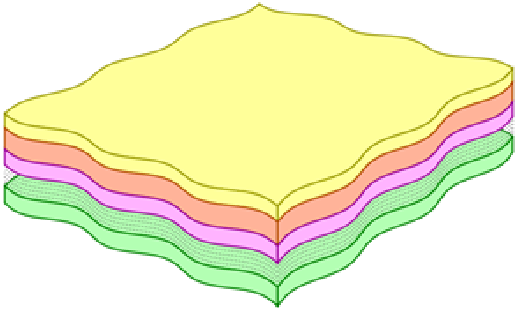
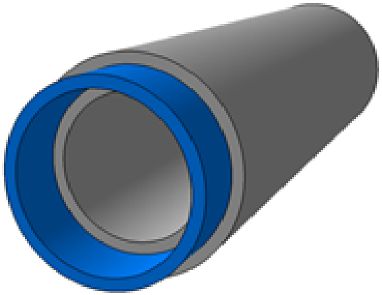
Transient Field Computation in multilayered structures





Time $t \to$ complex Laplace variable s

Space:

- material properties invariant with respect to $\mathbf{x} = (x, y)$
- Horizontal position $\mathbf{x} \to$ Horizontal wavevector $\mathbf{k}$
- Computation in the $(\mathbf{k}, z, s)$-domain
- ODEs with respect to the vertical position z

Space:

- material properties invariant with respect to θ and z
- Axial position $z \to$ ax. wavenumber k
Azimuthal position $\theta \to$ az. wavenumber n
- Computation in the (r, n, k, s)-domain
- ODEs with respect to the radial position r

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▷ The displacement vector $\tilde{\mathbf{U}}(z)$ satisfies in each plane layer ($\mathbf{n}$ unit vertical vector, $\mathbb{I}$ identity matrix):

$$(\mathbf{n} \diamond \mathbf{n})\, \tilde{\mathbf{U}}''(z) - \mathbb{i}\, \left[(\mathbf{n} \diamond \mathbf{k}) + (\mathbf{k} \diamond \mathbf{n})\right] \tilde{\mathbf{U}}'(z) - \left[(\mathbf{k} \diamond \mathbf{k}) + \rho\, s^2\, \mathbb{I}\right] \tilde{\mathbf{U}}(z) = -\tilde{\mathbf{F}}(z) \tag{1}$$

$$\text{Stress vector in the } z\text{-direction:} \quad \tilde{\mathbf{\Sigma}}_z(z) = (\mathbf{n} \diamond \mathbf{n})\, \tilde{\mathbf{U}}'(z) - \mathbb{i}\, (\mathbf{n} \diamond \mathbf{k})\, \tilde{\mathbf{U}}(z) \tag{2}$$

▷ The displacement vector $\tilde{\mathbf{U}}(r)$ satisfies in each tubular layer:

$$\left[\mathbb{T} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

$$\begin{aligned} & (\mathbf{n}_r \diamond \mathbf{n}_r)\, \tilde{\mathbf{U}}''(r) \; + \\ & \Big\{ -\mathbb{i}\, k\, (\mathbf{n}_r \diamond \mathbf{n}_z + \mathbf{n}_z \diamond \mathbf{n}_r) \; + \\ & \frac{1}{r} \left[\, (\mathbf{n}_r \diamond \mathbf{n}_r) - \mathbb{i}\, \left(\, (\mathbf{n}_r \diamond \mathbf{n}_\theta)\, (n\, \mathbb{I} + \mathbb{i}\, \mathbb{T}) + (n\, \mathbb{I} + \mathbb{i}\, \mathbb{T})\, (\mathbf{n}_\theta \diamond \mathbf{n}_r)\, \right) \right] \Big\}\, \tilde{\mathbf{U}}'(r) \; - \\ & \Big\{ \left[\, \rho\, s^2\, \mathbb{I} + k^2\, (\mathbf{n}_z \diamond \mathbf{n}_z)\, \right] + \\ & \frac{k}{r} \left[\, \mathbb{i}\, (\mathbf{n}_r \diamond \mathbf{n}_z) + (n\, \mathbb{I} + \mathbb{i}\, \mathbb{T})\, (\mathbf{n}_\theta \diamond \mathbf{n}_z) + (\mathbf{n}_z \diamond \mathbf{n}_\theta)\, (n\, \mathbb{I} + \mathbb{i}\, \mathbb{T})\, \right] + \\ & \frac{1}{r^2}\, (n\, \mathbb{I} + \mathbb{i}\, \mathbb{T})\, (\mathbf{n}_\theta \diamond \mathbf{n}_\theta)\, (n\, \mathbb{I} + \mathbb{i}\, \mathbb{T}) \Big\}\, \tilde{\mathbf{U}}(r) = -\tilde{\mathbf{F}}(r) \end{aligned} \tag{3}$$

$$\text{Radial stress:} \quad \tilde{\mathbf{\Sigma}}_r(r) = (\mathbf{n}_r \diamond \mathbf{n}_r)\, \tilde{\mathbf{U}}'(r) - \mathbb{i}\, \left[\frac{1}{r}\, (\mathbf{n}_r \diamond \mathbf{n}_\theta)\, (n\, \mathbb{I} + \mathbb{i}\, \mathbb{T}) + k\, (\mathbf{n}_r \diamond \mathbf{n}_z)\right] \tilde{\mathbf{U}}(r) \tag{4}$$

