Homework #5

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2.1

Problem 3.4.7. Let $E \subseteq \mathbb{R}^d$ be a measurable set such that $|E| < \infty$, and assume that f_n and f are measurable functions that are finite a.e. and satisfy $f_n \to f$ a.e. on E. Prove that there exist measurable sets $E_k \subseteq E$ such that $E \setminus (\cup_{k=1}^{\infty} E_k)$ has measure zero and for each individual k, we have that $f_n \to f$ uniformly on E_k. Even so, show by example that f_n need not converge uniformly to f on E.

Solution. By Egorov's Theorem, we can find a measurable set $A_n \subseteq E$ such that $|A_n| < 1/n$ and f_n converges to f uniformly on $|E \setminus A_n|$. We notice that because A_n is measurable, and E is measurable, then $E_n = E \setminus A_n$ is also measurable. We can see that by looking at $E \setminus (\cup_k E_k)$:

$$E \setminus (\cup_k E_k) = E \cap (\cup_k E_k)^c = E \cap (\cap_k E_k^c) = \cap_k (E \cap E_k^c) = \cap_k A_k$$

where we use the fact that because we define $E_k = E \setminus A_k$, then it follows that $E \setminus E_k = A_k$. But, we have that $\cap_k A_k \subseteq A_k$ for every k, so $|\cap_k A_k| < 1/n$ for all n, so $|E \setminus (\cup_k E_k)| = |\cap_k A_k| = 0$.

However, take for example the example in the book 3.4.1:

$$f_n(x) = \begin{cases} 0, & \text{at } x = 0 \\ \text{linear}, & \text{for } 0 < x < \frac{1}{2n} \\ 1, & \text{at } x = \frac{1}{2n} \\ \text{linear}, & \text{for } \frac{1}{2n} < x < \frac{1}{n} \\ 0, & \text{for } \frac{1}{n} \leq x \leq 1 \end{cases}$$

This is a family of functions indexed on n, finite a.e., and pointwise converges to $f = 0$ on $[0, 1]$. Fix an $x_0 \in [0, 1]$. If $x_0 = 0$, then $f_n(0) = 0$ for all n, so suppose $x_0 > 0$. Consider $|f_n(x_0)|$. By the Archimidean principle, there exists $N \in \mathbb{N}$ such that $x_0 > \frac{1}{N}$. In particular, then, take $n > N$. Since $\frac{1}{n} < \frac{1}{N} < x_0 < 1$, by construction of our function, we have $f(x_0) = 0$.

Further, we may take $A_k = [0, 1/k]$. We can see that, using the same argument, $f_n \to 0$ uniformly on $(1/k, 1]$ as if we take $n > k$, then, for every $x \in (1/k, 1]$, $x > 1/k > 1/n$, so $f_n(x) = 0$. So, $\sup_{x \in (1/k,1]}(f_n(x)) = 0$ for every $n > k$, and so $\lim_{n\to\infty}(\sup_{x\in(1/k,1]} |f_n(x) - 0|) = 0$. However, we are still not uniformly continuous on all of $[0, 1]$ because for any n, $\sup_{x\in[0,1]} |f_n - 0| \geq 1 > 0$, because for each n, $f_n(1/2n) = 1$, so the limit of the supremums is at least 1. Thus, we are not uniformly convergent on $[0, 1]$ to $f = 0$.

□

2.2

Problem 4.1.12. Let E be a measurable subset of $\mathbb{R}^d$. Suppose that f, g are measurable functions on E such that $0 \leq f \leq g$ and $\int_E f < \infty$. Prove that $g - f$ is measurable, $0 \leq \int_E (g - f) \leq \infty$, and, as extended real numbers:

$$\int_E (g-f) = \int_E g - \int_E f$$

Solution. Firstly, since $\int_E f < \infty$, by Lemma 4.1.8, we know that $f < \infty$ a.e. If $g < \infty$ a.e., then by 3.2.2 $g - f$ is measurable. Otherwise, suppose $g = \infty$ on $A \subseteq E$ with $|A| > 0$, and g is finite on $E \setminus A$. It should be clear that A is measurable, because $A = \{g = \infty\} = \cap_{n=1}^{\infty}\{g > n\}$, a countable intersection of measurable sets. Then, we have that $E \setminus A$ is also measurable. Then, by 3.2.2 again, $g - f$ is measurable on $E \setminus A$, as on $E \setminus A, f, g$ are finite a.e. Further, on A, we have that $g - f = \infty$. So, then, we can see that $\{g - f > a\}_E = \{g - f > a\}_{E\setminus A} \cup A$, which is the union of two measurable sets, thus measurable.

