Gauss-Hermite Quadrature

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1 One-dimensional quadrature

Guassian quadrature produces approximations of the form

$$\int_a^b f(x)w(x)dx \approx \sum_{i=1}^{N_{\text{Gauss}}} \omega_i f(\xi_i)$$

where $w(x)$ is a weighting function, ξ_i are quadrature nodes and ω_i are quadrature weights. The approximation is exact for polynomials of degree $2N_{\text{Gauss}} - 1$ using N_{Gauss} quadrature nodes.

Gauss-Hermite quadrature arises when one writes

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}dx = \sum_{n=1}^{N_{\text{Gauss}}} \omega_n f(\xi_n) + \frac{N_{\text{Gauss}}!\sqrt{\pi}}{2^{N_{\text{Gauss}}}} \cdot \frac{f^{(2N_{\text{Gauss}})}(z)}{(2N_{\text{Gauss}})!}, \qquad z \in (-\infty, \infty).$$

It is used in connection with Normal random variables. Let $Y \sim \mathcal{N}(m, \sigma^2)$, then

$$\mathbb{E}[f(Y)] = \left(2\pi\sigma^2\right)^{-1/2} \int_{-\infty}^{\infty} f(y)e^{-(y-m)^2/2\sigma^2}dy.$$

By the change of variable $x = (y - m)/\sqrt{2\sigma^2}$, one obtains

$$\mathbb{E}[f(Y)] = \pi^{-1/2} \int_{-\infty}^{\infty} f\left(m + \sqrt{2\sigma^2}x\right) e^{-x^2}dx \approx \pi^{-1/2} \sum_{n=1}^{N_{\text{Gauss}}} \omega_n f\left(m + \sqrt{2\sigma^2}\xi_n\right).$$

Proposition 1. *To compute the expectation of $f(Y)$ with $Y \sim \mathcal{N}(m, \sigma^2)$ using one-dimensional Gauss-Hermite quadrature, the nodes and associated weights are*

$$\left(m + \sqrt{2\sigma^2}\xi_n, \omega_n\right), n = 1, \ldots, N_{Gauss}$$

2 Multidimensional quadrature

See section 7.5 in Judd (1998). Let $m = [m_1, \ldots, m_d]'$ and Σ with generic element σ_{ij} and suppose $Y \sim \mathcal{N}_d(m, \Sigma)$, then

$$\mathbb{E}[f(Y)] = (2\pi)^{-d/2} (\det \Sigma)^{-1/2} \int_{\mathbb{R}^d} f(y)e^{-\frac{1}{2}(y-m)^\top \Sigma^{-1}(y-m)}dy.$$

Let $\Sigma = LL^\top$ be the Cholesky decomposition of the covariance matrix and recall that $L =$

$(l_{ij})_{1\le i,j\le d}$ is lower triangular, $\left(L^{-1}\right)^T L^{-1} = \Sigma^{-1}$ and $\det \Sigma = (\det L)^2$. Apply the change of variable $x = L^{-1}(y-m)/\sqrt{2}$. Then $y = m + \sqrt{2}Lx$ and $x^\top x = \frac{1}{2}(y-m)^\top \left(L^{-1}\right)^T L^{-1}(y-m) = \frac{1}{2}(y-m)^\top \Sigma^{-1}(y-m)$. We can also write

$$y_i = m_i + \sqrt{2}\sum_{k=1}^{d} l_{ik} x_k$$

so that

$$\frac{\partial}{\partial x_j} y_i = \sqrt{2} l_{ij}$$

hence the Jacobian of the change of variable is $\sqrt{2}L$ and its determinant is $2^{d/2}\det L$. The formula for the change of variable gives

$$\begin{aligned}\mathbb{E}\left[f(Y)\right] &= (2\pi)^{-d/2}\left(\det \Sigma\right)^{-1/2}\int_{\mathbb{R}^d} f(m+\sqrt{2}Lx)e^{-x^\top x}\left|2^{d/2}\det L\right| dx\\ &= \pi^{-d/2}\int_{\mathbb{R}^d} f(m+\sqrt{2}Lx)e^{-x^\top x}dx.\end{aligned}$$

