Homework #3

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Math 237: Homework #3

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Question 1. Show that the only function $f \in L^1(\mathbb{R})$ such that $f = f * f$ is $f = 0$ almost everywhere.

Solution. Let $f \in L^1(\mathbb{R})$. Suppose we have that $f = f * f$. We recall that the Fourier transform distributes pointwise over convolution, hence we have that:

$$\hat{f}(\zeta) = \widehat{f * f}(\zeta) = \hat{f}(\zeta)\hat{f}(\zeta)$$

Hence, we already have that at each ζ, $\hat{f} = 0, 1$, by solving the equation $\hat{f}(\zeta)^2 - \hat{f}(\zeta) = 0$.

From Lemma 9.2.3 in Heil, we have that $\hat{f}$ is continuous. Moreover, from the Riemann-Lesbesgue lemma, since $f \in L^1$, we already know that $\hat{f} \in C_0(\mathbb{R})$. Hence, by continuity, since $\hat{f}$ may only take on values $0, 1$, and that it is continuous, $\hat{f} = 0$ everywhere.

Thus, by Corollary 9.2.12 in Heil, since $\hat{f} = 0$ everywhere, it is true almost everywhere of course, and hence $f = 0$ almost everywhere.

□

Question 6. Suppose that $f \in \mathrm{AC}(\mathbb{T})$, that is, 1 periodic and absolutely continuous on $[0, 1]$.

6.1)

Prove that $\hat{f}'(n) = 2\pi i n \hat{f}(n)$ for all $n \in \mathbb{Z}$. Conclude that $\lim_{|n| \to \infty} n\hat{f}(n) = 0$.

6.2)

Show that if $\int_0^1 f(x)dx = 0$, then:

$$\int_0^1 |f(x)|^2 dx \leq \frac{1}{4\pi^2} \int_0^1 |f'(x)|^2 dx$$

Solution. 6.1)

First, we recall that by Corollary 6.1.5 in Heil, that since f is absolutely continuous on $[0, 1]$, we have that f is differentiable almost everywhere, and $f' \in L^1[0, 1]$. Since f' is at least L^1, we may look at its Fourier transform:

$$\hat{f}'(n) = \int_0^1 f'(x) \exp(-2\pi i n x) dx$$

We notice that for a fixed value of $n \in \mathbb{Z}$, that we may compute the derivative of $g = \exp(-2\pi i n x)$ as $g' = -2\pi i n \exp(-2\pi i n x)$. Hence, g is differentiable everywhere on $[0, 1]$, and of course, since $| - 2\pi i n \exp(-2\pi i n x)| = 2\pi n$ always, it is bounded. Hence, by Lemma 5.2.5, g is Lipschitz on $[0, 1]$. Thus, since Lipschitz functions are absolutely continuous (6.1.3), we may apply integration by parts on f, g. We have then that:

$$\int_0^1 f'(x) \exp(-2\pi i n x) dx = f(1) \exp(-2\pi i n) - f(0) \exp(0) - \int_0^1 f(x)[-2\pi i n \exp(-2\pi i n x)] dx$$

Now, because f is 1-periodic, we have that $f(0) = f(1)$. Further, $\exp(-2\pi in) = \exp(0)$, of course, since $2\pi n$ is a multiple of 2π for any n. Hence, the first terms vanish.

We are left then with, after using the linearity of the integral:

$$2\pi in \int_0^1 f(x)\exp(-2\pi inx)dx$$

However, we recognize the integral as exactly the Fourier transform of f at n. Hence, we have our result, that:

$$\hat{f}'(n) = 2\pi in\hat{f}(n)$$

Again, by the Riemann-Lebesgue lemma, we have that since $f' \in L^1$, that $\lim_{|n|\to\infty} \hat{f}'(n) = 0$. But, from our result, we have that:

$$\lim_{|n|\to\infty} 2\pi in\hat{f}(n) = \lim_{|n|\to\infty} \hat{f}'(n) = 0 \implies \lim_{|n|\to\infty} n\hat{f}(n) = 0$$

6.2)

First, we use the result from Heil Problem 9.3.24 (b) that the Plancherel equality holds for $f \in L^1(\mathbb{T})$. and prove that later.

