Transient field calculation in embedded multilayered anisotropic tubular structures

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1 Notations and transformed domains

1.1 Physical domain

The cylindrical coordinates of a point M are (r, θ, z) such that $\overrightarrow{OM} = r\,\mathbf{n}_r + z\,\mathbf{n}_z$, where the unit vector $\mathbf{n}_r$ depends on the azimuth θ (see Fig. 1).

t denotes time.

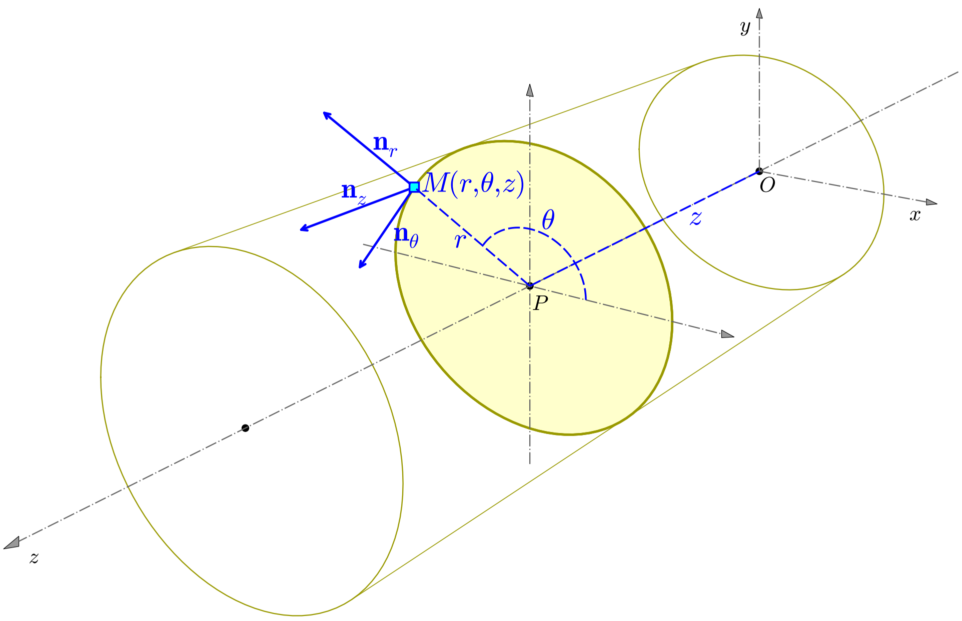

Figure 1: Cylindrical coordinates.

1.2 Axial wavenumber

The axial wavenumber is k, and corresponds to the variable of the Fourier transform with respect to the axial position z:

$$\mathcal{F}(\mathbf{u})(k) = \int_{-\infty}^{+\infty} \mathbf{u}(z)\ \exp(\mathbb{i}\,k\,z)\ \mathrm{d}z \quad \Longleftrightarrow \quad \mathbf{u}(z) = \frac{1}{2\,\pi} \int_{-\infty}^{+\infty} \mathcal{F}(\mathbf{u})(k)\ \exp(-\mathbb{i}\,k\,z)\ \mathrm{d}k \quad . \tag{1}$$

1.3 Azimuthal wavenumber

The azimuthal wavenumber is an integer n, and corresponds to the variable of the Fourier series with respect to the azimuth θ:

$$\mathcal{S}(\mathbf{u})(n) = \frac{1}{2\,\pi} \int_{-\pi}^{\pi} \mathbf{u}(\theta)\ \exp(\mathbb{i}\,n\,\theta)\ \mathrm{d}\theta \quad \Longleftrightarrow \quad \mathbf{u}(\theta) = \sum_{n=-\infty}^{+\infty} \mathcal{S}(\mathbf{u})(n)\ \exp(-\mathbb{i}\,n\,\theta) \quad . \tag{2}$$

	Time-domain t	notation	Laplace domain s	notation
Physical space r, θ, z	(r, θ, z, t)	$\mathbf{u}$	(r, θ, z, s)	$\mathbf{U}$
Azimuthal and axial wavenumbers n, z Radial position r	(r, n, k, t)	$\tilde{\mathbf{u}}$	(r, n, k, s)	$\tilde{\mathbf{U}}$
Azimuthal and axial wavenumbers n, z Radial wavenumber κ (see Appendix)	(κ, n, k, t)	$\hat{\mathbf{u}}$	(κ, n, k, s)	$\hat{\mathbf{U}}$

Table 1: Notations

1.4 Laplace domain

The complex s is the variable of the Laplace transform with respect to time t:

$$\mathcal{L}(\mathbf{u})(s) = \int_0^{+\infty} \mathbf{u}(t)\, \exp(-s\,t)\, \mathrm{d}t \quad \Longleftrightarrow \quad \underbrace{\forall a > 0\,,\; \mathbf{u}(t) = \frac{\exp(a\,t)}{2\,\pi} \int_{-\infty}^{+\infty} \mathcal{L}(\mathbf{u})(a + \mathrm{i}\,\omega)\, \exp(\mathrm{i}\,\omega\,t)\, \mathrm{d}\omega}_{\text{Bromwich-Mellin Formula}} \quad , \tag{3}$$

where the signal $\mathbf{u}$ is assumed to be causal, *i.e.*, $\forall t \leqslant 0$, $\mathbf{u}(t) = 0$.

2 Equations and background

Let us consider a multilayered medium made of a tubular system consisting of a number N of perfect cylindrical layers, stacked together. The layers are labeled β. The interface between layers β and β+1, also labeled β, is located at radial position $r = r_\beta$, as illustrated in Fig. 2. Each layer is an anisotropic solid, with a given thickness $h_\beta = r_\beta - r_{\beta-1}$. The areas inside and outside this tubular system can either be vacua, isotropic solid or fluid media. This tube is assumed to be infinite in the z-direction. This layered medium is submitted to external forces that can be located anywhere.