Due to the lower triangular nature of L, and denoting $l'_{ij} \coloneqq \sqrt{2}l_{ij}$, one can also write the change of variable as

$$\begin{aligned}y_1 &= m_1 + l'_{11}x_1\\ y_2 &= m_2 + l'_{21}x_1 + l'_{22}x_2\\ &\vdots\\ y_j &= m_j + \textstyle\sum_{i=1}^{j} l'_{ji}x_i\end{aligned}$$

hence the expectation as

$$\mathbb{E}\left[f(Y)\right] = \pi^{-d/2}\int_{\mathbb{R}^d} f\left(m_1 + l'_{11}x_1, \ldots, m_d + \textstyle\sum_{i=1}^{d} l'_{di}x_i\right)\prod_{i=1}^{d} e^{-x_i^2}dx_1\cdots x_d$$

This expression is now amenable to recursive use of univariate quadratures

$$\begin{aligned}\mathbb{E}\left[f(Y)\right] &= \pi^{-d/2}\int_{\mathbb{R}^{d-1}}\prod_{i=1}^{d-1} e^{-x_i^2}\left(\int_{\mathbb{R}} f\left(m_1 + l'_{11}x_1, \ldots, m_d + \textstyle\sum_{i=1}^{d} l'_{di}x_i\right)e^{-x_d^2}dx_d\right)dx_1\cdots dx_{d-1}\\ &\approx \pi^{-d/2}\int_{\mathbb{R}^{d-1}}\prod_{i=1}^{d-1} e^{-x_i^2}\left(\sum_{n_d=1}^{N_{\text{Gauss}}}\omega_{n_d} f\left(m_1 + l'_{11}x_1, \ldots, m_d + \textstyle\sum_{i=1}^{d-1} l'_{di}x_i + l'_{dd}\xi_{n_d}\right)\right)dx_1\cdots dx_{d-1}\\ &\approx \pi^{-d/2}\sum_{n_d=1}^{N_{\text{Gauss}}}\omega_{n_d}\int_{\mathbb{R}^{d-1}}\prod_{i=1}^{d-1} e^{-x_i^2} f\left(m_1 + l'_{11}x_1, \ldots, m_d + \textstyle\sum_{i=1}^{d-1} l'_{di}x_i + l'_{dd}\xi_{n_d}\right)dx_1\cdots dx_{d-1}\end{aligned}$$

then the inner integral is equal to

$$\int_{\mathbb{R}^{d-2}} \prod_{i=1}^{d-2} e^{-x_i^2} \left\{ \int_{\mathbb{R}} f \begin{pmatrix} m_1 + l'_{11} x_1 \\ \vdots \\ m_{d-1} + \sum_{i=1}^{d-2} l'_{d-1,i} x_i + l'_{d-1,d-1} x_{d-1} \\ m_d + \sum_{i=1}^{d-2} l'_{di} x_i + l'_{d,d-1} x_{d-1} + l'_{dd} \xi_{n_d} \end{pmatrix} e^{-x_{d-1}^2} dx_{d-1} \right\} dx_1 \cdots dx_{d-2}$$

$$\approx \int_{\mathbb{R}^{d-2}} \prod_{i=1}^{d-2} e^{-x_i^2} \left\{ \sum_{n_{d-1}=1}^{N_{\text{Gauss}}} \omega_{n_{d-1}} f \begin{pmatrix} m_1 + l'_{11} x_1 \\ \vdots \\ m_{d-1} + \sum_{i=1}^{d-2} l'_{d-1,i} x_i + l'_{d-1,d-1} \xi_{n_{d-1}} \\ m_d + \sum_{i=1}^{d-2} l'_{di} x_i + l'_{d,d-1} \xi_{n_{d-1}} + l'_{dd} \xi_{n_d} \end{pmatrix} \right\} dx_1 \cdots dx_{d-2}$$

so that

$$\mathbb{E}\left[f(Y)\right] = \pi^{-d/2} \sum_{n_d=1}^{N_{\text{Gauss}}} \omega_{n_d} \sum_{n_{d-1}=1}^{N_{\text{Gauss}}} \omega_{n_{d-1}} \int_{\mathbb{R}^{d-2}} \prod_{i=1}^{d-2} e^{-x_i^2} f \begin{pmatrix} m_1 + l'_{11} x_1 \\ \vdots \\ m_{d-1} + \sum_{i=1}^{d-2} l'_{d-1,i} x_i + l'_{d-1,d-1} \xi_{n_{d-1}} \\ m_d + \sum_{i=1}^{d-2} l'_{di} x_i + l'_{d,d-1} \xi_{n_{d-1}} + l'_{dd} \xi_{n_d} \end{pmatrix} dx_1 \cdots dx_{d-2}$$

