Homework #6

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Math 285: Homework #6

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Question 1. Show that the pullback of covectors by a linear map satisfies the two functorial properties:

(i) If $\mathbb{1}_V : V \to V$ is the identity map on V, then $\mathbb{1}_V^* = \mathbb{1}_{A_k(V)}$, the identity map on $A_k(V)$.

(ii) If $K : U \to V$ and $L : V \to W$ are linear maps on vector spaces, then:

$$(L \circ K)^* = K^* \circ L^* : A_k(W) \to A_k(U)$$

Solution. (i)

Let $v_1, ..., v_k \in V, f \in A_k(V)$, and by definition then, we have that:

$$\mathbb{1}_V^*(f)(v_1, ..., v_k) = f(\mathbb{1}_V(v_1), ..., \mathbb{1}_V(v_k)) = f(v_1, ..., v_k)$$

Then, since we have that $\mathbb{1}_V^*(f)$ and f act identically on arbitrary $v_1, ..., v_k \in V$, this implies that $\mathbb{1}_V^*(f) = f$. Since the choice of f was arbitrary, this is true for all $f \in A_k(V)$, and therefore, $\mathbb{1}_V^*$ acts as identity on $A_k(V)$, thus is equal to $\mathbb{1}_{A_k(V)}$.

(ii)

Let $f \in A_k(V)$, and let $v_1, ..., v_k \in V$. We may consider the action of $K^* \circ L^*$ on f:

$$K^* \circ L^*(f)(v_1, ..., v_k) = K^*(L^*(f))(v_1, ..., v_k) = L^*(f)(K(v_1), ..., K(v_k)) =$$
$$f(L(K(v_1)), ..., L(K(v_k))) = f(L \circ K(v_1), ..., L \circ K(v_k)) = (L \circ K)^*(f)(v_1, ..., v_k)$$

Again, since this is true for all $v_1, ..., v_k$, this is an equality of functions $K^* \circ L^*(f) = (L \circ K)^*(f)$. Since this is true for all $f \in A_k(V)$, this is an equality of maps $K^* \circ L^* = (L \circ K)^*$.

□

Question 2. Let $L : V \to V$ be a linear operator on a vector space with dimension n. Show that the pullback $L^* : A_n(V) \to A_n(V)$ acts as multiplication by the determinant of L.

Solution. We recall that from Proposition 3.36, that if $e_1, ..., e_n$ is a basis for V, and $\alpha^1, ..., \alpha^n$ is the dual basis in $V^\vee$, that for a multi-index $I = (i_1 < ... < i_k)$, the alternating k-linear functions have basis α^I. Then, of course, we say that the $A_n(V)$ are scalar multiples of $\alpha^1 \wedge ... \wedge \alpha^n$. Since the pullback is linear, we need only show that L^* acts as multiplication by its determinant on $\alpha^1 \wedge ... \wedge \alpha^n$.

Now, as a 1-linear function, consider the pullback $L * (\alpha^i)$. Considering an arbitrary vector $v = \sum_{j=1}^n v_j e_j \in V$, and writing A as a matrix in the e_j basis, we have that:

$$L^*(\alpha^i)(v) = \alpha^i(L(v)) = \alpha^i \left(\sum_{j=1}^n \left[\sum_{k=1}^n A_{jk} v_k \right] e_j \right) = \sum_{j=1}^n \left[\sum_{k=1}^n A_{jk} v_k \right] \alpha^i e_j = \sum_{j=1}^n \left[\sum_{k=1}^n A_{jk} v_k \right] \delta_i^j = \sum_{k=1}^n A_{ik} v_k$$

