Trapezoid o' trapezoid!

Problem Statement

Consider the scalar ordinary differential equation (ODE) $\dot{x} = f(t, x(t))$ starting from $x_0 = x(0)$. Then, the trajectory $x(t)$ over $t \in [0, T]$ satisfies:

$$x(t_{n+1}) = x(t_n) + \int_{\tau=t_n}^{\tau=t_{n+1}} f(\tau, x(\tau))d\tau$$

for $t_n = nh$. Let $g(\tau) \coloneqq f(\tau, (x(\tau)))$. Assume that g is $\mathcal{C}^2$ over $[0, T]$.

a

If $0 \leq a \leq b \leq T$, then prove that g satisfies:

$$\int_a^b g(\tau)d\tau = \frac{1}{2}(b-a)[g(a)+g(b)] - \frac{1}{2}\int_a^b \left[\left(\frac{b-a}{2}\right)^2 - \left(\tau - \frac{a+b}{2}\right)^2\right] g''(\tau)d\tau$$

Hint: Use integration by parts.

b

Then, use the identity in part a) to prove that there exists a constant M s.t.

$$\left|\int_a^b g(\tau)d\tau - \frac{1}{2}(b-a)[g(a)+g(b)]\right| \leq M\frac{(b-a)^3}{12}$$

c

Deduce that the local truncation error of the trapezoidal method is $\mathcal{O}(h^3)$; i.e.:

$$x(t_{n+1}) = x(t_n) + \frac{h}{2}[f(t_n, x(t_n)) + f(t_{n+1}, x(t_{n+1}))] + \mathcal{O}(h^3)$$

d (*optional*)

Under possibly additional assumptions, prove that the global error for the trapezoidal method is $\mathcal{O}(h^2)$; i.e., there exists some constant C s.t.

$$|x(t_n) - x_n| \leq Ch^2 \forall t \in [0, T]$$

Solution

a

Consider:

$$\int_a^b \left[\left(\frac{b-a}{2}\right)^2 - \left(\tau - \frac{a+b}{2}\right)^2\right] g''(\tau)d\tau$$

Letting:

$$u = \left[\left(\frac{b-a}{2}\right)^2 - \left(\tau - \frac{a+b}{2}\right)^2\right] \implies du = a + b - 2\tau d\tau$$

$$dv = g''(\tau)d\tau \implies v = g'(\tau)$$

$$\int_a^b u dv = uv|_a^b - \int_a^b v du \equiv \left[\left(\left(\frac{b-a}{2}\right)^2 - \left(\tau - \frac{a+b}{2}\right)^2\right) g'(\tau)\right]_a^b - \int_a^b g'(\tau)(a+b-2\tau)d\tau$$

The whole left term evaluated at the end points reduces to 0, so:

$$= -\int_a^b g'(\tau)(a+b-2\tau)d\tau$$

IBP again using:

$$u = a+b-2\tau \implies du = -2d\tau$$

$$dv = g'(\tau)d\tau \implies v = g(\tau)$$

$$= -\left([(a+b-2\tau)g(\tau)]_a^b - \int_a^b g(\tau)(-2d\tau)\right)$$

$$= -\left([(a+b-2b)g(b) - (a+b-2a)g(a)] - \int_a^b g(\tau)(-2d\tau)\right)$$

$$= -\left((a-b)(g(b)+g(a)) - \int_a^b g(\tau)(-2d\tau)\right)$$

$$= (b-a)(g(b)+g(a)) + \int_a^b g(\tau)(-2d\tau)$$

$$= (b-a)(g(b)+g(a)) - 2\int_a^b g(\tau)d\tau$$

Hokay, now sub this integral back into the original equation:

$$\int_a^b g(\tau)d\tau = \frac{1}{2}(b-a)\left[g(a)+g(b)\right] - \frac{1}{2}\left((b-a)(g(b)+g(a)) - 2\int_a^b g(\tau)d\tau\right)$$

$$\int_a^b g(\tau)d\tau = \frac{1}{2}(b-a)\left[g(a)+g(b)\right] - \frac{1}{2}(b-a)(g(b)+g(a)) + \int_a^b g(\tau)d\tau$$

Cancel the the terms $a, b, g(a), g(b)$ and we're left with:

$$\int_a^b g(\tau)d\tau = \int_a^b g(\tau)d\tau$$

$\square$

b

Based on the proof provided by [1]. Consider a partition of the Trapezoidal rule:

$$x_k = a + kh,\ k = 0, 1, \ldots, n,\ h = \frac{b-a}{n}$$

And the full trapezoidal rule is given as:

$$T = \frac{h}{2}(f(a)+f(b)) + h\sum_{k=1}^{n-1} f(a+kh)$$

Consider the sub-interval $[x_{k-1}, x_k]$ for $k = 1, \ldots, n$. An estimate of the error of this sub-interval is given by our function g:

$$g(x) = f(x) - f(x_{k-1}) - \frac{(f(x_k) - f(x_{k-1}))(x - x_{k-1})}{h}$$

We have previously proven that:

$$\int_{x_{k-1}}^{x_k} g(x)dx = -\frac{1}{2}\int_{x_{k-1}}^{x_k} (x - x_{k-1})(x_k - x)g''(x)dx$$