and finally

$$\mathbb{E}\left[f(Y)\right] = \pi^{-d/2} \sum_{n_1=1}^{N_{\text{Gauss}}} \cdots \sum_{n_d=1}^{N_{\text{Gauss}}} \omega_{n_1} \cdots \omega_{n_d} f \begin{pmatrix} m_1 + l'_{11} \xi_{n_1} \\ \vdots \\ m_{d-1} + \sum_{i=1}^{d-1} l'_{d-1,i} \xi_{n_i} \\ m_d + \sum_{i=1}^{d} l'_{di} \xi_{n_i} \end{pmatrix}$$

2.1 The bivariate case

Especially, when $d = 2$, one has

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \quad L = \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix}$$

hence the Gauss-Hermite quadrature formula is

$$\mathbb{E}\left[f(Y)\right] = \pi^{-1} \sum_{n_1=1}^{N_{\text{Gauss}}} \sum_{n_2=1}^{N_{\text{Gauss}}} \omega_{n_1}\omega_{n_2} f\left(m_1 + \sigma_1\sqrt{2}\xi_{n_1}, m_2 + \sigma_2\sqrt{2}\left(\rho\xi_{n_1} + \sqrt{1-\rho^2}\xi_{n_2}\right)\right) \tag{1}$$

Proposition 2. *To compute the expectation of* $f(Y_1, Y_2)$ *with* $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} m_1 \\ m_1 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}\right)$

using two-dimensional Gauss-Hermite quadrature, the nodes and associated weights are

$$\left(\begin{bmatrix} m_1+\sqrt{2\sigma_1^2}\xi_{n_1} \\ m_2+\sqrt{2\sigma_2^2}\left(\rho\xi_{n_1}+\sqrt{1-\rho^2}\xi_{n_2}\right)\end{bmatrix},\omega_{n_1}\omega_{n_2}\right),n_1=1,\dots,N_{Gauss},n_2=1,\dots,N_{Gauss}$$

2.2 Application: optimal portfolio for a CRRA investor and two risky assets

There are two risky assets with excess returns $r=(r_1,r_2)^\top$ and a risk-free asset with return r_f. The investor assumes that returns are jointly bivariate Normal $r\sim\mathcal{N}_2(m,\Sigma)$ and she has a Gaussian prior over the mean vector m, that is $m\sim\mathcal{N}_2(\mu,\Lambda)$ where $\Lambda=\operatorname{diag}\left(\lambda_1^2,\lambda_2^2\right)$. Her predictive probability p is then $r\sim\mathcal{N}_2(\mu,\Sigma+\Lambda)$. The investor has an initial wealth W_0 and chooses a portfolio where fractions $\theta=(\theta_1,\theta_2)^\top$ are invested in the risky assets and $\theta_0=1-\theta_1-\theta_2$ in the risk-free asset. Therefore the end of period wealth is $W_T=W_0(1+r_f+\theta_1r_1+\theta_2r_2)$ and its expected utility for a CRRA investor with risk-aversion parameter γ is

$$\mathbb{E}_p\left[u\left(W_T\right)\right]=\frac{W_0^{1-\gamma}}{1-\gamma}\mathbb{E}_p\left[(1+r_f+\theta_1r_1+\theta_2r_2)^{1-\gamma}\right]$$

The FOCs are

$$0=\mathbb{E}_p\left[(1+r_f+\theta_1r_1+\theta_2r_2)^{-\gamma}r_i\right]\qquad i=1,2.$$

Using GH quadrature:

1. Compute the N_{Gauss} vector $R^{(1)}\in\mathbb{R}$ such that

$$R_{n_1}^{(1)}=\mu_1+\sqrt{2(\sigma_1^2+\lambda_1^2)}\xi_{n_1}$$

2. Compute the $N_{\text{Gauss}}\times N_{\text{Gauss}}$ square matrix $R^{(1)}\in\mathbb{R}^2$ such that

$$R_{n_1,n_2}^{(2)}=\mu_2+\sqrt{2\left(\sigma_2^2+\lambda_2^2\right)}\left(c\xi_{n_1}+\sqrt{1-c^2}\xi_{n_2}^\top\right)$$

   where $c=\rho\sigma_1\sigma_2/\sqrt{\left(\sigma_1^2+\lambda_1^2\right)\left(\sigma_2^2+\lambda_2^2\right)}$.