We notice, that because $v_k = \alpha^k(v)$, that we may rewrite this as:

$$L^*(\alpha^i)(v) = \sum_{k=1}^n A_{ik}\alpha^k(v)$$

Since the choice of v were arbitrary, this is an equality of covectors:

$$L^*(\alpha^i) = \sum_{k=1}^n A_{ik}\alpha^k$$

Then, we consider $L^*(\alpha^1 \wedge ... \wedge \alpha^n)(v_1, ..., v_n)$, for arbitrary vectors $v_1, ..., v_n \in V$. We see that:

$$L^*(\alpha^1 \wedge ... \wedge \alpha^n)(v_1, ..., v_n) = (\alpha^1 \wedge ... \wedge \alpha^n)(L(v_1), ..., L(v_n)) = A(\alpha^1 \otimes ... \otimes \alpha^n)(L(v_1), ..., L(v_n)) =$$

$$\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\alpha^1(L(v_{\sigma(1)}))...\alpha^n(L(v_{\sigma(n)})) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) L^*(\alpha^1)(v_{\sigma(1)})...L^*(\alpha^n)(v_{\sigma(n)}) =$$

$$A(L^*(\alpha^1) \otimes ... \otimes L^*(\alpha^n))(v_1, ..., v_n) = L^*(\alpha^1) \wedge ... \wedge L^*(\alpha^n)(v_1, ..., v_n)$$

Again, varying over all $v_1, ..., v_n$, we see an equality of covectors:

$$L^*(\alpha^1 \wedge ... \wedge \alpha^n) = L^*(\alpha^1) \wedge ... \wedge L^*(\alpha^n)$$

Now, by homework 1, question 7, we have that if $\beta^i = \sum_{j=1}^k a^i_j \gamma^j$, for two sets of covectors $\{\beta^i\}, \{\gamma^j\}$, $1 \leq i, j \leq k$, we have that:

$$\beta^1 \wedge ... \wedge \beta^k = \det(A)\gamma^1 \wedge ... \wedge \gamma^k$$

Taking $\beta^i = L^*(\alpha^i)$, and $\gamma^i = \alpha^i$, we see that because $L^*(\alpha^i) = \sum_{k=1}^n A_{ik}\alpha^k$, we have that:

$$L^*(\alpha^1 \wedge ... \wedge \alpha^n) = L^*(\alpha^1) \wedge ... \wedge L^*(\alpha^n) = \det(A)\alpha^1 \wedge ... \wedge \alpha^n$$

Thus, L^* acts on $A_n(V)$ by multiplication by $\det(A)$, from the linear and basis considerations before.

$\square$

Question 3. (a) Let $i : S^1 \to \mathbb{R}^2$ be the inclusion map of the unit circle. Denote the standard coordinates on $\mathbb{R}^2$ as (x, y) and denote the restriction of these coordinates to S^1 as $(\overline{x}, \overline{y})$. Clearly, we have that $\overline{x} = i^*(x), \overline{y} = i^*(y)$.

On the upper semicircle $U = \{(a, b) \in S^1 : b > 0\}$, $\overline{x}$ is a local coordinate, so $\partial/\partial\overline{x}$ is well-defined. Prove that for $p \in U$, we have that:

$$i_*\left(\left.\frac{\partial}{\partial \overline{x}}\right|_p\right) = \left.\left(\frac{\partial}{\partial x} + \frac{\partial \overline{y}}{\partial \overline{x}}\frac{\partial}{\partial y}\right)\right|_p$$

(b) Let C be a smooth curve in $\mathbb{R}^2$. Let U be a chart on C such that $\overline{x}$, the restriction of the coordinate x on $\mathbb{R}^2$ is a local coordinate.

Solution. (a)

Let $\epsilon > 0$.

First, we start with a curve $c : (-\epsilon, \epsilon) \to S^1 \cap U$ such that $c(0) = p, c'(0) = \frac{\partial}{\partial \overline{x}}$.

$\square$

Question 4. Let $f : GL_n(\mathbb{R}) \to \mathbb{R}$ be the determinant map $A \mapsto \det(A)$. Consider a matrix $B \in SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det(A) = 1\}$. By example 9.10 (note: numbering based off of Chapter 3, v1-1 in Canvas), for $A = [a_{kl}]$, there exists a (k, l) such that the partial dervative $\frac{\partial f}{\partial a_{kl}}(A) \neq 0$.

