the following properties:*

(i) The points of P' are in general position (that is, no two points have the same x- or y-coordinate).

(ii) For any k, if there exists a k-point ε-net for P', then there exists a k-point ε-net for P.

Proof. Let P be a set of n points. Let X, Y be the sets of projections of points of P on the x- and y-axis respectively. Let the set D be defined as follows: for each pair $p, q \in P$, if the distance $|p_x - q_x|$ is non-zero, add $|p_x - q_x|$ to the set D, and do the same for y-distances. Choose δ to satisfy $0 < \delta < \frac{\min D}{2}$. In particular, all δ-neighborhoods are pairwise disjoint. We construct a new set P' as follows: for every point $p \in P$ consider its δ-neighborhood. Remove the point p and place q new points in that neighborhood. Those new points form a set P'. Observe, that one can always choose these points in such a way that no two points inside P' have the same x- or y-coordinate, that is, P' is in general position. Moreover, due to the definition of D, the projections of any two δ-neighborhoods on the x- or y-axis intersect if and only if the original points — centers of those neighborhoods — have the same x- or y-coordinate, respectively.

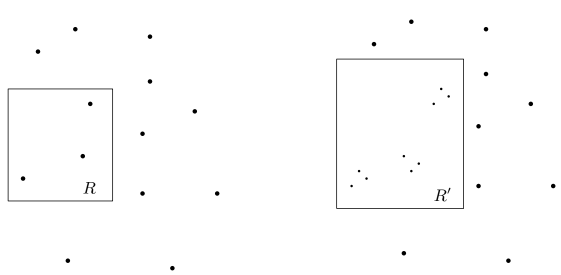

Figure 1: Rectangle R in P corresponds to R' in P' ($q = 3$).

For the proof of the second property, suppose that some net Q' is an ε-net for the set P'. We now construct an ε-net Q for P. Let $q' = (x', y') \in Q'$ and let $q = (x, y)$ be the point in Q, corresponding to q', defined as follows: if there exists $x_0 \in X$ such that $x' \in [x_0 - \delta, x_0 + \delta]$ we set $x = x_0$, otherwise we set $x = x'$. We do the same for y. We now show that Q is indeed an ε-net for P.

To see this, let R be a rectangle that contains more than ε-fraction points of P. This rectangle corresponds to a rectangle R', containing more than ε-fraction of points of P' — it suffices to take R' as the smallest rectangle which contains all points of P' placed instead of points inside of R — see Figure 1. Since Q' is an ε-net for P', there is some point $q' \in Q'$ which lies inside R'. Observe that because we have chosen R' to be the smallest rectangle, the point $q \in Q$ corresponding to q' must lie inside R, which shows that none of the rectangles containing more than the ε-fraction of points of P avoids Q. □

Using this lemma, from now on, whenever we consider a set of points P, we assume it is in general position. In nearly all proofs we will also assume that $|P|$ is divisible by a certain number.

1.2 Nets of size 1, 2, 3 and 5

Theorem 1.2. $\varepsilon_1^{\mathcal{R}} = \frac{1}{2}$.

Proof. Let P be a set of n points where n is divisible by 2. Let q denote the only point of a net. For the lower bound, observe that one of the two open half-spaces determined by the vertical line passing through q contains at least $\frac{n}{2} - 1 = (\frac{1}{2} - \frac{1}{n}) \cdot n$ points. Moreover, there exists a rectangle which contains all these points and avoids q. Since n can be chosen arbitrarily large, the constant in parentheses can be arbitrarily close to $\frac{1}{2}$.

For the upper bound, observe that since n is even, we can take q as the intersection of two lines each dividing the set P into two equal halves, horizontally and vertically. □

Theorem 1.3. $\varepsilon_2^{\mathcal{R}} = \frac{2}{5}$.

Proof. We first prove the lower bound, that is $\varepsilon_2^{\mathcal{R}} \geq \frac{2}{5}$. For a contradiction, suppose $\varepsilon_2^{\mathcal{R}} < \frac{2}{5}$. Let h_1, h_2, v_1 and v_2 be two horizontal and two vertical lines, respectively. These lines partition the plane into nine rectangles, which we denote by $A_{i,j}$ for $1 \leq i, j \leq 3$ as illustrated in Figure 2. For n a multiple of 5, place $\frac{n}{5}$ points in each of the rectangles $A_{1,1}, A_{1,3}, A_{2,2}, A_{3,1}, A_{3,3}$. Observe that each of the outer strips $A_{1,1} \cup A_{1,2} \cup A_{1,3}$, $A_{3,1} \cup A_{3,2} \cup A_{3,3}$, $A_{1,1} \cup A_{2,1} \cup A_{3,1}$ and $A_{1,3} \cup A_{2,3} \cup A_{3,3}$ contains exactly $\frac{2n}{5}$ points, so our two-point $\varepsilon_2^{\mathcal{R}}$-net Q should contain one point inside of each of the four strips. However, any three of these strips have empty intersection, thus $Q \subset A_{1,1} \cup A_{3,3}$ or $Q \subset A_{1,3} \cup A_{3,1}$. Assume w.l.o.g. the latter case. Then $A_{1,1} \cup A_{1,2} \cup A_{2,1} \cup A_{2,2}$ is an axis-parallel rectangle containing exactly $2n/5$ points of P and avoiding Q (see Figure 2), which contradicts the assumption that $\varepsilon_2^{\mathcal{R}} < \frac{2}{5}$.

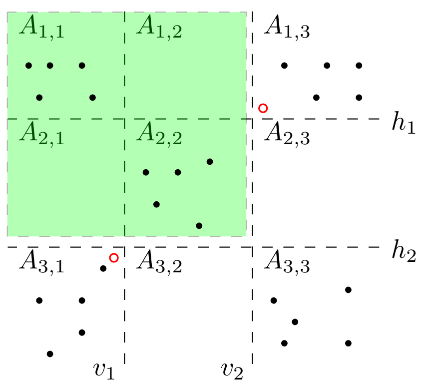

Figure 2: Red circles are points of Q, green rectangle contains $\frac{2n}{5}$ points and avoids Q.

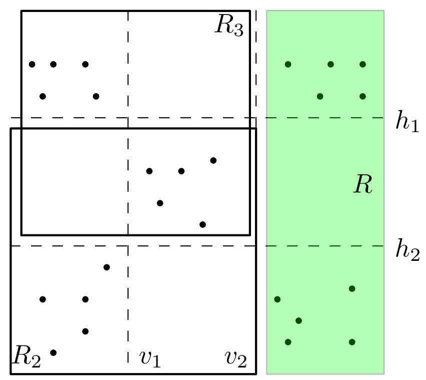

Figure 3: Both R_2 and R_3 contain more than $2n/5$ points of P, R contains exactly $2n/5$.

We proceed to the proof of the upper bound, that is $\varepsilon_2^{\mathcal{R}} \leq \frac{2}{5}$. Suppose we are given a set P of n points where n is a multiple of 5. Let v_1 and v_2 be two vertical lines such that to the left of v_1

and to the right of v_2 there are exactly $2n/5$ points of P, and there are exactly $n/5$ points of P in between these lines. Let h_1 and h_2 be horizontal lines defined in a similar manner. We now prove that one of the sets $Q_1 = \{h_1 \cap v_1, h_2 \cap v_2\}$ and $Q_2 = \{h_1 \cap v_2, h_2 \cap v_1\}$ is a $\frac{2}{5}$-net for P.

