0.1 Introduction

The basic Solow model assumed the following:

- Constant returns to scale production function
- Constant population growth n
- Constant technological progress g
- Fixed saving rate s

Althought it is a useful model, the choice of s is arbitrary and **does not obey any microeconomic foundation**. In particular, microeconomic theory relates present and futur consumption decisions through the interest rate. The Ramsey model ammends this shortcomming and the decisions of all agents are microeconomically founded. In particular, we assume:

Ramsey assumptions

- Large number of identical firms operating the same technology:
  - Rent capital and hire labour
- Large number of families:
  - Consume, supply labour, own and lend capital
- Constant returns to scale production function

Agents (families) optimally decide consumption and savings, depeding on the interest rate. Hence, the **saving rate** is definetely no longer exogenous and **need not be constant**.

0.2 References

Ramsey (1928) Cass (1965) Koopmans (1965)

Romer, Advanced Macroeconomics: Chapter 2[ˆ1]

0.3 [ˆ1]: Romer derives all the results in continuous time, we use discrete time.

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The economy is populated by a large number of identical households. Each household is composed on one individual who lives indefinitely.

1.2 Utility representation

Households exhibit the following utility representation:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t).$$

Assumptions on $u(c)$

The utility function satisfies the following properties:

- $u(c)$ is a continuous function defined over $[0, +\infty)$
- It has continuous derivatives of any required order defined over $(0, +\infty)$
- In particular: $u'(c) > 0$ and $u''(c) < 0$
- We also impose the Inada conditions: $\lim_{c \to 0} u'(c) = +\infty$ and $\lim_{c \to +\infty} u'(c) = 0$

1.3 Household's budget constraint

The household reveices labour income w which uses to save k and consume c. For simplicity, savings take the form of capital (instead of assets). Households lend capital to firms, obtaining a real interest r. However, capital is subject to a constant depreciation rate δ.

Therefore, each period of time, households face the following budget constraint:

$$c_t + k_{t+1} = w_t + r_t k_t + (1 - \delta) k_t.$$

The left-hand side represents expenditures during period t:

- Consumption
- Saving

The right-hand side includes all sources of income:

- Wages

- Interests
- Remaining capital after depreciation
  - Effectively, households can *eat* capital

Note: The budget constraint could have included *dividends* d_t in the right-hand side. However, as we shall see, firms operate in perfect competition, and make zero profits.

1.4 Solving the household's problem

First, we assume that households have perfect foresight. This means that a household is able to perfectly forecast all the future values of the relevant variables when deciding. For instance, at time t the household is able to correctly compute r_{t+1} and w_{t+1}.

Assumption H1: Househlds have perfect foresight.

In this problem, we want to obtain the path of c_t. Note that the problem is infinite, in the sense that we need to determine c_t for each and every period of time. Instead, we can try to solve for the optimal trajectory of c_t, this is, how it evolves over time: $c_{t+1} = G(c_t)$. The optimisation problem reads:

$$\max_{c_t, k_{t+1}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

$$c_t + k_{t+1} = w_t + r_t k_t + (1-\delta)k_t. \tag{2}$$

First, start off by writing the Langrangian equation corresponding to the problem:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t (w_t + r_t k_t + (1-\delta)k_t - c_t - k_{t+1}).$$

At time t, the households has *two* decisions to make:

- How much to consume at t: c_t
- How much to save at t: k_{t+1}

Therefore, we derive the Lagrangian function with respect to c_t and k_{t+1}.

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_t + \lambda_{t+1}(r_{t+1} + (1-\delta))$$

Set both equal to zero to obtain the maximum, and combine the equations to get:

$$u'(c_t) = \beta u'(c_{t+1})(r_{t+1} + 1 - \delta).$$

This condition is called the **Euler equation**. It tells us the optimal behaviour of the household. Rearranging the expression a little bit, we obtain:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(r_{t+1} + 1 - \delta).$$

Therefore, if the interest rate is to increase, the household would optimally postpone consumption: this is, decrease c_t and increase c_{t+1}. **Note:** Remember that $u'' < 0$: marginal utilities are higher the lower consumption is. Slightly more complex relationships can arise if population grows or there is technological progress, see Romer, Chapter 2.

1.4.1 The Euler equation

The Euler equation appears often in optimisation problems: both using discrete and continuous time. The discrete-time version is relatively easier to intepret and obtain. In our case, the Euler equation relates the present and future marginal utilities of consumption. Imagine that the household decreased the amount consumed today, c_t by an infinitesimally small amount Δ_{c_t}. This causes a utility loss of $\Delta_{c_t} u'(c_t)$. Saving Δ_{c_t} until tomorrow and consuming all the proceedings generates a utility gain of $\Delta_{c_t}(r_{t+1} + 1 - \delta)u'(c_{t+1})$. This gain *must* be, of course, discounted using the rate β to compare present-day equivalents.

Since the household is optimising, both the loss and the gain must be equal, otherwise, there would be an alternative consumption level that would maximise utility. Hence, putting everything together: $\Delta_{c_t} u'(c_t) = \beta \Delta_{c_t}(r_{t+1} + 1 - \delta)u'(c_{t+1})$. Cancel $\Delta_{c_t} > 0$ on both sides to obtain the Euler equation. Alternatively, we can interpret the Euler equation as stating that it is not optimal to slightly deviate from the optimal path, consuming slighly less for instance, and later returning to the optimal path. **Remark:** the scheme we are putting in place here does not modify the intertemporal budget constraint because all the additional proceedings from saving are consumed.

1.4.2 The Transversality Condition

The Euler equation is *not* enough to fully determine the optimal sequence of consumption and savings. For the moment, trajectories where the capital or consumption grow boundedless are feasible, but these should not be optimal.

- First, consider the economy continously accumulates capital. In that case, consumption must decrease towards zero. Since we have assumed that $\lim_{c\to 0} u'(c) = +\infty$ (Inada condition) a slighly increase in consumption raises utility by a large amount, this is, the marginal utility of consumption is very large. Hence, it cannot be optimal to accumulate capital and let consumption go to zero.
- Alternatively, let savings vanish. This case typically implies that k_t converges to zero in finite time (this is, for some $t < \infty$). The condition that capital must be positive prevents such cases.

Note: the preceding points are only descriptive of what the transversality condition implies. In fact, that transversality condition reads as:

$$\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0.$$

In general, we will obtain steady-state solution in which the path of capital and consumption is bounded. This is, in the optimal solution consumption $c_t \to c^\star$ and capital $k_t \to k^\star$. In this case, the fact that $\beta \in (0, 1)$ ensures the validity of the transversality condition.

We assume that a large number of idential firms populates the economy. Firms produce a single, homogeneous good using labour and capital. The production function has the following properties (assumptions):

1. $F(K_t, X_t)$ is continuous and defined on $[0, +\infty)^2$,
2. $F(K_t, X_t)$ has continuous derivatives of every requires order on $(0, +\infty)^2$,
3. $F_i(K_t, X_t) > 0, F_{ii}(K_t, X_t) < 0$ and $F_{ii}(K_t, X_t)F_{jj}(K_t, X_t) - F_{ij}(K_t, X_t)F_{ji}(K_t, X_t) > 0$: the function is stricly increasing in both arguments and strictly concave.
4. $F(K_t, X_t)$ is homogeneous of degree one.
5. The Inada conditions are satisfied: $\lim_{K\to 0} F_1(K_t, X_t) = \lim_{X\to 0} F_2(K_t, X_t) = +\infty$ and $\lim_{K\to +\infty} F_1(K_t, X_t) = \lim_{X\to +\infty} F_2(K_t, X_t) = 0$.