Use Lemma 9.9 and the implicit function theorem (9.8) to prove the following:

(a) There exists a neighborhood of A in $SL_n(\mathbb{R})$ such that a_{ij}, $(i, j) \neq (k, l)$ forms a coordinate system, and a_{kl} is a C^∞ function of the other entries.

(b) The group multiplication map:

$$\overline{\mu} : SL_n(\mathbb{R}) \times SL_n(\mathbb{R}) \to SL_n(\mathbb{R})$$

is C^∞.

Solution. (a)

Without too much trouble, it is easy to see f is a C^∞ map of manifolds, as we can view it as a subset of $\mathbb{R}^{n^2}$, so we may take charts compatible with standard coordinates being each matrix entry. Since the determinant is a degree n homogeneous polynomial in the matrix entries, it is C^∞ on this chart. Since we may take the open set of this chart to be all of $GL_n(\mathbb{R})$, we see f as a C^∞ map.

We notice that we can view $SL_n(\mathbb{R}) = f^{-1}(1)$. Thus, by Theorem 9.8, $SL_n(\mathbb{R})$ is a regular submanifold with dimension $n - 1$.

Fix some $A \in SL_n(\mathbb{R})$.

Now, following example 9.12 with the special linear group, defining m_{ij} as the determinant of the submatrix obtained by deleting the i-th row and the j-th column, we may rewrite the map f as, for a selected row $1 \leq i \leq n$:

$$f(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} m_{ij}$$

Since m_{ij}, varying across j, is obtained by deleting the i-th row, m_{ij} is not a function of a_{il} for any $1 \leq j, l \leq n$. Further, since the determinant of matrices in $SL_n(\mathbb{R})$ is exactly 1, by the determinental rank being n, there must exist (k, l) such that $m_{kl} \neq 0$

Then, for such a (k, l) we have that:

$$\frac{\partial f}{\partial a_{kl}} = \sum_{j=1}^{n} \frac{\partial}{\partial a_{kl}} (-1)^{k+j} a_{kj} m_{kj} = \sum_{j=1}^{n} (-1)^{k+j} \delta_j^l m_{kj} = (-1)^{k+j} m_{kl}$$

Since this is itself a C^∞, being a homogeneous polynomial of degree $n - 1$ in $GL_n(\mathbb{R})$, it is in particular, continuous. Thus, we may find some neighborhood U such that $A \in U$ and $\frac{\partial f}{\partial a_{kl}} \neq 0 \implies m_{kl} \neq 0$, by considering the open set $\mathbb{R} \setminus \{0\}$, and looking at the inverse image of the derivative.

Then, by Lemma 9.9, with a change of coordinates $F = f - 1$ and therefore $f^{-1}(1) = F^{-1}(0)$, but $\partial F \partial a_{kl} = \frac{\partial f}{\partial a_{kl}}$, we see that since on U, the Jacobian $J(F) = [\partial F \partial a_{kl}] \neq 0$, and therefore, we may replace the coordinate a_{kl} with $F = \det(A) - 1$ to obtain an adapted chart for $GL_n(\mathbb{R})$ relative to $SL_n(\mathbb{R})$.

Then, we have the chart $(U, a_{ij}, \det(A) - 1)$ with $1 \leq i, j \leq n, (i, j) \neq (k, l)$. Of course, due to the definition of $SL_n(\mathbb{R})$, we can see that $U \cap SL_n(\mathbb{R})$ is defined by the vanishing of $\det(A) - 1$, and therefore, the other a_{ij} coordinates form a coordinate system on this neighborhood.