Transient Field Computation in multilayered structures

Exact solutions of Eq. (1) without volumic source in each plane layer: $\forall z\,,\ z_{\beta-1} < z < z_\beta\,,$ six partial waves:

$$\tilde{\mathbf{U}}(z) = \underbrace{\sum_{i=1}^{3} a_{\beta,i}\ \exp[-\mathbb{i}\,\kappa_{\beta,i}\,(z-z_\beta)]\,\mathbf{p}_{\beta,i}+}_{\text{upgoing waves}} \underbrace{\sum_{i=4}^{6} a_{\beta,i}\ \exp[-\mathbb{i}\,\kappa_{\beta,i}\,(z-z_{\beta-1})]\,\mathbf{p}_{\beta,i}\,.}_{\text{downgoing waves}} \tag{5}$$

(General anisotropy)

Sources at interfaces:

$$\Delta\tilde{\mathbf{U}}(z_\beta) = \boldsymbol{\phi}_\beta \quad \text{or/and} \quad \Delta\tilde{\boldsymbol{\Sigma}}_z(z_\beta) = \boldsymbol{\psi}_\beta \tag{6}$$

Exact solutions of Eq. (3) without volumic source in each tubular layer: $\forall r\,,\ r_{\beta-1} < r < r_\beta\,,$ six partial waves:

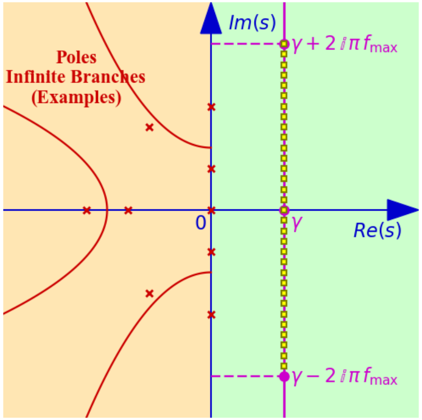
$$\tilde{\mathbf{U}}(r) = \underbrace{\sum_{i=1}^{3} a_{\beta,i}\ \mathbf{I}_{n,\beta,i}(\eta_{\beta,i}\,r)+}_{\text{ingoing waves}} \underbrace{\sum_{i=4}^{6} a_{\beta,i}\ \mathbf{K}_{n,\beta,i}(\eta_{\beta,i}\,r)\,.}_{\text{outgoing waves}} \tag{7}$$

(functions including modified Bessel functions and normalization by exponentials)
(Limitation: transversely isotropy with axial symmetry)
Sources at interfaces:

$$\Delta\tilde{\mathbf{U}}(r_\beta) = \boldsymbol{\phi}_\beta \quad \text{or/and} \quad \Delta\tilde{\boldsymbol{\Sigma}}_r(r_\beta) = \boldsymbol{\psi}_\beta \tag{8}$$

Transient Field Computation in multilayered structures

Some considerations on computation



LAPLACE transform: $H(s) = \displaystyle\int_0^{\infty} h(t)\, \mathbb{e}^{-s\,t}\, \mathbb{d}t$.

Bromwich-Mellin Formula:

$$\forall \gamma > 0\,,\ h(t) = \mathbb{e}^{\gamma\,t} \int_{-\infty}^{+\infty} H(\gamma + 2\,\mathbb{i}\,\pi\, f)\, \mathbb{e}^{2\,\mathbb{i}\,\pi\, f\, t}\, \mathbb{d}f\ . \qquad (9)$$

$f \mapsto H(\gamma + 2\,\mathbb{i}\,\pi\, f)$ is the Fourier Transform of the signal $t \mapsto h(t)\, \mathbb{e}^{-\gamma\,t}$.

1d time grid: $t_m = m\,\delta t$, $0 \leqslant m < 2\,N_t$, duration $d = 2\,N_t\,\delta t$

The *FFT* can be used while both the Nyquist-Shannon criterion and its dual are satisfied: [Cooley & Tukey, 1965], [Phinney, 1965]

- Band-limited spectrum: $\forall f > f_{\max} = \dfrac{1}{2\,\delta t}\,,\ H(\gamma + 2\,\mathbb{i}\,\pi\, f) \approx 0$
- Finite duration: $\forall t\,,\ t \notin [0, d[\,,\ h(t)\, \mathbb{e}^{-\gamma\,t} \approx 0$

Transient Field Computation in multilayered structures

Some considerations on computation

2d space grid:

$$\left\{\begin{array}{llll} x_i = i\,\delta x, & -N_x < i \leqslant N_x, & \text{Period: } 2\,N_x\,\delta x\ ; \\ y_j = j\,\delta y, & -N_y < j \leqslant N_y, & \text{Period: } 2\,N_y\,\delta y\ . \end{array}\right. \iff \left\{\begin{array}{lll} k_{x\,i} = i\,\delta k_x, & -N_x < i \leqslant N_x, & k_{x\ \max} = \pi/\delta x\ ; \\ k_{y\,j} = j\,\delta k_y, & -N_y < j \leqslant N_y, & k_{y\ \max} = \pi/\delta y\ . \end{array}\right.$$

or

$$\left\{\begin{array}{lll} \theta_i = i/(\pi\,N_\theta), & -N_\theta < i \leqslant N_\theta, & \text{Period: } 2\,\pi\ ; \\ z_j = j\,\delta z, & -N_z < j \leqslant N_z, & \text{Period: } 2\,N_z\,\delta z\ . \end{array}\right. \iff \left\{\begin{array}{lll} n, & -N_\theta < n \leqslant N_\theta & ; \\ k_{z\,j} = j\,\delta k_z, & -N_z < j \leqslant N_z, & k_{z\ \max} = \pi/\delta z\ . \end{array}\right.$$

