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Uniformization of Annuli via an Algorithmic Construction of a Harmonic Conjugate Function

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Notation: let $[a : b] := \{z \in \mathbb{Z}\colon a \leq z \leq b\}$.

Given a region $R \subset \mathbb{R}^2$ which is a topological annulus, we construct a triangulation $T = (T_0, T_1, T_2)$ where $T_0 = (v_0, v_1, \ldots, v_{n_v-1})$ is a tuple of triangulation vertices $v_i \in \mathbb{R}^2$, $T_1 = (e_0, e_1, \ldots, e_{n_e-1})$ is a tuple of triangulation edges $e_i \in T_0 \times T_0$, and $T_2 = (t_0, t_1, \ldots, t_{n_t-1})$ is a tuple of triangles $t_i \in T_0 \times T_0 \times T_0$. The triangulation is constructed to be acute, meaning that for all i, $t_i \in T_2$ is an acute triangle.

We also construct the Voronoi tesselation $\Lambda = (\Lambda_0, \Lambda_1, \Lambda_2)$ dual to the (Delaunay) triangulation, where $\Lambda_0 = (\nu_0, \nu_1, \ldots, \nu_{n_\nu-1})$ is a tuple of the Voronoi vertices $\nu_i \in \mathbb{R}^2$, $\Lambda_1 = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n_\varepsilon-1})$ is a tuple of the Voronoi edges $\varepsilon_i \in \Lambda_0 \times \Lambda_0$, and $\Lambda_2 = (\rho_0, \rho_1, \ldots, \rho_{n_t-1})$ is a tuple of the Voronoi polygons $\rho_i \in (\Lambda_0) \cup (\Lambda_0 \times \Lambda_0) \cup \cdots \cup (\Lambda_0 \times \ldots_{V\text{-times}} \times \Lambda_0)$ where V is the maximum number of vertices of the constructed polygons. Duality means that each polygon ρ_i corresponds to triangle $t_i \in T_2$.

Quite often, we will limit attention to the contained Voronoi tesselation $\tilde{\Lambda} = (\tilde{\Lambda}_0, \tilde{\Lambda}_1, \tilde{\Lambda}_2)$, where we only keep polygons $\tilde{\Lambda}_2 = \{\rho_i \in \Lambda_2 \colon \operatorname{geo}(\rho_i) \subsetneq R\}$, and we only keep the necessary vertices and edges of Λ_0 and Λ_1 in order to compose the polygons in $\tilde{\Lambda}_2$, i.e. if a vertex ν_i only appears in a non-contained polygon $\rho \in \tilde{\Lambda_2} \setminus \Lambda_2$ then it will not be contained in $\tilde{\Lambda_0}$, and similarly with edges to form $\tilde{\Lambda_1}$. Note that the contained polygons $\tilde{\Lambda}_2$ correspond to the interior vertices of T_0. Since $\tilde{\Lambda}_2 \subseteq \Lambda_2$ if they are regarded as sets rather than tuples, we can define a reindexing map $h\colon [1\colon n_\rho] \to [1\colon n_t]$ defined by $h(i) = j$ if $v_i \in \operatorname{geo}(\rho_i)$, i.e. h maps the index in the contained polygons $\tilde{\Lambda}_2$ to the index of that polygon in the original tuple of polygons Λ_2.

Next, we solve the PDE g on the triangulation vertices T_0.

Now, we arbitrarily choose a base cell from the contained vertices, denoted by $\rho_\omega \in \tilde{\Lambda_2}$. If desired, this can be done by choosing a base point $p_\omega \in R$ and then choosing the base cell to be the contained cell $\rho_\omega \in \tilde{\Lambda_2}$ such that $\operatorname{geo}(\rho_\omega) \ni p_\omega$. This will not always work, since typically $\operatorname{geo}(\tilde{\Lambda}_2) := \cup_{\rho \in \tilde{\Lambda}_2} \{\operatorname{geo}(\rho)\} \subsetneq R$, but if the triangulation is refined enough it will generally cause $\operatorname{geo}(\tilde{\Lambda}_2)$ to approach R.

We now use the base cell to begin constructing topological slit of the annulus. Choose an arbitrary point $h \in \mathbb{R}^2$ in the hole of the annulus R. This "point in the hole" of R will serve the role of the origin in defining an angle measure on R. To this end, we take the triangular vertex v_ω corresponding to the base cell ρ_ω and construct the ray $r_0 = \overrightarrow{hv_\omega}$.

