Probability Theory I
Assignment 1

Claire He

12 septembre 2022

Notations

Exercise 1

We proceed inductively. For $n = 1$, the formula holds. For $n = 2$, the formula holds as well straightforwardly : for any two events A, B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. For that we used the following that uses the property for disjoint sets :

$$P(A) + P(B) = (P(A \cap B^c) + P(A \cap B)) + (P(B \cap A^c) + P(B \cap A)) = P(A \cup B) + P(B \cap A)$$

Now assume by induction the generalised result for a certain $n \geq 1$. Let $A_1, \ldots, A_{n+1}$ be events on $(\Omega, \mathcal{F}, P)$.

$$\begin{aligned}
P(\bigcup_{i=1}^{n+1} A_i) &= P(A_{n+1} \cup (\bigcup_{i=1}^{n} A_i) \\
&= P(\bigcup_{i=1}^{n} A_i) + P(A_{n+1}) - P(A_{n+1} \cap \bigcup_{i=1}^{n} A_i) \text{ using formula for } \ n = 2 \\
&= \sum_{i=1}^{n} P(A_i) - \sum_{i<j\leq n} P(A_i \cap A_j) + \sum_{i<j<k\leq n} P(A_i \cap A_j \cap A_k) + \cdots + \\
&\quad (-1)^{n+1} P(\bigcap_{i=1}^{n} A_i) + P(A_{n+1}) - P(A_{n+1} \cap \bigcup_{i=1}^{n} A_i) \text{ using the induction hypothesis for } \ n
\end{aligned}$$

Now let's work on the last term of the LHS.

$$\begin{aligned}
P(A_{n+1} \cap \bigcup_{i=1}^{n} A_i) &= P(\bigcup_{i=1}^{n} (A_i \cap A_{n+1})) \text{ then using induction hypothesis we get :} \\
&= \sum_{i=1}^{n} P(A_i \cap A_{n+1}) - \sum_{i<j\leq n} P(A_i \cap A_j \cap A_{n+1}) + \cdots + (-1)^{n+1} P(\bigcap_{i=1}^{n} A_i \cap A_{n+1}) \\
&= \sum_{i=1}^{n} P(A_i \cap A_{n+1}) - \sum_{i<j\leq n} P(A_i \cap A_j \cap A_{n+1}) + \ldots \\
&\quad + (-1)^n \sum_{\substack{\mathcal{J} \subset \{1,\ldots,n\} \\ |\mathcal{J}|=n-1}} P(\bigcap_{j\in\mathcal{J}} A_j \cap A_{n+1}) + (-1)^{n+1} P(\bigcap_{i=1}^{n} A_i \cap A_{n+1})
\end{aligned}$$

Now piecing the correspondingly colored terms together :

$$P(\bigcup_{i=1}^{n+1} A_i) = \sum_{i=1}^{n+1} P(A_i) - \sum_{1\leq i<j\leq n+1} P(A_i \cap A_j) + \sum_{i<j<k\leq n+1} P(A_i \cap A_j \cap A_k) + \dots$$
$$+(-1)^n \sum_{\substack{\mathcal{J}'\subset\{1,\dots,n+1\}\\ |\mathcal{J}'|=n}} P(\bigcap_{j\in\mathcal{J}'} A_j) + (-1)^{n+2} P(\bigcap_{i=1}^{n+1} A_i) \quad \text{as required for} \quad n+1.$$

Exercise 2

Take $\mathcal{C}^* = \bigcap_{\substack{\mathcal{C}\subset\mathcal{A}\\ \mathcal{A}\ \sigma-\text{algebra}}} \mathcal{A}$. Note that $\mathcal{C}^*$ is a σ-algebra as intersection of σ-algebrae.
Take any $\mathcal{G}$ σ-algebra containing $\mathcal{C}$,

$$\mathcal{C}^* \cap \mathcal{G} = \bigcap_{\substack{\mathcal{C}\subset\mathcal{A}\\ \mathcal{A}\ \sigma-\text{algebra}}} \mathcal{A} \cap \mathcal{G} = \bigcap_{\substack{\mathcal{C}\subset\mathcal{A}\\ \mathcal{A}\ \sigma-\text{algebra}}} \mathcal{A} = \mathcal{C}^*$$

So $\mathcal{C}^* \cap \mathcal{G} = \mathcal{C}^*$ hence $\mathcal{C}^* \subset \mathcal{G}$ and we conclude that $\mathcal{C}^* = \sigma(\mathcal{C})$ as required.