3. Compute the $N_{\text{Gauss}}\times N_{\text{Gauss}}$ square matrix $R(\theta)\in\mathbb{R}^4$ such that

$$R(\theta)=\left(1+r_f+\theta_1R^{(1)}+\theta_2R^{(2)}\right)**(-\gamma)$$

   where scalars and the vector $R^{(1)}$ are broadcasted to $N_{\text{Gauss}}\times N_{\text{Gauss}}$ square matrices.

4. Compute the two components of the function FOC:

$$FOC^{(1)}(\theta)=\pi^{-1}\sum_{n_1,n_2=1}^{N_{\text{Gauss}}}\omega_{n_1}\omega_{n_2}\left\{R_{n_1,n_2}(\theta)*R_{n_1}^{(1)}\right\}$$

$$FOC^{(2)}(\theta)=\pi^{-1}\sum_{n_1,n_2=1}^{N_{\text{Gauss}}}\omega_{n_1}\omega_{n_2}\left\{R_{n_1,n_2}(\theta)*R_{n_1,n_2}^{(2)}\right\}$$

   where $*$ denotes element-wise multiplication.

3 Double integration with a Bayesian prior

Assume that μ and Λ are the parameters of the Normal prior over the vector of means m. We now want to compute $\mathbb{E}_\mu\left[\phi\left(\mathbb{E}_m\left[f(Y)\right]\right)\right]$. We stay in the bivariate case so that the inner expectation is given by equation (1) and $\psi\left(m_1, m_2\right) := \phi\left(\mathbb{E}_m\left[f(Y)\right]\right)$ is now seen as a function of m_1 and m_2. Let

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1^2 & \varrho\lambda_1\lambda_2 \\ \rho\lambda_1\lambda_2 & \lambda_2^2 \end{bmatrix} \quad L = \begin{bmatrix} \lambda_1 & 0 \\ \varrho\lambda_2 & \upsilon_2\sqrt{1-\varrho^2} \end{bmatrix}$$

Then

$$\mathbb{E}_\mu\left[\psi\left(m_1, m_2\right)\right] \approx \pi^{-1} \sum_{p_1, p_2 = 1}^{N_{\text{Gauss}}} \omega_{p_1}\omega_{p_2}\psi\left(\mu_1 + \lambda_1\sqrt{2}\xi_{p_1}, \mu_2 + \lambda_2\sqrt{2}\left(\varrho\xi_{p_1} + \sqrt{1-\varrho^2}\xi_{p_2}\right)\right)$$

and

$$\begin{aligned}\mathbb{E}_\mu\left[\phi\left(\mathbb{E}_m\left[f(Y)\right]\right)\right] \approx \pi^{-1} \sum_{p_1, p_2 = 1}^{N_{\text{Gauss}}} \omega_{p_1}\omega_{p_2}\phi\Bigg\{ \pi^{-1} \sum_{n_1, n_2 = 1}^{N_{\text{Gauss}}} \omega_{n_1}\omega_{n_2} \\ f\begin{pmatrix} \mu_1 + \lambda_1\sqrt{2}\xi_{p_1} + \sigma_1\sqrt{2}\xi_{n_1} \\ \mu_2 + \lambda_2\sqrt{2}\left(\varrho\xi_{p_1} + \sqrt{1-\varrho^2}\xi_{p_2}\right) + \sigma_2\sqrt{2}\left(\rho\xi_{n_1} + \sqrt{1-\rho^2}\xi_{n_2}\right) \end{pmatrix}\Bigg\}\end{aligned}$$

3.1 Application: Bivariate Normal with Normal conjugate Bayesian prior

Let $\theta = \left(\theta_1, \theta_2\right)^\top$ and $r = \left(r_1, r_2\right)^\top$. We need to write the function $FOC \colon \mathbb{R}^2 \to \mathbb{R}^2$

$$FOC\left(\theta\right) = \mathbb{E}_\mu\left[\left(\mathbb{E}_m\left[\left(R_f + \theta_1 r_1 + \theta_2 r_2\right)^{1-\gamma}\right]\right)^{\frac{\gamma-\eta}{1-\gamma}} \mathbb{E}_m\left[\left(R_f + \theta_1 r_1 + \theta_2 r_2\right)^{-\gamma} r\right]\right]$$

using GH quadrature.