Then, since $f' \in L^1(\mathbb{T})$, we have that:

$$\sum_{n\in\mathbb{Z}} |\hat{f}'(n)|^2 = \|f'\|_2^2 = \int_0^1 |f'(x)|^2 dx$$

Now, from 6.1, we have that $\hat{f}'(n) = 2\pi in\hat{f}(n)$ for each n. Hence, we have that:

$$\sum_{n\in\mathbb{Z}} |\hat{f}'(n)|^2 = \sum_{n\in\mathbb{Z}} 4\pi^2 n^2 |\hat{f}(n)|^2 = 4\pi^2 \sum_{n\in\mathbb{Z}} n^2 |\hat{f}(n)|^2$$

Now, we notice that since $\int_0^1 f(x)dx = 0$, that evidently, $\hat{f}(0) = 0$, as $\hat{f}(0) = \int_0^1 f(x)\exp(-2\pi i0x)dx = \int_0^1 f(x)dx = 0$.

Because this is 0, and for all $n \in \mathbb{Z}, n \neq 0$, we have that $|\hat{f}(n)| \leq |n\hat{f}(n)|$, we can conclude that:

$$4\pi^2 \sum_{n\in\mathbb{Z}} n^2 |\hat{f}(n)|^2 \geq 4\pi^2 \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$$

Finally, applying Plancherel's equaity again for f, as $f \in L^1(\mathbb{T})$, we see that:

$$4\pi^2 \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = 4\pi^2 \|f\|_2^2 = 4\pi^2 \int_0^1 |f(x)|^2 dx$$

And rewriting all of these inequalities together, we have that:

$$4\pi^2 \int_0^1 |f(x)|^2 \leq \int_0^1 |f'(x)|^2 dx \implies \int_0^1 |f(x)|^2 \leq \frac{1}{4\pi^2} \int_0^1 |f'(x)|^2 dx$$

as desired.

Now, we return to proving 9.3.24.

First, we wish to show that if $f \in L^1(\mathbb{T})$, and $\hat{f} \in L^2(\mathbb{Z})$, then $f \in L^2(\mathbb{T})$.

Next, we want to show that the Plancherel equality holds, either both sides being finite or infinite. If $f \in L^2 \cap L^1$, then we're fine, by Corollary 9.3.14.

Then, we assume $f \notin L^2(\mathbb{T})$ but $f \in L^1(\mathbb{T})$. Then, by the previous result, we have that $\hat{f} \notin L^2(\mathbb{Z})$. Hence, we have that both sides of the Plancherel equality are infinite, and we are done.

□

Question 12. Fix a $g \in L^2(\mathbb{R})$. Let $k \in \mathbb{Z}$, and define the operator T_k on g that sends $T_k(g(x)) \mapsto g(x-k)$. Prove that the family $\{T_k g\}_{k\in\mathbb{Z}}$ is an orthonormal sequence if and only if $\sum_{k\in\mathbb{Z}} |\hat{g}(\zeta - k)|^2 = 1$ almost everywhere.

Solution. First, suppose $T_k g$ is orthonormal. Consider the function $h(\zeta) = \sum_{k\in\mathbb{Z}} |\hat{g}(\zeta - k)|^2$. Since k varies over all $\mathbb{Z}$, evidently, h is 1-periodic. Moreover, since we know that the Fourier transform brings L^2 to L^2, we must have that $\hat{g} \in L^2$, and hence, $|\hat{g}|^2$ is in L^1. Since this integral converges, by the integral test then, so too must h at each point $\zeta \in [0, 1]$. In fact, since the integral over $[0, 1]$ is actually:

$$\int_0^1 |h(\zeta)| = \int_0^1 \sum_k |\hat{g}(\zeta - k)|^2 d\zeta = \sum_k \int_0^1 |\hat{g}(\zeta - k)|^2 d\zeta = \sum_k \int_k^{k+1} |\hat{g}(\zeta)|^2 d\zeta = \int_{\mathbb{R}} |\hat{g}(\zeta)|^2 d\zeta$$

Where we justify interchanging the sum and integral by taking partial sums, and noting that of course, for the partial sums, their integral over $[0, 1]$ is bounded by above by the integral of $|\hat{g}|^2$ over all of $\mathbb{R}$. Thus, $h \in L^1(\mathbb{T})$.