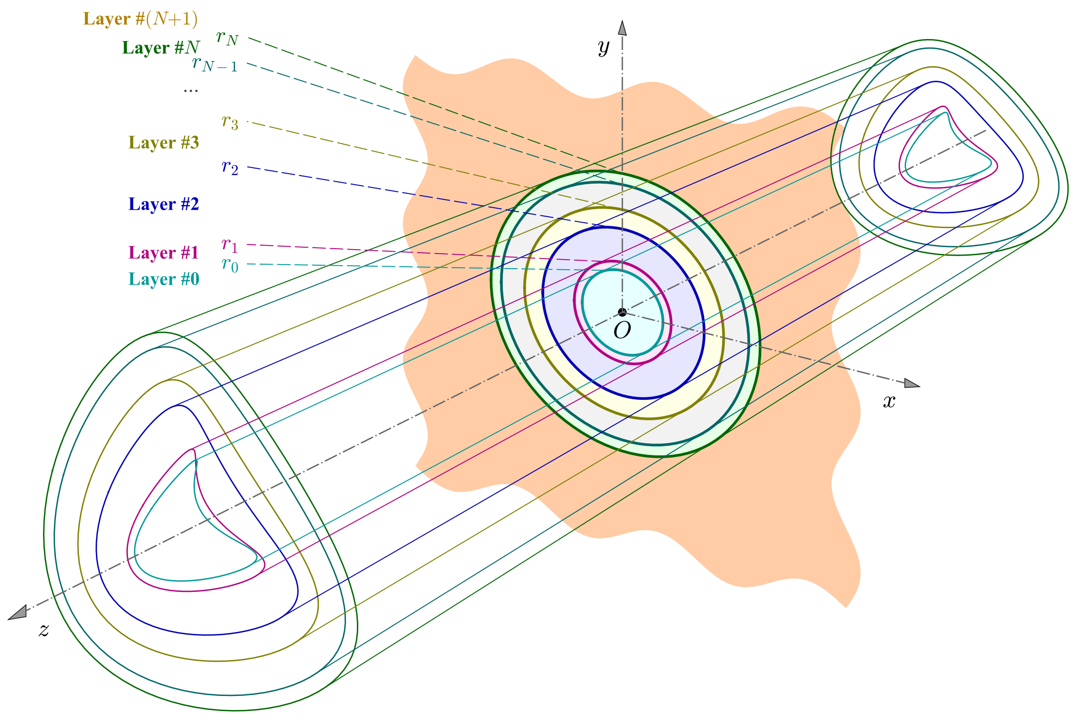

Figure 2: A multilayered infinite cylinder of n different layers.

2.1 Basic equations in the physical space

The mechanical properties, *i.e.*, the mass density ρ_β and the stiffnesses c^β_{ijkm} are constant in each layer. With our notations, let us first state the wave equations in each layer. Combining the causality principle, Newton's second law:

$$\rho_\beta\, \partial_t^2 \mathbf{u}(r,\theta,z,t) - \left\{ \left[\frac{1}{r} + \partial_r\right] \boldsymbol{\sigma}_r(r,\theta,z,t) + \frac{1}{r}\left[\mathbb{T} + \partial_\theta\right] \boldsymbol{\sigma}_\theta(r,\theta,z,t) + \partial_z \boldsymbol{\sigma}_z(r,\theta,z,t) \right\} = \mathbf{f}_\beta(r,\theta,z,t)\ , \tag{4}$$

where $\mathbb{T} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and Hooke's law which expresses the stress $\boldsymbol{\sigma}_{\mathbf{d}}$ in direction $\mathbf{d}$ (unit vector) with respect to displacement $\mathbf{u}$:

$$\boldsymbol{\sigma}_{\mathbf{d}}(r,\theta,z,t) = (\mathbf{d} \overset{\beta}{\diamond} \nabla)\mathbf{u}(r,\theta,z,t) + \frac{1}{r}\,(\mathbf{d} \overset{\beta}{\diamond} \mathbf{n}_\theta)\, \mathbb{T}\, \mathbf{u}(r,\theta,z,t)\,,\ \text{where } \nabla \text{ is the gradient operator } \left(\partial_r, \frac{1}{r}\partial_\theta, \partial_z\right)^{\mathrm{T}}\ , \tag{5}$$

the displacement field $\mathbf{u}(r,\theta,z,t)$ is given, at any time t and any location (r,θ,z), by the following system expressed in the medium β:

$$\begin{cases} \rho_\beta\, \partial_t^2 \mathbf{u}(r,\theta,z,t) - \left\{ \left[\left(\nabla + \frac{1}{r}\mathbf{n}_r\right) \overset{\beta}{\diamond} \nabla \right] + \left[\left(\nabla + \frac{1}{r}\mathbf{n}_r\right) \overset{\beta}{\diamond} \mathbf{n}_\theta \right] \frac{\mathbb{T}}{r} + \frac{\mathbb{T}}{r} \left[(\mathbf{n}_\theta \overset{\beta}{\diamond} \nabla) + (\mathbf{n}_\theta \overset{\beta}{\diamond} \mathbf{n}_\theta)\, \frac{\mathbb{T}}{r} \right] \right\} \mathbf{u}(r,\theta,z,t) = \mathbf{f}_\beta(r,\theta,z,t), \\ \hfill \text{for } t > 0, \\ \mathbf{u}(r,\theta,z,t) = 0, \hfill \text{for } t < 0, \\ \boldsymbol{\sigma}_r(r,\theta,z,t) = \left[(\mathbf{n}_r \overset{\beta}{\diamond} \nabla) + (\mathbf{n}_r \overset{\beta}{\diamond} \mathbf{n}_\theta)\, \frac{\mathbb{T}}{r} \right] \mathbf{u}(r,\theta,z,t) \quad \text{(radial stress)}\ . \end{cases} \tag{6}$$

The bilinear product $\overset{\beta}{\diamond}$ has been defined in [2] by a three-by-three matrix $(\mathbf{a} \overset{\beta}{\diamond} \mathbf{b})$ such that $(\mathbf{a} \overset{\beta}{\diamond} \mathbf{b})_{im} = c^\beta_{ijkm}\, a_j\, b_k$, with the Einstein summation convention. The above equations depend on each layer through the values of the elastic constants c^β_{ijkm}, *i.e.* through the operator $\overset{\beta}{\diamond}$. The field $\mathbf{f}_\beta(r,\theta,z,t)$ denotes the force per unit volume exerted by the part of the sources located in layer β.

The stiffness tensor can be represented by a 6-by-6 symmetric matrix, by using the Voigt notation:

$$1 \leftrightarrow rr \mid 2 \leftrightarrow \theta\theta \mid 3 \leftrightarrow zz \mid 4 \leftrightarrow \theta z \mid 5 \leftrightarrow rz \mid 6 \leftrightarrow r\theta \quad .$$

For a transversely isotropic medium, the stiffness tensor is:

$$\begin{pmatrix} c_{11} & c_{11} - 2\,c_{66} & c_{13} & 0 & 0 & 0 \\ & c_{11} & c_{13} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & \text{(sym)} & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ & & & & & c_{66} \end{pmatrix}\ , \tag{7}$$

while for an isotropic medium: $c_{33} = c_{11}$, $c_{13} = c_{11} - 2\,c_{66}$, and $c_{44} = c_{66}$.

