Homework #3

Eric Tao
Math 233: Homework #3

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Question 1. Let u be a harmonic function on a region Ω. What can we say about the set of points such that $\nabla u = 0$, that is, the set of points where $u_x = u_y = 0$?

Solution. Recall that if u is a real harmonic function, then we may identify it as the real part of a holomorphic function $f(x, y) = u(x, y) + iv(x, y)$ locally. Suppose $u_x = u_y = 0$. Then, by the Cauchy-Riemann equations, we have that at these points, $v_x = v_y = 0$. Further, identifying $f'(z) = \partial f(z)$ for $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, we have that:

$$f'(z) = \partial f(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}\left[(u_x + v_y) + i(v_x - u_y)\right]$$

So, we have that at points where $u_x = u_y = 0$, we have that $f'(z) = 0$. But, since f is holomorphic on this neighborhood, so is f'. Therefore, $\{(x, y) : \nabla u(x, y) = 0\}$ is either all of the neighborhood, or has no limit points. Since Ω is a region, we can always patch our entire region with overlapping neighborhoods, so this extends to all of Ω.

Now, if u is a complex-valued harmonic function, we simply identify it as $u = w + iv$, where w, v are the real and imaginary portions. It should be clear that if u is harmonic, so must w, v as:

$$u_{xx} + u_{yy} = w_{xx} + iv_{xx} + w_{yy} + v_{yy} = (w_{xx} + w_{yy}) + i(v_{xx} + v_{yy}) = 0 \implies w_{xx} + w_{yy} = 0, v_{xx} + v_{yy} = 0$$

Then, suppose $u_x = u_y = 0$. At such points, we would have that $u_x = w_x + iv_x = 0, u_y = w_y + iv_y = 0 \implies w_x = w_y = 0, v_x = v_y = 0$. But, by the previous work, since v, w are real harmonic functions, they either have no limit points, or are the full space. It should be clear then, that the set of points where $\nabla u = 0$ is simply the union of these sets. It too may only be the full space or not have limit points, as if it did, then we could construct a subsequence of points coming from either the set where $\nabla v = 0$, or $\nabla w = 0$, which would imply that the original set had a limit point, a contradiction. □

Question 2. Let u, v be real harmonic functions on a plane region Ω. Under what conditions is uv harmonic?

Further, show that u^2 may not be harmonic on Ω, unless u is constant.

Further, for which $f \in \mathcal{H}(\Omega)$ is $|f|^2$ harmonic?

Solution. We start by proving that if we take the Laplacian of uv, $\Delta(uv)$, then this is equal to $2\nabla u \cdot \nabla v$:

$$\Delta(uv) = (uv)_{xx} + (uv)_{yy} = (u_x v + uv_x)_x + (u_y v + uv_y)_y = u_{xx}v + u_x v_x + u_x v_x + uv_{xx} + u_{yy}v + u_y v_y + u_y v_y + uv_{yy}$$

Because u, v are harmonic, we know that $u_{xx} + u_{yy} = 0, v_{xx} + v_{yy} = 0$, so:

$$= v(u_{xx} + v_{xx}) + 2u_x v_x + u(v_{xx} + v_{yy}) + 2u_y v_y = 2(u_x v_x + u_y v_y) = 2\langle u_x, u_y\rangle \cdot \langle v_x, v_y\rangle = 2\nabla u \cdot \nabla v$$

Here, it should be clear then that if u^2 is not constant, then u^2 is not harmonic. We have that $\Delta(u^2) = \Delta(uu) = 2\nabla u \cdot \nabla u = 2|\nabla u|^2$. So, suppose u is harmonic, then for $\Delta(u^2) = 0$, this implies that $|\nabla u| = 0$ for all $z \in \Omega$. However, this implies immediately that u is constant, and we have the contrapositive.

Now, of course, if u or v is constant, suppose $u = a$ is constant, then of course $uv = av$ is harmonic, being a scalar multiple of a harmonic function. So, assume u, v both non-constant.