Now, we wish to just see that we may define a_{kl} as a C^∞ function of the other entries on U. With some algebraic manipulation:

$$f(A) = \det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} m_{kj} \implies \det(A) - \sum_{j=1, j \neq l}^{n} (-1)^{k+j} a_{kj} m_{kj} = (-1)^{k+l} a_{kl} m_{kl} \implies$$

$$a_{kl} = (-1)^{k+l}\frac{1}{m_{kl}}\left(\det(A) - \sum_{j=1, j\neq l}^{n} (-1)^{k+j} a_{kj} m_{kj}\right)$$

Since $\det(A) = 1$ on $U \cap SL_n(\mathbb{R})$, we have that:

$$a_{kl} = (-1)^{k+l}\frac{1}{m_{kl}}\left(1 - \sum_{j=1, j\neq l}^{n} (-1)^{k+j} a_{kj} m_{kj}\right)$$

Of course, m_{ij} is a polynomial without a_{kl}, hence C^∞ in the other coordinates. Further, m_{kl} is non-0 on U here, and polynomial, hence $\frac{1}{m_{kl}}$ is C^∞. Thus, this is a sum and product of C^∞ functions, hence C^∞ on this domain.

(b)

Fix a $(A, B) \in SL_n(\mathbb{R}) \times SL_n(\mathbb{R})$.

By part (a), we may find $U \in SL_n(\mathbb{R})$ such that $A \in U$, and is defined as a submanifold chart with coordinates $a_{ij}, (i, j) \neq (k, l)$. Similarly, we may find V with $B \in V$ and with coordinates $b_{ij}, (i, j) \neq (k', l')$.

We may look at the neighborhood $(A, B) \in U \times V \subseteq SL_n(\mathbb{R}) \times SL_n(\mathbb{R})$. Further, for $C = A * B$, we may find a neighborhood $W \subseteq SL_n(\mathbb{R})$, such that $C \in W$ and C takes on the coordinates $c_{i''j''}, (i'', j'') \neq (k'', l'')$. Using this to look at the components $\overline{\mu}$, on $U \times V$, the natural matrix multiplication has the form, for $A * B = C = [c_{ij}]$ and $(m, n) \neq (k'', l'')$:

$$\overline{\mu}^{mn}(A, B) = c_{mn} = \sum_{p=1}^{n} a_{mp} b_{pn}$$

For $m \neq k, n \neq l'$, we can see that c_{mn} is familiarly a homogeneous polynomial of the coordinates, hence C^∞. On the other hand, when either $m = k$ or $n = l'$ we have a sum of degree 2 polynomials, as well as a term of the form $a_{kl} b_{ln}$ or $a_{mk'} b_{k'l'}$. By the considerations of part (a), a_{kl} is a C^∞ function of the other $n^2 - 1$ coordinates, and so is $b_{k'l'}$. Thus, in these cases, the overall sum is a sum and product of C^∞ functions, hence C^∞.

Thus, by this argument, each of the entries of C is a C^∞ function on the $a_{ij}, b_{i'j'}, (i, j) \neq (k, l), (i', j') \neq (k', l')$. Hence, each component of $\overline{\mu}$ is a C^∞ function on these coordinates. Since these are a set of coordinates for $U \times V$, this implies that $\overline{\mu}$ is a C^∞ function at (A, B). Since we may repeat this procedure for any $(A, B) \in SL_n(\mathbb{R}) \times SL_n(\mathbb{R})$, this implies that $\overline{\mu}$ is C^∞ on the entire set.

Note that technically, we don't need to exclude k'', l'' from the components (m, n) but since we merely need to verify from a coordinate neighborhood of $SL_n(\mathbb{R}) \times SL_n(\mathbb{R})$ to a suitable coordinate neighborhood of $SL_n(\mathbb{R})$, it is enough to look at the relevant coordinate functions.

□

Question 5. Let M be a manifold, and let $(U, \phi) = (U, x^1, ..., x^m), (V, \psi) = (V, y^1, ..., y^m)$ be charts such that $U \cap V \neq \emptyset$.

Consider the induced charts $(TU, \tilde{\phi}), (TV, \tilde{\psi})$ on TM, the total space of the tangent bundle, with transition map $\tilde{\psi} \circ \tilde{\phi}^{-1}$ that sends:

$$(x^1, ..., x^m, a^1, ..., a^m) \mapsto (y^1, ..., y^m, b^1, ..., b^m)$$

(a) Compute the Jacobian matrix of the transition map at $\phi(p)$.