Note: the last two inequalities in Assumption 3 determine that the Hessian matrix of the production function is negative definite, hence the function is sctrictly concave.

Firms maximise real profits. Since there are many firms competing, in equilibrium they make exactly zero profits. Moreover, also in equilibrium factors are paid their marginal productivity. Since the production function F is homogeneous of degree one wecan write it in *intensive* terms:

$$f(k) \equiv F\left(\frac{K}{X}, 1\right),$$

where $k \equiv \frac{K}{X}$.

Because markets are competitive, capital earns its marginal product $\partial F(K,X)/\partial K$ or, equivalently, $f'(k)$ in intensive terms. Thus, the real interest rate at time t is:

$$r_t = f'(k_t).$$

The marginal product of labour is given by $\partial F(K,X)/\partial X$. In intensive terms, it is equal to:

$$w_t = f(k) - f'(k)k.$$

At the equilibrium, we have that factors are paid their marginal productivities, thus:

$$r_t = f'(k)$$

and

$$w_t = f(k) - f'(k)k.$$

1.5 Definition of intertemporal equilibrium

An intertemporal equilibrium with perfect foresight is **a sequence** $(k_t, c_t) \in \mathbb{R}^2_{++}, t = 0, \ldots, +\infty$, such that given $k_0 > 0$, **households and firms optimise**, **markets clear**, and **capital accumulation follows** $k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$. Thus, all the following equations must be satisfied:

$$u'(c_t) = \beta(r_{t+1} + 1 - \delta)u'(c_{t+1}), \tag{3}$$

$$k_{t+1} = f(k) + (1-\delta)k_t - c_t, \tag{4}$$

$$w_t = f(k_t) - f'(k_t)k_t, \tag{5}$$

$$r_t = f'(k_t), \tag{6}$$

$$\lim_{t\to\infty} \beta^t u'(c_t)k_{t+1} = 0. \tag{7}$$

The first equation is the Euler equation, which implies that households are optimising. Equations 3 and 4 denote market clearing conditions for labour and capital. The second equation is the law of motion for capital: capital next period equals production net of consumption plus the remaining capital that has not depreciated. Finally, the traversality condition allows us to pick a non-explosive path.

After some substitutions, we can express the intertemporal equilibrium —forgetting for a moment about the transversality condition— as:

$$u'(c_t) = \beta(f'(k_{t+1}) + 1 - \delta)u'(c_{t+1}), \tag{8}$$

$$k_{t+1} = f(k) + (1-\delta)k_t - c_t. \tag{9}$$

This is a two-dimensional dynamic system with a unique predetermined state variable: k_t. To analyse it, we first compute its steady state and then check the dynamics.

The steady state of an N-dimensional dynamic system is a configuration of the system variables' such that the system remains unchanged over time. This is, upon reaching the steady state, the system remains there forever. Hence, if the system is given by:

$$x^1_{t+1} \quad = f_1(x^1_t, \ldots, x^N_t), \tag{10}$$

$$\vdots \tag{11}$$

$$x^N_{t+1} \quad = f_N(x^1_t, \ldots, x^N_t). \tag{12}$$

at the steady state we have that

$$
\begin{aligned}
x_{t+1}^1 &= x_t^1, && (13)\\
&\vdots && (14)\\
x_{t+1}^N &= x_t^N. && (15)
\end{aligned}
$$

In many cases, the steady state is only reached after some periods, unless we intentionally start at the steady state. Moreover, the steady state is often approached asymptotically: it is never fully reached, but we get infinitesimally close to it. Finally, steady states can be (asymptotically) stable or unstable. In the first sort, variables approach the steady-state level. In the second, the system *diverges* from the steady state.

In our case, denote by $(\bar{k}, \bar{c}) \in \mathbb{R}_+^2$ the capital and consumption levels at the steady state. Then, a steady state solution satifies the following system of equations:

$$
\begin{aligned}
u'(\bar{c}) &= \beta(f'(\bar{k}) + 1 - \delta)u'(\bar{c}), && (16)\\
\bar{k} &= f(\bar{k}) + (1-\delta)\bar{k} - \bar{c}. && (17)
\end{aligned}
$$

Simplifying leads to:

$$
\begin{aligned}
f'(\bar{k}) &= \frac{\theta}{\beta}, && (18)\\
\bar{c} &= f(\bar{k}) - \delta\bar{k}, && (19)
\end{aligned}
$$

where $\theta \equiv 1 - \beta(1-\delta) \in (0,1)$.

From the Inada conditions for the production function we have that

$$
0 = \lim_{k\to+\infty} f'(k) < \frac{\theta}{\beta} < \lim_{k\to 0} f'(k) = +\infty.
$$

Moreover, $f'(k)$ is monotonically decreasing. Then, by the intermediate value theorem, there exists a unique level $\bar{k} \in (0, +\infty)$ that solves the equation.

This level is a *modified golden rule.* Consequently, $\bar{c} = f(\bar{k}) - \delta\bar{k}$ uniquely determines the steady-state consumption level.

We can study the dynamics of the model using the [phase diagram].

1.6 The Golden rule

The Golden rule level of capital is the level of capital, $k^{\mathcal{GR}}$ that maximises consumption, c at the steady state, thus achieving maximum consumption: $c^{\mathcal{GR}}$. We already know that, at the steady state:

$$c = f(k) - \delta k.$$

Hence, the level of capital that maximies c is given by:

$$\frac{\partial c(k)}{\partial k} = f'(k) - \delta = 0.$$

To maximise consumption at the steady state we must have:

$$f'(k) = \delta \implies k^{\mathcal{GR}} = f'^{-1}(\delta).$$

However, at the *competitive* steady state we have:

$$f'(k) = \frac{1-\beta(1-\delta)}{\beta} \implies \bar{k} = f'^{-1}\left(\frac{1-\beta(1-\delta)}{\beta}\right).$$

Hence, $k^{\mathcal{GR}} = \bar{k}$ is only possible if $\frac{1-\beta(1-\delta)}{\beta} = \delta \implies \beta = 1$.

1.6.1 Under- or over-accumulation of capital in the Ramsey model

According to our result before, the steady state does not correspond, in general, with a level of capital that maximises consumption at the steady state.

$$f'(\bar{k}) = \frac{1-\beta(1-\delta)}{\beta}$$

$$f'(k^{\mathcal{GR}}) = \delta.$$

So, unless $\beta = 1$, the economy is not in the Golden-rule level of capital.

Since $\frac{1-\beta(1-\delta)}{\beta} > \delta$, we conclude that $\bar{k} < k^{\mathcal{GR}}$: the level of capital is below its Golden-rule level.

