ECE 530 (Fall 2023) **Instructor:** S. Bose

Homework 8

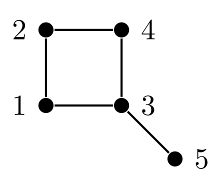
Assigned: Nov 9 Due: Nov 17

Instructions: *Problems marked (**) are optional. You are encouraged to discuss with your peers. You need to, however, turn in your own solutions. Mention the names of your collaborators. Submissions made after but within one week of the due date will be marked on 50% of the total points. You don't need to submit after that. Please turn in any code.*

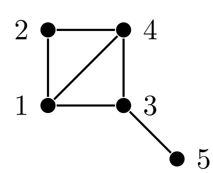
Problem 1. Appreciate Cholesky graphically [6+7+7 = 20 points]

Preamble: We need a few definitions for this problem.

- For any symmetric matrix $\mathbf{B} \in \mathbb{R}^{m \times m}$, define $\mathcal{G}(\mathbf{B})$ as a graph on m nodes that describes the sparsity pattern of $\mathbf{B}$. That is, two different nodes i and j share an edge if and only if $\mathbf{B}_{ij} = \mathbf{B}_{ji} \neq 0$. Extend the above to similarly define the graph describing the sparsity pattern of a lower triangular matrix.
- We say a node *is eliminated* from a graph to denote the operation where said node is removed, and possibly new edges are introduced to connect each of the neighbors of the deleted node. For example, eliminating node 2 from G_1 in Figure 1(a) will define a graph on nodes 1, 3, 4, and 5, with all edges of G_1 between these nodes and an extra edge between 1 and 4.
- A graph is said to be *chordal*, if all its minimal cycles are triangles. By minimal cycles, we mean cycles that do not have a chord in them. See Figure 1 for an example.



(a) Graph G_1



(b) Graph G_2

Figure 1: G_1 is not a chordal graph because it has a 4-membered minimal cycle $(1, 2, 4, 3)$. On the other hand, G_2 is a chordal graph because its minimal cycles $(1, 2, 4)$ and $(1, 3, 4)$ are triangles.

(a) Find a matrix $\mathbf{A} \in \mathbb{R}^{5\times 5}$ such that $\mathcal{G}(\mathbf{A}) = G_1$ in Figure 1(a) and $\mathbf{A} \succ 0$.
Hint: *Associate positive numbers or 'weights' to each node and each edge of G_1. Define $\mathbf{A}_{ij} = \mathbf{A}_{ji}$ as the negative of the weight on the edge (i, j) of G_1. Define $\mathbf{A}_{ii}$ to be the sum of the weight of node i and the weights of all edges connected to node i. Such a matrix is called a Laplacian matrix for graph G_1. Compute the eigenvalues of your matrix to verify that it is indeed positive definite.*

(b) Compute the lower triangular Cholesky factor $\mathbf{L} \in \mathbb{R}^{5\times 5}$ with positive diagonals such that $\mathbf{L}\mathbf{L}^\top$ equals $\mathbf{A}$ in part (a). Draw a graph on 5 nodes that describes the sparsity pattern of $\mathbf{L}$, i.e., draw $\mathcal{G}(\mathbf{L})$. Verify

that $\mathcal{G}(\mathbf{L})$ is a chordal graph.
Hint: *The MATLAB command for obtaining the Cholesky factorization of* $\mathbf{A}$ *is* `chol(A, 'lower')`.

(c) Notice that G_2 in Figure 1(b) is a chordal graph. Find a permutation matrix $\mathbf{P} \in \mathbb{R}^{5\times 5}$ such that when you compute the Cholesky factor $\mathbf{L}'$ of $\mathbf{PAP}^\top$ with your $\mathbf{A}$ from part (a), then $\mathcal{G}(\mathbf{L}') = G_2$.
Hint: *Eliminate the nodes of* G_1 *in the sequence* $(5, 2, 3, 1, 4)$. *Encode that elimination order in a permutation matrix.*