Continuity equations at the interface β are given by the following system:

$$\begin{pmatrix} \mathbf{u}(r_\beta^+,\theta,z,t) \\ \boldsymbol{\sigma}_r(r_\beta^+,\theta,z,t) \end{pmatrix} - \begin{pmatrix} \mathbf{u}(r_\beta^-,\theta,z,t) \\ \boldsymbol{\sigma}_r(r_\beta^-,\theta,z,t) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{p}_\beta(\theta,z,t) \end{pmatrix}, \tag{8}$$

where r_β^- and r_β^+ indicate the fact that the field under consideration is calculated in layers $\beta{-}1$ and β, respectively. The *interface source term* $\mathbf{p}_\beta(\theta,z,t)$ defines the normal stress jump at interface β, and corresponds to an applied force per unit surface if one of the media is a vacuum. Notice that if $\mathbf{p}_\beta = \mathbf{0}$, then Eq. (8) merely expresses the continuity of displacement and normal stress.

For simplicity, the β index will be omitted below and will be reintroduced only when necessary to avoid ambiguity. The volumic sources described by $\mathbf{f}_\beta(r,\theta,z,t)$ in Eq. (6) can straddle two or more layers. In such case, the volumic source term is considered as zero at each interface ($r = r_\beta$).

Eq. (6) and (8) are to be solved using Fourier and Laplace transforms on invariant dimensions. In a first time (see sections 3 and 4), Eq. (6) is solved separately in each layer containing a source term. This defines an *incident field* in each layer, which corresponds to the field that a source would radiate in this layer considered to be unbounded. Then (see section 5), the *reflected field* can be obtained, which is the contribution of all the interfaces, and is calculated by taking into account the continuity relationships of Eq. (8).

General case	$(\mathbf{n}_r \diamond \mathbf{n}_r)$	$(\mathbf{n}_\theta \diamond \mathbf{n}_\theta)$	$(\mathbf{n}_z \diamond \mathbf{n}_z)$	$(\mathbf{n}_\theta \diamond \mathbf{n}_z)$	$(\mathbf{n}_r \diamond \mathbf{n}_z)$	$(\mathbf{n}_r \diamond \mathbf{n}_\theta)$
Trans. Isotrop.	$\begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{66} & 0 \\ 0 & 0 & c_{44} \end{pmatrix}$	$\begin{pmatrix} c_{66} & 0 & 0 \\ 0 & c_{11} & 0 \\ 0 & 0 & c_{44} \end{pmatrix}$	$\begin{pmatrix} c_{44} & 0 & 0 \\ 0 & c_{44} & 0 \\ 0 & 0 & c_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{13} \\ 0 & c_{44} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & c_{13} \\ 0 & 0 & 0 \\ c_{44} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_{11}-2\,c_{66} & 0 \\ c_{66} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Table 2: Diamond product of the cylindrical basis vectors, for transversely isotropic media, remembering that in all cases: $(\mathbf{b} \diamond \mathbf{a}) = (\mathbf{a} \diamond \mathbf{b})^{\mathrm{T}}$

3 Equation to be solved in the (r, n, k, s)-domain

3.1 General case

After a Fourier transform with respect to the axial position z, a Fourier series expansion with respect to the angular position θ, and a Laplace transform with respect to time t, the wave equation (6) becomes the following ordinary differential system in the (r, n, k, s)-domain (see Table 1):

$$
\begin{aligned}
&(\mathbf{n}_r \diamond \mathbf{n}_r)\,\tilde{\mathbf{U}}''(r) \; + \\
&\left\{ -\mathbb{i}\,k\,(\mathbf{n}_r \diamond \mathbf{n}_z + \mathbf{n}_z \diamond \mathbf{n}_r) + \frac{1}{r}\left[\,(\mathbf{n}_r \diamond \mathbf{n}_r) - \mathbb{i}\;\big(\,(\mathbf{n}_r \diamond \mathbf{n}_\theta)\,(n\,\mathbb{I} + \mathbb{i}\,\mathbb{T}) + (n\,\mathbb{I} + \mathbb{i}\,\mathbb{T})\;(\mathbf{n}_\theta \diamond \mathbf{n}_r)\,\big)\,\right] \right\} \tilde{\mathbf{U}}'(r) \; - \\
&\Big\{ \left[\,\rho\, s^2\,\mathbb{I} + k^2\,(\mathbf{n}_z \diamond \mathbf{n}_z)\,\right] + \\
&\frac{k}{r}\left[\,\mathbb{i}\,(\mathbf{n}_r \diamond \mathbf{n}_z) + (n\,\mathbb{I} + \mathbb{i}\,\mathbb{T})\;(\mathbf{n}_\theta \diamond \mathbf{n}_z) + (\mathbf{n}_z \diamond \mathbf{n}_\theta)\;(n\,\mathbb{I} + \mathbb{i}\,\mathbb{T})\,\right] + \\
&\frac{1}{r^2}\,(n\,\mathbb{I} + \mathbb{i}\,\mathbb{T})\;(\mathbf{n}_\theta \diamond \mathbf{n}_\theta)\;(n\,\mathbb{I} + \mathbb{i}\,\mathbb{T}) \Big\}\; \tilde{\mathbf{U}}(r) = -\tilde{\mathbf{F}}(r) \quad ,
\end{aligned} \tag{9}
$$

where $\mathbb{I}$ denotes the 3-by-3 identity matrix.

The radial stress becomes in the (r, n, k, s)-domain:

$$
\tilde{\mathbf{\Sigma}}_r(r) = (\mathbf{n}_r \diamond \mathbf{n}_r)\,\tilde{\mathbf{U}}'(r) - \mathbb{i}\,\left[\frac{1}{r}\,(\mathbf{n}_r \diamond \mathbf{n}_\theta)\;(n\,\mathbb{I} + \mathbb{i}\,\mathbb{T}) + k\,(\mathbf{n}_r \diamond \mathbf{n}_z)\right]\,\tilde{\mathbf{U}}(r) \quad . \tag{10}
$$

Unfortunately, in the most general case, an analytical solution of Eq. (9) is not known yet. On the contrary, an analytical solution is known for transversely isotropic medium as detailed below.

