Effective Mordell for Curves via the Structure of the Mordell–Weil Lattice

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Contents

1 Introduction

Hilbert's Tenth Problem asks whether there exists an algorithm to decide if a general Diophantine equation has integer solutions. While the answer is negative, a natural variation of this question posed over the rationals remains unresolved in full generality. However, for curves of $g \geq 2$, Faltings' Theorem guarantees that the set of rational points is finite.

The catch: Faltings' result is ineffective—it confirms finiteness but offers no method for computing or even bounding the rational points. This project contributes to ongoing efforts to make such finiteness results explicit.

In particular, we build on the García–Fritz–Pasten framework ("Effective Mordell for curves with enough automorphisms" [GFP25]), whose bound requires the curve's automorphism group to be large. In practice, many genus-2, Mordell-Weil rank-2 curves have few rational symmetries - often of order 4 or 6 - and hence lie outside its scope.

To address this issue, we replace their requirement of "sufficiently many automorphisms relative to the Mordell–Weil rank of the Jacobian" [GFP25] by instead exploiting the lattice structure of the free part of the Mordell–Weil group of the Jacobian. Our new criteria—developed in depth in Section 5—yield explicit height bounds even when a curve's automorphism group is small.

2 Background Summary

In this section, we review relevant background in algebraic and arithmetic geometry, including divisors, Jacobian varieties, and height theory. We then recall the main theorems proved in [GFP25].

2.1 Divisors

The following definitions come from [HS00]

Definition 2.1. (Weil Divisors) Let X be an algebraic variety. The *group of Weil divisors* on X is the free abelian group generated by the closed subvarieties of codimension one on X. It is denoted by $\mathrm{Div}(X)$.

More simply, if we let X be a curve, we can view Weil Divisors to be the formal sum of closed points with integer coefficients, and can be denoted as $D = \sum n_i P_i \in \mathrm{Div}(X)$, where n_i are integers. This is a finite sum, so $n_i = 0$ for all but a finite number of P_i.

Definition 2.2. (Principal Divisors) Let X be a variety, and let $f \in K(X)$ be a rational function on X. The *divisor of f* is the divisor

$$\operatorname{div}(f) = \sum_Y \operatorname{ord}_Y(f)\, Y \in \operatorname{Div}(X).$$

We call $D \in \operatorname{Div}(X)$ to be principal if $D = \operatorname{div}(f)$ for some $f \in K(X)^*$.

We can define a linear equivalence of divisors D and D' to be $D \sim D'$, where $D' = D + \operatorname{div}(f)$ for some f.

Definition 2.3. (Divisor Class Group)

1. The divisor class group of X is the group of divisor classes modulo linear equivalence. It is denoted by $\operatorname{Cl}(X)$. The linear equivalence class of a divisor D will be denoted by $\operatorname{Cl}(D)$. Then
$$\operatorname{Cl}(D) = \{D' \in \operatorname{Div}(X) | D' - D = \operatorname{div}(f)\}$$
for some rational function f.

2. Let $\deg \operatorname{Cl}(D)$ be the degree of the divisor class. Then
$$\deg \operatorname{Cl}(D) = \deg(D).$$

Definition 2.4. (Pullback of Divisors) For a nonconstant morphism of curves $f : X \to Y$, if $P \hookrightarrow Y$ is a closed point, then its pullback f^*P is the preimage with multiplicities. For general divisors, this pullback extends linearly.

Definition 2.5. (The Canonical Divisor) Let ω be a nonzero differential n-form on X. Any other nonzero differential n-form ω' on X has the form

$$\omega' = f\,\omega$$

for some rational function $f \in K(X)^*$. It follows that

$$\operatorname{div}(\omega') = \operatorname{div}(\omega) + \operatorname{div}(f),$$

so the divisor class associated to an n-form is independent of the chosen form. This divisor class is called the *canonical class* of X. Any divisor in the canonical class is called a *canonical divisor* and is denoted by K_X.

2.2 Height Functions

Morally, height functions measure the arithmetic "complexity" of a point. The following definitions come from [HS00]

As an example of an height function, consider the projective space $\mathbb{P}^n$. Let $P = [x_0, \ldots, x_n]$, and $x_0, \ldots, x_n \in \mathbb{Z}$, and $\gcd(x_0, \ldots, x_n) = 1$. Then

$$h(P) = \max\{|x_0|, \ldots, |x_n|\}.$$

Definition 2.6. (Height Functions) Let $\varphi : X \to \mathbb{P}^n$ be a morphism. The *(absolute logarithmic) height* on X relative to φ is the function

$$h_\varphi : X(\overline{\mathbb{Q}}) \longrightarrow [0, \infty), \quad h_\varphi(P) = h\big(\varphi(P)\big),$$

where

$$h : \mathbb{P}^n(\overline{\mathbb{Q}}) \longrightarrow [0, \infty)$$

is the height function on projective space defined in the examples above.

An important property that all height functions should satisfy is the Northcott Property. This property is used in a variety of finiteness theorems.

Theorem 2.1 (Northcott Property)**.** *For any constant B, the set $\{P \in X(K) | h(P) < B\}$ is finite.*

2.2.1 Weil Height Machine

We can connect divisors on a curve and height functions with the following theorem.

Theorem 2.2 (Weil Height Machine)**.** *Let K be a number field. For every smooth projective variety X/K there exists a map*

$$h_X\colon \operatorname{Div}(X) \longrightarrow \big\{\textit{functions } X(K) \to \mathbb{R}\big\}$$

such that

1. *(Functoriality) Let $\phi : Y \to X$ be a morphism and $D \in \operatorname{Div}(X)$. Then*
$$h_{Y,\phi^* D}(P) = h_{X,D}(\phi(P)) + O(1).$$
2. *(Additivity) Let $D, E \in \operatorname{Div}(X)$. Then*
$$h_{X,D+E}(P) = h_{X,D}(P) + h_{X,E}(P) + O(1).$$
3. *(Linear Equivalence) Let $D, E \in \operatorname{Div}(X)$, with D linearly equivalent to E. Then*
$$h_{X,D}(P) = h_{X,E}(P) + O(1).$$
4. *(Positivity) For an effective divisor $D \in \operatorname{Div}(X)$, and B be the base locus, then*
$$h_{X,D}(P) \geq O(1) \qquad \textit{for all } P \in (X \backslash B)(\bar{K}).$$

2.2.2 Canonical Heights

Theorem 2.3. *Let A/K be an abelian variety defined over a number field, and let $D \in$ $\mathrm{Div}(A)$ be a divisor whose divisor class is* symmetric, *(i.e., $[-1]^*D \sim D$). Then there exists a height function*

$$\hat{h}_{A,D} : A(K) \longrightarrow \mathbb{R},$$

called the canonical height *on A relative to D. It satisfies the following properties:*

(a) $\hat{h}_{A,D}(P) = \hat{h}_{A,D}(P) + O(1)$ *for all* $P \in A(K)$.