Massively Parallel Computation

Diffraction by a Delamination

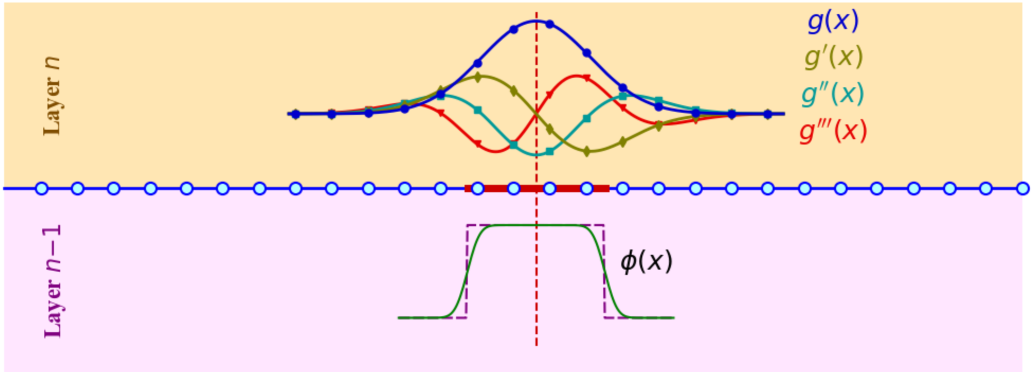


Figure 1: A 2D defect between two layers The ϕ function describing the defect is here the indicator function $\mathbb{1}_{\mathcal{D}}$, smoothed or not.

Incident field (vertical stress) on the interface (at $z = z_I$): $\boldsymbol{\sigma}_z^{[\mathrm{inc}]}(\mathbf{x}, s)$

Diffracted field (vertical stress) on the interface: $\boldsymbol{\sigma}_z^{[\mathrm{dif}]}(\mathbf{x}, s) = \displaystyle\int \mathcal{G}(\mathbf{x} - \boldsymbol{\xi}, s)\, \Delta\mathbf{u}(\boldsymbol{\xi}, s)\, \mathrm{d}\boldsymbol{\xi}$

Equation to be solved: $\forall \mathbf{x} \in \mathcal{D}\,,\ \boldsymbol{\sigma}_z(\mathbf{x}, s) = \boldsymbol{\sigma}_z^{[\mathrm{inc}]}(\mathbf{x}, s) + \boldsymbol{\sigma}_z^{[\mathrm{dif}]}(\mathbf{x}, s) = \mathbf{0}$

Numerically, we want to minimize: $\displaystyle\int \boldsymbol{\sigma}_z(\mathbf{x}, s) \cdot \boldsymbol{\sigma}_z(\mathbf{x}, s)^\star\, \phi(\mathbf{x})\, \mathrm{d}\mathbf{x}$

Diffraction by a Delamination

An inner product associated to the defect shape

The Shannon's interpolation formula: $\mathbf{u}(\mathbf{x}) = \sum_n \mathrm{sc}(\mathbf{x}-\mathbf{x}_n)\;\mathbf{u}_n$, where $\mathbf{u}_n = \mathbf{u}(\mathbf{x}_n)$, $\mathrm{sc}(x) = \sin(\pi\,x/\delta x)\,\delta x/(\pi\,x)$ (2D), and $\mathrm{sc}[(x,y)] = \sin(\pi\,x/\delta x)\,\sin(\pi\,y/\delta y)\,\delta x\,\delta y/(\pi^2\,x\,y)$ (3D).

We can define a *positive semi-definite inner product*[a] of 2 discretized fields:

$$\begin{aligned}\langle\mathbf{u},\mathbf{v}\rangle &= \int \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})^\star\,\phi(\mathbf{x})\;\mathrm{d}\mathbf{x} = \sum_{n,m}\left[\int \mathrm{sc}(\mathbf{x}-\mathbf{x}_n)\,\mathrm{sc}(\mathbf{x}-\mathbf{x}_m)\,\phi(\mathbf{x})\;\mathrm{d}\mathbf{x}\right](\mathbf{u}_n\cdot\mathbf{v}_m^\star) = \mathbf{u}\cdot(\mathcal{A}\,\mathbf{v}^\star)\\ &= (\mathcal{R}\,\mathbf{u})\cdot(\mathcal{R}\,\mathbf{v}^\star)\;.\end{aligned} \tag{10}$$

The inner product $\langle\bullet,\bullet\rangle$ is completely characterized by the $\mathcal{R}$ matrix which is substantially less expensive to store than the $\mathcal{A}$ matrix ($r\times N$ against N^2 without using sparse matrix storage).

[a] (i) $\langle\mathbf{u},\mathbf{v}\rangle = \langle\mathbf{v},\mathbf{u}\rangle$; (ii) $\langle\alpha\,\mathbf{u}_1+\beta\,\mathbf{u}_2,\mathbf{v}\rangle = \alpha\,\langle\mathbf{u}_1,\mathbf{v}\rangle+\beta\,\langle\mathbf{u}_2,\mathbf{v}\rangle$; (iii) $\langle\mathbf{u},\mathbf{u}\rangle \geqslant 0$; but $\langle\mathbf{u},\mathbf{u}\rangle = 0 \not\Rightarrow \mathbf{u}=\mathbf{0}$.