For a contradiction, suppose that neither Q_1 nor Q_2 is a $\frac{2}{5}$-net for P. Since Q_1 is not a $\frac{2}{5}$-net for P, at least one of the rectangles $R_1 = A_{1,2} \cup A_{1,3} \cup A_{2,2} \cup A_{2,3}$ and $R_2 = A_{2,1} \cup A_{2,2} \cup A_{3,1} \cup A_{3,2}$ contains more than $\frac{2n}{5}$ points of P. Similarly, because Q_2 is not a $\frac{2}{5}$-net for P, at least one of the rectangles $R_3 = A_{1,1} \cup A_{1,2} \cup A_{2,1} \cup A_{2,2}$ and $R_4 = A_{2,2} \cup A_{2,3} \cup A_{3,2} \cup A_{3,3}$ contains more than $\frac{2n}{5}$ points of P. For the sake of the argument, suppose that R_2 and R_3 each contain at least $2n/5$ points of P. Note that the rightmost strip R contains exactly $2n/5$ points — see Figure 3. Finally, observe that the intersection of R_2 and R_3 contains at most $n/5$ points, because it contains only points from the middle horizontal strip (between h_1 and h_2), which contains exactly $\frac{n}{5}$ points. It follows that R, R_2 and R_3 altogether contain strictly more than $\frac{2n}{5} + (\frac{2n}{5} + \frac{2n}{5} - \frac{n}{5}) = n$ points, a contradiction. $\square$

Theorem 1.4. $\varepsilon_3^{\mathcal{R}} = \frac{2}{6}$.

Proof. We first prove that $\varepsilon_3^{\mathcal{R}} \geq \frac{2}{6}$. For a contradiction, suppose that $\varepsilon_3^{\mathcal{R}} < \frac{2}{6}$. Consider lines h_1, h_2, v_1, v_2 and rectangles $A_{i,j}$, defined similarly as in the proof of Theorem 1.3 — see Figure 4. Let P be the set of n points where n is a multiple of 6, defined as follows: place $\frac{n}{6}$ points inside each of the four rectangles $A_{1,1}, A_{1,3}, A_{3,1}, A_{3,3}$, and place the remaining $\frac{2n}{6}$ points inside $A_{2,2}$. Suppose there exists a 3-point $\varepsilon_3^{\mathcal{R}}$-net Q for P. Since $\varepsilon_3^{\mathcal{R}} < \frac{2}{6}$, one of the points of the net, say $q_1 \in Q$ should lie inside $A_{2,2}$. Furthermore, because each of the outer strips contains exactly $\frac{2n}{6}$ points, the net Q has at least one point inside each of those strips. Since q_1 is already in $A_{2,2}$, it follows that the remaining two points of Q are placed in $A_{1,1}$ and $A_{3,3}$ or in $A_{1,3}$ and $A_{3,1}$. Assume the latter w.l.o.g. Since $\frac{2n}{6}$ is even, either above or below the horizontal line defined by q_1 there are at least $\frac{n}{6}$ points from $A_{2,2}$. That way, at least one of two rectangles marked in green in Figure 4 contains no fewer than $2n/6$ points of P and avoids Q, a contradiction.

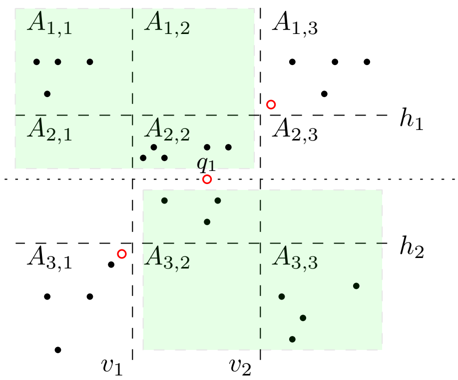

Figure 4: Red circles denote points of Q. One of the green rectangles contains at least $2n/6$ points.

The proof of the upper bound is rather simple. Let P be a set of n points where n is divisible by 6. Choose $q_1 \in Q$ so that the vertical line v passing through q_1 has exactly $n/3$ points of P on its left, and the horizontal line h passing through q_1 has exactly $n/3$ points of P below. Next, let P_{above} be the set of all points of P above h, and P_{right} be the set of all points of P to the right of v. Let q_2 be a point which forms a $\frac{1}{2}$-net for P_{above}, and q_3 — a point which forms a $\frac{1}{2}$-net for P_{right}. Consult Figure 5 for an example.

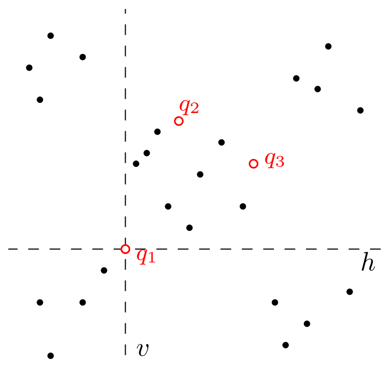

Figure 5: A construction which shows that $\varepsilon_3^{\mathcal{R}} \leq \frac{2}{6}$.

Every rectangle avoiding Q avoids q_1 and thus lies entirely inside one of the four open half-planes determined by h or v. Clearly, any rectangle lying fully below h or fully to the left of v contains at most $\frac{2n}{6}$ points, due to the definition of these lines. Any other rectangle which avoids q_1 must lie either fully above h or fully to the right of v, hence the set of points it contains is a subset of P_{above} or a subset of P_{right}, respectively. Since q_2 and q_3 are $\frac{1}{2}$-nets for P_{above} and P_{right}, respectively, no such rectangle can contain more than $\frac{2n}{3} \cdot \frac{1}{2} = \frac{n}{3}$ points of P, which shows that $Q = \{q_1, q_2, q_3\}$ is a $\frac{2}{6}$-net for P. □

Theorem 1.5. $\varepsilon_5^{\mathcal{R}} \leq \frac{2}{8}$.

Proof. Let P be a set of n points where n is divisible by 8. Let v and h be vertical and horizontal lines, respectively, such that half of the points of P lies to the left and half lies to the right of v (above and below h, respectively). Let $q_{\text{center}} = h \cap v$. Every rectangle avoiding q_{center} lies entirely inside one of the four open half-planes determined by h or v. Now we define $P_{\text{left}}, P_{\text{right}}, P_{\text{top}}, P_{\text{bottom}}$ to be the points of P which lie in the respective half-planes, and $q_{\text{left}}, q_{\text{right}}, q_{\text{top}}, q_{\text{bottom}}$ to be $\frac{1}{2}$-nets for those point sets. Let $Q = \{q_{\text{center}}, q_{\text{left}}, q_{\text{right}}, q_{\text{top}}, q_{\text{bottom}}\}$. It follows that each of the sets $P_{\text{left}}, P_{\text{right}}, P_{\text{top}}, P_{\text{bottom}}$ contains at most $\frac{n}{2}$ points and if any rectangle R avoids Q, then the set of points it contains is a subset of one of the sets $P_{\text{left}}, \ldots, P_{\text{bottom}}$. Since Q contains a $\frac{1}{2}$-net for each of the latter sets, any such rectangle contains at most $\frac{1}{2} \cdot \frac{n}{2} = \frac{n}{4}$ points of P. Therefore our set Q is a $\frac{1}{4}$-net for P.