Define the set $A = \{z \in \Omega : \nabla u(z) = 0 \text{ or } \nabla v(z) = 0\}$. By the first problem, we know that neither of those sets have limit points in Ω. Since both of those are closed conditions, A is the union of two closed sets, and thus closed. Thus, consider $\Omega' = \Omega \setminus A$.

This is an open set, of course, being open minus closed, or equivalently, open intersect open. Further, it must be connected, since the points of A have no limit points, and are at most countable. Suppose $x, y \in \Omega'$, and consider a path between them in Ω. This may have at most countably many disconnections when we move to Ω'. Since A has no limit points, we may restrict down into a small enough punctured disk around any connection and take a path there - this punctured disk must be completely contained within Ω' due to A having no limit points. Since we have merely countably many of these issues, we are assured that we can patch this. Finally, this must be dense because let U be any open set in Ω. Choose any $a \in U$. There exists a disk $D(a, r) \subset U$, with uncountable cardinality. But, A is merely countable, thus $D(a, r) \setminus A \neq \emptyset$. Thus, since $A \cup \Omega' = \Omega$, we must have that $D(a, r) \cap \Omega' \neq \emptyset$. Thus, we have that Ω' is a region.

Now, we have that since $\Delta(uv) = 0$, we must have that $u_x v_x + u_y v_y = 0 \implies u_x v_x = -u_y v_y$. Since we wish uv to be harmonic, this must hold for all $z \in \Omega'$, which leads us to two cases, since $u_x, u_y, v_x, v_y \neq 0$ on Ω':

Case 1:

$$\begin{cases} v_x = -\lambda u_y \\ v_y = \lambda u_x \end{cases}$$

It should be clear that due to the definition of Ω', that $\lambda \neq 0$. In particular, since u, v are harmonic on Ω, they are continuous on all of Ω, with continuous first derivatives. Thus, these must actually hold for all of Ω, since u_x, u_y, v_x, v_y. Thus, we can say that the function

$$f = \lambda u + iv$$

is holomorphic, since these are exactly the Cauchy-Riemann equations for $u' = \lambda u, v' = v$. Thus, in this case, uv is harmonic if we may find a λ such that u, v are real and imaginary parts of a holomorphic function.

Case 2:

$$\begin{cases} u_x = -\lambda u_y \\ v_y = \lambda v_x \end{cases}$$

Consider the first equation. This implies that $u_{xx} = -\lambda u_{yx}$ and $u_{yy} = -\frac{1}{\lambda} u_{xy}$. Thus, in such a case, since u is harmonic, we must have that:

$$u_{xx} + u_{yy} = 0 \implies -\lambda u_{yx} - \frac{1}{\lambda} u_{xy} = 0 \implies u_{xy} = 0$$

Similarly:

$$v_{xx} + v_{yy} = 0 \implies \lambda v_{yx} + \frac{1}{\lambda} v_{xy} = 0 \implies v_{xy} = 0$$

However, since $u_x, u_y \neq 0$ on Ω', this implies that $u_x = f(x)$ since $u_{xy} = 0$ and $u_y = g(y)$ since $u_{yx} = 0$. Then, we must have that $u = F(x) + G(y)$ for $F' = f, G' = g$, and due to harmonicity, we further have that $f'(x) + g'(y) = 0$. This can only be true on all of Ω' if f', g' are constant, which implies that F, G are at most quadratics. However, since we started with $u_x = -\lambda u_y$, this implies that $F'(x) = -\lambda G'(y)$, and if F, G are polynomials, this implies then that F', G' are constants and thus F, G are linear. Thus, we have that:

$$u = -\lambda ax + ay + b$$

Running through the same logic with v, we see that:

$$v = cx + \lambda cy + d$$

However, here, we notice that:

$$\begin{cases} u_x = -\lambda a \\ u_y = a \\ v_x = c \\ v_y = \lambda c \end{cases}$$

Choosing $\lambda' = -\frac{c}{a}$, we see that:

$$\begin{cases} -\lambda' u_y = \frac{c}{a} a = c = v_x \\ \lambda' u_x = -\frac{c}{a} \cdot -\lambda a = \lambda c = v_y \end{cases}$$

and thus we are back in case 1. Thus, in either case, we see that uv is harmonic for u, v non-constant if there exists a $\lambda \neq 0$ such that $\lambda u + iv$ is holomorphic.