(b) For all integers m,

$$\hat{h}_{A,D}([m]P) = m^2\,\hat{h}_{A,D}(P), \quad \forall P \in A(K).$$

(c) The canonical height map is a quadratic form. The associated pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_D : A(K) \times A(K) \longrightarrow \mathbb{R}$$

is defined by

$$\langle P, Q\rangle_D = \frac{\hat{h}_{A,D}(P+Q) - \hat{h}_{A,D}(P) - \hat{h}_{A,D}(Q)}{2},$$

and is bilinear. Moreover, it satisfies

$$\langle P, P\rangle_D = \hat{h}_{A,D}(P).$$

2.3 Jacobians

Definition 2.7 ([HS00])**.** Let X be a smooth projective curve of genus $g > 1$. There exists an abelian variety $J(X)$ called the *Jacobian* of X, and an injection

$$j : X \hookrightarrow J(X),$$

called the *Jacobian embedding* of X.
A property of J is that it is a group of linear equivalence class of divisors of X of degree 0.

For the paper, we instead use the following map j

$$j : X \to J, \qquad P \mapsto \mathrm{Cl}((2g-2)(P) - K_X). \tag{1}$$

Note that $\mathrm{Cl}((2g-2)(P) - K_X)$ has degree 0, since $\deg K_X = (2g-2)$.

Theorem 2.4. *Mordell-Weil Theorem*

The group of K-rational points on the Jacobian (abelian variety) is finitely generated. So,

$$J(\mathbb{Q}) \cong \mathbb{Z}^r \oplus T,$$

where r is the Mordell Weil rank and T is the torsion subgroup.

Since J is an abelian variety, we can define a canonical height, $\hat{h}_{J,D}$ over J, for a divisor $D \in \mathrm{Div}(X)$ with a symmetric divisor class. Some useful properties of doing this are the following

1. For $\sigma \in \mathrm{Aut}(X)$, then $\hat{h}_{J,D}(j(P)) = \hat{h}_{J,D}(j(\sigma(P)))$ for all $P \in X(\mathbb{Q})$
2. Defines a positive definite quadratic for on a real vector space: $J(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$

2.4 Intersection Theory

We introduce the relevant lemmas and theorems from Intersection Theory.

Lemma 2.5. *Let $\mathscr{X} \to B$ be a regular arithmetic surface and $s \in B$ a closed point. Let C_i and C_j be distinct, irreducible components of the special fiber $\mathscr{X}_s$. If the local intersection number $i_s(C_i, C_j) = 1$, then C_i and C_j are geometrically reduced and irreducible.*

Proof. Let $D = C_i$, and $E = C_j$. From [Liu02, Theorem 9.1.12], we see that

$$i_s(D, E) = \deg_{k(s)} \mathcal{O}_{\mathscr{X}}(D)|_E.$$

By [Liu02, Exercise 9.1.9], we see that $\deg_{\mathbb{F}_2} \mathcal{O}_{\mathscr{X}}(D)|_E$ is a multiple of $r_D e_D$, where r_E is the number of irreducible components of D and e_D is the geometric multiplicity of D. $e_D = 1$ if and only if D is geometrically reduced. In addition, $e_D = 1$ if and only if the component is geometrically irreducible and e_D divides $\deg_s D$.

Then if $i_s(D, E) = 1$, this implies that $r_D = 1$ and $e_D = 1$. $e_D = 1$ implies that the components are geometrically reduced, and $r_D = 1$ implies that the components are geometrically irreducible. □

Lemma 2.6. [Liu02, §9.3] *Let C_i be an irreducible component of $\mathscr{X}_s$. Then if $i_s(C_i, C_i) \neq -1$ for all components C_i, then the regular model is minimal.*

Lemma 2.7. *Let $\mathscr{X} \to B$ be a regular arithmetic surface and $s \in B$ a closed point. Let C_i be a irreducible components of the special fiber $\mathscr{X}_s$. We define the arithmetic genus $g_i := g_a(C_i)$. Then if we let d_i be the multiplicity of the component and g be the genus of the curve X, then*

$$2g - 2 = -\sum_j d_j C_j^2 + \sum_j 2(g_j - 1)d_j.$$

Proof. We use the adjunction formula as defined in [Liu02, Theorem 1.37], which is

$$p_a(E) = 1 + \frac{1}{2}(E^2 + K_{\mathscr{X}/B} \cdot E)$$

From this we compute the arithmetic genus g_j to be

$$g_j = g_a(C_j) = 1 + \frac{1}{2}(C_j^2 + K_{\mathscr{X}/B} \cdot C_j)$$

We also use [Liu02, Proposition 1.35], which states

$$2g - 2 = \sum_j d_j K_{\mathscr{X}/B} \cdot C_j$$

From this, we can substitute the canonical divisor into the arithmetic genus, g_j, to find

$$K_{\mathscr{X}/B} C_j = 2g_j - 2 - C_j^2$$

$$2g - 2 = \sum_j d_j(2g_j - 2 - C_j^2) = -\sum_j d_j C_j^2 + \sum_j 2(g_j - 1)d_j$$

$\square$

2.5 Main Results from [GFP25]

SETUP:

From now on we fix the following set-up.

For $n \geq 3$, define:

$\theta(r, n) =$ the largest possible *minimum angle* between any two unit vectors in a set of n vectors in $\mathbb{R}^r$.

For $g \geq 2$ and $n \geq 3$, define:

$$\tau(g, r, n) = \begin{cases} \cos\theta(r, n) - \frac{1}{g} & \text{if } r \geq 2 \\ 1 - \frac{1}{g} & \text{if } r = 1 \end{cases}$$

For example, here is a table for $g = 2$ and $r = 2$

n	3	4	5	$\boxed{6}$	7	8	9	10	11	12
τ	-1	-0.5	-0.19	0	0.12	0.20	0.26	0.30	0.34	0.36

Let K be a number field, let X be a geometrically integral, smooth, projective curve over K of genus $g \geq 2$ whose group of automorphisms defined over K is

$$G := \mathrm{Aut}_K(X).$$

Let J be its Jacobian and j be the map defined in Equation (1) earlier.

Let $\hat{h}$ be the Néron–Tate height on J, normalized to K, associated to 2Θ where Θ is the theta divisor class on J.

Given a prime $\mathfrak{p}$ in $\mathcal{O}_K$, we will define a quantity $\phi_{\mathfrak{p}}(X)$ which comes from intersection theory on the fibre at $\mathfrak{p}$ of the minimal regular model of X over $\mathcal{O}_K$. This number $\phi_{\mathfrak{p}}(X)$ is easily computed from such a model, and it vanishes whenever the special fibre at $\mathfrak{p}$ has only one component (in particular, for good reduction). Furthermore, given an embedding $v\colon K \hookrightarrow \mathbb{C}$ we let X_v be the Riemann surface of the $\mathbb{C}$-points of X via v, and we write $\delta(Y)$ for the Faltings δ-invariant of a Riemann surface Y of positive genus.