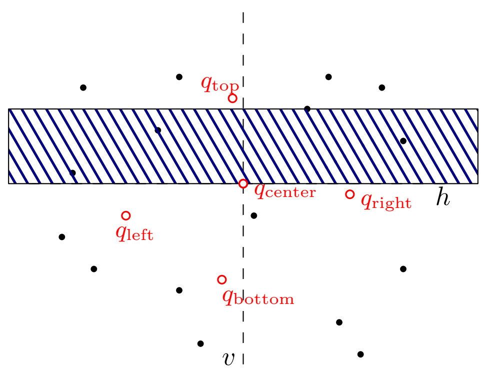

Figure 6: All the points that tiled rectangle contains belong to P_{top}.

□

1.3 Nets of size 4

Let P be a set of n points. We aim at proving that we can construct a $\frac{2}{7}$-net for P, using 4 points for the net $Q = \{q_1, \ldots, q_4\}$. For simplicity we assume that n is divisible by 7 (see Lemma 1.1).

Define vertical lines $v_{\frac{2}{7}}, v_{\frac{3}{7}}, v_{\frac{4}{7}}, v_{\frac{5}{7}}$ such that exactly the δ-fraction of points lie to the left of v_δ and exactly the $(1-\delta)$-fraction of points lie to its right for each $\delta \in \{\ldots\}$. Let horizontal lines $h_{\frac{2}{7}}, h_{\frac{3}{7}}, h_{\frac{4}{7}}, h_{\frac{5}{7}}$ be defined similarly (instead of the fraction to the left, we take the fraction of the points above). In particular, any of the defined lines does not pass through any point of P.

In the following two lemmas by the "corner" we mean one of the four regions of the plane, determined by four given lines, two horizontal and two vertical ones, as in Figure 7.

Lemma 1.6. *If one of the corners determined by the four lines $v_{\frac{2}{7}}, v_{\frac{5}{7}}, h_{\frac{2}{7}}$ and $h_{\frac{5}{7}}$ contains at least $\frac{n}{7}$ points of P, then there exists a 4-point $\frac{2}{7}$-net for P.*

Proof. Suppose w.l.o.g. that the upper left corner contains at least $\frac{n}{7}$ points of P, and let the set of points in that corner be C. Take $q_1 = v_{\frac{2}{7}} \cap h_{\frac{2}{7}}$ — see Figure 8. Observe that every rectangle avoiding q_1 should lie entirely inside one of the four half-planes defined by $v_{\frac{2}{7}}$ and $h_{\frac{2}{7}}$. All rectangles which lie to the left of $v_{\frac{2}{7}}$ or above $h_{\frac{2}{7}}$ contain at most $2n/7$ points. All other rectangles avoiding q_1 should lie to the right of $v_{\frac{2}{7}}$ or below $h_{\frac{2}{7}}$, thus containing only points from the set $P \setminus C$. Let points q_2, q_3 and q_4 be chosen so that they form a $\frac{2}{6}$-net for $P \setminus C$, and $Q = \{q_1, \ldots, q_4\}$. Since $|P \setminus C| \leq \frac{6n}{7}$, any rectangle which intersects only points from $P \setminus C$ and avoids Q contains at most $\frac{6n}{7} \cdot \frac{2}{6} = \frac{2n}{7}$ points of P. Thus, any rectangle avoiding Q contains at most $\frac{2n}{7}$ points of P. $\square$

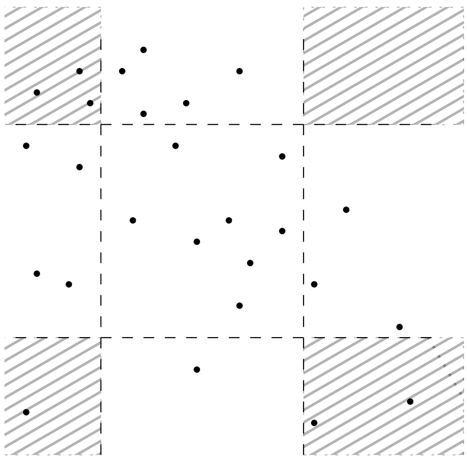

Figure 7: Four tiled corners — unbounded rectangular regions of the plane.

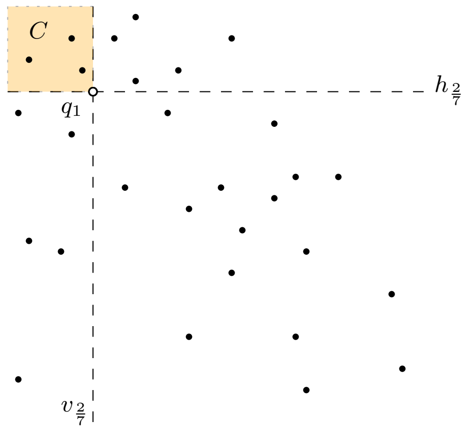

Figure 8: Choice of the first point of Q.

Lemma 1.7. *If one of the corners determined by the four lines $v_{\frac{3}{7}}, v_{\frac{4}{7}}, h_{\frac{3}{7}}$ and $h_{\frac{4}{7}}$ contains at least $\frac{2n}{7}$ points of P, then there exists a 4-point $\frac{2}{7}$-net for P.*

Proof. Suppose w.l.o.g. that the upper left corner contains at least $\frac{2n}{7}$ points of P, and let the set of points in that corner be C. Take $q_2 = v_{\frac{3}{7}} \cap h_{\frac{3}{7}}$ and q_1 to be a $\frac{1}{2}$-net for the set of points inside the green region in Figure 9, that is, for points of P to the left of $v_{\frac{3}{7}}$ or above $h_{\frac{3}{7}}$. Finally, take q_3 and q_4 to form a $\frac{2}{5}$-net for the set of points $P \setminus C$.

Observe that the green region in Figure 9 contains at most $\frac{3n}{7} + \frac{3n}{7} - \frac{2n}{7} = \frac{4n}{7}$ points, and the set $P \setminus C$ contains at most $n - |C| \leq n - \frac{2n}{7} = \frac{5n}{7}$ points.

Let R be any rectangle avoiding Q. Since R avoids q_2, it should lie entirely inside one of the four open half-planes determined by $v_{\frac{3}{7}}$ and $h_{\frac{3}{7}}$. Now, either R lies fully inside the green region, or R

contains only points from the set $P \setminus C$. In the former case, since R avoids q_1, it contains at most $\frac{4n}{7} \cdot \frac{1}{2} = \frac{2n}{7}$ points of P. In the latter case, R avoids q_3 and q_4 which form a $\frac{2}{5}$-net for $P \setminus C$, so R contains at most $|P \setminus C| \cdot \frac{2}{5} \leq \frac{5n}{7} \cdot \frac{2}{5} = \frac{2n}{7}$ points of P. Our set $Q = \{q_1, \ldots, q_4\}$ is therefore a $\frac{2}{7}$-net for P.

□

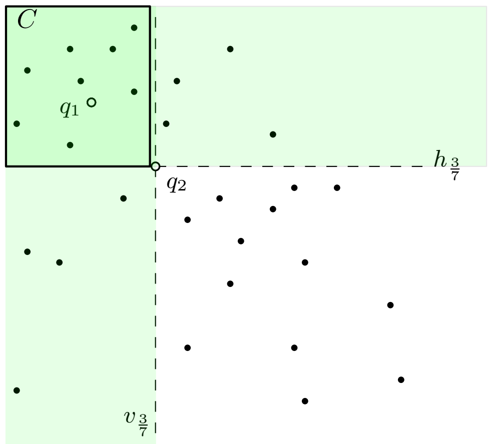

Figure 9: Choice of the two first points of Q.