3.2 Transversely isotropic medium

For transversely isotropic media, Eqs. (9) and (10) can be simplify by using the properties summarized in Table 2 to obtain the following differential equation:

$$
\begin{aligned}
&\tilde{\mathbf{U}}''(r) + \begin{bmatrix} \dfrac{1}{r} & \dfrac{-\mathbb{i}\,n\,(c_{11}-c_{66})}{r\,c_{11}} & \dfrac{-\mathbb{i}\,k\,(c_{13}+c_{44})}{c_{11}} \\ \dfrac{-\mathbb{i}\,n\,(c_{11}-c_{66})}{r\,c_{66}} & \dfrac{1}{r} & 0 \\ \dfrac{-\mathbb{i}\,k\,(c_{13}+c_{44})}{c_{44}} & 0 & \dfrac{1}{r} \end{bmatrix} \tilde{\mathbf{U}}'(r) \; - \\
&\begin{bmatrix} \dfrac{\rho\,s^2}{c_{11}} + \dfrac{1}{r^2}\left(1 + \dfrac{c_{66}}{c_{11}}\,n^2\right) + \dfrac{c_{44}}{c_{11}}\,k^2 & \dfrac{-\mathbb{i}\,n\,(c_{11}+c_{66})}{r^2\,c_{11}} & 0 \\ \dfrac{\mathbb{i}\,n\,(c_{11}+c_{66})}{r^2\,c_{66}} & \dfrac{\rho\,s^2}{c_{66}} + \dfrac{1}{r^2}\left(1 + \dfrac{c_{11}}{c_{66}}\,n^2\right) + \dfrac{c_{44}}{c_{66}}\,k^2 & \dfrac{n\,k\,(c_{13}+c_{44})}{r\,c_{66}} \\ \dfrac{\mathbb{i}\,k\,(c_{13}+c_{44})}{r\,c_{44}} & \dfrac{n\,k\,(c_{13}+c_{44})}{r\,c_{44}} & \dfrac{\rho\,s^2}{c_{44}} + \dfrac{n^2}{r^2} + \dfrac{c_{33}}{c_{44}}\,k^2 \end{bmatrix} \tilde{\mathbf{U}}(r) = \begin{bmatrix} \dfrac{-\tilde{F}_r(r)}{c_{11}} \\ \dfrac{-\tilde{F}_\theta(r)}{c_{66}} \\ \dfrac{-\tilde{F}_z(r)}{c_{44}} \end{bmatrix} .
\end{aligned} \tag{11}
$$

and the radial stress:

$$
\tilde{\mathbf{\Sigma}}_r(r) = \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{66} & 0 \\ 0 & 0 & c_{44} \end{pmatrix} \tilde{\mathbf{U}}'(r) - \mathbb{i}\,\left[\frac{1}{r}\begin{pmatrix} \mathbb{i}\,(c_{11}-2\,c_{66}) & n\,(c_{11}-2\,c_{66}) & 0 \\ n\,c_{66} & -\mathbb{i}\,c_{66} & 0 \\ 0 & 0 & 0 \end{pmatrix} + k\begin{pmatrix} 0 & 0 & c_{13} \\ 0 & 0 & 0 \\ c_{44} & 0 & 0 \end{pmatrix}\right] \tilde{\mathbf{U}}(r) \quad . \tag{12}
$$

4 Incident waves

This section coupled with Appendix A is necessary to introduce volumic sources inside layers and to understand the concept of ingoing and outgoing waves. It can be skipped in a first reading.

Only the part of the source inside Layer β is considered here, as for multilayered plates [6]. This source is assumed to be located between $r_{\min} \geqslant r_{\beta-1}$ and $r_{\max} \leqslant r_\beta$ (see Fig. 2) and generates the *incident field* in Layer β.

This incident field is obtained by convolution, with respect to the radius, of the Green tensor $\tilde{\mathbf{G}}_a$ by the volumic source term $\tilde{\mathbf{F}}$, as follows:

$$\tilde{\mathbf{U}}(r) = \int_{r_{\min}}^{r_{\max}} \tilde{\mathbf{G}}_a(r)\, \tilde{\mathbf{F}}(a)\, \mathbb{d}a\,, \tag{13}$$

the Green tensor $\tilde{\mathbf{G}}_a$ being solution of Eq. (9), where the vectors $\tilde{\mathbf{U}}(r)$ and $\tilde{\mathbf{F}}(r)$ are replaced by the matrices $\tilde{\mathbf{G}}_a(r)$ and $\delta(r-a)\,\mathbb{I}$, respectively. This Green tensor is the response to a point-source located at $r = a$ and can be separated into ingoing waves $\tilde{\mathbf{G}}_a^{[\text{in}]}(r)$ $(r < a)$ and outgoing waves $\tilde{\mathbf{G}}_a^{[\text{out}]}(r)$ $(r > a)$ radiated into the cylinder of radius a and the surrounding space, respectively.

Three cases have to be distinguished:

- $\boxed{r_{\beta-1} \leqslant r \leqslant r_{\min}}$ $\tilde{\mathbf{U}}(r) = \displaystyle\int_{r_{\min}}^{r_{\max}} \tilde{\mathbf{G}}_a^{[\text{in}]}(r)\, \tilde{\mathbf{F}}(a)\, \mathbb{d}a$ (ingoing waves only).
- $\boxed{r_{\min} < r < r_{\max}}$ $\tilde{\mathbf{U}}(r) = \displaystyle\int_{r_{\min}}^{r} \tilde{\mathbf{G}}_a^{[\text{out}]}(r)\, \tilde{\mathbf{F}}(a)\, \mathbb{d}a + \int_{r}^{r_{\max}} \tilde{\mathbf{G}}_a^{[\text{in}]}(r)\, \tilde{\mathbf{F}}(a)\, \mathbb{d}a$ (both outgoing and ingoing waves).
- $\boxed{r_{\max} \leqslant r \leqslant r_\beta}$ $\tilde{\mathbf{U}}(r) = \displaystyle\int_{r_{\min}}^{r_{\max}} \tilde{\mathbf{G}}_a^{[\text{out}]}(r)\, \tilde{\mathbf{F}}(a)\, \mathbb{d}a$ (outgoing waves only).

4.1 Transversely isotropic medium: exact solution

(see Appendix A)

4.2 General anisotropic case: numerical solution

(to be completed)

5 Refracted waves

This section is sufficient to treat the problem of surface sources located at interfaces.

The refracted waves result from both refraction of incident waves and emission of sources at interfaces. In each tubular layer, they satisfied Eq. (9) without source-term ($\tilde{\mathbf{F}}(r) = \mathbf{0}$).

5.1 Transversely isotropic medium: exact solution

5.1.1 Six partial waves: displacement vectors

In transversely isotropic media, six partial waves are solution of Eq. (11) without source-term.

