Homework #2

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Math 233: Homework #2

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Question 1. Suppose f is an entire function, and that for every power series:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

at least one coefficient is 0. Prove that f is polynomial.

Solution. Consider the family of sets:

$$Z_n = \{a \in \mathbb{C} : f^{(n)}(a) = 0\}$$

We recall, that for the power series centered on a, that we have $n!c_n = f^{(n)}(a)$. Because we have that at least one coefficient is 0, that implies that for every $a \in \mathbb{C}$, $a \in Z_m$ for some m, because suppose c_m is the coefficient that equals 0, then we have that $f^{(m)}(a) = n!c_m = 0$.

We notice here that this family of sets is countable, and the complex numbers are uncountable. Thus, at least one Z_n is uncountably large. In particular, we consider $f^{(n)}$. Since f is entire, then so must be $f^{(n)}$. Since Z_n is uncountably large, by Theorem 10.18, we must have that $Z_n = \mathbb{C}$. But then, this implies that $f^{(n)}$ is identically 0; further, taking derivatives, we have that $f^{(n+k)}$ for every $k \geq 1$ is also identically 0.

Then, we have, for every $m \geq n$, that:

$$c_m = \frac{1}{m!} f^{(m)}(a) = 0$$

and thus, f is polynomial, since in its formal power series, every coefficient after a certain point is identically 0.

Here, we'll show that if S is a set of zeros of a holomorphic function f, and is uncountable in $\mathbb{C}$, then it must have a limit point. Due to the σ-finite nature of $\mathbb{C}$, we may consider the sets $S \cap D(0,n)$, for $n \in \mathbb{N}$. Clearly, we have that $\cup_n (S \cap D(0,n)) = S$. But further, since there are only a countable number of sets $D(0,n)$, and S is uncountable, there must exist some n such that $S \cap D(0,n)$ is at least countably infinite, as otherwise, we could only have a countable infinity times a finite number of objects at most, which is less than uncountably many.

Fix such an n, and denote $S_n = D \cap D(0,n)$. Define a sequence $\{s_i\}_{i=1}^{\infty} \subset S_n$, where each s_i is unique. This is possible, of course, because S_n is at least countable. But, since f is holomorphic, f is continuous, thus, since $f^{-1}(0) = S$, S is closed. Thus, S_n is the intersection of a compact set and a closed set, and is thus compact. Therefore, there exists some x such that $s_{i_j} \to x$, as in a compact set, every sequence has a convergent subsequence. If $x \notin \{s_i\}$, then we are done. Otherwise, suppose $x = s_{i_k}$ for some i_k. We notice, of course, that if we consider the sequence $\{s_{i_j}\}_{j \neq k}^{\infty}$, this is a sequence without x, that converges to x. Thus, S contains a limit point.

$\square$

Question 2. Suppose P, Q are polynomials, with $\deg(Q) \geq \deg(P) + 2$ and such that the rational function $R = P/Q$ has no pole on the real line. Prove that the integral of R over $(-\infty, \infty)$ is equal to $2\pi i$ times the sum of the residues of R on the upper half plane. What is the analogous statement for the lower half plane? Use this method to compute:

$$\int_{[-\infty,\infty]} \frac{x^2}{1+x^4} dx$$

Solution. We begin by noticing that becuase Q is polynomial, it has exactly $\deg(Q) = m$ zeros, with no limit points in the set of zeros. Thus, R has no limit points on $Z(Q)$, R is certainly holomorphic on $\mathcal{H}(\mathbb{C} \setminus Z(Q))$, and R may only have poles at a subset of A. Thus, R is meromorphic. In particular, denote the set of points where R has poles on the upper half plane as $\tilde{Z}(Q)$

Then, we may apply the residue theorem. Since $Z(Q)$ is at most countable, take $r > \max\{|z| : z \in \tilde{Z}(Q)\}$, finite. Take the chain Γ as the upper-semicircle traversed along the real line from $(-r, 0) \to (r, 0)$, and then via re^{it} for $0 \leq t \leq \pi$. Clearly, this is actually a cycle, and further, since R has no poles on the real line, and because r is larger than the modulus of any zero of Q, we must have that $\Gamma^* \subset \mathbb{C} \setminus \tilde{Z}(Q)$. Vacuously, we have that $\mathrm{Ind}_\Gamma(\alpha) = 0$ for $\alpha \notin \mathbb{C}$, since this set is empty.