We are now ready to prove the theorem.

Theorem 1.8. $\varepsilon_4^{\mathcal{R}} \leq \frac{2}{7}$.

Proof. To begin with, note that Lemma 1.6 and Lemma 1.7 apply. That is, we can assume that the conditions mentioned in those lemmas are not met, and proceed further. Denote by $M_{3\times3}$ the rectangle lying between the lines $v_{\frac{2}{7}}, v_{\frac{5}{7}}, h_{\frac{2}{7}}, h_{\frac{5}{7}}$ — see Figure 10. We break down the proof into two cases: either $M_{3\times3}$ contains at most $2n/7$ points, or it contains more than $2n/7$ points. In the former case, we use the fact that there is no need to choose any of the points of a net inside $M_{3\times3}$. In the latter case, we have to take at least one point of a net inside $M_{3\times3}$.

Let us consider the first case, that is when $M_{3\times3}$ contains at most $2n/7$ points. Let the first two points of a net be q_1 and q_2, placed as in Figure 10. Note that there are exactly $3n/7$ points to the left of $v_{\frac{3}{7}}$, which passes through q_1 and q_2, and that the two rectangles depicted in green in Figure 10 contain strictly fewer than $2n/7$ points each — because of Lemma 1.7. Therefore we can choose the third point of a net q_3 in a way that any rectangle to the left of $v_{\frac{3}{7}}$ and above or below q_3 contains at most $2n/7$ points. The point q_4 is taken to be a $\frac{1}{2}$-net for those points of P which lie to the right of $v_{\frac{3}{7}}$.

We now prove that each rectangle R avoiding Q contains at most $\frac{2n}{7}$ points. There are exactly $\frac{4n}{7}$ points of P to the right of $v_{\frac{3}{7}}$, so if R lies entirely to the right of $v_{\frac{3}{7}}$ and avoids q_4 then it contains at most $\frac{1}{2} \cdot \frac{4n}{7} = \frac{2n}{7}$ points of P. If R lies entirely to the left of $v_{\frac{3}{7}}$ and avoids q_3, it clearly cannot contain more than $\frac{2n}{7}$ points. Therefore, if R contains more than $\frac{2n}{7}$ points, it should intersect $v_{\frac{3}{7}}$. Note that we can consider only inclusion-maximal rectangles which avoid Q. In an ongoing argument R is an inclusion-maximal rectangle which contains more than $\frac{2n}{7}$ points of P and intersects $v_{\frac{3}{7}}$.

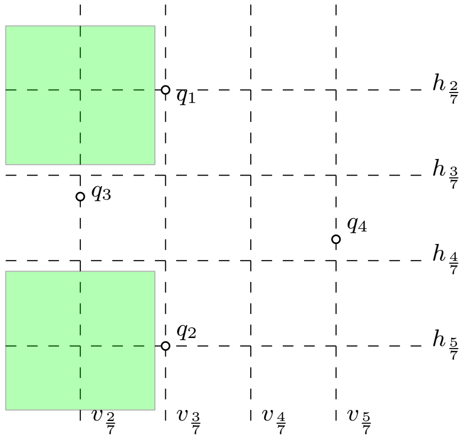

Figure 10: Choice of Q.

Clearly, rectangles lying entirely above $h_{\frac{2}{7}}$ or below $h_{\frac{5}{7}}$ contain at most $\frac{2n}{7}$ points. Thus R lies between $h_{\frac{2}{7}}$ and $h_{\frac{5}{7}}$. Moreover, R cannot lie inside inside $M_{3\times3}$, because we assumed that $M_{3\times3}$ contains at most $\frac{2n}{7}$ points. It follows that R lies above or below q_4, almost touches $v_{\frac{2}{7}}$ with its left side and crosses $v_{\frac{5}{7}}$ with its right side. Of course, these two cases (above/below q_4) are symmetrical, so we consider the former — that is rectangle R in Figure 11.

Observe that there are at most $n/7$ points of P in the left upper corner. Therefore, adding together the numbers of points in the uppermost strip and the leftmost strip we get at least $\frac{2n}{7}+\frac{2n}{7}-\frac{n}{7}=\frac{3n}{7}$ points of P (region Γ in Figure 11). Moreover, due to the choice of q_4, rectangle R_2 which is depicted in Figure 11 contains exactly $2n/7$ points. Adding the numbers of points in R, R_2 and Γ we get strictly more than $\frac{2n}{7}+\frac{2n}{7}+\frac{3n}{7}=n$ points, a contradiction.

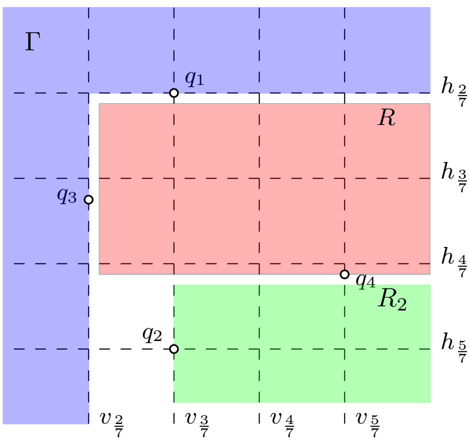

Figure 11: Green rectangle contains exactly $2n/7$ points, Γ contains at least $3n/7$ points.

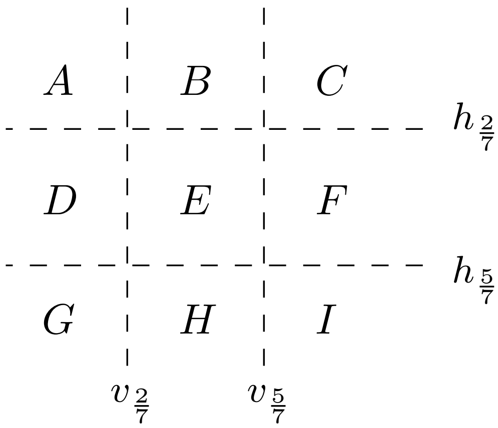

Figure 12: Letters denote fractions of P inside each cell.

For the second case, suppose that $M_{3\times3}$ contains strictly more than $2n/7$ points of P. Let us leave only lines $v_{\frac{2}{7}}, h_{\frac{2}{7}}, v_{\frac{5}{7}}$ and $h_{\frac{5}{7}}$. Let $A, B, \ldots, I$ denote the fractions of points of P in the corresponding cells as illustrated in Figure 12. We do some calculations first, and then construct the net itself. The following clearly holds:

$$A+B+C+D+E+F+G+H+I=1 \tag{1}$$

Because we have assumed that $M_{3\times3}$ contains more than $2n/7$ points, we have:

$$E>\tfrac{2}{7} \tag{2}$$

Since each of the outer strips contains exactly $2n/7$ points, we have:

$$2(A+C+G+I)+B+D+G+H=\tfrac{8}{7} \tag{3}$$

For the sum of three middle strips, we clearly have:

$$D+E+F=\tfrac{3}{7} \tag{4}$$

Subtracting (2) from (1) we get:

$$A+B+C+D+F+G+H+I<\tfrac{5}{7} \tag{5}$$

Subtracting (5) from (3) we get:

$$A+C+G+I>\tfrac{3}{7} \tag{6}$$

Subtracting (2) from (4) we obtain:

$$D+F<\tfrac{1}{7} \tag{7}$$

and similarly:

$$B+H<\tfrac{1}{7} \tag{8}$$

Due to Lemma 1.6, we have:

$$A,C,G,I<\tfrac{1}{7} \tag{9}$$

And because of (6), the sum of any two among these four corners should be greater than $\frac{1}{7}$. Finally, because of (9) we have:

$$A+D+G+H+I\geq\tfrac{3}{7} \tag{10}$$

so subtracting that from (1) we have:

$$B+C+E+F\leq\tfrac{4}{7} \tag{11}$$

We start constructing the net Q as follows: take $q_1=v_{\frac{2}{7}}\cap h_{\frac{2}{7}}$, $q_2=v_{\frac{5}{7}}\cap h_{\frac{5}{7}}$. The point q_3 is chosen to be a $\frac{1}{2}$-net for those points of P which lie to the left of $v_{\frac{2}{7}}$ and above $h_{\frac{5}{7}}$. There are at most $4n/7$ such points, because of (11). Note that rectangle which contains all points to the right of q_1 and above q_3, as well as rectangle containing all points to the right of q_3 and above q_2 contain exactly $2n/7$ points each. We call these two rectangles $(BC)'$ and $(CF)'$ respectively — see Figure 13.

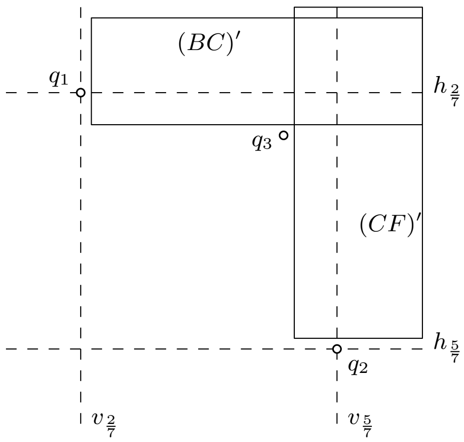

Figure 13: Placement of q_3.

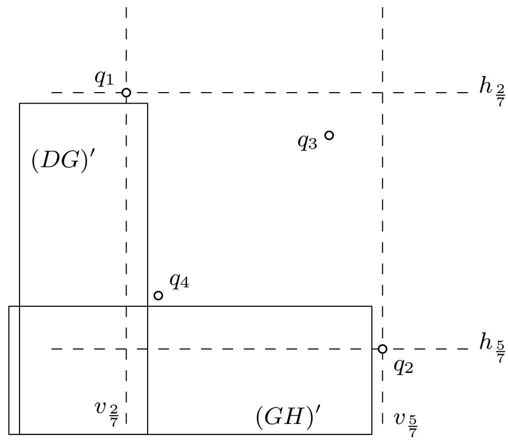

Figure 14: Placement of q_4.

We place q_4 in a similar manner: it is chosen to be a $\frac{1}{2}$-net for the points of P which lie to the right of $v_{\frac{5}{7}}$ and below $h_{\frac{2}{7}}$. There are at most $4n/7$ such points, and it could be proved in a similar manner to the proof of(11). Note that two rectangles containing all points to the left of q_4 and below q_1 and all points to the left of q_2 and below q_4 contain exactly $2n/7$ points each. We call these rectangles $(DG)'$ and (GH') respectively — see Figure 14.

We now want to argue, that any rectangle avoiding Q contains at most $2n/7$ points. Obviously, rectangles lying fully inside outer strips contain at most $2n/7$ points. Moreover, any rectangle which entirely lies inside the region $B \cup C \cup E \cup F$ or inside $D \cup E \cup G \cup H$ contains at most $\frac{4n}{7} \cdot \frac{1}{2} = \frac{2n}{7}$ points — because of the choice of q_3 and q_4 respectively. That means that we should consider only full-width horizontal or full-height vertical rectangles. Due to symmetry, we consider only the former rectangles, that is those which stretch from the left to right and contain the maximum number of points.

The horizontal strip between q_1 and q_3 contains at most $2n/7$ points, since it contains a part from $(BC)'$ which is no greater than $(BC)' - B - C = A < 1/7$, and a part to the left, which is no greater than D, which by (7) is no greater than $\frac{1}{7}$, so we have $2/7$ in total — see Figure 15. A horizontal strip between q_4 and q_2 cannot contain more than $2n/7$ points due to the same reasoning.

Note that if it occurs that q_4 is above q_3, then there are no more rectangles that could contain more than $2n/7$ points. However, it might be the case depicted in Figure 16. That is, we need to show that if q_4 occurs to be below q_3, then the rectangle R_1 cannot contain more than the $\frac{2}{7}$-fraction of P.

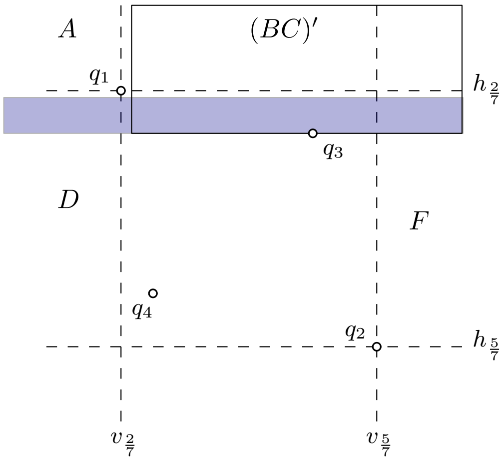

Figure 15: A horizontal strip between q_1 and q_3 cannot contain more than $2n/7$ points.

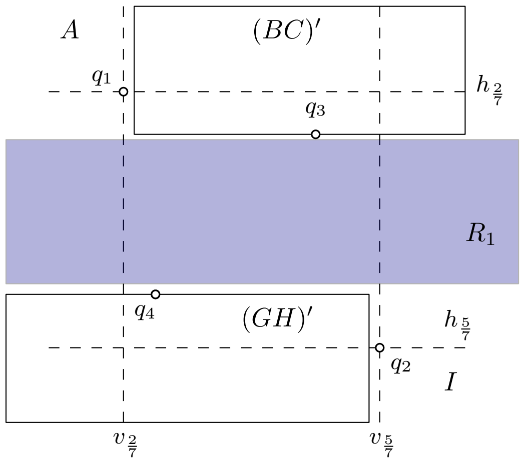

Figure 16: The number of points in R_1 is bounded from above.

But because by (9) we have $A + I > \frac{1}{7}$, and $(BC)' = (GH)' = \frac{2}{7}$, from (1), the number of points in R_1 is bounded by

$$n \cdot ((1) - A - I - (BC)' - (GH)') \leq \frac{2n}{7}.$$

That means that in the second case (when $M_{3\times3}$ contains strictly more than $2n/7$ points of P) the set $Q = \{q_1, \ldots, q_4\}$ is indeed a $\frac{2}{7}$-net for P, which completes the proof of the theorem. □

2 Grid conjecture

In this section we address a conjecture about a specific property of the nets for $\mathcal{R}$, which was raised in [7] and inspired by results from [3] and [6]. We show that the conjecture is false. In particular, we disprove the results claimed in [6].

In Section 2.1 we introduce all necessary notation and the conjecture itself. Then in sections 2.2 through 2.4 we describe our approach which gave the results described in Section 2.5. An impatient reader could jump to Section 2.5 immediately after 2.1, as sections 2.2 to 2.4 are not necessary to understand counterexamples themselves, but are provided to show our way to get to them.