Using the ray r, we construct a slit path of Voronoi cells S_ρ in $\tilde{\Lambda}_2$ by the following constructive algorithm:

1. Given: Voronoi polygon topology A_ρ, the index of the base cell ω, the ray r defining the slit of the plane.
2. First, we construct the outward component of the cell path slit.
3. Initialize $p = \omega$.
4. (1) For `edge` in `edges`(p): if `edge` intersects r with a positive orientation, then update $p \leftarrow A_\rho[p, \texttt{edge}]$. If $p \neq -1$, goto step (1).
5. Finally, we perform a similar process to construct the inward component of the cell path slit, reversing the orientation of the intersection being searched for. Now the two components can be appropriately concatenated to form the entire cell path slit from inner boundary to outer boundary.

Algorithm 1: Algorithm to Compute the Slit Cell Path

Input: Voronoi polygon topology A_ρ, the index of the base cell ω, the ray r defining the slit of the plane
Output: The slit path

$p \leftarrow \omega$
f = solve_pdf(T)
singular_vertex = find_singular_vertex(T)
singular_height = f(singular_vertex)
singular_level_curve = compute_level_curve(T, singular_height)

Note: it is simpler (and avoids a possible issue with having the base cell not in the first connected component of the cells intersected by the ray r) to restrict the selection of the base cell ρ_ω to the cells with are hit first by the ray r.

5 Computation of the Harmonic Conjugate

The harmonic conjugate function g^* is approximated by the combinatorial harmonic conjugate function g_n^* which is initially defined on Λ_0. For each $\omega \in \Lambda_0$, we compute a discrete path γ from ω_0 to ω, represented as a list where the first element γ_0 is ω_0 and the last element γ_{n_γ} is ω, where n_γ is the number of elements of the path. To be a path, γ must have $(\gamma_i, \gamma_{i+1}) \in \Lambda_1$ or $(\gamma_{i+1}, \gamma_i) \in \Lambda_1$ for all $i = 1, 2, \ldots, n_\gamma - 1$. The path γ is computed by constructing a weighted graph where the edges of Λ_1 that intersect the slit have weight infinity, and then finding the shortest path from ω_0 to ω on the path. Since ω_0 was chosen to have the smallest angle of all $\omega \in \lambda_0$, this ensures that the path γ is counter-clockwise in the sense that $\theta(\omega) > \theta(\omega_0)$, where θ is the angle of the point with respect to the slit. The function that computes this path will be denoted `shortest_cck_path`, since it computes the shortest counter-clockwise path from ω_0 to ω.

To compute the values of the harmonic conjugate function g^* on each $\omega \in \Lambda_0$, we need to compute the flux contributing edges along the path $\gamma = \texttt{shortest\_cck\_path}(\omega_0, \omega)$. For each edge $e \in \Lambda_0 \times \Lambda_0$ in the path γ, we compute the perpendicular edge $e^\perp \in T_0 \times T_0$. This can be quickly computed using the information stored in the topology objects. The perpendicular edges $\gamma^\perp = \{e_i^\perp\}_{i=1}^{n_\gamma}$ corresponding to the edges $\gamma = \{e_i\}_{i=1}^{n_\gamma}$ are the flux contributing edges for the path γ.

The flux itself gives the value $g^*(\omega)$, and is computed given the flux contributing edges $\gamma^\perp$ by the formula:

$$\sum_{e^\perp \in \gamma^\perp} c(e^\perp) |g(e_0^\perp) - g(e_1^\perp)|$$

Note that this formula for the flux uses the absolute value of the difference in g along the edge $e^\perp$ to avoid the unnecessary step of orienting all of the flux contributing edges in the same way. Recall that c denotes the conductance of an edge.

Algorithm 2: Algorithm to Compute the Discrete Harmonic Conjugate on Λ_0

Input: Λ_0

Output: A function g_n^* defined on the indices $\omega \in \Lambda_0$

$g_n^*(\omega) \leftarrow 0$

for $\omega \in \Lambda_0$ **do**

  $\gamma =$ `shortest_cck_path`(ω_0, ω)

  for $e \in \gamma$ **do**

    $e^\perp =$ `perpendicular_edge`(e)

    $g_n^*(\omega) \leftarrow g_n^*(\omega) + c(e^\perp)|g(e_0^\perp) - g(e_1^\perp)|$

  end

  return g_n^*

end

6 Computation of the Period

7 Construction of the Uniformizing Function

Linear interpolation of the harmonic function g, which is defined on T_0, using barycentric average to define an interpolated g defined on Λ_0

8 Previous Work and Numerical Results