Thus, we have that:

$$\frac{1}{2\pi i} \int_\Gamma R(z)dz = \sum_{a \in \tilde{Z}(Q)} \mathrm{Res}(R; a)\mathrm{Ind}_\Gamma(a)$$

Since we oriented Γ positively, or, via 10.37, since by going into the interior of the semi-circle, we go from the right to the left of the path, we have that $\mathrm{Ind}_\Gamma(a) = 1$ for every a. Thus, we have that:

$$\frac{1}{2\pi i} \int_\Gamma R(z)dz = \sum_{a \in \tilde{Z}(Q)} \mathrm{Res}(R; a)$$

However, now we examine the left side a bit more. Letting $\gamma_1 = [-r, r]$ and letting $\gamma_2 = re^{it}, 0 \leq t \leq \pi$, we see that:

$$\int_\Gamma R(z)dz = \int_{\gamma_1} R(z)dz + \int_{\gamma_2} R(z)dz = \int_{-r}^{r} R(z)dz + \int_{\gamma_2} R(z)dz$$

In particular, we want to look at $\int_{\gamma_2} R(z)dz \leq \|R\|_\infty \int_0^\pi |\gamma_2'(t)dt| = \|R\|_\infty 2\pi r$. Taking an estimate, we look at $\|R\|_\infty 2\pi r$ on γ_2, as $r \to \infty$. Well, letting the degree of Q be m:

$$\|R\|_\infty 2\pi r \leq \sup\{\frac{|P|}{|Q|} : z \in \gamma_2\} * 2\pi r = \sup\{\frac{2\pi r|P|}{|Q|} : z \in \gamma_2\}$$

Here, we let $Q = \sum_{i=0}^{m} b_i z^i$, and $P = \sum_{j=0}^{m-2} a_j z^j$, where we note that since P has degree at most $m+2$, it could have less, so many of the a_j may be 0. This is valid because polynomials are entire, so we may take power series centered at 0.

Well, then we have that:

$$2\pi r \frac{|P|}{|Q|} = 2\pi r \frac{\left|\sum_{j=0}^{m-2} a_j z^j\right|}{\left|\sum_{i=0}^{m} b_i z^i\right|}$$

Applying the triangle inequality, and reverse triangle inequalities, we have that:

$$\begin{cases} \left|\sum_{j=0}^{m-2} a_j z^j\right| \leq \sum_{j=0}^{m-2} |a_j z^j| = \sum_{j=0}^{m-2} |a_j||z^j| \\ \left|\sum_{i=0}^{m} b_i z^i\right| \geq \left||b_m||z^m| - \sum_{i=0}^{m-1} |b_i||z^i|\right| \end{cases}$$

Thus, we have that:

$$2\pi r\frac{\left|\sum_{j=0}^{m-2}a_jz^j\right|}{\left|\sum_{i=0}^{m}b_iz^i\right|}\leq 2\pi r\frac{\sum_{j=0}^{m-2}|a_j||z^j|}{\left||b_m||z^m|-\sum_{i=0}^{m-1}|b_i||z^i|\right|}$$

Applying the fact that we're on $z=re^{it}$, we find that:

$$2\pi r\frac{\sum_{j=0}^{m-2}|a_j||z^j|}{\left||b_m||z^m|-\sum_{i=0}^{m-1}|b_i||z^i|\right|}=2\pi r\frac{\sum_{j=0}^{m-2}|a_j|r^j}{\left||b_m|r^m-\sum_{i=0}^{m-1}|b_i|r^i\right|}$$

We note here that if we take r large enough, then of course $|b_m|r^m>\sum_{i=0}^{m-1}|b_i|r^i$ for any parameters $b_1,...,b_m$, due to a quick application of a ratio test. Thus, we may drop the absolute values, and then divide through the entire fraction by r^m to obtain:

$$2\pi r\frac{\sum_{j=0}^{m-2}|a_j|r^j}{|b_m|r^m-\sum_{i=0}^{m-1}|b_i|r^i}=2\pi\frac{\sum_{j=0}^{m-2}|a_j|r^{j+1}}{|b_m|r^m-\sum_{i=0}^{m-1}|b_i|r^i}=2\pi\frac{\sum_{j=0}^{m-2}|a_j|r^{j+1-m}}{|b_m|-\sum_{i=0}^{m-1}|b_i|r^{i-m}}$$

Here, we notice that since $0\leq j\leq m-2\implies j+1-m<0$ and $0\leq i\leq m-1\implies i-m<0$, that when we take the limit of this expression as $r\to\infty$, that the limit of this is 0.