Now, let $f \in \mathcal{H}(\Omega)$, and consider $|f|^2$. Explicitly taking derivatives:

$$\frac{\partial^2}{\partial x^2}|f|^2 = \frac{\partial^2}{\partial x^2}(u^2 + v^2) = \frac{\partial}{\partial x}(2uu_x + 2vv_x) = 2(u_x^2 + uu_{xx} + v_x^2 + vv_{xx})$$

Of course then, the same equation will hold for the y, just switching the labels. Thus:

$$2(u_x^2+uu_{xx}+v_x^2+vv_{xx})+2(u_y^2+uu_{yy}+v_y^2+vv_{yy}) = 2(u(u_{xx}+u_{yy})+u_x^2+u_y^2+v(v_{xx}+v_{yy})+v_x^2+v_y^2) = 2(u_x^2+u_y^2+v_x^2+v_y^2)$$

where we've used the fact that because u, v come from the real, imaginary parts of a holomorphic function, u, v are harmonic.

Now, applying the Cauchy-Riemann equations, we obtain:

$$2(u_x^2 + u_y^2 + v_x^2 + v_y^2) = 2(2v_x^2 + 2v_y^2) = 4(v_x^2 + v_y^2) = 4(u_x^2 + u_y^2)$$

However, since u is a real-valued function, so must be u_x, u_y. Then, since $u_x^2, u_y^2 \geq 0$, for this to be harmonic, we must have $u_x, u_y = 0$. But that implies that u and thus v, are constants. Thus, we have that $|f|^2$ is harmonic iff f is constant. $\square$

Question 3. Suppose f is a complex function on a region Ω, and both f, f^2 are harmonic on Ω. Prove that either $f, \overline{f}$ must be holomorphic on Ω.

Solution. It is clear that if $f = a \in \mathbb{C}$, that is, constant, then f, f^2 are harmonic and $f, \overline{f}$ are both holomorphic. Thus, we restrict ourselves to f non-constant.

Now, we see that:

$$\Delta(f^2) = (2ff_x)_x + (2ff_y)_y = 2[f_x^2 + f_y^2] = 2[(f_x + if_y)(f_x - if_y)] = 2\overline{\partial} f \partial f$$

where, as in the text, we identify:

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Where we've used the fact that f is harmonic to say that $f(f_{xx} + f_{yy}) = 0$ Now, since f^2 is harmonic, we have that $\Delta(f^2) = 0$, which implies that at every point in Ω, either $\partial f = 0$ or $\overline{\partial} f = 0$. Now, consider $\partial f, \overline{\partial} f$. In particular, consider the quanitity $\overline{\partial}(\partial f)$:

$$\overline{\partial}(\partial f) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\frac{1}{2}(f_x - if_y) = \frac{1}{4}(f_{xx} + if_{xy} - if_{yx} + f_{yy}) = 0$$

That is, for f harmonic, ∂f is holomorphic on Ω, because the Cauchy-Riemann equations hold. In particular, its zero set is either all of Ω, or a countable subset without limit points. If its zero set is all of Ω, we are done, since if we expand out ∂f, we find that:

$$\partial f = \frac{1}{2}(f_x - if_y) = \frac{1}{2}(u_x + iv_x - iu_y + v_y) = 0 \implies \begin{cases} u_x = -v_y \\ u_y = v_x \end{cases}$$

and thus we have that:

$$\overline{\partial}(\overline{f}) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u - iv) = \frac{1}{2}(u_x - iv_x + iu_y + v_y) = 0$$

