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CENTRE FOR AUTONOMOUS SYSTEMS

41014 Sensors and Control for Mechatronic Systems

Dr. **Liang Zhao**

Centre for Autonomous Systems
University of Technology,
Sydney

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0. Quiz 2

- 15%
- 40 mins
- Restrict open book: one hand writing A4 paper

0. Lecture 6-8

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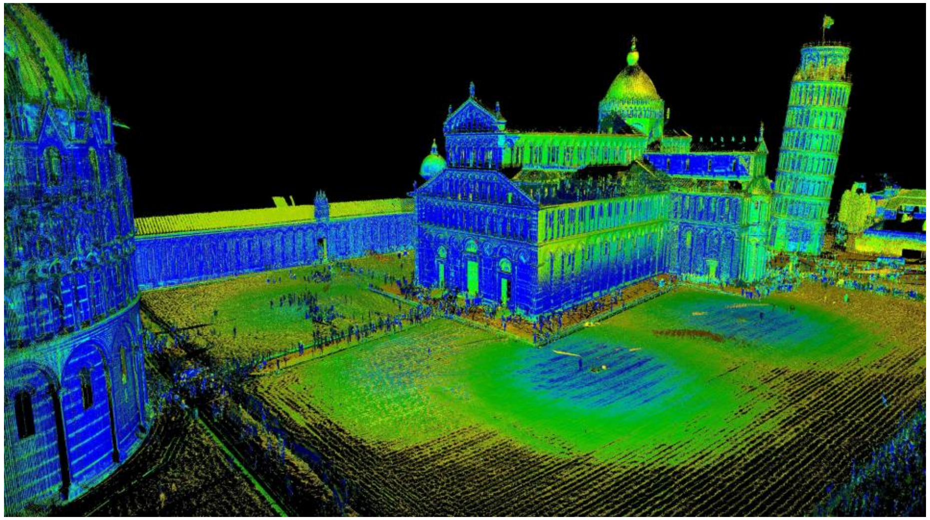
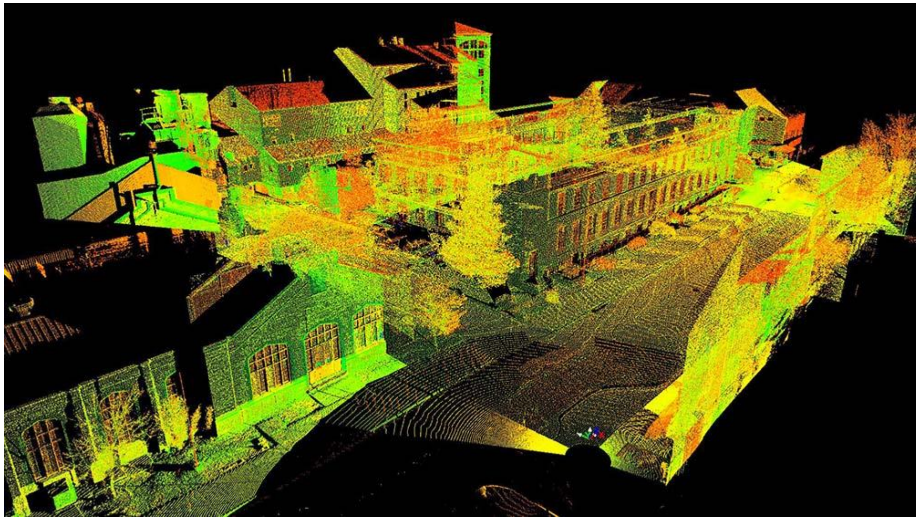
❖ Control:

- Linear Continuous-Time Systems
- Linear Discrete-Time Systems
- **Nonlinear Discrete-Time Systems**





