Solutions to the exercise.

First, using the utility representation, we have to compute the savings function. Households utility is given by

$$u(c) = \frac{c^{1-\frac{1}{2}}}{1-\frac{1}{2}}.$$

It is sbubject to the intertemporal budget constraint, which we can write as:

$$w_t = c_t + s_t, \quad d_{t+1} = s_t R_{t+1}$$

or

$$w_t = c_t + \frac{d_{t+1}}{R_{t+1}}.$$

We maximise the total discounted utility subject to the constraint:

$$\max_{s_t} u(\underbrace{w_t - s_t}_{c_t}) + \beta u(\underbrace{s_t R_{t+1}}_{d_{t+1}}).$$

In our case, it becomes:

$$\max_{s_t} \frac{(w_t - s_t)^{1-\frac{1}{2}}}{1-\frac{1}{2}} + \beta \frac{(R_{t+1}s_t)^{1-\frac{1}{2}}}{1-\frac{1}{2}}.$$

The first order condition is:

$$(w_t - s_t)^{-\frac{1}{2}} = \beta R_{t+1}(R_{t+1}s_t)^{-\frac{1}{2}}.$$

If we rearrange, we can obtain a closed-form solution for the savings. Notice that we use a CIES utility function. Therefore, the interest rate *must* appear in the expression for savings.

$$w_t - s_t = (\beta R_{t+1})^{-2} R_{t+1} s_t \implies s_t = \frac{w_t}{1+\beta^{-2}R^{-1}} = w_t \frac{\beta^2 R_{t+1}}{\beta^2 R_{t+1} + 1}.$$

We can check several things, for instance, that savings increase in the wages and that the marginal propensity to save is between 0 and 1 (as is the case in the OLG model):

$$\frac{\partial s_t}{\partial w_t} = \frac{\beta^2 R_{t+1}}{\beta^2 R_{t+1} + 1} \in (0,1) > 0.$$

And also, savings increase with the interest rate. *Note:* savings increase in with the interest rate because $\sigma > 1$. If we had, instead $0 < \sigma < 1$ savings would have been decreasing.

$$\frac{\partial s_t}{\partial R_{t+1}} = w_t \frac{\beta^2}{(1+\beta^2 R_{t+1})^2} > 0.$$

Next, we can use the market clearing conditions for interests and wages. In particular, we were given a Cobb-Douglas production function with $\alpha = \frac{1}{2}$. Therefore, we have the following:

$$w_t(k_t) = f(k_t) - f'(k_t)k_t = (1-\alpha){k_t}^{\alpha} = \frac{1}{2}{k_t}^{\frac{1}{2}},$$

$$R_t(k_t) = f'(k_t) = \alpha {k_t}^{\alpha-1} = \frac{1}{2}{k_t}^{-\frac{1}{2}}.$$

Some notes: I am not presenting the temporary equilibrium here, you can always do this. Also, the existence of a temporary equilibrium is guaranteed because the functions that define it are single-valued. In the OLG model, the existence of an intertemporal equilibrium is also guaranteed. In particular, finding an intertemporal equilibrium means finding a value k_{t+1} such that

$$k_{t+1} = \frac{1}{1+n} s\left(\omega(k_t), R_{t+1}\right) = \frac{1}{1+n}\left(\omega(k_t) f'k(t_{t+1})\right)$$

for a given value of capital k_t. In words, the idea is that if we are given a value k_t, can we compute k_{t+1}? The answer is yes! To see this, let $H(k_{t+1}) = k_{t+1} - \frac{1}{1+n}\left(\omega(k_t), f'k(t_{t+1})\right)$. We can show that

$$\lim_{k_{t+1}\to 0} H(k_{t+1}) < 0 \quad \text{and} \quad \lim_{k_{t+1}\to +\infty} H(k_{t+1}) > 0.$$

Therefore, by the intermediate value theorem we know that at least one solution exists. We can further show that the solution is unique by showing that the function $H(k_{t+1})$ is increasing in k_{t+1}, this is, that the function is monotonous. *Note:* the function coul have been monotonously decreasing and we would also have had a unique solution. However, we know that it goes from negative to positive, if anything, it must be monotonically increasing. We can check it:

$$\frac{\partial H(k_{t+1})}{\partial k_{t+1}} = 1 - \frac{1}{1+n} s'_R, f''(k_{t+1}).$$

In our case, this expression becomes:

$$\frac{\partial H(k_{t+1})}{\partial k_{t+1}} = 1 - \frac{1}{1+n} w_t \frac{k_{t+1}^{\alpha}(\alpha-1)\alpha\beta^2}{(1+\beta^2\alpha {k_{t+1}}^{\alpha-1})^2} > 0.$$

Next, we can compute the steady states of the economy. First, notice that we have a Cobb-Douglas production function. Hence, $f(0) = 0$ and we know that an autarky steady state with $\bar{k} = 0$ exists. So we should find it. We solve for the steady states using the functional forms we have, and imposing $n = 0, \alpha = \frac{1}{2}, \sigma = 2$.