We put together these invariants by defining the following quantity:

$$\begin{aligned} M(X) \; &= \frac{(g-1)^2}{3}\,\max\{6, g+1\} \sum_{v:K\to\mathbb{C}} \delta(X_v) \;+\; 2\,(g+1)\sum_{\mathfrak{p}} \phi_{\mathfrak{p}}(X)\log N\mathfrak{p} \\ &\quad +\; 2\,[K:\mathbb{Q}]\,g\,(g-1)^2\big(3g\log g + 16\big). \end{aligned}$$

We say X has **enough automorphisms** if the number $n := \#\mathrm{Aut}(X)$ satisfies

Theorem 2.8 (Main result). *Let $H \le G$ be a subgroup of the K-rational automorphism group of X, let $n = \#H$, and let $r =$ rank $J(K)$. If*

$$\tau = \tau(g,r,n) > 0,$$

then every $P \in X(K)$ with trivial H-stabilizer satisfies

$$\hat{h}\big(j(P)\big) \;\le\; \frac{M(X)}{2\,g\,\tau}.$$

Theorem 2.9 (Explicit gap principle). *For all pairs of different points $P, Q \in X(K)$ we have*

$$\hat{h}\big(j(P)\big) \;+\; \hat{h}\big(j(Q)\big) \;-\; 2\,g\,\langle j(P),\, j(Q)\rangle \;\ge\; -\,M(X).$$

In particular, if $\hat{h}(j(P))$ and $\hat{h}(j(Q))$ are non-zero, and if θ is the angle between $j(P)$ and $j(Q)$ in $J(K)\otimes_{\mathbb{Z}}\mathbb{R}$ using the Néron–Tate pairing, then

$$\cos\theta \;\le\; \frac{M(X)}{2g\sqrt{\hat{h}(j(P))\,\hat{h}(j(Q))}} \;+\; \frac{1}{2g}\left(\sqrt{\frac{\hat{h}(j(P))}{\hat{h}(j(Q))}} \;+\; \sqrt{\frac{\hat{h}(j(Q))}{\hat{h}(j(P))}}\right).$$

2.6 Mumford Gap Principle

Let K be a number field and X be a curve defined over K of genus $g \geq 2$. Let K_X be a canonical divisor on X, and let $j : X \to J = J(X)$ be the map defined in Equation (1) earlier.

$\boxed{\text{Claim 1:}}$ The map j is at most $(2g-2)^{2g}$-to-1.

Proof. For P and Q to match to the same point in J, $(2g-2)(P) - K_X \sim (2g-2)(Q) - K_X$. This implies that $(2g-2)(P - Q) \sim 0$, which means that $P - Q$ is a torsion point of J. Since J is an abelian variety, we know that $J[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$. In this case where $n = 2g-2$, we find that the map j is at most $(2g-2)^{2g}$-to-1.

□

$\boxed{\text{Claim 2:}}$ For $P, Q \in X(\mathbb{Q})$,$P \neq Q$, then

$$\|j(P) - j(Q)\|_\Theta^2 \geq \left(1 - \frac{1}{g}\right)(\|j(P)\|_\Theta^2 + \|j(Q)\|_\Theta^2) + O(1).$$

Proof. Let $\Theta \in \mathrm{Div}(J)$ be a theta divisor and $\tilde{\Theta} = \Theta + [-1]^*\Theta$. Let $\langle \cdot, \cdot \rangle_\Theta$ be the canonical height pairing attached to Θ, and let $\| \cdot \|_\Theta$ be the associated norm, i.e., $\|x\| = \sqrt{\langle x, x \rangle_\Theta}$.

We also use the following linear equivalence on $X \times X$ [HS00, Exercise A.8.2(c)]

$$(j \times j)^*(gs_{12}^*\tilde{\Theta} - (g+1)p_1^*\tilde{\Theta} - (g+1)p_2^*\tilde{\Theta}) \sim -8g(g-1)^2\Delta$$

where $s_{12}, p_1, p_2 : J \times J \to J$ are defined by $s_{12}(x, y) = x + y$, $p_1(x, y) = x$, $p_2(x, y) = y$, $j \times j : X \times X \to J \times J$ is defined by $(j \times j)(P, Q) = (j(P), j(Q))$, and $\Delta \subset X \times X$ is the diagonal.
Let $P, Q \in X(\mathbb{Q})$, and $P \neq Q$. Consider the following equation

$$\begin{aligned}
\lambda =& g h_{J,\tilde{\Theta}}(j(P) + j(Q)) - (g+1)h_{J,\tilde{\Theta}}(j(P)) - (g+1)h_{J,\tilde{\Theta}}(j(Q)) \\
=& h_{C\times C, -8g(g+1)^2\Delta}(P, Q) + O(1) \\
=& -8g(g+1)^2 h_{X\times X,\Delta}(P, Q) + O(1)
\end{aligned}$$

We also notice that

$$\hat{h}_{J,\tilde{\Theta}}(x) = \hat{h}_{J,\Theta}(x) + \hat{h}_{J,\Theta}(-x) = 2\hat{h}_{J,\Theta}(x)$$

And

$$\begin{aligned}
2\langle j(P), j(Q)\rangle_\Theta &= \hat{h}_{J,\Theta}(j(P) + j(Q)) - \hat{h}_{J,\Theta}(j(P)) - \hat{h}_{J,\Theta}(j(Q)) \\
&= \frac{1}{2}\hat{h}_{J,\tilde{\Theta}}(x)(j(P) + j(Q)) - \frac{1}{2}\hat{h}_{J,\tilde{\Theta}}(x)(j(P)) - \frac{1}{2}\hat{h}_{J,\tilde{\Theta}}(x)(j(Q)) \\
4\langle j(P), j(Q)\rangle_\Theta &= \hat{h}_{J,\tilde{\Theta}}(x)(j(P) + j(Q)) - \hat{h}_{J,\tilde{\Theta}}(x)(j(P)) - \hat{h}_{J,\tilde{\Theta}}(x)(j(Q)) \\
&= h_{J,\tilde{\Theta}}(x)(j(P) + j(Q)) - h_{J,\tilde{\Theta}}(x)(j(P)) - h_{J,\tilde{\Theta}}(x)(j(Q)) + O(1)
\end{aligned}$$