2.1 Nets chosen from a grid

Let P be a set of n points in the plane, where n is divisible by $k+3$ for some $k \geq 1$. Define vertical lines $v_0, v_{\frac{2}{k+3}}, v_{\frac{3}{k+3}}, \ldots, v_{\frac{k+1}{k+3}}, v_1$, such that exactly δ-fraction of points lie to the left of v_δ and exactly $(1-\delta)$-fraction of points lie to its right. Let horizontal lines $h_0, h_{\frac{2}{k+3}}, h_{\frac{3}{k+3}}, \ldots, h_{\frac{k+1}{k+3}}, h_1$ be defined similarly (instead of the fraction to the left, we take the fraction of the points above). In particular, any of the defined lines does not pass through any point of P, and all points of P lie inside the area determined by h_0, h_1, v_0 and v_1.

Let $\mathcal{I}$ be the set of pairwise intersections of all lines of the grid except h_0, h_1, v_0 and v_1. We will refer to points of $\mathcal{I}$ as to *grid points*.

Let $A_{i,j}$ be the intersection of P with (i,j)-th cell of the grid for $1 \leq i, j \leq k+1$. We also define $A_{*,j} := \bigcup_i A_{i,j}$ and $A_{i,*} := \bigcup_j A_{i,j}$. Due to the definition of the grid, the following conditions are satisfied:

$$|A_{1,*}| = \tfrac{2}{k+3}n,\ |A_{k+1,*}| = \tfrac{2}{k+3}n,\ |A_{*,1}| = \tfrac{2}{k+3}n,\ |A_{*,k+1}| = \tfrac{2}{k+3}n,$$
$$|A_{i,*}| = |A_{*,j}| = \tfrac{1}{k+3}n \quad i,j \in \{2,\ldots,k\}.$$

In other words, each of the four utter strips contain exactly $\frac{2n}{k+3}$ points, and all other strips contain exactly $\frac{n}{k+3}$ points each. From now on we will call these bounds *grid conditions*.

For the purpose of illustration here and in future we will use $k=4$ — see Figure 17.

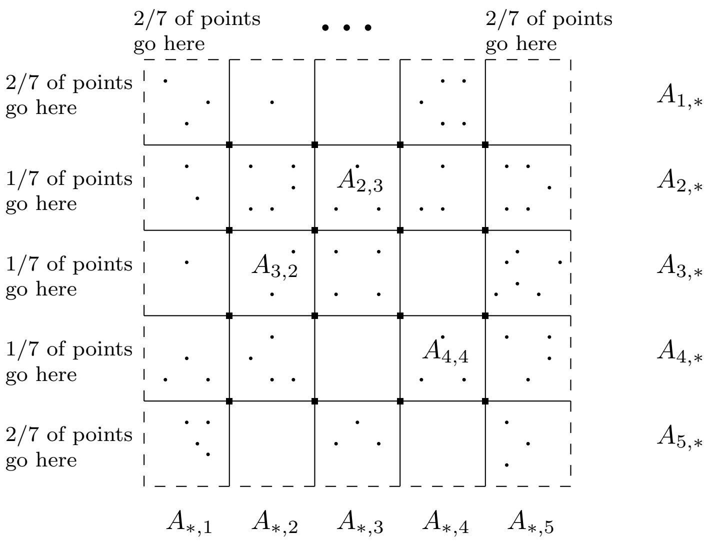

Figure 17: Grid for $k=4$. Small black points are points of P. Square points form set $\mathcal{I}$.

Speaking of grid conditions, one can consider them in the reverse direction: first, take a grid on the plane, and then place points of the set P to meet the above-mentioned conditions. Now when we have the connection between the point set P and the grid, we are ready to state the conjecture.

Conjecture 2.1. *Let $k \in \mathbb{N}$ and P be a point set of size n where n is divisible by $k+3$. There exists a k-point $\frac{2}{k+3}$-net for P with respect to $\mathcal{R}$, where the net is chosen from the set of grid points.*

Note that the proofs of Theorem 1.2 and Theorem 1.3 show that this conjecture is true for $k \in \{1, 2\}$.

Our objective in the following part of this work is a verification of this conjecture for other small values of k. Our approach is the exhaustive search of a counterexample. If this search fails, the conjecture will occur to be true for that particular value of k. That said, Conjecture 2.1 is false if and only if there exists a point set P of size n, meeting grid conditions, such that for each k-point set Q chosen from the set of grid points there exists a rectangle which contains strictly more than $\frac{2n}{k+3}$ points and avoids Q. We will search for such point sets P. In the following section, we explore their properties and argue that we might as well search for $(k+1) \times (k+1)$ matrices with certain properties.

When it will not create a misunderstading, we will write a "net" instead of a "k-point set".

2.2 Structure of a counterexample

Suppose we have some set of points P of size n which is a counterexample to Conjecture 2.1. Let us restrict our attention only to rectangles which could witness that P is a counterexample. Note that despite $\mathcal{R}$ is infinite, we should not consider all rectangles in $\mathcal{R}$ — for example, a rectangle that lies entirely outside the grid is not worth considering. Clearly, we can restrict our attention only to rectangles which lie fully inside the grid (that is, inside the area determined by h_0, h_1, v_0 and v_1). We now argue that in fact, it suffices to consider only a finite set of rectangles.

The fact that P is a counterexample means that for any choice of a net $Q \subset \mathcal{I}$ there exists a rectangle R which lies fully inside the grid, avoids Q, and contains more than $\frac{2n}{k+3}$ points of P. Now observe, that instead of R we can consider an open rectangle R', which contains all points from all cells which R intersects and is stretched to the boundary of these cells. The word "open" means that points lying on the border of R' are not considered to be in R'. Clearly R' contains more than $\frac{2n}{k+3}$ points, and if R avoids Q, then R' avoids Q too.

In further discussion, we consider only open rectangles defined by the grid lines. Obviously, there are finitely many such rectangles — $\binom{k+2}{2}^2$. Slightly abusing notation, we will use the word "rectangle" to denote "open axis-parallel rectangle, defined by the grid lines". We will refer to the set of these rectangles as *grid rectangles*.

Suppose that $k \in \mathbb{N}$ is fixed, we have a grid drawn on the plane and we want to find a counterexample P. Note that due to the grid conditions, some of the grid rectangles cannot contain strictly more than $\frac{2n}{k+3}$ points of P for any choice of P. In fact, grid rectangles are rather a property of the grid, not the point set. This allows us to classify grid rectangles as follows (suppose the set P is yet to be chosen):

Definition 3. Let $k \in \mathbb{N}$ and the respective grid be drawn on the plane. Consider all choices of a point set P of size n meeting grid conditions. We classify all grid rectangles as follows:

- *Insufficient rectangles* - those which **cannot** contain more than $\frac{2n}{k+3}$ points of P.
- *Sufficient rectangles* - those which **sometimes** contain more than $\frac{2n}{k+3}$ points of P.
- *Oversufficient rectangles* - those which **always** contain more than $\frac{2n}{k+3}$ points of P.

An example is given in Figure 18.