that is, $\overline{f}$ is holomorphic. Otherwise, suppose $Z = \{z \in \Omega : \partial f(z) = 0\}$ has no limit points. Since f harmonic, at least one of $\partial f, \overline{\partial} f = 0$ so on $\Omega \setminus Z$, $\overline{\partial} f = 0$. But, because Z has no limit points, by continuity, $\overline{\partial} f = 0$ actually on all of Ω, where we know this must be continuous, because it is the linear combination of continuous functions. Then, we would have that:

$$\overline{\partial} f = \frac{1}{2}(f_x + if_y) = \frac{1}{2}(u_x + iv_x + iu_y - v_y) \implies \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

and thus

$$\overline{\partial}(f) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}(u_x + iv_x + iu_y - v_y) = 0$$

that is, f is holomorphic.

□

Question 4. Let Ω be a region, and $f_n \in \mathcal{H}(\Omega)$ for all n. Set $u_n = \Re(f_n)$, and suppose u_n converges uniformly on compact subsets of Ω and that there exists $z \in \Omega$ such that $f_n(z)$ converges. Prove that f_n converges uniformly on compact subsets of Ω.

Solution. By hypothesis, there exists a $z_0 \in \Omega$ such that $f_n(z_0)$ converges. Since Ω is open, we may choose an $R > 0$ such that $\overline{D}(z_0, R) \subset \Omega$, since if the disk $D(a, r)$ is contained in Ω, the closed disk $\overline{D}(a, r/2)$ is as well.

Since this is a compact set, and u_n converges uniformly on compact sets, if we set $u = \lim_{n\to\infty} u_n(z)$ for $z \in \overline{D}(z_0, R)$, by theorem 11.11, we have that u is harmonic. Since u is harmonic on $D(z_0, R)$ and continuous on the boundary, we have that by 11.9 that u is the real part of a holomorphic function defined by:

$$f(z_0 + z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{Re^{it} + z}{Re^{it} - z} u(z_0 + Re^{it})dt$$

for $|z| < R$.

In the same way, we see that since each u_n is harmonic on the same disk, we may find a sequence of holomorphic functions g_n such that:

$$g_n(z_0+z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{Re^{it}+z}{Re^{it}-z}u_n(z_0+Re^{it})dt$$

But, u_n is also the real part of f_n, also a holomorphic function. Thus, by 11.10, these holomorphic functions may only differ by an imaginary additive constant, and we may say that there exists $c_n \in \mathbb{R}$ such that $f_n = g_n + ic_n$.

First, we wish to show that $g_n \to f$ uniformly for any $r < R$, the closed disk $\overline{D}(z_0, r)$. Let $\epsilon > 0$ be given. Well, by definition, we have that for any point $|z| < r$:

$$|f-g_n|=\left|\frac{1}{2\pi}\int_0^{2\pi}\frac{Re^{it}+z}{Re^{it}-z}u(z_0+Re^{it})dt-\frac{1}{2\pi}\int_0^{2\pi}\frac{Re^{it}+z}{Re^{it}-z}u_n(z_0+Re^{it})dt\right|=$$

$$\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{Re^{it}+z}{Re^{it}-z}u(z_0+Re^{it})-\frac{Re^{it}+z}{Re^{it}-z}u_n(z_0+Re^{it})dt\right|\le$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{Re^{it}+z}{Re^{it}-z}\right|\left|u(z_0+Re^{it})-u_n(z_0+Re^{it})\right|dt$$

