Probability Theory I
Assignment 7

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Exercise 1

Let $\mathcal{C}$ be a monotone class and a field. Since it is a field, it is closed under complementation and finite union or intersections. Since it is monotone class, it is closed under increasing countable unions and countably decreasing intersections.

- $\varnothing \in \mathcal{C}$ since $\mathcal{C}$ is a field.
- $\mathcal{C}$ is closed under complementation from $\mathcal{C}$ being a field.
- Now let's take $(A_n) \in \mathcal{C}^{\mathbb{N}}$. Define $B_n = \bigcup_{k=1}^{n} A_k$ for $n \in \mathbb{N}$. Since $\mathcal{C}$ is a field, $B_n \in \mathcal{C}$ as finite union of elements of $\mathcal{C}$, and moreover (B_n) defines an increasing sequence. Since $\mathcal{C}$ is a monotone class, $\bigcup_{n=1}^{\infty} B_n \in \mathcal{C}$. Now this writes also $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{\infty} A_k \in \mathcal{C}$ as required for a σ-field.

Exercise 2

Now let $\mathcal{C}$ be a π-system and a λ-system. Since it is a π-system it is closed under finite intersections and as a λ-system , $\Omega \in \mathcal{C}$, it is closed under countable increasing unions and monotone differences.

- $\varnothing = \Omega \backslash \Omega \in \mathcal{C}$ since $\Omega \in \mathcal{C}$
- Want to show $\mathcal{C}$ is closed under complementation. Take $A \in \mathcal{C}$, then $A \subset \Omega$ and $A^c = \Omega \backslash A \in \mathcal{C}$ by property of λ-systems.
- Now let's take $(A_n) \in \mathcal{C}^{\mathbb{N}}$. By the previous assertion, note that for n, $A_n^c \in \mathcal{C}$. Define $B_n = \bigcup_{k=1}^{n} A_k$. Since $\mathcal{C}$ is a λ-system, $B_n^c = \bigcap_{k=1}^{n} A_k^c \in \mathcal{C}$ as finite intersections of elements of $\mathcal{C}$. By complementation again, $B_n \in \mathcal{C}$. As before, (B_n) defines an increasing sequence. Since $\mathcal{C}$ is closed under countable increasing unions, $\bigcup_{n=1}^{\infty} B_n \in \mathcal{C}$. Now this writes also $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{\infty} A_k \in \mathcal{C}$ as required for a σ-field.

Exercise 3

Suppose $\mathcal{I}, \mathcal{J} \subset \mathcal{F}$ are two π-systems on $(\Omega, \mathcal{F}, P)$. Following the hint, for $I \in \mathcal{I}$, let's consider $\pi_1 : H \mapsto P(I \cap H)$ and $\pi_2 : H \mapsto P(I) \cdot P(H)$ measures on $\mathcal{J}$. We need to check they are indeed measures.

- $\pi_1(\varnothing) = P(\varnothing \cap I) = P(\varnothing) = 0$ and $\pi_2(\varnothing) = P(\varnothing) \cdot P(I) = 0$
- Let $A_1, A_2, \cdots \in \mathcal{J}$ pairwise disjoint,

$$\begin{aligned}
\pi_1(\bigcup_{n\in\mathbb{N}} A_n) &= P(I \cap \bigcup_{n\in\mathbb{N}} A_n) \\
&= P(\bigcup_{n\in\mathbb{N}} A_n \cap I) \\
&= \sum_{n\in\mathbb{N}} P(A_n \cap I) \quad \text{since P is a measure} \\
&= \sum_{n\in\mathbb{N}} \pi_1(A_n)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\pi_2(\bigcup_{n\in\mathbb{N}} A_n) &= P(I) \cdot P(\bigcup_{n\in\mathbb{N}} A_n) \\
&= P(I) \cdot \sum_{n\in\mathbb{N}} P(A_n) \quad \text{since P is a measure} \\
&= \sum_{n\in\mathbb{N}} P(A_n) \cdot P(I) \\
&= \sum_{n\in\mathbb{N}} \pi_2(A_n)
\end{aligned}$$

So π_1 and π_2 are measures on $\mathcal{J}$.

By determination of measures, if $\pi_1 = \pi_2$ on $\mathcal{J}$, since it is a π-system, then $\pi_1 = \pi_2$ on $\sigma(\mathcal{J})$. Indeed, for any $H \in \mathcal{J}$,

$$\begin{aligned}
\pi_1(H) &= P(I \cap H) \quad \text{independence of } \mathcal{I}, \mathcal{J} \\
&= P(I) \cdot P(H) \\
&= \pi_2(H)
\end{aligned}$$

In particular, this yields that $\mathcal{I}$ and $\sigma(\mathcal{J})$ are independent.

Now fix $H \in \sigma(\mathcal{J})$ and consider the measures $\tilde{\pi}_1 : I \mapsto P(I \cap H)$ and $\tilde{\pi}_2 : I \mapsto P(I) \cdot P(H)$ on $\mathcal{I}$, we show similarly that the measures coincide on $\mathcal{I}$ hence by determination of measure on $\sigma(\mathcal{I})$. Indeed, for any $I \in \mathcal{I}$,

$$\begin{aligned}
\tilde{\pi}_1(I) &= P(I \cap H) \quad \text{independence of } \mathcal{I}, \sigma(\mathcal{J}) \\
&= P(I) \cdot P(H) \\
&= \tilde{\pi}_2(I)
\end{aligned}$$

In particular this means that $\sigma(\mathcal{I})$ and $\sigma(\mathcal{J})$ are independent.

Exercise 4