We proceed with

$$\begin{aligned}
\lambda &= g(h_{J,\tilde{\Theta}}(j(P)+j(Q)) - h_{J,\tilde{\Theta}}(j(P)) - h_{J,\tilde{\Theta}}(j(Q))) - h_{J,\tilde{\Theta}}(j(P)) - h_{J,\tilde{\Theta}}(j(Q)) \\
&= 4g\langle j(P), j(Q)\rangle_\Theta - 2h_{J,\Theta}(j(P)) - 2h_{J,\Theta}(j(Q)) + O(1) \\
\langle j(P), j(Q)\rangle_\Theta &= \frac{1}{4g}\lambda + \frac{1}{2g}(\hat{h}_{J,\Theta}(j(P)) + \hat{h}_{J,\Theta}(j(Q))) + O(1) \\
&= \frac{1}{4g}\lambda + \frac{1}{2g}(||j(P)||_\Theta^2 + ||j(Q)||_\Theta^2) + O(1) \\
&\leq \frac{1}{2g}(||j(P)||_\Theta^2 + ||j(Q)||_\Theta^2) + O(1)
\end{aligned}$$

The divisor Δ is effective, which means that λ is bounded above by a $O(1)$ term, since $P \neq Q$.
Note that $||j(P) - j(Q)||_\Theta^2 = ||j(P)||_\Theta^2 + ||j(Q)||_\Theta^2 - 2\langle j(P), j(Q)\rangle_\Theta$. We can say that

$$\begin{aligned}
||j(P) - j(Q)||_\Theta^2 &\geq ||j(P)||_\Theta^2 + ||j(Q)||_\Theta^2 - \frac{1}{g}\left(||j(P)||_\Theta^2 + ||j(Q)||_\Theta^2\right) + O(1) \\
&\geq \left(1 - \frac{1}{g}\right)\left(||j(P)||_\Theta^2 + ||j(Q)||_\Theta^2\right) + O(1)
\end{aligned}$$

□

This defines a lower bound for $||j(P) - j(Q)||^2$. We can further rewrite the bound in terms of the heights of $j(P)$ and $j(Q)$, as the following

$$||j(P) - j(Q)||_\Theta^2 \geq \left(1 - \frac{1}{g}\right)\left(\hat{h}(j(P)) + \hat{h}(j(Q))\right) + O(1).$$

2.6.1 Comparison with the Explicit Gap Principle

Recall the explicit gap principle [GFP25, Theorem 1.2] to be

$$\hat{h}(j(P)) + \hat{h}(j(Q)) - 2g\langle j(P), j(Q)\rangle \geq -M(X).$$

The explicit gap principle provides a stronger bound compared to the Mumford gap principle. The explicit gap principle also does not depend on the points P and Q. In addition, the explicit gap principle defines the $O(1)$ in the Mumford gap principle.

3 Example Curves Investigated

Here, we apply the method from [GFP25] on two curves as a proof of concept. We also compare this method with the Chabauty-Coleman Method by doing another computation.

3.1 1st curve over $\mathbb{Q}$

The hyperellitpic curve with LMFDB label `38416.a.614656.1`

$$X : y^2 = x^6 - 3x^5 - x^4 + 7x^3 - x^2 - 3x + 1$$

3.1.1 Curve Data: (from LMFDB [LMF25])

$$\begin{aligned} g &= 2 \quad \text{(genus)} \\ \mathrm{Aut}_{\mathbb{Q}}(X) &\cong D_6 \quad \text{(order 12)} \\ r &= 2 \quad \text{(Mordell–Weil rank)} \\ \text{Known } X(\mathbb{Q}) &= \{(1 : \pm 1 : 0), (0 : \pm 1 : 1), (1 : \pm 1 : 1)\} \end{aligned}$$

From the table in the paper, $\tau(g = 2, r = 2, n = 12) = 0.36$. So, we can apply Theorem 1.1 [GFP25].

The smooth projective curve defined by the equation would be over the weighted projective space $\mathbb{P}^2[1, 3, 1]$. This results in the homogenized curve

$$y^2 = x^6 - 3x^5 z - x^4 z^2 + 7x^3 z^3 - x^2 z^4 - 3xz^5 + z^6$$

3.1.2 Finding $M(X)$

Step 1: Arithmetic contribution

The minimal discriminant $= 2^8.7^4$ so the only bad primes are 2 and 7.

Intersection matrix at p = 7 is [0]
Intersection matrix at p = 2 is:

$$\begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}$$

From RegularModel command in MAGMA ([BCP97]) we get, $m_j = 1$ for all j.

From $\mathcal{M}$, we see that every component has one intersection number of 1. By Lemma 2.5, this implies that the fiber components are geometrically irreducible. In addition, since none of the diagonal entries in $\mathcal{M}$ are -1, by Lemma 2.6, the model is minimal.

Next we find the arithmetic genus g_j of each component. Recall from Lemma 2.7 that

$$2g-2=\sum_j d_j\left(2g_j-2-C_j^2\right)=-\sum_j d_jC_j^2+\sum_j 2(g_j-1)d_j$$

Note that this is the same formula found in Lemma 4.6 in [GFP25]

We say that $C_j^2=[C_j.C_j]$. Then to find the arithmetic genus g_j, we find that

$$\begin{gathered}\sum_j d_jC_j^2=12\\ 2(g-1)=2=12+\sum_j 2(g_j-1)d_j\\ \sum_j(g_j-1)d_j=\sum_j g_jd_j-\sum_j d_j=-5\\ \sum_j g_jd_j=0\end{gathered}$$

Because the curve is geometrically irreducible, $g_j\geq 0$. In addition, since $d_j>0$, this means that $g_j=0$ for all j.

Next we find an upper bound ϕ_2 for $[\Psi_{\mathfrak{p},P},.\Psi_{\mathfrak{p},P}]$, where $\Psi_{\mathfrak{p},P}:=\sum_{j=1}^s a_jC_j$ and $[C_i.\Psi_{\mathfrak{p},P}]=[C_i.E_P]$, where $E_P:=K_{\mathscr{X}/B}-2(g-1)D_P$ and $P\in X(\mathbb{Q})$. First we find $[C_i.E_P]$ as such

$$\begin{aligned}[C_j.E_P]=[C_j.K_{\mathscr{X}/B}-2(g-1)D_P]&=[C_j.K_{\mathscr{X}/B}]-2(g-1)[C_j.D_P]\\ &=[C_j.K_{\mathscr{X}/B}]-2(g-1)d_{j,k}\end{aligned}$$

By the adjunction formula and that $g=2$ and $g_j=0$ that

$$\begin{aligned}[C_j.K_{\mathscr{X}/B}]-2(g-1)d_{j,k}&=2(g_j-1)-[C_j,C_j]-2(g-1)d_{j,k}\\ &=-2-[C_j,C_j]-2d_{j,k}\end{aligned}$$

To find $\Psi_{\mathfrak{p},P}$, we set up the system of linear equations as such for components C_k where C_k has a multiplicity of 1. [GFP25, Corollary 4.5]

$$\begin{cases} \sum_j a_j[C_1.C_j] & = [C_1.E_P] = -2 - [C_1.C_1] - 2d_{1,k} \\ & \vdots \\ \sum_j a_j[C_s.C_j] & = [C_s.E_P] = -2 - [C_s.C_s] - 2d_{s,k} \\ \sum_j a_j & = 0 \end{cases}$$

The multiplicities have to be 1 because the intersection number of the divisor E_p and the special fiber is 1. Then if the multiplicity of the component is not 1, then the intersection number is no longer 1, which is not possible.

