Bezout's Theorem for Plane Curves, First Consequences, and into Multiprojective Spaces

Eric Tao
Math 240

December 8, 2022

1 Preliminaries

Here are a few preliminaries we will start with while talking about Bezout's Theorem for a projective plane curve.

First, we will want to talk about the intersection number of two affine plane curves. Let $F, G \in \mathbb{A}^2$ be plane curves. We wish to define a quantity known as the intersection number at P, that is, $I(P, F \cap G)$, which, intuitively, we want to encode the degree to which two curves intersect.

We start by talking about the properties that we wish this quantity to have. They are:

(1) $I(P, F \cap G) \geq 0$, and if F, G share an irreducible component that passes through P, that $I(P, F \cap G) = \infty$

(2) $I(P, F \cap G) = 0 \iff P \notin F \cap G$, that is, it is a local quantity that depends only on the components of F, G that pass through P, and that for empty plane curves F, G non-0 constants, that $I(P, F \cap G) = 0$.

(3) If T is an linear change of variables, and $T(P) = Q$, then $I(P, F \cap G) = I(Q, T(F) \cap T(G))$.

(4) $I(P, F \cap G) = I(P, G \cap F)$

(5) For the multiplicities $m_p(F), m_p(G)$, that is, the multiplicity of F, G at p, we have that $I(P, F \cap G) \geq m_p(F)m_p(G)$, with equality holding if and only if F, G do not have common tangent lines at P

(6) If $F = \Pi F_i^{r_i}$ and $G = \Pi G_j^{s_j}$, then $I(P, F \cap G) = \Sigma_{i,j} r_i s_j I(P, F_i \cap G_j)$, that is, the intersection number should sum over the union of curves.

(7) $I(P, F \cap G) = I(P, F \cap (G + fF))$ for any $f \in k[x, y]$, that is, sums of G with multiples of our original curve do not affect the intersection number.

(8) If P is a non-singular point on F, then $I(P, F \cap G) = \text{ord}_P^F(G)$, that is, it is the order of the local parameter for F, of the image of G in the coordinate ring

(9) If F, G have no common components, then:

$$\Sigma_P I(P, F \cap G) = \dim_k(k[X, Y]/(F, G))$$

Then, we define the intersection number as:

$$I(P, F \cap G) = \dim_k(\mathcal{O}_P/(F, G))$$

where we have $\mathcal{O}_P$ as our familiar local ring at the point P.

We will not go into detail on the proof of how this number is unique and satisfies the above conditions, but a reference would be Fulton's Algebraic Curves, 3.3.

Additionally, we will need the following Proposition:

(1) Let

$$0 \to V' \xrightarrow{\psi} V \xrightarrow{\phi} V' \to 0$$

be an exact sequence of finite-dimensional vector spaces over a field k. Then $\dim V' + \dim V' = \dim V$.

(2) Let:

$$0 \to V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} V_4 \to 0$$

be an exact sequence of finite-dimensional vector spaces. Then we have that $\dim V_4 = \dim V_3 - \dim V_2 + \dim V_1$.

(1) We notice, by the application of the rank-nullity theorem, that we have that $\dim V = \dim V' + \dim \ker(\phi)$. But, the $\ker(\phi)$ is exactly the image of V_1 in V_2. But, since ϕ_1 is injective, that is exactly the dimensionality of V', so we are done.

(2) Define $W = \mathrm{Im}(\phi_2) = \ker(\phi_3)$. Then, we may split this into short sequences:

$$0 \to V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} W \to 0$$
$$0 \to W \xrightarrow{\psi} V_3 \xrightarrow{\phi_3} V_4 \to 0$$

where ψ is simply inclusion, naturally injective. Then, from (1), we have that:

$$\dim V_2 = \dim W + \dim V_1$$
$$\dim V_3 = \dim W + \dim V_4$$

So, solving for $\dim W$, we get that

$$\dim V_3 = \dim V_2 - \dim V_1 + \dim V_4 \implies \dim V_4 = \dim V_3 - \dim V_2 + \dim V_1$$

as desired.