Now, because $0 \leq f \leq g$, we have that $0 \leq g - f$. Then, taking the integrals of both sides, we have that $0 = \int_E 0 \leq \int_E g - f$. On the other side, since $\int_E \phi \in [0, \infty]$ for arbitrary simple functions, and $\int_E (g - f) = \sup(\{\int_E \phi : 0 \leq \phi \leq (g - f), \phi \text{ simple }\})$, then $\int_E (g - f) \leq \infty$. So we have $0 \leq \int_E (g - f) \leq \infty$.

Now, first, assume $g \not< \infty$ for a.e. $x \in E$. Then, by 4.1.8, by the contrapositive of (d), we have that $\int_E g = \infty$. Further, suppose $g = \infty$ on exactly $A \subseteq E$, $|A| > 0$. Suppose we take $Z = \{x \in E : f(x) = \infty\}$. Then, we look at $g - f$ on $A \setminus Z$. By construction, we have that g is infinite and f is finite here. Further, since f finite a.e., we have that $|Z| = 0$, so $|A \setminus Z| > 0$. Then, again by (d), we have that $\int_E g - f = \infty$, and since $\int_E f < \infty$ by hypothesis, we are done, as we have $\infty = \infty - \int_E f$, which is true.

Now, suppose $g < \infty$ a.e. as well. Consider $\int_E (g - f) + \int_E f = \sup(\{\int_E \phi : 0 \leq \phi \leq (g - f), \phi \text{ simple }\}) + \sup(\{\int_E \phi : 0 \leq \psi \leq f, \psi \text{ simple }\})$, where we notice $g - f$ and f are nonnegative measurable functions. Let ϕ_n, ψ_n be simple functions $0 \leq \phi_n \leq (g - f)$ and $0 \leq \psi_n \leq f$, and $\int_E \phi_n \to \int_E (g - f), \int_E \psi_n \to \int_E f$. We notice that $\phi_n + \psi_n$ is a simple function as well, and $0 \leq \phi_n + \psi_n \leq (g - f) + f = g$. Since this is true for any arbitrary ϕ_n, ψ_n simple functions approximating $g - f, f$ and we can find a simple function that approximates g by their sum, we have that $\int_E (g - f) + \int_E f \leq \int_E g$. However, by properties of the sup, we also have then that for ϕ, ψ as above, we have that $\sup(\phi_n + \psi_n) \leq \sup_n \phi_n + \sup_n \psi_n$, so we have that $\int_E g \leq \int_E (g - f) + \int_E f$. So we have equality, and since $\int_E f < \infty$, we may rearrange to recover $\int_E (g - f) = \int_E g - \int_E f$. □

2.3

Problem 4.2.11. Assume $E \subseteq \mathbb{R}^d$ and $f : E \to [0, \infty]$ are measurable, and $\int_E f < \infty$. Given $\epsilon > 0$, prove that there exists a measurable set $A \subseteq E$ such that $|A| < \infty$ and $\int_A f \geq \int_E f - \epsilon$.

Solution. Construct the sets $A_n = E \cap [-n, n]^d$. First of all, these are measurable, being the intersection of a box and a measurable set. Clearly, we have that since $A_n \subseteq [-n, n]^d$, $|A_n| \leq (2n)^d < \infty$. Further, since $\cup_n^{\infty}[-n, n]^d = \mathbb{R}^d$, we have that $\cup_n^{\infty} A_n = E$. Further, since $[-n, n]^d \subseteq [-(n + 1), n + 1]^d$, we have that $A_1 \subset A_2 \subset ....$ Then, we may apply the result of exercise 4.2.5, and we have that $\int_E f = \lim_{n\to\infty} \int_{A_n} f$. Now, let $\epsilon > 0$ be given. Then by the definition of the limit, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|\int_{A_n} f - \int_E f| < \epsilon$. Take any $n > N$. Since $A_n \subseteq E$, from 4.1.8, we know that $\int_{A_n} f \leq \int_E f < \infty$, so we have that $\int_E f - \int_{A_n} f < \epsilon$, and because they are both finite, we may rearrange to find $\int_E f - \epsilon < \int_{A_n} f$ as desired.

□

Problem 4.2.14. Let f be a continuous, non-negative function on the interval $[a, b]$. Prove that the Riemann integral on $[a, b]$ coincides with its Lebesgue integral $\int_a^b f(x)dx$.

