Probability Theory I
Assignment 3

Claire He

27 septembre 2022

Exercise 1

We are proving (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) Suppose events $\{A_\alpha\}_{\alpha\in I}$ are independent. Note that $\sigma(A_\alpha) := \{A_\alpha, A_\alpha^c, \varnothing, \Omega\}$. Also, for any set A, $A\cap\varnothing = \varnothing$ and $A\cap\Omega = A$ so the property of independence works on these types of sets. Moreover, for $\alpha_1,\dots,\alpha_n \in I$ $P(\bigcap_{i=1}^n A_{\alpha_i}) = \prod_{i=1}^n P(A_{\alpha_i})$ by hypothesis. Therefore, to show that our $\{\sigma(A_\alpha)\}_{\alpha\in I}$ are independent, it suffices to show that for A_1, A_2 in our events collection, $P(A_1 \bigcap A_2^c) = P(A_1)P(A_2^c)$. Will follow that for any $\alpha_0,\dots,\alpha_n\in I$, $P(\bigcap_{j=1}^n A_{\alpha_j}\cap A_{\alpha_0}^c) = \prod_{j=1}^n P(A_{\alpha_j})P(A_{\alpha_0}^c)$.

$$\begin{aligned}
P(A_1\cap A_2^c) &= P(A_1\setminus(A_2\cap A_1))\\
&= P(A_1) - P(A_2\cap A_1) \text{ by independence of events}\\
&= P(A_1) - P(A_2)P(A_1)\\
&= P(A_1)(1-P(A_2)) = P(A_1)P(A_2^c) \text{ as required.}
\end{aligned}$$

(ii) $\Rightarrow$ (i) Is straightforward using that $\sigma(A_\alpha) := \{A_\alpha, A_\alpha^c, \varnothing, \Omega\}$.

(ii) $\Rightarrow$ (iii) Suppose $\sigma(\{A_\alpha\})_{\alpha\in I}$ is independent. Let $\alpha_1,\dots,\alpha_n\in I$, note the following, for $u_i\in\mathbb{R}$

$$\{\mathbf{1}_{A_{\alpha_i}} = u_i\} = \begin{cases} A_{\alpha_i} & \text{if } u_i = 1\\ A_{\alpha_i}^c & \text{if } u_i = 0\\ \varnothing & \text{otherwise}\end{cases}$$

So for any $u_i\in\mathbb{R}$, $\mathbf{1}_{A_{\alpha_i}}^{-1}(u_i)\in\sigma(A_{\alpha_i})$. By assumption, since $\sigma(\{A_\alpha\})_{\alpha\in I}$ is independent, the random variables $(\mathbf{1}_{A_{\alpha_i}})_{i\in I}$ are independent.

(iii) $\Rightarrow$ (i) Assume events $\{\mathbf{1}_{A_\alpha}\}_{\alpha\in I}$ are independent. Note that $\mathbf{1}_{A_\alpha}$ is in L^1 since $1\in L^1$ and $0\le \mathbf{1}_{A_\alpha}\le 1$, for $\alpha_1,\dots,\alpha_n\in I$

$$\begin{aligned}
P(\bigcap_{i=1}^n A_{\alpha_i}) &= E(\mathbf{1}_{\bigcap_{i=1}^n A_{\alpha_i}}) = E(\mathbf{1}_{A_{\alpha_1}}\cdots\mathbf{1}_{A_{\alpha_n}})\\
&= E(\mathbf{1}_{A_{\alpha_1}})\cdots E(\mathbf{1}_{A_{\alpha_n}}) \text{ using independence of } \mathbf{1}_{A_{\alpha_i}}\\
&= P(A_{\alpha_1})\cdots P(A_{\alpha_n})\\
&= \prod_{i=1}^n P(A_{\alpha_i}) \text{ as required}
\end{aligned}$$

Exercise 2

($\Rightarrow$) Suppose $\sigma(Y)\subset\sigma(X)$.

