Homework #1

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Math 233: Homework #1

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Question 1. The following fact was tacitly used in this chapter: if A, B are disjoint subsets of the plane, A is compact, B is closed, then there exists a $\delta > 0$ such that, for all $\alpha \in A$, $\beta \in B$, $|\alpha - \beta| \geq \delta > 0$. Prove this for $A, B \subset X$ for X an arbitrary metric space.

Solution. Let X be a metric space, $A \subseteq X$ compact, $B \subseteq X$ closed, $A \cap B = \emptyset$

Suppose not. Then, there exist pairs of points (α_n, β_n) such that $d(\alpha_n, \beta_n) < \frac{1}{n}$. Now, consider the sequence of points $\{\alpha_n\}_{n=1}^{\infty}$. Since A is compact, we know that there exists a subsequence $\{\alpha_{n_k}\}_{k=1}^{\infty}$, convergent to α.

Let $\epsilon > 0$ be given. Since $\alpha_{n_k} \to \alpha$, choose N_k such that $d(\alpha, \alpha_{n_k}) < \frac{\epsilon}{2}$ for all $n_k > N_k$. Choose N such that $\frac{1}{n} < \frac{\epsilon}{2}$ for all $n > N$. Choose M_k such that $M = \max(N, N_k)$. Assume $m > M, m \in \{n_k\}_{k=1}^{\infty}$. Consider the sequence of $\{\beta_{n_k}\}_{k=1}^{\infty}$, and in particular, consider:

$$d(\alpha, \beta_m) \leq d(\alpha, \alpha_m) + d(\alpha_m, \beta_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, we have that $\beta_{n_k} \to \alpha$. Since $\{\beta_{n_k}\}_{k=1}^{\infty} \subset B$, a closed set, $\alpha \subset B$, because closed sets contain its limit points. But, this is a contradiction. Thus, $\delta > 0$ exists. □

Question 2.

Solution. □

Question 3. Suppose f, g are entire functions, and suppose that for all $z \in \mathbb{C}$, that $|f(z)| \leq |g(z)|$. What conclusion can you draw?

Solution. Claim: for some $m \in \mathbb{C}$, $f = mg$.

First suppose $g = 0$. Then, since $|f| \leq |g| = 0$, this implies that $f = 0$ everywhere. Then, of course $f = mg$, for actually any m.

Now, suppose not. Then, define $Z(g) = \{z \in \mathbb{C} : g(z) = 0\}$, that is, the zero set of g, and consider the function $h = \frac{f}{g}$. By the algebra of holomorphic functions, we have that h is holomorphic on at least $\mathbb{C} \setminus Z(g)$.

Because $\mathbb{C}$ is of course a connected open set, we have the result that $Z(g)$ has no limit points in $\mathbb{C}$. Then, let $a \in Z(g)$. Because a is not a limit point, there exists $r > 0$ such that $D(a, r) \cap Z(g) = \emptyset$. We have then that h is holomorphic on $D(a, r) \setminus \{a\}$, a region. Further, on $\mathbb{C} \setminus Z(g)$, we have that $|h| = \frac{|f|}{|g|} \leq 1$. So, in particular, on $D'(a, \frac{r}{2}) = \{z \in \mathbb{C} : 0 < |z - a| < \frac{r}{2}\} \subseteq \mathbb{C} \setminus Z(g)$, we have that h is bounded. Then, by Theorem 10.20 from Rudin, we have that f has a removable singularity at a.

Now, we recall from Theorem 10.18, that $Z(g)$ is at most countable. So, we may patch h countably many times at each point in $Z(g)$ to produce a holomorphic function everywhere, which we call $\tilde{h}$. Further, since $\tilde{h}$ is holomorphic, it must be continuous everywhere. Thus, since $|\tilde{h}(z)| \leq 1$ at every point other than $z \in Z(g)$, we must have that $|\tilde{h}(z)| \leq 1$ everywhere by continuity. Thus, we have that $\tilde{h}$ is a bounded, entire function, and by Liouville's Theorem, it must be constant, that is, $\tilde{h} = k$ for some $k \in \mathbb{C}$. Then, we have that at least on $\mathbb{C} \setminus Z(g)$, that $f(z) = kg(z)$.

However, $kg(z)$ is certainly holomorphic, and it agrees with $f(z)$ almost everywhere, which of course is a set with limit points in Ω. Thus, $f = kg$ everywhere.

□

Question 4. Suppose that f is an entire function, and

$$|f(z)| \leq A + B|z|^k$$

for all z, where A, B, k are positive real numbers. Prove that f must be polynomial.

Solution. Because f is entire, it is analytic, specifcally at $a = 0$, with infinite radius of convergence. Then, we may rewrite f as:

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

Now, we apply Theorem 10.22. We have that:

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta$$

Here, we use our hypothesis. Since we have that $|f(z)| \leq A + B|z|^k$, we must have that:

$$|f(re^{i\theta})| \leq A + B|re^{i\theta}|^k = A + Br^k$$

Thus, using our first equation then, we have a bound:

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} (A + Br^k)^2 d\theta = (A + Br^k)^2$$

Now, suppose we have that $c_n \neq 0$ for some $n > k$. Then, we would have that:

$$\frac{|c_n| r^{2n}}{(A + Br^k)^2} = \frac{|c_n| r^{2(n-k)}}{(\frac{A}{r^k} + B)^2}$$

Now, since f is entire and thus the radius of convergence is infinite, we may take the limit as $r \to \infty$. But, since $n > k$, we have that:

$$\lim_{r \to \infty} \frac{|c_n| r^{2(n-k)}}{(\frac{A}{r^k} + B)^2} = \infty$$

Then, $c_n = 0$ for every $n > k$. Then, this implies that we have that

$$f(z) = \sum_{n=0}^{\lfloor k \rfloor} c_n z^n$$

and since this holds everywhere, with finite degree, f is polynomial.

□

Question 5.

Solution. □