1. Compute the $\left(N_{\text{Gauss}}\right)^4$ square matrix $R^{(1)} \in \mathbb{R}^4$ such that

$$R^{(1)}_{p_1, p_2, n_1, n_2} = \mu_1 + \sqrt{2\lambda_{11}}\xi_{p_1} + \sqrt{2s_{11}}\xi_{n_1}.$$

2. Compute the $\left(N_{\text{Gauss}}\right)^4$ square matrix $R^{(2)} \in \mathbb{R}^4$ such that

$$R^{(2)}_{p_1, p_2, n_1, n_2} = \mu_2 + \varrho\sqrt{2\lambda_{22}}\xi_{p_1} + \sqrt{2\lambda_{22}\left(1-\varrho^2\right)}\xi_{p_2} + \rho\sqrt{2s_{22}}\xi_{n_1} + \sqrt{2s_{22}\left(1-\rho^2\right)}\xi_{n_2}.$$

3. Compute the $\left(N_{\text{Gauss}}\right)^4$ square matrix $R\left(\theta\right) \in \mathbb{R}^4$ such that

$$R\left(\theta\right) = R_f + \theta_1 R^{(1)} + \theta_2 R^{(2)}$$

where R_f, θ_1, θ_2 are scalars and R_f is broadcasted to an $\left(N_{\text{Gauss}}\right)^4$ square matrix.

4. Compute the $N_{\text{Gauss}} \times N_{\text{Gauss}}$ square matrix $E^{(1)} \in \mathbb{R}^2$ such that

$$E^{(1)}_{p_1,p_2} = \pi^{-1} \sum_{n_1,n_2=1}^{N_{\text{Gauss}}} \omega_{n_1}\omega_{n_2} \left(R_{p_1,p_2,n_1,n_2}(\theta) ** (1-\gamma)\right)$$

   where $**$ denotes the element-wise power operation.

5. Compute the matrix $E^{(21)} \in \mathbb{R}^2$ such that

$$E^{(21)}_{p_1,p_2} = \pi^{-1} \sum_{n_1,n_2=1}^{N_{\text{Gauss}}} \omega_{n_1}\omega_{n_2} \left\{\left(R_{p_1,p_2,n_1,n_2}(\theta) ** (-\gamma)\right) * R^{(1)}\right\}$$

   where $*$ denotes element-wise multiplication.

6. Compute the matrix $E^{(22)} \in \mathbb{R}^2$ such that

$$E^{(22)}_{p_1,p_2} = \pi^{-1} \sum_{n_1,n_2=1}^{N_{\text{Gauss}}} \omega_{n_1}\omega_{n_2} \left\{\left(R_{p_1,p_2,n_1,n_2}(\theta) ** (-\gamma)\right) * R^{(2)}\right\}.$$

7. Compute the two components of the function FOC:

$$FOC^{(1)}(\theta) = \pi^{-1} \sum_{p_1,p_2=1}^{N_{\text{Gauss}}} \omega_{p_1}\omega_{p_2} \left\{\left(E^{(1)}_{p_1,p_2} ** \left(\frac{\gamma-\eta}{1-\gamma}\right)\right) * E^{(21)}_{p_1,p_2}\right\}$$
$$FOC^{(2)}(\theta) = \pi^{-1} \sum_{p_1,p_2=1}^{N_{\text{Gauss}}} \omega_{p_1}\omega_{p_2} \left\{\left(E^{(1)}_{p_1,p_2} ** \left(\frac{\gamma-\eta}{1-\gamma}\right)\right) * E^{(22)}_{p_1,p_2}\right\}$$

References

Judd, K. L. (1998): *Numerical Methods in Economics*, MIT Press.