Diffraction by a Delamination

An inner product associated to the defect shape

$$\langle \mathbf{u}, \mathbf{v} \rangle = (\mathcal{R}\,\mathbf{u}) \cdot (\mathcal{R}\,\mathbf{v}^\star) \,.$$

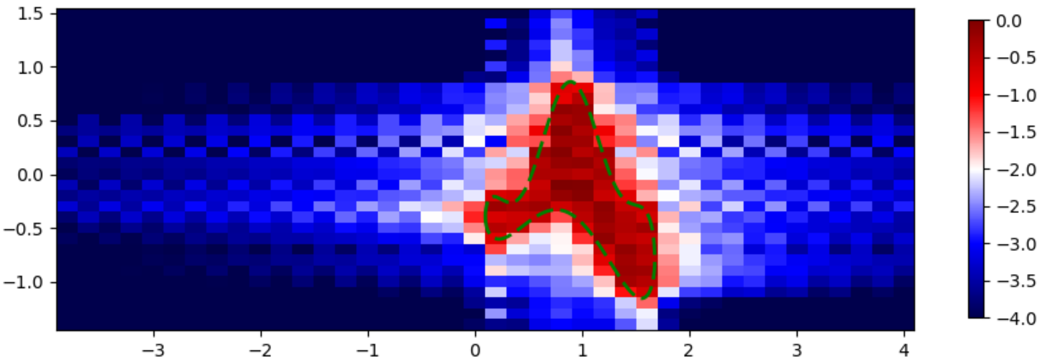


Figure 2: Map of $2\,\log_{10}[\,(\,\max_{\ell} |r_{\ell ij}|\,)_{i,j} / \max_{\ell\,i\,j} |\mathcal{R}|\,]$. The dashed line is the boundary of the defect.

Diffraction by a Delamination

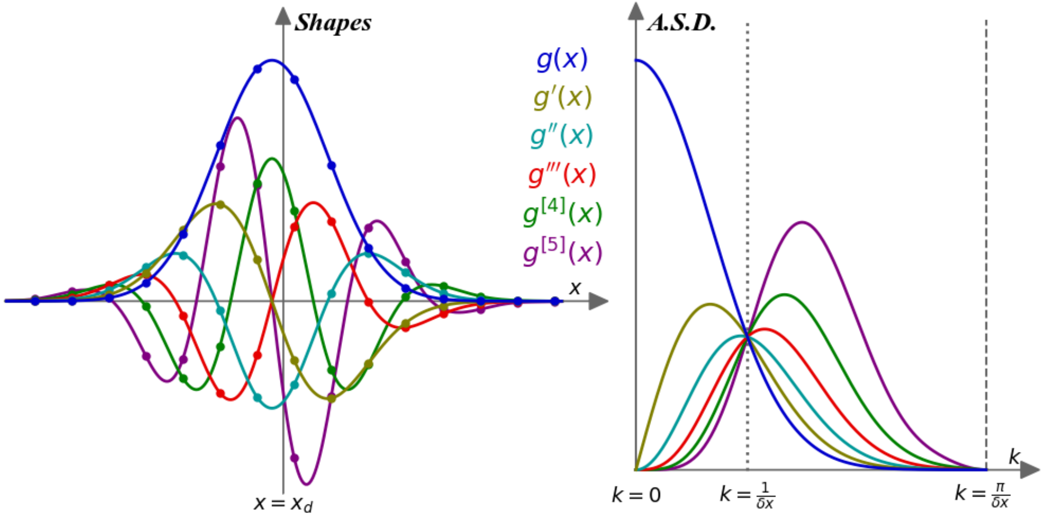
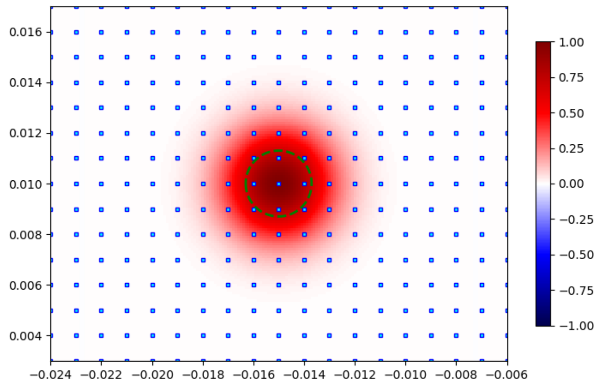
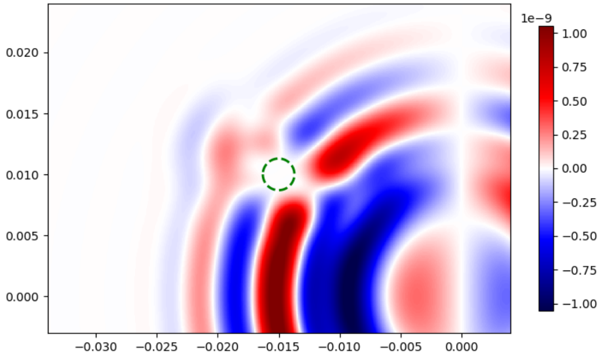


Figure 3: The Gaussian shape $x \mapsto \exp\left\{-2\left[(x - x_d)/\delta x\right]^2/9\right\}$ and its derivatives, which corresponds to approximations of the Dirac delta function and its derivatives, satisfying the Nyquist-Shannon Criterion (wavenumbers less than $\pi/\delta x$, where δx denotes the discretization step). x and k denote the position and the wavenumber, respectively.

