Department of Mathematical Sciences
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Study Guide

Applied Industrial Mathematics

The Master's Examination in Applied Industrial Mathematics covers the following topics.

1. Statistical Reasoning (Probability Distributions)
    (a) Random Variables
    (b) Uniform Distributions
    (c) Gaussian Distributions
    (d) The Binomial Distribution
    (e) The Poisson Distribution
2. Monte Carlo methods
    (a) Computing Integrals
    (b) Mean Time Between Failure
    (c) Average Number of Rejected Requests
    (d) Average Waiting Time
3. Data Acquisition and Manipulation
    (a) The z-Transform
    (b) Linear Recursions
    (c) Filters
4. The Discrete Fourier Transform (DFT)
    (a) Real-time Processing
    (b) Properties of the DFT
    (c) The Fast Fourier Transform
    (d) Image Processing
5. Linear Programming
    (a) Optimization
    (b) The Diet Problem
    (c) Using the Simplex Algorithm to Solve Diet Problems

6. Regression
    (a) Best fit to discrete data
    (b) Norms on $\mathbb{R}^n$
    (c) Hilbert space
7. Cost Benefit Analysis
    (a) Present Value
    (b) Life-Cycle Saving

Examples

1. If x and y are drawn randomly from the interval $[0, 2]$, what is the probability that $x^2+y < 1$?

   Solution: The question can be rephrased as: How likely is a random point in the square $0 \leq x, y \leq 2$ to fall in the area under the curve $y = 1 - x^2$? We first calculate the area under the curve $x^2 + y = 1$ or $y = 1 - x^2$ bounded by the positive x- and y- axes of the Cartesian coordinate system. This area is given by

   $$A_1 = \int_0^1 (1 - x^2)\, dx = \frac{2}{3}.$$

   We next calculate the area of the square $[0, 2] \times [0, 2]$, which is $A_2 = 4$. Thus, the probability is

   $$\frac{A_1}{A_2} = \frac{1}{6}.$$

2. On average, one sodium vapor tunnel light burns out every week. How often each year will two or more lights burn out in a day?

   Solution: Use the Poisson distribution

   $$p(k) = \frac{(\lambda T)^k\, e^{-\lambda T}}{k!}.$$

   $p(k)$ gives the probability that k events occur in the time period T, when λ is the average number of events occurring during a unit time. The mean $\mu = \lambda T = \dfrac{1}{7}$. Thus, for this problem,

   $$p(k) = \frac{(1/7)^k e^{-1/7}}{k!}.$$

   The probability that there are two or more failing lights in a day is

   $$1 - p(0) - p(1) \approx 0.0092824.$$

   Multiply this probability by 365 (number of days in a year) to get

   $$(0.0092824)(365) = 3.3880760.$$

   Therefore, we will have about three times a year for two or more lights to burn out in a day.

3. A normally distributed random variable X has mean $\mu = 72$ and standard deviation $\sigma = 11$. Find the probability that X lies between a = 54 and b = 75.

   Solution: The probability density function is $f(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(x-\mu)^2/2\sigma^2}$. Thus, the required probability is

   $$\text{prob}(54 \leq X \leq 75) = \int_{54}^{75} f(x)\, dx \approx 0.5565868.$$

   Hence, we have a probability of 55.66% that X lies between a and b.

4. Find the z-transform of the signal 1,2,3,1,2,3,1,2,3,...

Solution: The Z-transform of a constant signal $x = \{\alpha, \alpha, \alpha, \ldots\}$ is given by

$$X = \sum_{k=0}^{\infty} \alpha z^{-k} = \alpha \sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^k = \frac{\alpha}{1 - \dfrac{1}{z}} = \frac{\alpha z}{z-1}.$$

Thus, the z-transforms of the signals $x_1 = \{1, 1, 1, \ldots\}$, $x_2 = \{2, 2, 2, \ldots\}$, and $x_3 = \{3, 3, 3, \ldots\}$ are

$$X_1 = \frac{z}{z-1}, \quad X_2 = \frac{2z}{z-1}, \quad \text{and } X_1 = \frac{3z}{z-1} \tag{1}$$