(b) Show that the determinant of the transition map at $\phi(p)$ takes on the value:

$$\left(\det\left[\frac{\partial y^i}{\partial x^j}\right]\right)^2$$

Proof. (a)

By definition, the Jacobian matrix of a map $F : N \to M$ relative to a chart $(x^1, ..., x^n)$ of N is simply $J(F) = \left[\frac{\partial F^i}{\partial x^j}\right]$, where F^i is the i-th component of F in a chart of M.

Then, recalling section 12.2, we have that the action of the transition map $\tilde{\psi} \circ \tilde{\phi}^{-1}$ has the following form, where we recall that $x^i = r^i \circ \phi$ is the i-th component of ϕ and similar for y^j and ψ:

$$(\phi(p), a^1, ..., a^m) = (x^1(p), ..., x^m(p), a^1, ..., a^m) \mapsto (y^1(p), ..., y^m(p), b^1, ..., b^m) = (\psi \circ \phi^{-1}(\phi(p)), b^1, ..., b^n)$$

To compute the transformation that takes the a^i to a specified b^j, we recall that at a point $p \in U \cap V$, we may describe a fixed tangent vector $v \in T_pM$ by the bases $\left\{\frac{\partial}{\partial x^i}\right\}$ or equivalently by $\left\{\frac{\partial}{\partial y^j}\right\}$. Thus, we have the equality:

$$\sum_i a^i \frac{\partial}{\partial x^i} = \sum_j b^j \frac{\partial}{\partial y^j}$$

Using the standard trick and applying both sides onto y^k, we see that:

$$\sum_i a^i \frac{\partial y^k}{\partial x^i} = \sum_j b^j \frac{\partial y^k}{\partial y^j} = \sum_j b^j \delta^k_j = b^k$$

Thus, we have that:

$$(x^1(p), ..., x^m(p), a^1, ..., a^m) \mapsto \left(y^1(p), ..., y^m(p), \sum_i a^i \frac{\partial y^1}{\partial x^i}, ..., \sum_i a^i \frac{\partial y^m}{\partial x^i}\right)$$

Now, we are equipped to describe the Jacobian of this map. We see that for $1 \le i \le m$, $F^i = y^i$, and so, for $1 \le j \le m$, the derivatives correspond to the x^i, and so we have that via this numbering and denoting $\frac{\partial}{\partial x^i} y^j = y^j_i$, the upper left $m \times m$ submatrix A has the form:

$$A = \begin{bmatrix} y^1_1 & \dots & y^1_m \\ \vdots & \ddots & \vdots \\ y^m_1 & \dots & y^m_m \end{bmatrix}$$

On the other hand, the coordinates from $m+1$ to $2m$ represent the a^i. Since the transition map $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is induced via the transition map $\psi \circ \phi^{-1}$, we must have that the y^i are independent of the a^j. Thus, the Jacobian matrix has a $m \times m$ zero matrix in the top right.

Now, using the explicit description of the b^j, we may compute each block matrix of the lower $m+1, ..., 2m$ rows. In the first m coordinates, we see that for b^j, and denoting $\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} y^k = y^k_{ji}$:

$$\frac{\partial}{\partial x^k} \sum_i a^i y^j_i = \sum_i a^i \frac{\partial}{\partial x^k} y^j_i = \sum_i a^i y^j_{ik}$$

Thus, the lower left $m \times m$ block matrix has the form:

$$C = \begin{bmatrix} \sum_i a^i y^1_{i1} & \dots & \sum_i a^i y^1_{im} \\ \vdots & \ddots & \vdots \\ \sum_i a^i y^m_{i1} & \dots & \sum_i a^i y^m_{im} \end{bmatrix}$$

Lastly, with the same argument that y^j is independent of a^i for all i, j the lower right block matrix has the form, for b^j:

$$\frac{\partial}{\partial a^k} b^j = \frac{\partial}{\partial a^k} \sum_i a^i y_i^j = \sum_i \frac{\partial a^i}{\partial a^k} y_i^j = \sum_i \delta_k^i y_i^j = y_k^j$$