Three ingoing waves (see Appendix A for more details) are expressed as combinations of modified Bessel functions of the first kind $\mathcal{I}_i$ (see *e.g.*, [9, §10.25]). The fist two ingoing waves contains axial displacement:

$$\text{for } j = 1, 2\,, \quad \tilde{\mathbf{U}}_j(r) = \begin{bmatrix} \dfrac{\mathcal{I}_{n-1}(\eta_j\, r) + \mathcal{I}_{n+1}(\eta_j\, r)}{2} \\ -\mathbb{i}\, \dfrac{\mathcal{I}_{n-1}(\eta_j\, r) - \mathcal{I}_{n+1}(\eta_j\, r)}{2} \\ \mathbb{i}\, b_j\, \mathcal{I}_n(\eta_j\, r) \end{bmatrix} = \begin{bmatrix} \mathcal{I}_n'(\eta_j\, r) \\ \dfrac{-\mathbb{i}\, n}{\eta_j\, r}\, \mathcal{I}_n(\eta_j\, r) \\ \mathbb{i}\, b_j\, \mathcal{I}_n(\eta_j\, r) \end{bmatrix}\,, \tag{14}$$

where η_j is the square root with positive real part of the j^{th} root of the following polynomial of the second degree in X:

$$\left[c_{11}\, X - \left(\rho\, s^2 + c_{44}\, k^2\right)\right] \left[c_{44}\, X - \left(\rho\, s^2 + c_{33}\, k^2\right)\right] + \left(c_{13} + c_{44}\right)^2\, k^2\, X\,, \tag{15}$$

and the coefficient b_j satisfies:

$$b_j = \frac{c_{11}\,\eta_j^2 - \left(\rho\,s^2 + c_{44}\,k^2\right)}{-\left(c_{13}+c_{44}\right)\,k\,\eta_j} = \frac{\left(c_{13}+c_{44}\right)\,k\,\eta_j}{c_{44}\,\eta_j^2 - \left(\rho\,s^2 + c_{33}\,k^2\right)}\,, \tag{16}$$

while the third ingoing wave contains no axial dispacement:

$$\tilde{\mathbf{U}}_3(r) = \begin{bmatrix} \mathbb{i}\,\dfrac{\mathcal{I}_{n-1}(\eta_3\,r) - \mathcal{I}_{n+1}(\eta_3\,r)}{2} \\ \dfrac{\mathcal{I}_{n-1}(\eta_3\,r) + \mathcal{I}_{n+1}(\eta_3\,r)}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\mathbb{i}\,n}{\eta_3\,r}\,\mathcal{I}_n(\eta_3\,r) \\ \mathcal{I}_n'(\eta_3\,r) \\ 0 \end{bmatrix}, \text{ where } \eta_3 = \sqrt{\frac{\rho\,s^2 + c_{44}\,k^2}{c_{66}}}\,. \tag{17}$$

Three outgoing waves are expressed as combinations of modified Bessel functions of the second kind $\mathcal{K}_i$:

$$\text{for } j=1,2\,,\quad \tilde{\mathbf{U}}_{j+3}(r) = \begin{bmatrix} \dfrac{-\mathcal{K}_{n-1}(\eta_j\,r) - \mathcal{K}_{n+1}(\eta_j\,r)}{2} \\ -\mathbb{i}\,\dfrac{-\mathcal{K}_{n-1}(\eta_j\,r) + \mathcal{K}_{n+1}(\eta_j\,r)}{2} \\ \mathbb{i}\,b_j\,\mathcal{K}_n(\eta_j\,r) \end{bmatrix} = \begin{bmatrix} \mathcal{K}_n'(\eta_j\,r) \\ \dfrac{-\mathbb{i}\,n}{\eta_j\,r}\,\mathcal{K}_n(\eta_j\,r) \\ \mathbb{i}\,b_j\,\mathcal{K}_n(\eta_j\,r) \end{bmatrix}, \tag{18}$$

and

$$\tilde{\mathbf{U}}_6(r) = \begin{bmatrix} \mathbb{i}\,\dfrac{-\mathcal{K}_{n-1}(\eta_3\,r) + \mathcal{K}_{n+1}(\eta_3\,r)}{2} \\ \dfrac{-\mathcal{K}_{n-1}(\eta_3\,r) - \mathcal{K}_{n+1}(\eta_3\,r)}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\mathbb{i}\,n}{\eta_3\,r}\,\mathcal{K}_n(\eta_3\,r) \\ \mathcal{K}_n'(\eta_3\,r) \\ 0 \end{bmatrix}. \tag{19}$$

Note that $\eta_{1,2,3}$ are not exactly radial wavenumbers, even if they have the same unit, because $\eta_j^2 = -\kappa^2$, where κ is a radial wavenumber (see Appendix A).

5.1.2 Six partial waves: radial stress vectors

The radial stress vectors corresponding to these six partial waves are:

$$\text{for } j=1,2\,,\quad \tilde{\mathbf{\Sigma}}_{r\,j}(r) = \begin{bmatrix} \left(c_{11}\,\eta_j + c_{13}\,b_j\,k\right)\,\mathcal{I}_n(\eta_j\,r) + c_{66}\,\eta_j\,\dfrac{\mathcal{I}_{n-2}(\eta_j\,r) - 2\,\mathcal{I}_n(\eta_j\,r) + \mathcal{I}_{n+2}(\eta_j\,r)}{2} \\ -\mathbb{i}\,c_{66}\,\eta_j\,\dfrac{\mathcal{I}_{n-2}(\eta_j\,r) - \mathcal{I}_{n+2}(\eta_j\,r)}{2} \\ \mathbb{i}\,c_{44}\,\left(b_j\,\eta_j - k\right)\,\dfrac{\mathcal{I}_{n-1}(\eta_j\,r) + \mathcal{I}_{n+1}(\eta_j\,r)}{2} \end{bmatrix}, \tag{20}$$

$$\tilde{\mathbf{\Sigma}}_{r\,3}(r) = \begin{bmatrix} \mathbb{i}\,c_{66}\,\eta_3\,\dfrac{\mathcal{I}_{n-2}(\eta_3\,r) - \mathcal{I}_{n+2}(\eta_3\,r)}{2} \\ c_{66}\,\eta_3\,\dfrac{\mathcal{I}_{n-2}(\eta_3\,r) + \mathcal{I}_{n+2}(\eta_3\,r)}{2} \\ c_{44}\,k\,\dfrac{\mathcal{I}_{n-1}(\eta_3\,r) - \mathcal{I}_{n+1}(\eta_3\,r)}{2} \end{bmatrix}, \tag{21}$$

$$\text{for } j=1,2\,,\quad \tilde{\mathbf{\Sigma}}_{r\,j+3}(r) = \begin{bmatrix} \left(c_{11}\,\eta_j + c_{13}\,b_j\,k\right)\,\mathcal{K}_n(\eta_j\,r) + c_{66}\,\eta_j\,\dfrac{\mathcal{K}_{n-2}(\eta_j\,r) - 2\,\mathcal{K}_n(\eta_j\,r) + \mathcal{K}_{n+2}(\eta_j\,r)}{2} \\ -\mathbb{i}\,c_{66}\,\eta_j\,\dfrac{\mathcal{K}_{n-2}(\eta_j\,r) - \mathcal{K}_{n+2}(\eta_j\,r)}{2} \\ \mathbb{i}\,c_{44}\,\left(b_j\,\eta_j - k\right)\,\dfrac{-\mathcal{K}_{n-1}(\eta_j\,r) - \mathcal{K}_{n+1}(\eta_j\,r)}{2} \end{bmatrix}, \tag{22}$$

$$\tilde{\mathbf{\Sigma}}_{r\,6}(r) = \begin{bmatrix} \mathbb{i}\,c_{66}\,\eta_3\,\dfrac{\mathcal{K}_{n-2}(\eta_3\,r) - \mathcal{K}_{n+2}(\eta_3\,r)}{2} \\ c_{66}\,\eta_3\,\dfrac{\mathcal{K}_{n-2}(\eta_3\,r) + \mathcal{K}_{n+2}(\eta_3\,r)}{2} \\ c_{44}\,k\,\dfrac{-\mathcal{K}_{n-1}(\eta_3\,r) + \mathcal{K}_{n+1}(\eta_3\,r)}{2} \end{bmatrix}. \tag{23}$$