There is an important remark to be made here.

- The level of capital in the *competitive equilibrium* maximises life-time utility: we obtained it solving the utility maximisation problem.
- This level of capital does **not** maximise steady-state consumption.

In that sense, achieving the Golden Rule level of capital $k^{\mathcal{GR}}$ is **not** desirable from the viewpoint of utility maximisation.

1.7 Stability of the steady state

The economy is represented by a 2×2 system of first-order difference equations. We can analyse the stability of the steady-state analysing the the eigenvalues of the Jacobian matrix evaluated at the steady state. **Note:** the [Appendix] discusses this in more detail.

Our dynamic equations are:

$$u'(c_t) = \beta(f'(k_{t+1}) + 1 - \delta)u'(c_{t+1}),\ k_{t+1} = f(k_t) + (1-\delta)k_t - c_t.$$

First, compute the Jacobian matrix:

$$\bar{A} = \begin{pmatrix} \frac{\partial c_{t+1}}{\partial c_t} & \frac{\partial c_{t+1}}{\partial k_t} \\ \frac{\partial k_{t+1}}{\partial c_t} & \frac{\partial k_{t+1}}{\partial k_t} \end{pmatrix}.$$

In our case:

$$\bar{A} = \begin{pmatrix} \frac{u''(c_t)+\beta f''(k_{t+1})u'(c_{t+1})}{\beta[f'(k_{t+1})+1-\delta]u''(c_{t+1})} & -\frac{\beta f''(k_{t+1})[f'(k_t)+1-\delta]u'(c_{t+1})}{\beta[f'(k_{t+1})+1-\delta]u''(c_{t+1})} \\ -1 & f'(k_t)+1-\delta \end{pmatrix}.$$

Then, we evaluate the Jacobian matrix $\bar{A}$ at the steady state, using the information we know:

$$k_{t+1} = k_t = \bar{k},$$

$$c_{t+1} = c_t = \bar{c},$$

$$1 = \beta \left(f'(\bar{k}) + 1 - \delta\right).$$

Going one step at a time, we should first substitute all t and $t+1$ variables for the steady-state levels $\bar{k}$ and $\bar{c}$.

$$\bar{A}\Big|_{\substack{k_t=k_{t+1}=\bar{k} \\ c_t=c_{t+1}=\bar{c}}} = \begin{pmatrix} \frac{u''(\bar{c})+\beta f''(\bar{k})u'(\bar{c})}{\beta\left[f'(\bar{k})+1-\delta\right]u''(\bar{c})} & -\frac{\beta f''(\bar{k})\left[f'(\bar{k})+1-\delta\right]u'(\bar{c})}{\beta\left[f'(\bar{k})+1-\delta\right]u''(\bar{c})} \\ -1 & f'(\bar{k})+1-\delta \end{pmatrix} =$$

$$\begin{pmatrix} \frac{u''(\bar{c})+\beta f''(\bar{k})u'(\bar{c})}{\beta\left[f'(\bar{k})+1-\delta\right]u''(\bar{c})} & -\frac{\beta f''(\bar{k})\left[f'(\bar{k})+1-\delta\right]u'(\bar{c})}{\beta\left[f'(\bar{k})+1-\delta\right]u''(\bar{c})} \\ -1 & f'(\bar{k})+1-\delta \end{pmatrix} =$$

$$\begin{pmatrix} \frac{u''(\bar{c})+\beta f''(\bar{k})u'(\bar{c})}{u''(\bar{c})} & -\frac{f''(\bar{k})u'(\bar{c})}{u''(\bar{c})} \\ -1 & \frac{1}{\beta} \end{pmatrix} =$$

$$\begin{pmatrix} 1-\beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}} & f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}} \\ -1 & \frac{1}{\beta} \end{pmatrix},$$

where $\epsilon_c = -\frac{u''(c)}{u'(c)}c$ represents the degree of relative risk aversion.

Finally, lets compute the roots of the characteristic equation (the eigenvalues) of this matrix. We can use several techniques:

1.7.1 Direct computation of the eigenvalues

In our case, it boils down to solving the determinant of the matrix $|\bar{A}-\lambda I| = 0$ Hence, we have a second-order equation in λ:

$$\begin{vmatrix} 1-\beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}}-\lambda & f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}} \\ -1 & \frac{1}{\beta}-\lambda \end{vmatrix} = \lambda^2 + \lambda\left(\beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}} - 1 - \frac{1}{\beta}\right) + \frac{1}{\beta} = 0.$$

The roots of this equation are:

$$\lambda_1 = \frac{1+\frac{1}{\beta}-\beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}} + \sqrt{\left(1+\frac{1}{\beta}-\beta f''(\bar{k})\right)^2 - \frac{4}{\beta}}}{2} > 1$$

It is clear that $\lambda_1 > 1:$ the term $1+\frac{1}{\beta}-\beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}} > 1$ because $f''(\cdot) < 0$.

For λ_2 we have:

$$\lambda_2 = \frac{1+\frac{1}{\beta}-\beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}} - \sqrt{\left(1+\frac{1}{\beta}-\beta f''(\bar{k})\right)^2 - \frac{4}{\beta}}}{2}.$$

Let $\phi \equiv 1+\frac{1}{\beta}-\beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}}$ and $\kappa \equiv \frac{4}{\beta}$. Assume that $\lambda_2 > 1$, then we must have:

$$\frac{\phi - \sqrt{\phi^2-\kappa}}{2} > 1 \implies \kappa > 4\phi - 4.$$

Substituting:

$$4 + \frac{4}{\beta} - 4\beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}} - 4 < \frac{4}{\beta} \implies -4\beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}} < 0.$$

But this is impossible because $f''(\cdot) < 0$. Hence, $\lambda_2 < 1$. Moreover, we also know that $\lambda_2 > 0$. Hence, $\lambda_1 > 1, \lambda_2 \in (0,1)$ and the **steady state is a saddle. Note:** It is a saddle because the *absolute value* of one eigenvalue is larger than one, and the *absolute value* of the secon eigevalue is between 0 and 1.

1.7.2 Use the intermediate value theorem

The characteristic equation associated with the Jacobian evaluated at the steady state is:

$$G(\lambda) = \lambda^2 + \lambda\left(\beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}} - 1 - \frac{1}{\beta}\right) + \frac{1}{\beta} = 0.$$

First, notice this is a continuous function on λ. We are interested in checking whether the roots (or at least one root) lies in the interval $(-1,1)$. Then, compute the following:

$$\lim_{\lambda\to-\infty} G(\lambda) = +\infty$$

$$\lim_{\lambda\to+\infty} G(\lambda) = +\infty$$

$$G(-1) = \frac{2}{\beta} - \beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}} > 0$$

$$G(0) = \frac{1}{\beta} > 0$$

$$G(1) = \beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}} < 0.$$

Hence, $\exists \lambda_2 \in (-1,0)$ such that $G(\lambda_2) = 0$. Similarly, the second root lies beyond 1. Therefore, $\lambda_1 > 1, \lambda_2 \in (0,1)$ and the **steady state is a saddle.**

1.7.3 Use eigenvalues' properties

For any matrix, we have the following:

- The product of eigenvalues equals the **determinant** of the matrix: $\det(M) = \lambda_1\lambda_2 \dots \lambda_N$,
- The sum of eigenvalues equals the **trace** of matrix: $\text{tr}(M) = \lambda_1 + \lambda_2 + \dots + \lambda_N$.

