Assignment

Eric Tao
Math 240: Homework #10

December 1, 2022

Problem 10.1. Let C be a projective, non-singular curve, D a divisor on C with degree $d > 0$ such that $\mathcal{L}(D)$ is base point-free, of dimension r. Let $\phi : C \to \mathbb{P}^r$ be the morphism associated to D.

(a) Projecting from a point $P \notin C$ induces a morphism $\phi_P : C \to \mathbb{P}^{r-1}$. Show that this morphism is associated to subseries of $\mathcal{L}(D)$.

(b) Projecting from a point $P \in C$ induces a rational map $\phi_P : C \setminus \{P\} \to \mathbb{P}^{r-1}$. Show that it extends to a morphism $\overline{\phi}_P : C \to= \mathbb{P}^{r-1}$. Identify a linear series that this morphism is associated to.

Solution. (a)

Viewing this as projecting from a point in the image space, we pick a linear transformation such that $P = (1, 0, ...0)$ is not on the curve, and the morphism is realized by $(f_1, ..., f_r)$ as a basis in $\mathcal{L}(D)$. We may do this since we can always take a change of variables so that we miss this point, and linear changes of variables merely permute the non-degenerate morphisms generated by $\mathcal{L}(D)$.

Now, if we view this as projecting onto the hyperplane $X_0 = 0$ in these coordinates, we notice that this leads to a projection onto $(0, f_2, ..., f_r)$. In particular, since these are members of $\mathcal{L}(D)$, that they cannot simultaneously vanish on the curve. Then, this cuts out a regular map onto a $\mathbb{P}^{r-1}$ subspace of $\mathbb{P}^r$ of the form where we lose, arbitrarily, one of the basis vectors, which makes this a subseries of $\mathcal{L}(D)$.

(b)

If we believe in part (a), we can do the same procedure, which is only ill-defined at the point $P \in C$. Then, this is a rational map from $C \setminus \{P\} \to \mathbb{P}^{r-1}$. However, in class, we proved that projective varieties cannot admit holes in the map. Then, this must extend to a regular map from all of $C \to \mathbb{P}^{r-1}$.

Looking at the action of the map, if we take a linear transformation to let $P = (1, 0...0)$ as before, then we see the same action as happening, where the subspace resembles us losing a basis polynomial.

□

Problem 10.2. (a) Show that any two effective divisors of degree d in $\mathbb{P}^1$ are linearly equivalent.

(b) Let C be a projective non-singular curve, D a divisor on C of degree $d > 0$, and such that $l(D) = \dim \mathcal{L}(D) = d + 1$. Show that $C = \mathbb{P}^1$.

(c) Show that if C is a projective non-singular curve that is not isomorphic to $\mathbb{P}^1$, then for any $d > 1$, there are effective divisors of degree d that are not linearly equivalent.

Solution. (a)

Fix a $d > 0$. Let $D = \Sigma_i^n c_i[P_i], D' = \Sigma_j^m d_j[Q_j]$ be effective divisors of $\mathbb{P}^1$. Consider $f = \Pi_k^m (x - Q_k)^{d_k}, g = \Pi_l^n (x - P_l)^{c_l}$, where, for my sanity, we take (x - y) to mean X_0 if $y_0 = 0$, and otherwise, $y_1/y_0 X_1 - X_0$ where X_0, X_1 are the formal variables for the 0th and 1st coordinates and y_0, y_1 are the coordinates of the point y. We notice that since D, D' have the same degree d, then f, g are homogenous polynomials of degree d. Then, we may look at g/f as a rational function. Since f has finitely many zeros, exactly $\{Q_1, ..., Q_m\}$, we can look at this quotient on the open set $\mathbb{P}^1 \setminus \{Q_1, ..., Q_m\}$, open because individual points are closed. Then, we notice that:

$$D(g/f) = \Sigma_l^n c_l[P_l] + \Sigma_k^m - d_m[Q_k] = D - D'$$

Thus, D, D' are linearly equivalent.

(b)

First, suppose $d = 1$. Then, we have a morphism $\phi : C \to \mathbb{P}^1$. Since we know morphisms of projective varieties are closed maps, this must land in a closed set in $\mathbb{P}^1$. In particular, we notice from the fact that we have dimension 2 in $l(D)$ that we have two linearly independent regular maps as our function into $\mathbb{P}^1$. Since they are linearly independent, this cannot be a finite set of points, and thus must be the whole thing.