To show that $[C_i.\Psi_{\mathfrak{p},P}] = [C_i, E_P]$, we expand out $[C_i.\Psi_{\mathfrak{p},P}]$.

$$[C_i.\Psi_{\mathfrak{p},P}] = \left[C_i.\sum_{j=1}^{s} a_j C_j\right] = \sum_{j=1}^{s} a_j[C_i.C_j]$$

From the systems of equations, we find that for each component C_i that $\sum_{j=1}^{s} a_j[C_i.C_j] = [C_i.E_P]$

To bound ϕ_2, we consider the components C_i of multiplicities 1 and the systems of equations that come from $[C_i, \Psi_{\mathfrak{p},P}]$. By symmetry, we only consider the two following systems.

$$\mathcal{M}\vec{a}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathcal{M}\vec{a}^{(5)} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

We solve these systems of equations to solve for $\vec{a}^{(1)}$ and $\vec{a}^{(5)}$ and find the following results.

$$\vec{a}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{a}^{(5)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2/3 \\ 4/3 \end{bmatrix}$$

To find the self-intersection number, we compute $\vec{a}^{\top}\mathcal{M}\vec{a}$. We do this to find

$$\left[\sum_j \vec{a}^{(1)}C_j.\sum_j \vec{a}^{(1)}C_j\right] = 0$$

$$\left[\sum_j \vec{a}^{(5)} C_j . \sum_j \vec{a}^{(5)} C_j\right] = -\frac{8}{3}$$

Therefore $\phi_2 = \frac{8}{3}$.

Step 2: Analytic contribution
Using the command RichelotIsogenousSurfaces in MAGMA ([BCP97]) on the Jacobian of this curve, We get that J has an isogeny of degree 4 to the elliptic curves (minimal form),

$$E_1 : y^2 = x^3 - x^2 - 142x + 701,$$

$$E_2 : y^2 = x^3 - x^2 - 2x + 1.$$

Using command FaltingsHeight in MAGMA ([BCP97]) we obtain,

$$h(E_1) = -0.20503,$$

$$h(E_2) = -0.75434.$$

Using Theorem 3.3 [GFP25],

$$h(J) \leq h(E_1) + h(E_2) + \frac{\log 4}{2},$$

$$h(J) \leq -0.2662.$$

Putting this all together,

$$M(X) < 24h(J) + 6\phi_2 \log 2 + 110.05 = 109$$

3.1.3 Height Bound

Using Theorem 1.1 [GFP25],

$$\widehat{h}(j(P)) \leq \frac{109}{2 \cdot 2 \cdot 0.36} = \frac{109}{1.44} = 76$$

3.1.4 Determining all of the rational points on X

Finally, the rational points found using this height bound is:

$$\{(1 : \pm 1 : 0), (0 : \pm 1 : 1), (1 : \pm 1 : 1)\}$$

3.2 2nd curve over $\mathbb{Q}$

The hyperelliptic curve with LMFDB label `614656.a.614656.1`

$$X : y^2 = -x^6 - 3x^5 + x^4 + 7x^3 + x^2 - 3x - 1$$

3.2.1 Curve Data: (from LMFDB [LMF25])

$$\begin{aligned} g &= 2 \quad \text{(genus)} \\ \mathrm{Aut}_{\mathbb{Q}}(X) &\cong D_6 \quad \text{(order 12)} \\ r &= 2 \quad \text{(Mordell–Weil rank)} \\ \text{Known } X(\mathbb{Q}) &= \{(1 : \pm 1 : 1),\, (-2 : \pm 1 : 1),\, (-1 : \pm 1 : 2)\} \end{aligned}$$

From the table, $\tau = 0.36 > 0$, then Theorem 1.1 [GFP25] applies.

3.2.2 Finding $M(X)$

Using command BadPrimes in MAGMA ([BCP97]), we find two bad primes $p = 2$ and $p = 7$.

The intersection matrix at both $p = 2$ and $p = 7$ is $[0]$. Then the arithmetic part of $M(X)$, $\sum_{\mathfrak{p}} \varphi_{\mathfrak{p}}(X) \log \mathrm{N}\mathfrak{p}$, is zero.

Using the command RichelotIsogenousSurfaces in MAGMA ([BCP97]) on the Jacobian of this curve, We get that J has an isogeny of degree 4 to the elliptic curves,

$$E_1 : y^2 = x^3 - \frac{83}{16}x^2 + \frac{19}{256}x - \frac{1}{4096},$$

$$E_2 : y^2 = x^3 + \frac{19}{4}x^2 + \frac{83}{16}x + \frac{1}{64}.$$

Using command FaltingsHeight in MAGMA ([BCP97]) we obtain,

$$h(E_1) = -0.20503,$$

$$h(E_2) = -0.75434.$$

Using Theorem 3.3 [GFP25],

$$h(J) \leq h(E_1) + h(E_2) + \frac{\log 4}{2}$$
$$h(J) \leq -0.2662$$

Putting all together,

$$M(X) < 24h(J) + 6\sum_p \phi_p \log p + 110.05 = 103.66$$

3.2.3 Height Bound

Using Theorem 1.1 [GFP25]:

$$\widehat{h}(j(P)) \leq \frac{M(X)}{2g\tau} < \frac{103.66}{2 \cdot 2 \cdot 0.36} = 71.986$$

3.2.4 Determining all of the rational points on X

Finally, the rational points found using this height bound is:

$$\{(1 : \pm 1 : 1),\ (-2 : \pm 1 : 1),\ (-1 : \pm 1 : 2)\}$$

3.3 A Comparison: Chabauty-Coleman Method [MP12]

One of the main approach to bound rational points is Chabauty-Coleman method (see for instance [MP12]). We are going to apply both Chabauty-Coleman method and the method from [GFP25] to the curve $X : y^2 = x^6 + 2x^5 + 7x^4 + 8x^3 + 7x^2 + 2x + 1$.

