ECE 530 (Fall 2023) **Instructor:** S. Bose

Homework 6

Assigned: Oct 21 Due: Oct 31

Instructions: *Problems marked (**) are optional. You are encouraged to discuss with your peers. You need to, however, turn in your own solutions. Mention the names of your collaborators. Submissions made after but within one week of the due date will be marked on 50% of the total points. You don't need to submit after that.*

Problem 1. Trapezoid o' trapezoid! [12+5+3 = 20 points]

Consider a scalar ODE $\dot{x}(t) = f(t, x(t))$ starting from $x(0)$. Then, the trajectory $x(t)$ over $t \in [0, T]$ satisfies

$$x(t_{n+1}) = x(t_n) + \int_{\tau=t_n}^{\tau=t_{n+1}} f(\tau, x(\tau))d\tau$$

for $t_n = nh$. Let $g(\tau) := f(\tau, x(\tau))$. Assume that g is twice continuously differentiable over $[0, T]$.

(a) If $0 \leq a < b \leq T$, then prove that g satisfies

$$\int_a^b g(\tau)d\tau = \frac{1}{2}(b-a)[g(a)+g(b)] - \frac{1}{2}\int_a^b \left[\left(\frac{b-a}{2}\right)^2 - \left(\tau - \frac{a+b}{2}\right)^2\right] g''(\tau)d\tau \qquad (1)$$

Hint: *Use integration by parts.*

(b) Then, utilize the identity in part (a) to prove that there exists a constant M such that

$$\left|\int_a^b g(\tau)d\tau - \frac{1}{2}(b-a)[g(a)+g(b)]\right| \leq \frac{(b-a)^3}{12}M. \qquad (2)$$

(c) Deduce that the local truncation error of trapezoidal method is $\mathcal{O}(h^3)$, i.e.,

$$x(t_{n+1}) = x(t_n) + \frac{h}{2}\left[f(t_n, x(t_n)) + f(t_{n+1}, x(t_{n+1}))\right] + \mathcal{O}(h^3).$$

(d) (**) [15 points] Under possibly additional assumptions, prove that the global error for the trapezoidal method is $\mathcal{O}(h^2)$, i.e., there exists a constant C, such that

$$|x(t_n) - x_n| \leq Ch^2$$

for all $0 \leq t_n \leq T$.

Problem 2. Deriving Adams-Moulton, not 1, but 2. [6+12+4+3= 25 points]

Consider a scalar ODE $\dot{x}(t) = f(t, x(t))$ with $x(0)$ as the initial point. With a step-size of $h > 0$ and $t_n = nh$ for $n \geq 0$, we have

$$x(t_{n+1}) = x(t_n) + \int_{\tau=t_n}^{\tau=t_{n+1}} f(\tau, x(\tau)) d\tau.$$

Adams-Moulton method seeks to compute x_n recursively as

$$x_{n+1} = x_n + \int_{\tau=t_n}^{\tau=t_{n+1}} g(\tau) d\tau, \tag{3}$$

where $g(\tau)$ is a polynomial approximation to $f(\tau, x(\tau))$. Define $G_n := f(t_n, x_n)$. Then, AM(2) method seeks a quadratic approximation to g that takes the values

$$g(t_{n-1}) = G_{n-1}, \quad g(t_n) = G_n \quad g(t_{n+1}) = G_{n+1}. \tag{4}$$

Consider g of the form

$$g(\tau) = G_{n-1} L_{n-1}(\tau) + G_n L_n(\tau) + G_{n+1} L_{n+1}(\tau), \tag{5}$$

where L's are the Legendre polynomials defined by

$$L_{n-1}(\tau) := \frac{(\tau - t_n)(\tau - t_{n+1})}{(t_{n-1} - t_n)(t_{n-1} - t_{n+1})},\ L_n(\tau) := \frac{(\tau - t_{n-1})(\tau - t_{n+1})}{(t_n - t_{n-1})(t_n - t_{n+1})},\ L_{n+1}(\tau) := \frac{(\tau - t_{n-1})(\tau - t_n)}{(t_{n+1} - t_{n-1})(t_{n+1} - t_n)}.$$

(a) Verify that g defined in (5) satisfies (4).

(b) Compute $\int_{\tau=t_n}^{\tau=t_{n+1}} L_k(\tau) d\tau$ for $k = n-1, n, n+1$.

(c) Plug your results from part (b) into the relation

$$x_{n+1} = x_n + \sum_{k=n-1}^{n+1} G_k \int_{\tau=t_n}^{\tau=t_{n+1}} L_k(\tau) d\tau, \tag{6}$$

Replace G_k by $f(t_k, x_k)$ for $k = n-1, n, n+1$ in the above equation to find the implicit relation among x_{n+1}, x_n and x_{n-1} for the AM(2) method. Compare your result to AM(2) in class notes or Wikipedia.

(d) Design an approximate AM(2) method by utilizing one step of a method of your choice to solve the implicit equation you derived at each iteration, starting from the forward Euler solution.