OLG

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2025-08-26

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1 Syllabus

This course provides a comprehensive introduction to the Overlapping Generations (OLG) model, one of the most versatile and influential frameworks in modern macroeconomics. Unlike the Ramsey model with infinitely-lived agents, the OLG framework captures the realistic life-cycle dynamics where individuals live for finite periods, making decisions about consumption, savings, and intergenerational transfers.

The OLG model is particularly powerful for analyzing issues that involve intergenerational considerations, such as social security systems, public debt sustainability, educational policies, and long-run economic development. The course combines theoretical foundations with applications to contemporary economic research, demonstrating how this framework helps us understand fundamental questions about economic growth, inequality, and policy design.

Through the study of cutting-edge research papers, we explore how economists use the OLG framework to investigate diverse phenomena from the evolution of social preferences to environmental sustainability.

1.1 Course Objectives

Upon completion of this course, students will be able to:

Theoretical Foundations: - Master the core OLG model with its key assumptions and derive competitive equilibria - Understand the fundamental differences between OLG and Ramsey models, particularly regarding dynamic efficiency - Analyze steady-state properties, stability conditions, and the possibility of multiple equilibria - Comprehend the golden rule and its welfare implications

Technical Skills: - Solve household optimization problems in a life-cycle context with perfect foresight - Characterize intertemporal equilibria and analyze their existence and uniqueness - Evaluate the welfare properties of competitive equilibria and identify potential market failures

Research Applications: - Critically analyze how contemporary researchers extend the basic OLG framework - Understand applications to cultural evolution, environmental economics, and development - Develop skills to formulate and solve original research questions using the OLG framework

1.2 Prerequisites

- Intermediate Microeconomics (consumer theory, optimization)
- Mathematical Methods for Economics (differential calculus, basic dynamic systems)
- Introduction to Macroeconomics
- Basic knowledge of econometrics is helpful but not required

1.3 Course Format

- Duration: 10 sessions (2 hours per session)

1.4 Assessment

Student evaluation consists of:

- **Final Examination (100%)**: Comprehensive exam covering theoretical concepts, model applications, and analysis of research papers discussed in class

1.5 Contact Information

- Instructor: Eric Roca (eric.roca_fernandez@uca.fr)
- Office Hours: By appointment

1.6 Bibliography

1.6.1 Core Textbooks

- Croix and Michel (2009)
- Campante, Sturzenegger, and Velasco (2021) Freely available online

1.6.2 Research Papers

- Diamond (1965)
- Galor and Moav (2006)
- Galor and Özak (2016)
- Croix and Dottori (2008)

2 The OLG model

2.1 Introduction

Based on Croix and Michel (2009).

We have studied the Ramsey model, which predicts that the path derived from the competitive equilibrium is optimal. One of the main features of this model is that agents have an infinite horizon: they live forever and optimise considering an infinite horizon.

The overlapping generations model changes this hypothesis and focuses on the life-cycle: agents make decisions regarding how to consume, and how much to save for retirement. This is, the OLG model assumes that agents work until some age, and then retire.[1] A focus of the OLG model is the intergenerational redistribution, allowing to study:

- social security,
- education policies and
- public debt.

The main departure with respect to the Ramsey model is that in OLG, agents are heterogeneous. Individuals live for two periods of time, and then die. In the first period, they are young and work. When old, they retire and live from savings. Hence, at any point in time, *two types of agents with **different** budget constraints exist*: young and adults.[2]

[1] Bisin and Verdier also model the case when parental indoctrination and how it affects the evolution of cultural traits.

[2] The assumption $\ln(R^1) > 2\ln(R^0)$ is important to establish that $\hat{\beta} \in (0, 1)$.

As we shall see, in this model the competitive path *may not be optimal.* Consequently, there may be instances in which the utility of all individuals can be increased. Therefore, the OLG model opens the door to government intervention, to reallocate consumption and savings efficiently. The OLG framework also permits the existence of bubbles and fluctuations.

A basic reference for this model is Diamond (1965) Numerical exercises Sample exam

2.2 Preliminaries

In this model, time is discrete and extends from $t = 0, 1, \dots, \infty$. Individuals make decisions at points in time. We shall have initial conditions detailing the state of the economy at $t = 0$.

2.2.1 Individuals live for two periods

The main difference with respect to the Ramsey model is that in the OLG model, individuals live for two periods. **Note:** this means that, at every point in time, two generations are alive and overlap.

This is relevant: the economy goes on forever but individuals only operate during some periods. Hence, there will be infinite two-period-lived generations. In particular, at $t = 0$, we will have a young and an adult generation. This adult generation will die at the end of $t = 1$, the young generation will become adults and have children: the *new* young generation of $t = 2$. Hence, we can represent the generations diagrammatically —in brackets I have denoted the year in which each generation was born.

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
Old (t=-1)	Die				

$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
Young(t=0)	Old (t=0)	Die			
Young(t=1)	Old (t=1)	Die			
	Young(t=2)	Old (t=2)	Die		
		Young(t=3)	Old (t=3)	Die	
			Young(t=4)	Old (t=4)	
				Young(t=5)	

To simplify the model, we assume that each adult born in $t-1$ has $n > -1$ children. **Note:** we assume the number of children to be constant. More complex set-ups include endogenous fertility. These are the young population at time t Therefore, the total population N at time t is composed of adults and young people.

$$N_t = \underbrace{N_{t-1}}_{\text{Adults}} + \underbrace{N_{t-1}n}_{\text{Youngs}} = N_{t-1}(1+n).$$

The total population at any time t is:

$$N_t = N_0(1+n)^t.$$

2.3 Assumptions

2.3.1 Firms

We assume that a large number of identical firms populate the economy. Firms produce a single, homogeneous good using labour and capital. The production function $F(K, L)$ has the following properties:

Assumption OLG 1: The production function satisfies the following properties:

- **OLG 1.1** $F(K, L)$ is continuous and defined on $[0, +\infty)^2$,
- **OLG 1.2** $F(K, L)$ has continuous derivatives of every required order on $(0, +\infty)^2$,
- **OLG 1.3** The production function is strictly increasing in both arguments: $F_i(K, L) > 0$,
- **OLG 1.4** The production function is strictly concave:

$$\begin{aligned}&F_{ii}(K, L) < 0\\&F_{ii}(K, L)F_{jj}(K, L) - F_{ij}(K, L)F_{ji}(K, L) > 0\end{aligned}$$

- **OLG 1.5** $F(K, L)$ is homogeneous of degree one.
- **OLG 1.6** $F(K, L)$ satisfies the Inada conditions:

$$\begin{gathered}\lim_{K\to 0} F'_K(K, L) = \lim_{L\to 0} F'_L(K, L) = +\infty\\ \lim_{K\to +\infty} F'_K(K, L) = \lim_{L\to +\infty} F'_L(K, L) = 0.\end{gathered}$$

Note: Assumption OLG 1.4 implies that the Hessian matrix of the production function is negative definite, hence the function is strictly concave.

Firms maximise real profits. Since there are many firms competing, in equilibrium they make exactly zero profits. Moreover, in equilibrium factors are paid their marginal productivity. Since the production function F is homogeneous of degree, we can write it in *intensive* terms:

$$f(k) \equiv F\left(\frac{K}{L}, 1\right), \quad k \equiv \frac{K}{L}. \tag{2.1}$$

Because markets are competitive, capital earns its marginal product $\partial F(K, L)/\partial K$ or, equivalently, $f'(k)$ in intensive terms. Thus, the real interest rate at time t is:

$$r_t = f'(k_t). \tag{2.2}$$

The marginal product of labour is given by $\partial F(K, L)/\partial L$. In intensive terms, it is equal to:

$$w_t = f(k) - f'(k)k. \tag{2.3}$$

2.3.2 Households

Individuals live for two periods. As before, we assume perfect foresight for individuals. **Assumption OLG 2** Individuals have perfect foresight.

When **young**, they are endowed with *one unit of labour* that they *supply inestastically.* **Assumption OLG 3** Individuals supply one unit of labour inelastically when young. They receive the ongoing wage rate w_t and allocate this income between:

- current consumption c_t,
- savings s_t that are invested in the firms.

Therefore, the budget constraint of a *young* individual in period t is:

$$w_t = c_t + s_t.$$

Once an individual reaches old age the next period, he consumes his savings (plus the interest rate received), reproduces —exogenous fertility at rate n— and dies. Old people *do not care* about anything happening after death. Therefore, an agent has one unique choice:

- consumption when adult, d_{t+1}.

The budget constraint for this period is:

$$s_t(1 - \delta + r_{t+1}) = d_{t+1}.$$

with $\delta \in (0,1)$ being the capital depreciation rate.

Hence, an individual faces two budget constraints. However, we can collapse both into a unique *intertemporal budget constraint.*

2.3.2.1 The intertemporal budget constraint

In the economy, we have consumption as the numeraire. It is more convenient for us to combine the two budget constraints corresponding to young and old ages into one single constraint. Starting from

$$\begin{cases} w_t = c_t + s_t \\ d_{t+1} = s_t(1 - \delta + r_{t+1}) = s_t R_{t+1} \end{cases} \tag{2.4}$$

where $R_t \equiv 1 - \delta + r_{t+1}$ represents the return on savings, isolate s_t in the second equation and plug it in the first one:

$$w_t = c_t + \frac{d_{t+1}}{R_{t+1}}. \tag{2.5}$$

The intertemporal budget constraint indicates that the total present value of income (w_t, the only source of income) equals the total present value of expenditures. The present value of consumption when old d_{t+1} is discounted using the interest rate R_{t+1}.

It is clear that savings, as usual, will be a function of wages w and interests R. So will consumption at all periods of time.

2.3.2.2 Utility function

We suppose that the life-cycle utility function is additively separable:

$$U(c, d) = u(c) + \beta u(d),\ \beta \in (0, 1) \tag{2.6}$$

where $\beta \in (0, 1)$ is the psychological discount factor. We assume that $u(c)$ has the properties

Assumption OLG 4

- **OLG 4.1** $u'(c) > 0$,
- **OLG 4.2** $u''(c) < 0$,
- **OLG 4.3** $\lim_{c \to 0} u'(c) = +\infty$.

The last assumption $\lim_{c \to 0} u'(c) = +\infty$ implies that an individual will always have a positive consumption —as long as he has enough income to finance it.

Another important implication of the choice of the utility formulation is that c and d are normal goods: the demand is *not decreasing* in wealth. It follows from additive separability and concavity.

2.3.3 The behaviour of individuals

At time t, young individuals receive their wages, consume and save while maximising the utility function.

$$\begin{aligned} &\max u(c_t) + \beta u(d_{t+1}) \\ \text{s.t.} \quad & w_t = c_t + s_t \\ & d_{t+1} = R_{t+1} s_t \\ & c_t \geq 0, d_{t+1} \geq 0. \end{aligned} \tag{2.7}$$

We have two possibilities to solve this problem:

2.3.3.1 Substitution

First, we can substitute c_t and d_{t+1} in the utility function, leading to:

$$u(w_t - s_t) + \beta u(R_{t+1} s_t).$$

This function is strictly concave with respect to s_t because of our assumptions. The solution is *the savings function*:

$$s_t = s(w_t, R_{t+1}).$$

The solution is interior as a consequence of the assumptions, and it is characterised by the first-order condition:

$$u'(w_t - s_t) = \beta R_{t+1} u'(R_{t+1} s_t). \tag{2.8}$$

2.3.3.2 Lagrangian

Instead, we can use the intertemporal budget constraint and build the Lagrangian:

$$\mathcal{L} = u(c_t) + \beta u(d_{t+1}) + \lambda_t (w_t - c_t - \frac{d_{t+1}}{R_{t+1}}).$$

The first order conditions imply that:

$$u'(c_t) = \lambda_t, \quad \beta u'(d_{t+1}) = \frac{\lambda_t}{R_{t+1}}.$$

Combining both, we obtain Equation 2.8 again:

$$u'(c_t) = \beta R_{t+1} u'(d_{t+1}).$$

2.4 Intertemporal elasticity of substitution

The intertemporal elasticity of substitution is defined as:

$$\epsilon_{d_{t+1},c_t} = \frac{\partial\left(\frac{d_{t+1}}{c_t}\right)}{\partial R_{t+1}} \frac{R_{t+1}}{\left(\frac{d_{t+1}}{c_t}\right)}. \tag{2.9}$$

From the Euler equation we have $\frac{u'(c_t)}{u'(d_{t+1})} = \beta R_{t+1}$. So, consumption when old d_{t+1} is going to be proportional to consumption when young c_t.

Let $d_{t+1} = x_{t+1}c_t$. Then, the intertemporal elasticity of substitution becomes:

$$\epsilon_{d_{t+1},c_t} = \frac{\partial x_{t+1}}{\partial R_{t+1}} \frac{R_{t+1}}{x_{t+1}}.$$

We can compute it using the Euler equation $u'(c_t) = \beta R_{t+1} u'(x_{t+1}c_t)$. In fact, totally differentiating on both sides yields

$$0 = \beta u'(x_{t+1}c_t)\mathrm{d}R_{t+1} + \beta R_{t+1} u''(x_{t+1}c_t)c_t \mathrm{d}x_{t+1}.$$

Hence,

$$\frac{\mathrm{d}x_{t+1}}{\mathrm{d}R_{t+1}} = -\frac{u'(x_{t+1}c_t)}{R_{t+1}u''(x_{t+1}c_t)c_t} = -\frac{u'(d_{t+1})}{R_{t+1}u''(d_{t+1})c_t}.$$

The intertemporal elasticity of substitution is then given by:

$$\epsilon_{d_{t+1},c_t} = \frac{\mathrm{d}x_{t+1}}{\mathrm{d}R_{t+1}} \frac{R_{t+1}}{x_{t+1}} = -\frac{u'(d_{t+1})}{R_{t+1}u''(d_{t+1})c_t} \frac{R_{t+1}}{\underbrace{x_{t+1}}_{\frac{d_{t+1}}{c_t}}} = -\frac{u'(d_{t+1})}{u''(d_{t+1})d_{t+1}}.$$

Note: the intertemporal elasticity of substitution is equal to the inverse of the coefficient of relative risk aversion. Risk averse individuals do *not* want large changes in consumption, so they have a large risk aversion and are *not* willing to trade off future and present consumption. Conversely, risk neutral individuals do not mind abrupt changes in consumption, so they are willing to trade off future and present consumption.

Hence, in this model, the intertemporal elasticity of substitution coincides with the inverse of the coefficient of relative risk aversion. **Note:** If you read Croix and Michel (2009, 12) , they take a different approach and also refuse to use the concept or risk aversion: the model does not feature risk.

2.4.1 CIES example

We illustrate the previous concept using a constant intertemporal elasticity of substitution utility function:

$$u(c) = \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}.$$

In this case, we have the following Euler condition:

$$u'(c_t) = \beta R_{t+1} u'(d_{t+1}) \implies \frac{d_{t+1}}{c_t} = (\beta R_{t+1})^{\sigma}.$$

Therefore, the intertemporal elasticity of substitution is

$$\epsilon_{d_{t+1},c_t} = \frac{\partial \frac{d_{t+1}}{c_t}}{\partial R_{t+1}} \frac{R_{t+1}}{\frac{d_{t+1}}{c_t}} = \sigma(\beta R_{t+1})^{\sigma-1} \beta R_{t+1} \frac{1}{(\beta R_{t+1})^{\sigma}} = \sigma.$$

Meanwhile, the coefficient of relative risk aversion

$$RRA(c) = -\frac{u''(c)}{u'(c)}c = \frac{1}{\sigma} = \epsilon^{-1}_{d_{t+1},c_t}$$

is the inverse of the intertemporal elasticity of substitution.