Now, let $d > 1$. We apply 10.1(b) multiple times to reduce down by projecting from points in $C \to \mathbb{P}^d \to \mathbb{P}^{d-1} \to ... \to \mathbb{P}^1$ and we use the fact that we lose only one basis function at each step to conclude that we still have 2 basis vectors at the end, and thus still must be the whole thing.

(c)

Suppose $C \ncong \mathbb{P}^1$. Then, by part (b), we know that for any degree $d > 0$, we have that $\dim \mathcal{L}(D) \neq d+1$. Then, since we know that $\dim \mathcal{L}(D) \leq d+1$ in generality, this implies that $\dim \mathcal{L}(D) < d+1$. I'm not sure where to go from here, it seems like I should show that there are trivially dimension 1 and 2 divisors of degree $d > 1$, but I dont' see how the sections arise.

□

Problem 10.3. Let C be the twisted cubic parametrized by (s^3, s^2t, st^2, t^3).

(a) Show that the projection of the curve from the point $(1, 0, 0, 0)$ to the plane $X_0 = 0$ is a conic.

(b) Show that the projection from the point $(0, 1, 0, 0)$ onto the plane $X_1 = 0$ is a cuspidal cubic.

Solution. (a)

We consider first the image in the plane $X_0 = 0$. Let $A = (s^3, s^2t, st^2, t^3), B = (1, 0, 0, 0)$. In a projective space, a line is exactly $xA + yB$ for $x, y \in k$, our base field. Then, to be in our plane $X_0 = 0$, we solve for x, y. In particular, we look at the first coordinate, and extract the condition:

$$xs^3 + y = 0$$

If $x = 0$, then we have $y = 0$, so our point is identically 0, which is not allowed. Then, suppose $y = 0$. Then, this is only reasonable if $s = 0$, so that we are coming from the point $(0, 0, 0, t^3) = (0, 0, 0, 1)$, which we notice is already in our plane, which is fine. Then, assume $x, y \neq 0$. Then, we look at $y = -xs^3$. Substituting into the equation of our line, we find the point in the plane as being:

$$x(s^3, s^2t, st^2, t^3) + (-xs^3)(1, 0, 0, 0) = (0, xs^2t, xst^2, xt^3) = (0, s^2t, st^2, t^3)$$

Since we know that from our original curve that s, t cannot be both 0, as that would not be a valid point in $\mathbb{P}^3$, we are guaranteed that the last 3 coordinates never become identically 0. Then, we can project down into a $\mathbb{P}^2$ copy and retrieve the coordinates (s^2t, st^2, t^3). Looking at the parametrization, we notice that we can realize this as the zero locus of the polynomial: $V(Y_1^2 - Y_0Y_2)$, where we name the coordinates Y_0, Y_1, Y_2, which we identify as a conic, as it is a the zero locus of a degree 2 homogeneous polynomial.

(b)

In the same vein, we do the same procedure, and look at the condition from the second coordinate: $xs^2t + y = 0$. First, we see if $s = 0$, we're looking at the point $(0, 0, 0, t^3) = (0, 0, 0, 1)$ which is already in the hyperplane. Similarly, if $t = 0$, we're looking at $(s^3, 0, 0, 0) = (1, 0, 0, 0)$, also in the hyperplane. And, we see that if $x = 0$, then $y = 0$, and vice versa, so we may not allow either of those, if we assume $s, t \neq 0$. Then, in that case, we take $y = -xs^2t$. Substituting, we find:

$$x(s^3, s^2t, st^2, t^3) + (-xs^2t)(0, 1, 0, 0) = (xs^3, 0, xst^2, xt^3) = (s^3, 0, st^2, t^3)$$

Again, since we know s, t cannot be identically 0, we may look at this as a point in a $\mathbb{P}^2$, (s^3, st^2, t^3), and, by the shape of the parametrization, we notice that we may realize this as the zero locus of the polynomial $Y_1^3 - Y_0 Y_2^2$. Analyzing this polynomial for singular points, we compute the Jacobian as:

$$\mathcal{J} = [-Y_2^2, 3Y_1^2, -2Y_0 Y_2]$$

Looking for actual points, we notice by the first two entries that that forces $Y_2 = 0, Y_1 = 0$, but Y_0 remains free, so we expect $(1, 0, 0)$ to be a singular point.

Now, analyzing the singular point, we look at the tangent cone here. In particular, we look at the affine version where we delete $Y_0 = 0$. Then, we can take $Y_0 = 1$, since we can always scale to achieve this, and then this implies in this affine plane, we are looking at the polynomial $Y_1^3 - Y_2^2$. Looking at the tangent cone, this has form $-Y_2^2$, which has multiplicity 2, which is a cusp. Thus, this is a cuspidal cubic.

□