2 Bezout's Theorem for Plane Curves

Now, we are in a good place to state and prove Bezout's Theorem:

We will be taking a projective plane curve to mean a hypersurface in $\mathbb{P}^2$, that is, a surface cut out by a single polynomial. Note that in the projective setting, we define the intersection number to be an affine copy of F, G in our projective space.

Bezout's Theorem states the following:

Let F, G be projective plane curves of degree m, n respectively, such that they do not share any irreducible component. Then, we have that:

$$\Sigma_P I(P, F \cap G) = mn$$

Proof:

Firstly, since $F \cap G$ is finite, because the intersection of two closed sets are closed, and since they share no common component, they may only intersect in a finite number of discrete points, we may take a change of coordinates such that $F \cap G$ contains no points such that $X_2 = 0$.

Then, we may consider the affine version of these curves $F_* = F(X_0, X_1, 1), G_* = G(X_0, X_1, 1)$, where we look at the affine copy on $\mathbb{P}^2 \setminus V(X_2)$.

Using property (9) as stated above, we have that:

$$\Sigma_P I(P, F \cap G) \equiv \Sigma_P I(P, F_* \cap G_*) = \dim_k k[X_0, X_1]/(F, G)$$

Define $\Gamma_* = k[X_0, X_1]/(F_*, G_*), \Gamma = k[X_0, X_1, X_2]/(F, G), R = k[X_0, X_1, X_2$, and Γ_d to be the vector space of homogeneous polynomials of degree d in Γ. We notice then that it is sufficient to show that $\dim \Gamma_* = \dim \Gamma_d$, and that for some d, $\dim \Gamma_d = mn$.

First, we will show that $\dim \Gamma_d = mn$ for every $d \geq m + n$. Define $\pi : R \to \Gamma$ be the natural projection into the quotient space, $\phi : R \times R \to R$ via $\phi(r, r') = rF + r'G$, and $\psi : R \to R \times R$ via $\psi(r) = (Gr, -Fr)$. We claim that the following sequence is exact:

$$0 \to R \xrightarrow{\psi} R \times R \xrightarrow{\phi} R \xrightarrow{\pi} \Gamma \to 0$$

ψ we see as injective because it is a ring hom with trivial kernel, since for F, G non-0, they are both 0 only if $r = 0$. We see that $\psi \circ \phi(r) = \phi(Gr, -Fr) = rGF - rFG = 0$, and that if $\phi = 0$, then $rF + r'G = 0$, so $rF = -r'G$. Then, $F|r'G$, but since F, G share no factors, this implies that $F|r'$. In a similar fashion, $G|r$. Then, if we say that $r = \overline{r}G, r' = \overline{r}'F$, then we have that $\overline{r}GF + \overline{r}'FG = 0 \implies FG(\overline{r} + \overline{r}') = 0 \implies \overline{r} = -\overline{r}'$. So, the kernel is composed of objects that look like $(\overline{r}G, -\overline{r}F)$, which is exactly objects that come from the image of ψ.

The exactness of the other side is trivial, as everything in the image of ϕ is a multiple $rF + r'G$, and since Γ is exactly R modulo (F, G), that is exactly what goes to 0 in the quotient, and that's surjective due to being a projection.

Now, if we look in particular on the action of these maps on the homogenous polynomials of degree d on R, denoted as R_d, we can see these as exact sequences of form:

$$0 \to R_d \xrightarrow{\psi} R_{d-m} \times R_{d-n} \xrightarrow{\phi} R_d \xrightarrow{\pi} \Gamma_d \to 0$$

However, we notice here that because we live in the space of polynomials with 2 variables, as a vector space, we have have that $\dim R_d = (d+1)(d+2)/2$, so then by our proposition earlier, we have that

$$\dim \Gamma_d = \dim R_d - \dim(R_{d-m} \times R_{d-n}) + \dim R_d = d - (d - m + d - n) + d = m + n$$

for at least $d \geq m + n$