Back to our original expression, we had that:

$$\frac{1}{2\pi i}\left(\int_{-r}^{r}R(z)dz+\int_{\gamma_2}R(z)dz\right)=\sum_{a\in\tilde{Z}(Q)}\mathrm{Res}(R;a)\implies\int_{-r}^{r}R(z)dz+\int_{\gamma_2}R(z)dz=2\pi i\sum_{a\in\tilde{Z}(Q)}\mathrm{Res}(R;a)$$

Taking the limit of both sides as $r\to\infty$, and noticing the right side is a constant, we find that:

$$\lim_{r\to\infty}\left(\int_{-r}^{r}R(z)dz+\int_{\gamma_2}R(z)dz\right)=2\pi i\sum_{a\in\tilde{Z}(Q)}\mathrm{Res}(R;a)\implies\int_{-\infty}^{\infty}R(z)dz=2\pi i\sum_{a\in\tilde{Z}(Q)}\mathrm{Res}(R;a)$$

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as desired. It should be clear that the analogous statement for the lower half plane is that

$$\int_{-\infty}^{\infty}R(z)dz=-2\pi i\sum_{a\in\tilde{Z}(Q)}\mathrm{Res}(R;a)$$

where we understand $\tilde{Z}(Q)$ here to be on the lower half plane. This is because although we can construct Γ in the same way for the upper half plane, the difference is that to traverse the real line from $-\infty\to\infty$, we would be negatively oriented, and we would have that $\mathrm{Ind}_\Gamma(a)=-1$ for every residue on the lower half plane.

Now, using this to compute

$$\int_{[-\infty,\infty]}\frac{x^2}{1+x^4}dx$$

we notice that we have poles on the upper half plane at $z=e^{\pi i/4},e^{3\pi i/4}$. Computing the residues at these points, we notice that if we factor $1+z^4$, that these must be simple poles. So we compute these via taking limits:

$$\mathrm{Res}(f;e^{\pi i/4})=\lim_{z\to e^{\pi i/4}}(z-e^{\pi i/4})\frac{z^2}{(z-e^{\pi i/4})(z^3+e^{\pi i/4}z^2+e^{2\pi i/4}z+e^{3\pi i/4})}=\frac{e^{2\pi i/4}}{4e^{3\pi i/4}}=\frac{1}{4}e^{-\pi i/4}$$

and

$$\text{Res}(f; e^{3\pi i/4}) = \lim_{z \to e^{3\pi i/4}} (z - e^{3\pi i/4}) \frac{z^2}{(z - e^{3\pi i/4})(z^3 + e^{3\pi i/4} z^2 + e^{6\pi i/4} z + e^{9\pi i/4})} = \frac{e^{6\pi i/4}}{4e^{9\pi i/4}} = \frac{1}{4} e^{-3\pi i/4}$$

Then, we have that:

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = 2\pi i \left(\frac{1}{4} e^{-\pi i/4} + \frac{1}{4} e^{-3\pi i/4} \right) = \frac{\pi}{2} (e^{\pi i/4} + e^{-\pi i/4}) = \frac{\pi}{\sqrt{2}}$$

$\square$

Question 3. Compute

$$\int_0^\infty \frac{dx}{1+x^n}$$

for $n \geq 2$.

Solution. First, we work in the abstract. Suppose that z_0 is a pole of a rational function g/h, with g, h being holomorphic in an open set U containing z_0. Suppose further that $g(z_0) \neq 0, h(z_0) = 0, h'(z_0) \neq 0$.

Looking at a power series of h around z_0, we see that since $h(z_0) = 0$, this must have form:

$$h(z) = \sum_{i=0}^{\infty} a_n (z - z_0)^i = \sum_{i=1}^{\infty} a_n (z - z_0)^i$$

and thus we may factor this as:

$$h(z) = \sum_{i=1}^{\infty} a_n (z - z_0)^i = (z - z_0) \sum_{i=0}^{\infty} a_{n+1} (z - z_0)^i = (z - z_0)\psi(z)$$

where we notice $\psi(z_0) = a_1 = h'(z_0)$.

