ECE 530 (Fall 2023) **Instructor:** S. Bose

Homework 5

Assigned: Oct 12 Due: Oct 20

Instructions: *You are encouraged to discuss with your peers. You need to, however, turn in your own solutions. Mention the names of your collaborators. Submissions made after but within one week of the due date will be marked on 50% of the total points. You don't need to submit after that.*

Problem 1. The Burden of Stability. [3+4+8+5 = 20 points]

Consider the scalar ordinary differential equation (ODE)

$$\dot{x}(t) = f(t, x(t)) := \lambda x(t) + (1-\lambda)\cos(t) - (1+\lambda)\sin(t), \qquad x(0) = 1. \tag{1}$$

The file `run_ODE.m` provides a framework to run various different methods for numerical integration of (1) with step-size $h > 0$ over time $t \in [0, T = 10]$. Let $N = \lfloor \frac{T}{h} \rfloor$, where the notation $\lfloor z \rfloor$ stands for the largest integer not exceeding z. Specifically, it computes a vector $(x_0 = x(0), x_1, \ldots, x_N)$, where x_n's are the proxies for $x(t_n)$ computed recursively via different methods with $t_n = nh$. Define the average error of any numerical integration method applied to this ODE as

$$\mathcal{E} := \frac{1}{N+1} \sum_{n=0}^{N} |x(t_n) - x_n| .$$

Use the code or otherwise to generate the plots required below and answer the questions. Please submit your code at least for parts c) and d).

(a) Show that the analytical solution of the ODE is given by $x(t) = \cos(t) + \sin(t)$.

(b) Plot the result of numerical integration via forward Euler method with $h = 0.15, 0.30, 0.45$ over $[0, T]$ together with the analytical solution. Comment how the average error for forward Euler $\mathcal{E}_{\text{FE}}$ varies with h. Is forward Euler method stable for all values of h you simulated?

(c) To implement backward Euler method for a given step-size $h > 0$, one needs to solve the nonlinear equation

$$x_{n+1} := x_n + hf(t_{n+1}, x_{n+1}),$$

in each iteration $n \geq 0$. Implement Newton-Raphson to solve the equation $F(y) = 0$, where

$$F(y) := y - x_n - hf(t_{n+1}, y),$$

starting from the forward Euler solution, given by $y^{(0)} := x_n + hf(t_n, x_n)$. Iterate till $|F(y)| < 10^{-5}$. For the same values of h used in part (b), numerically integrate (1) using backward Euler method and plot the results together with the analytical solution on the same graph. Comment how the average error for backward Euler $\mathcal{E}_{\text{BE}}$ varies with h. Is backward Euler method stable for all values of h you simulated?

(d) To implement the trapezoidal method with a step-size $h > 0$, one needs to solve the nonlinear equation

$$x_{n+1} := x_n + \frac{h}{2}\left[f(t_n, x_n) + f(t_{n+1}, x_{n+1})\right]$$

in each iteration $n \geq 0$. Implement Newton-Raphson to solve the equation $F(y) = 0$ with

$$F(y) := y - x_n - \frac{h}{2}\left[f(t_n, x_n) + f(t_{n+1}, y)\right],$$

starting from the forward Euler solution, given by $y^{(0)} := x_n + hf(t_n, x_n)$. Iterate till $|F(y)| < 10^{-5}$. For the same values of h used in part (b), numerically integrate (1) using trapezoidal method and plot the results together with the analytical solution on the same graph. Comment how the average error for trapezoidal method $\mathcal{E}_{\text{trap}}$ varies with h. Is trapezoidal method stable for the values of h you simulated?

Problem 2. Conjugacy is Independence. [4 points]

If $\{d^0, \ldots, d^{n-1}\}$ are pairwise Q-conjugate vectors in $\mathbb{R}^n \setminus \{0\}$ for $Q \succ 0$, then prove that these vectors are linearly independent.

Problem 3. The Price of Laziness. [3+3 = 6 points]

For the ODE given by $\dot{x} = f(x, t)$, investigate the absolute stability of the following numerical integration schemes:

(a) Trapezoidal rule with one step fixed-point iteration from forward Euler, given by

$$x_{n+1} = x_n + \frac{h}{2}\left[f(x_n, t_n) + f(x_n + hf(x_n, t_n), t_{n+1})\right].$$

(b) Backward Euler with one step fixed-point iteration from forward Euler, given by

$$x_{n+1} = x_n + hf\left(x_n + hf(x_n, t_n), t_{n+1}\right).$$