2.5 The savings function

The savings function arises by solving:

$$s(w, R) = \arg\max \left[u(w - s) + \beta u(Rs)\right].$$

Taking the first derivative with respect to s and solving provides an implicit function: the savings function.

$$u'(w - s)(-1) + \beta u'(Rs)R = 0.$$

Denote this function $\phi(s, w, R)$:

$$\phi(s, w, R) = -u'(w - s) + \beta R u'(Rs) = 0.$$

Following from the assumption on the utility function, the savings function is continuous and continuously differentiable.

We are interested in determining whether savings increase or decrease with wealth and the interest rate.

2.5.1 The effect of wages

We begin by analysing the effect of wages on savings:

$$\frac{\partial s}{\partial w} = -\frac{\frac{\partial \phi(\cdot)}{\partial w}}{\frac{\partial \phi(\cdot)}{\partial s}} = -\frac{-u''(w-s)}{u''(w-s) + \beta R^2 u''(Rs)} = \frac{1}{\underbrace{1 + \beta R^2 \underbrace{\frac{u''(Rs)}{u''(w-s)}}_{>0}}_{>1}} \in (0,1).$$

We thus have that the marginal propensity to save out of income is between 0 and 1: $0 < s'_w < 1$, which reflects the fact that consumption goods are normal goods.

2.5.2 The effect of the interest rate

Similarly, we compute the derivative of $\frac{\partial s(w,R)}{\partial R}$:

$$\frac{\partial s}{\partial R} = -\frac{\beta u'(d) + \beta R s u''(d)}{u''(c) + \beta R^2 u''(d)} = -\frac{\beta u'(d)\left[1 - \frac{1}{\sigma(d)}\right]}{u''(c) + \beta R^2 u''(d)}.$$

Hence,

$$\frac{\partial s}{\partial R} \gtreqless 0 \quad \text{if} \quad \sigma(d) \gtreqless 1.$$

Two effects interact in this derivative:

- Wealth effect: if the interest rate rises, we obtain more out of the same savings, hence become wealthier. We consume more of all goods —c_t and d_{t+1} are normal goods.
- Substitution effect: it is more profitable to save and consume d_{t+1}, inducing savings.

When the inter-temporal elasticity of substitution is lower than 1, the substitution effect is dominated by the income effect. In that case, a rise in the rate of return has a negative effect on savings. When the inter-temporal elasticity of substitution is higher than 1, households are ready to exploit the rise in the remuneration of savings by consuming relatively less today. In this case, raising the rate of return boosts savings. When the inter-temporal elasticity of substitution is equal to 1 (log-utility), the income effect exactly compensates the substitution effect and there is no effect of the rate of return on savings.

2.5.2.1 Example using a CIES function

Under a CIES utility, we have that:

$$s(w, R) = \frac{1}{1 + \beta^{-\sigma} R^{1-\sigma}} w,$$

and

$$s'_w = \frac{1}{1 + \beta^{-\sigma} R^{1-\sigma}} > 0.$$

However, $s'_R = -\frac{w\beta^{-\sigma} R^{-\sigma}}{(1+\beta^{-\sigma} R^{1-\sigma})^2}(1-\sigma) \gtreqless 0$ depending on $\sigma \lesseqgtr 1$.

Log-utility:

In the case of logarithmic utility, savings are independent of the interest rate R and are linear in wages. Log-utility is a special case of the CIES function when $\sigma = 1$. In this case, the wealth and substitution effects cancel each other:

$$s(w, R) = \frac{\beta}{1 + \beta} w.$$

2.5.2.2 Example using a CES function

Note: This is a transformation of the CIES function, so things will look rather similar.

$$U(c,d) = \left[\alpha c^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha) d^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}.$$

From here, substituting $c = w - s$ and $d = Rs$, and solving for the optimal savings we obtain:

$$\begin{aligned}
&\alpha(w-s)^{-\frac{1}{\epsilon}} = (1-\alpha)R(Rs)^{-\frac{1}{\epsilon}} \\
&s = \frac{\left[\frac{\alpha}{1-\alpha}\right]^{-\epsilon} w}{R^{1-\epsilon} + \left[\frac{\alpha}{1-\alpha}\right]^{-\epsilon}} = \frac{w}{1 + \left[\frac{\alpha}{1-\alpha}\right]^{\epsilon} R^{1-\epsilon}}.
\end{aligned}$$

Therefore,

$$s'_R = -\frac{w}{\left(1 + \left[\frac{\alpha}{1-\alpha}\right]^{\epsilon} R^{1-\epsilon}\right)^2} \left(\frac{\alpha}{1-\alpha}\right)^{\epsilon} (1-\epsilon) R^{-\epsilon}.$$

Then, when $\epsilon > 1$ savings increase with the interest rate: $s'_R > 0$. In that case, *the substitution effect dominates.* **Alternative interpretation:** ϵ is the elasticity of intertemporal substitutability, hence if it is large, individuals are willing to substitute present consumption for future consumption. **Alternative interpretation:** $\frac{1}{\epsilon}$ measures the relative risk aversion. Hence, for large ϵ individuals accept more risk, meaning that they are willing to accept more changes in consumption. Smoothing is less important.

Conversely, when $\epsilon < 1$, an increase in the interest rate lowers savings: $s'_R < 0$. In that case, the wealth effect dominates. $\epsilon < 1$ also means that c and d are intertemporally complementary, and individuals want to consume them in constant proportions.

2.6 Temporary equilibrium

Before turning to the intertemporal equilibrium and the analysis of the steady state, we study the temporary equilibrium that takes place every period.

We have not discussed the firms, but they follow the same setup as in Ramsey: use capital and labour in a perfectly competitive environment.

We shall work in intensive form: $k_t \equiv \frac{K_t}{N_t}$.

1. **Labour market equilibrium:** *Only* young individuals supply labour. Moreover, they do so *inelastically.* During period t there are N_t young agents and, hence, the supply of labour is N_t. Equating this to labour demand from firms L_t gives the wage rate:

$$w_t = \omega\left(\frac{K_t}{N_t}\right) = \omega(k_t).$$

2. **Capital market:** *Only* old individuals own capital. Since firms operate competitively, they make zero profits. Hence, $f'(k_t)K_t$ is distributed as interests. Old households receive $N_{t-1}R_t s_{t-1} = R_t K_t$. So, what old households receive must equal what firms distribute In other terms: $R_t K_t = f'(k_t)K_t$ and $R_t = f'(k_t)$.

3. **Good market:** Finding the equilibrium for this market departs from the otherwise similar Ramsey case. Remember that we have *two* types of agents: young and old. Total production is given by:

$$Y_t = F(K_t, N_t) = N_t f(k_t).$$

   Total demand for goods combines the consumption of d_t, the old generation living in period t, and the demand for consumption *and* investment from the young: c_t, s_t. Therefore, counting how many old and young individuals live during period t we have that total demand equals:

$$N_{t-1}d_t + N_t(c_t + s_t).$$

The equilibrium on the goods market implies:

$$Y_t = N_{t-1}d_t + N_t(c_t + s_t).$$

Also, we can check that total production equals total consumption:

$$N_t(c_t + s_t) = N_t w_t = N_t(f(k_t) - k_t f'(k_t)) = Y_t - K_t f'(k_t),$$

$$N_{t-1}d_t = N_{t-1}R_t s_{t-1} = R_t K_t = K_t f'(k_t).$$

A temporary equilibrium is a set
$w_t, R_t, K_t, L_t, Y_t, k_t, I_t, c_t, s_t, d_t$ that satisfies:

$$\begin{aligned}
w_t &= \omega(k_t), \\
R_t &= f'(k_t), \\
L_t &= N_t, \\
Y_t &= N_t f(k_t), \\
Y_t &= N_{t-1}d_t + N_t(c_t + s_t) \\
I_t &= N_t s_t, \\
c_t &= w_t - s_t, \\
s_t &= s(\omega(k_t), R_{t+1}), \\
d_t &= R_t s_{t-1}.
\end{aligned}$$

The existence of a temporary equilibrium is guaranteed because the functions are single-valued.

2.7 Intertemporal equilibrium with perfect foresight

The equilibrium equation that links two consecutive periods is the capital accumulation equation. In particular, the savings of young individuals at period t are transformed into productive capital at $t+1$.

Note: in this model, it is useful to write the equations first in aggregate terms and then convert them to its intensive-form representation.

$$K_{t+1} = N_t s_t = N_t s\left(\omega(k_t), R_{t+1}\right).$$

In intensive form:

$$k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{N_t}{N_{t+1}} s\left(\omega(k_t), R_{t+1}\right) = \frac{1}{1+n} s\left(\omega(k_t), R_{t+1}\right).$$

Finally, we can incorporate the equilibrium in the capital market (together with perfect foresight) and replace $R_{t+1} = f'(k_{t+1})$:

$$k_{t+1} = \frac{1}{1+n} s\left(\omega(k_t), f'(k_{t+1})\right).$$

Intertemporal equilibrium (for perfect foresigt): Given an initial capital stock $k_0 = K_0 \big/ N_{-1}$, an intertemporal equilibrium (for perfect foresight) is a sequence of temporary equilibria that satisfies for all $t > 0$ the equation:

$$k_{t+1} = \frac{1}{1+n} s\left(\omega(k_t), f'(k_{t+1})\right).$$

Note: Croix and Michel (2009, 20–27) discuss on pp. 20–27 the existence and uniqueness of the intertemporal equilibrium.

2.7.1 Existence of an intertemporal equilibrium

The existence of at least one temporary equilibrium is guaranteed by the properties of the functions. The proof is quite involved, though, as is presented below. Having an intertemporal equilibrium means having a solution for k_{t+1} in the equation

$$k_{t+1} = \frac{1}{1+n} s(\omega(k_t), f'(k_{t+1})),$$

where k_t is predetermined at t.

Proof

The proof uses the following equation about savings:

$$0 < s\left(w, f'(k)\right) < w.$$

In words, it indicates that individuals have positive savings, and they save *only* part of their total income.
Next, define

$$H(k, w) = (1+n)k - s(w, f'(k)) = 0.$$

Basically, here we are using the definition of an intertemporal equilibrium. Having an intertemporal equilibrium is then equivalent to finding a k satisfying the previous equation.
We use the intermediate value theorem to show that at least one solution exists. First, we analyse the behaviour of $H(k, w)$ when k tends to $+\infty$.
From $0 < s\left(w, f'(k)\right) < w$, we have

$$0 < \frac{s(w, f'(k))}{k} < \frac{w}{k}.$$

Limit when $k \to +\infty$

Keeping w fixed, the limit of $\frac{w}{k}$ when $k \to +\infty$ is 0. Then,

$$\lim_{k \to +\infty} \frac{s(w, f'(k))}{k} = 0.$$

Consequently,

$$\lim_{k \to +\infty} \frac{H(w,k)}{k} = \lim_{k \to +\infty} (1+n) - \frac{s(w, f'(k))}{k} = 1 + n > 0.$$

Limit when $k \to 0$
When k tends to zero, we shall distinguish two cases regarding $f'(k)$. It can be that either

- Case $\lim_{k \to 0} f'(k) = f'(0) > 0$ and finite. Then, the function $s(w, f'(k))$ is well defined and positive and we have:

$$\lim_{k \to 0} H(w,k) = \lim_{k \to 0} [(1+n)k - s(w, f'(k))] = -s(w, f'(k)) < 0.$$

- Case $\lim_{k \to 0} f'(k) = +\infty$. Two things can occur in that case.
  - Savings are positive: $\lim_{k \to 0} s(w, f'(k)) > 0$. This case is analogous to the previous one:

$$\lim_{k \to 0} H(w,k) = \lim_{k \to 0} [(1+n)k - s(w, f'(k))] = -s(w, f'(k)) < 0.$$

  - Savings tend to zero: $\lim_{k \to 0} s(w, f'(k)) = 0$. In this case, since the interest rate goes to infinity individuals tend to save zero. This implies that consumption in the second period, d, tends to infinity. To see this, first notice that

$d = f'(k)s(w, f'(k))$. Moreover, from the first order conditions we know that:

$$u'(d) = \frac{u'(w-s)}{\beta f'(k)}$$

and hence

$$\lim_{k\to 0} u'(d) = \lim_{k\to 0} \frac{u'(w-s)}{\beta f'(k)} = 0.$$

Hence, $u'(d)$ tends to zero which means that d goes to infinity. Therefore:

$$\lim_{k\to 0} f'(k)s(w, f'(k)) = +\infty.$$

We also know that $0 < f'(k)k < f(k)$ because $f'(k)k + \omega(k) = f(k)$ and $\omega(k) > 0$. Therefore, for a bounded $0 < k < 1$ we also have $f'(k)k < f(1)$. Hence, for $k < 1$:

$$\frac{s(w, f'(k))}{k} = \frac{f'(k)s(w, f'(k))}{f'(k)k} > \frac{f'(k)s(w, f'(k))}{f(1)}.$$

Then,

$$\lim_{k\to 0} \frac{s(w, f'(k)}{k} = +\infty$$

and

$$\frac{H(w,k)}{k} = 1 + n - \frac{s(w, f'(k)}{k} < 0$$

for small k.

2.7.2 Uniqueness of the intertemporal equilibrium

Having established that at least one level k_{t+1} exists that solves the equation, we would like it to be unique, thus having a unique equilibrium. Otherwise, for some level k there would be multiple values for k_{t+1} (meaning multiple values for the savings). Individuals lack a coordinating mechanism to select any of these equilibria.

We show that the intertemporal elasticity of substitution σ is important in determining uniqueness. In particular, if $\sigma > 1$, then the equilibrium is unique. We will revisit this issue later.

We have used the intermediate value theorem to show that solutions exist. Moreover, it shows that $\lim_{k\to 0} \frac{H(w,k)}{k} < 0$ and $\lim_{k\to +\infty} \frac{H(w,k)}{k} > 0$. Therefore, if $H(w,k)$ is monotonically increasing in k, then there is one and only one solution to $H(w,k) = 0$. Hence, we would have a unique solution. This means that given a value k_t, we can find a unique valu k_{t+1}.

If we take the derivative of $H(w,k)$ with respect to k, we obtain:

$$1 + n - s_R' f''(k).$$

Consequently, in order to have a monotonous function it is sufficient to impose:

Assumption OLG.4 :

$$1 + n - s_R' f''(k) > 0.$$

And then, $k_{t+1} = g(k_t)$.

In particular, this condition indicates that the effect of the interest rate on savings should *not* be too negative.