Then, consider the Fourier transform of h. We have that:

$$\hat{f}(n) = \int_0^1 h(\zeta) \exp(-2\pi i n\zeta) d\zeta = \int_0^1 \left(\sum_k |\hat{g}(\zeta - k)|^2 \right) \exp(-2\pi i n\zeta) d\zeta$$

Here, since we know that $\exp(-2\pi i n k) = 1$ for every $k \in \mathbb{Z}$ regardless of n, we may scale the phase such that $\zeta = \zeta - k$ when distributing the exponential. Thus, we have that this equals:

$$\int_0^1 \sum_k \left(|\hat{g}(\zeta - k)|^2 \exp(-2\pi i n(\zeta - k)) \right) d\zeta$$

Playing the same trick with interchanging sums and integrals, by using the fact that the exponential has modulus 1, we see that this is simply an integral over all of $\mathbb{R}$ again, and doing some algebra with the modulus, we see:

$$\int_0^1 \sum_k \left(|\hat{g}(\zeta - k)|^2 \exp(-2\pi i n(\zeta - k)) \right) d\zeta = \int_{\mathbb{R}} |\hat{g}(\zeta)|^2 \exp(-2\pi i n\zeta) d\zeta = \int_{\mathbb{R}} [\hat{g}(\zeta) \exp(-2\pi i n\zeta)] \overline{\hat{g}(\zeta)} d\zeta$$

Since $\hat{g} \in L^2$, of course $\overline{\hat{g}} \in L^2$, and so too is $\hat{g} \exp(-2\pi i n\zeta)$. Thus, we may view this as, in the lanugage of Heil 9.2.21, if M_a is the modulation operator that sends $f(x)$ to $\exp(2\pi i a x) f(x)$, as:

$$\int_{\mathbb{R}} [\hat{g}(\zeta) \exp(-2\pi i n\zeta)] \overline{\hat{g}(\zeta)} d\zeta = \langle M_{-n}\hat{g}, \hat{g} \rangle$$

Now, by Theorem 9.4.6, the Fourier transform on L^2 functions obeys the Parseval Identity. Moreover, by Heil 9.2.21, if T_b is the translation operator that sends $f(x) \mapsto f(x - b)$, then $\hat{T_a f} = M_{-a}\hat{f}$. (We will prove this separately). Thus, we have that:

$$\langle M_{-n}\hat{g}\hat{g} \rangle = \langle T_n g, g \rangle$$

By hypothesis then, this is exactly δ_n^0, as we had that $\{T_k g\}$ was orthonormal. We know that the function 1 has 1 as the zeroth coefficient, and 0 else. By the uniqueness theorem (9.3.12) then, we must have that $h = 1$ almost everywhere.

Now, suppose that $\sum_{k\in\mathbb{Z}} |\hat{g}(\zeta - k)|^2 = 1$ almost everywhere. Consider $\langle T_k g, T_{k'} g\rangle$ for some $k \in \mathbb{Z}$. Again, by the Parseval identity, we have that:

$$\langle T_k g, T_{k'} g\rangle = \langle M_{-k}\hat{g}, M_{-k'}\hat{g}\rangle = \int_{\mathbb{R}} \exp(-2\pi i\zeta k)\hat{g}(\zeta)\overline{\exp(-2\pi i\zeta k')\hat{g}(\zeta)}d\zeta = \int_{\mathbb{R}} \exp(-2\pi i\zeta(k-k'))|\hat{g}(\zeta)|^2$$