In our case:

$$\lambda_1\lambda_2 = \frac{1}{\beta}$$

$$\lambda_1 + \lambda_2 = \underbrace{1 + \frac{1}{\beta} - \beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}}}_{>1}.$$

The first equation implies that either a) $\lambda_1 > 0, \lambda_2 > 0$ or b) $\lambda_1 < 0, \lambda_2 < 0$. However, since $\lambda_1 + \lambda_2 > 1$ b) is impossible. Then, $\lambda_1 > 0, \lambda_2 > 0$. Substitute $\lambda_1\lambda_2 = \frac{1}{\beta}$ in the second equation:

$$\lambda_1 + \lambda_2 = 1 + \lambda_1\lambda_2 - \beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}}.$$

Rearranging:

$$\lambda_1 + \lambda_2 - \lambda_1\lambda_2 = 1 + -\beta f''(\bar{k})\frac{\bar{c}}{\epsilon_{\bar{c}}} > 1 \implies \lambda_1 + \lambda_2 - \lambda_1\lambda_2 - 1 > 0.$$

Factorisation leads to:

$$(1 - \lambda_2)(\lambda_1 - 1) > 0.$$

Therefore:

$$\lambda_2 < 1$$

$$\lambda_1 > 1$$

$$\lambda_1 > 0, \lambda_2 > 0$$

and **the steady state is saddle.**

Note: based on Section 2.3 of *Romer, Advanced Macroeconomics.*

We can describe the dynamics of the economy using two equations:

$$u'(c_t) = \beta(f'(k_{t+1}) + 1 - \delta)u'(c_{t+1}), \tag{20}$$

$$k_{t+1} = f(k_t + (1-\delta)k_t - c_t. \tag{21}$$

$$

1.8 The dynamics of c

The first equation describes the dynamics of consumption.

$$u'(c_t) = \beta(f\prime(k_{t+1}) + 1 - \delta)u'(c_{t+1}).$$

Rearranging it we arrive at:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(f'(k_{t+1}) + 1 - \delta).$$

If the left-hand side term equals 1 ($\frac{u'(c_t)}{u'(c_{t+1})} = 1$), then consumption remains constant over time. This is, when

$$1 = \beta(f'(k_{t+1}) + 1 - \delta)$$

consumption is constant over time: $c_{t+1} = c_t$. This condition depends only on the level of capital. Denote $k^\star$ such level. When $k < k^\star$, $f'(k) > f'(k^\star)$, implying that $u'(c_t) > u'(c_{t+1})$ and hence consumption will raise over time. Similarly, if $k > k^\star$ implies that consumption falls. This information is summarised in the following Figure.

The arrows show the direction in which consumption c evolves. As discussed before, consumtion raises when capital is below $k^\star$ and raises when above. Note the vertical line: it denotes the level of capital $k^\star$ such that c is constant. The value $k^\star$ can be easily computed: $k^\star = f'^{-1}\left(\frac{1-\beta(1-\delta)}{\beta}\right)$.

1.9 The dynamics of k

We can proceed simiarly with the motion of capital. The relevant equation in this case is:

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t.$$

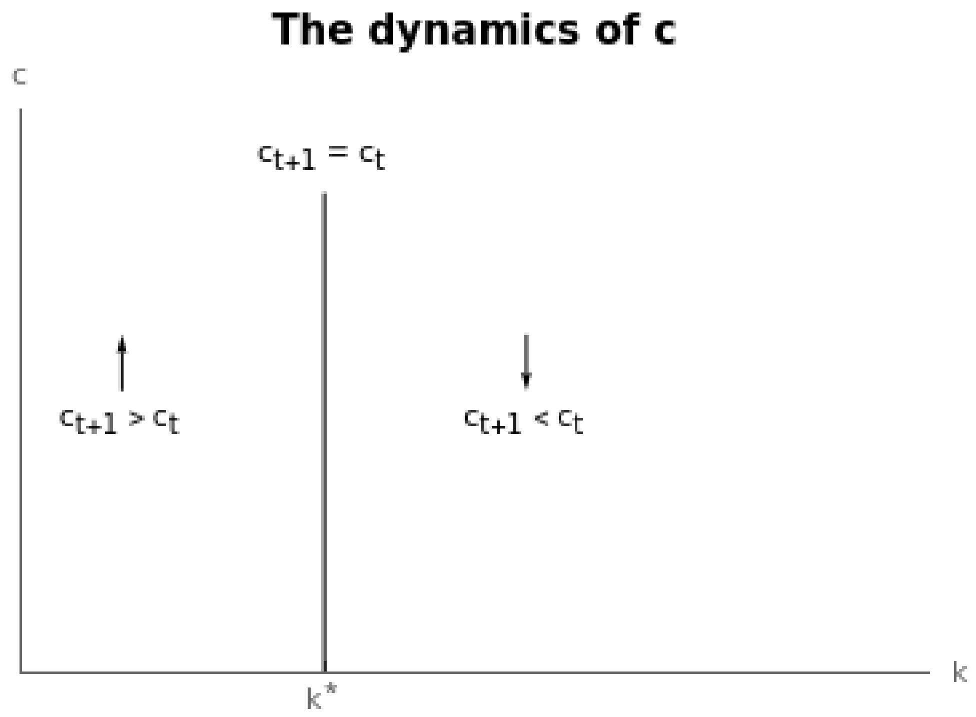

Figure 1: dynamics of c

As before, we are interested by the combinations of (k, c) such that capital remains constant over time. In this case, though, and contrasting with the previous, we will obtain that both capital and consumption affect capital levels in the future: production uses current capital, and current consumption depletes current production. We can rearrange the previous equation to obtain the following:

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t.$$

Setting $k_{t+1} = k_t \implies k_{t+1} - k_t = 0$ so capital is constant yields:

$$0 = f(k_t) - \delta k_t - c_t \implies c_t = f(k_t) - \delta k_t.$$

In the (k, c) space, the equation takes a form similar to a parabola: it combines production —with decreasing marginal returns— with a linear depreciation rate. Technically, it can be seen that $\frac{\partial f(k) - \delta k}{\partial k} = f'(k) - \delta$, which admits a unique maximum: $f'(k)$ is a decreasing function. Moreover, $\frac{\partial^2 f(k) - \delta k}{\partial k^2} = f''(k) < 0$, confirming that the optimal point is a maximum.