Moreover, if **OLG.4** is verified, then k_{t+1} is increasing in k_t. Indeed,

$$k_{t+1} = g(k_t, k_{t+1}) = \frac{1}{1+n} s(\omega(k_t), R(k_{t+1})).$$

and

$$\frac{\partial k_{t+1}}{\partial k_t} = -\frac{\frac{\partial g(k_t, k_{t+1})}{\partial k_t}}{\frac{\partial g(k_t, k_{t+1})}{\partial k_{t+1}}} = \frac{\overbrace{s'_\omega \omega'(k_t)}^{>0}}{1+n-s'_R \underbrace{R'(k_{t+1})}_{=f''(k_{t+1})<0}} > 0,$$

and k_{t+1} increases in k_t.

> **!** Important
>
> Hence, if the function H is monotonously increasing, the solution is unique. This condition depends on the sign of:
>
> $$\frac{\partial H(k_{t+1})}{\partial k_{t+1}} = 1 + \frac{1}{1+n} s'_R f''(k_{t+1}).$$

Finally, we can use a more restrictive —but easier to work with— assumption that ensures a unique solution for the intertemporal equilibrium. Assumption **OLG.4** states:

$$1 + n - s'_R f''(k) > 0.$$

Therefore, it is sufficient to have

$$s'_R \geq 0$$

meaning that savings increase with the interest rate. This behaviour is readily verified when the intertemporal elasticity of substitution is at least one. Then, to have monotonic dynamics and unique temporal equilibrium it is sufficient to have $\sigma \geq 1$.

We have different cases:

1. $s'_R = 0$, which happens under log-utility. In this case, $s'_R = 0 > \frac{1+n}{R'(k_{t+1})}$ and $H(k_{t+1})$ is always increasing.
2. $s'_R > 0$, the intertemporal elasticity of substitution is greater than 1 and individuals are willing to trade off higher future consumption against present consumption. Savings increase to consume more in the future. Then, clearly $s'_R > 0 > \frac{1+n}{R'(k_{t+1})}$.
3. $s'_R < 0$, we can have a non-monotonous capital path as k_{t+1} can be increasing or decreasing with k_t.

2.7.3 Multiple Equilibria and Non-monotonous Dynamics

Under case 3, the condition for uniqueness is not satisfied, multiple levels of k_{t+1} solve the intertemporal equilibrium. This section is based on the Groth (2016, 92–93)

In that case, the function $g(k)$ is backwards bending for some values of k. This feature implies that there is more than one *intemporal equilibrium*. We ruled out such a possibility assuming that $\sigma(c) \geq 1$, this is, we imposed a large enough intertemporal elasticity of substitution.

For instance, take an isoelastic utility function $u(c) = \frac{c^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}}$ and a CES production with $A = 20$, $\alpha = \frac{1}{2}$, $\beta = 0.3$, $n = 1.097$, $\sigma = 0.1$, $\rho = -2$. First, since $\sigma < 1$, it allows for the possibility of having multiple temporary equilibria. Second, we can verify that this is the case. For example, if $k_t = 1$, then:

$$k_{t+1} = \frac{1}{1+n} s(\omega(k_t), R(k_{t+1})) \implies k_{t+1} = \begin{cases} 0.26 \\ 1.234 \\ 5.832 \end{cases} .$$

In fact, for the entire range $k_t \in [0.76, 1.37]$ there is multiplicity of equilibria. Consequently, in all these cases we also have that $\frac{\mathrm{d}k_{t+1}}{\mathrm{d}k_t} < 0$.

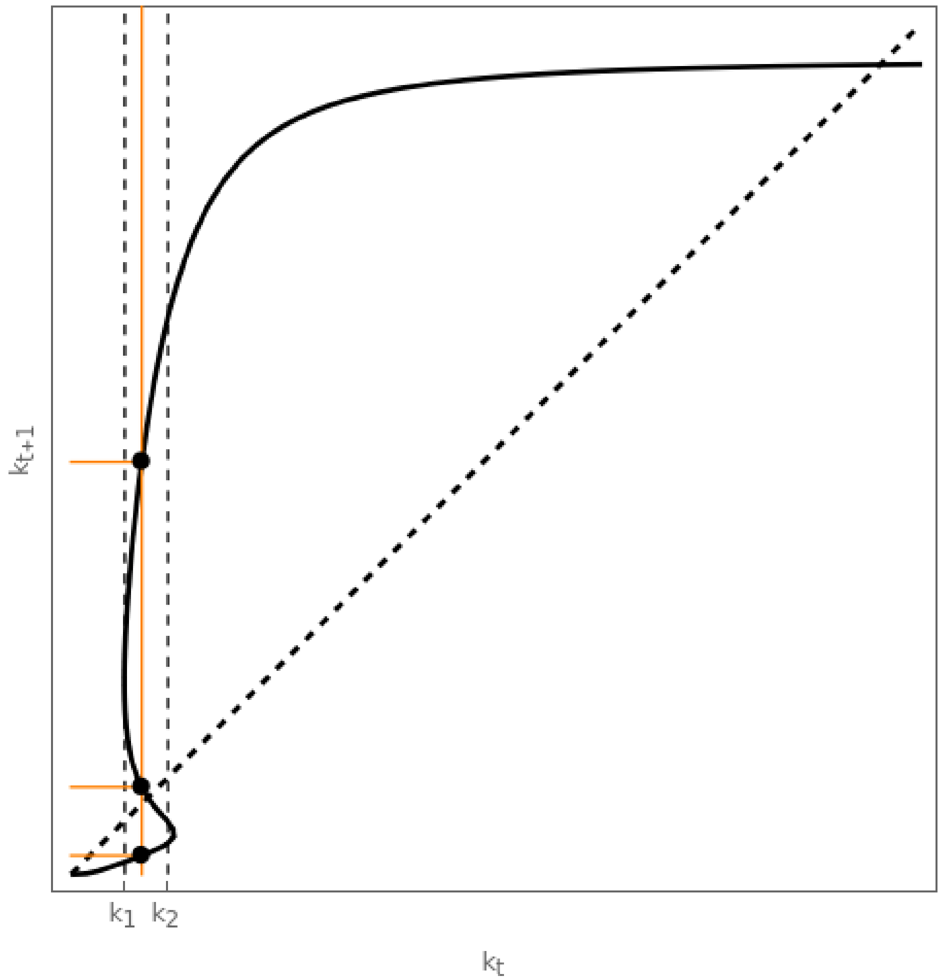


Figure 2.1: Multiple equilibria

This implies that, when young individuals observe the current capital level k_t, they may expect different future capital levels k_{t+1} depending on their beliefs about the economy: some individuals will expect $k_{t+1} = 0.26$, others $k_{t+1} = 1.234$, and still others $k_{t+1} = 5.832$. Consequently, the capital that will be realized will, in most cases, not coincide with any of these expectations. This violates the *perfect foresight* assumption, as individuals are not able to accurately predict future capital levels based on current information.

2.8 Steady states

We assume that the intertemporal equilibrium exists and is unique. As noted when characterising the uniqueness of the temporary equilibrium, assuming that $\sigma > 1$ is a sufficient condition. Therefore, the dynamics of capital are given by:

$$k_{t+1} = \frac{1}{1+n} s(\omega(k_t), f'(k_{t+1})).$$

Hence, k_{t+1} is an implicit function of k_t:

$$k_{t+1} = g(k_t).$$

The steady states of this economy can be found solving the previous equation for $k_{t+1} = k_t = \bar{k}$.

$$\bar{k} = \frac{1}{1+n} s\left(\omega(\bar{k}), f'(\bar{k})\right).$$

2.8.1 Steady state with $\bar{k} = 0$

A steady state with $\bar{k} = 0$ is only possible if $f(0) = 0$ or, equivalently, $\omega(0) = 0$. We may call this steady state an *autarky steady state.* In that case, since $\omega(0) = 0$ and we know that individuals save a fraction of their income:

$$s(\omega(0), R(0)) < \omega(0) = 0.$$

Hence, given zero capital, savings are equally zero and capital remains there, constituting a steady state.

Alternatively, if it is possible to produce without capital (CES production function with large enough substitutability), the autarkic steady state does not exist.

2.8.1.1 Example of the autarkic steady state

Take $F(K, L) = (\alpha K^\rho + (1-\alpha)L^\rho)^{\frac{1}{\rho}}$ and $u(c) = \log(c)$. Furthermore, assume a low elasticity of substitution, $\rho << 0$. In intensive form, production equals:

$$f(k_t) = (\alpha k_t^\rho + (1-\alpha))^{\frac{1}{\rho}}.$$

Wages and savings are:

$$\omega(k_t) = (1-\alpha)(\alpha k^\rho + (1-\alpha))^{\frac{1}{\rho}-1}$$
$$s_t = \frac{\beta}{1+\beta} w_t.$$

Then, capital accumulation becomes:

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)\left[\alpha k_t^{\rho} + 1 - \alpha\right]^{\frac{1-\rho}{\rho}}.$$

With low substitutability, $\rho < 0$, zero is always a possible steady state. There can be two additional, positive steady states depending on the curvature of k_{t+1}.

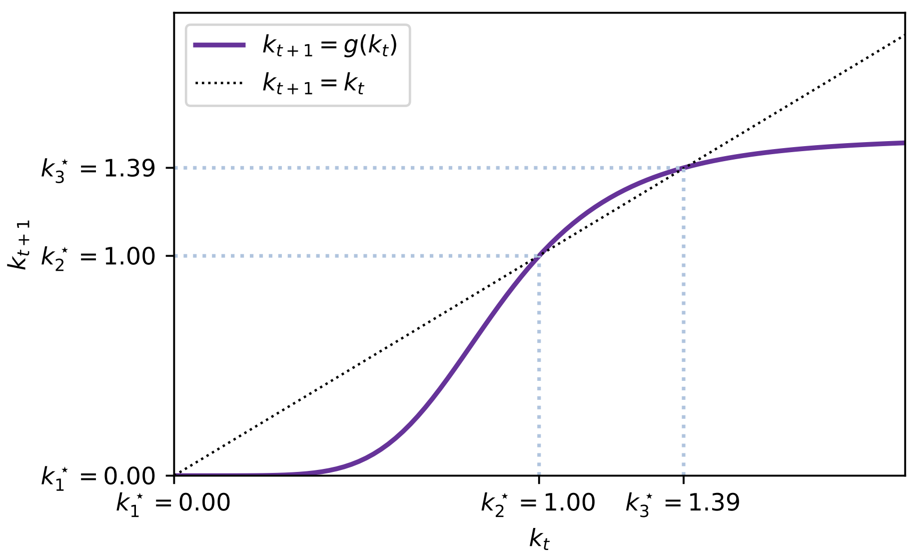


Figure 2.2: Autarky steady state and two positive steady states

Similarly, the Cobb-Douglas case $\rho \to 0$ also has zero as a steady state. However, it is unstable.

2.8.2 Other steady state

The economy can also present interior steady states with $\bar{k} > 0$. Unlike the Ramsey model, here we can have stable and unstable steady states, as we shall see. Moreover, the configuration varies depending on the production and utility functions.

Two steady states, autarky and positive This is a common configuration that arises, for instance, under a Cobb-Douglas production function and log-utility. Other functions can also exhibit this behavior, particularly those with similar functional forms.

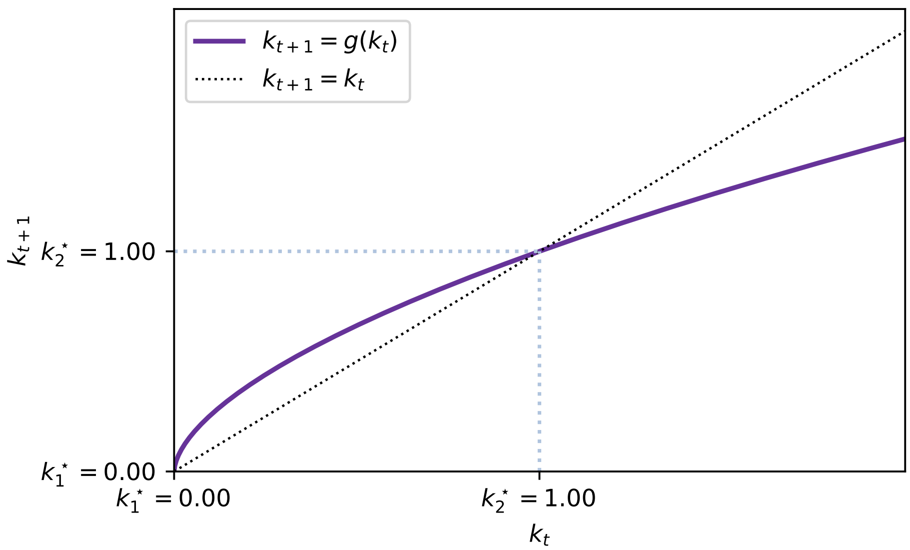


Figure 2.3: Autarky steady state and a positive steady state

Single, non-autarky steady state This steady state is characterized by a unique positive capital level $\bar{k} > 0$. This configuration requires $f(0) > 0$. With a CES production function, capital and labour must be substitutes, this is, $< 0\rho < 1$: $(\alpha k^\rho + (1-\alpha))^{\frac{1}{\rho}}$.

Only an autarky steady state In this case, $f(0) = 0$ is required so that the autarky steady state can exist. Moreover, it requires $f(k_{t+1}) < f(k_t)$. This type of setup requires $\rho < 0$, this is, complementarity between capital

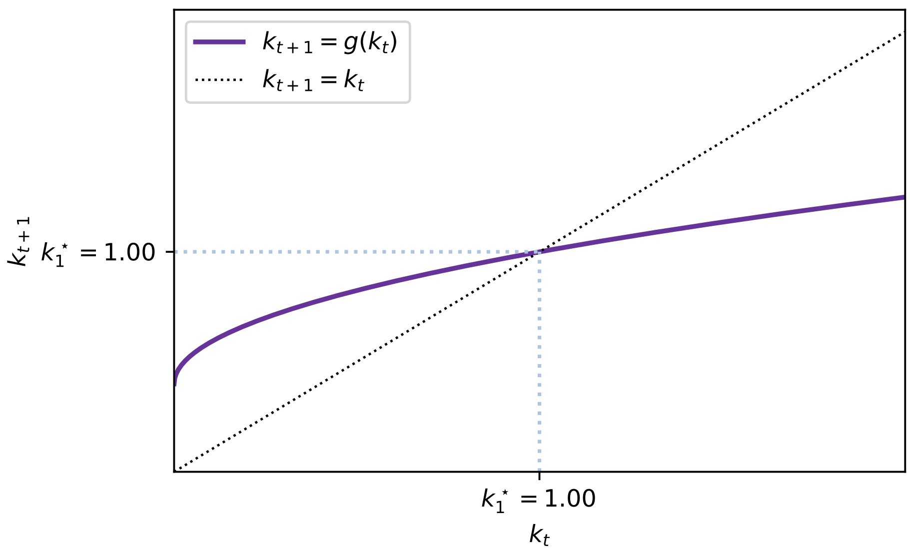


Figure 2.4: Single, non-autarky steady state

and labor and low levels of output given the inputs.