5.2 General anisotropic case: numerical solution

(to be completed)

References

[1] Ducasse E., Yaacoubi S., *The Hankel transform of first- and second-order tensor fields: definition and use for modeling circular symmetric leaky waveguides*, Physics Procedia **3**(1) (2010) 515-521 (International Congress on Ultrasonics, Santiago de Chile, January 2009). DOI 10.1016/j.phpro.2010.01.067

[2] Ducasse E., Deschamps M., *A nonstandard wave decomposition to ensure the convergence of Debye series for modeling wave propagation in an immersed anisotropic elastic plate*, Wave Motion **49** (2012) 745-764. DOI 10.1016/j.wavemoti.2012.05.001

[3] Ducasse E., Deschamps M., *Time-domain computation of the response of composite layered anisotropic plates to a localized source*, Wave Motion **51** (2014) 1364-1381. DOI 10.1016/j.wavemoti.2014.08.003

[4] Erdélyi A., Bateman H., *Table of Integral Transforms* (vol. II), McGraw-Hill, 1954, LCCN 54-6214, ISBN 07-019549-8.

[5] Gradshteyn I. S., Ryzhik I. M., *Table of Integrals, Series and Products* (4$^{\text{th}}$ ed.), Academic Press, 1965, LCCN 65-29097, ISBN 0-12-294750-9.

[6] Mora P., Ducasse E., Deschamps M., *Transient 3D elastodynamic field in an embedded multilayered anisotropic plate*, Ultrasonics **69** (2016) 106-115. DOI 10.1016/j.ultras.2016.03.020

[7] Mora P., Ducasse E., Deschamps M., *Interaction of a guided wave with a crack in an embedded multilayered anisotropic plate: global matrix with Laplace transform formalism*, Proceedings of ICU 2015, Physics Procedia **70** (2015) 326-329. DOI 10.1016/j.phpro.2015.08.217

[8] Olver F. W. J., Lozier D. W., Boisvert R. F., Clark C. W., *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010, Print companion to [9].

[9] *NIST Digital Library of Mathematical Functions*, `http://dlmf.nist.gov`, Online companion to [8].

[10] Stroh A.N., *Steady state problems in anisotropic elasticity*, J. Math. Phys. **41** (1962) 77–103. DOI 10.1002/sapm196241177

Appendix A Response of an infinite transversely isotropic medium to a cylindrical source

We want to solve the following equation, for a source radius a and given wavenumbers n and k:

$$\rho\,\partial_t^2\tilde{\mathbf{g}}_a(r,t) - \begin{bmatrix} c_{11} & 0 & 0 \\ 0 & c_{66} & 0 \\ 0 & 0 & c_{44} \end{bmatrix} \partial_r^2\tilde{\mathbf{g}}_a(r,t) - \begin{bmatrix} \dfrac{c_{11}}{r} & \dfrac{-\mathbb{i}\,n\,(c_{11}-c_{66})}{r} & -\mathbb{i}\,k\,(c_{13}+c_{44}) \\ \dfrac{-\mathbb{i}\,n\,(c_{11}-c_{66})}{r} & \dfrac{c_{66}}{r} & 0 \\ -\mathbb{i}\,k\,(c_{13}+c_{44}) & 0 & \dfrac{c_{44}}{r} \end{bmatrix} \partial_r\tilde{\mathbf{g}}_a(r,t) +$$
$$\begin{bmatrix} \dfrac{c_{11}}{r^2}\left(1+\dfrac{c_{66}}{c_{11}}\,n^2\right)+c_{44}\,k^2 & \dfrac{-\mathbb{i}\,n\,(c_{11}+c_{66})}{r^2} & 0 \\ \dfrac{\mathbb{i}\,n\,(c_{11}+c_{66})}{r^2} & \dfrac{c_{66}}{r^2}\left(1+\dfrac{c_{11}}{c_{66}}\,n^2\right)+c_{44}\,k^2 & \dfrac{n\,k\,(c_{13}+c_{44})}{r} \\ \dfrac{\mathbb{i}\,k\,(c_{13}+c_{44})}{r} & \dfrac{n\,k\,(c_{13}+c_{44})}{r} & \dfrac{c_{44}}{r^2}\,n^2+c_{33}\,k^2 \end{bmatrix} \tilde{\mathbf{g}}_a(r,t) = \delta(r-a)\,\delta(t)\,\mathbb{I}\,. \tag{A.1}$$

By taking the n-order Tensor Hankel Transform of $\tilde{\mathbf{g}}_a(r,t)$ with respect to r (see [1]):

$$\hat{\mathbf{g}}_a(\kappa,t) = \int_0^\infty \mathbb{J}_n(\kappa\,r)\,\tilde{\mathbf{g}}_a(r,t)\,r\,\mathrm{d}r \quad\Longleftrightarrow\quad \tilde{\mathbf{g}}_a(r,t) = \int_0^\infty \mathbb{J}_n(\kappa\,r)\,\hat{\mathbf{g}}_a(\kappa,t)\,\kappa\,\mathrm{d}\kappa\,, \tag{A.2}$$

where the integral kernel is:

$$\mathbb{J}_n(X) = \begin{bmatrix} \dfrac{\mathcal{J}_{n+1}(X)-\mathcal{J}_{n-1}(X)}{2} & -\mathbb{i}\,\dfrac{\mathcal{J}_{n+1}(X)+\mathcal{J}_{n-1}(X)}{2} & 0 \\ \mathbb{i}\,\dfrac{\mathcal{J}_{n+1}(X)+\mathcal{J}_{n-1}(X)}{2} & \dfrac{\mathcal{J}_{n+1}(X)-\mathcal{J}_{n-1}(X)}{2} & 0 \\ 0 & 0 & \mathcal{J}_n(X) \end{bmatrix}, \tag{A.3}$$