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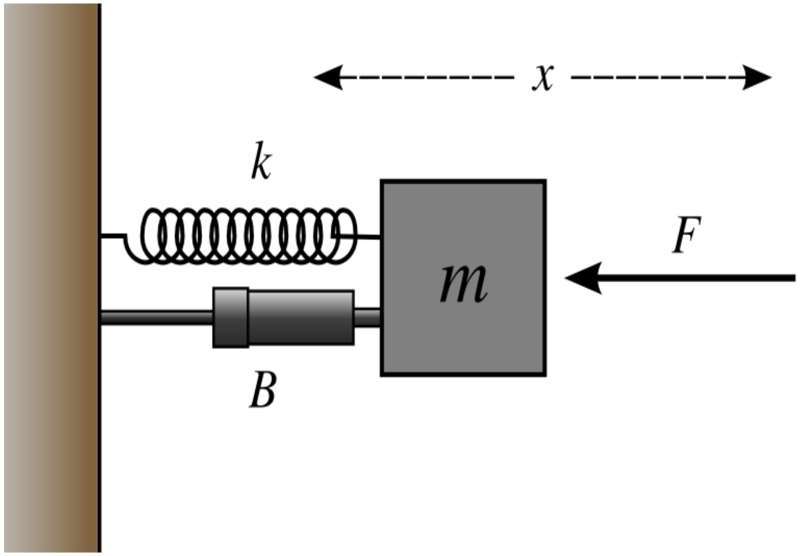
❖ Control Part 1:

- **Linear Continuous-Time Systems**
- Formulation
- Stability
- Pole Placement
- Controllability
- Linear quadratic optimal control



0. Lecture-7

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Lecture:

- Linear discrete-time Systems
- Discretization
- Stability
- State feedback control
- Optimal control

Active hands on:

- Examples
- Solve problem using Matlab

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41014 Sensors and Control for Mechatronic Systems

Lecture-8: Control Part 3 Nonlinear Discrete-Time Systems

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1. Nonlinear Discrete-Time System

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Introduction:

- A system of difference equations as

$$x(k+1) = f(x(k), u(k))$$

- *f(x,u)* is a vector function with *n* components:

$$f(x,u) = \begin{bmatrix} f_1(x,u) \\ \vdots \\ f_n(x,u) \end{bmatrix}$$

1. Nonlinear Discrete-Time System

❖ Introduction:

- The functions $f_i(x, u)$ are in turn functions of $n + m$ variables, the components of the x and u vectors.

$$
\begin{aligned}
x_1(k+1) &= f_1(x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k)) \\
x_2(k+1) &= f_2(x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k)) \\
&\vdots \\
x_n(k+1) &= f_n(x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k))
\end{aligned}
$$

- $f_i(x, u)$ is nonlinear.

Introduction:

- The outputs of the model:

$$y(k) = h(x(k), u(k))$$

- can then be calculated from the internal variables $x_i(k)$ and the inputs $u_i(k)$:

$$\begin{aligned}
y_1(k) &= h_1(x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k)) \\
y_2(k) &= h_2(x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k)) \\
&\vdots \\
y_p(k) &= h_p(x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k))
\end{aligned}$$

- $h_i(x, u)$ is nonlinear.

❖ Compare to Linear discrete-time system:

- Consider the set of n first-order linear difference equations forced by the input $u(k) \in R^n$

$$
\begin{cases}
x_1(k+1) &= a_{11}x_1(k) + \ \dots \ + a_{1n}x_n(k) & +b_1u(k) \\
x_2(k+1) &= a_{21}x_1(k) + \ \dots \ + a_{2n}x_n(k) & +b_2u(k) \\
\vdots & \vdots & \vdots \\
x_n(k+1) &= a_{n1}x_1(k) + \ \dots \ + a_{nn}x_n(k) & +b_nu(k) \\
x_1(0) = x_{10}, \quad \dots \quad x_n(0) = x_{n0}
\end{cases}
$$

- In compact matrix form:

$$
\begin{cases}
x(k+1) &= Ax(k) + Bu(k) \\
x(0) &= x_0
\end{cases}
$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$

1. Nonlinear Discrete-Time System

❖ Nonlinear discrete-time state-space models:

- Nonlinear discrete-time model:

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \qquad\qquad k = 0, 1, 2, \ldots \\ y(k) &= h(x(k), u(k)) \end{aligned}$$

- *u(k)*: input at time k , an m-dimensional column vector.
- *y(k)*: output at time k , a p-dimensional column vector.
- *x(k)*: state at time k , an n-dimensional column vector.
- The model is said to be *n-th* order.
- For a given initial value $x(k_0) = x_0$, always has a unique solution.