Because g is $\mathcal{C}^2$ on the finite interval, we know that it's second derivative must be bounded. Let us say:

$$M = \max_{x\in[a,b]} |g''(x)|$$

By definition, $g''(x) = f''(x)$ so:

$$\left|\int_{x_{k-1}}^{x_k} g(x)dx\right| \leq \frac{1}{2}\int_{x_{k-1}}^{x_k} (x - x_{k-1})(x_k - x)|f''(x)|dx$$

$$\leq \frac{M}{2}\int_{x_{k-1}}^{x_k} (x - x_{k-1})(x_k - x)dx$$

$$= \frac{M}{2}\int_{x_{k-1}}^{x_k} (-x^2 + (x_{k-1} + x_k)x - x_{k-1}x_k)dx$$

$$= \frac{M}{12}(x_k - x_{k-1})^3$$

Noting that we've used $x_k - x_{k-1}$ here instead of $b - a$, we have:

$$= \frac{M}{12}(b - a)^3$$

c

Note that $b - a = h$. So:

$$T = \frac{M}{12}h^3$$

which is clearly $\mathcal{O}(h^3)$.

d

Using the prior work (i.e. considering the sum of all partitions for global error), we can further say:

$$\left|\int_a^b f(x)dx - T\right| \leq \sum_{k=1}^{n} \frac{1}{12}Mh^3 = \frac{1}{12}Mh^3 n = \frac{1}{12}M(b - a)h^2$$

Problem 2: Deriving Adams-Moulton, not 1, but 2

Problem Statement

Consider a scalar ODE $\dot{x}(t) = f(t, x(t))$ with $x(0)$ as the initial point. With a step-size of $h > 0$ and $t_n = nh$ for $n \geq 0$ we have:

$$x(t_{n+1}) = x(t_n) + \int_{\tau=t_n}^{\tau=t_{n+1}} f(\tau, x(\tau))d\tau$$

Adams-Moulton seeks to compute x_n recursively as:

$$x_{n+1} = x_n + \int_{\tau=t_n}^{\tau=t_{n+1}} g(\tau)d\tau \tag{1}$$

where $g(\tau)$ is a polynomial approximation to $f(\tau, x(\tau))$. Define $G_n := f(t_n, x_n)$. Then, the AM(2) method seeks a quadratic approximation to g that takes the values

$$g(t_{n-1}) = G_{n-1},\ g(t_n) = G_n,\ g(t_{n+1}) = G_{n+1} \tag{2}$$

Consider g of the form

$$g(\tau) = G_{n-1}L_{n-1}(\tau) + G_nL_n(\tau) + G_{n+1}L_{n+1}(\tau) \tag{3}$$

where L's are the Legendre polynomials defined by:

$$L_{n-1}(\tau) := \frac{(\tau - t_n)(\tau - t_{n+1})}{(t_{n-1} - t_n)(t_{n-1} - t_{n+1})},\ L_n(\tau) := \frac{(\tau - t_{n-1})(\tau - t_{n+1})}{(t_n - t_{n-1})(t_n - t_{n+1})},\ L_{n+1}(\tau) := \frac{(\tau - t_{n-1})(\tau - t_n)}{(t_{n+1} - t_{n-1})(t_{n+1} - t_n)}$$

a

Verify that g defined in (3) satisfies (2).

b

Compute $\int_{\tau=t_n}^{\tau=t_{n+1}} L_k(\tau)d\tau$ for $k = n-1, n, n+1$.

c

Plug your results from part (b) into the relation

$$x_{n+1} = x_n + \sum_{k=n-1}^{n+1} G_k \int_{\tau=t_n}^{\tau=t_{n+1}} L_k(\tau)d\tau$$

Replace G_k with $f(t_k, x_k)$ for $k = n-1, n, n+1$ in the above equation to find the implicit relation among x_{n+1}, x_n, and x_{n-1} for the AM(2) method. Compare your result to AM(2) in the class notes or Wikipedia.

d

Design an approximate AM(2) method by utilizing one step of a method of your choice to solve the implicit equation you derived at each iteration, starting from the forward Euler solution.