(d) (**)[10 points] Let $\mathbf{X} \in \mathbb{R}^{n\times n}$ be a positive definite matrix and $\mathbf{\Gamma} \in \mathbb{R}^{n\times n}$ be a lower triangular matrix with positive diagonals, such that $\mathbf{X} = \mathbf{\Gamma}\mathbf{\Gamma}^\top$. Prove that $\mathcal{G}(\mathbf{X})$ is a subgraph of $\mathcal{G}(\mathbf{\Gamma})$ and $\mathcal{G}(\mathbf{\Gamma})$ is a chordal graph.
Hint: *Use an induction argument on the size of the matrix. Characterize how the steps in Cholesky factorization shapes the sparsity pattern of* $\mathbf{\Gamma}$. *Finally, utilize the following fact that if a node and a set of edges from that node is added to a chordal graph G, then the obtained graph is chordal if and only if every two neighbors of the new node in G already shared an edge in G.*

Problem 2. Split 'em matrices for linear systems. **[8+6+8+8 = 30 points]**

Consider a linear system of equations $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is invertible. The matrix splitting method splits $\mathbf{A}$ as $\mathbf{A} = \mathbf{M} - \mathbf{N}$ and successively solves the equation

$$\mathbf{x}^{k+1} = \mathbf{M}^{-1}\left(\mathbf{N}\mathbf{x}^k + \mathbf{b}\right) \tag{1}$$

for $k \geq 0$ to perform a fixed-point iteration in order to solve the linear system $\mathbf{Ax} = \mathbf{b}$. Identify $\mathbf{L}$, $\mathbf{D}$ and $\mathbf{U}$ as the strictly lower triangular, diagonal, and the strictly upper triangular part of $\mathbf{A}$. Then, $\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$. When $\mathbf{M} = \mathbf{L} + \mathbf{D}$, then the fixed-point iteration in (1) is equivalent to

$$\mathbf{x}_i^{k+1} := \frac{1}{\mathbf{A}_{ii}}\left[b_i - \sum_{j<i}\mathbf{A}_{ij}\mathbf{x}_j^{k+1} - \sum_{j>i}\mathbf{A}_{ij}\mathbf{x}_j^{k}\right] \tag{2}$$

for $i = 1, \ldots, n$ and $k \geq 0$.

(a) Write the update equation for each element of $\mathbf{x}$ as in (2) when $\mathbf{M} = \mathbf{D} + \mathbf{U}$ using backward induction to compute the iterates in (1) successively.

(b) Call the method you devised in part (a) as the *backward* Gauss-Siedel method. Which among backward Gauss-Siedel and Jacobi methods would you expect to converge faster to the solution of $\mathbf{Ax} = \mathbf{b}$? Which one will you choose if distributed computation is involved, and why?

(c) Implement both backward Gauss-Siedel and Jacobi methods to solve the linear system of equations $\mathbf{Ax} = \mathbf{b}$ with

$$\mathbf{A} := \begin{pmatrix} 10 & 5 & 3 & 4 \\ 4 & 10 & 2 & 1 \\ 1 & 3 & 8 & 2 \\ 1 & 6 & 3 & 9 \end{pmatrix}, \qquad \mathbf{b} := \begin{pmatrix} 4 \\ -5 \\ 4 \\ -11 \end{pmatrix},$$

starting from $\mathbf{x}^0 := (1,1,1,1)^\top$. Plot the residues and errors in the successive iterates on a semilog plot. Iterate till the residue falls below 10^{-5}. The error and the residue, respectively, at iteration k are given by

$$\varepsilon^k := \|\mathbf{x} - \mathbf{x}^*\|, \qquad r^k := \left\|\mathbf{b} - \mathbf{A}\mathbf{x}^k\right\|,$$

where $\mathbf{x}^*$ is a solution of the linear system of equations.

(d) Recall that the method of successive over relaxation (SOR) uses $\mathbf{M} = \frac{1}{\omega}\mathbf{D} + \mathbf{L}$. Any matrix splitting method converges to the solution of a linear system, if the *spectral radius* of the matrix $\mathbf{I} - \mathbf{M}^{-1}\mathbf{A}$ is less than 1. Plot the spectral radius of $\mathbf{I} - \mathbf{M}^{-1}\mathbf{A}$ for SOR as a function of ω in the range $[0.01, 2.20]$ for 20 randomly generated positive definite $\mathbf{A}$'s on the same graph. Based on your plot, can you guess for what values of ω, SOR converges for positive definite matrices?
Hint: *To randomly generate positive definite matrices, first create* $\mathbf{G} \in \mathbb{R}^{n\times n}$ *using the MATLAB commands 'randn' or 'rand', and then define* $\mathbf{A} = \mathbf{G}\mathbf{G}^\top$.