Numerical Analysis Comprehensive Examination Study Guide

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Updated: September 2016

1 Topics

The topics below are covered in Chapter 1-6 of Numerical Analysis by Burden and Faires (Ref. 1 below)

1. *Error Analysis:* Round-off errors, error propagation, algorithms and convergence.
2. *Solutions of equations in one variable:* the Bisection method, fixed-point iteration, Newtons method, Secant method, and error analysis for iterative methods.
3. *Interpolation and polynomial approximation:* Lagrange polynomial, divided differences, cubic splines, and error formulas.
4. *Numerical differentiation and integration:* numerical differentiation, Richardson's extrapolation, numerical integration and composite numerical integration, degree of precision of quadrature formulas, Romberg integration, adaptive and Gaussian quadrature methods.
5. *Numerical methods for Ordinary Differential Equations–Initial Value Problems:* Eulers method, Higher order Taylor series methods, Runge-Kutta methods, multistep methods, Error control for Runge-Kutta type methods, Stability.
6. *Numerical Linear Algebra:* Gaussian elimination, matrix factorization (A = LU), pivoting strategies, matrix norms, condition number of a matrix, iterative methods (Jacobi, Gauss-Seidel, and relaxation techniques)

2 References

1. Richard L. Burden and J.Douglas Faires, ISBN-10: 0-538-73351-9 Numerical Analysis, Ninth Edition, Brooks/Cole, 2011 **or** Richard L. Burden, J. Douglas Faires, and Annette M. Burden ISBN-10: 1305253663, Numerical Analysis, Tenth Edition 2016
2. Ward Cheney and David Kinkaid, ISBN-10: 1133103715, Numerical Mathematics and Computing, Seventh Edition, Brooks/Cole, 2013.

3 Sample Problems

The problems below are meant to provide a general idea of what the student is expected to know. All problems on the comprehensive exam may not follow the pattern in any of the problems below. **A student appearing for the comprehensive examination is expected to have a comprehensive knowledge of the topics listed above.**

1. Define what it means for a sequence $\{\alpha_n\}$ to converge to α with a rate of convergence $\mathcal{O}(\beta_n)$; Determine the rate of convergence of $\lim_{n\to\infty} \sin\frac{1}{n^2} = 0$
2. Suppose $\lim_{h\to 0} G(h) = 0$ and $\lim_{h\to 0} F(h) = L$; Define $F(h) = L + \mathcal{O}(G(h))$. Find the rate of convergence for $\lim_{h\to 0} \frac{\sin h - h\cos h}{h} = 0$
3. Suppose the sequence $\{p_n\}$ converges to p. Define $\{p_n\}$ **converges linearly** to p and $\{p_n\}$ **converges quadratically** to p

4. Show that the equation $x^3 + 4x^2 - 10 = 0$ has a solution in the interval $[1, 2]$. Use the Bisection method to determine the approximation p_4 of the solution at the fourth iteration. Estimate the number of iterations needed to get an approximation of the solution to within 10^{-5} error accuracy.

5. Assume that $f \in C[a, b]$ and $f(a) \cdot f(b) < 0$ Show that the bisection method generates a sequence $\{p_n\}$ for approximating a zero p of f with $p_n = p + \mathcal{O}(1/2^n)$

6. Use Newtons method to determine the approximation p_4 of the solution to the equation $x^3 + 4x^2 - 10 = 0$ using the initial approximation $p_0 = 1$.

7. Let $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$. Further assume that there exists a positive constant $0 < L < 1$ such that $\dfrac{|g(x) - g(y)|}{|x - y|} \leq L$. Show that $g(x)$ has a unique fixed point in $[a, b]$.

8. Determine a function $g(x)$ that has a fixed point at the unique root of the equation $x^3 + 4x^2 - 10 = 0$ in the interval $[1, 2]$, such that the convergence of the fixed-point iteration to the root is quadratic.

