Numerical Analysis Comprehensive Examination Study Guide

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The following list of topics is not meant to be comprehensive. The exam may cover topics that are not listed on this study guide but were covered in the corresponding course or covered in prerequisite undergraduate courses. The sample questions are merely representative of content that may be covered on the exam.

1 Topics

The topics below are covered in Chapter 1-6 of Numerical Analysis by Burden and Faires (Ref. 1 below)

1. *Error Analysis:* Round-off errors, error propagation, algorithms and convergence.

2. *Solutions of equations in one variable:* the Bisection method, fixed-point iteration, Newtons method, Secant method, and error analysis for iterative methods.

3. *Interpolation and polynomial approximation:* Lagrange polynomial, divided differences, cubic splines, and error formulas.

4. *Numerical differentiation and integration:* numerical differentiation, Richardson's extrapolation, numerical integration and composite numerical integration, degree of precision of quadrature formulas, Romberg integration, adaptive and Gaussian quadrature methods.

5. *Numerical methods for Ordinary Differential Equations–Initial Value Problems:* Eulers method, Higher order Taylor series methods, Runge-Kutta methods, multistep methods, Error control for Runge-Kutta type methods, Stability.

6. *Numerical Linear Algebra:* Gaussian elimination, matrix factorization (A = LU), pivoting strategies, matrix norms, condition number of a matrix, iterative methods (Jacobi, Gauss-Seidel, and relaxation techniques)

2 References

1. Richard L. Burden and J.Douglas Faires, ISBN-10: 0-538-73351-9 Numerical Analysis, Ninth Edition, Brooks/Cole, 2011 **or** Richard L. Burden, J. Douglas Faires, and Annette M. Burden ISBN-10: 1305253663, Numerical Analysis, Tenth Edition 2016

2. Ward Cheney and David Kinkaid, ISBN-10: 1133103715, Numerical Mathematics and Computing, Seventh Edition, Brooks/Cole, 2013.

3 Sample Problems

1. Define what it means for a sequence $\{\alpha_n\}$ to converge to α with a rate of convergence $\mathcal{O}(\beta_n)$; Determine the rate of convergence of $\lim_{n\to\infty} \sin\dfrac{1}{n^2} = 0$

2. Suppose $\lim_{h\to 0} G(h) = 0$ and $\lim_{h\to 0} F(h) = L$; Define $F(h) = L + \mathcal{O}(G(h))$. Find the rate of convergence for $\lim_{h\to 0} \dfrac{\sin h - h\cos h}{h} = 0$

3. Suppose the sequence $\{p_n\}$ converges to p. Define $\{p_n\}$ **converges linearly** to p and $\{p_n\}$ **converges quadratically** to p

4. Show that the equation $x^3 + 4x^2 - 10 = 0$ has a solution in the interval $[1, 2]$. Use the Bisection method to determine the approximation p_4 of the solution at the fourth iteration. Estimate the number of iterations needed to get an approximation of the solution to within 10^{-5} error accuracy.

5. Assume that $f \in C[a, b]$ and $f(a) \cdot f(b) < 0$ Show that the bisection method generates a sequence $\{p_n\}$ for approximating a zero p of f with $p_n = p + \mathcal{O}(1/2^n)$

6. Use Newtons method to determine the approximation p_4 of the solution to the equation $x^3 + 4x^2 - 10 = 0$ using the initial approximation $p_0 = 1$.

7. Let $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$. Further assume that there exists a positive constant $0 < L < 1$ such that $\dfrac{|g(x) - g(y)|}{|x - y|} \leq L$. Show that $g(x)$ has a unique fixed point in $[a, b]$.

8. Determine a function $g(x)$ that has a fixed point at the unique root of the equation $x^3 + 4x^2 - 10 = 0$ in the interval $[1, 2]$, such that the convergence of the fixed-point iteration to the root is quadratic.