Now, playing the same trick to convert this into h and changing the integral into an integral over $[0,1]$ by slicing across intervals like $[k, k+1]$, we have this to be equal to:

$$\sum_n \int_0^1 \exp(-2\pi i\zeta(k-k'-n))|\hat{g}(\zeta-n)|^2 d\zeta = \int_0^1 \sum_n \left(\exp(-2\pi i\zeta(k-k'-n))|\hat{g}(\zeta-n)|^2\right) d\zeta$$

Factoring out the exponential, and using again that integer shifts are equivalent to multiplying by 1, we have that:

$$\int_0^1 \sum_n \left(|\hat{g}(\zeta-n)|^2\right) \exp(-2\pi i\zeta(k-k'))d\zeta = \hat{h}(k-k') = \delta^0_{k-k'}$$

where we get that $\hat{h}(k-k')$ aligns with $\delta^0_{k-k'}$ almost everywhere, but since we're working over $\mathbb{Z}$, this is true everywhere as the only sets of measure 0 are empty.

Hence, we have that $\{T_k g\}$ is orthonormal.

Now, we need to prove that $\hat{T_a f} = M_{-a}\hat{f}$, for $f \in L^1$, as the L^2 part comes from the continuity of these operators, and convergence in the L^2 Fourier transform sense.

Well, computing this directly, we see that by a change of variables $x \mapsto x + a$

$$\hat{T_a f}(\zeta) = \int_{\mathbb{R}} f(x-a)\exp(-2\pi in\zeta x)dx = \int_{\mathbb{R}} f(x)\exp(-2\pi in\zeta(x+a))dx =$$

$$\exp(-2\pi in\zeta a)\int_{\mathbb{R}} f(x)\exp(-2\pi in\zeta x)dx = \exp(-2\pi in\zeta a)\hat{f}(\zeta) = M_{-a}\hat{f}(\zeta)$$

as desired.

□

Question 14. Let $p(x) = \chi_{[0,1)}(x)$, $h(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x)$. Let $j, k \in \mathbb{Z}$, and define $I_{jk} = [2^{-j}k, 2^{-j}(k+1))$. Further define the following functions:

$$\begin{cases} p_{jk} = 2^{j/2}p(2^j x - k) \\ h_{jk} = 2^{j/2}h(2^j x - k) \end{cases}$$

14.1)

Prove that $\{h_{jk}\}$ is an orthonormal sequence in L^2.

14.2)

For each fixed $j \in \mathbb{Z}$, prove that $\{p_{jk}\}_{k\in\mathbb{Z}}$ is an orthonormal sequence in L^2.

14.3)

Fix a $j \in \mathbb{Z}$. Let g_j be any step function, constant on each interval I_{jk} for $k \in \mathbb{Z}$. Show that we may express $g_j(x) = g_{j-1}(x) + r_{j-1}(x)$, where

$$r_{j-1}(x) = \sum_{k\in\mathbb{Z}} a_{j-1}(k)h_{j-1,k}(x)$$

for some coefficients $a_{j-1}(k)$ and some step function $g_{j-1}(x)$, constant on intervals $I_{j-1,k}$.

14.4)

Fix a $J \geq 0$. Consider the set:

$$\{p_{Jk} : 0 \leq k \leq 2^J - 1\} \cup \{h_{j,k} : j \geq J, 0 \leq k \leq 2^j - 1\}$$

Prove that this set is an orthonormal sequence in $L^2[0,1]$.

14.5)

For $f \in L^2[0,1]$, and a fixed $J \geq 0$, show that we may find g_j step functions for $j \geq J$, such that they are constant on each $I_{j,k}$, and that g_j approximates f in the L^2 norm.

Use this result and the result of 14.4 to show that the set in 14.4 is an orthonormal basis for $L^2[0,1]$.