We claim that this has residue $g(z_0)/h'(z_0)$. We look at the limit:

$$\lim_{z \to z_0} \left(\frac{g(z)}{h(z)} - \frac{g(z_0)}{h'(z_0)(z - z_0)} \right) = \lim_{z \to z_0} \left(\frac{g(z)h'(z_0)(z - z_0) - g(z_0)h(z)}{h(z)h'(z_0)(z - z_0)} \right)$$

We notice that if we evaluate $z = z_0$, the fraction has form $\frac{0}{0}$. Thus, we apply L'Hopital's once:

$$\lim_{z \to z_0} \left(\frac{g(z)h'(z_0)(z - z_0) - g(z_0)h(z)}{h(z)h'(z_0)(z - z_0)} \right) = \lim_{z \to z_0} \left(\frac{g'(z)h'(z_0)(z - z_0) + g(z)h'(z_0) - g(z_0)h'(z)}{h'(z)h'(z_0)(z - z_0) + h(z)h'(z_0)} \right)$$

This still has form $\frac{0}{0}$, so applying L'Hopital's again:

$$\lim_{z \to z_0} \left(\frac{g'(z)h'(z_0)(z - z_0) + g(z)h'(z_0) - g(z_0)h'(z)}{h'(z)h'(z_0)(z - z_0) + h(z)h'(z_0)} \right) =$$
$$\lim_{z \to z_0} \left(\frac{g''(z)h'(z_0)(z - z_0) + g'(z)h'(z_0)(z - z_0) + g'(z)h'(z_0) - g(z_0)h''(z)}{h''(z)h'(z_0)(z - z_0) + h'(z)h'(z_0) + h'(z)h'(z_0)} \right) =$$
$$\frac{g'(z_0)h'(z_0) - g(z_0)h''(z_0)}{h'(z_0)^2}$$

which, because we assumed $h'(z_0) \neq 0$ is not of indeterminant form. Thus, for this value of $c_1 = g(z_0)/h'(z_0)$, $f - c_1(z-z_0)^{-1}$ has a removable singularity at z_0, and g/h has a simple pole at z_0.

Thus, we now look at $f(z) = \frac{1}{1+z^n}$. We see that since $1+z^n$ splits as the n-th roots of -1, that they must all be simple poles. Further, from the work done above, if we identify $g(z) = 1, h(z) = 1+z^n$, the residue at the pole z_0 must be:

$$\frac{g(z_0)}{h'(z_0)} = \frac{1}{nz_0^{n-1}}$$

Then, in the same general strategy as question 2, first, we fix an n. We consider the path Γ that goes $[0,r], \{re^{ti} : 0 \leq t \leq 2\pi/n\}, [re^{2\pi i/n}, 0]$. Call these paths $\gamma_1, \gamma_2, \gamma_3$ respectively. We see that since the poles are the n-th roots of -1, that they must be of form $e^{\pi i/n + 2\pi k/n}$ for $0 \leq k \leq n-1$. Then, the only residue within Γ is at $e^{i\pi/n}$, with the value of $(ne^{i\pi(n-1)/n})^{-1}$, so long as $r > 1$. Thus, by the residue theorem, we have that since $\mathrm{Ind}_\Gamma(a) = 1$ for everything inside Γ:

$$\frac{1}{2\pi i}\left(\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz\right) = \frac{1}{ne^{i\pi(n-1)/n}}$$

Now, since we will be taking $r \to \infty$, we may neglect $\int_{\gamma_2}$ for the same reason as the last problem, that as $r \to \infty$, $\|f\|_\infty * \int \gamma'(t)dt \to 0$.

We will look at $\int_{\gamma_3} f(z)dz$ first.

Evaluating this integral, we find that using the parametrization $\gamma(t) = tre^{2\pi i/n}, 0 \leq t \leq 1$:

$$\int_{\gamma_3} f(z)dz = \int_{\gamma_3} \frac{1}{1+z^n}dz = \int_1^0 \frac{1}{1+(tre^{2\pi i/n})^n} re^{2\pi i/n}dt = re^{2\pi i/n}\int_0^1 \frac{1}{1+t^nr^ne^{2\pi i}}dt = -re^{2\pi i/n}\int_1^0 \frac{1}{1+t^nr^n}dt$$

Ok, let's now look at $\int_{\gamma_1} f(z)dz$, with the parametrization $\gamma(t) = rt, 0 \leq t \leq 1$:

$$\int_{\gamma_1} f(z)dz = \int_0^1 \frac{1}{1+r^nt^n}rdt = r\int_0^1 \frac{1}{1+t^nr^n}dt$$