2. Equilibrium

- A constant trajectory, generated by a constant input function, is called *equilibrium*.
  - Equilibrium point -- a point where the system can stay forever without moving.

- Given a constant input $\bar{u}$, the equilibria are solutions of the following equations:

$$\bar{x} = f(\bar{x}, \bar{u})$$
$$\bar{y} = h(\bar{x}, \bar{u})$$

2. Equilibrium

- How to calculate equilibria?
  - Solve the solution of *x(k+1)-x(k)=0.*
- Example:

$$x(k+1) = \frac{1}{4}x(k)^2 + x(k) - 2u(k)$$

  - When $u(k) \equiv 0.5$, compute the equilibrium points.
- Solution:

$$x(k+1) - x(k) = \frac{1}{4}x(k)^2 - 1 = 0$$

  - The solutions are $\bar{x} = \pm 2$.

2. Equilibrium

- Activity 1:

- Calculate the equilibria?

$$x_1(k+1) = \frac{1}{4}x_1(k)^2 + x_1(k) - 2u_1(k)$$
$$x_2(k+1) = \frac{1}{4}x_2(k)^2 + x_1(k) - 2u_2(k)$$

  - When $u_1(k) \equiv 0.5$ and $u_2(k) \equiv 1$, compute the equilibrium points.

2. Equilibrium

- Activity 1:

- Calculate the equilibria?

$$x_1(k+1) = \frac{1}{4}x_1(k)^2 + x_1(k) - 2u_1(k)$$
$$x_2(k+1) = \frac{1}{4}x_2(k)^2 + x_1(k) - 2u_2(k)$$

  - When $u_1(k) \equiv 0.5$ and $u_2(k) \equiv 1$, compute the equilibrium points.

- Solution:

$$x_1(k+1) - x_1(k) = \frac{1}{4}x_1(k)^2 - 1 = 0$$
$$x_2(k+1) - x_2(k) = \frac{1}{4}x_2(k)^2 + x_1 - x_2 - 2 = 0$$

2. Equilibrium

- Activity 1:

- Solution:

$$x_1(k+1) - x_1(k) = \frac{1}{4}x_1(k)^2 - 1 = 0$$

$$x_2(k+1) - x_2(k) = \frac{1}{4}x_2(k)^2 + x_1 - x_2 - 2 = 0$$

  - The solutions are $\bar{x}_1 = \pm 2$.
  - When $\bar{x}_1 = 2$.

$$\frac{1}{4}x_2(k)^2 - x_2 = 0$$

  - The solutions are $\bar{x}_2 = 0 \; or \; \bar{x}_2 = 4$.

2. Equilibrium

- Activity 1:

- Solution:

$$x_1(k+1) - x_1(k) = \frac{1}{4}x_1(k)^2 - 1 = 0$$

$$x_2(k+1) - x_2(k) = \frac{1}{4}x_2(k)^2 + x_1 - x_2 - 2 = 0$$

  - The solutions are $\bar{x}_1 = \pm 2$.
  - When $\bar{x}_1 = -2$.

$$\frac{1}{4}x_2(k)^2 - x_2 - 4 = 0$$

  - The solutions are $\bar{x}_2 = 2 \pm 2\sqrt{5}$.

3. Linearization

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Given a nonlinear time invariant system

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)) \qquad \mathbf{x} \in \mathbb{R}^n \quad \mathbf{u} \in \mathbb{R}^m$$

$$\mathbf{y}(k) = \mathbf{g}(\mathbf{x}(k), \mathbf{u}(k)) \qquad \mathbf{y} \in \mathbb{R}^p$$

and an equilibrium

$$\mathbf{u}(k) = \bar{\mathbf{u}} \quad \mathbf{x}(k) = \bar{\mathbf{x}} \quad \mathbf{y}(k) = \bar{\mathbf{y}} \quad \forall k$$