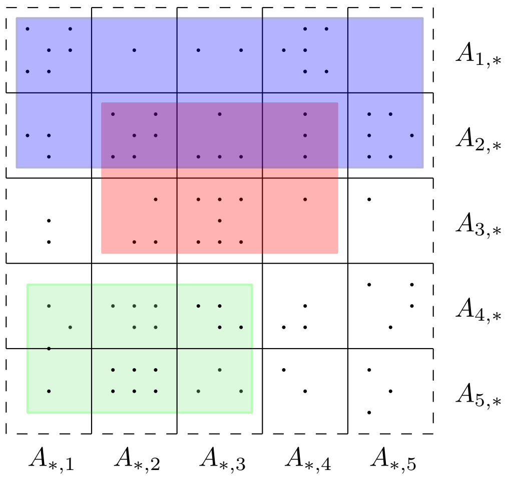

Figure 18: Top-down: oversufficient, insufficient and sufficient rectangles respectively ($k = 4$).

We will denote the set of all sufficient rectangles by R_{suf} and the set of all oversufficient rectangles by R_{oversuf}. The following is a direct consequence of the Definition 3:

Observation 2.2. *Any $\frac{2}{k+3}$-net must hit all oversufficient rectangles.*

From now on, when we refer to the notion of a net we mean a net that covers all oversufficient rectangles. We denote the set of all such nets by $\mathcal{Q}$. The following is a crucial observation for this and the next two sections:

Observation 2.3. *A counterexample to Conjecture 2.1 exists if and only if there exists a subset $S \subseteq R_{\text{suf}}$ such that the following conditions are met:*

(i) No net from $\mathcal{Q}$ hits all rectangles from S (for each net $Q \in \mathcal{Q}$ there exist some rectangle $R \in S$ which avoids Q).

(ii) There exist a point set P meeting grid conditions such that every rectangle in S contains strictly more than $\frac{2}{k+3}$-fraction of points of P.

Proof. Suppose that some set P is a counterexample. Let $S \subseteq R_{\text{suf}}$ be the set of all rectangles from R_{suf} which contain more than the $\frac{2}{k+3}$-fraction of P. By the definition of P, S is not covered by any net from $\mathcal{Q}$, because only rectangles from $\mathcal{S}$ can contain more than the $\frac{2}{k+3}$-fraction of P and avoid nets from $\mathcal{Q}$. □

Because we are interested in a counterexample, we must somehow check all possible sets of points. Observation 2.3 helps us to look at this from a different point of view — we can take a primal look on some subsets of R_{suf}, and then try to find a set P from the second condition of this observation.

2.3 Matrices and a naive algorithm

Let us remind that we narrowed the set of nets to some set $\mathcal{Q}$ of nets which cover all oversufficient rectangles, and in Observation 2.3 we argued that the only condition that some counterexample P should meet is to have more than the $\frac{2}{k+3}$-fraction of its points in each of the rectangles from some $S \subseteq R_{\text{suf}}$, such that S cannot be hit by any net. This allows us to employ the following algorithm:

1: **procedure** FIND-SET
2: **for** $S \subseteq R_{\text{suf}}$ **do** ▷ $O(2^{|R_{\text{suf}}|})$ iterations
3: **if** $\forall Q \in \mathcal{Q}\ \ \exists R \in S$ such that $R \cap Q = \varnothing$ **then** ▷ no net can hit all rectangles from S
4: **if** $\exists P\ :\ \forall R \in S$ R contains more than the $\frac{2}{k+3}$-fraction of points of P **then**
5: **return** P ▷ P is a counterexample
6: **end if**
7: **end if**
8: **end for**
9: **return** $\varnothing$
10: **end procedure**

The main problem in this algorithm is line 4: we are looking for some subset of points that must satisfy given conditions. However, because we are considering only stretched open rectangles, the geometrical aspects of P do not matter much: it is sufficient to know how many points of P are contained in each cell. Thus, instead of looking for a point set, we can look for $(k+1) \times (k+1)$ matrices which encode the density of points of P in each cell. To sum up, here are the properties which matrix $X = (x_{i,j})_{1 \le i,j \le k+1}$ we are looking for must satisfy:

(P1) $x_{i,j} \ge 0$ for any i, j

(P2) $\sum_{i,j} x_{i,j} = 1$

(P3) $\sum_j x_{1,j} = \sum_j x_{k+1,j} = \sum_i x_{i,1} = \sum_i x_{i,k+1} = \frac{2}{k+3}$, and $\sum_i x_{i,h} = \sum_j x_{h,j} = \frac{1}{k+3}$ for $2 \le h \le k$

Note that (P1, P2) basically say that X represents some set of points P, and (P3) forces the grid conditions on any set which X represents. Now it is easy to add another condition. Fix some $S \subseteq R_{\text{suf}}$. For any $R \in S$, let $C_R := \{(i,j)\ :\ R$ intersects the (i,j)-cell of a grid$\}$. Now, let

(P4(S)) For all $R \in S$ $\sum\limits_{i,j \in C_R} x_{i,j} > \frac{2}{k+3}$

This condition is exactly what we look for in line 4 in our `FIND-SET` algorithm. We are now ready to restate the algorithm:

1: **procedure** FIND-SET
2: **for** $S \subseteq R_{\text{suf}}$ **do** ▷ $O(2^{|R_{\text{suf}}|})$ iterations
3: **if** $\forall Q \in \mathcal{Q}\ \ \exists R \in S$ such that $R \cap Q = \varnothing$ **then** ▷ no net can hit all rectangles from S
4: **if** $\exists X\ :\ X$ satisfies (P1-P3) and (P4(S)) **then**
5: **return** X ▷ X encodes some counterexample
6: **end if**
7: **end if**
8: **end for**
9: **return** $\varnothing$
10: **end procedure**

Since we iterate over all subsets of R_{suf}, we are exhaustively checking all counterexamples — see Observation 2.3.

Observe that now in line 4, given fixed $S \in R_{\text{suf}}$, we can solve the following linear program to find a matrix X (since property (P4) is dependent on S we denote this LP by $LP(S)$):

$$
\begin{array}{ll}
\text{maximize} & \delta \\
\text{subject to} & \displaystyle\sum_{1 \le i,j \le k+1} x_{i,j} = 1 \\
& x_{i,j} \ge 0, \qquad\qquad \text{for } 1 \le i,j \le k+1 \\
& \text{(P3)} \\
& \displaystyle\sum_{i,j \in C_R} x_{i,j} \ge \tfrac{2}{k+3} + \delta \text{ for all } R \in S
\end{array}
$$

The matrix X we are looking for in line 4 must be a solution to $LP(S)$ for some $S \subseteq R_{\text{suf}}$. That is, we will have a separate LP task for each S, and if for at least one LP it occurs that $\delta > 0$ it would mean that X encodes some subset of points P which contains strictly more than the $\frac{2}{k+3}$-fraction of its points in each of $R \in S$ and hence is a counterexample. Our LP has rational coefficients and hence if a solution exists, all $x_{i,j}$ and δ should be rationals, therefore we will be able to obtain set P by multiplying all numbers by the product of their denominators.

2.4 Optimizing the algorithm

Now that we have the algorithm, the problem is that the complexity of it is prohibitive. One can check that already for $k = 4$ we have 49 rectangles in R_{suf}, and $\mathcal{Q}$ contains 348 nets. Thus, we have $348 \cdot 2^{49}$ iterations in two outer cycles. Of course, not all of them will reach LP, but that is already too much. To be more efficient we need to somehow narrow our search.