$\mathcal{J}_n$ denoting the Bessel function of the first kind[1] (*e.g.*, [9, §10.2]), and κ the radial wavenumber, we obtain the following simple ordinary differential system with respect to time:

$$\rho\,\partial_t^2\hat{\mathbf{g}}_a(\kappa,t)+\underbrace{\begin{bmatrix} c_{11}\,\kappa^2+c_{44}\,k^2 & 0 & -\mathbb{i}\,(c_{13}+c_{44})\,\kappa\,k \\ 0 & c_{66}\,\kappa^2+c_{44}\,k^2 & 0 \\ \mathbb{i}\,(c_{13}+c_{44})\,\kappa\,k & 0 & c_{44}\,\kappa^2+c_{33}\,k^2 \end{bmatrix}}_{\rho\,\Omega^2}\hat{\mathbf{g}}_a(\kappa,t)=a\,\delta(t)\,\mathbb{J}_n(\kappa\,a)\,. \tag{A.4}$$

It is noticeable that in the latter equation, only the source-term is dependent from the azimuthal wavenumber n.

The positive eigenvalues of the Hermitian matrix Ω^2 – except if $\kappa=k=0$ (rigid-body motion for $n=0$) – are:

$$\begin{aligned}\omega_{1,2,3}^2=\frac{(c_{11}+c_{44})\;\kappa^2+(c_{33}+c_{44})\;k^2\pm d}{2\,\rho},\frac{c_{66}\,\kappa^2+c_{44}\,k^2}{\rho}\,,\\ \text{where } d=\sqrt{\left[(c_{11}-c_{44})\;\kappa^2-(c_{33}-c_{44})\;k^2\right]^2+4\,\left[(c_{13}+c_{44})\,\kappa\,k\right]^2}\,,\end{aligned} \tag{A.5}$$

which gives $\omega_{1,2,3}^2/\left(\kappa^2+k^2\right)=c_{11}/\rho,c_{66}/\rho,c_{66}/\rho$ in the standard isotropic case, *i.e.*, c_L^2, c_T^2, c_T^2, where c_L and c_T are the celerities of the longitudinal and transverse waves, respectively.

Equation (A.4) leads to the response in the (κ,n,k,s)-domain:

$$\hat{\mathbf{G}}_a=a\begin{bmatrix} \rho\,s^2+c_{11}\,\kappa^2+c_{44}\,k^2 & 0 & -\mathbb{i}\,(c_{13}+c_{44})\,\kappa\,k \\ 0 & \rho\,s^2+c_{66}\,\kappa^2+c_{44}\,k^2 & 0 \\ \mathbb{i}\,(c_{13}+c_{44})\,\kappa\,k & 0 & \rho\,s^2+c_{44}\,\kappa^2+c_{33}\,k^2 \end{bmatrix}^{-1}\mathbb{J}_n(\kappa\,a)\,, \tag{A.6}$$

which gives in the (r,n,k,s)-domain:

$$\begin{aligned}\tilde{\mathbf{G}}_a(r)=a\int_0^\infty\mathbb{J}_n(\kappa\,r)\begin{bmatrix} \dfrac{\rho\,s^2+c_{44}\,\kappa^2+c_{33}\,k^2}{D} & 0 & \dfrac{\mathbb{i}\,(c_{13}+c_{44})\,\kappa\,k}{D} \\ 0 & \dfrac{1}{\rho\,s^2+c_{66}\,\kappa^2+c_{44}\,k^2} & 0 \\ \dfrac{-\mathbb{i}\,(c_{13}+c_{44})\,\kappa\,k}{D} & 0 & \dfrac{\rho\,s^2+c_{11}\,\kappa^2+c_{44}\,k^2}{D} \end{bmatrix}\mathbb{J}_n(\kappa\,a)\,\kappa\,\mathrm{d}\kappa\,,\\ \text{where } D=\left(\rho\,s^2+c_{11}\,\kappa^2+c_{44}\,k^2\right)\left(\rho\,s^2+c_{44}\,\kappa^2+c_{33}\,k^2\right)-\left[(c_{13}+c_{44})\,\kappa\,k\right]^2\,.\end{aligned} \tag{A.7}$$

The roots of D, considered as a polynomial in $-\kappa^2$, are:

$$\eta_{1,2}^2=\frac{b_1\mp\Lambda}{2\,b_2}\text{ with }\left\{\begin{array}{lcl} b_2 & = & c_{11}\,c_{44} \\ b_1 & = & (c_{11}+c_{44})\,\rho\,s^2+\left(c_{11}\,c_{33}-c_{13}^2-2\,c_{13}\,c_{44}\right)\,k^2 \\ b_0 & = & \left(\rho\,s^2+c_{33}\,k^2\right)\left(\rho\,s^2+c_{44}\,k^2\right) \\ \Lambda & = & \sqrt{b_1^2-4\,b_2\,b_0} \end{array}\right.\,, \tag{A.8}$$

and we define $\eta_3^2=\left(\rho\,s^2+c_{44}\,k^2\right)/c_{66}$ such that:

$$\tilde{\mathbf{G}}_a(r)=a\int_0^\infty\mathbb{J}_n(\kappa\,r)\begin{bmatrix} \dfrac{d_{111}}{\kappa^2+\eta_1^2}+\dfrac{d_{112}}{\kappa^2+\eta_2^2} & 0 & \dfrac{\mathbb{i}\,k\,d_{131}}{\kappa\,(\kappa^2+\eta_1^2)}+\dfrac{\mathbb{i}\,k\,d_{132}}{\kappa\,(\kappa^2+\eta_2^2)} \\ 0 & \dfrac{c_{66}^{-1}}{\kappa^2+\eta_3^2} & 0 \\ \dfrac{-\mathbb{i}\,k\,d_{131}}{\kappa\,(\kappa^2+\eta_1^2)}+\dfrac{-\mathbb{i}\,k\,d_{132}}{\kappa\,(\kappa^2+\eta_2^2)} & 0 & \dfrac{d_{331}}{\kappa^2+\eta_1^2}+\dfrac{d_{332}}{\kappa^2+\eta_2^2} \end{bmatrix}\mathbb{J}_n(\kappa\,a)\,\kappa\,\mathrm{d}\kappa\,, \tag{A.9}$$

where the coefficients defined by

$$\left\{\begin{array}{lcl} d_{11j} & = & \dfrac{1}{2\,c_{11}\,\Lambda}\,\left\{\Lambda+(-1)^j\left[\rho\,(c_{11}-c_{44})\;s^2+\left(c_{11}\,c_{33}+c_{13}^2+2\,c_{13}\,c_{44}\right)\;k^2\right]\right\} \\ d_{33j} & = & \dfrac{1}{2\,c_{44}\,\Lambda}\,\left\{\Lambda-(-1)^j\left[\rho\,(c_{11}-c_{44})\;s^2+\left(c_{11}\,c_{33}-c_{13}^2-2\,c_{13}\,c_{44}-2\,c_{44}^2\right)\;k^2\right]\right\} \\ d_{13j} & = & \dfrac{c_{13}+c_{44}}{2\,c_{11}\,c_{44}\,\Lambda}\,\left[\Lambda-(-1)^j\left[\rho\,(c_{11}+c_{44})\;s^2+\left(c_{11}\,c_{33}-c_{13}^2-2\,c_{13}\,c_{44}\right)\;k^2\right]\right\} \end{array}\right.$$

