ECE 530 (Fall 2023) **Instructor:** S. Bose

Homework 1

Assigned: Aug 29 Due: Sep 6

Instructions: *You are encouraged to discuss with your peers. You need to, however, turn in your own solutions. Mention the names of your collaborators. Crediting other classmates does not take away any points from you. Any submissions made after but within one week of the due date will be marked on 50% of the total points. You don't need to submit after that.*

Problem 1. Playing with Newton Rhapson. [10+10+5+10+5 points]

(a) Using Newton Rhapson method, solve the following system of equations in $\mathbf{x} := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, given by

$$f_1(\mathbf{x}) := 4x_2^2 + 4x_2 + 52x_1 - 19 = 0,$$
$$f_2(\mathbf{x}) := 169x_1^2 + 3x_2^2 + 111x_1 - 10x_2 - 10 = 0,$$

starting from $\mathbf{x}^0 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

- Report $\mathbf{x}^1$ and $\mathbf{x}^2$.
- Terminate your iteration when
$$\left\| \begin{pmatrix} f_1(\mathbf{x}^k) \\ f_2(\mathbf{x}^k) \end{pmatrix} \right\|_2 < \varepsilon,$$
where choose $\varepsilon = 10^{-10}$. Report if the process does not terminate within 100 iterations. If it terminates, call the last iterate $\mathbf{x}^*$. Report $\mathbf{x}^*$ and the number of iterations till convergence, i.e., k for which $\mathbf{x}^k = \mathbf{x}^*$.
- Plot the error magnitudes $\|\mathbf{x}^k - \mathbf{x}^*\|_2$ as a function of k on a semilog plot.

(b) Repeat part (a), but use *forward difference approximation* to numerically evaluate each entry of the Jacobian at each iteration. Use a step-size of 0.5 for the approximation. That is, use $[f(z + 0.5) - f(z)]/0.5$ in place of $f'(z)$ for any scalar function f of interest.

(c) Repeat part (a) with *center difference approximation* to numerically evaluate the Jacobian. Use the same step-size as in part (b).

(d) Repeat part (a), but use Jacobian surrogates defined by Broyden's method. Clearly state how you start the sequence of the Jacobian surrogates and how you choose $\mathbf{x}^1$.

(e) Vary the starting point appropriately and discuss qualitatively how the iterative processes in parts (a), (b), (c), (d) behave. If you had to choose between the methods in parts (b), (c), or (d), which one would you choose? Will your answer change if evaluating f_1, f_2 is computationally much more intensive?

Problem 2. Understanding numerical differentiation. [9+9+2 points]

We have empirically seen in class that the approximation quality of center difference approximation is far superior to that of forward and backward difference ones, when performing numerical differentiation. In this problem, we gain analytical insight into this behavior.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a thrice continuously differentiable function over an interval $[a, b]$. The following Taylor's expansion of f might be useful in answering the questions.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f^{''}(x) + \frac{h^3}{6}f^{'''}(\xi),$$

for all x and $x + h$ in (a, b), where ξ lies on the line segment joining x and $x + h$.

(a) Show that

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \le M|h|$$

for all $x \in (a, b)$ for some positive constant M.

(b) Show that

$$\left| \frac{f(x+h) - f(x-h)}{2h} - f'(x) \right| \le N|h|^2$$

for all $x \in (a, b)$ for some positive constant N.

(c) Qualitatively discuss what the results in parts (a) and (b) mean.