Exercise 3

Let $\mathcal{C} = \{\{\omega\} : \omega \in \Omega\}$ and let $\mathcal{A} := \{A \subset \Omega : A \ \text{ or } \ A^c \ \text{ is countable } \}$. We want to show that $\mathcal{A} = \sigma(\mathcal{C})$. Note that $\sigma(\mathcal{C})$ contains any countable union of singletons. It is easy to see that $\mathcal{C} \subset \mathcal{A}$ since singletons are countable.

Also straightforward that $\mathcal{A}$ is a σ-algebra : $\varnothing \in \mathcal{A}$, for $A \in \mathcal{A}$, $A^c \in \mathcal{A}$ in either cases, if A is countable, or if it isn't, by construction A^c is countable and is in $\mathcal{A}$, and if $(A_n)_n$ are events of $\mathcal{A}$, let's denote A_n^* the countable events, and $A_n^{\#}$ the ones where A_n^c is countable. Then $\bigcup_n A_n = \bigcup_n A_n^* \cup \bigcup_n ((A_n^{\#})^c)^c$. Using the stability by complementary property, the stability by union property is also verified.

Now for any $A \in \mathcal{A}$, if A is countable, then A is the union of a countable number of singletons, so $A \in \sigma(\mathcal{C})$. If A isn't, then A^c is countable, and is the union of a countable number of singletons and therefore $A^c \in \sigma(\mathcal{C})$. Using the stability by complementary, $A \in \sigma(\mathcal{C})$. Therefore, $\mathcal{A} \subset \sigma(\mathcal{C})$ and by minimality, $\mathcal{A} = \sigma(\mathcal{C})$.

Exercise 4

- $\mathcal{C}_1 = \{(a, b) : a \in \mathbb{R}, b \in \mathbb{R}, a < b\}$
  We claim that any open set in $\mathbb{R}$ can be written as countable union of open intervals and that we can even choose those intervals to be disjoint ie $\forall O \in \mathcal{B}(\mathbb{R}),\ O = \bigsqcup_{n\in\mathbb{N}}(a_n, b_n)$ where $a_n < b_n$. Therefore, $\mathcal{B}(\mathbb{R}) \subset \mathcal{C}_1$.
  Conversely, any open interval of $\mathcal{C}_1$ is an open set. Since the borelians are generated by open sets of $\mathbb{R}$, $\mathcal{C}_1 \subset \mathcal{B}(\mathbb{R})$.
  Hence $\mathcal{C}_1 = \mathcal{B}(\mathbb{R})$.
- $\mathcal{C}_2 = \{[a, b] : a \in \mathbb{R}, b \in \mathbb{R}, a < b\}$
  For this, see that for any $a < b$, $[a, b] = \bigcup_{n\geq1}\{(a - n, a) \cup (b, b + n)\}^c$ Using the previous point, $(a - n, a)$ and $(b, b + n)$ are open intervals, therefore, $[a, b] \in \mathcal{C}_1$ by stability by countable union and stability of complementary since we have shown that $\mathcal{C}_1 = \mathcal{B}(\mathbb{R})$ is a σ-algebra. Therefore, $\mathcal{C}_2 \subset \mathcal{C}_1$.
  Conversely, for any $a < b$, $(a, b) = \bigcap_{n\geq1}[a+\frac{1}{n}, b-\frac{1}{n}]$ so similarly, since we can write any open interval as a countable intersection of closed intervals, $\mathcal{C}_1 \subset \mathcal{C}_2$.
  Hence, $\mathcal{C}_2 = \mathcal{B}(\mathbb{R})$.
- $\mathcal{C}_3 = \{(a, b] : a \in \mathbb{R}, b \in \mathbb{R}, a < b\}$
  We start from the following : for any $a < b$, $(a, b] = \bigcap_{n\geq1}(a, b + \frac{1}{n})$. Since $(a, b + \frac{1}{n}) \in \mathcal{C}_1$, and this is a countable intersection, $\mathcal{C}_3 \subset \mathcal{C}_1$.