$$\bar{k} = w_t \frac{\beta^2 R_{t+1}}{1+\beta^2 R_{t+1}} = \underbrace{\frac{1}{2}\bar{k}^{\frac{1}{2}}}_{w} \frac{\beta^2 \overbrace{\frac{1}{2}\bar{k}^{-\frac{1}{2}}}^{R}}{1+\beta^2 \underbrace{\frac{1}{2}\bar{k}^{-\frac{1}{2}}}_{R}} = \frac{1}{4}\frac{\beta^2}{1+\beta^2\frac{1}{2}\bar{k}^{-\frac{1}{2}}} = \frac{\beta^2 \bar{k}^{\frac{1}{2}}}{4\bar{k}^{\frac{1}{2}} + 2\beta^2}$$

Taking the denominator to right-hand side and subtrating we can arrive at:

$$2\beta^2\bar{k} + 4\bar{k}^{\frac{3}{2}} - \beta^2\bar{k}^{\frac{1}{2}} = \bar{k}^{\frac{1}{2}}\left(2\beta^2\bar{k}^{\frac{1}{2}} + 4\bar{k} - \beta^2\right) = 0.$$

Hence, either $bark = 0$ or $2\beta^2\bar{k}^{\frac{1}{2}} + 4\bar{k} - \beta^2 = 0$. For the second case, we can find a solution for $\bar{k}$, the actual value is not crucial in what follows but it would be nice if you could compute it.

Finally, we check the stability of each of the *two* steady states. First, let's write the derivative of k_{t+1} with respect to k_t using the fact that

$$k_{t+1} = \underbrace{(1-\alpha){k_t}^{\alpha}}_{w_t} \frac{R^2 \overbrace{\alpha {k_{t+1}}^{\alpha-1}}^{R_{t+1}}}{1+\beta^2 \underbrace{\alpha {k_{t+1}}^{\alpha-1}}_{R_{t+1}}} = {k_t}^{\frac{1}{2}} \frac{\beta^2 {k_{t+1}}^{-\frac{1}{2}}}{4+2\beta^2 {k_{t+1}}^{-\frac{1}{2}}} = {k_t}^{\frac{1}{2}} \frac{\beta^2}{4{k_{t+1}}^{\frac{1}{2}}+2\beta^2}$$

or

$$k_{t+1}\left(4{k_{t+1}}^{\frac{1}{2}}+2\beta^2\right) = {k_t}^{\frac{1}{2}}\beta^2.$$

Then:

$$\frac{\partial k_{t+1}}{\partial k_t} = \frac{\beta^2 \frac{1}{2} {k_t}^{-\frac{1}{2}}}{4{k_{t+1}}^{\frac{1}{2}}+2\beta^2+2{k_{t+1}}^{\frac{1}{2}}}.$$

Evaluating it at $k_t = k_{t+1} = \bar{k} = 0$ yields

$$\frac{\beta^2 \frac{1}{2}\, \overbrace{0^{-\frac{1}{2}}}^{\infty}}{4\underbrace{0^{\frac{1}{2}}}_{0}+2\beta^2+2\underbrace{0^{\frac{1}{2}}}_{0}} = \infty,$$

and the steady state $\bar{k}=0$ is unstable.

For the second steady state we can use the capital accumulation equation (we tweaked it a little bit)

$$k_{t+1}\left(4{k_{t+1}}^{\frac{1}{2}}+2\beta^2\right) = {k_t}^{\frac{1}{2}}\beta^2$$

evaluated at the steady state, this is,

$$\bar{k}\left(4\bar{k}^{\frac{1}{2}}+2\beta^2\right) = \bar{k}^{\frac{1}{2}}\beta^2$$

Notice that the part within parentheses appears in the denominator of $\frac{\partial k_{t+1}}{\partial k_t}$ when it is evaluated at the steady state. Hence, we can substitute

$$\left(4\bar{k}^{\frac{1}{2}}+2\beta^2\right) = \frac{\bar{k}^{\frac{1}{2}}\beta^2}{\bar{k}} = \bar{k}^{-\frac{1}{2}}\beta^2.$$

So $\frac{\partial k_{t+1}}{\partial k_t} = \frac{\beta^2 \frac{1}{2} {k_t}^{-\frac{1}{2}}}{4{k_{t+1}}^{\frac{1}{2}}+2\beta^2+2{k_{t+1}}^{\frac{1}{2}}}$, once evaluated at the steady state becomes:

$$\frac{\beta^2 \frac{1}{2}\bar{k}^{-\frac{1}{2}}}{4\bar{k}^{\frac{1}{2}}+2\beta^2+2\bar{k}^{\frac{1}{2}}}.$$

Substituting $\bar{k}\left(4\bar{k}^{\frac{1}{2}}+2\beta^2\right) = \bar{k}^{\frac{1}{2}}\beta^2$ we obtain:

$$\frac{\frac{1}{2}\beta^2\bar{k}^{-\frac{1}{2}}}{2\bar{k}^{\frac{1}{2}}+\beta^2\bar{k}^{-\frac{1}{2}}}=\frac{\frac{1}{2}\beta^2}{2\bar{k}+\beta^2}=\frac{\beta^2}{4\bar{k}+2\beta^2}<1,$$

and the second steady state is stable.