Let $\mathcal{P}$ define a family of sets of rectangles which reach the 4th line:

$$S \in \mathcal{P} \Leftrightarrow \nexists Q \in \mathcal{Q} : Q \text{ hits all rectangles from } S$$

The following observation implies that it suffices to consider only inclusion-minimal elements of $\mathcal{P}$:

Observation 2.4. *Take any $S_1, S_2 \in \mathcal{P}$ such that $S_1 \subset S_2$. If $LP(S_2)$ has a solution, then $LP(S_1)$ has a solution.*

Proof. Obviously, if there exists a matrix X which fulfils (P1-P3) and (P4(S_2)) then X also fulfils (P4(S_1)). This follows from the fact, that in (P4(S_1)) there are the same inequalities as in (P4(S_2)), maybe except a few. Thus, if we have some $S \in \mathcal{P}$ which is inclusion-minimal, there is no need to check its supersets in $\mathcal{P}$ — if $LP(S)$ has a solution, this solution encodes a counterexample P, and if not, then none of the supersets of S has a solution either. □

Let $\mathcal{G} \subset \mathcal{P}$ denote the set of all inclusion-minimal elements of $\mathcal{P}$:

$$\mathcal{G} := \{S \in \mathcal{P} : \nexists S' \in \mathcal{P} \text{ such that } S' \subset S\}$$

Here is the final version of the algorithm:

```
procedure FIND-SET
    for S ∈ G do
        if LP(S) has a solution X then
            return X                          ▷ X encodes some counterexample
        end if
    end for
    return ∅
end procedure
```

The following observation allows us to optimize the procedure of generating the set $\mathcal{G}$:

Observation 2.5. *For any $S \in \mathcal{G}$, S cannot contain two rectangles R_1 and R_2 such that R_1 lies fully inside R_2. Thus, S must form an antichain in the set $\mathcal{S}$ ordered by inclusion (in a geometrical sense).*

Proof. Suppose by contradiction that S contains R_1 and R_2 such that R_1 lies fully inside R_2. By definition of $\mathcal{G}$ we know that there does not exist a net in $\mathcal{Q}$ which hits all rectangles from S. Observe that the same applies to the set $S \setminus \{R_2\}$, and thus S is not an inclusion-minimal set with this property. $\square$

Using this observation, we implemented a procedure of generating the set $\mathcal{G}$ in C++ language. Our results are presented in the following subsection.

2.5 Results for $1 \leq k \leq 6$

Obviously, for the nets of size 1 and 2 the conjecture holds — recall our proofs from Section 1.2. For $k = 3, 4, 5, 6$ we provide counterexamples. The workhorse to find a counterexample for $k = 4$ was the approach discussed above. To solve our LP tasks $LP(S)$ for each $S \in \mathcal{G}$ we used Google OR-tools software suite [8]. Our own code is published at [9]. After evaluating, we came out with a bunch of counterexamples, one of which is depicted in Figure 20 (that said, there exist a lot more than one counterexample for $k = 4$). To point out, the counterexample for $k = 3$ (see Figure 19) was found ad-hoc, and counterexamples for $k = 5, 6$ were found by modifying that for $k = 4$. We start with a counterexample for $k = 3$. On this the following pictures, numbers denote the number of points to be put inside a correspondent cell. For example, a point set P which is represented in Figure 19 contains 12 points.

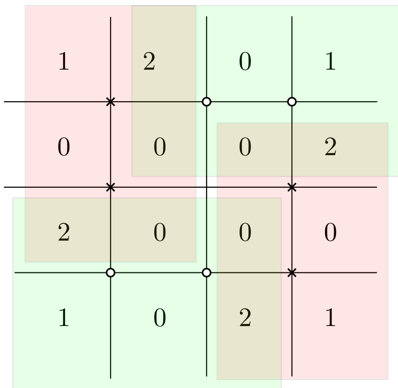

Figure 19: Four presented rectangles contain more than $\frac{2}{6} \cdot 12$ points each.

Obviously, no two among presented rectangles cannot be covered simultaneously, if one has to choose 3-point $\frac{2}{6}$-net only from the set of grid points. This shows, that for $k = 3$ the conjecture is false. The remaining counterexamples are provided below, and a reasonings for them follow the idea of that for $k = 3$. That is, they use the fact that provided rectangles do not intersect in any of the grid points.

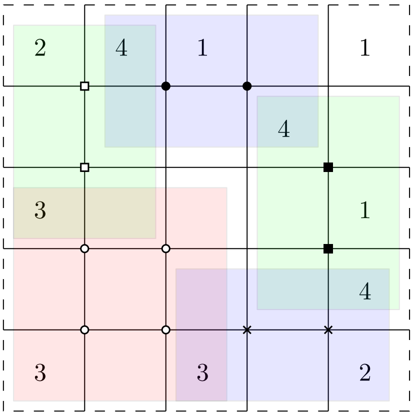

Figure 20: Five presented rectangles contain more than $\frac{2}{7} \cdot 28$ points each.

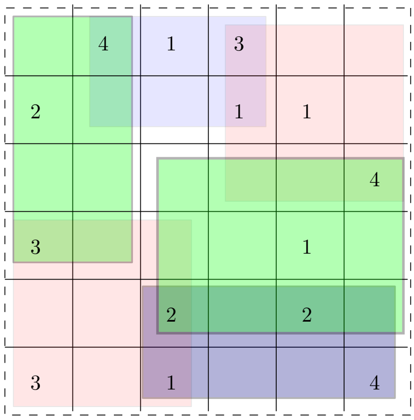

Figure 21: Six presented rectangles contain more than $\frac{2}{8} \cdot 32$ points .each

It was claimed in [6] that $\varepsilon_6^{\mathcal{R}} \leq \frac{2}{9}$, and the proof given there was via a program which chooses a net from the set of grid points. Our result shows that that proof was incorrect, so whether $\varepsilon_6^{\mathcal{R}} \leq \frac{2}{9}$ remains unsolved.

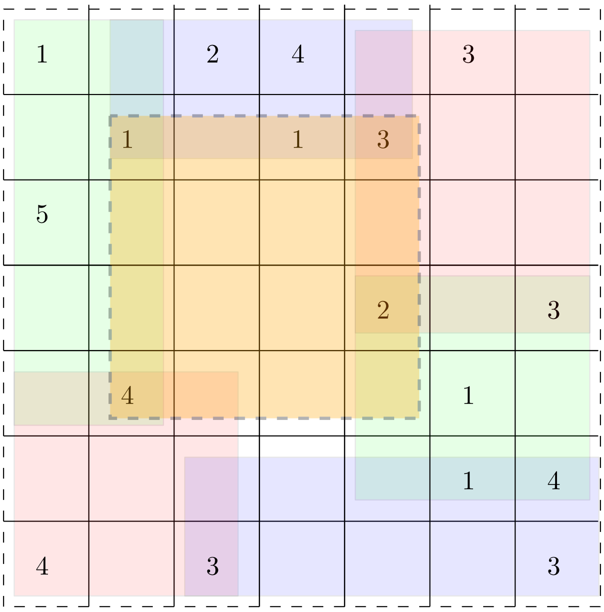

Figure 22: Seven presented rectangles contain more than $\frac{2}{9} \cdot 45$ points each

All of this shows that there is no evidence (except for cases with $k = 1$ and 2) that $\frac{2}{k+3}$-nets can be chosen from a grid. Besides, the question about the optimal size of a net still remains open.

Acknowledgements

I would like to thank Prof. Bartosz Walczak for his patience and efforts put into this work. I would also like to thank Dr. Lech Duraj, my tutor for the first two years of studies, for his support and lending a helping hand.

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