[1] Satisfying $\mathcal{J}_{-n}(z)=(-1)^n\,\mathcal{J}_n(z)$ [5, §8.404-2].

satisfy:

$$d_{11j}\, d_{33j} + \frac{k^2}{\eta_j^2}\, d_{13j}^2 = 0\ .$$

Thanks to the following property [4, §8.11 Eq. (16)], assuming $Re(\eta) > 0$ and $n \geqslant \nu \geqslant 0$, and $\mathcal{I}_n$, $\mathcal{K}_n$ denoting the *modified Bessel functions* of first and second kind[2], respectively:

$$\int_0^{\infty} \frac{\mathcal{J}_{n\pm 2\,\nu}(\kappa\, r)\, \mathcal{J}_n(\kappa\, a)}{\kappa^2 + \eta^2}\, \kappa\, \mathrm{d}\kappa = \left| \begin{array}{ll} (-1)^{\nu}\, \mathcal{K}_n(\eta\, a)\, \mathcal{I}_{n\pm 2\,\nu}(\eta\, r)\ , & 0 < r < a \\ (-1)^{\nu}\, \mathcal{I}_n(\eta\, a)\, \mathcal{K}_{n\pm 2\,\nu}(\eta\, r)\ , & 0 < a < r \end{array} \right. , \tag{A.10}$$

we obtain after some algebra for $r < a$ (ingoing waves):

$$\begin{aligned} \tilde{\mathbf{G}}_a^{[\mathrm{in}]}(r) \;&=\; a \sum_{j=1}^{2} d_{11j} \begin{bmatrix} \dfrac{\mathcal{I}_{n-1}(\eta_j\, r) + \mathcal{I}_{n+1}(\eta_j\, r)}{2} \\ -\mathbb{i}\, \dfrac{\mathcal{I}_{n-1}(\eta_j\, r) - \mathcal{I}_{n+1}(\eta_j\, r)}{2} \\ \dfrac{\mathbb{i}\, d_{13j}\, k}{d_{11j}\, \eta_j}\, \mathcal{I}_n(\eta_j\, r) \end{bmatrix} \begin{bmatrix} \dfrac{\mathcal{K}_{n-1}(\eta_j\, a) + \mathcal{K}_{n+1}(\eta_j\, a)}{2} \\ \mathbb{i}\, \dfrac{\mathcal{K}_{n-1}(\eta_j\, a) - \mathcal{K}_{n+1}(\eta_j\, a)}{2} \\ \dfrac{\mathbb{i}\, d_{13j}\, k}{d_{11j}\, \eta_j} \mathcal{K}_n(\eta_j\, a) \end{bmatrix}^{\mathrm{T}} \\ &+ \frac{a}{c_{66}} \begin{bmatrix} \dfrac{\mathcal{I}_{n-1}(\eta_3\, r) - \mathcal{I}_{n+1}(\eta_3\, r)}{2} \\ -\mathbb{i}\, \dfrac{\mathcal{I}_{n-1}(\eta_3\, r) + \mathcal{I}_{n+1}(\eta_3\, r)}{2} \\ 0 \end{bmatrix} \begin{bmatrix} \dfrac{\mathcal{K}_{n-1}(\eta_3\, a) - \mathcal{K}_{n+1}(\eta_3\, a)}{2} \\ \mathbb{i}\, \dfrac{\mathcal{K}_{n-1}(\eta_3\, a) + \mathcal{K}_{n+1}(\eta_3\, a)}{2} \\ 0 \end{bmatrix}^{\mathrm{T}} , \end{aligned} \tag{A.11}$$

and for $r > a$ (outgoing waves):

$$\begin{aligned} \tilde{\mathbf{G}}_a^{[\mathrm{out}]}(r) \;&=\; a \sum_{j=1}^{2} d_{11j} \begin{bmatrix} \dfrac{\mathcal{K}_{n-1}(\eta_j\, r) + \mathcal{K}_{n+1}(\eta_j\, r)}{2} \\ -\mathbb{i}\, \dfrac{\mathcal{K}_{n-1}(\eta_j\, r) - \mathcal{K}_{n+1}(\eta_j\, r)}{2} \\ \dfrac{-\mathbb{i}\, d_{13j}\, k}{d_{11j}\, \eta_j}\, \mathcal{K}_n(\eta_j\, r) \end{bmatrix} \begin{bmatrix} \dfrac{\mathcal{I}_{n-1}(\eta_j\, a) + \mathcal{I}_{n+1}(\eta_j\, a)}{2} \\ \mathbb{i}\, \dfrac{\mathcal{I}_{n-1}(\eta_j\, a) - \mathcal{I}_{n+1}(\eta_j\, a)}{2} \\ \dfrac{-\mathbb{i}\, d_{13j}\, k}{d_{11j}\, \eta_j} \mathcal{I}_n(\eta_j\, a) \end{bmatrix}^{\mathrm{T}} \\ &+ \frac{a}{c_{66}} \begin{bmatrix} \dfrac{\mathcal{K}_{n-1}(\eta_3\, r) - \mathcal{K}_{n+1}(\eta_3\, r)}{2} \\ -\mathbb{i}\, \dfrac{\mathcal{K}_{n-1}(\eta_3\, r) + \mathcal{K}_{n+1}(\eta_3\, r)}{2} \\ 0 \end{bmatrix} \begin{bmatrix} \dfrac{\mathcal{I}_{n-1}(\eta_3\, a) - \mathcal{I}_{n+1}(\eta_3\, a)}{2} \\ \mathbb{i}\, \dfrac{\mathcal{I}_{n-1}(\eta_3\, a) + \mathcal{I}_{n+1}(\eta_3\, a)}{2} \\ 0 \end{bmatrix}^{\mathrm{T}} , \end{aligned} \tag{A.12}$$

where $Re(\eta_j) > 0\,,\ j = 1, 2, 3\,.$

[2] Satisfying $\mathcal{I}_{-n}(z) = \mathcal{I}_n(z)$ and $\mathcal{K}_{-n}(z) = \mathcal{K}_n(z)$, from [5, §8.406,§8.407].