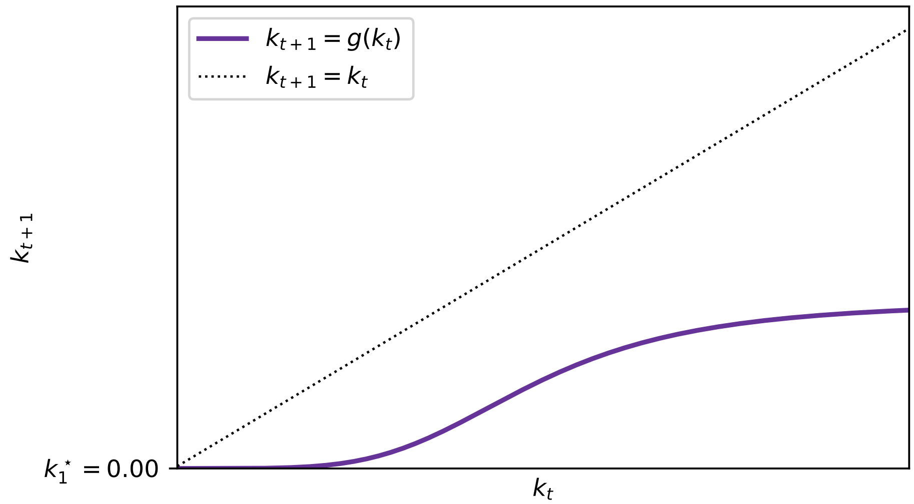


Figure 2.5: Only an autarky steady state

2.9 Stability of the steady state

This model can feature one or several steady states. This depends on the characteristics of the production and utility functions.

First, the dynamics of k are monotonic: either continuously increase or decrease. This follows from assuming a unique intertemporal equilibrium. This extra assumption also implies that the intermporal elasticity of substitution is *not too small*. In particular, we operationalise (Croix and Michel 2009, 25) it as:

$$\forall c > 0, \quad u'(c) + cu''(c) \geq 0 \Leftrightarrow \sigma(c) \geq 1.$$

Due to the fact the monotonicity of the dynamics, capital can converge either to $0, \bar{k} \in (0, +\infty)$, or $+\infty$.

2.9.1 The economy ***never*** goes to $k = +\infty$

This is equivalent to saying that starting with a a high level of capital k_0, the dynamics imply $k_{t+1} < k_t$.

First, we have that:

$$g(k) = \frac{1}{1+n} s(\omega(k), f'(\underbrace{g(k)}_{k_{t+1}})) < \frac{\omega(k)}{1+n}.$$

This holds because savings s are *always* a fraction of wages ω.

Second, we show that:

$$\lim_{k \to +\infty} \frac{g(k)}{k} = 0.$$

To see the intuition for this limit, assume for a moment it holds. Then, it must be that for large enough k, $g(k) = k_{t+1} < k_t$ and the dynamics are decreasing and converge to some $\bar{k}$. In this case, $\bar{k}$ is the largest steady state, and because the dynamics are monotonic, the economy never goes to the other side of the steady state.

We show next that $\lim_{k \to +\infty} \frac{g(k)}{k} = 0$. We do so by showing that $\lim_{k \to +\infty} \frac{\omega(k)}{k} = 0.$

Proof

We start by noting that:

$$\frac{\omega(k)}{k} = \frac{f(k)}{k} - f'(k) \leftarrow \omega(k) = f(k) - f'(k)k.$$

When $k \to +\infty$, **the first term** $\frac{f(k)}{k}$ admits a positive limit, let's call it l_1. This is because:

1. $\frac{f(k)}{k} > 0.$
2. $\frac{f(k)}{k}$ is decreasing:

$$\frac{\partial \frac{f(k)}{k}}{\partial k} = \frac{f'(k)k - f(k)}{k^2} = -\frac{f(k) - f'(k)k}{k^2} = -\frac{\omega(k)}{k^2} < 0.$$

The **second term** $f'(k)$ is decreasing and positive, thus it admits a limit l_2 when $k \to +\infty$.
Apply the mean value theorem evaluated at k and $2k$:

$$f(2k) - f(k) = (2k - k)f'(k(1+\theta)), \quad \text{with}\, \theta \in (0,1).$$

Then,

$$\frac{2f(2k)}{2k} - \frac{f(k)}{k} = f'(k(1+\theta))$$

Taking the limit when $k \to +\infty$ we obtain $2l_1 - l_1 = l_2 \implies l_1 = l_2$.
Finally,

$$\lim_{k\to+\infty} \frac{\omega(k)}{k} = \lim_{k\to+\infty} \frac{f(k)}{k} - f'(k) = l_1 - l_2 = 0.$$

Since $\lim_{k\to+\infty} \frac{\omega(k)}{k} = 0$ and $\frac{g(k)}{k} < \frac{\omega(k)}{k}$ then $\lim_{k\to+\infty} \frac{g(k)}{k} = 0$. In conclusion, if the inital value of capital is large enough, the dynamics are decreasing and k converges to an interior steady state $\bar{k}$.

2.10 Examples

2.10.1 Example 1: Log-utility and Cobb-Douglas

Suppose that $u(c, d) = \log(c) + \beta \log(d)$ and $f(k_t) = Ak_t^\alpha$.

The wage rate is given by

$$w_t = f(k_t) - f'(k_t) = (1-\alpha)Ak_t^\alpha$$

and the savings function is obtained solving

$$u'(w_t - s_t) = \beta R_{t+1} u'(R_{t+1} s_t).$$

Rearranging and isolating s we are left with the savings function

$$s_t = \frac{\beta}{1+\beta} w_t.$$

The dynamics are given by the capital accumulation law

$$k_{t+1} = \frac{1}{1+n} s_t = \frac{1}{1+n} \frac{\beta}{1+\beta} w_t = \frac{1}{1+n} \frac{\beta}{1+\beta} (1-\alpha)Ak_t^\alpha.$$

The model with log-utility and Cobb-Douglas production functions has two steady states: zero and a unique positive steady state. We can solve for them:

$$\bar{k} = \frac{1}{1+n} \frac{\beta}{1+\beta} (1-\alpha)A\bar{k}^\alpha \implies \bar{k} = \begin{cases} 0 \\ \\ \left(\frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)A\right)^{\frac{1}{1-\alpha}} \equiv \phi^{\frac{1}{1-\alpha}} \end{cases}.$$

The dynamics are depicted in Figure 2.3.

Local dynamics: stability

We check the stability of the two steady states using the first derivative evaluated at the steady state. **Note:** in general we would use the Jacobian matrix, but the OLG model has only one equation in one variable.

Remember that, for a discrete-time system, a *non-hyperbolic* steady state is stable if it the determinant of the Jacobian (here the absolute value of the derivative) evaluated at the steady state is in the unit circle $(-1, 1)$.

In our case:

$$g(k) = k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)Ak_t^{\alpha}$$
$$g'(k) = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)\alpha Ak_t^{\alpha-1} > 0.$$

Then

$$\bar{k} = 0 \implies g'(0) = +\infty \quad \to \text{unstable}$$
$$\bar{k} = \phi^{\frac{1}{1-\alpha}} \implies g'(\phi^{\frac{1}{1-\alpha}}) = \phi\alpha\phi^{\frac{\alpha-1}{1-\alpha}} = \alpha \in (0,1) \quad \to \text{stable.}$$

Only an autarky steady state In this case, $f(0) = 0$ is required so that the autarky steady state can exist. Moreover, it requires $f(k_{t+1}) < f(k_t)$. This type of setup requires $\rho < 0$, this is, complementarity between capital and labor and low levels of output given the inputs.

2.10.2 Log-utility and CES technology

Under a CES specification, $y_t = A\left[\alpha k_t^{\rho} + (1-\alpha)\right]^{\frac{1}{\rho}}$, $\rho \leq 1$ wages are given by

$$\omega_t = (1-\alpha)\left[\alpha k^{\rho} + 1 - \alpha\right]^{\frac{1-\rho}{\rho}}.$$

Hence, the dynamics of capital are governed by:

$$k_{t+1} = g(k_t) = \frac{1}{1+n}s_t = \frac{1}{1+n}\frac{\beta}{1+\beta}A(1-\alpha)\left[\alpha k_t^{\rho} + 1 - \alpha\right]^{\frac{1-\rho}{\rho}}.$$

The curve $g(k)$ is increasing:

$$g'(k) = \frac{\beta A(1-\alpha)}{(1+n)(1+\beta)}(1-\rho)\left[\alpha k^{\rho} + 1 - \alpha\right]^{\frac{1-2\rho}{\rho}} k^{\rho-1} > 0.$$

Depending on the value of ρ we can have different configurations regarding the steady states. Ultimately, what matters is the concavity of the function $g(k_t)$.

$$g''(k_t) = -\frac{1}{1+n}\frac{\beta}{1+\beta}A(1-\rho)\alpha\left[\alpha k^{\rho} + 1 - \alpha\right]^{\frac{1-3\rho}{\rho}} k^{\rho-2}\left(\rho\alpha k^{\rho} + (1-\alpha)(1-\rho)\right).$$

2.10.2.1 Unique steady state, $\bar{k} > 0$

If $\rho \in (0,1)$, the second derivative is clearly negative, and g is concave. Moreover, $g(0) > 0$, so $\bar{k} = 0$ cannot be a steady state:

$$g(k_t) = \frac{\beta A(1-\alpha)}{(1+n)(1+\beta)}(\alpha k^{\rho} + 1 - \alpha)^{\frac{1-\rho}{\rho}} \implies g(0) = \frac{\beta A(1-\alpha)^{\frac{1}{\rho}}}{(1+n)(1+\beta)} > 0.$$

In that case, the economy displays a unique steady state, and it is globally stable.

See Figure 2.4.

2.10.2.2 Unique $\bar{k} = 0$ or multiple steady states

If $\rho < 0$, the function $g(k)$ changes concavity. In particular, there exists a level $\hat{k}$ such that:

$$g''(k) > 0 \,\text{if}\, k < \hat{k}.$$
$$g''(k) < 0 \,\text{if}\, k > \hat{k}.$$

with

$$\hat{k} = \left(-\frac{(1-\alpha)(1-\rho)}{\rho\alpha}\right)^{\frac{1}{\rho}} > 0 \,\text{since}\, \rho < 0.$$

Moreover, $g(0) = 0$ so $\bar{k} = 0$ can be a steady state.

In this case, there are two sub-cases:

1. If $k_{t+1} < k_t \implies \frac{\omega(k)}{k} < (1+n)\frac{1+\beta}{\beta}$ then the trajectory of k_{t+1} is always below k. Then, there is one unique steady state: $\bar{k} = 0$. The economy converges towards this unique steady state. We can derive the condition above from:

$$k_{t+1} < k_t \implies \frac{1}{1+n} s\left(\omega(k_t), R_{t+1}\right) < k_t \implies$$
$$\frac{1}{1+n}\frac{\beta}{1+\beta} w_t < k_t \implies \frac{w_t}{k_t} < (1+n)\frac{1+\beta}{\beta}.$$

2. If there is some $k_{t+1} > k_t \implies \frac{\omega(k)}{k} > (1+n)\frac{1+\beta}{\beta}$ then there are two positive steady states $\bar{k}_a < \bar{k}_b$ plus the zero steady state $\bar{k} = 0$.

1. All trajectories starting at $k_0 < \bar{k}_a$ converge to $\bar{k} = 0$ and $\bar{k} = 0$ is locally stable in the range $[0, \bar{k}_a]$. The steady state with $\bar{k} = 0$ is a **poverty trap**: whenever the economy starts with a low enough level of capital, it will always converge towards the $\bar{k} = 0$ steady state.
2. Trajectories starting at $k_0 = \bar{k}_a$ remain at $\bar{k}_a$. This steady state is unstable.
3. Finally, trajectories starting with $k_0 > \bar{k}_a$ converge towards $\bar{k}_b$, which is a locally stable steady state in $(\bar{k}_a, +\infty)$.

See Figure 2.5 and Figure 2.2.

2.11 Solved example

Suppose that production can be parametrized using the following Cobb-Douglas specification:

$$Y = F\left(K_t, L_t\right) = K_t^{\frac{1}{2}} L_t^{\frac{1}{2}}.$$

Furthermore, assume the following utility function:

$$u(c_t) = \frac{c_t^{1-\frac{1}{3}} - 1}{1 - \frac{1}{3}}$$

where the intertemporal elasticity of substitution is -3.

We can compute production function, the wage and the interest in intensive form as:

$$\begin{aligned} f(k_t) &= k_t^{\alpha} \\ w_t &= f(k_t) - f'(k_t)k_t = (1-\alpha)k_t^{\alpha} \\ r_t &= f'(k_t) = \alpha k_t^{\frac{-1}{2}} \end{aligned}$$

Similarly, the savings function can be derived from the Euler equation as follows:

$$\begin{aligned} u'(w_t - s_t) &= \beta R_{t+1} u'(s_t R_{t+1}) \implies \\ \implies (w_t - s_t)^{\frac{-1}{3}} &= \beta R_{t+1} (R_{t+1} s_t)^{\frac{-1}{3}} \implies \\ \implies (w_t - s_t) &= \beta^{-3} R_{t+1}^{-2} s_t \implies \\ s_t &= \frac{w_t}{1 + \beta^{-3} R_{t+1}^{-2}} \end{aligned}$$

Therefore, the capital accumulation equation is:

$$\begin{aligned} k_{t+1} &= \frac{1}{1+n} \frac{w_t}{1 + \beta^{-3} R_{t+1}^{-2}} = \frac{1}{1+n} \frac{\frac{1}{2} k_t^{\frac{1}{2}}}{1 + \beta - 3 \left(\frac{1}{2} k_{t+1}^{\frac{-1}{2}} \right)^{-2}} \implies \\ k_{t+1} &= \frac{1}{1+n} \frac{\frac{1}{2} k_t^{\frac{1}{2}}}{1 + 4\beta^{-3} k_{t+1}} \implies \\ \frac{1}{2} k_t^{\frac{1}{2}} &= (1+n) k_{t+1} \left(1 + 4\beta^{-3} k_{t+1}\right) \implies \\ k_{t+1} &= \frac{-(1+n) + \sqrt{(1+n)^2 + 8(1+n)\beta^{-3} k_t^{\frac{1}{2}}}}{8(1+n)\beta^{-3}} \end{aligned}$$

We can then show that the autarky steady state exists. Indeed, because we have $f(0) = 0$, then $\bar{k} = 0$ is a steady state. The economy has a second steady state, but solving for it explicitly is challenging becuase it involves a third degree polynomial. Nevertheless, let's sketch the process, knowing that a steady state solves $k_t = k_{t+1} = \bar{k}$. To simplify the notation, in what follows I will write k instead of $\bar{k}$.

$$
\begin{aligned}
k = \frac{-(1+n) + \sqrt{(1+n)^2 + 8(1+n)\beta^{-3}k^{\frac{1}{2}}}}{8(1+n)\beta^{-3}} &\implies \\
1 + 8(1+n)\beta^{-3}k = \sqrt{1 + 8^{-3}k^{\frac{1}{2}}} &\implies \\
16\beta^{-3}k + 64\beta^{-6}k^2 = 8\beta^{-3}k^{\frac{1}{2}} &\implies \\
k^{\frac{1}{2}}\left(8\beta^{-3}k^{\frac{3}{2}} + k^{\frac{1}{2}} - 1\right) &= 0
\end{aligned}
$$

From here, it is obvious that $k = 0$ is a steady state, as we already knew. In principle, the part in brackes may have up to three solutions, but as said, these are hard to get. Instead, we can show that one and only one (real) solution can exist by studying the stability of the steady states. Moreover, the functional form we have is similar to a square root: we have the square root of something to the power $\frac{1}{2}$, so this means that the function will behave similar to $x^{\frac{1}{4}}$. Thus, we can expect only a second steady state to exist.

