Trapezoid o' trapezoid!

Problem Statement

Consider the scalar ordinary differential equation (ODE) $\dot{x} = f(t, x(t))$ starting from $x_0 = x(0)$. Then, the trajectory $x(t)$ over $t \in [0, T]$ satisfies:

$$x(t_{n+1}) = x(t_n) + \int_{\tau=t_n}^{\tau=t_{n+1}} f(\tau, x(\tau))d\tau$$

for $t_n = nh$. Let $g(\tau) \coloneqq f(\tau, (x(\tau)))$. Assume that g is $\mathcal{C}^2$ over $[0, T]$.

a

If $0 \leq a \leq b \leq T$, then prove that g satisfies:

$$\int_a^b g(\tau)d\tau = \frac{1}{2}(b-a)[g(a) + g(b)] - \frac{1}{2}\int_a^b \left[\left(\frac{b-a}{2}\right)^2 - \left(\tau - \frac{a+b}{2}\right)^2\right] g''(\tau)d\tau$$

Hint: Use integration by parts.

b

Then, use the identity in part a) to prove that there exists a constant M s.t.

$$\left|\int_a^b g(\tau)d\tau - \frac{1}{2}(b-a)[g(a)+g(b)]\right| \leq M\frac{(b-a)^3}{12}$$

c

Deduce that the local truncation error of the trapezoidal method is $\mathcal{O}(h^3)$; i.e.:

$$x(t_{n+1}) = x(t_n) + \frac{h}{2}[f(t_n, x(t_n)) + f(t_{n+1}, x(t_{n+1}))] + \mathcal{O}(h^3)$$

d (*optional*)

Under possibly additional assumptions, prove that the global error for the trapezoidal method is $\mathcal{O}(h^2)$; i.e., there exists some constant C s.t.

$$|x(t_n) - x_n| \leq Ch^2 \forall t \in [0, T]$$

Solution

a

Recall that IBP is:

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$$

Letting $u :== g(\tau)$ and $vd\tau$, then:

$$\int_a^b g(\tau)d\tau = g(\tau)\tau\Big|_a^b - \int_a^b \tau g'(\tau)d\tau$$

Using IBP again for the right-most term, this time letting $u = g'(\tau)$ and $dv = \tau d\tau$:

$$= g(\tau)\tau\Big|_a^b - \left(\left[g'(\tau) - \frac{1}{2}\tau^2\right]_a^b - \int_a^b \tau g''(\tau)d\tau\right)$$

$$= g(\tau)\tau\Big|_a^b - \left[g'(\tau) - \frac{1}{2}\tau^2\right]_a^b + \int_a^b \tau g''(\tau)d\tau$$

I used [1] for reference.

b

Re-arrange the identity to:

$$\int_a^b g(\tau)d\tau - \frac{1}{2}(b-a)[g(a)+g(b)] = -\int_a^b \left[\left(\frac{b-a}{2}\right)^2 - \left(\tau - \frac{a+b}{2}\right)^2\right] g''(\tau)d\tau$$

Note that, because $g(\tau)$ is $\mathcal{C}^2$ over $[0,T]$ (a bounded domain), this implies that $|g''(\tau)|$ is bounded. Using the power of foresight, let us define M as follows:

$$M = 2g''\left(\arg\max_{\tau\in[0,T]} |g''(\tau)|\right)$$

Or, in English, the most "extreme" value that g'' can take (rather than the maximum or minimum). Then: $\int^{max}$

c

d

Globally, the error will be the sum of each local interval; i.e.:

$$\int_a^b g(\tau)d\tau = \int_{a_0}^{a_1} g(\tau)d\tau + \ldots + \int_{a_n}^b g(\tau)d\tau$$

The error on a subinterval

$$\int_{a_i}^{a_{i+1}} g(\tau)d\tau = -\frac{a_{i+1}-a_i}{12} g''(\tau)$$

Problem 2: Deriving Adams-Moulton, not 1, but 2

Problem Statement

Consider a scalar ODE $\dot{x}(t) = f(t, x(t))$ with $x(0)$ as the initial point. With a step-size of $h > 0$ and $t_n = nh$ for $n \geq 0$ we have:

$$x(t_{n+1}) = x(t_n) + \int_{\tau=t_n}^{\tau=t_{n+1}} f(\tau, x(\tau))d\tau$$

Adams-Moulton seeks to compute x_n recursively as:

$$x_{n+1} = x_n + \int_{\tau=t_n}^{\tau=t_{n+1}} g(\tau)d\tau \tag{1}$$

where $g(\tau)$ is a polynomial approximation to $f(\tau, x(\tau))$. Define $G_n := f(t_n, x_n)$. Then, the AM(2) method seeks a quadratic approximation to g that takes the values

$$g(t_{n-1}) = G_{n-1},\ g(t_n) = G_n,\ g(t_{n+1}) = G_{n+1} \tag{2}$$

Consider g of the form

$$g(\tau) = G_{n-1}L_{n-1}(\tau) + G_nL_n(\tau) + G_{n+1}L_{n+1}(\tau) \tag{3}$$

where L's are the Legendre polynomials defined by:

$$L_{n-1}(\tau) := \frac{(\tau - t_n)(\tau - t_{n+1})}{(t_{n-1} - t_n)(t_{n-1} - t_{n+1})},\ L_n(\tau) := \frac{(\tau - t_{n-1})(\tau - t_{n+1})}{(t_n - t_{n-1})(t_n - t_{n+1})},\ L_{n+1}(\tau) := \frac{(\tau - t_{n-1})(\tau - t_n)}{(t_{n+1} - t_{n-1})(t_{n+1} - t_n)}$$

a

Verify that g defined in (3) satisfies (2).

b

Compute $\int_{\tau=t_n}^{\tau=t_{n+1}} L_k(\tau)d\tau$ for $k = n-1, n, n+1$.

c

Plug your results from part (b) into the relation

$$x_{n+1} = x_n + \sum_{k=n-1}^{n+1} G_k \int_{\tau=t_n}^{\tau=t_{n+1}} L_k(\tau)d\tau$$

Replace G_k is a polynomial approximation to $f(\tau, x(\tau))$.

Solution

Bibliography

[1] Ed Bender. URL: https://mathweb.ucsd.edu/~ebender/20B/77_Trap.pdf.