Solution. 14.1)

Let $j,k \in \mathbb{Z}$, and take two $h_{j,k}, h_{j',k'}$. We first notice by the definition of h_{jk}, that we may rewrite h_{jk} in terms of characteristic functions of $I_{j+1,k}$:

$$h_{jk} = 2^{j/2}(\chi_{I_{j+1,2k}} - \chi_{I_{j+1,2k+1}})$$

Consider $\langle h_{j,k}, h_{j',k'} \rangle$, where I will drop the complex conjugate as these are real-valued functions. In terms of characteristic functions of $I_{j,k}$, we have that:

$$\langle h_{j,k}, h_{j',k'} \rangle = \int_{\mathbb{R}} h_{j,k} h_{j',k'} = \int_{\mathbb{R}} 2^{j/2}(\chi_{I_{j+1,2k}} - \chi_{I_{j+1,2k+1}}) 2^{j'/2}(\chi_{I_{j'+1,2k'}} - \chi_{I_{j'+1,2k'+1}})$$

Without loss of generality, suppose $j \geq j'$, and switch labels if this is not true. First, suppose $j = j'$. Then, looking at the integrand and factoring out the $2^{j/2}$, we have that:

$$(\chi_{I_{j+1,2k}} - \chi_{I_{j+1,2k+1}})(\chi_{I_{j+1,2k'}} - \chi_{I_{j+1,2k'+1}}) = \chi_{I_{j+1,2k}}\chi_{I_{j+1,2k'}} - \chi_{I_{j+1,2k+1}}\chi_{I_{j+1,2k'}} - \chi_{I_{j+1,2k}}\chi_{I_{j+1,2k'+1}} + \chi_{I_{j+1,2k+1}}\chi_{I_{j+1,2k'+1}}$$

First, suppose $k = k'$. Then, we have that this is equal to:

$$\chi_{I_{j+1,2k}} + \chi_{I_{j+1,2k+1}}$$

as the square of a characteristic function is itself, and the cross terms vanish as $I_{j,k}$ and $I_{j,k'}$ are disjoint unless $k = k'$.

Then, in the case $j = j', k = k'$, we have that the integral evaluates as $2^j \int_{\mathbb{R}} \chi_{I_{j+1,2k}} + \chi_{I_{j+1,2k+1}} = 2^j(2^{-j-1} + 2^{-j-1}) = 1$.

Else, suppose $k \neq k'$. Then, we have that $\chi_{I_{j+1,2k}}\chi_{I_{j+1,2k'}}, \chi_{I_{j+1,2k+1}}\chi_{I_{j+1,2k'+1}}$ vanish. Looking at the remaining terms, we have:

$$-\chi_{I_{j+1,2k+1}}\chi_{I_{j+1,2k'}} - \chi_{I_{j+1,2k}}\chi_{I_{j+1,2k'+1}}$$

which, again, may be non-0 if and only if either $2k+1 = 2k'$ or $2k = 2k'+1$. However, since $k, k' \in \mathbb{Z}$, this is impossible. Hence, for all $k \neq k', j = j'$, this integral vanishes as desired.

Now, suppose $j < j'$. Due to the nested structure of dyadic intervals, for each $I_{j',2k'}, I_{j',2k'+1}$, these fit exactly within a single $I_{j,l}$ for some l. Hence, we can say that at least one of:

$$\begin{cases} \chi_{I_{j+1,2k}}[\chi_{I_{j'+1,2k'}} - \chi_{I_{j'+1,2k'+1}}] \\ \chi_{I_{j+1,2k+1}}[\chi_{I_{j'+1,2k'}} - \chi_{I_{j'+1,2k'+1}}] \end{cases}$$

vanishes.

If both vanish, then we are done, as the integral disappears as desired. WLOG, suppose the first term survives. Then, we have our integral as:

$$2^{j/2+j'/2}\int_{\mathbb{R}} \chi_{I_{j+1,2k}}\chi_{I_{j'+1,2k'}} - \chi_{I_{j+1,2k}}\chi_{I_{j'+1,2k'+1}}$$

Evidently, then, the first term takes on 1 on an interval of measure $2^{-j'-1}$ and the second term takes on -1 on an interval of measure $2^{-j'-1}$. Hence, they cancel out, and we have that for all $j < j'$, $k \in \mathbb{Z}$, that $\langle h_{j,k}, h_{j',k'}\rangle$. Hence, we have that:

$$\langle h_{j,k}, h_{j',k'}\rangle = \delta_j^{j'}\delta_k^{k'}$$

as desired.