Thus, we notice that:

$$\int_{\gamma_3} f(z)dz = -e^{2\pi i/n}\int_{\gamma_1} f(z)dz$$

So, taking the limit as $r \to \infty$, we get that:

$$\frac{1}{2\pi i}\left(\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz\right) = \frac{1}{ne^{i\pi(n-1)/n}} \implies \int_0^\infty f(x)dx - e^{2\pi i/n}\int_0^\infty f(x)dx = \frac{2\pi i}{ne^{i\pi(n-1)/n}}$$

Therefore:

$$\int_0^\infty f(x)dx = \frac{\pi}{n}\frac{2i}{e^{i\pi(n-1)/n}(1-e^{2i\pi/n})} = \frac{\pi}{n}\frac{2i}{e^{i\pi(n-1)/n} - e^{i\pi(n+1)/n}} = \frac{\pi}{n}\frac{2i}{e^{i\pi/n} - e^{i\pi/n}}$$

where we use the fact that $e^{i\pi} = -1$. Now, here, we use the fact that $e^{iz} = \cos(z) + i\sin(z)$ to see that:

$$\frac{e^{iz} - e^{-iz}}{2i} = \frac{\cos(z) + i\sin(z) - [\cos(-z) + i\sin(-z)]}{2i} = \frac{\cos(z) + i\sin(z) - \cos(z) + i\sin(z)]}{2i} = \frac{2i\sin(z)}{2i} = \sin(z)$$

Thus:

$$\int_0^\infty f(x)dx = \frac{\pi}{n}\frac{2i}{e^{i\pi/n} - e^{i\pi/n}} = \frac{\pi}{n}\frac{1}{\sin \pi/n}$$

□

Question 4. Suppose $\Omega_1, \Omega_2 \subset \mathbb{C}$ are plane regions, f, g non-constant complex functions defined on Ω_1 and Ω_2 respectively, and $f(\Omega_1) \subset \Omega_2$. Let $h = g \circ f$. If f, g are holomorphic, we know that h is holomorphic. Suppose that f, h are holomorphic. Can we conclude anything about g? How about if g, h are holomorphic?

Solution. First, we look at the case where g, h are holomorphic, but f need not be.

Take $\Omega_1 = \Omega_2 = \mathbb{C}$. Take $f = \sqrt{r}e^{i\theta/2}$, for $-\pi \geq \theta < \pi$, $g = z^2$, and $h = z$.

It should be clear, that we have:

$$h = g(f(z)) = g(\sqrt{r}e^{i\theta/2}) = (\sqrt{r}e^{i\theta/2})^2 = re^{i\theta} = z$$

and, since g, h are polynomial, they are entire. However, looking at, say, $r = 1$, if we take the path to $z = -1$, via $\theta \to \pi$, we get that:

$$f^+(r, \theta) = \sqrt{r}e^{i\theta/2} = \sqrt{r}(\cos(\pi/2) + i\sin(\pi/2))$$

On the other hand, if we take the path the other way, as $\theta \to -\pi$, we find:

$$f^-(r, \theta) = \sqrt{r}e^{i\theta/2} = \sqrt{r}(\cos(-\pi/2) + i\sin(-\pi/2)) = \sqrt{r}(\cos(\pi/2) - i\sin(\pi/2))$$

Thus, f is not continous as $z \to -1$, and thus, f may not be holomorphic. So, we may find f, g, h such that g, h are holomorphic, $h = g(f(z))$, but f need not be holomorphic.

Now, suppose f, h are holomorphic over Ω_1. We claim that g is holomorphic on $f(\Omega_1)$. Since holomorphicity is defined via a limit, or equivalently, at a small enough neighborhood, it suffices to show that for any point $w_0 \in f(\Omega_1)$, that g is holomorphic in some neighborhood of w_0.

First, fix some point $w_0 \in f(\Omega_1)$, and fix some point $z_0 \in \Omega_1$ such that $f(z_0) = w_0$. We notice that since constants are holomorphic, and the sum and composition of holomorphic functions are also holomorphic, that we may assume that $w_0, z_0 = 0$, since we may consider the related function $\tilde{f}(z) = f(z+z_0) - w_0, \tilde{g}(z) = g(w + w_0) - g(w_0)$. Certainly, $\tilde{f}$ is holomorphic, since $\tilde{f} = (f \circ z + z_0) - w_0$, and if $\tilde{g}$ were holomorphic, we may express $g = (\tilde{g} \circ w - w_0) + g(w_0)$ and thus g would be holomorphic as well.