Here, we notice that in terms of moduli, $|Re^{it}+z| \le |Re^{it}|+|z| \le R+r$, and similarly, $|Re^{it}-z| \ge R-r$. Thus, we have the estimate:

$$\left|\frac{Re^{it}+z}{Re^{it}-z}\right|\le\frac{R+r}{R-r}$$

for all $|z| < r$. Further, since $\overline{D}(a,r)$ is compact, we have that $u_n \to u$ uniformly.Then, choose N such that for all $n > N$, $|u(z)-u_n(z)| < \epsilon\frac{R-r}{R+r}$. Then, for any $n > N$ and for every $z \in \overline{D}(a,r)$, we have that:

$$|f(z)-g_n(z)|\le\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{Re^{it}+z}{Re^{it}-z}\right|\left|u(z_0+Re^{it})-u_n(z_0+Re^{it})\right|dt\le\frac{1}{2\pi}\int_0^{2\pi}\frac{R+r}{R-r}\epsilon\frac{R-r}{R+r}dt=$$

$$\frac{1}{2\pi}\int_0^{2\pi}\epsilon dt=\frac{1}{2\pi}\epsilon 2\pi=\epsilon$$

Thus, we have that $g_n \to f$ uniformly for every $\overline{D}(a,r), r < R$.

Next, we restrict our focus to z_0. We have that $f_n = g_n + ic_n$. Thus, at z_0, since $g_n(z_0) \to f(z_0)$ because of what we showed above, we have that:

$$f(z_0)=\lim_{n\to\infty}f_n(z_0)=\lim_{n\to\infty}(g_n(z_0)+ic_n)=\lim_{n\to\infty}g_n(z_0)+i\lim_{n\to\infty}c_n=f(z_0)+i\lim_{n\to\infty}c_n$$

Thus, we have that $\lim_{n\to\infty} c_n$ exists and is equal to 0.

Now, we look at any closed disk $\overline{D}(z_0,r), r < R$ again, and look at f_n this time. Let $\epsilon > 0$ be given. We have that:

$$\|f-f_n\|=\|f-g_n-ic_n\|\le\|f-g_n\|+\|c_n\|$$

Since we have that $g_n \to f$ uniformly,there exists N_g such that for all $n > N_g$, $\|f-g_n\| < \epsilon/2$. Since $c_n \to 0$ is a sequence of constant numbers, there exists N_c such that for all $n > N_c$, $\|c_n\| < \epsilon/2$. Then, for any $n > \max(N_g, N_c)$:

$$\|f-f_n\|\le\|f-g_n\|+\|c_n\|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

Thus, we have that $f_n \to f$ uniformly for any $\overline{D}(z_0,r), r < R$.

Define Ω_1, Ω_2 via the following:

$$\Omega_1 = \{z \in \Omega : \{f_n(z)\} \text{ converges } \}$$

$$\Omega_2 = \{z \in \Omega : \{f_n(z)\} \text{ does not converge } \}$$

From what we've shown above, Ω_1 must be open, since we've shown that there exists a disk around a convergent point z_0 such that for any concentric, closed disk contained within this neighborhood, f_n converges uniformly on the closed disk.

However, we notice that Ω_2 must also be open, because we chose a closed disk $\overline{D}(z_0, R)$ to analyze and the complement of such in Ω is open. Thus, the union of such complements is also open. Further, by definition, $\Omega_1 \cap \Omega_2 = \emptyset$.

Thus, because Ω is a region, we must have that either $\Omega = \Omega_1$, or $\Omega = \Omega_2$, and by hypothesis, we see that $z_0 \in \Omega_1 \implies \Omega = \Omega_1$, and thus $f_n \to f$ on all of Ω.

Then, the result is clear. Let K be any compact subset of Ω. For each $k \in K$, we may find $r_k > 0$ such that $D(k, r_k) \subset \Omega$. Consider $\cup_k D(k, r_k)$. Clearly, this is a open cover of K, so by the compactness of K, there exists a finite subcover