The derivative of the dynamic function with respect to k_t is:

$$
\frac{\partial k_{t+1}}{\partial k_t} = \frac{1}{4\sqrt{k_t}\sqrt{(1+n)\left(1+n+\frac{8\sqrt{k_t}}{\beta^3}\right)}}
$$

When $k_t = k_{t+1} = 0$, the value of the derivative is $+\infty$, and thus, that steady state is non-stable. Moreover, for any value $k > 0$, the derivative is between 0 and 1. Therefore, any other possible steady state is stable. Because the economy cannot converge to $k \to \infty$, and because all the potential steady states with positive capital are stable, this implies that only can exist. Otherwise, all of the would attract the dynamics, but this is impossible in this model: we must converge somewhere. In fact, the plot corresponds to Figure 2.3.

2.12 The golden-rule level of capital

The first welfare theorem states that a competitive equilibrium is Pareto optimal. However, the first welfare theorem has two requisites the Diamond model does not meet: a finite number of goods and a finite number of agents. Therefore, the allocation resulting from the competitive equilibrium may *not* be Pareto optimal. This section is based on Croix and Michel (2009, Ch. 2) and Groth (2016).

In the Diamond model —and different from the Ramsey model—, the capital stock on the balanced growth path may exceed the golden-rule level. This implies that a permanent increase in consumption is possible.

When discussing the golden-rule level of capital, we are *not* interested in how consumption is divided between the old and the young. Rather, we seek to maximise total consumption during one period. In fact, if we maximise total consumption, the ensuing allocations between young and old will be efficient. First, let $\mathfrak{C}_t$ represent total consumption:

$$\mathfrak{C}_t \equiv C_t + D_t.$$

The economy-wide resource constraint dictates that total production is either consumed or invested:

$$\mathfrak{C}_t + K_{t+1} = F(K_t, N_t) + (1-\delta)K_t.$$

Alternatively, aggregate consumption per unit of labour is:

$$\begin{aligned} \mathfrak{c}_t \equiv \frac{\mathfrak{C}_t}{N_t} &= \frac{F(K_t, N_t) + (1-\delta)K_t - K_{t+1}}{N_t} = \\ &= f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}. \end{aligned}$$

The *golden-rule level of the capital-labour ratio* $k^{\mathcal{GR}}$ is the value of the capital-labour ratio k that results in the highest possible *sustainable* level of consumption per unit of labour. The fact that it is sustainable requires that it is replicable forever. For this reason, we consider the steady state with $k_{t+1} = k_t = \bar{k}$. The resource constraint at the steady state simplifies to:

$$\bar{\mathfrak{c}} = f(\bar{k}) - (\delta + n)\bar{k} \equiv \mathfrak{c}(k).$$

We maximise this function with respect to $\bar{k}$:

$$\mathfrak{c}'(\bar{k}) = f'(\bar{k}) - (n + \delta) = 0$$

The fact that $\mathfrak{c}''(\bar{k}) = f''(\bar{k}) < 0$ assures that we have the maximum.

Assuming that $n + \delta > 0$ and that f satisfies:

$$\lim_{k \to +\infty} f'(k) < n + \delta < \lim_{k \to 0} f'(k),$$

then $\mathfrak{c}'(\bar{k}) = f'(\bar{k}) - (n+\delta) = 0$ has a solution in $\bar{k}$ and it is unique. Hence, the golden-rule level of capital is given by:

$$f'(k^{\mathcal{GR}}) = n + \delta.$$

The highest sustainable consumption level per unit of labour is obtained when, at steady state, the net marginal productivity of capital equals the growth rate of the economy.

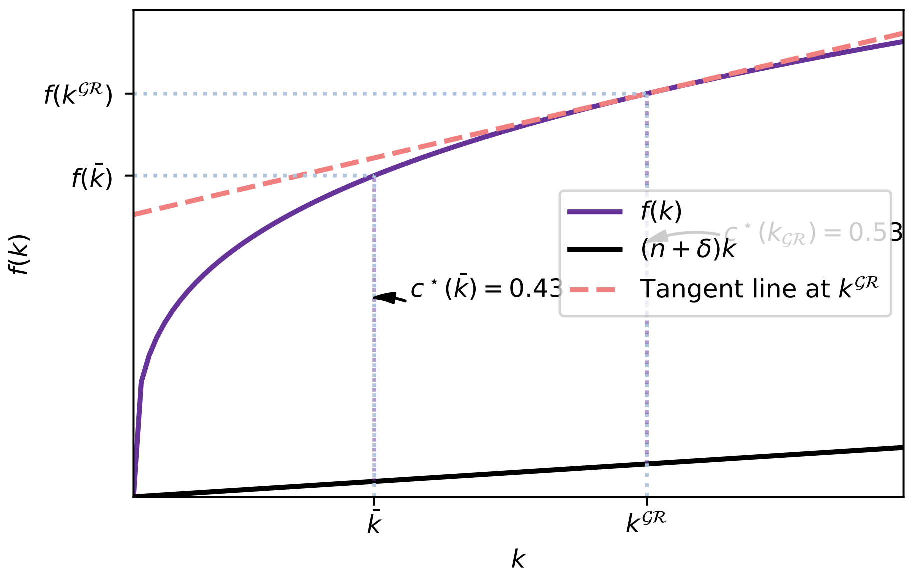


Figure 2.6: The golden-rule level of capital

2.12.1 Over- and under-accumulation of capital

The golden-rule level of capital is given by $f'(k) = n + \delta$. However, in the decentralised economy, the steady-state level of savings is highly *unlikely* to coincide with it. For instance, a log-utility function coupled with a Cobb-Douglas production function leads to a steady-state value of capital such that $f'(k) = \frac{\beta}{(1+n)(1+\beta)}^{\frac{1}{1-\alpha}}$. In general, $\frac{\beta}{(1+n)(1+\beta)}^{\frac{1}{1-\alpha}} \neq n + \delta$.

If the steady-state level of the economy is such that $f'(\bar{k}) < n + \delta$, the economy has accumulated too much capital: if $f'(\bar{k}) < f'(k^{\mathcal{GR}}) \implies \bar{k} > k^{\mathcal{GR}}$. Conversely, if the interest rate at the competitive steady state is such that $f'(\bar{k}) > n + \delta$, then it lacks capital compared to the golden rule.

2.12.1.1 Improving the situation: dynamic inefficiency

Let's suppose for the moment that the decentralised economy reaches a situation of over-accumulation of capital, this is, $f'(\bar{k}) < n+\delta$. It is possible to improve upon it by means of redistribution. Simply, a planner could order young individuals to save less, as to reach $k^{\mathcal{GR}}$. Doing so increases the available total output per worker, which can be consumed either by the old or the young, improving their utility. In subsequent periods, the economy would remain at $k^{\mathcal{GR}}$, which maximises total output per worker.

In particular, assume we impose a lump-sum tax on all young individuals and we transfer it to the old generation. The transfer is fully consumed (old individuals gain utility from consuming it). Suppose the transfer is one good from each young to the old. There are $1 + n$ young individuals per old individual, so the old receive $1 + n$ goods. Let's repeat this scheme every period.

- The young transfer one good to the old
- Consumption of the young does not change because they refrain from saving.

- In the future, they receive $1+n$, which represents a return rate of $1+n$ on their "investment".
- In the competitive market, they would have obtained $f'(k) < n+\delta$.
- Clearly, the return young individuals achieve under the tax system is larger, and hence they are better off.

At the same time, old individuals are also better off. In the first period, when the reform is introduced, they consume more. Later generations obtain a higher return for the tax they pay, leaving them better off, too.

When organising such scheme is possible, we say that the competitive economy was **dynamically inefficient.**

Note: it is possible to solve the dynamic inefficiency introducing money. Young individuals would exchange goods for paper notes with the expectation of exchanging them back again when old.

2.12.1.2 Dynamic efficiency

In the opposite case, when $\bar{k} < k^{\mathrm{GR}}$, reaching the golden-rule level of capital requires increasing it. Suppose we are at time t_0. Increasing the capital to $k^{\mathcal{GR}}$ would certainly improve consumption for all generations $t > t_0$. However, young individuals at t_0 should forgo some consumption to save more and build the necessary capital. Hence, the consumption of the young generation at t_0 necessarily decreases. Our policy of improving utility for all individuals fails in this case.

We call such situations **dynamically efficient**: it is impossible to increase the utility of all generations.

2.12.2 Example

We use a log-utility and Cobb-Douglas case to illustrate over- and under-accumulation of capital.

$$U(c,d) = \log(c) + \beta \log(d)$$
$$f(k) = Ak^{\alpha}.$$

The golden-rule level of capital dictates that total consumption per worker should be maximised. Using the resource constraint:

$$\max c_t = f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}.$$

Evaluating it at the steady state yields:

$$\max \mathfrak{c} = f(k) - (n+\delta)k \implies f'(k^{\mathcal{GR}}) = n + \delta.$$

Hence, in our case:

$$A\alpha {k^{\mathcal{GR}}}^{\alpha-1} = n + \delta \implies k^{\mathcal{GR}} = \left(\frac{A\alpha}{n+\delta}\right)^{\frac{1}{1-\alpha}}.$$

In the decentralised economy, we have that:

$$k_{t+1} = \frac{1}{1+n} s(\omega(k_t), f'(k_{t+1})) = \frac{1}{1+n}\frac{\beta}{1+\beta}A(1-\alpha)k_t^{\alpha}.$$

Hence, the steady-state level of capital is:

$$\bar{k} = \frac{A\beta(1-\alpha)}{(1+n)(1+\beta)}\bar{k}^{\alpha} \implies \bar{k} = \frac{A(1-\alpha)\beta}{(1+n)(1+\beta)}^{\frac{1}{1-\alpha}}.$$

We have capital over-accumulation if: $\bar{k} > k^{\mathcal{GR}} \implies \frac{(n+\delta)\beta}{(1+n)(1+\beta)} > \frac{\alpha}{1-\alpha}$.

Check Romer (2018, 89–90) for an empirical discussion about the dynamic efficiency.

2.13 The Central Planner

Suppose now, instead, that a central planner is in charge of organising consumption and investment for all individuals. The planner operates by aggregating individual utility, discounting future generations at the rate γ. **Note:** γ is the discount rate of future *generations*, not how young people discount old-age utility.

Her utility considers the utility of all generations, including the initial old generation who owns the initial stock of capital k_0.

2.13.1 Planner's utility

Planner's utility reads:

$$\sum_{t=-1}^{\infty} \gamma^t U(c_t, d_{t+1}).$$

Note that, although the planner can decide any allocation, she must respect the resource constraint:

$$f(k_t) + (1-\delta)k_t = c_t + \frac{1}{1+n} d_t + (1+n)k_{t+1}.$$

Assume that $U(c_t, d_{t+1})$ is separable: $U(c_t, d_{t+1}) = u(c_t) + \beta u(d_{t+1})$. Expanding the utility of the planner, we can reformulate it in more convenient terms:

$$
\begin{aligned}
&\sum_{t=-1}^{\infty} \gamma^t \left(u(c_t) + \beta u(d_{t+1}) \right) = \\
&= \gamma^{-1} u(c_{-1}) + \gamma^{-1} \beta u(d_0) + \gamma^0 u(c_0) + \gamma^0 \beta u(d_1) + \gamma^1 u(c_1) + \gamma^1 \beta u(d_2) + \ldots \\
&= \sum_{t=0}^{\infty} \gamma^t \left(u(c_t) + \frac{\beta}{\gamma} u(d_t) \right) + \gamma^{-1} u(c_{-1}).
\end{aligned}
$$

The term $\gamma^{-1}u(c_{-1})$ represents the consumption of the generation born at $t = -1$, but since it is a constant it will not affect the maximisation.

Hence, the planner's problem is now how to allocate consumption between the young and old that are alive during period t.

2.13.2 Maximisation

We use a substitution to obtain the planner's optimal allocation, namely, the Euler equation;

$$
\max_{k_{t+1}, c_t, d_t} \sum_{t=0}^{\infty} \gamma^t \left(u(c_t) + \frac{\beta}{\gamma} u(d_t) \right)
$$

$$
\text{s.t. } f(k\_t) + (1-\delta)k\_t = c\_t + \frac{1}{1+n} d\_t + (1+n)k\_t + 1.
$$

$$
\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \gamma^t \left(u(c_t) + \frac{\beta}{\gamma} u \left(\underbrace{[1+n]\left[f(k_t) + (1-\delta)k_t - (1+n)k_{t+1} - c_t \right]}_{d_t} \right) \right).
$$

Taking derivatives and equating them to zero yields:

$$\gamma^t u'(c_t) = \gamma^t \frac{\beta}{\gamma} u'(d_t)(1+n)$$

$$\gamma^t \frac{\beta}{\gamma} u'(d\_t)(1+n)(1+n) = \gamma^{t+1} \frac{\beta}{\gamma}(1+n)u'(d\_t+1)\left(f'(k\_t+1)+1-\delta\right).$$

Combining both equations we get the Euler equation:

$$u'(c_t) = \beta u'(d_{t+1})\left(f'(k_{t+1})+1-\delta\right).$$

The planner's Euler equation coincides with the decentralised one, where we had $R_{t+1} = f'(k_{t+1}) + 1 - \delta$.

However, the planner also allocates consumption between the young and the old at time t.

$$\gamma^t u'(c_t) = \gamma^t \frac{\beta}{\gamma} u'(d_t)(1+n)$$

In the centralised equilibrium, individuals do *not* arbitrage between young and old consumption at time t. We missed this equation because individuals are short-sighted, and only derive utility while alive. Hence, they have no interest in trading off utility with the young generation once they are old.

2.13.3 Steady state and modified golden rule

Using the fact that

$$\gamma^t \frac{\beta}{\gamma} u'(d_t)(1+n)(1+n) = \gamma^{t+1} \frac{\beta}{\gamma}(1+n)u'(d_{t+1})\left(f'(k_{t+1})+1-\delta\right)$$

we can easily characterise the steady state knowing that $d_t = d_{t+1} = \bar{d}, k_t = k_{t+1} = \bar{k}$.

$$f'(\bar{k}) = \frac{1+n}{\gamma} + 1 - \delta.$$

This equation provides us with the modified golden rule: the level of capital that maximises the planner's utility. Clearly, if $\gamma = 1$, the planner attributes the same weight to all generations and we recover the golden rule: $f'(k) = n + \delta$. As we discussed before, it is quite *unlikely* the decentralised equilibrium converges towards the golden rule (modified or not).

3 de la Croix and Dottori (2008)

This section discusses Croix and Dottori (2008), using a simplified model to illustrate the key points.