— Step 1
Let's consider as the hint suggests that we take Y to be an indicator. Then, it follows that $Y = \mathbf{1}_A$ where $A \in \sigma(X)$, hence there exists $B \in \mathcal{B}(\mathbb{R})$ such that $A = X^{-1}(B)$. Then, for any ω,

$$Y(\omega) = \mathbf{1}_A(\omega) = \mathbf{1}_{X^{-1}(B)}(\omega) = \mathbf{1}_B(X(\omega))$$

Hence $Y = \mathbf{1}_B \circ X$ where $\mathbf{1}_B$ is Borel by construction.

— Step 2
Now let's take Y to be a simple function, we can write $Y = \sum_{k=1}^n a_k \mathbf{1}_{A_k}$ where (A_k) events on $(\Omega, \mathcal{F})$, and $(a_k) \in \mathbb{R}^n$, by the previous construction, can see that there exists for each A_k a $B_k \in \mathcal{B}(\mathbb{R})$ such that $A_k = X^{-1}(B_k)$. And we see that for $\omega \in \Omega$, setting $h = \sum_{k=1}^n a_k \mathbf{1}_{B_k}$ which is Borel,

$$Y(\omega) = \sum_{k=1}^n a_k \mathbf{1}_{A_k}(\omega) = \sum_{k=1}^n a_k \mathbf{1}_{X^{-1}(B_k)}(\omega) = \sum_{k=1}^n a_k \mathbf{1}_{B_k}(X(\omega)) = h(X(\omega))$$

So $Y = h \circ X$ for Y simple function.

— Step 3
Now we can take any Y real valued non negative random variable, by simple approximation theorem we can find a sequence of simple functions $Y_n \uparrow Y$ as taken as before and denoting each Borel function h_n for Y_n, so $Y_n = h_n \circ X$. And by taking Y_n increasing, we have h_n increasing and Borel. Take $h = \limsup_n h_n$ so h is also Borel. For any $\omega \in \Omega$,

$$Y(\omega) = \lim_n Y_n(\omega) = \lim_n h_n(X(\omega)) = h(X(\omega))$$

Hence, $Y = h \circ X$

— Step 4
Now taking any real valued Y, we can write $Y = Y^+ - Y^-$ where Y^+ and Y^- are non negative. So we can write $Y^+ = h_+ \circ X$ and $Y^- = h_- \circ X$ and setting $h = h_+ - h_-$, $Y = h \circ X$ as required.

($\Leftarrow$) Now suppose that $Y = h \circ X$ with h Borel. Let's show that $\sigma(Y) \subset \sigma(X)$. Take $A \in \sigma(Y)$, there exists $B \in \mathcal{B}(\mathbb{R})$ such that $A = Y^{-1}(B) = (h \circ X)^{-1}(B) = X^{-1}(h^{-1}(B))$ since h is Borel, $h^{-1}(B) \in \mathcal{B}(\mathbb{R})$ as well, and setting $B' = h^{-1}(B)$, $A = X^{-1}(B')$ hence $A \in \sigma(X)$ as required.

Exercise 3

Let $X : \Omega \to [0, \infty)$ be a random variable with $E(X) = 1$ on $(\Omega, \mathcal{F}, P)$ and the mapping $Q : \mathcal{F} \to [0, 1]$ defined by $Q(A) := \int_A X dP = E(X\mathbf{1}_A)$ for $A \in \mathcal{F}$.

Let's show it is a probability measure.

1. $Q(\varnothing) = E(X\mathbf{1}_\varnothing) = E(0) = 0$
2. For $(A_n)_{n \in \mathbb{N}}$ pairwise disjoint sets,

$$\begin{aligned} Q(\bigcup_{n\in\mathbb{N}} A_n) &= E(X\mathbf{1}_{\bigcup_{n\in\mathbb{N}} A_n}) = E(X(\sum_{n\in\mathbb{N}} \mathbf{1}_{A_n})) \\ &= E(\sum_{n\geq 1} X\mathbf{1}_{A_n}) \text{ then using exercise 5} \\ &= \sum_{n\geq 1} E(X\mathbf{1}_{A_n}) \\ &= \sum_{n\geq 1} Q(A_n) \end{aligned}$$

3. $Q(\Omega) = E(X) = 1$

Hence Q is a probability measure.