Solution

a

Consider $g(t_{n-1})$ and note that $\tau = t_{n-1}$ causes the numerators of $L_n(\tau)$ and $L_{n+1}(\tau)$ to go to zero. Furthermore, note that:

$$L_{n_1}(t_{n-1}) = \frac{(t_{n-1}-t_n)(t_{n-1}-t_{n+1})}{(t_{n-1}-t_n)(t_{n-1}-t_{n+1})} = 1$$

$$\implies g(t_{n-1}) = G_{n-1}\checkmark$$

Considering $g(t_n)$, we see that for $\tau = t_n$ makes the numerators of L_{n-1}, L_{n+1} go to zero. Furthermore, note that:

$$L_n(t_n) = \frac{(t_n-t_{n-1})(t_n-t_{n+1})}{(t_n-t_{n-1})(t_n-t_{n+1})} = 1$$

$$\implies g(t_n) = G_n\checkmark$$

To save on some typing, we can see that this pattern continues for t_{n+1} and, indeed:

$$g(t_{n+1}) = G_{n+1}\checkmark$$

b

Using Wolfram (using a,b, and c instead of t_{n-1}, t_n, t_{n+1} and then making the appropriate replacements...), we find:

$$\int_{t_n}^{t_{n+1}} L_{n-1}(\tau)d\tau = \frac{(t_n-t_{n+1})^3}{6(t_{n-1}-t_n)(t_{n-1}-t_{n+1})}$$

$$\int_{t_n}^{t_{n+1}} L_n(\tau)d\tau = \frac{(t_n-t_{n+1})(3t_{n-1}-2t_n-t_{n+1})}{6(t_n-t_{n-1})}$$

$$\int_{t_n}^{t_{n+1}} L_{n+1}(\tau)d\tau = -\frac{(t_n-t_{n+1})(3t_{n-1}-t_n-2t_{n+1})}{6(t_{n-1}-t_{n+1})}$$

Note that:

$$t_n - t_{n-1} = h,\ t_{n+1}-t_n = h,\ t_{n+1}-t_{n-1} = t_{n+1}-t_n+t_n-t_{n-1} = 2h$$

$$\frac{(t_n-t_{n+1})^3}{6(t_{n-1}-t_n)(t_{n-1}-t_{n+1})} = \frac{(-h)^3}{6(-h)(-2h)} = \frac{-h}{12}$$

The next two are slightly trickier, consider $\int L_n$:

$$\frac{(t_n-t_{n+1})(3t_{n-1}-2t_n-t_{n+1})}{6(t_n-t_{n-1})}$$

The left term in the numerator is $-h$, the denominator is $6h$. The right numerator is:

$$3t_{n-1}-2t_n-t_{n+1} = -(t_{n+1}+2t_n-3t_{n-1}) = -(t_{n+1}-t_n+t_n+2t_n-3t_{n-1})$$

$$= -(t_{n+1}-t_n+3t_n-3t_{n-1}) = -(h+3h) = -4h$$

Recombining we get:

$$= \frac{-h(-4h)}{6h} = \frac{2h}{3}$$

Finally, consider the last integral:

$$-\frac{(t_n - t_{n+1})(3t_{n-1} - t_n - 2t_{n+1})}{6(t_{n-1} - t_{n+1})}$$

Starting with the top-right portion of the numerator:

$$3t_{n-1} - t_n - 2t_{n+1} = -(2_{n+1} + t_n - 3t_{n-1}) = -(2_{n+1} - 2t_n + 3t_n - 3t_{n-1})$$

$$= -(2h + 3h) = -5h$$

The left term of the numerator becomes $-h$, and the denominator becomes $-12h$, so the full fraction becomes:

$$-\frac{-h(-5h)}{-12h} = \frac{5h}{12}$$

c

$$x_{n+1} = x_n + \sum_{k=n-1}^{n+1} G_k \int_{\tau=t_n}^{\tau=t_{n+1}} L_k(\tau)d\tau = x_n + f(t_{n-1}, x_{n-1})\left(\frac{-h}{12}\right) + f(t_n, x_n)\left(\frac{2h}{3}\right) + f(t_{n+1}, x_{n+1})\left(\frac{5h}{12}\right)$$

$$x_{n+1} = x_n + \sum_{k=n-1}^{n+1} G_k \int_{\tau=t_n}^{\tau=t_{n+1}} L_k(\tau)d\tau = x_n + \frac{h}{12}\left(5f(t_{n+1}, x_{n+1}) + 8f(t_n, x_n) - f(t_{n-1}, x_{n-1})\right)$$

Which indeed matches (with a change in indexing convention) the Wikipedia page about Linear multi-step methods.

d

Based upon the prior homework, we note that for the Trapezoidal rule (i.e. AM(1)) and Backward Euler (i.e. AM(0)) with one-step fixed-point iteration, we substitute $x_{n+1} = x_n + hf(x_n, t_n)$ So:

$$x_{n+1} = x_n + \frac{h}{12}\left(5f(t_{n+1}, x_n + hf(x_n, t_n)) + 8f(t_n, x_n) - f(t_{n-1}, x_{n-1})\right)$$

Use your desired solver (I'd probably go with Newton-Raphson) to solve the sub-problems. The only other quirk is how to initialize x_{n-1} for step 0. I'd either set it to zero or set it equal to x_n.

Bibliography

[1] Andre Heck and Marthe Schut. *Sowiso - Uva*. URL: https://uva.sowiso.nl/courses/theory/128/95/1604/en.