3.1 Introduction

This paper analyses the very distinct population patterns of two remote islands in the Pacific Ocean: Eastern Island and Tikopia. While Tikopians managed to control population growth and natural resources usage, the inhabitants of Eastern Island engaged in clan competition for the control of resources, leading to overpopulation and overexplotation of resources.

A key aspect of this paper is the especial role of fertility. Most papers assume that parents derive some utility from having children. However, in de la Croix and Dottori, fertility is the result of a Nash bargaining process for the control of resources. In particular, a larger population, facilitated by having more children, raises the value of the fallback option during the negotiation process. Consequently, individuals optimally decide to have more children, because this implies a better bargaining position. In this sense, equilibrium-level fertility rates result from the complementarities between different groups' fertility decisions. However, the externalities of a higher fertiltiy rate are not internalised and, in the long run, population explodes, leading to a natural catastrophe.

3.1.1 Historical data

Based on data from archaeological studies, it has been estimated that the population of Eastern Island increased very little between 400CE (100 people by that time) and 110CE, and from then on, it exploded, reaching 10000 people during 1400-1600CE. The effects of the population race could be perceived by 1600CE: food consumption declined and population plummeted during the 17th century. By 1772, when Europeans arrived at Eastern Island, the total population was around 3000 people. In parallel, data about forests in Eastern Island indicate that upon the arrival of the first settlers during 400CE, tree-cutting begun. By 1400CE, deforestation had reached its peak and when the Europeans arrived, there were basically no trees in the island.

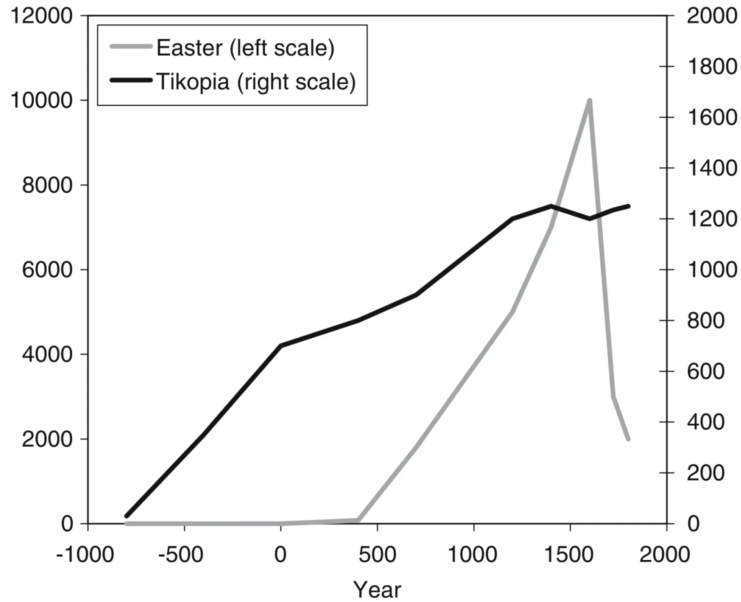


Fig. 1 Population of Easter Island and Tikopia

Figure 3.1: Figure 1 in de la Croix and Dottori, 2008

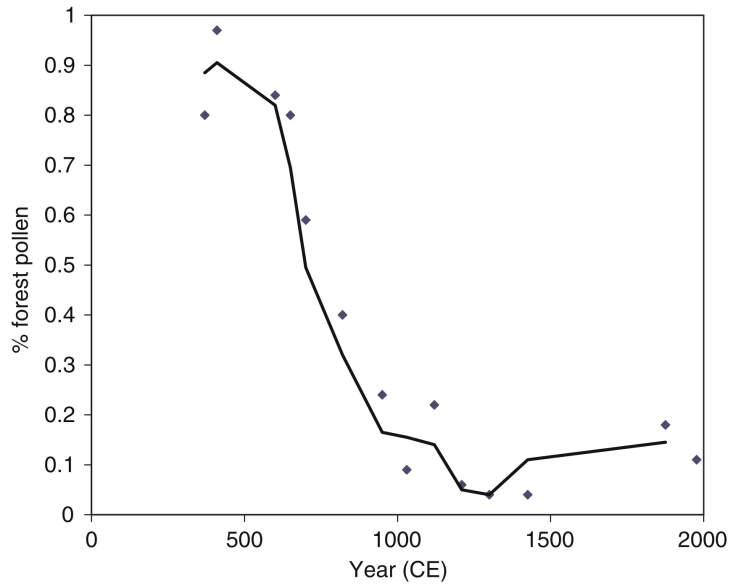


Fig. 2 Forest coverage on Easter Island

Figure 3.2: Figure 2 in de la Croix and Dottori, 2008

Meanwhile, Tikopia was settled around 900BCE and people lived by slash and burn agriculture. By 100BCE, due to decreasing returns from natural resources, pig breeding began, lasting until the 17th century, when Tikopians abandoned it because pigs required too many resources. Total population stabilized at around 1200 people and was kept at that level by purposeful mechanisms: celibacy, abortion, infanticide, sea exploration by young males, etc.

3.2 The model

De la Croix and Dottori work using the OLG framework. In the paper, agents live for two periods. However, important decisions are taken at the clan level, which acts as a representative agent. Clans (and individuals) are rational, have perfect foresight and take the actions of the other clans as given. The timing is as follows:

1. Each clan chooses its fertility level,
2. A Nash-Cournot fertility equilibrium level arises,
3. Crops are cultivated and shared between clans following a non-cooperative bargaining process.

For simplicity, the island is populated by two opposed clans, all individuals belong to one clan only and they cannot change clan.

3.2.1 Preferences

Clan i at time t consists of $N_{i,t}$ adults. Adults work, support their parents and have children. Old agents only consume what their children provide for them.[1] Total utility is given by:

$$U_{i,t} = c_{i,t} + \beta d_{i,t+1},$$

where $c_{i,t}, d_{i,t+1}$ represents consumption when adult and old, respectively.

[1] Bisin and Verdier also model the case when parental indoctrination and how it affects the evolution of cultural traits.

3.2.2 Budget

The income of an adult agent is $y_{i,t}$. Each adult has to support his parents by giving them some resources. However, support for parents is not linear, but rather it depends on the number of siblings. In particular, each sibling contributes the following share of his income:

$$\frac{\tau}{1+n_{i,t-1}},$$

where $\tau \in (0,1)$. Clearly, the contribution decreases with the number of siblings.

Consequently, an agent in his old age who had $n_{i,t}$ children receives, as **total** old age support,

$$d_{i,t+1} = n_{i,t}\frac{\tau}{1+n_{i,t}}y_{i,t+1}.$$

Lastly, since total income is distributed between consumption and supporting parents $\left(y_{i,t} = c_{i,t} + \frac{\tau}{1+n_{i,t-1}}y_{i,t}\right)$, consumption when young is just:

$$c_{i,t} = \left(1 - \frac{\tau}{1+n_{i,t-1}}\right)y_{i,t}.$$

3.2.3 Population

The population of each clan evolves according to the chosen fertility level:

$$N_{i,t+1} = N_{i,t}n_{i,t}.$$

3.2.4 Production

Production depends only on land. The amount of land is fixed at L, and total factor productivity depends on the available natural resources R_t.

$$Y_{i,t} = A(R_t)L.$$

The dynamics of resources follow the paper by Matsumoto (2002):

$$R_{t+1} = \left(1 + \delta - \delta\frac{R_t}{K} - b(N_{1,t} + N_{2,t})\right) R_t,$$

where $K > 0$ is the carrying capacity (maximum possible number of resources), $\delta > 0$ is the growth rate of resources while $b > 0$ measures the effect of population on resources.

3.2.4.1 Crop-sharing

We denote by θ_t the share of crops Y_t that Group 1 appropriates. Therefore, each adult in Groups 1 and 2 obtains:

$$y_{1,t} = \theta_t \frac{Y_t}{N_{1,t}},$$

$$y_{2,t} = (1 - \theta_t)\frac{Y_t}{N_{2,t}}.$$

There are no property rights on the island, and groups have to bargain to decide how to split the total porduction Y_t. This bargaining process is non-cooperative and, if no agreement is reached, clans will battle to appropriate the entire production.

3.3 Bargaining

Bargaining takes place under Nash-bargaining, and the outcome of the process solves:

$$(U_{1,t} - \bar{U}_{1,t})^{\gamma}(U_{2,t} - \bar{U}_{2,t})^{1-\gamma},$$

where $U_{1,t}$ is what Group 1 shall receive and $\bar{U}_{1,t}$ is the fall-back option of Group 1, this is, what Group 1 receives is there is no agreement. When there is no agreement between Group 1 and Group 2, the clans fight and the winner takes all. The probability that Group 1 wins the war, denoted by π_t, depends on its size.

$$\pi_t = \frac{N_{1,t}}{N_{1,t} + N_{2,t}}.$$

So, the more adults in one group, the more likely it is to win the war. From the equation, it is clear that clans have an incentive to increase their population: it helps win the war (if it happens) and provides them with a better bargaining position by raising $\bar{U}_{i,t}$.

Suppose than clans reach an agreement θ_t on how to share crops: θ_t goes to Group 1, and the remining $1 - \theta_t$ goes to Group 2. Then, the indirect utility of an individual is given by:

$$U_{1,t} = \left(1 - \frac{\tau}{1 + n_{1,t-1}}\right)\frac{\theta_t Y_t}{N_{1,t}} + \beta\frac{n_{1,t}\tau}{1 + n_{1,t}}\frac{\theta_{t+1} Y_{t+1}}{N_{1,t+1}},$$
$$U_{2,t} = \left(1 - \frac{\tau}{1 + n_{2,t-1}}\right)\frac{(1-\theta_t) Y_t}{N_{2,t}} + \beta\frac{n_{2,t}\tau}{1 + n_{2,t}}\frac{(1-\theta_{t+1}) Y_{t+1}}{N_{2,t+1}}.$$

If, instead, there is no agreement, clans fight, and the winner takes the entire production. Since π_t denotes the probability that Group 1 wins the fight, the fall-back utilities are given by:

$$\bar{U}_{1,t} = \pi_t \left(1 - \frac{\tau}{1+n_{1,t-1}}\right)\frac{Y_t}{N_{1,t}} + \beta \frac{n_{1,t}\tau}{1+n_{1,t}} \frac{\theta_{t+1}Y_{t+1}}{N_{1,t+1}},$$
$$\bar{U}_{2,t} = (1-\pi_t) \left(1 - \frac{\tau}{1+n_{2,t-1}}\right)\frac{Y_t}{N_{2,t}} + \beta \frac{n_{2,t}\tau}{1+n_{2,t}} \frac{(1-\theta_{t+1})Y_{t+1}}{N_{2,t+1}}.$$

The difference between $U_{i,t}$ and $\bar{U}_{i,t}$ is:

$$U_{1,t} - \bar{U}_{1,t} = \left(1 - \frac{\tau}{1+n_{1,t-1}}\right)(\theta_t - \pi_t)\frac{Y_t}{N_{1,t}},$$
$$U_{2,t} - \bar{U}_{2,t} = \left(1 - \frac{\tau}{1+n_{2,t-1}}\right)(1 - \theta_t - (1-\pi_t))\frac{Y_t}{N_{2,t}}.$$

Since, at the time of bargaining, $\left(1 - \frac{\tau}{1+n_{i,t}}\right)\frac{Y_t}{N_{i,t}}$ has already been determined, we can abstract from it in the maximisation.

After substituting, θ_t, is the optimal sharing rule which solves

$$\theta_t = \arg\max\left(\theta_t - \pi_t\right)^{\gamma}\left(1 - \theta_t - (1-\pi_t)\right)^{1-\gamma}.$$

The optimal level θ_t is then:

$$\theta_t = \frac{N_{1,t}}{N_{1,t} + N_{2,t}}.$$

3.4 Fertility

Finally, we can compute the optimal fertility levels for each group by maximising utility. So, Group 1 and Group 2 maximise:

$$\max_{n_{1,t}} \overbrace{\left(1-\frac{\tau}{1+n_{1,t-1}}\right)\frac{\theta_t Y_t}{N_{1,t}}}^{constant\, at\, t}+$$

$$+\frac{\beta\tau n_{1,t}}{1+n_{1,t}}\left[\frac{\overbrace{N_{1,t}n_{1,t}}^{N_{1,t+1}}}{\underbrace{N_{1,t}n_{1,t}}_{N_{1,t+1}}+\underbrace{N_{2,t}n_{2,t}}_{N_{2,t+1}}}\right]\frac{A(R_{t+1})}{\underbrace{N_{t,1}n_{t,1}}_{N_{1,t+1}}}.$$

$$\max_{n_{2,t}} \overbrace{\left(1-\frac{\tau}{1+n_{2,t-1}}\right)\frac{\theta_t Y_t}{N_{2,t}}}^{constant\, at\, t}+$$

$$+\frac{\beta\tau n_{2,t}}{1+n_{2,t}}\left[1-\frac{\overbrace{N_{2,t}n_{2,t}}^{N_{2,t+1}}}{\underbrace{N_{2,t}n_{2,t}}_{N_{2,t+1}}+\underbrace{N_{2,t}n_{2,t}}_{N_{2,t+1}}}\right]\frac{A(R_{t+1})}{\underbrace{N_{t,2}n_{t,2}}_{N_{2,t+1}}}.$$

The optimum levels of fertility satisfy:

$$n^{\star}_{1,t}=\left(\frac{N_{2,t}}{N_{1,t}}\right)^{\frac{1}{3}},$$

$$n^{\star}_{2,t}=\left(\frac{N_{1,t}}{N_{2,t}}\right)^{\frac{1}{3}},$$

So, the best course of action for Group 1 is to increase its fertility as Group 2 becomes more populous, and a race for population occurs. Of course, this has implications for the environment, because larger populations are destructive:

$$R_{t+1}=\left(1+\delta-\delta\frac{R_t}{K}-b(N_{1,t}+N_{2,t})\right)R_t,$$

3.5 Steady state level of population and natural resources

Lastly, we can compute the steady state level of population using the two dynamical equations:

$$n^{\star}_{1,t} = \left(\frac{N_{2,t}}{N_{1,t}}\right)^{\frac{1}{3}} \implies N_{1,t+1} = N_{1,t}\left(\frac{N_{2,t}}{N_{1,t}}\right)^{\frac{1}{3}}$$
$$n^{\star}_{2,t} = \left(\frac{N_{1,t}}{N_{2,t}}\right)^{\frac{1}{3}} \implies N_{2,t+1} = N_{2,t}\left(\frac{N_{1,t}}{N_{2,t}}\right)^{\frac{1}{3}},$$

It is simpler to solve the corresponding linearised system, which we can obtain by taking logarithms:

$$\tilde{N}_{i,t+1} = \frac{2}{3}\tilde{N}_{i,t} + \frac{1}{3}\tilde{N}_{j,t},$$

where $\tilde{N} = \log N$.