9. Assume that $x_0, x_1, \ldots, x_n$ are $(n + 1)$ distinct real numbers and a function f is defined in some interval containing these points. If $f(x_i) = y_i$ for $0 \leq i \leq n$, define the *Lagrange basis polynomials* $L_i(x)$ corresponding to the given data; Use the Lagrange basis to define the **n-th Lagrange interpolating polynomial** $P_n(x)$. Construct the Lagrange interpolating polynomial for $(1, 1), (2, 3)$ and $(4, 5)$.

10. Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Show that for each $x \in [a, b]$, there exists a number $\xi(x) \in (a, b)$ such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i), \text{ where } P_n(x) \text{ is the } n\text{-th Lagrange interpolating polynomial}$$

Use this result to find an error bound for approximating $f(0.45)$ by $P_2(0.45)$ using $x_0 = 0, x_1 = 0.6$, and $x_2 = 0.9$ and $f(x) = \sqrt{1 + x}$.

11. A clamped cubic spline s for a function f is defined on $[1, 3]$ by

$$s(x) = \begin{cases} s_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & 1 \leq x \leq 2; \\ s_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3, & 2 \leq x \leq 3. \end{cases}$$

Given $f'(1) = f'(3)$, find a, b, c, and d.

12. Suppose that $N(h)$ is an approximation to M for every $h > 0$ and that $M = N(h) + k_1 h^2 + k_2 h^4 + k_3 h^6 + \cdots$ for some constants $k_1, k_2, k_2, \ldots$. Use the values of $N(h), N(h/3)$, and $N(h/9)$ to produce an $\mathcal{O}(h^6)$ approximation to M.

13. Define *degree of accuracy or precision* of a quadrature formula.

    (a) Determine constants a, b, c, and d so that the quadrature formula $I_f = af(-1) + bf(1) + cf'(-1) + df(1)$ approximating $\displaystyle\int_{-1}^{1} f(x)\,dx$ has a degree of accuracy three.

    (b) Show that a quadrature formula has a degree of precision n if and only if the error $E(P(x)) = 0$ for all polynomials $P(x)$ of degree n or less but $E(P(x)) \neq 0$ for some polynomial $P(x)$ of degree $n + 1$

14. Derive the simple trapezoidal rule for approximating $\displaystyle\int_a^b f(x)\,dx$ using a linear Lagrange interpolating polynomial to approximate $f(x)$. If $x_0 = a, x_1 = b$, and $h = b - a$, show that the simple trapezoidal approximation has an error of $\mathcal{O}(h^3)$

15. Assume f is continuous and satisfies a Lipschitz condition on $D = \{(t, y) | a \leq t \leq b, -\infty < y < \infty\}$ with a constant L. If there exists a constant M with $|y''(t)| \leq M$ for all $t \in [a, b]$, $y(t)$ is the unique solution to the initial value problem $y'(t) = f(t, y)$, $a \leq t \leq b$, $y(a) = \alpha$, and $w_0, w_1, \ldots, w_N$ are the approximations generated by Euler's method for some positive integer N, then for each $i = 0, 1, 2, \ldots, N$, show that $|y(t_i) - w_i| \leq \dfrac{hM}{2L}\left[e^{L(t_i-\alpha)} - 1\right]$.

16. Use the modified Euler method to approximate the solution to the initial value problem $y' = \dfrac{1+t}{1+y}$, $1 \leq t \leq 2$ and $y(1) = 2$ given $h = 0.2$; compare your computed values with the actual solution $y(t) = \sqrt{t^2 + 2t + 6} - 1$.

17. Consider the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix}$.

(a) Find a unit lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

(b) Use the result of the previous part to find $\det(A)$.

(c) If Gaussian elimination with partial pivoting were used to solve $Ax = b$, where b is an arbitrary 3-vector, what would be the first elementary row operation?

(d) Give an example of a nonsingular 2×2 matrix B for which an LU decomposition does not exist, and explain why it does not exist.