Notice that

$$X_1(z) = \frac{z}{z-1} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \frac{1}{z^6} + \frac{1}{z^7} + \cdots$$

$$X_2(z) = \frac{2z}{z-1} = 2 + \frac{2}{z} + \frac{2}{z^2} + \frac{2}{z^3} + \frac{2}{z^4} + \frac{2}{z^5} + \frac{2}{z^6} + \frac{2}{z^7} + \cdots$$

$$X_3(z) = \frac{3z}{z-1} = 3 + \frac{3}{z} + \frac{3}{z^2} + \frac{3}{z^3} + \frac{3}{z^4} + \frac{3}{z^5} + \frac{3}{z^6} + \frac{3}{z^7} + \cdots$$

The coefficients of the Laurent series can be computed by observing that a Laurent series corresponds to a Taylor series after substituting z by $1/z$.

It takes three steps to repeat each of the numbers (1, 2, and 3) in the given signal. Thus, substitute z by z^3 in 1 to get the z-transforms of the signals:

$$\{1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots\}, \quad \{2, 0, 0, 2, 0, 0, 2, 0, 0, 2 \ldots\}, \quad \{3, 0, 0, 3, 0, 0, 3, 0, 0, 3 \ldots\}$$

The results after substituting z^3 for z in (1) are

$$Z_1 = \frac{z^3}{z^3-1}, \quad Z_2 = \frac{2z^3}{z^3-1} \quad \text{and } Z_3 = \frac{3z^3}{z^3-1} \tag{2}$$

We now introduce the delay factors to the transforms in (2) and add them to get the z transform of the given signal, which is given as follows.

$$X(z) = Z_1 + \frac{Z_2}{z} + \frac{Z_3}{z^2} = \frac{z^3}{z^3-1} + \frac{2z^2}{z^3-1} + \frac{3z}{z^3-1} = \frac{z(z^2+2z+3)}{z^3-1}. \tag{3}$$

Note that the Laurent series of (3) is given below, which is what is the expected result.

$$\frac{z(z^2+2z+3)}{z^3-1} = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{1}{z^3} + \frac{2}{z^4} + \frac{3}{z^5} + \frac{1}{z^6} + \frac{2}{z^7} + \frac{3}{z^5} + \frac{1}{z^6} + \frac{2}{z^7} + \frac{3}{z^8} + \frac{1}{z^9} + \frac{2}{z^{10}} + \cdots$$

5. Solve the recursion $x_k = x_{k-1} + 2x_{k-2}$, where $x_{-1} = 1/2$ and $x_{-2} = -1/4$. Use the z-transform method. **Answer:** $x_k = \dfrac{1}{3}\left[2^k - (-1)^k\right]$

6. Let $\xi = e^{\frac{2\pi}{n}i} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$, which is the primitive nth root of unity of least positive angle. The Fourier matrix

$$F_n = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \xi^{-1} & \xi^{-2} & \xi^{-3} & \cdots & \xi^{-(n-1)} \\ 1 & \xi^{-2} & \xi^{-4} & \xi^{-6} & \cdots & \xi^{-2(n-1)} \\ 1 & \xi^{-3} & \xi^{-6} & \xi^{-9} & \cdots & \xi^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \xi^{-(n-1)} & \xi^{-2(n-1)} & \xi^{-3(n-1)} & \cdots & \xi^{-(n-1)^2} \end{bmatrix}$$

satisfies the linear transformation $\hat{x} = Fx$, where the accumulated data $x = (x_0, x_1, x_2, \ldots, x_{n-1})$ is transformed into the frequency domain object $\hat{x} = (\hat{x}_0, \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{n-1})$ by the Discrete Fourier Transform $x \longmapsto \hat{x}$. Note that $\hat{x}_k = \sum_{j=0}^{n-1} x_j\, \xi^{-jk}$. Show that $F_n^{-1} = (1/n)\overline{F}_n$, where the overbar is complex conjugation.

7. Consider the data (x, y) given in the following table

x	-2	-1	1	2	3
y	2	1	-1	0	2

Find the best-fitting quadratic $y = ax^2 + bx + c$ and graph the results.

8. Compute the present value of \$1,000 five years from now, using a discount rate of 5%.
Answer: \$778.80

References

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