Let's now analyse how capital evolves when consumption is above and below the level that makes it constant. From $k_{t+1} - k_t = f(k_t) - \delta k_t - c_t$, if consumption is above the level we have that $k_{t+1} - k_t < 0$, meaning that capital decreases over time. The opposite applies when

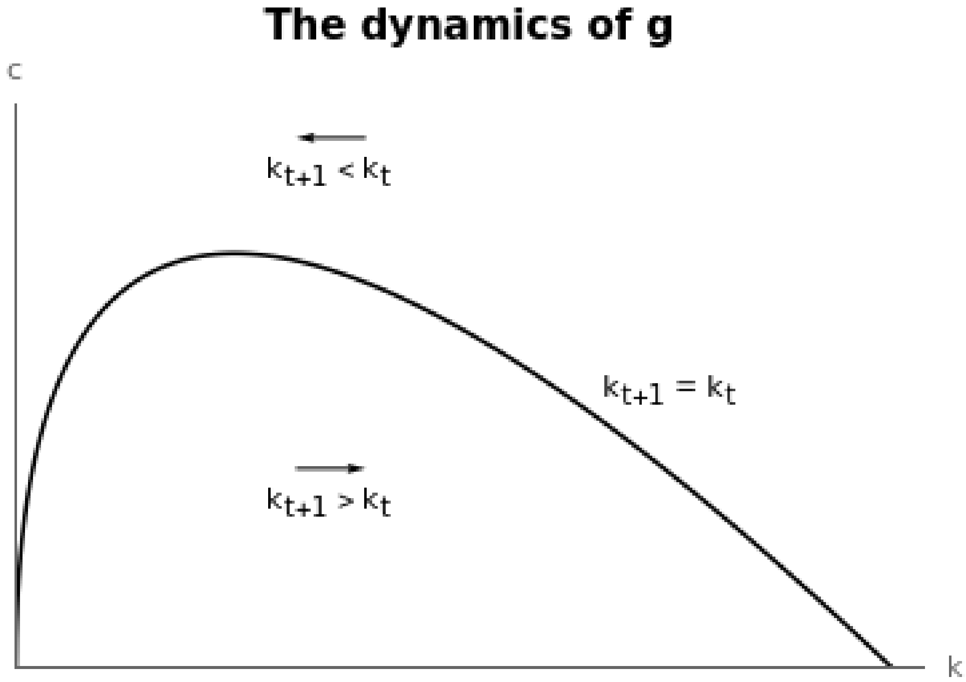

Figure 2: dynamics of g

consumption is below the level that guarantees a constant level of capital. The Figure below summarises these findings:

1.10 The phase diagram

We can combine the information above to produce the *phase diagram*. It indicates the motion of variables at different points of the (k, c) space.

The arrows show the motion of each variable at different conbinations of (k, c). For instance, to the left of the $c_{t+1} = c_t$ locus and above the $_{t+1} - k_t = 0$ locus consumption increases and capital decreases. On top of each curve only capital or consumption moves, the other remains constant. For example, on the $c_{t+1} - c_t$ locus, consumption is constant but capital changes. Finally, the point indicating the intersection of both curves is the steady state: all variables remain constant at their values. — title: Additional material - trajectory linktitle: Trajectory toc: true type: docs date: "2019-07-11T00:00:00Z" lastmod: "2019-07-11T00:00:00Z" draft: false menu: Ramsey: parent: The Ramsey model weight: 6

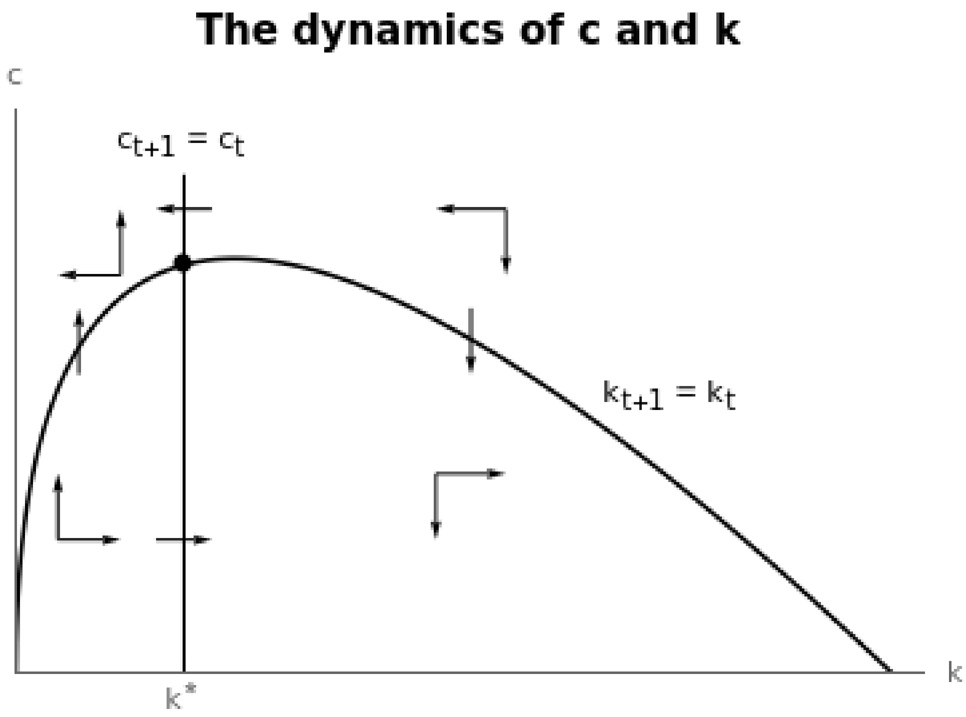

Figure 3: phase diagram

2 Prev/next pager order (if `docs_section_pager` enabled in `params.toml`)

2.1 weight: 6

2.2 The trajectory around the steady state

For the moment, we have established that the model converges towards a unique steady state following a saddle path. **Note:** This means that there is only one combination of inital capital and consumption, (k_0, c_0), such that the ecomomy converges. Any other initial value consumption at $t = 0$ (capital is pre-determined and thus we *cannot* change it) has a diverging trajectory.

We now compute the exact behaviour of (k_t, c_t) around the steady state. In general, though, the notion of *around the steady state* is quite generous and we extend it quite far from the steady state.

The model is higgly non-linear, so we study a simplied linearised version around the steady state. First, let's approximate the dynamics of capital and consumption [around the steady state.]

2.2.1 The approximation around the steady state

The behaviour of the dynamical system around the steady state can be approximated using a first-order Taylor expansion around it. In that sense:

$$\begin{pmatrix} c_{t+1} - \bar{c} \\ k_{t+1} - \bar{k} \end{pmatrix} \approx \bar{A} \begin{pmatrix} c_t - \bar{c} \\ k_t - \bar{k} \end{pmatrix}.$$

This dynamics are governed by 2–dimensional system of difference equations. To solve it, we begin by studying a simpler system.

2.2.2 Diagonalising the matrix: eigenvalues and eigenvectors

Note: based on the notes by Benjamin Moll.

To simplify notation, let $y_t = \begin{pmatrix} c_t - \bar{c} \\ k_t - \bar{k} \end{pmatrix}$. Our problem can be written as: $y_{t+1} = \bar{A} y_t$.