The hyperelliptic curve with LMFDB label `847.a.847.1`

$$X : y^2 = x^6 + 2x^5 + 7x^4 + 8x^3 + 7x^2 + 2x + 1$$

3.3.1 Curve Data: (from LMFDB [LMF25])

$$\begin{aligned} g &= 2 \quad \text{(genus)} \\ \mathrm{Aut}_{\mathbb{Q}}(X) &\cong C_2^2 \quad \text{(order 4)} \\ r &= 1 \quad \text{(Mordell–Weil rank)} \\ \text{Known } X(\mathbb{Q}) &= \{(1 : \pm 1 : 0),\ (0 : \pm 1 : 1),\ (-1 : \pm 1 : 1)\} \end{aligned}$$

3.3.2 Example of Chabauty-Coleman Method [MP12]

rank$J(\mathbb{Q}) < g$, then this method applies

3.3.2a Finding $\#X(\mathbb{F}_p)$

This curve has good reduction at $p = 5$, we get $X(\mathbb{F}_5)$:

$$\{(1:1:0), (1:4:0), (0:1:1), (0:4:1), (4:3:1), (4:2:1)\}$$

$\#X(\mathbb{F}_5) = 6$

3.3.2b Bounding $\#X(\mathbb{Q})$

Using Theorem 5.3b [MP12], $\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_5) + (2g-2) = 6 + 2 = 8$, which is larger than the true number of rational point. The finding of rational points will never terminate based on this method.

3.3.3 Example of [GFP25] Method

Since $r = 1$, $\tau = 1 - 1/2 = 0.5 > 0$, then Theorem 1.1 [GFP25] applies.

3.3.3a Finding $M(X)$

Using command BadPrimes in MAGMA ([BCP97]), we find three bad primes $p = 2$, $p = 7$ and $p = 11$.

The intersection matrix at all the bad primes are [0]. Then the arithmetic part of $M(X)$, $\sum_{\mathfrak{p}} \varphi_{\mathfrak{p}}(X) \log \mathrm{N}\mathfrak{p}$, is zero.

Using the command RichelotIsogenousSurfaces in MAGMA ([BCP97]) on the Jacobian of this curve, We get that J has an isogeny of degree 4 to the elliptic curves,

$$E_1 : y^2 = x^3 + \frac{3}{16}x^2 + \frac{35}{256}x + \frac{49}{4096}$$
$$E_2 : y^2 = x^3 + \frac{5}{16}x^2 + \frac{3}{256}x + \frac{7}{4096}$$

Using command FaltingsHeight in MAGMA ([BCP97]) we obtain,

$$h(E_1) = -0.78665$$

$$h(E_2) = -1.11273$$

Using Theorem 3.3 [GFP25],

$$h(J) \leq h(E_1) + h(E_2) + \frac{\log 4}{2}$$

$$h(J) \leq -1.598$$

Putting all together,

$$M(X) < 24h(J) + 6\sum_p \phi_p \log p + 110.05 = 71.70$$

3.3.3b Height Bound

$$\widehat{h}(j(P)) \leq \frac{M(X)}{2g\tau} < \frac{71.70}{2 \cdot 2 \cdot 0.5} = 35.85$$

3.3.3c Determining all of the rational points on X

Finally, the rational points found using this height bound is:

$$\{(1 : \pm 1 : 0), (0 : \pm 1 : 1), (-1 : \pm 1 : 1)\}$$

3.3.4 Comparison

In contrast to Chabauty–Coleman method [MP12], which only applies when the rank is less than the genus and gives a bound on the number of rational points, the approach of [GFP25] has no such rank restriction and provides an explicit height bound. This reduces the problem to a finite search for rational points, whereas Chabauty–Coleman may lead to a search that never terminates.

4 Our Generalizations and Results

In Section 5 we develop new criteria on the real Mordell–Weil space

$$V := J(K) \otimes_{\mathbb{Z}} \mathbb{R},$$

that recover explicit height bounds even when a curve's automorphism group is too small for the "enough automorphisms" hypothesis of the García–Pasten [GFP25] method. We view the free part of the Jacobian as a Euclidean lattice in V, and derive height bounds from how automorphisms act as isometries:

- **Section 5.1 (Spectral-Gap Criterion).** By symmetrizing each automorphism's action on V, we obtain a self-adjoint operator whose smallest nontrivial eigenvalue α_H measures how strongly a subgroup $H \subset \mathrm{Aut}_K(X)$ "mixes" lattice directions. When $\alpha_H > 1/g$, the vectors in V are forced into a bounded radius around the origin.
- **Section 5.2 (Injectivity Criterion).** If the natural map

  $$\varphi : \mathrm{Aut}_K(X) \to O(V)$$

  fails to be injective, some nontrivial automorphism fixes all of V. This hidden symmetry gives an explicit height bound, regardless of group size.
- **Section 5.3 (Proof of Concept).** We use a genus-2 curve to show how to test kernel-injectivity in practice by computing pullbacks of generators and checking which automorphisms act trivially on the lattice.
- **Section 5.4 (Bravais-Lattice Classification).** The Gram matrix of the Néron–Tate pairing on free generators determines whether the Mordell–Weil lattice is oblique, rectangular, square, or hexagonal, predicting which criterion applies.
- **Section 5.5 (Further Questions).** We outline directions for extending these methods.

4.1 Spectral Gap Criterion:

SETUP:
Let X/K be a smooth projective curve of genus $g \geq 2$, and let $J(X)$ denote its Jacobian. Let $V = J(K) \otimes \mathbb{R}$ denote the real Mordell–Weil space, equipped with the canonical Néron–Tate height pairing

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}.$$

Fix a finite subgroup $H \subset \mathrm{Aut}_K(X)$. Each $\sigma \in H$ induces an isometric linear map (Equation (5) in [Fuj97]) $\sigma^* : V \to V$, extended from pullback on divisor classes. For each nontrivial $\sigma \in H$, define the symmetric operator

$$S_\sigma = \frac{1}{2}(\sigma^* + (\sigma^*)^\dagger) : V \to V.$$

where $\dagger$ denotes the adjoint of the linear map.
Let

$$\alpha_H := \max_{\substack{\sigma \in H \\ \sigma \neq id}} \lambda_{\min}(S_\sigma),$$

where $\lambda_{\min}(S_\sigma)$ is the smallest eigenvalue of S_σ.