We can locally approximate the nonlinear system, around the equilibrium, with the linearized system

$$\delta\mathbf{x}(k+1) = \mathbf{A}\delta\mathbf{x}(k) + \mathbf{B}\delta\mathbf{u}(k)$$

$$\delta\mathbf{y}(k) = \mathbf{C}\delta\mathbf{x}(k) + \mathbf{D}\delta\mathbf{u}(k)$$

where

$$\delta\mathbf{x}(k) = \mathbf{x}(k) - \bar{\mathbf{x}} \quad \delta\mathbf{u}(k) = \mathbf{u}(k) - \bar{\mathbf{u}} \quad \delta\mathbf{y}(k) = \mathbf{y}(k) - \bar{\mathbf{y}}$$

and

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \quad \mathbf{B} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \quad \mathbf{C} = \left.\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \quad \mathbf{D} = \left.\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}$$

3. Linearization

- Example:

- Linearize the nonlinear discrete-time system

$$x_1(k+1) = \frac{1}{4}x_1(k)^2 + x_1(k) - 2u_1(k)$$
$$x_2(k+1) = \frac{1}{4}x_2(k)^2 + x_1(k) - 2u_2(k)$$

- Solution:

$$\frac{\partial f_1}{\partial x_1} = \frac{1}{2}x_1 + 1, \frac{\partial f_1}{\partial x_2} = 0$$
$$\frac{\partial f_2}{\partial x_1} = 1, \frac{\partial f_2}{\partial x_2} = \frac{1}{2}x_2$$

3. Linearization

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- Example:

- Solution:

$$\frac{\partial f_1}{\partial x_1} = \frac{1}{2}x_1 + 1, \frac{\partial f_1}{\partial x_2} = 0$$

$$\frac{\partial f_2}{\partial x_1} = 1, \frac{\partial f_2}{\partial x_2} = \frac{1}{2}x_2$$

- At equilibrium point $\bar{x}_1 = 2, \bar{x}_2 = 4$:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

- At equilibrium point $\bar{x}_1 = 2, \bar{x}_2 = 0$:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

3. Linearization

- Activity 2:

$$x(k+1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

  - When $u(k) \equiv 0.5$, the equilibrium points are $\bar{x} = \pm\sqrt{2}$.

- Linearization?

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- Activity 2:

$$x(k+1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

  - When $u(k) \equiv 0.5$, the equilibrium points are $\bar{x} = \pm\sqrt{2}$.

- Linearization?

  - $\mathrm{A} = x(k) + 1, B = -2$
  - When at equilibrium points $\bar{x} = \sqrt{2}$:

$$\delta x(k+1) = (1+\sqrt{2})\delta x(k) - 2\delta u(k)$$

  - When at equilibrium points $\bar{x} = -\sqrt{2}$:

$$\delta x(k+1) == (1-\sqrt{2})\delta x(k) - 2\delta u(k)$$

4. Stability

- As the linearized system is a linear discrete-time system, we can assess the stability of the equilibrium point of the nonlinear system by analyzing the state matrix

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}$$

- We can state the following results:
  - if all the eigenvalues of matrix $\boldsymbol{A}$ lie inside the unit circle (λ_i < 1), the equilibrium point is asymptotically stable,
  - if at least one eigenvalue of matrix $\boldsymbol{A}$ lies outside the unit circle ($\exists i$:λ_i > 1), the equilibrium point is unstable
  - If the eigenvalues of matrix $\boldsymbol{A}$ lie inside the unit circle and there is at least one eigenvalue on the circumference the linearization, that is a first order approximation, is too rough to assess the stability of the equilibrium point.

4. Stability

- Example:

$$x(k+1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

  - When $u(k) \equiv 0.5$, the equilibrium points are $\bar{x} = \pm\sqrt{2}$.

- Linearization?