It $\bar{A}$ was a diagonal matrix, the solution to the system would be straighforward. In fact, if that were the case (I change variables to avoid confussion):

$$\begin{pmatrix} \phi_{t+1} \\ \\ \omega_{t+1} \end{pmatrix} = \begin{pmatrix} a_{1,1} & 0 \\ 0 & a_{2,2} \end{pmatrix} \begin{pmatrix} \phi_t \\ \omega_t \end{pmatrix}.$$

Then, it is clear that $\phi_{t+1} = a_{1,1}\phi_t \implies \phi_t = a_{1,1}^t \phi_0$. And we would find a similar, equivalent expression for ω.

$\bar{A}$ is not diagonal, but we can apply a Jordan decomposition to it and obtain an equivalent system governed by a diagonal matrix. In particular, we look for an 2×2 invertible matrix X such that $X^{-1}\Lambda X = \bar{A}$, where Λ is diagonal. Then, we apply the following transformation to our system (pre-mulitiplying by X^{-1} on both sides and multiplying by XX^{-1} on the righ-hand side).

$$X^{-1} y_{t+1} = X^{-1}\bar{A}(XX^{-1})y_t = X^{-1}y_{t+1} = X^{-1}\bar{A}X(X^{-1}y_t) =$$

$$= \Lambda X^{-1} y_t.$$

Denote $z \equiv X^{-1}y$. Thus, the system becomes:

$$z_{t+1} = \Lambda z_t.$$

Since Λ is diagonal, the soltions are of the form:

$$z_t = \Lambda^t z_0.$$

Note: in general, the power of a matrix is complex. However, for a *diagonal matrix* it is simply the power of each component.

Once we have the solution for the transformed system we must undo the transformation: in fact, we do not care abut the evolution of z. This step involves using that $X^{-1}y = z \implies y = Xz$.

The matrix Λ In a Jordan decomposition, the diagonal matrix Λ consists of the matrix whose entries are its eigenvalues. Thus (we have a 2×2 system),

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

The matrix X The matrix X contains the eigenvectors of $\bar{A}$ in columns. There is one eigenvector associated to each eigenvalue. **Note:** it is important to correctly relate each eigenvetor to its eigenvalue.

To find an eigenvector associated with λ_1 (*there are infinitely many possible eigenvectors, we only want one*) we solve the following system:

$$\bar{A}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{or} \quad (\bar{A} - \lambda_1 \mathbb{I}) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The matrix X then becomes:

$$X = \begin{pmatrix} v_{1,1} & v_{2,1} \\ 1 & 1 \end{pmatrix}.$$

2.2.3 Solving the system

We begin with the solution for the transformed variable z. We already know its takes the form:

$$z_t = \Lambda^t z_0 \quad \text{or} \quad \begin{pmatrix} z_t^1 \\ z_t^2 \end{pmatrix} = \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} \begin{pmatrix} v_0^1 \\ v_0^2 \end{pmatrix}.$$

Next, we reverse the transformation, this is, we obtain the dynamics of y : $y_t = Xz_t$. Therefore, we obtain:

$$y_t = Xz_t = X\Lambda^t z_0 = \begin{pmatrix} v_{1,1} & v_{2,1} \\ v_{1,2} & v_{2,2} \end{pmatrix} \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} \begin{pmatrix} z_0^1 \\ z_0^2 \end{pmatrix}.$$

Alternatively, multiplying the matrices and recalling that $y_t = \begin{pmatrix} c_t - \bar{c} \\ k_t - \bar{k} \end{pmatrix}$:

$$\begin{pmatrix} c_t - \bar{c} \\ k_t - \bar{k} \end{pmatrix} = z_0^1 \lambda_1^t \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} + z_0^2 \lambda_2^t \begin{pmatrix} v_{2,1} \\ v_{2,2} \end{pmatrix}.$$

The eigenvalues appear clearly in the solution. **Remember** that we know that one eigenvalue is bigger than one, while the second lies within the unit circle. Without lose of generality, assume that $\lambda_1 \in (-1, 1)$ and $\lambda_2 > 1$. According to the solution before, this implies an explosive behaviour: $\lambda_2 > 1 \implies \lim_{t \to +\infty} \lambda_2^t = +\infty$. To have a well-behaved dynamics we must impose $z_0^2 = 0$, which eliminates the explosive behaviour.

Technical remark: Denote by l the number of eigenvalues in the unit circle, and denote by m the number of pre-determinted state variables:

- if $l = m$ (standard case): saddle-path, *unique* optimal trajectory. The eigenvalues within the unit circle govern the speed of convergence.
- if $l < m$: unstable
- if $l > m$: multiple optimal trajectories.

After imposing $z_0^2 = 0$ the system becomes:

$$\begin{pmatrix} c_t - \bar{c} \\ k_t - \bar{k} \end{pmatrix} = z_0^1 \lambda_1^t \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix}.$$

Closing the model with initial values We know that the economy begins with a level of capital $k_{t=0} = k_0 > 0$. Substituting $t = 0$ in the equation above, and focusing on capital, we obtain:

$$k_0 - \bar{k} = z_0^1 v_{1,2} \implies z_0^1 = \frac{k_0 - \bar{k}}{v_{1,2}}.$$

Therefore, we can substitute the value for z_0^1 to obtain the dynamics of capital:

$$k_t - \bar{k} = z_0^1 \lambda_1^t v_{1,2} = \left(k_0 - \bar{k}\right) \lambda^t.$$

Similarly, we have that for consumption:

$$c_t - \bar{c} = z_0^1 \lambda_1^t v_{1,1} = \frac{v_{1,1}}{v_{1,2}} \lambda_1^t (k_0 - \bar{k}).$$

Finally, the initial value of consumption c_0 that puts the economy in the saddle path is obtained by setting $t = 0$ in the previous equation:

$$c_0 - \bar{c} = \frac{v_{1,1}}{v_{1,2}} (k_0 - \bar{k}).$$

We can now completely solve an example.

2.3 Utility and production functions

We assume that utility is logarithmic and we take a Cobb-Douglas production function.

$$u(c_t) = \log(c_t), \tag{22}$$

$$F(K_t, X_t) = A K_t^{\alpha} X_t^{1-\alpha}, \tag{23}$$

$$A > 0, \alpha \in (0,1). \tag{24}$$

$$ We can check that both functions satisfy the Inada conditions:

$$\lim_{c_t \to 0} u'(c_t) = \lim_{c_t \to 0} \frac{1}{c_t} = +\infty, \quad \lim_{c_t \to +\infty} u'(c_t) = \lim_{c_t \to +\infty} \frac{1}{c_t} = 0, \quad \lim_{K_t \to 0} F'_{K_t}(K_t, X_t) = \lim_{K_t \to 0} A \alpha K_t^{\alpha - 1} X_t^{1-\alpha} = +\infty, \quad K_t$$

The production function, expressed in intensive terms, becomes:

$$F(K_t, X_t) = X_T F\left(\frac{K_t}{X_t}, 1\right) = X_t f(k_t) = X_t k_t^{\alpha}, k_t \equiv \frac{K_t}{X_t}.$$

with total production $Y_t = F(K_t, X_t)$ and production per capita equal to $y_t \equiv \frac{Y_t}{X_t} = f(k_t)$.