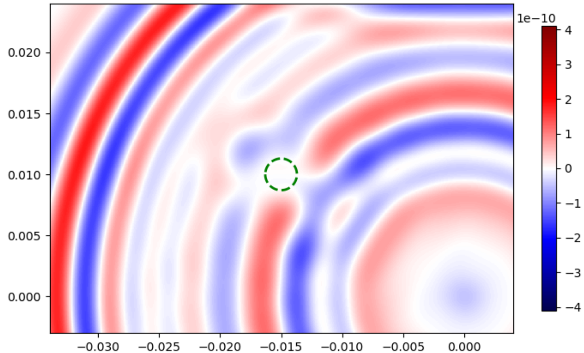


(a) Gaussian shape, defect and grid.

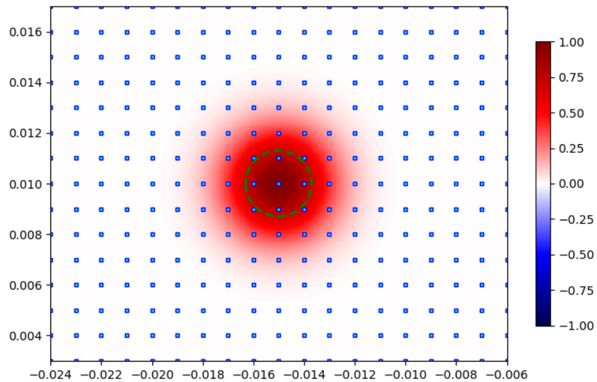
Radius: $1.3\,\delta x$
Number of nodes in the defect: 5
Number of lines in the $\mathcal{R}$ matrix: 19
Degrees of freedom: $57 = 19{\times}3$
Size of the basis: $30 = 3{\times}(1+2+3+4)$
Derivatives of orders 1 to 3



(b) Total field $\sigma_{xz}(x, y, z, t)$.



(c) Total field $\sigma_{zz}(x, y, z, t)$.



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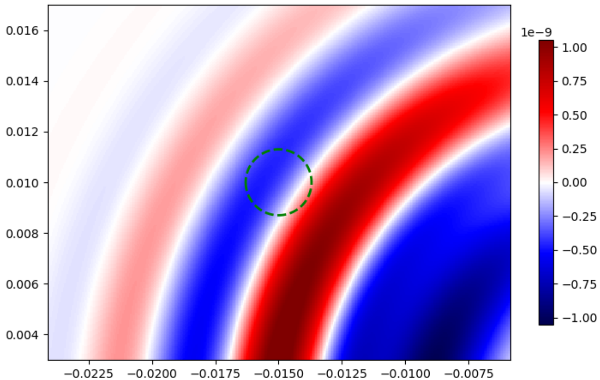
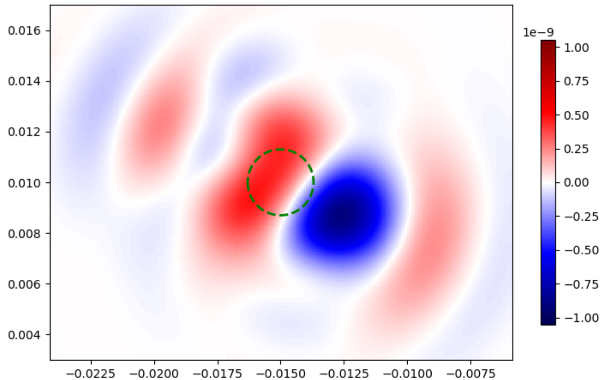
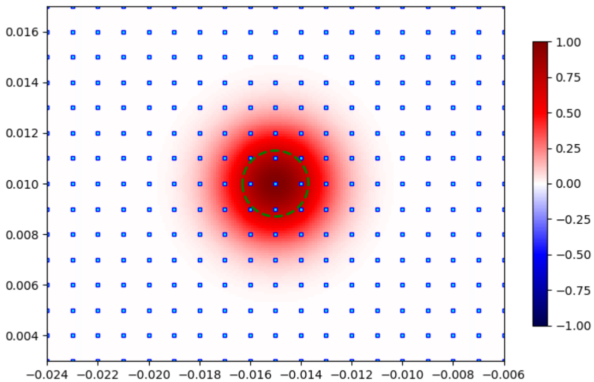


(d) Incident field $\sigma_{xz}^{[\mathrm{inc}]}(x, y, z, t)$.



(e) Diffracted field $\sigma_{xz}^{[\mathrm{dif}]}(x, y, z, t)$.