  - $\mathrm{A} = x(k) + 1, B = -2$
  - When at equilibrium points $\bar{x} = \sqrt{2}$: **(*unstable*)**

$$\delta x(k+1) = (1+\sqrt{2})\delta x(k) - 2\delta u(k)$$

  - When at equilibrium points $\bar{x} = -\sqrt{2}$: **(stable)**

$$\delta x(k+1) == (1-\sqrt{2})\delta x(k) - 2\delta u(k)$$

4. Stability

Matlab Example:

$$x(k+1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When at equilibrium points $\bar{x} = \sqrt{2}$: **(*unstable*)**

- When at equilibrium points $\bar{x} = -\sqrt{2}$: **(stable)**

❖ Example 1:

$$x(k+1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When at equilibrium points $\bar{x} = \sqrt{2}$: ***(unstable)***

$$\delta x(k+1) = (1+\sqrt{2})\delta x(k) - 2\delta u(k)$$

- Apply state feedback control law

$$\delta u(k) = -K\delta x(k)$$

$$\delta x(k+1) = (A - KB)\delta x(k) = (1 + \sqrt{2} + 2K)\delta x(k)$$

- Let $(A - KB) = 1 - \sqrt{2}$, then $K = -\sqrt{2}$
- $\delta u(k) = -K\delta x(k) = \sqrt{2}\delta x(k)$

5. State Feedback Control

- Example 1:

$$x(k+1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When at equilibrium points $\bar{x} = \sqrt{2}$: **(*unstable*)**

$$\delta x(k+1) = (1+\sqrt{2})\delta x(k) - 2\delta u(k)$$

- Apply state feedback control law
- $\delta u(k) = -K\delta x(k) = \sqrt{2}\delta x(k)$
- As $\delta u(k) = u(k) - \bar{u}, \delta x(k) = x(k) - \bar{x}$,

$$u(k) - \bar{u} = -K(x(k) - \bar{x}), u(k) = -K(x(k) - \bar{x}) + \bar{u}$$

$$u(k) = \sqrt{2}x(k) - 1.5$$

- Matlab example

5. State Feedback Control

- Example 2:

$$x(k+1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

  - When at equilibrium points $\bar{x} = \sqrt{2}$: **(*unstable*)**

$$\delta x(k+1) = (1+\sqrt{2})\delta x(k) - 2\delta u(k)$$

  - Apply state feedback control law

$$\delta u(k) = -K\delta x(k)$$

$$\delta x(k+1) = (A - KB)\delta x(k) = (1 + \sqrt{2} + 2K)\delta x(k)$$

  - Let $(A - KB) = 0$, then $u = ?$
  - **Activity 3**

5. State Feedback Control

- Example 2:

$$x(k+1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When at equilibrium points $\bar{x} = \sqrt{2}$: ***(unstable)***

$$\delta x(k+1) = (1+\sqrt{2})\delta x(k) - 2\delta u(k)$$

- Apply state feedback control law

$$\delta u(k) = -K\delta x(k)$$

$$\delta x(k+1) = (A-KB)\delta x(k) = (1+\sqrt{2}+2K)\delta x(k)$$

- Let $(A-KB)=0$, then $K = -\frac{1+\sqrt{2}}{2}$
- $\delta u(k) = -K\delta x(k) = \frac{1+\sqrt{2}}{2}\delta x(k)$

5. State Feedback Control

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❖ Example 2:

$$x(k+1) = \frac{1}{2}x(k)^2 + x(k) - 2u(k)$$

- When at equilibrium points $\bar{x} = \sqrt{2}$: ***(unstable)***

$$\delta x(k+1) = (1+\sqrt{2})\delta x(k) - 2\delta u(k)$$

- Apply state feedback control law
- $\delta u(k) = -K\delta x(k) = \frac{1+\sqrt{2}}{2}\delta x(k)$
- As $\delta u(k) = u(k) - \bar{u}, \delta x(k) = x(k) - \bar{x}$,

$$u(k) - \bar{u} = -K(x(k) - \bar{x}), u(k) = -K(x(k) - \bar{x}) + \bar{u}$$

$$u(k) = \frac{1+\sqrt{2}}{2}x(k) - \frac{1+\sqrt{2}}{2}$$

- Matlab example

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Next Lectures: Integrating Sensor and Control

Dr. **Liang Zhao**

Centre for Autonomous Systems
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THANK YOU

Questions?