2.4 Household's optimisation

Instead of using the results from previous sections, we develop again the utility maximisation. First, we write down the household's budget constraint:

2.4.1 Household's budget constraint

At each period, total received income is composed of *wages* and *interests*. It can be spend on *consumption* or *saving*. Remember that capital depreciates at a rate $\delta \in [0,1]$. Hence, we receive back from firms $1-\delta$ times the capital we lend. Therefore:

$$w_t + r_t k_t + (1-\delta)k_t = c_t + k_{t+1}.$$

2.4.2 Intertemporal utility maximistion: Lagrangean

The intertemporal utility maximisation problem is:

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{25}$$

$$\text{s.t.} \quad w_t + r_t k_t + (1-\delta)k_t = c_t + k_{t+1}, \tag{26}$$

$$c_t, k_{t+1} > 0, k_{t=0} = k_0 > 0. \tag{27}$$

$$

The Lagrangean becomes:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t (w_t + (r_t + 1 - \delta)k_t - c_t - k_{t+1}).$$

We use the first-order conditions with respect to c_t and k_{t+1} to find the optimal consumption path. In particular, in this step we shall obtain the Euler equation, which we combine latter with the transversality condition.

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \lambda_t = \beta^t \frac{1}{c_t} - \lambda_t = 0 \; \frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_t + \lambda_{t+1}(r_{t+1} + 1 - \delta) = 0.$$

Note: the term k_{t+1} appears within the summation at $t = t+1$. In fact expanding it reveals this fact clearly:

$$+\dots\lambda_t(w_t+(r_t+1-\delta)k_t-c_t-k_{t+1})+ \qquad +\lambda_{t+1}(w_{t+1}+(r_{t+1}+1-\delta)k_{t+1}-c_{t+1}-k_{t+2}+\dots$$

Combining both, we obtain the Euler equation:

$$\beta^t\frac{1}{c_t}=r_{t+1}\beta^{t+1}\frac{1}{c_{t+1}} \implies c_{t+1}=\beta(r_t+1-\delta)c_t.$$

Finally, we should remember to impose the transversality condition:

$$\lim_{t\to+\infty}\beta^t u'(c_t)k_{t+1}=\lim_{t\to+\infty}\beta^t\frac{1}{c_t k_{t+1}}=0.$$

2.5 Firm's optimisation

In this model, firms operate under pefect competition, making zero profits. Moreover, factors are paid their marginal productivity.

$$r_t=F'_{K_t}(K_t,X_t)=\alpha K_t^{\alpha-1}X_t^{1-\alpha}=\alpha\left(\frac{K_t}{X_t}\right)^{\alpha-1}=\alpha k_t^{\alpha-1}.\ w_t=F'_{X_t}(K_t,X_t)=(1-\alpha)K_t^{\alpha}X_t^{-\alpha}=(1-\alpha)\left(\frac{K_t}{X_t}\right)^{\alpha}$$

Alternatively, using the information in the [Appendix] and working directly with the intensive-form production function:

$$r_t=f'(k_t)=\frac{\partial f(k_t)}{\partial k_t}=\alpha k_t^{\alpha-1},\ w_t=f(k_t)-f'(k_t)k_t=k_t^{\alpha}-\alpha k_t^{\alpha-1}k_t=(1-\alpha)k_t^{\alpha}.$$

2.6 The dynamic system

We are now in a position to solve the model. We have the following equations:

$$c_{t+1}=\beta(r_{t+1}+1-\delta)c_t,\ w_t+r_tk_t+(1-\delta)k_t=c_t+k_{t+1},\ w_t=(1-\alpha)k_t^{\alpha},\ r_t=\alpha k^{\alpha-1},\ \lim_{t\to+\infty}\beta^t\frac{1}{c_t}k_{t+1}=0.$$

Substituing the wage and interest rate into the budget constraint allows us to retrieve the dynamics of capital:

$$c_{t+1} = \beta(\alpha k^{\alpha-1} + 1 - \delta)c_t,\ k_t^\alpha + (1-\delta)k_t = c_t + k_{t+1},\ \lim_{t\to+\infty} \beta^t \frac{1}{c_t} k_{t+1} = 0.$$

2.6.1 Steady state

Before trying to solve for the optimal trajectory, we analyse the steady states of this economy. A steady-state ($\bar{k}, \bar{c}$ is such that capital and consumption remain constant over time: $c_{t+1} = c_t$ and $k_{t+1} = k_t$. Applying this idea to the equations above we get —here we can temporarily forget about the transversality condition.

$$\bar{c} = \beta(\alpha \bar{k}^{\alpha-1} + 1 - \delta)\bar{c},\ \bar{k}^\alpha + (1-\delta)\bar{k} = \bar{c} + \bar{k}.$$

Hence, using the first equation we can easily get the steady level of capital:

$$\bar{k} = \left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}}.$$

Using this solution in the second equation yields the steady-state level of consumption:

$$\bar{c} = \left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}}.$$

2.7 Stability

First, we analyse the stability of the steady state, as we did in the [general case] Since we are using log-utility, the coefficient of relative risk aversion $\epsilon_c = 1$.

Instead of plugging the correct values in the Jacobian matrix $\bar{A}$, for the sake of clarity, we derive everything again:

$$\bar{A} = \begin{pmatrix} \frac{\partial c_{t+1}}{\partial c_t} & \frac{\partial c_{t+1}}{\partial k_t} \\ \frac{\partial k_{t+1}}{\partial c_t} & \frac{\partial k_{t+1}}{\partial k_t} \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} \beta\left(\alpha k_{t+1}^{\alpha-1} + 1 - \delta\right) - \frac{\beta c_t \alpha(\alpha-1)}{k_{t+1}^{2-\alpha}} & \frac{\beta c_t \alpha(\alpha-1)}{k_{t+1}^{2-\alpha}}\left(\alpha k_t^{\alpha-1} + 1 - \delta\right) \\ -1 & \alpha k_t^{\alpha-1} + 1 - \delta \end{pmatrix}$$

Evaluating it at the steady state:

$$\bar{A}\Big|_{\substack{k_t=k_{t+1}=\bar{k}\\ c_t=c_{t+1}=\bar{c}}} = \begin{pmatrix} 1-\beta\bar{c}\alpha(\alpha-1)\bar{k}^{\alpha-2} & \alpha(\alpha-1)\bar{c}\bar{k}^{\alpha-2} \\ -1 & \frac{1}{\beta} \end{pmatrix},$$

where we have used $1 = \beta\left(\alpha\bar{k}^{\alpha-1} + 1 - \delta\right)$ at the steady state.