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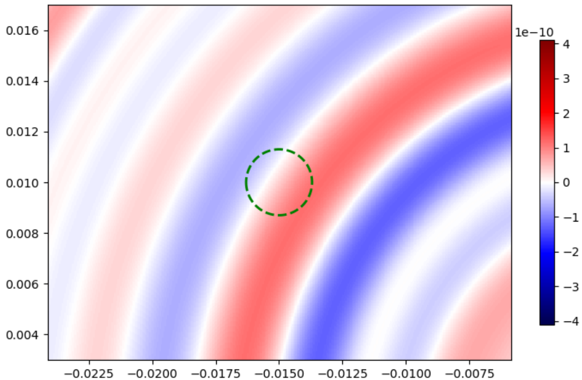
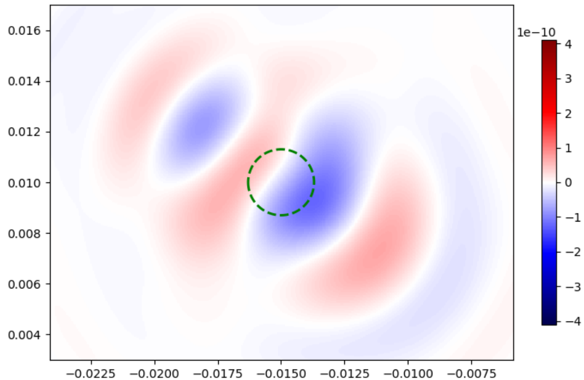
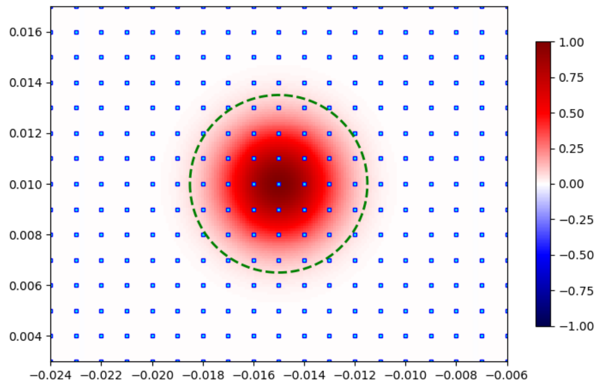


(f) Incident field $\sigma_{zz}^{[\mathrm{inc}]}(x,y,z,t)$.

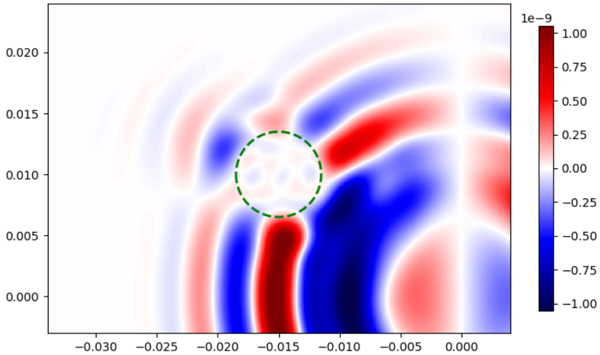


(g) Diffracted field $\sigma_{zz}^{[\mathrm{dif}]}(x,y,z,t)$.

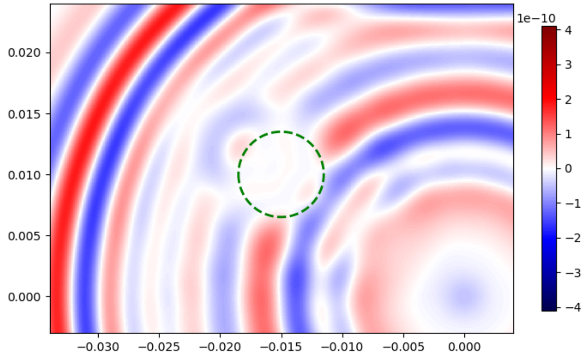


(a) Gaussian shape, defect and grid.

Radius: $3.5\,\delta x$
Number of nodes in the defect: 37
Number of lines in the $\mathcal{R}$ matrix: 80
Degrees of freedom: $240 = 80\times3$
Size of the basis: $63 = 3\times(1+2+3+4+5+6)$
Derivatives of orders 1 to 5



(b) Total field $\sigma_{xz}(x, y, z, t)$.



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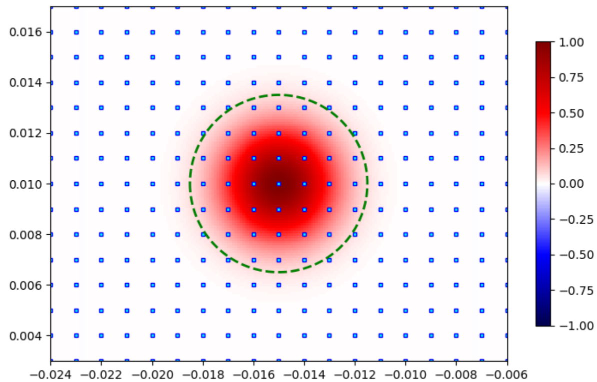