We can obtain the dynamics of the logarithmic system:

$$\tilde{N}_{i,t} = \frac{\tilde{N}_{i,0} + \tilde{N}_{j,0}}{2} + \frac{1}{2}3^{-t}\left(\tilde{N}_{i,0} - \tilde{N}_{j,0}\right).$$

The steady state level of population, for each group is

$$\bar{N}_i = \bar{N}_j = \sqrt{N_{1,0}}\sqrt{N_{2,0}},$$

and the corresponding level of natural resources at the steady state is

$$\bar{R} = K\left(1 - \frac{b(\bar{N}_i + \bar{N}_j)}{\delta}\right).$$

3.5.1 Simulated trajectory

Lastly, we can compute the trajectory of the system for a set of parameters to visualize the evolution of the main variables. For instance, if we take the following parametrisation $N_{1,0} = 9, N_{2,0} = 20, \delta = 0.08, K = 400, b = 0.0012, R_0 = 300$

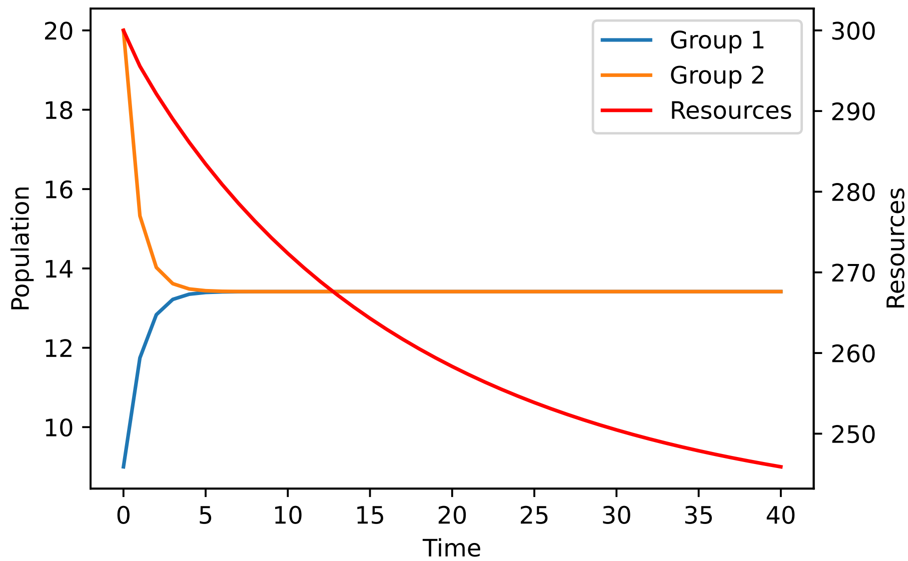


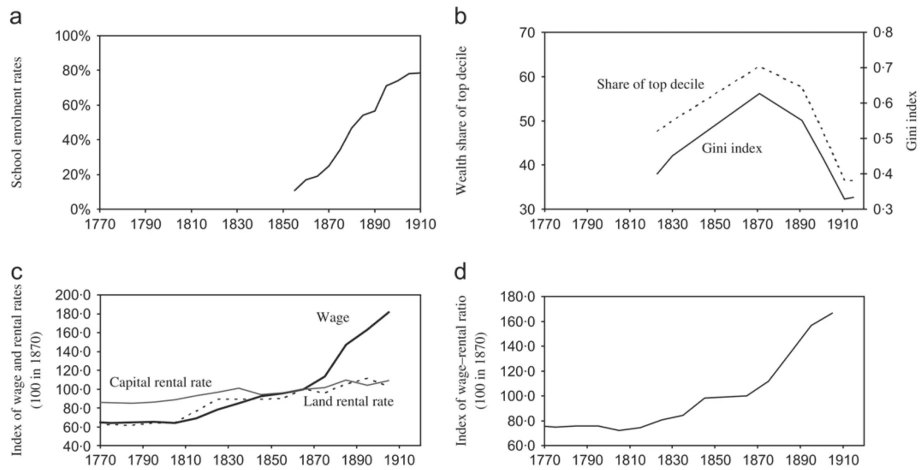
Figure 3.3: Simulated trajectory of populations and resources over time

4 Galor and Moav (2006)

This section discusses the work of Galor and Moav (2006), which presents an OLG model to explain the rise of publicly financed education and the transition from a class structure characterised by capitalists and workers to another one where all individuals are capitalists.

The papers proposes that the demise of the class structure was a deliberate action from the part of the capitalists to sustain their profits as human capital became more and more important in the production process. This is, at some point in time, human capital becomes *really* necessary to produce, and the capitalists find it optimal to tax themselves to finance the education of workers. Doing so, raises the human capital level and allows them to keep their profits.

The paper is set-up after the industrial revolution, let's say around 1850 and is meant to describe the transition of the modern economies between that time and the beginning of the 20th century. The authors motivate the paper by showing that during this time, school enrolment rates increased while, at the same time, inequality decreased: the demise of classes. The theory the authors propose parallels this evolution. Furthermore, they supplement the model with econometrics that are compatible with the predictions of the model.



Source: Flora *et al.* (1983), real wage (Clark, 2005), rental rates (Clark, 2002), real wage (Clark, 2005), and return to capital (Clark, 2002).

FIGURE 1

Schooling, factor prices, and inequality, England 1770–1920. The evolution of (a) the fraction of children aged 5–14 in public primary schools: England, 1855–1920, (b) earnings inequality: England, 1820–1913, (c) wages and rental rates: England, 1770–1920, and (d) the wage–rental ratio: England, 1770–1920

Figure 4.1: Figure 1 in Galor and Moav (2006)

4.1 The model

We present a simplified version of the model using precise functional forms for the human capital accumulation process and the production function. This paper comprises two separate sections: * The general model * The application of the model to a society with two classes. We will follow this approach.

For the moment, the economy comprises *only* one type of individuals. Individuals live for two periods of time: young and adult. Young individuals do not produce and use their time to acquire human capital. If education is provided, human capital accumulation is faster.

4.1.1 Production and prices

A single homogenous good is produced using physical and human capital according to a Cobb-Douglas production function. In particular:

$$Y_t = F(H_t, K_t) = AK_t^{\alpha} H_t^{1-\alpha} = Ak_t^{\alpha}, \quad k_t \equiv \frac{K_t}{H_t}.$$

Given the wage rate per efficiency unit of labour w_t and the return to capital r_t, producers maximise profits by choosing the level of capital K_t and efficiency units of labour H_t, this is,
$K_t, H_t = \arg\max \left[AH_t k_t^{\alpha} - w_t H_t - r_t K_t\right]$. Considering perfect competition, the inverse demand for each factor is:

$$\begin{aligned} r_t &= f'(k_t) = \alpha A k_t^{\alpha-1} = r(k_t), \\ w_t &= f(k_t) - f'(k_t)k_t = (1-\alpha)Ak_t^{\alpha} = w(k_t). \end{aligned}$$

4.1.2 Individuals and preferences

Every period, a new generation of size 1 is born. Individuals have only parent and each has only one son. Individuals live for two periods: during their youth, they accumulate human capital; and education improves human capital accumulation. Young individuals may receive a positive bequest from their parents on which they earn interests (the bequest is physical capital that is lent to producers). When adults, they supply their human capital as efficiency units of labour; receive interests on their assets and allocate the total income between consumption and a bequest.

The bequest b_t is transferred from parents to children, and the government collects a tax $\tau_t \geq 0$ on it. The remaining $1 - \tau_t$ is saved for future consumption. Physical capital **fully depreciates** between periods.

As mentioned, individuals accumulate human capital during their youth, and if they are provided with education (e_t), human capital accumulation is enhanced. However, even if no education is provided, all young individuals manage to obtain a minimum level of human capital: we set it equal to one. We model human capital accumulation as follows:[1]

$$h_{t+1} = 1 + \frac{e_t}{1 + e_t} = h(e_t).$$

This type of function ensures that under some conditions, investment in education is not optimal; while guaranteeing a minimum level of human capital.

When adults, individuals receive wages on their human capital, as well as the rental price on the bequest (the part not taxed by the government). Therefore, an individual with a bequest b_t and education level e_t has income equal to:

$$I^i_{t+1} = w_{t+1}h(e_t) + (1 - \tau)b^i_t R_{t+1},$$

and, since capital fully depreciates, $R_{t+1} = r_{t+1} = r(k_{t+1})$.

Lastly, the preferences of adults include consumption and a taste for giving bequests. Bequests in the model are a type of luxury good: only when income is large enough, $b_t > 0$. In particular:

$$u^i_t = (1 - \beta)\log(c^i_{t+1}) + \beta \log(\bar{\theta} + b^i_{t+1}),$$

where $\bar{\theta} > 0$ and $\beta \in (0, 1)$. The budget constraint is simple:

$$c^i_{t+1} + b^i_{t+1} = I^i_{t+1}.$$

[1] Bisin and Verdier also model the case when parental indoctrination and how it affects the evolution of cultural traits.

4.1.3 Optimisation

We can easily compute the value of bequests, as a function of the income level. Replacing c_{t+1} in the objective function and taking the derivative with respect to b_{t+1} yields:

$$\frac{\partial}{\partial b^i_{t+1}} = 0 \implies \frac{1-\beta}{b^i_{t+1} - I^i_{t+1}} + \frac{\beta}{b^i_{t+1} + \bar{\theta}} = 0 \implies$$
$$b^i_{t+1} = \begin{cases} \beta(I^i_{t+1} - \theta) & \text{if} \quad I^i_{t+1} > \theta \\ 0 & \text{if} \quad I^i_{t+1} \leq \theta \end{cases}$$

where $\theta \equiv \bar{\theta}\frac{1-\beta}{\beta}$. Hence, when income is relative low, individuals do not give bequests.

4.1.4 Evolution of physical and human capital

Remember that bequests left during period t are the capital of period $t+1$. If B_t is the total amount of bequests left during t, then

$$K_{t+1} = (1-\tau)B_t.$$

The remaining τB_t goes to the government, which uses it to fund education. Population is normalised to 1, therefore, each individual receives education equal to: $e_t = \tau B_t$, and human capital evolves as:

$$H_{t+1} = h(e_t) = h(\tau_t B_t) = 1 + \frac{\tau_t B_t}{1 + \tau_t B_t}.$$

Last, the level of $k_t = \frac{K_t}{H_t}$ is:

$$k_{t+1} = \frac{K_{t+1}}{H_{t+1}} = \frac{(1-\tau_t)B_t}{h(\tau_t B_t)} = \frac{(1-\tau_t)B_t}{1 + \frac{\tau_t B_t}{1+\tau_t B_t}} = k(\tau_t, B_t).$$

4.2 Optimal level of taxation

The paper assumes that the government selects the tax rate that maximises individuals' utility. One important feature of the model is that utility is increasing in income I^i_{t+1}. We can easily check this by rewriting the indirect utility:

$$
\begin{aligned}
u^i_t &= (1-\beta)\log(c^i_{t+1}) + \beta\log(\bar{\theta} + b^i_{t+1}) = \\
&= \begin{cases} (1-\beta)\log(I^i_{t+1} - \beta I^i_{t+1} + \beta\theta) + \beta\log(\beta I^i_{t+1} - \beta\theta) & \text{if} \quad I^i_{t+1} > \theta \\ (1-\beta)\log(I^i_{t+1}) + \beta\log(\bar{\theta}) & \text{if} \quad I^i_{t+1} \leq \theta \end{cases}
\end{aligned}
$$

which is increasing in I^i_{t+1} because $\beta \in (0,1)$.

Therefore, instead of maximising the indirect utility, the government can maximise second-period income I^i_{t+1}, which in turn will maximise utility. Second-period income is: $w_{t+1}h(\tau^i_t B_t) + (1-\tau^i_t)b^i_t R_{t+1}$ where $w_{t+1} = w(k_{t+1}$ and $R_{t+1} = R(k_{t+1}$. At the same time, $k_{t+1} = \frac{(1-\tau_t)B_t}{h(\tau_t B_t)} = \frac{(1-\tau_t)B_t}{1+\frac{\tau_t B_t}{1+\tau_t B_t}}$. Putting everything together,

$$
\begin{aligned}
\tau^i_t &= \arg\max w_{t+1}h(\tau^i_t B_t) + (1-\tau^i_t)b^i_t R_{t+1} \\
&= \arg\max (1-\tau^i_t)^\alpha h(\tau^i_t B_t)^{1-\alpha} B^\alpha_t \left(1 - \alpha + \alpha\frac{b^i_t}{B_t}\right).
\end{aligned}
$$

Maximising with respect to τ^i_t yields:

$$\frac{\partial}{\partial \tau_t^i} = 0 \implies$$

$$B_t^\alpha \left(1-\alpha+\alpha\frac{b_t^i}{B_t}\right)\left[\alpha(1-\tau_t^i)^{\alpha-1}h(\tau_t^i B_t)^{1-\alpha}(-1)+\right.$$
$$\left.+(1-\alpha)h'(\tau_t^i B_t)h(\tau_t^i B_t)^{-\alpha}B_t(1-\tau_t^i)^\alpha\right]=0$$

$$\alpha(1-\tau)^{\alpha-1}h(\tau_t^i B_t)^{1-\alpha}=$$
$$=(1-\alpha)h'(\tau_t^i B_t)B_t(1-\tau)^\alpha h(\tau_t^i B_t)^\alpha$$

$$\alpha(1-\tau)^{\alpha-1}h(\tau_t^i B_t)^{1-\alpha}B_t^{\alpha-1}=$$
$$=(1-\alpha)h'(\tau_t^i B_t)B_t^\alpha(1-\tau)^\alpha h(\tau_t^i B_t)^\alpha$$

$$\underbrace{\alpha(1-\tau)^{\alpha-1}h(\tau_t^i B_t)^{1-\alpha}B_t^{\alpha-1}}_{R(k_{1+1})}=$$
$$=\underbrace{(1-\alpha)B_t^\alpha(1-\tau)^\alpha h(\tau_t^i B_t)^\alpha}_{w(k_{t+1})}h'(\tau_t^i B_t)$$

$$R(k_{t+1})=w(k_{t+1})h'(\tau_t^i B_t)$$

The optimal condition for τ_t^i does not involve the bequest received b_t^i. Consequently, everybody will agree on the optimality of the tax rate and it will be implemented. In our case, substituting and solving for τ_t:

$$\tau_t = \begin{cases} \frac{-B_t(1+2\alpha)+\sqrt{B_t^2(1+4(1+2B_t)(1-\alpha)\alpha)}}{4B_t^2\alpha} & \text{if} \quad B_t > \frac{\alpha}{1-\alpha} \\ 0 & \text{if} \quad B_t \leq \frac{\alpha}{1-\alpha} \end{cases} = \tau(B_t)$$

Alternatively, it is possible to re-express the condition for positive taxation in terms of k_{t+1} :

$$\tau_t = \begin{cases} \frac{-B_t(1+2\alpha)+\sqrt{B_t^2(1+4(1+2B_t)(1-\alpha)\alpha)}}{4B_t^2\alpha} & \text{if} \quad k_{t+1} > \frac{\alpha}{1-\alpha} \\ 0 & \text{if} \quad k_{t+1} \leq \frac{\alpha}{1-\alpha} \end{cases} = \tau(B_t)$$

4.3 One economy, two groups

We suppose now that the economy, at time $t = 0$, consists of two groups: capitalists (C) and workers (W). The share that capitalists represent is denoted by λ_t. However, since all individuals always have one child, shares remain constant, this is, $\lambda_t = \lambda$. The unique difference between the two groups is the initial endowment of capital: * Capitalists own the initial stock of capital (which we assume is sufficiently large as to be able to bequest). * Workers have no capital, and thus give no bequests.