14.2)

In a similar fashion to 14.1, but maybe slightly cleaner, we do the same procedure. Fix a choice of $j \in \mathbb{Z}$. First, we reexpress p_{jk} in terms of characteristic functions. We see that, in analogy to 14.1, that

$$p_{jk} = 2^{j/2}\chi_{I_{jk}}$$

as we can see that $2^j x - k$ takes $2^{-j}k$ to 0 and $2^{-j}k + 1$ to 1, hence takes $[2^{-j}k, 2^{-j}k+1)$ to $[0,1)$ as this is linear in x.

Then, considering $\langle p_{jk}, p_{jk'}\rangle$, dropping the complex conjugate again, we see that:

$$\langle p_{jk}, p_{jk'}\rangle = \int_{\mathbb{R}} 2^{j/2}\chi_{I_{jk}} 2^{j/2}\chi_{I_{jk'}} = 2^j\int_{\mathbb{R}} \chi_{I_{jk}}\chi_{I_{jk'}}$$

By definition, we may look at $I_{jk} \cap I_{jk'}$. We have that:

$$I_{jk} \cap I_{jk'} = [2^{-j}k, 2^{-j}(k+1)) \cap [2^{-j}k', 2^{-j}(k'+1))$$

Since $j, k \in \mathbb{Z}$, these intervals have endpoints at multiple of 2^{-j}, and hence these have overlap if and only if $k = k'$. Hence, we can say that:

$$2^j\int_{\mathbb{R}} \chi_{I_{jk}}\chi_{I_{jk'}} = 2^j\int_{\mathbb{R}} \chi_{I_{jk}}\delta_k^{k'} = 2^j|2^{-j}(k+1) - 2^{-j}k|\delta_k^{k'} = 2^j 2^{-k}\delta_k^{k'} = \delta_k^{k'}$$

Therefore, $\{p_{jk}\}_{k\in\mathbb{Z}}$ is an orthonormal sequence in L^2 for fixed $j \in \mathbb{Z}$.

14.3)

Fix a j, and a step function g_j, and consider the dyadic intervals one larger, $I_{j-1,k}$. Of course, each $I_{j-1,k}$ may be broken into $I_{j,2k} \cup I_{j,2k+1}$, as we may split the interval as:

$$[2^{-j+1}k, 2^{-j+1}(k+1)) = [2^{-j}2k, 2^{-j}(2k+2))) = [2^{-j}2k, 2^{-j}2k+1) \cup [2^{-j}2k+1, 2^{-j}2k+2)$$

Call $g_j(I_{j,2k}) = c$ and $g_j(I_{j,2k+1}) = d$. Define $g_{j-1}(I_{j-1,k}) = \frac{c+d}{2}$ and define $a_{j-1}(k) = \frac{c-d}{2}$. Then, on $I_{j,2k}, I_{j,2k+1}$, we have that:

$$\begin{cases} g_{j-1}(I_{j,2k}) + r_{j-1}(I_{j,2k}) = \frac{c+d}{2} + \frac{c-d}{2}\chi_{I_{j,2k}}(I_{j,2k}) = c \\ g_{j-1}(I_{j,2k+1}) + r_{j-1}(I_{j,2k+1}) = \frac{c+d}{2} - \frac{c-d}{2}\chi_{I_{j,2k}}(I_{j,2k}) = d \end{cases}$$

We may continue this construction for each $I_{j,k}$, and determine $g_{j-1}, r_{j-1}, a_{j-1}$ on the entire real line, with g_{j-1} a step function constant on $I_{j-1,k}$, $a_{j-1}(k)$ coefficients, and $r_{j-1} = \sum a_{j-1}(k)h_{j-1,k}$

14.4)

We have already shown $\{p_{Jk} : 0 \leq k \leq 2^J - 1\}$ to be orthonormal for $k \in \mathbb{Z}$, so of course this is orthonormal on its own, and similarly for $\{h_{j,k} : j \geq J, 0 \leq k \leq 2^j - 1\}$ for $j, k \in \mathbb{Z}$. So, we only need to show that $\langle p_{Jk}, h_{j,k'} \rangle = 0$ for any $j \geq J, 0 \leq k \leq 2^J - 1, 0 \leq k' \leq 2^j - 1$