Solution. Take Γ be a partition of $[a, b]$, $\Gamma = \{a = x_0 < ... < x_n = b\}$. Consider the lower Riemann sum $L_\Gamma = \Sigma_{j=1}^n m_j(x_j - x_{j-1})$ where if we call $E_j = [x_j - x_{j-1}]$, $m_j = \inf_{x\in E_j} f(x)$. , and construct the simple function $\phi = \Sigma_j^n m_j \chi_{E_j}$, which must be simple because we have only finitely many intervals. By the definition of the m_j's, we have that $0 \leq \phi \leq f$. Further, we see that $L_\Gamma = \int_E \phi \leq \int_E f(x)$, where $E = [a, b]$ because if we look at what the Lebegue integral of ϕ is, it is exactly L_Γ as $|\chi_{E_j}| = (x_j - x_{j-1})$. Now, take the upper Riemann sum, $U_\Gamma = \Sigma_j^n M_j(x_j - x_{j-1}), M_j = \sup_{x\in E_j} f(x)$, and construct the simple function $\psi = \Sigma_j^n M_j \chi_{E_j}$, simple for the same reason as ϕ. In particular, this is a simple function where we have $f \leq \psi$. Then, we have that $\int_E f \leq \int_E \psi = U_\Gamma$. So, we have that for a fixed partition, $L_\Gamma \leq \int_E f \leq U_\Gamma$. However, now, we use that because f is continuous, f is integrable. Then, we know that $\sup_\Gamma L_\Gamma = \inf_\Gamma U_\Gamma = I \in \mathbb{R}$. Since $L_\Gamma \leq \int_E f$ was true for any Γ, and $\int_E f \leq U_\Gamma$ as well, we can say that $\sup_\Gamma L_\Gamma \leq \int_E f \leq \inf_\Gamma U_\Gamma$. But, by the squeeze theorem then, $\int_E f = \sup_\Gamma L_\Gamma = \inf_\Gamma U_\Gamma = I$ and we are done. □

Problem 4.2.17. Let $f : E \to [0, \infty]$ be a nonnegative, measurable function defined on a measurable set $E \subseteq \mathbb{R}^d$.

(a) Define the graph of f as:

$$\Gamma_f = \{(x, f(x)) : x \in E, f(x) < \infty\}$$

Show that $|\Gamma_f| = 0$.

(b) The region under the graph of f is the set R_f that consists of all points $(x, y) \in \mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$ such that $x \in E$, and y satisfies:

$$\begin{cases} 0 \leq y \leq f(x), & \text{if } f(x) < \infty, \\ 0 \leq y < \infty, & \text{if } f(x) = \infty. \end{cases}$$

Show that R_f is a measurable subset of $\mathbb{R}^{d+1}$ and its Lebesgue measure is:

$$|R_f| = \int_E f(x)dx$$

Solution. (a)

First, define $A = \{x \in E : f(x) = \infty\}$, and define $g = f|_{E\setminus A}$ to be the restriction of f on g. We notice that the graph is only defined on $E \setminus A$, so this is a reasonable restriction to make. Now, consider, for $n \geq 1$, the sets $E_n = E \cap ([-(n+1), n+1]^d \setminus [-n, n]^d)$. We notice that for each E_n, since $E_n \subseteq ([-(n+1), n+1]^d \setminus [-n, n]^d)$, that is, between the cube of side length 2n and 2n+2 centered on 0, that E_n is in particular bounded, and measurable as they are the intersection of two measurable sets. Further, by our restriction, g is still measurable, and finite a.e. So, for each E_n, we apply Luzin's theorem as follows: take the sequence $\{1/m\}_{m\in\mathbb{N}}$, and construct $F_{E_n,m}$ to be a closed set such that $|E_n \setminus F_{E_n,m}| < 1/m$ and g is continuous on $F_{E_n,m}$. In particular then, consider $\cup_m F_{E_n,m}$ and call this B_n. This is a union of measurable sets, thus measurable. Further, we have that $|E_n \setminus \cup_m F_{E_n,m}| < 1/m$ for all $m \geq 1$, so $|E_n \setminus \cup_m F_{E_n,m}| = 0$. Call this set Z_n. Now, via our identification of $E = \cup_n E_n$, we may divide E into $E = \cup_n B_n \cup \cup_n Z_n$. Since g is continous on each B_n, it must be continous on their union, we have that the graph of a continous function has measure 0. Now, consider $\cup_n Z_n \times \mathbb{R}$. Clearly, the graph of g is a subset of this set, and this set, being a Cartesian product of spaces, has measure $|\cup_n Z_n||\mathbb{R}|$. But, this is a union of sets of measure 0, and each disjoint by the construction of E_n, so $|\cup_n Z_n| = 0$, and by our convention, we have that $0 * \infty = 0$. So, this is a set of measure 0, and thus the graph on $\cup_n Z_n$ has measure 0. Since the graph has measure 0 both on $\cup_n Z_n$ and $\cup_n B_n$, it has measure 0 on $E \setminus A$, which is exactly its domain, and so we are done.