Thus, we have the lower right matrix takes on the form:

$$D = \begin{bmatrix} y_1^1 & \dots & y_m^1 \\ \vdots & \ddots & \vdots \\ y_1^m & \dots & y_m^m \end{bmatrix}$$

Thus, the Jacobian has the following block form:

$$J(F) = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$$

with 0 denoting a $m \times m$ matrix with entries identically 0, and A, C, D as computed above. In particular, we notice that $A = D$, and thus, we may rewrite this as:

$$J(F) = \begin{bmatrix} A & 0 \\ C & A \end{bmatrix}$$

with:

$$\begin{cases} A = \left[\frac{\partial y^i}{\partial x^j}\right]_{i,j} \\ C = \left[\sum_l a^l y_{lj}^i\right]_{i,j} \end{cases}$$

(b)

First, we will prove the following lemma:

Lemma. *Let M be a $n \times n$ matrix, $n \geq 2$. Suppose that in block matrix form, we have that:*

$$M = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$$

where A, D are square submatrices, $i \times i, j \times j$ respectively, $i + j = n$ and C is a $j \times i$ submatrix, and 0 a $i \times j$ submatrix with entries identically 0.

Then, $\det(M) = \det(A)\det(D)$*.*

Proof. Proceed by induction on n.

In the base case, $n = 2$. Then, the only case is that A, C, D are exactly scalar values, and we have that:

$$M = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$$

for $a, c, d \in k$, our base field.

Then, by direct computation, $\det(M) = ad - c0 = ad = \det(A)\det(D)$.

Now, suppose this is true for all $n \leq k - 1$, and consider M a $k \times k$ matrix. Let A be square of shape $i \times i$ and D be square of shape $j \times j$.

First, suppose $i = 1$. Then, $A = [a]$, and we have that:

$$M = \begin{bmatrix} a & 0 \\ C & D \end{bmatrix}$$

Computing the determinant by expanding along the first row and denoting the determinant of the submatrix obtained by deleting the i-th row and j-th column by m_{ij}, we find that since the only non-0 term in the first row is a, that:

$$\det(M) = am_{11} = a\det(D) = \det(A)\det(D)$$

Now, suppose $i > 1$.

Computing the determinant by expanding along the first row and denoting the determinant of the submatrix obtained by deleting the i-th row and j-th column by m_{ij}, we find that since the $i+1, ..., k$ entries in the first row are 0:

$$\det(M) = \sum_{l=1}^{i} (-1)^{l+1} a_{1l} m_{1l}$$

However, we notice, the submatrix S_{1l} obtained by deleting the first row and l-th column of M is a matrix of dimension $k-1 \times k-1$, and has the shape

$$S_{1l} = \begin{bmatrix} A_{1l} & 0 \\ C_l & D \end{bmatrix}$$

where we denote A_{1l} as the submatrix of A obtained by deleting the first row, and l-th column, and C_l from deleting the l-th column. In particular, by the induction hypothesis, we have that:

$$m_{1l} = \det(S_{1l}) = \det(A_{1l})\det(D)$$

Therefore, we may rewrite $\det(M)$ as

$$\det(M) = \sum_{l=1}^{i} (-1)^{l+1} a_{1l} m_{1l} = \sum_{l=1}^{i} (-1)^{l+1} a_{1l} \det(A_{1l})\det(D) = \det(D)\left(\sum_{l=1}^{i} (-1)^{l+1} a_{1l}\det(A_{1l})\right)$$

However, we recognize the sum as exactly the expansion computation for $\det(A)$, viewed as an $i \times i$ square matrix and expanded along its first row. Thus, we have that:

$$\det(M) = \det(D)\left(\sum_{l=1}^{i} (-1)^{l+1} a_{1l}\det(A_{1l})\right) = \det(D)\det(A)$$

as desired.

$\square$

Now, using this lemma and the results from part (a), we have that

$$\det(J(F)) = \det(A)\det(A) = \left(\det\left[\frac{\partial y^i}{\partial x^j}\right]\right)^2$$

as desired. $\square$