Probability Theory I
Assignment 2

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Exercise 1

— First let's construct a distribution function that is continuous on the irrationals, discontinuous on the rationals.
Let's consider a bijection of $\mathbb{N}$ onto $\mathbb{Q}$ so that we can enumerate $r_1, \dots, r_n, \dots$ all rationals of $[0,1]$.
Let $f_n(x) = 2^{-n}\mathbf{1}_{\{x \geq r_n\}}$ and $F(x) := \sum\limits_{n\geq 1} f_n(x) = \sum\limits_{r_n \leq x} \frac{1}{2^n}$.
Note that $F : \mathbb{R} \to \mathbb{R}$ is strictly increasing by construction and right continuous.
Also, note that $F(x) \underset{x+\infty}{\longrightarrow} \sum\limits_{n\geq 1} \frac{1}{2^n} = 1$ and $F(x) \underset{x-\infty}{\longrightarrow} 0$.
So F is a distribution function.
Remains to prove it is discontinuous at every rational and continuous at every irrational.
Let $x \in \mathbb{Q}$, then using our bijection of $\mathbb{N}$ onto $\mathbb{Q}$, there exists a r_{n_0} such that $x = r_{n_0}$ and
$F(x^-) = \sum\limits_{r_n \geq x^-} \frac{1}{2^n} = \sum\limits_{r_n \geq r_{n_0-1}} \frac{1}{2^n} = F(x) + \frac{1}{2^{n_0-1}} > F(x)$.
Similarly, $F(x^+) = \sum\limits_{r_n \geq x^+} \frac{1}{2^n} = \sum\limits_{r_n \geq r_{n_0+1}} \frac{1}{2^n} = F(x) - \frac{1}{2^{n_0+1}} < F(x)$.
So F is discontinuous at every rational.
On the other hand, let x be an irrational, by density of irrationals, we can find a sequence of rationals a_m converging towards x from below.
$F(x) - F(a_m) = \sum\limits_{r_n \leq x} \frac{1}{2^n} - \sum\limits_{r_n \leq a_m} \frac{1}{2^n} = \sum\limits_{a_m < r_n \leq x} \frac{1}{2^n} < \sum\limits_{n \geq a_m} \frac{1}{2^n} = \frac{1}{2^{a_m-1}} \underset{m\to\infty}{\to} 0$ Hence F is continuous at every irrational.

— Conversely, assume F is a distribution function discontinuous at every irrational, continuous at every rational. Let μ be the underlying measure density corresponding to F. Since F is measurable, for x irrational, $\mu^{-1}(\{F(x) - F(x-)\}) \in \mathcal{B}(\mathbb{R})$, therefore we would have, $\mu^{-1}(\bigcup\limits_{x \text{ irrational}} \{F(x) - F(x-)\}) = \bigcup\limits_{x \text{ irrational}} \{x\} \in \mathcal{B}(\mathbb{R})$. This implies that $\bigcup\limits_{x \text{ irrational}} \{x\}$ is a Borel set, meaning irrationals are the unions of countable closed/open sets. Then we'd be able to write the union of irrationals as countable union of intervals, but this is impossible since the irrationals overlap no interval by density of rationals.

Exercise 2 : *Continuity of Measures*

Let (A_n) be an increasing sequence of measurable sets (for inclusion ie $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$) on $(\Omega, \mathcal{F}, P)$ probability space. Let $A = \bigcup_{n\in\mathbb{N}} A_n$. We can set $A_0 = \varnothing$.
Now consider $B_n := A_n \backslash A_{n-1}$ for $n \geq 1$, by construction, we see that (B_n) are disjoint measurable sets

and $A_n = \bigcup_{k=1}^{n} B_k$ for all $n \geq 1$. Now combining this and $\sigma-$additivity property of P, we have :

$$\begin{aligned} P(A) &= P(\bigcup_{n\geq 1} A_n) = P(\bigcup_{n\geq 1}\bigcup_{k=1}^{n} B_k) \\ &= \sum_{n\geq 1} P(B_n) = \lim_n \sum_{k=1}^{n} P(B_k) \\ &= \lim_n P(\bigcup_{k=1}^{n} B_k) \\ P(A) &= \lim_n P(A_n) \quad \text{as required} \end{aligned}$$

Now similarly,, if (B_n) is a decreasing sequence of measurable sets, and $B = \bigcap_n B_n$. Let's consider $A_n = B_n^c$ and $A = B^c$. Then (A_n) is an increasing sequence of measurable sets increasing towards $A = \bigcup_n A_n$ and applying the previous steps with the property that $P(B_n) = P(A_n^c) = 1 - P(A_n)$ for all n, we get the following :

$$\begin{aligned} P(B) &= 1 - P(A) = 1 - P(\bigcup_n A_n) \\ &= 1 - \lim_n P(A_n) = \lim_n 1 - P(A_n) \\ P(B) &= \lim_n P(B_n) \quad \text{as required} \end{aligned}$$

Exercise 3

— F is non decreasing :
Let $a \leq b \in \mathbb{R}$, see that $(-\infty, a] \subseteq (-\infty, b]$ hence using monotonicity of μ, $F(a) = \mu((-\infty, a]) \leq \mu((-\infty, b]) = F(b)$.