Theorem 4.1. *If $\alpha_H > 1/g$, then for any $P \in X(K)$ with trivial H-stabilizer, setting $v =$ image of P in V and $h = \|v\|^2 = \hat{h}(j(P))$, we have*

$$h \leq \frac{M(X)}{2(g\alpha_H - 1)},$$

where $M(X)$ is the constant from the paper [GFP25]

Proof. By the **Courant–Fischer theorem**, the smallest eigenvalue of S_σ satisfies

$$\lambda_{\min}(S_\sigma) = \min_{v \in V, v \neq 0} \frac{\langle v, S_\sigma v \rangle}{\langle v, v \rangle} = \inf_{\|v\|=1} \langle v, S_\sigma v \rangle.$$

But for any unit v, by symmetry of $\langle \cdot, \cdot \rangle$

$$\langle v, (\sigma^*)^\dagger v \rangle = \langle \sigma^* v, v \rangle = \langle v, \sigma^* v \rangle$$

So,

$$\langle v, S_\sigma v \rangle = \frac{1}{2} \left(\langle v, \sigma^* v \rangle + \langle v, (\sigma^*)^\dagger v \rangle \right) = \langle v, \sigma^* v \rangle$$

Hence,

$$\inf_{\|v\|=1} \langle v, \sigma^* v \rangle = \lambda_{\min}(S_\sigma).$$

Since $|H| \geq 2$ and the stabilizer of P is trivial, by construction, there exists $\sigma \neq id$ in H such that $\langle v, \sigma^*(v) \rangle \geq h\alpha_H$

Finally, applying Theorem 1.2 from [GFP25] to the points $v, \sigma^*(v)$ (each of height h) gives

$$2g(h\alpha_H) - 2h \leq 2g\langle v, \sigma^*(v) \rangle - 2h \leq M(X),$$

so,

$$h \leq \frac{M(X)}{2(g\alpha_H - 1)}.$$

Since $\alpha_H > 1/g$, we get the result.

□

4.2 Injectivity Criterion

SETUP:

Consider, the natural homomorphism which takes automorphisms in $G := \mathrm{Aut}_K(X)$ to its pullback (on divisor classes extended to V).

$$\phi\colon G \to O(V), \qquad \sigma \mapsto \sigma^*,$$

Let $M(X)$ be the "gap" constant appearing in Theorem 1.2 of the paper [GFP25]

Let $H := \ker(\phi)$

Theorem 4.2. *If ϕ is not injective, then:*

$$\hat{h}(j(P)) \leq \frac{M(X)}{2g-2}.$$

for all $P \in X(K)$ with trivial H-stabilizer,

Proof. Since ϕ is not injective, there exists $\sigma \in G \setminus \{1\}$ with $\phi(\sigma) = id$. Let $h = \hat{h}\big(j(P)\big)$.

Then:

- $\sigma(P) \neq P$, since P is not stabilized by σ.
- $\phi(\sigma) = id$ implies
$$j\big(\sigma(P)\big) = \phi(\sigma)\big(j(P)\big) = j(P),$$
so the two distinct points $P, \sigma(P) \in X(K)$ have the same images in V.

Applying Theorem 1.2 ([GFP25]) to the pair $P \neq \sigma(P)$.

Since
$$\langle j(P), j(\sigma(P))\rangle = \langle j(P), j(P)\rangle = \hat{h}(j(P)),$$
we get,
$$2\hat{h}(j(P)) - 2g\,\langle j(P), j(\sigma(P))\rangle = 2\hat{h}(j(P)) - 2g\hat{h}(j(P)) \geq -M(X).$$

$$(2-2g)h \geq -M(X) \quad \implies \quad h \leq \frac{M(X)}{2g-2}.$$

□

For simplicity, we use $K = \mathbb{Q}$ for the remainder of this discussion.

Recall: By the Mordell-Weil Theorem

$$J(\mathbb{Q}) \;\cong\; \mathbb{Z}^r \;\oplus\; T.$$

$\boxed{\text{Claim 1:}}$ If $\hat{h}(\Delta_i) = \hat{h}(F_i - G_i) = 0$ for all i, then ϕ must be non-injective.

Proof. Direct consequence of the defintion of the identity map. □

4.3 Proof Of Concept: Test for Injectivity

Consider the curve $y^2 = x^6 - 12x^4 + 6x^3 - 284x^2 + 1488x - 1815$

Genus = 2, Rank = 2, $\mathrm{Aut}(X_{\mathbb{Q}}) = C_2^2$

Mordell-Weil group of the Jacobian: $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$

Using MAGMA [BCP97], we get the following:

G_2 and G_3 are the free generators of the Mordell-Weil group of the Jacobian.

For the automorphism, $[X : Y : Z] \longmapsto [\,X : Y : -Z\,]$

$$\begin{aligned}
G_2 \text{ (Generator 2)} &= \big(x+6,\ -x^3-216,\ 2\big),\\
F_2 \text{ (Pullback of } G_2) &= \big(x-6,\ -x^3+216,\ 2\big),\\
\widehat{h}(G_2) &= 0.53292782791729530361440257343586,\\
\widehat{h}(F_2) &= 0.53292782791729530361440257343586,\\
\widehat{h}\big(G_2 - F_2\big) &= 0.
\end{aligned}$$

$$\begin{aligned}
G_3 \text{ (Generator 3)} &= \left(x^2 - \tfrac{75}{2},\ \tfrac{7}{10}x,\ 2\right),\\
F_3 \text{ (Pullback of } G_3) &= \left(x^2 - \tfrac{75}{2},\ \tfrac{7}{10}x,\ 2\right),\\
\widehat{h}(G_3) &= 4.0682006734939169341912769267765,\\
\widehat{h}(F_3) &= 4.0682006734939169341912769267765,\\
\widehat{h}\big(G_3 - F_3\big) &= 0.
\end{aligned}$$

Hence, this automorphism acts trivially on the real vector space V.

4.4 Bravais-Lattice Classification

This example motivates the following:

$$\phi\colon G \longrightarrow O(\Lambda) \subset O\big(J(\mathbb{Q}) \otimes \mathbb{R}\big), \qquad \sigma \mapsto \sigma^*, \qquad \Lambda = J(\mathbb{Q})/T$$

By Lagrange's theorem, for any finite G we have $|G| = \big|\ker\phi\big| \times \big|im\phi\big| \leq \big|\ker\phi\big| \times \big|O(\Lambda)\big|$.

$$\text{Rearranging gives} \quad \big|\ker\phi\big| \geq \frac{|G|}{\big|O(\Lambda)\big|}.$$

In particular, if $|G| > |O(\Lambda)|$, then $\ker\phi$ is non-trivial.

Let us focus on the specific case where rank of $J(\mathbb{Q}) = 2$,

$$\Lambda \;=\; J(\mathbb{Q})/T \;\cong\; \mathbb{Z}^2,$$

Using the 2D-Bravais lattice classification, the lattice Λ can be one of the following:

Lattice type		$O(\Lambda)$	$\lvert O(\Lambda)\rvert$
Oblique	no special relations	$C_2 = \{\pm I\}$	2
Rectangular	sides unequal, angles 90 degrees	$D_2 = C_2 \times C_2$	4
Square	all sides equal, angles 90 degrees	D_4	8
Hexagonal	equal lengths, angles 60 degrees	D_6	12

Intuitively, this tells us that the sharper bound from our Theorem 4.2 is more likely to apply for curves whose Mordell - Weil lattice (of the Jacobian) has less intrinsic symmetry!