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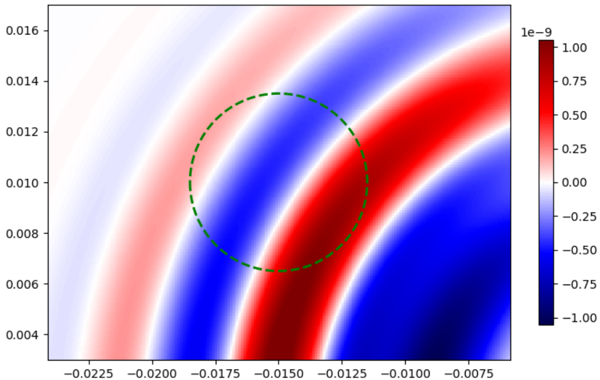
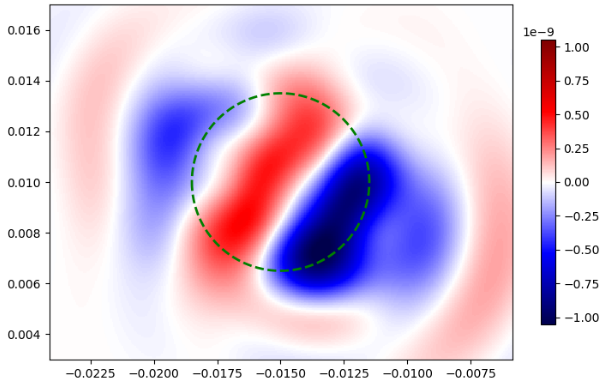
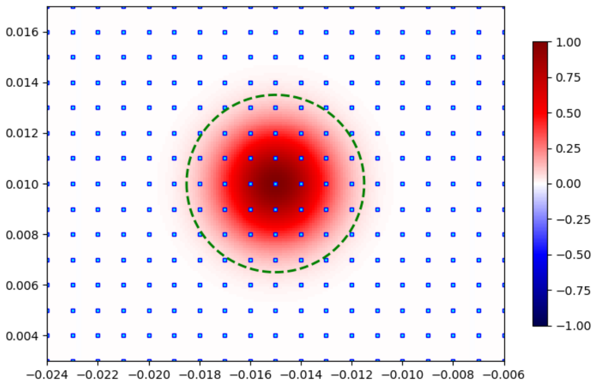


(d) Incident field $\sigma_{xz}^{[\mathrm{inc}]}(x,y,z,t)$.



(e) Diffracted field $\sigma_{xz}^{[\mathrm{dif}]}(x,y,z,t)$.



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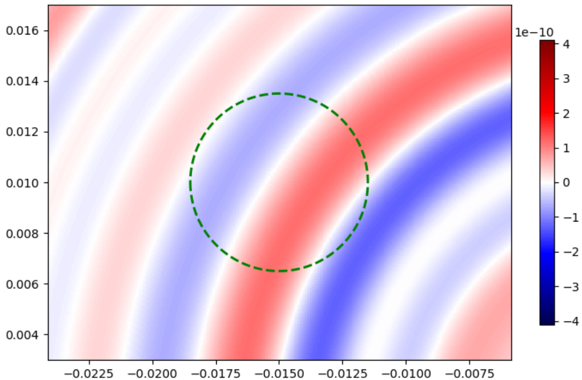

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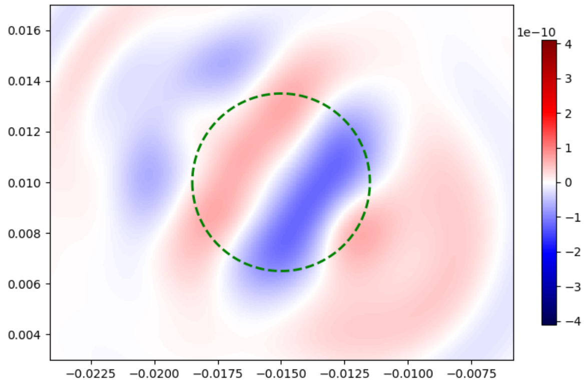

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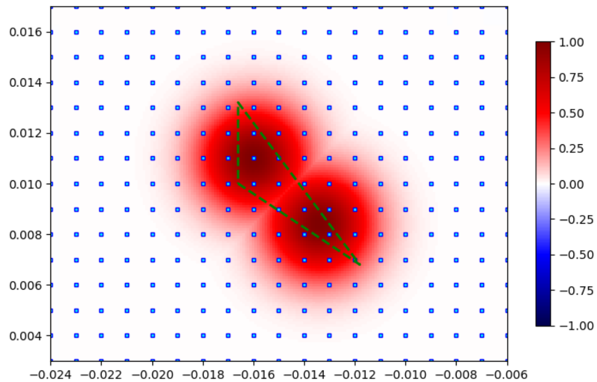
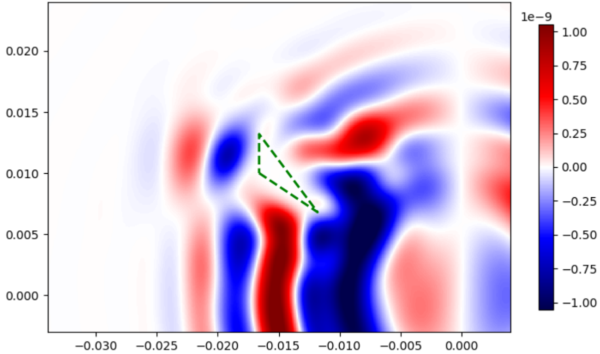


(a) Gaussian shape, defect and grid.