— F is right continuous :
Let $a \in \mathbb{R}$, see that $((-\infty, a + \frac{1}{n}])_{n\geq 1}$ is a decreasing sequence of measurable sets on the real line and $((-\infty, a+\frac{1}{n}])_{n\geq 1} \downarrow (-\infty, a]$. Using the property shown in *exercise 2*, $\lim_n F(a+\frac{1}{n}) = \lim_n \mu((-\infty, a+\frac{1}{n}]) = \mu((-\infty, a]) = F(a)$.

— $F(\infty) = 1$ and $F(-\infty) = 0$:
$F(\infty) = \mu((-\infty, \infty)) = \mu(\mathbb{R}) = 1$ by construction of μ probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$. Therefore, $F(-\infty) = \mu(\varnothing) = 1 - \mu(\mathbb{R}) = 0$.

Now let $a < b$ be any real numbers. Let's write $F(a-) = \lim_n F(a - \frac{1}{n}) = \mu(\bigcup_n (-\infty, a - \frac{1}{n}]) = \mu((-\infty, a))$ the left limit of F at a. This equality is obtained using continuity of measure for increasing sequences of measurable sets where $(-\infty, a - \frac{1}{n}] \uparrow \bigcup_n (-\infty, a - \frac{1}{n}] = (-\infty, a)$.

$$
\begin{aligned}
\mu((a,b]) &= \mu((-\infty,b]\backslash(-\infty,a]) \\
&= \mu((-\infty,b]) - \mu((-\infty,a]) \\
&= F(b) - F(a) \\
\mu(\{a\}) &= \mu((-\infty,a])\backslash(-\infty,a)) \\
&= \mu((-\infty,a]) - \mu((-\infty,a)) \\
&= F(a) - F(a-) \\
\mu([a,b)) &= \mu((-\infty,b)\backslash(-\infty,a)) \\
&= \mu((-\infty,b)) - \mu((-\infty,a)) \\
&= F(b-) - F(a-) \\
\mu([a,b]) &= \mu([a,b)\cup\{b\}) \quad \text{disjoint} \\
&= F(b-) - F(a-) + F(b) - F(b-) \\
&= F(b) - F(a-) \\
\mu((a,b)) &= \mu((a,b]\backslash\{b\}) \\
&= \mu((a,b]) - \mu(\{b\}) \\
&= F(b) - F(a) - F(b) - F(b-) \\
&= F(b-) - F(a)
\end{aligned}
$$

Exercise 4

Let's consider $\{\omega_1,\omega_2,\omega_3,\omega_4\} := \Omega$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $P(A) = \frac{|A|}{|\Omega|}$ as probability space. Consider events $A_1 = \{\omega_1,\omega_2\}$, $A_1 = \{\omega_1,\omega_3\}$, and $A_3 = \{\omega_2,\omega_3\}$.

$$
\begin{aligned}
P(A_1,A_2) &= P(\{\omega_1\}) = \frac{1}{4} \\
&= P(A_1)P(A_2) = (\frac{1}{2})^2 = \frac{1}{4} \\
P(A_1,A_3) &= P(\{\omega_2\}) = \frac{1}{4} \\
&= P(A_1)P(A_3) = (\frac{1}{2})^2 = \frac{1}{4} \\
P(A_2,A_3) &= P(\{\omega_3\}) = \frac{1}{4} \\
&= P(A_3)P(A_2) = (\frac{1}{2})^2 = \frac{1}{4} \\
P(A_1,A_2,A_3) &= P(\varnothing) = 0 \\
P(A_1)P(A_2)P(A_3) &= \frac{1}{2^3} = \frac{1}{8} \neq 0
\end{aligned}
$$

Hence A_1, A_2, A_3 are pairwise independent but not independent.

Exercise 5

Let $\mathcal{J}$ be at most countable, $(\Omega, \mathcal{F}, P)$ a probability space. Let $(A_j)_{j\in\mathcal{J}}$, $(B_j)_{j\in\mathcal{J}}$ be measurable sets with $B_j \subset A_j$ for any $j \in \mathcal{J}$.

$$
\begin{aligned}
P(\bigcup_{j\in\mathcal{J}} A_j) - P(\bigcup_{j\in\mathcal{J}} B_j) &= P(\bigcup_{j\in\mathcal{J}} B_j \bigsqcup (A_j \backslash B_j)) - P(\bigcup_{j\in\mathcal{J}} B_j) \\
&= P(\bigcup_{j\in\mathcal{J}} A_j \backslash B_j) + P(\bigcup_{j\in\mathcal{J}} B_j) - P(\bigcup_{j\in\mathcal{J}} B_j) \\
&\leq \sum_{j\in\mathcal{J}} P(A_j \backslash B_j) \quad \text{using} \quad \sigma\text{-additivity} \\
&\leq \sum_{j\in\mathcal{J}} P(A_j) - P(B_j) \quad \text{as required.}
\end{aligned}
$$