Now let $A \in \mathcal{F}$ such that $P(A) = 0$. Since X is non negative, there exists a sequence of simple functions X_n such that $X_n \uparrow X$ and $X_n = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$

$$Q(A) = \int_A X dP = \int_A \lim_n X_n dP \underset{MCT}{=} \lim_n \int \sum_{i=1}^n a_i \mathbf{1}_{A_i \cap A} dP = \lim_n \sum_{i=1}^n a_i P(A_i \cap A) \underset{P(A)=0}{=} 0$$

Now let Y be a non negative random variable. Using our simple approximation theorem, have $Y_n \uparrow Y$ sequence of simple functions such that $Y_n = \sum_{i=1}^n b_i \mathbf{1}_{B_i}$ and $Y = \lim_n Y_n$. Now define as usual

$$\int_\Omega Y dQ \quad = \quad \lim_n \sum_{i=1}^n b_i Q(B_i)$$

Using the observation that X is non negative and the simple sequence used previously, for $i = 1, \dots, n$

$$\begin{aligned}
Q(B_i) &= \int_{B_i} X dP = \lim_n \sum_{j=1}^n a_j P(A_j \cap B_i) \\
\int_\Omega Y dQ &= \lim_n \sum_{i=1}^n b_i \big(\lim_n \sum_{j=1}^n a_j P(A_j \cap B_i) \big) \\
&= \lim_n \sum_{i=1}^n b_i \big(\int_{B_i} \lim_n \sum_{j=1}^n a_j \mathbf{1}_{A_j} dP \big) \text{ by MCT} \\
&= \lim_n \sum_{i=1}^n b_i \big(\int_{B_i} \lim_n X_n dP \big) = \lim_n \sum_{i=1}^n b_i \int_\Omega \mathbf{1}_{B_i} X dP \\
&= \int_\Omega X \lim_n \sum_{i=1}^n b_i \mathbf{1}_{B_i} dP = \int_\Omega X \lim_n Y_n dP \quad \text{using MCT} \\
&= \int_\Omega XY dP \text{ as required}
\end{aligned}$$

Exercise 4

Take $(\Omega, \mathcal{F})$ to be $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and let μ the Lebesgue measure. Let $X_n(\omega) := \frac{1}{n} \mathbf{1}_{[0,n]}(\omega)$ for any $\omega \in \Omega$. Note that $X_n(\omega) \to 0$ for all $\omega \in \Omega$ and $E(X_n) = \frac{1}{n} \mu([0, n]) = 1 < \infty$. Now, see that $E(X_n) = \int_{\mathbb{R}} X_n d\mu = 1 \to 1 \neq 0$.

Exercise 5

(a) Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of non negative rv. Note that $(\sum\limits_{k=1}^n X_k)_{n \in \mathbb{N}}$ is a non negative non decreasing sequence of random variables. Also see that $(\sum\limits_{k=1}^n X_k) \uparrow X =: \sum\limits_{n \geq 1} X_n$. Using monotone convergence, $E(\sum\limits_{n \geq 1} X_n) = \sum\limits_{n \geq 1} E(X_n)$.

(b) Now assume X takes values in $\mathbb{N}$. Note that $X_n := \mathbf{1}(X \geq n)$ is a non negative sequence of random variables and $X = \sum_{n \geq 1} \mathbf{1}(X \geq n)$. Using (a),

$$\begin{aligned} E(X) &= E(\sum_{n \geq 1} \mathbf{1}(X \geq n)) \\ &= \sum_{n \geq 1} E(\mathbf{1}(X \geq n)) \\ &= \sum_{n \geq 1} P(X \geq n) \end{aligned}$$