Montclair State University
Department of Mathematical Sciences

Comprehensive Exam Study Guide – Algebra

Linear Algebra

Topics

Systems of Linear Equations

Matrix Algebra, Determinants

Vector Spaces

Linear Transformations

Eigenvalues and eigenvectors

Jordan Canonical Form

Inner Product Spaces

Gram-Schmidt Process

References

Friedberg, Insel, Spence, Linear Algebra, Prentice Hall
Hoffman, K., Linear Algebra, Prentice Hall
Halmos, P. R., Finite Dimensional Vector Spaces, Springer Verlag
Larson, Elementary Linear Algebra, Cengage
Nering, E., Linear Algebra and Matrix Theory, Wiley

Abstract Algebra

Topics

Groups, Subgroups, Lagranges Theorem

Normal Subgroups, Homomorphisms, Homomorphism Theorems

Permutation Groups, Cayleys Theorem

Class Equation, Sylows Theorems

Rings, Ideals, Ring Homomorphisms

Polynomial Rings

The Field of Quotients of an Integral Domain

Extension Fields

Roots of Polynomials, Galois Theory

References

Beachy & Blair, Abstract Algebra, Waveland Press
Fraleigh, J. B., A First Course in Abstract Algebra, Addison-Wesley
Gallian, J, A., Contemporary Abstract Algebra, Houghton Mifflin
Goldstein, L. J., Abstract Algebra: A First Course
Hungerford, T. W., Algebra, Springer Verlag
Herstein, I. N., Topics in Algebra, Wiley

Sample Questions

1. (a) Define each of the following terms: G is a group; G is a cyclic group; H is a normal subgroup of G; the quotient group of G by H; S is a p-Sylow subgroup of G; the symmetric group S_5; the alternating group A_5.

   (b) Prove that S_5 has a cyclic subgroup of order 5.

2. State the following definitions. Assume that V and W are finite-dimensional vector spaces over a common field F with ordered bases β and γ respectively. Let T be a linear transformation from V into W.

   (a) Z is a subspace of V.

   (b) S is a linearly independent set of vectors in W.

   (c) The kernel (null space) of T.

   (d) The characteristic polynomial of T.

   (e) The j-th column of $[T]_\beta^\gamma$.

3. Let S_n be the group of permutations of $\{1, 2, ..., n\}$.

   (a) Define: transposition, cycle, even permutation.

   (b) Write $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$ as a product of disjoint cycles and as a product of transposition. What is the order of α? Is it an even permutation?

4. (a) If V is an n-dimensional vector space and S is a set of n orthonormal vectors in V, prove that S is a basis for V.

   (b) Use the Gram-Schmidt process to construct an orthonormal set from $\{(1, 0, 1), (0, 1, 1), (1, 1, 1)\}$.

5. Find the Jordan canonical form of

$$A = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

6. Give specific examples of the following, describing them briefly:

   (a) a non-abelian group of order 8

   (b) a non-commutative ring

   (c) a ring which has zero divisors (give elements which are zero divisors)

   (d) an integral domain which is not a field

   (e) a division ring (which is not a field)

   (f) a finite field

   (g) an infinite field which is of characteristic 2

7. (a) Let F and K be fields, $F \subseteq K$, $a \in K$. Define the following: K is an extension field F; the degree of K over F; $F(a)$ where $a \in K$, and K is an extension field of F; $a \in K$ is algebraic over F .

   (b) Let $\mathbb{Q}$ be the field of rational numbers. Find the degree of $\mathbb{Q}(i+\sqrt{3})$ over $\mathbb{Q}$, where $i^2 = 1$.

   (c) Is the field of complex numbers a finite extension of the field of rational numbers? Explain.

8. Prove the following two statements.

   (a) Let V and W be finite-dimensional vector spaces over the same field F. Furthermore, assume that $\dim(V) = \dim(W)$ and that $T : V \to W$ is a linear transformation. Prove that T is one to one if and only if T is onto.

   (b) If A is a matrix in row echelon form, then $\operatorname{rank}(A)$ equals the number of nonzero rows in A.

9. Find the eigenvalues and a basis for each eigenspace of $T(x, y, z) = (3x - 2y, -2x + 3y, 5z)$, where T is regarded as a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$. Is T diagonalizable? Explain briefly.

10. (a) Define: I is an ideal of the ring R.

    (b) Define: The ring of integers modulo n, $\mathbb{Z}_n$. Be sure to define addition and multiplication in $\mathbb{Z}_n$.

    (c) Prove that multiplication (mod n) is well defined.

    (d) Prove: If n is a prime integer and I is an ideal of $\mathbb{Z}_n$, then either $I = (0)$ or $I = \mathbb{Z}_n$.

11. Prove the following two statements.

    (a) If W_1 and W_2 are both subspaces of a vector space V, prove that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

    (b) Similar matrices have the same determinant.