To decide which of the following types the Mordell - Weil lattice falls into, we can compute the canonical height gram matrix using the free generators.

Let G_1 and G_2 be the free generators (minimal length) of the MW lattice of the Jacobian,

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}, \quad H_{ij} = \langle G_i, G_j \rangle.$$

This encodes the squared lengths $\|G_i\|^2 = H_{ii}$ and the inner product $\langle G_1, G_2\rangle = H_{12}$.

- Oblique lattice:
  No special relations among the entries, i.e.

  $$H_{12} \neq 0, \quad H_{11} \neq H_{22}, \quad \frac{H_{12}}{\sqrt{H_{11}H_{22}}} \neq \pm\frac{1}{2}.$$

  Then $O(\Lambda) \cong C_2$

- Rectangular lattice:
  Orthogonal but unequal lengths,

  $$H_{12} = 0, \quad H_{11} \neq H_{22}.$$

  Then $O(\Lambda) \cong D_2$

- Square lattice:
  Equal lengths and orthogonal,

  $$H_{12} = 0, \quad H_{11} = H_{22}.$$

  Then $O(\Lambda) \cong D_4$

- Hexagonal lattice:
  Equal lengths with 60 degree angle,

  $$H_{11} = H_{22}, \quad H_{12} = \tfrac{1}{2} H_{11}.$$

  Then $O(\Lambda) \cong D_6$

Notice, in our example the canonical height gram matrix is the following:

$$H = \begin{pmatrix} 3.3240939319624476 & -0.9138961045080984 \\ -0.9138961045080984 & 2.1168567476109618 \end{pmatrix}.$$

Here

$$H_{11} \neq H_{22}, \quad H_{12} \neq 0, \quad \frac{H_{12}}{\sqrt{H_{11}H_{22}}} \approx -0.344 \neq \pm\tfrac{1}{2}.$$

Hence Λ is an oblique lattice, and

$$O(\Lambda) \simeq C_2.$$

Since $|G| = 4 > |O(\Lambda)| = 2$, ϕ is non-injective!

4.5 Further Questions

- Develop algorithms to compute α_H efficiently.
- Test both criteria for genus ≥ 3 curves.
- What proportion of genus atleast 2 curves have Mordell–Weil lattices of each Bravais type? Can one predict which curves will satisfy the spectral or kernel criterion "generically"?

5 Appendix

Included below are some of our MAGMA[BCP97] scripts:

5.1 Canonical Height Gram Matrix of the free generators

```
R<x> := PolynomialRing(Rationals());
C     := HyperellipticCurve( x^6 - 12*x^4 + 6*x^3 - 284*x^2 + 1488*x - 1815 );

// 2) Form its Jacobian and compute its Mordell{Weil group
```

```
J     := Jacobian(C);
MW, mp := MordellWeilGroup(J);


// Mordell Weil Generators ( returns minimal length generators )
gens  := [ mp(g) : g in Generators(MW) ];

// pick out the free generators
freegens := [ gens[1], gens[2] ];

// compute the ⟨·,·⟩-matrix on just those
M := HeightPairingMatrix(freegens);

print("The height pairing matrix of the free generators is \n");
M;
```

5.2 Example used in Section 5.3

```
R<x> := PolynomialRing(Rationals());
C := HyperellipticCurve(x^6 - 12*x^4 + 6*x^3 - 284*x^2 + 1488*x - 1815);

K := CanonicalDivisor(C);
Aut := Automorphisms(C);
J := Jacobian(C);

print("Consider the following automorphism");
Aut[3];
phi := Aut[3];

MW, psi := MordellWeilGroup(J);
gens := [psi(MW.i) : i in [1..Ngens(MW)]];


// Generator 1
D := gens[1];
print "G1 (Generator 1) =", D;
F1 := Pullback(phi, D);
print "F1 (Pullback of G1 along the automorphism) =", F1;
hD := CanonicalHeight(J!D);
print "Canonical height of G1 =", hD;
hF1 := CanonicalHeight(J!F1);
print "Canonical height of F1 =", hF1;
if hD eq 0 then
```

```
    print "Since the canonical height is 0, G1 is a torsion point in the Jacobian";
end if;
print "";

// Generator 2
E := gens[2];
print "G2 (Generator 2) =", E;
F2 := Pullback(phi, E);
print "F2 (Pullback of G2 along the automorphism) =", F2;
hE := CanonicalHeight(J!E);
print "Canonical height of G2 =", hE;
hF2 := CanonicalHeight(J!F2);
print "Canonical height of F2 =", hF2;
hDiff2 := CanonicalHeight(J!E - J!F2);
print "Canonical height of (G2 - F2) =", hDiff2;
print "";

// Generator 3
F := gens[3];
print "G3 (Generator 3) =", F;
F3 := Pullback(phi, F);
print "F3 (Pullback of G3 along the automorphism) =", F3;
hF := CanonicalHeight(J!F);
print "Canonical height of G3 =", hF;
hF3 := CanonicalHeight(J!F3);
print "Canonical height of F3 =", hF3;
hDiff3 := CanonicalHeight(J!F - J!F3);
print "Canonical height of (G3 - F3) =", hDiff3;
```

5.3 Computing ϕ_2 in Section 4.1.2

```
R<x> := PolynomialRing(Rationals());
X := HyperellipticCurve(R![1, -3, -1, 7, -1, -3, 1], R![]);

RegModel := RegularModel(X, 2);
M := IntersectionMatrix(RegModel);
print(M);

g := 2;
s := 5;

BaseRing(M);
```

```
M_Q := Matrix(Rationals(), Nrows(M), Ncols(M), Eltseq(M));
BaseRing(M_Q);
b := Vector(Rationals(), [2,0,0,0,-2]);
a := Solution(M_Q, b);
a;

selfInt := &+[ a[i]*(&+[ M[i][j]*a[j] : j in [1..s] ]) : i in [1..s] ];
selfInt;
```

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References

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265. Computational algebra and number theory (London, 1993). MR1484478

[Fuj97] Masanari Fujimori, *Rational points of a curve which has a nontrivial automorphism*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **24** (1997), no. 3, 551–569.

[GFP25] Natalia Garcia-Fritz and Hector Pasten, *Effective mordell for curves with enough automorphisms*, 2025.

[HS00] Marc Hindry and Joseph H. Silverman, *Diophantine geometry*, Graduate Texts in Mathematics, vol. 201, Springer-Verlag, New York, 2000. An introduction. MR1745599

[Liu02] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications. MR1917232

[LMF25] The LMFDB Collaboration, *The L-functions and modular forms database*, 2025.

[MP12] William McCallum and Bjorn Poonen, *The method of Chabauty and Coleman*, Explicit methods in number theory, 2012, pp. 99–117. MR3098132