Therefore, the total amount of bequests in a given period is:

$$B_t = \lambda b_t^C + (1-\lambda) b_t^W.$$

The remainder of the model is the same as before, in particular,

$$k_{t+1} = \frac{(1-\tau(B_t))B_t}{h(\tau(B_t)B_t)}.$$

Such an economy shifts from having to classes of people to only one, where everybody owns capital. The critical transition occurs because of, eventually, capitalists find it optimal to impose a tax on themselves to finance public education. With it, and as wages keep increasing, workers are eventually able to give bequests, thus becoming capitalists. Instead of detailing the exact process (check the reference), we will simulate the economy for a set of parameters.

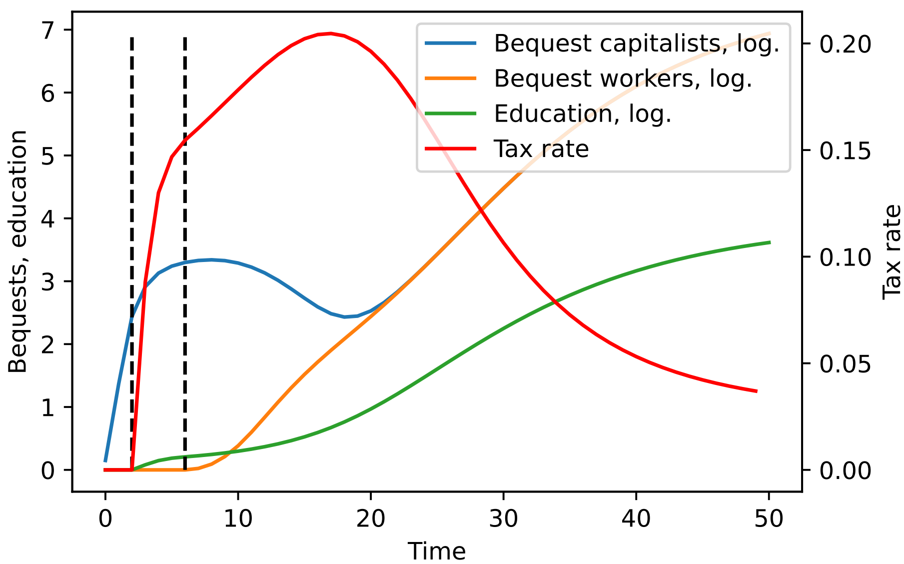


Figure 4.2: Simulated path of the economy proposed in Galor and Moav, 2006

5 Galor and Özak (2016)

Galor and Özak (2016) show empirically how differences in time preference (how much people discount the future) has an agricultural origin. This is important for development, because those who save more are the individuals who are more patient, this is, who are more future-oriented. Although the major contribution of the paper is empirical, the authors develop an OLG model from which they derive implications that are tested.

The theoretical model shows how the composition of a population can be modeled using the OLG framework. The most influential model in that regard is Bisin and Verdier (2001). However, the model in Galor and Özak is relatively simple and illustrates well some population dynamics.

5.1 The model

We work under the OLG framework, and we assume that the economy is agricultural and at the very early stages of development. In every period, the economy consists of individuals who live for **three** periods.

- During the first period of life, individuals are children and are economically passive. Consumption during this period is provided by parents.
- In the second period and third periods, individuals work
- All individuals can choose between two modes of production:

- Endowment mode: it provides an equal pay-off during the second and third periods of live. For instance, individuals may be hunters.
- Investment mode: it pays little during the second period of live, but the pay-off during the third period is much larger. This represents farmers: they must seed and wait for crops to grow.

Lastly, a crucial assumption of the mode is the **lack of financial markets and long-term storage technology**. This implies that individuals **cannot** transfers consumption between periods two and three. Hence, production in the second period has to be consumed in the second period; and consumption in the third period must be consumed in the third period.

5.1.1 Production modes

All adults must decide between the *endowment mode* or the *investment mode*. The endowment mode provides a constant level of output,$R^0 > 1$ in each of the working periods. If investment is instead chosen, it requires an investment during the first working period, which implies that less resources are available for consumption. In particular, we assume that it leaves individuals with 1 unit of consumption during their second period of life. However, the output it yields during the third period R^1 is higher than under *endowment*: $\ln(R^1) > 2\ln(R^0)$.

Finally, depending on the chosen production mode, the income of individual i is given by:

$$(y_{i,t}, y_{i,t+1}) = \begin{cases} (R^0, R^0) & \text{if endowment} \\ (1, R^1) & \text{if investment} \end{cases}$$

5.2 Preferences

A key aspect of the model to generate dynamics in the evolution of individual traits is the fertility decision. This approach is common: typically, fertility is linked to a trait through income. This is, individuals with more income will be able to have more children. If the trait is transmitted from parents to children, then the trait that allows generating more income will become more and more prevalent in the economy.[1] In any case, preferences are really important in this type of models, because fertility decisions are derived from them.

Every period t, a generation of size L_t becomes economically active, this is, reaches the second period of life. Those individuals were born in period $t-1$. At this stage, each individual will live for two periods. Remember that financial markets do not exist, and also that it is **impossible** to transfer resources between periods by storing them.

We assume that, during the second period of life, individuals only consume what they produce. Lastly, during the third and last period individuals consume and also have children. In particular, utility is given by:

$$u^{i,t} = \ln c_{i,t} + \beta_t^i[\gamma \ln n_{i,t+1} + (1-\gamma)\ln c_{i,t+1}], \quad \gamma \in (0,1),$$

where $c_{i,t}$ and $c_{i,t+1}$ are the levels of consumption in the second and third periods of life and $n_{i,t}$ is the number of children. It is important to comment on $\beta_t^i \in (0,1]$: it represents *individual i*'s discount factor, this is, how much he values the future with respect to the present. The larger β_t^i, the more the value of future and, hence, the more patient the individual is. Notice that β_t^i changes over time and also by individual.

During the second period, individuals do not really make any decision: since resources cannot be transferred, all production must be consumed.

[1] Bisin and Verdier also model the case when parental indoctrination and how it affects the evolution of cultural traits.

Hence, $c_{i,t} = y_{i,t}$. However, during the last period individuals can trade-off utility from consumption and utility from children. The paper assumes that each child costs τ units of consumption, which gives rise the last-period budget constraint:

$$y_{i,t+1} = c_{i,t+1} + \tau n_{i,t+1}.$$

Considering the preferences, utility maximisation implies:

$$c_{i,t+1} = (1-\gamma) y_{i,t+1},$$

$$n_{i,t+1} = \frac{\gamma}{\tau} y_{i,t+1}.$$

Lastly, the indirect utility ($v_{i,t}$) of individual i is given by:

$$v_{i,t} = \ln y_{i,t} + \beta_t^i [\ln y_{i,t+1} + \xi], \quad \xi \equiv \gamma \ln \left(\frac{\gamma}{\tau}\right) + (1-\gamma)\ln(1-\gamma).$$

5.3 Hunters or farmers

Since individuals can decide on their mode of production, they are free to choose to become either hunters or farmers. This is, each individual will decide the mode of production (endowment or investment) that maximises lifetime utility. Hence,:

$$v_{i,t} = \begin{cases} \ln R^0 + \beta_t^i (\ln(R^0) + \xi) & \text{if endowment} \\ \ln 1 + \beta_t^i (\ln(R^1) + \xi) & \text{if investment} \end{cases}.$$

An individual is indifferent between modes of production if he obtains the same utility from both. This is, the individual with $\beta_t^i = \hat{\beta}$ is indifferent between becoming a hunter or a farmer if and only if:

$$\ln R^0 + \hat{\beta}(\ln(R^0) + \xi) = \ln 1 + \hat{\beta}(\ln(R^1) + \xi).$$

Solving for $\hat{\beta}$ allows us to identify such individual:[2]

$$\hat{\beta} = \frac{\ln R^0}{\ln R^1 - \ln R^0} \in (0, 1).$$

Lastly, all individuals with $\beta^i_t < \hat{\beta}$ will optimally choose the endowment technology while those with $\beta^i_t > \hat{\beta}$ find the investment technology optimal. Note that, as the return to agriculture increases (R^1 increases), the cutoff value $\hat{\beta}$ decreases:

$$\frac{\partial \hat{\beta}}{\partial R^1} = \frac{-\ln R^0}{R^1(\ln R^1 - \ln R^0)^2} < 0,$$

this is, as agriculture becomes more and more profitable, more individuals will find it optimal to become hunters.

Hence, we can rewrite the income of an individual as a function of his β^i_t:

$$(y_{i,t}, y_{i,t+1}) = \begin{cases} (R^0, R^0) & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} \leq \hat{} \\ (1, R^1) & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} > \hat{} \end{cases}.$$

Of course, since income in the last period of life is different, hunters and farmers will have different number of children. In particular, using the optimal number of children derived above:

$$n_{i,t+1} = \frac{\gamma}{\tau} y_{i,t+1} = \begin{cases} \frac{\gamma}{\tau} R^0 \equiv n^E & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} \leq \hat{} \\ \frac{\gamma}{\tau} R^1 \equiv n^I & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} > \hat{} \end{cases}.$$

Because $R^1 > R^0$, farmers have more children than hunters.

[2]The assumption $\ln(R^1) > 2\ln(R^0)$ is important to establish that $\hat{\beta} \in (0, 1)$.

5.4 The evolution of preferences

Finally, we can compute how preferences change over time due to the differential fertility between farmers and hunters. This is, because farmers have more children than hunters, if we assume that preferences about time β_t^i are transmitted between parents and children, the share of farmers will increase over time. The paper assumes almost that, although modifies slightly the transmission of preferences for individuals engaging in farming. In particular: * β_t^i is perfectly transmitted if an individual is a hunter. * Farmers transmit a larger value of β_t^i to their children, reflecting an acquired tolerance to waiting and delaying reward. This is:

$$\beta_{i,t+1}^i = \begin{cases} \beta_t^i & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} \leq \hat{} \\ \phi(\beta_t^i, R^1) & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} > \hat{} \end{cases},$$

with

- $\beta_t^i \leq \phi(\beta_t^i) < 1$: the transmitted β is always more than the one the parent had,
- $\phi(\hat{\beta}, R^1) > \beta_t^i$,
- $\phi_\beta(\beta_t^i, R^1) > 0$: the higher the value of β_{t+1}^i, the more it increases,
- $\phi_{\beta\beta}(\beta_t^i, R^1) < 0$: but at a decreasing rate,
- $\phi_R(\beta_t^i, R^1) > 0$: the higher the value of R, the more β_{t+1}^i increases.

Suppose an individual at the beginning of time who has $\beta_0^i < \hat{\beta}$. This individual will optimally decide to be a hunter, and according to the process for the transmission of preferences, his sons will inherit $\beta_1^i = \beta_0^i$. Because $\hat{\beta}$ is constant over time, all sons will decide to be hunters as well and transmit the same time preferences, over and over again. Hence, if $\beta_0^i \leq \hat{\beta} \implies \lim_{t\to\infty} \beta_t^i = \beta_0^i$.

Suppose instead that $\beta_0^i > \hat{\beta}$. The individual will become a farmer and transmit $\phi(\beta_0^i, R^1) > \beta_0^i$. Accordingly, all his sons will also be farmers and keep transmitting an ever-increasing value of β^i_{t+1}. However, because $\phi_{\beta\beta}(\beta^i_t, R^1) < 0$, the transmission process has a steady-state, this is, $\lim_{t\to\infty} \beta^i_t = \bar{\beta}^I$. Notice that $\bar{\beta}^I$ is the maximum level β^i_t can reach.

5.4.1 Proof (not in the paper)

We want to show that $\beta^i_{t+1} = \phi(\beta^i_t, R^1)$ has a unique steady-state. This amounts to showing that $\bar{\beta}^I = \phi(\bar{\beta}^I, R^1)$ for a unique value $\bar{\beta}^I$. Define $G(\beta) = \phi(\beta, R) - \beta$. Since we focus on farmers, we know that the very initial one in the dynasty had $\beta^i_0 > \hat{\beta}$. So, for our purposes, the function G has as domain $\beta \in [\hat{\beta}, \infty)$. We know that $G(\hat{\beta}) = \phi(\hat{\beta}, R^1) - \hat{\beta} > 0$ because $\phi(\hat{\beta}, R^1) > \beta^i_t$. Moreover, the function G has unique maximum at $\phi_\beta = 1$ because $\phi_{\beta\beta} < 0$, and the maximum is positive because it must be larger than $\phi(\hat{\beta}, R^1) - \hat{\beta} > 0$. After the maxima, the function continuously decreases, thus crossing only one the horizontal axis, this is, there is a unique value $\bar{\beta}$ such that $G(\bar{\beta}^I) = 0$, which constitutes the unique steady state.

5.5 Evolution of traits over time

Lastly, suppose that initially, at time $t = 0$, the initial population presents different levels of time preference. We assume that, initially, traits are characterised by some distribution of $\eta(\beta^i_0)$ with support $[0, \bar{\beta}^I]$. Furthermore, we normalise the initial generation to be of size one: $L_0 = 1$. Alternatively:

$$L_0 = \int_0^{\bar{\beta}^I} \eta(\beta^i_0) \mathrm{d}\beta^i_0 = 1.$$

We also know that all individuals whose $\beta_0^i \leq \hat{\beta}$ decide to use the endowment technology, and the remaining opt for the investment technology. Therefore, the size of hunters (E) and farmers (I) is given by:

$$L_0^E = \int_0^{\hat{\beta}} \eta(\beta_0^i) \mathrm{d}\beta_0^i,$$

$$L_0^I = \int_{\hat{\beta}}^{\bar{\beta}^I} \eta(\beta_0^i) \mathrm{d}\beta_0^i.$$

The number of individuals evolves according to the fertility rate of each group, this is,

$$L_t^E = L_0^E {n^E}^t = (\frac{\gamma}{\tau} R^0)^t L_0^E$$

$$L_t^I = L_0^I {n^I}^t = (\frac{\gamma}{\tau} R^1)^t L_0^I$$

and total population is $L_t = L_t^E + L_t^I$.

Finally, notice that the distribution of β_t^i does not change for those individuals with $\beta_t^i \leq \bar{\beta}$: in fact, they all have the same number of children, and each child inherits the trait of his parent. Therefore, the average value $\bar{\beta}^E$ is constant over time. In contrast, the average value $\bar{\beta}_t^I$ increases. In any case, at any given period t, the overall average value for time preference is given by:

$$\bar{\beta}_t = \theta_t^E \bar{\beta}_t^E + (1 - \theta_t^E) \bar{\beta}_t^I,$$

where θ_t^E is the fraction of individuals who engage in the endowment production process, and $\bar{\beta}_t$ is just the weighted average.

$$\theta_t^E = \frac{L_t^E}{L_t^E + L_t^I} = \frac{{R^0}^t}{{R^0}^t + {R^1}^t \frac{L_0^I}{L_0^E}}.$$

Hence, as time advances, the share of the population engaged in the endowment production process shrinks towards zero:

$$\lim_{t \to \infty} \theta_t^E = 0.$$

This process reflects their lower reproductive success.

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