9. Assume that $x_0, x_1, \ldots, x_n$ are $(n + 1)$ distinct real numbers and a function f is defined in some interval containing these points. If $f(x_i) = y_i$ for $0 \leq i \leq n$, define the *Lagrange basis polynomials* $L_i(x)$ corresponding to the given data; Use the Lagrange basis to define the **n-th Lagrange interpolating polynomial** $P_n(x)$. Construct the Lagrange interpolating polynomial for $(1, 1), (2, 3)$ and $(4, 5)$.

10. Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Show that for each $x \in [a, b]$, there exists a number $\xi(x) \in (a, b)$ such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i), \text{ where } P_n(x) \text{ is the } n\text{-th Lagrange interpolating polynomial}$$

Use this result to find an error bound for approximating $f(0.45)$ by $P_2(0.45)$ using $x_0 = 0, x_1 = 0.6$, and $x_2 = 0.9$ and $f(x) = \sqrt{1 + x}$.

11. A clamped cubic spline s for a function f is defined on $[1, 3]$ by

$$s(x) = \begin{cases} s_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & 1 \leq x \leq 2; \\ s_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3, & 2 \leq x \leq 3. \end{cases}$$

Given $f'(1) = f'(3)$, find a, b, c, and d.

12. Suppose that $N(h)$ is an approximation to M for every $h > 0$ and that $M = N(h) + k_1 h^2 + k_2 h^4 + k_3 h^6 + \cdots$ for some constants $k_1, k_2, k_2, \ldots$. Use the values of $N(h), N(h/3)$, and $N(h/9)$ to produce an $\mathcal{O}(h^6)$ approximation to M.

13. Define *degree of accuracy or precision* of a quadrature formula.

    (a) Determine constants a, b, c, and d so that the quadrature formula $I_f = af(-1) + bf(1) + cf'(-1) + df(1)$ approximating $\displaystyle\int_{-1}^{1} f(x)\,dx$ has a degree of accuracy three.

    (b) Show that a quadrature formula has a degree of precision n if and only if the error $E(P(x)) = 0$ for all polynomials $P(x)$ of degree n or less but $E(P(x)) \neq 0$ for some polynomial $P(x)$ of degree $n + 1$

14. Derive the simple trapezoidal rule for approximating $\displaystyle\int_a^b f(x)\,dx$ using a linear Lagrange interpolating polynomial to approximate $f(x)$. If $x_0 = a, x_1 = b$, and $h = b - a$, show that the simple trapezoidal approximation has an error of $\mathcal{O}(h^3)$

15. Assume f is continuous and satisfies a Lipschitz condition on $D = \{(t, y) | a \leq t \leq b, -\infty < y < \infty\}$ with a constant L. If there exists a constant M with $|y''(t)| \leq M$ for all $t \in [a, b]$, $y(t)$ is the unique solution to the initial value problem $y'(t) = f(t, y)$, $a \leq t \leq b$, $y(a) = \alpha$, and $w_0, w_1, \ldots, w_N$ are the approximations generated by Euler's method for some positive integer N, then for each $i = 0, 1, 2, \ldots, N$, show that $|y(t_i) - w_i| \leq \dfrac{hM}{2L}\left[e^{L(t_i - \alpha)} - 1\right]$.

16. Use the modified Euler method to approximate the solution to the initial value problem $y' = \dfrac{1+t}{1+y}$, $1 \leq t \leq 2$ and $y(1) = 2$ given $h = 0.2$; compare your computed values with the actual solution $y(t) = \sqrt{t^2 + 2t + 6} - 1$.

17. Consider the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix}$.

    (a) Find a unit lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

    (b) Use the result of the previous part to find $\det(A)$.

    (c) If Gaussian elimination with partial pivoting were used to solve $Ax = b$, where b is an arbitrary 3-vector, what would be the first elementary row operation?

    (d) Give an example of a nonsingular 2×2 matrix B for which an LU decomposition does not exist, and explain why it does not exist.