Thus, we have that:

$$\langle p_{J,k}, h_{j,k'} \rangle = \int_{\mathbb{R}} p_{J,k} h_{j,k'} = \int_{\mathbb{R}} 2^{J/2} \chi_{I_{Jk}} [2^{j/2} (\chi_{I_{j+1,2k'}} - \chi_{I_{j+1,2k'+1}}) =$$

$$2^{J/2+j/2} \int_{\mathbb{R}} \chi_{I_{Jk}} \chi_{I_{j+1,2k'}} - \chi_{I_{Jk}} \chi_{I_{j+1,2k'+1}}$$

Now, in the same vein as 14.1 and the nesting of dyadic intervals, since $j+1 > j \geq J$, we must have that either both $I_{j+1,2k'+1}, I_{j+1,2k'+1} \subseteq I_{Jk}$ or neither are. If neither are, then this integral vanishes, and we are done.

Suppose then that both terms survive. Then, the first term takes on 1 on the interval $I_{j+1,2k'}$, and the second term takes on -1 on the interval $I_{j+1,2k'+1}$. Since these intervals have the same measure, the integral vanishes due to the opposite sign. Hence, we have that $\langle p_{Jk}, h_{j,k'} \rangle = 0$ for any $p_{Jk}, h_{j,k'}$ and thus, $\{p_{Jk} : 0 \leq k \leq 2^J - 1\} \cup \{h_{j,k} : j \geq J, 0 \leq k \leq 2^j - 1\}$ is a orthonormal sequence.

14.5)

Since $[0, 1]$ is compact, any step functions on dyadic intervals is aleady square integrable, including p_{jk}, h_{jk}.

First, suppose f is at least continuous. Construct g_j in the following way. Define $g_J = \sum_k \langle f, p_{J,k} \rangle p_{J,k}$. Then, define $g_{j+1} = g_j + \sum_k \langle f, h_{j,k} \rangle h_{j,k}$.

□

Question 16. Let ϕ be a non-0 function in $L^2(\mathbb{R})$. For any $f \in L^2(\mathbb{R})$, define $V_\phi f$ via:

$$V_\phi(f)(x, \zeta) = \int_{\mathbb{R}} f(t) \overline{\phi(t-x)} \exp(-2\pi i t \zeta) dt$$

For $a, b \in \mathbb{R}$, let T_a be the translation operator that sends $T_a(f(x)) \mapsto f(x-a)$ and let M_b the modulation operator that sends $M_b(f(x)) \mapsto \exp(2\pi b x) f(x)$.

16.1)

Prove that for each $f \in L^2$, $V_\phi f$ is uniformly continuous on $\mathbb{R}^2$, and that $\lim_{|(x,\zeta)| \to \infty} V_\phi f = 0$.

16.2)

Recall that the Schwarz space $\mathcal{S}(\mathbb{R})$ is defined as:

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R}) \text{ for all } m, n \geq 0\}$$

Prove that if $f \in \mathcal{S}(\mathbb{R})$, then $V_\phi \in S(\mathbb{R}^2)$.

16.3)

Prove that V_ϕ acts as an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^2)$, and that $\|V_\phi f\|_{L^2(\mathbb{R}^2)} = \|\phi\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}$ for every $f \in L^2$.

16.4)

Show that the operator V_ϕ^* defined by:

$$V_\phi^* F(t) - \|\phi\|_2^{-2} \iint_{\mathbb{R}^2} F(x, \zeta) \exp(2\pi i \zeta t) \phi(t-x) dx d\zeta$$

takes $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R})$, and that for each $f \in L^2(\mathbb{R})$, we can make sense of the following inversion formula:

$$f(t) = \|\phi\|_2^{-2} \iint_{\mathbb{R}^2} V_\phi f(x, \zeta) \exp(2\pi i \zeta t) \phi(t-x) dx d\zeta$$

Solution. □