Conversely, $(a,b) = \bigcap_{n\geq 1}(a, b-\frac{1}{n}]$, hence, $\mathcal{C}_1 \subset \mathcal{C}_3$.
Therefore $\mathcal{C}_3 = \mathcal{B}(\mathbb{R})$.

— $\mathcal{C}_4 = \{[a,b) : a \in \mathbb{R}, b \in \mathbb{R}, a < b\}$
For any $a < b$, $[a,b) = \bigcap_{n\geq 1}(a-\frac{1}{n}, b)$. Since $(a-\frac{1}{n}, b) \in \mathcal{C}_1$, and this is a countable intersection, $\mathcal{C}_4 \subset \mathcal{C}_1$.
Conversely, for any $a < b$, $(a,b) = \bigcap_{n\geq 1}[a+\frac{1}{n}, b)$, hence, $\mathcal{C}_1 \subset \mathcal{C}_4$.
Therefore $\mathcal{C}_4 = \mathcal{B}(\mathbb{R})$.

— $\mathcal{C}_5 = \{(a,\infty) : a \in \mathbb{R}\}$
Since $(a,\infty) = \bigcup_{n\geq 1}(a, a+n)$, $\mathcal{C}_5 \subset \mathcal{C}_1$.
Conversely, for any $a < b$, $(a,b] = (a,\infty) \cap (b,\infty)^c$, hence $\mathcal{C}_3 \subset \mathcal{C}_5$.
Threfore $\mathcal{C}_5 = \mathcal{B}(\mathbb{R})$.

Exercise 5

Let $B \in \mathcal{G}$, first notice that $(g\circ f)^{-1}(B) = f^{-1} \circ g^{-1}(B)$. Since g is measurable, $A := g^{-1}(B) \in \mathcal{E}$. Also, since f is measurable $f^{-1}(A) = f^{-1}(g^{-1}(B)) \in \mathcal{F}$. Therefore, if $B \in \mathcal{G}$, $(g\circ f)^{-1}(B) \in \mathcal{F}$ meaning $g \circ f$ is measurable.

Now let $X_1, \dots, X_n$ be measurable, and $\varphi : \mathbb{R}^n \to \mathbb{R}$ continuous. Following the hint, we take $O_1 \times \dots O_n \in \mathbb{R}^n$ open sets since it generates $\mathcal{B}(\mathbb{R}^n)$.

First, we claim that $(X_1, \dots, X_n)^{-1}(O_1 \times \cdots \times O_n) = X_1^{-1}(O_1) \cap \cdots \cap X_n^{-1}(O_n)$

Since $(O_i)_{i=1}^n$ are open sets in $\mathbb{R}$ and $(X_i)_{i=1}^n$ are measurable, for all $i = 1, \dots, n$, $X_i^{-1}(O_i) \in \mathcal{F}$, and since $\mathcal{F}$ is a σ-algebra, it is stable by finite intersection, therefore $(X_1, \dots, X_n)^{-1}(O_1 \times \cdots \times O_n) \in \mathcal{F}$. And we have shown that $(X_1, \dots, X_n) : \Omega^n \to \mathbb{R}^n$ is measurable.

Now, since $(X_1, \dots, X_n)$ is measurable and φ is continuous therefore measurable, using the first part of the question, $\varphi \circ (X_1, \dots, X_n)$ is measurable.