$$K \subset \cup_{i=1}^n D(k_i, r_{k_i}) \subset \Omega$$

Then, let $\epsilon > 0$ be given. For each i, choose r_i such that $r_i < r_{k_i}$, but that $\cup_{i=1}^n D(k_i, r_i)$ remains a cover of K. We may do this because of homework 1, finding the minimum distance between K and the complement of $\cup_{i=1}^n D(k_i, r_{k_i})$, a closed set. Then, by the work above, we have that on each $\overline{D}(k_i, r_i)$, that since $f_n \to f$ uniformly, there exists N_i such that for all $n > N_i$, $\|f - f_n\| < \epsilon$ on $D(k_i, r_i)$. We notice, that there are only finitely many N_i and thus it achieves a maximum. Thus, of course, we have that for $N = \max_i N_i$, for any $n > N$:

$$\|f - f_n\|_K = \|f - f_n\|_{\overline{D}(k_j, r_j)} < \epsilon$$

for some k_j, r_j since they cover K. Thus, $f_n \to f$ uniformly on compact subsets of Ω.

$\square$

Question 5. Let Ω be a region, K a compact subset of Ω, and fix some $z_0 \in \Omega$. Let u be any positive harmonic function. Prove that there exists $\alpha, \beta > 0$ such that

$$\alpha u(z_0) \leq u(z) \leq \beta u(z_0)$$

for all $z \in K$.

If $\{u_n\}$ is a sequence of positive harmonic functions in Ω, and $u_n(z_0) \to 0$, describe the behavior of $\{u_n\}$ on the rest of Ω. Repeat this process for if $u_n(z_0) \to \infty$. Show that $\{u_n\}$ must be positive.

Solution. First, fix some $z_0 \in \Omega$. Let u be any positive harmonic function on Ω. Let $z \in \Omega$ be any other point, and let γ be a path, $\gamma^* \subset \Omega$ such that $\gamma(0) = z_0, \gamma(1) = z$, which exists since Ω is connected. Further, assume that γ has a finite length. Such a path must exist. Since the path is a compact set, and the complement of Ω is closed, this implies that we may find a $R > 0$ such that $D(\zeta, R) \subset \Omega$ for all $\zeta \in \gamma^*$.

Now, consider $\overline{D}(z_0, R/2) \subset \Omega$. If $z \in D(z_0, R/3)$, then we can say that, for $r = |z - z_0|$, that:

$$\frac{R/2 - r}{R/2 + r} u(z_0) \leq u(z) \leq \frac{R/2 + r}{R/2 - r} u(z_0) \implies \frac{R/2 - r}{R/2 + r} \leq \frac{u(z)}{u(z_0)} \leq \frac{R/2 + r}{R/2 - r} \implies \frac{1}{5} \leq \frac{u(z)}{u(z_0)} \leq 5$$

Where we use the fact that $\frac{R/2-r}{R/2+r}$ is a decreasing function, since as $r \to R/2$, $R/2 - r$ decreases, and $R/2 + r$ increases, so the fraction decreases, so it takes on its minimum value at $r = R/3$. The same logic applies for the upper bound, as an increasing function.

Otherwise, take the boundary $\partial D(z_0, R/3)$. Since z is not contained within $D(z_0, R/2) \supset D(z_0, R/3)$, there exists at least some point $\zeta \in \gamma^*$ such that $\zeta \in \gamma^* \cap \partial D(z_0, R/3)$. If there are multiple such ζ, we choose the one corresponding to the largest parameter in $\gamma(t)$. Calling this point ζ_1, we have that:

$$\frac{R/2 - R/3}{R/2 + R/3} u(z_0) \le u(\zeta_1) \le \frac{R/2 + R/3}{R/2 - R/3} u(z_0) \implies \frac{1}{5} u(z_0) \le u(\zeta_1) \le 5u(z_0)$$

Now, we repeat this process for ζ_1 taking the role of z_0. If z is contained within $D(\zeta_1, R/3)$, then we have that, for $r = |z - \zeta_1|$ again:

$$\frac{R/2 - r}{R/2 + r} u(\zeta_1) \le u(z) \le \frac{R/2 + r}{R/2 - r} u(\zeta_1) \implies \frac{1}{5}^2 u(z_0) \le u(z) \le 5^2 u(z_0) \implies \frac{1}{5}^2 \le \frac{u(z)}{u(z_0)} \le 5^2$$