We can use the fact $\mathrm{tr}(\bar{A}) = \lambda_1 + \lambda_2, \det(\bar{A}) = \lambda_1\lambda_2$. Therefore:

$$\lambda_1\lambda_2 = \frac{1}{\beta}$$

$$\lambda_1 + \lambda_2 = \frac{1}{\beta} + 1 - \beta\bar{c}\alpha(\alpha-1)\bar{k}^{\alpha-2}.$$

The first equation implies that either a) $\lambda_1 > 0, \lambda_2 > 0$ or b) $\lambda_1 < 0, \lambda_2 < 0$. However, since $\lambda_1 + \lambda_2 > 1$ b) is impossible. Then, $\lambda_1 > 0, \lambda_2 > 0$. Substitute $\lambda_1\lambda_2 = \frac{1}{\beta}$ in the second equation:

$$\lambda_1 + \lambda_2 = 1 + \lambda_1\lambda_2 - \beta(\alpha(\alpha-1)\bar{k}^{\alpha-2}\frac{\bar{c}}{\epsilon_{\bar{c}}}.$$

Rearranging:

$$\lambda_1 + \lambda_2 - \lambda_1\lambda_2 = 1 + -\beta\alpha(\alpha-1)\bar{k}^{\alpha-2}\frac{\bar{c}}{\epsilon_{\bar{c}}} > 1 \implies \lambda_1 + \lambda_2 - \lambda_1\lambda_2 - 1 > 0.$$

Factorisation leads to:

$$(1-\lambda_2)(\lambda_1-1) > 0.$$

Therefore:

$$\lambda_2 < 1$$

$$\lambda_1 > 1$$

$$\lambda_1 > 0, \lambda_2 > 0$$

and **the steady state is saddle.**

2.8 Trajectory

Describing the trajectory around the steady state analitycally would be cumbersome. Introducing an alternative production function of the Ak family alleviates this problem, see the notes by

Instead, we solve this section numerically. In particular, assume that $\alpha = 0.3$,

, $\beta = 0.9$,

, $\delta = 0.1$. The initial level of capital $k_{t=0} = 1$.

With these values, we have the following:

$$\bar{k} = 1.65202, , \bar{c} = 0.997329, , \bar{A} = \begin{pmatrix} 1.08029 & -0.089214 \\ -1 & 1.11111 \end{pmatrix}.$$

The eigenvalues of the matrix $\bar{A}$ are

$$\lambda_1 = 0.796618 \in (-1, 1)$$

$$\lambda_2 = 1.39479 > 1.$$

One possible set of eigenvectors is given by:

$$v_1 = (0.314494, 1), v_2 = (-0.283675, 1)$$

which are associated to λ_1 and λ_2, respectively. Hence, the matrix

$$X = \begin{pmatrix} 0.314494 & -0.283675 \\ 1 & 1 \end{pmatrix}.$$

2.8.1 Solving the system

Applying the results from before the system is:

$$\begin{pmatrix} c_t - \bar{c} \\ k_t - \bar{k} \end{pmatrix} = z_0^1 0.796618^t \begin{pmatrix} 0.314494 \\ 1 \end{pmatrix} + z_0^2 1.39479 \begin{pmatrix} -0.283675 \\ 1 \end{pmatrix}.$$

Since $\lambda_2 > 1$ we must set the value $z_0^2 = 0$ to avoid an explosive behaviour. Consequently:

$$\begin{pmatrix} c_t - \bar{c} \\ k_t - \bar{k} \end{pmatrix} = z_0^1 0.796618^t \begin{pmatrix} 0.314494 \\ 1 \end{pmatrix}.$$

We can solve the equation first by setting $t = 0$ to obtain the value z_0^1 :

$$\underbrace{k_0}_{=1} - \underbrace{\bar{k}}_{=1.65202} = z_0^1 \times 1 = -0.652017.$$

Consequently, the dynamic equation for capital becomes:

$k_t - \bar{k} = -0.652017 \times 0.796618^t$

And for consumption:

$c_t - \bar{c} = -0.652017 \times 0.314494 \times 0.796618^t = -0.205055 \times 0.786618^t$. Finally, we solve for the initial level of consumption compatible with the saddle path. For $t = 0$ we have:

$c_0 - \underbrace{\bar{c}}_{=0.997329} = \underbrace{z_0^1}_{=-0.652017} \times 0.314494 \implies c_0 = 0.792273.$

2.9 Optimality

Before [we claimed] that the solution is *optimal* despite the fact that it does not maximise consumption at the steady state. However, what we claimed was that the *entire path* was optimal. We can proof it solving the Ramsey allocation problem from the perspective of the social planner.

2.9.1 Social planner

A social planner maximises total welfare, given by:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t).$$

The social planner directly allocates resource between consumption and investment, so he is not bothered by wages or interest rates. Hence, his intertemporal budget constraint reads. In particular, he has to allocate total production and capital net of depreciation between c_t and k_{t+1}.

$$f(k_t) = c_t + k_{t+1} + (1 - \delta)k_t.$$

He operates under a certain level $k_0 > 0$ given.

Note how the budget constraint is equivalent to the one we found in the competitive equilibrium. Moreover, total welfare coincides, too. Hence, the solutions will be the same, proving that the competitive equilibrium was optimal.

The two main equations of the model are relevant for the planner as well. First, he must follow the Euler equation. In fact, he can reduce present consumption by Δc at time t and save it. The present-day cost is $u'(c_t)\Delta_c$. He obtainins $f'(k_t)\Delta_c + (1-\delta)\Delta_c$ at time $t+1$, which raises utility by $(f'(k_t)\Delta_c + (1-\delta)\Delta_c)\, u'_{c_{t+1}}$. Optimality implies that this trade off is not possible anymore, hence, we obtain the Euler equation again.

Secondly, although the planner directly allocates consumption and investments, he must follow the technological constraint displayed above. This does not represent preferences, and coincides with the househld's budget constraint after clearing the markets.

Finally, observe that the *first welfare theorem* directly applies: if markets are competitve and there re no externalities, then the decentralised equilibrium is Pareto-efficient. All conditions hold in the Ramsey model.

2.10 The Golden rule

We have discussed before that, in the Ramsey model, the economy converges to a steady state with $\bar{k}$ below the golden-rule level of capital $k^{\mathcal{GR}}$.

This constrasts with the Solow model: in the Solow model, a sufficietly high saving rate causes capital over-accumulation. This opens the possibility of finding alternative paths that yield higher consumption levels in each and every period —meaning that such a saving rate was not Pareto-efficient.

In the Ramsey model, savings are optimally derived from the utility maximisation, and the level of utility depends on consumption. Moreover, the model features *no* externalities. Consequently, the model cannot produce a situation whereby it is possible to increase consumption at each and every period: this contradicts optimisation.

However, $\bar{k} < k^{\mathcal{GR}}$ implies that it is possible to increase the saving rate today, build up the extra capital needed to reach $k^{\mathcal{GR}}$ and then be able to consume $c^{\mathcal{GR}} > \bar{c}$ at the steady state —which, remember, lasts forever. Suppose that we had reached the steady-state. Why do the households not change their saving rate to reach $c^{\mathcal{GR}}$? The reason is that households value present consumption more than future consumption. Therefore, the current utility cost of a reduction in consumption is larger the future gains once discounted. Because of the discount rate, the utility gains from an eventual permanent increase in consumption are bounded.

All in all, the economy converges to a value $\bar{k}$ below the steady state. Because $\bar{k}$ is the optimal level of k for the economy to converge to, it is known as the *modified golden-rule* capital stock.

We [had already noticed] that the discount factor β prevents the economy from reaching the golden-rule level of captal. According to our computation, the Golden-rule level of capital is only attained if $\beta = 1$. This implies that present and future consumption are valued the same, which complements the discussion above.