Triangular Defect, 2 Gaussians Number of nodes in the defect: 9
Number of lines in the $\mathcal{R}$ matrix: 30
Degrees of freedom: $90 = 30{\times}3$
Size of the basis: $60 = 2 \times [3{\times}(1 + 2 + 3 + 4)]$
Derivatives of orders 1 to 3



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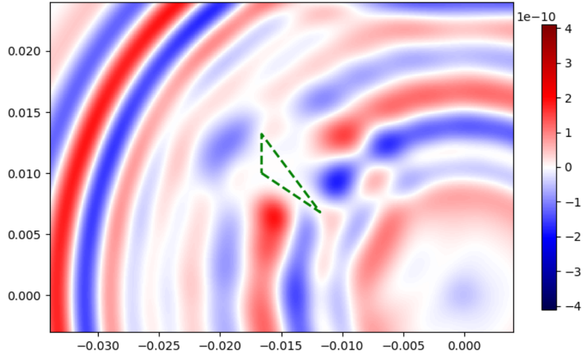

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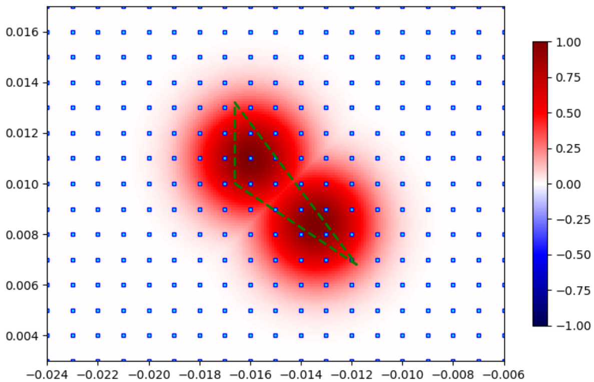

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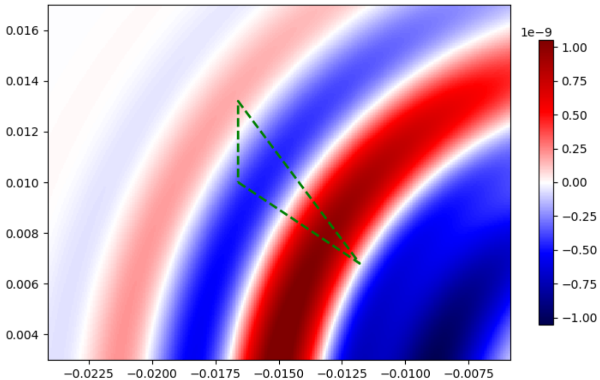

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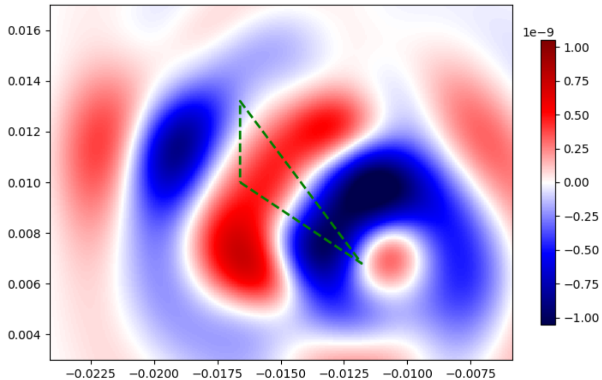

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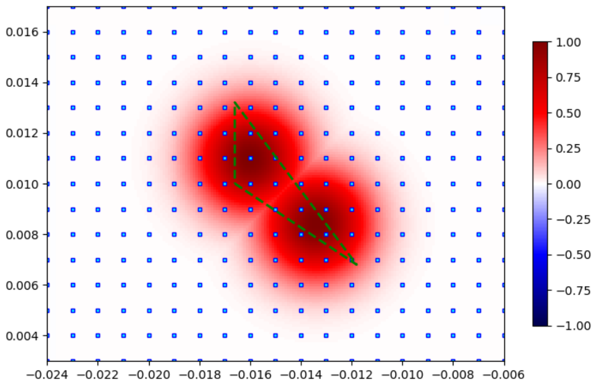

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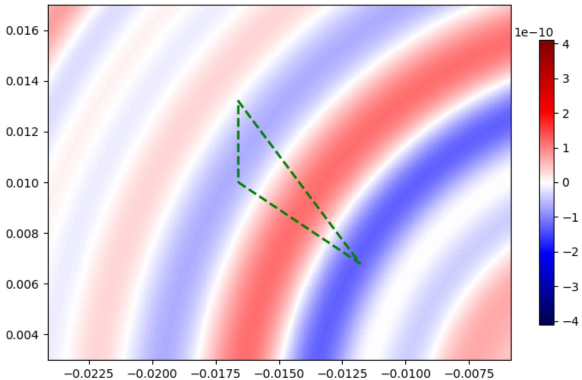
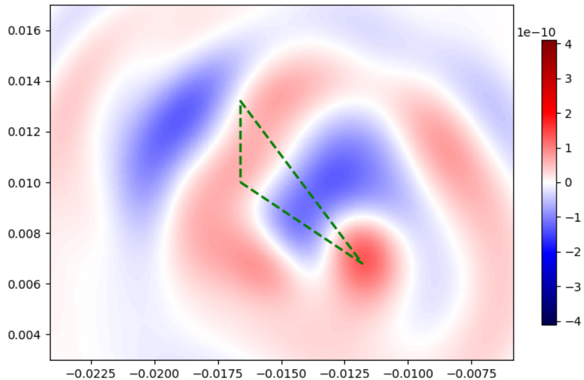


(f) Incident field $\sigma_{zz}^{[\mathrm{inc}]}(x, y, z, t)$.



(g) Diffracted field $\sigma_{zz}^{[\mathrm{dif}]}(x, y, z, t)$.

Hybrid computation: example of a stiffener