Otherwise, choose ζ_2 in the same fashion as ζ_1, by looking at the boundary $\partial D(\zeta_1, R/3)$. Importantly,this process must terminate in a finite amount of steps - in particular, it must terminate in at most $\lceil L/(R/3) \rceil = \lceil \frac{3L}{R} \rceil$ steps, where L is the path length of γ. Thus, letting $n = \lceil \frac{3L}{R} \rceil$, we have for our estimate, that:

$$\frac{1}{5}^n \le \frac{u(z)}{u(z_0)} \le 5^n$$

We notice that this is actually independent of u, and is strictly a function of the geometry.

Now, consider the function:

$$f(z) = \inf_{\gamma} \int_0^1 |\gamma'(t)| dt, \gamma(0) = z_0, \gamma(1) = z$$

that is, the length of the shortest path from $z_0 \to z$. This function is continuous. Let $\epsilon > 0$ be given. Consider the disk $D(z, \epsilon) \subset \Omega$ and fix some z' in the disk. For any path γ from $z_0 \to z$, consider the new path γ_1 defined by:

$$\gamma_1 = \begin{cases} \gamma(2t) & \text{for } t \in [0, 1/2] \\ z'(2t - 1) + z(2 - 2t) & \text{for } t \in [1/2, 1] \end{cases}$$

that is, the path that traverses γ twice as fast, concatenated with the straight line $z \to z'$. Then, we have that:

$$f(z') \le \int_0^1 |\gamma_1'(t)| dt = \int_0^{1/2} |\gamma'(t)| dt + \int_{1/2}^1 |2z' - 2z| dt < \int_0^{1/2} |\gamma'(t)| dt + \epsilon$$

Since f is defined as the infimum, and we have that for any point z', that the length from $z_0 \to z'$ is at most the length from $z_0 \to z$ plus ϵ, since the infimum is linear, we have that $f(z') \le f(z) + \epsilon$. Further, we see that this function is positive if z_0 is not in our set, being the integral of a real or complex modulus.

Now, let K be any compact set on Ω. Since f is continuous, f attains a maximum on K. Then, we can find a maximum distance $N = \max\{f(z) : z \in K\}$. But, applying our analysis on the path length, we have then that the largest value of $\lceil \frac{3L}{R} \rceil$ happens when $L = N$, at $\lceil \frac{3N}{R} \rceil$. Thus, we have that on K, that:

$$\frac{1}{5}^{\lceil \frac{3N}{R} \rceil} \le \frac{u(z)}{u(z_0)} \le 5^{\lceil \frac{3N}{R} \rceil}$$

Now, suppose u_n is a sequence of positive harmonic functions, and $u_n(z_0) \to 0$. Then, to maintain the ratio $u_n(z)/u_n(z_0)$, we must have that $u_n(z) \to 0$ on all of Ω, since we can always find a compact subset (i.e., a closed disk) around any point in Ω. Similarly, it should be clear that if $u_n(z_0) \to \infty$, we must have that $u_n(z) \to \infty$.

Easy issue if u is not necessarily positive. Consider $u_n((x,y)) = x + \frac{1}{n}$, which is clearly a harmonic function, and take the region $\Omega = \{(x,y) : x^2 + y^2 < 1\}$. Clearly, we see that $u_n((0,0)) = \frac{1}{n}$, and as $n \to \infty$, of course we have that $1/n \to 0$. However, we see that for any point, (x,y), that $u_n((x,y)) = x + \frac{1}{n} \to x$, which does not align with what we've said.

In a similar fashion, we can think of the function $u_n((x,y)) = u_n(x) = nx$. Clearly, for any $x > 0$, we have that $\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} nx = \infty$. However, consider any $(0,y)$. On this line, we have that $u_n(x) = n \cdot 0 = 0$. Thus, when we do not have positivity on the region, we need not have the inequalities stated above. $\square$