(b)

By Theorem 3.2.14, we have that there exists simple functions ϕ_n such that $0 \leq \phi_1 \leq ...$ and $\lim_{n\to\infty} \phi_n(x) = f(x)$ for all $x \in E$. Consider the graph of each function R_{ϕ_n} defined in the same way as R_f, that is, $R_{\phi_n} = \{(x, y) : x \in E, 0 \leq y \leq \phi_n(x)\}$. Look at any $\phi_n = \Sigma_{k=1}^m c_k \chi_{E_k}$. By the defintion of R_{ϕ_n} then, we have that $R_{\phi_n} = \cup_{k=1}^m E_k \times [0, c_k]$. However, we see that because the E_k are disjoint, each of these pieces are disjoint. Further, by 2.3.7, we have that $E_k \times [0, c_k]$ is measurable and takes on the value $|E_k \times [0, c_k]| = |E_k|c_k$. Then, R_{ϕ_n} is measurable, as it is the union of measurable sets, and since they are disjoint, we have that $|R_{\phi_n}| = \Sigma_{k=1}^m c_k|E_k|$, but that is exactly the definition of $\int_E \phi_n$, so $|R_{\phi_n}| = \int_E \phi_n$.

Now, because we have that $0 \leq \phi_1 \leq ...$, we have that $R_{\phi_1} \subseteq R_{\phi_2} \subseteq ...$ because suppose $(x, y) \in R_{\phi_k}$. Then, because $(x, y) \in x \times [0, \phi_k(x)]$. Since we have that $\phi_k \leq \phi_{k+1}$, we must have that, at x, $\phi_{k+1}(x) \geq$

$\phi_k(x)$. Then, $y \in [0, \phi_k(x)] \subseteq [0, \phi_{k+1}(x)]$, and so $(x, y) \in R_{\phi_{k+1}}$. Since this is true for arbitrary (x, y), we have the nested sets that we desired. Then, we have that by continuity from below, we have that $|\cup_{i=1}^{\infty} R_{\phi_i}| = \lim_{n\to\infty} |R_{\phi_n}| = \lim_{n\to\infty} \int_E \phi_n$. On one side, we have that because $\phi_n \nearrow f$ by the construction of 3.2.14, by the monotone convergence theorem, we have that $\lim_{n\to\infty} \int_E \phi_n = \int_E f$. On the other side, we claim that $\cup_{i=1}^{\infty} R_{\phi_i} = R_f$. It is clear that because $0 \leq \phi_1 \leq ...$ and $\phi_n \nearrow f$, that $R_{\phi_i} \subseteq R_f$ for all i. Then, we have that $\cup_{i=1}^{\infty} R_{\phi_i} \subseteq R_f$. Now, suppose we have a point $(x, y) \in \{x\} \times [0, f(x)]$ if $f(x) < \infty$ or $(x, y) \in \{x\} \times [0, \infty)$. In the first case, since $\lim_{n\to\infty} \phi_n(x) = f(x)$ for all $x \in E$, we may find $N \in \mathbb{N}$ such that $f(x) - \phi_n(x) < f(x) - y$ for all $n > N$. Then, we see that rearranging this inequality, we have that $y < \phi_n(x)$ for such an n, so we have that $(x, y) \in \{x\} \times [0, \phi_n(x)] \subseteq R_{\phi_n}$. Now, suppose we're in the second case. Then, we have that by the convergence, since $\lim_{n\to\infty} \phi_n(x) = f(x)$ still, that we can find $M \in \mathbb{M}$ such that for all $m > M$, $\phi_m(x) > y$. Then, for such an m, we have that $(x, y) \in R_{\phi_m}$. Therefore, regardless of the finiteness of $f(x)$, we have that there exists some n such that $(x, y) \in R_{\phi_n}$. Since the choice of (x, y) was arbitrary, we have that $R_f \subseteq \cup_{i=1}^{\infty} R_{\phi_i}$. Then, that means we have:

$$|R_f| = |\cup_{i=1}^{\infty} R_{\phi_i}| = \lim_{n\to\infty} |R_{\phi_n}| = \lim_{n\to\infty} \int_E \phi_n = \int_E f$$

$\square$