Now, first suppose $f'(0) \neq 0$. Then, by 10.30, we may find a holomorphic inverse ψ, defined on an open set V containing w_0 that maps back to a neighborhood U of z_0. Because ψ is a holomorphic inverse, we have that $f(\psi(z)) = z$ for $z \in V$. Thus, we have that, for $w_0 \in V$:

$$g(w_0) = g(f(\psi(w_0))) = h(\psi(w_0))$$

Thus, since on a neighborhood of w_0, we can identify g as a composition of holomorphic functions ψ, h, g is holomorphic as well.

Now, supose $f'(0) = 0$. Then, by 10.32, and by the hint, since we can identify:

$$f = 0 + [\phi(z)]^m$$

for some holomorphic function ϕ on a neighborhood of z_0, we can take this as a local holomorphic change of coordinates, and examine the related function $f(\phi(z)) = z^m$, it is sufficient to consider $f = z^m$. Note that we have that $m \geq 2$, since because $f(0) = 0, f'(0) = 0$, the order of the zero is at least 2.

Since h is a holomorphic function, we can identify a neighborhood of $z_0 = 0$ such that the power series

$$h(z) = \sum_{k=1}^{\infty} c_k z^k$$

converges. Now, take a neighborhood that we may let $f = z^m$ and that the power series for h converges. On such a neighborhood, we have that for a m-th root of unity α:

$$h(\alpha z) = g(f(\alpha z)) = g((\alpha z)^m) = g(z^m) = g(f(z)) = h(z)$$

Then applying this to the power series:

$$h(\alpha z) = \sum_{k=1}^{\infty} c_k(\alpha z)^k = \sum_{k=1}^{\infty} c_k(\alpha z)^k = \sum_{k=1}^{\infty} c_k \alpha^k z^k = \sum_{k=1}^{\infty} c_k z^k$$

but, for this to be true on the entire neighborhood, we must have then that:

$$c_k = c_k \alpha^k$$

for each k. But, this can only be true if $c_k = 0$ if $m \nmid k$. Then, we can express h as:

$$h(z) = \sum_{k=1}^{\infty} c_{km} z^{km}$$

But, since $h(z) = g(z^m)$, we have that:

$$h(z) = \sum_{k=1}^{\infty} c_{km} z^{km} \implies g(z^m) = \sum_{k=1}^{\infty} c_{km} z^{km} \implies g(z) = \sum_{k=1}^{\infty} c_{km} z^k$$

Then, g has a power series representation on this neighborhood, and thus, by 10.6, is holomorphic.

□

Question 5. Suppose Ω is a region, $f_n \in \mathcal{H}(\Omega)$ for $n \geq 1$. Suppose further that none of the f_n has a zero in Ω, and $f_n \to f$ uniformly on compact subsets of Ω. Prove that either f has no zero in Ω or $f(z) = 0$ on all of Ω.

Solution. Suppose that $f(z) \neq 0$, but there exists $z_0 \in \Omega$ such that $f(z_0) = 0$. Then, by 10.18, z_0 is an isolated point. Thus, we can find an $r > 0$ such that on the closed disk (and thus, compact) $\overline{D}(z_0, r)$, $f(z) = 0 \iff z = z_0$.

Clearly, the (positively-oriented) boundary $\gamma = \delta\overline{D}$ is a closed path in Ω, with $\mathrm{Ind}_\gamma(\alpha) = 0$ for all $\alpha \notin \Omega$. Further, $\mathrm{Ind}_\gamma(\alpha) = 0, 1$ for any $\alpha \in \Omega \setminus \gamma^*$. Further, since $f_n \to f$ uniformly on compact subsets, by 10.28, we have that $f \in \mathcal{H}(\Omega)$. Thus, we may apply Rouche's Theorem.

By hypothesis, since $\overline{D}(z_0, r)$ is compact, we have that $f_n \to f$ uniformly here. Further, since the boundary is compact, $|f|$ obtains a minimum on $\delta\overline{D}$, which we will call δ_0. Moreover, since $|f| > 0$ except at z_0, we must have that $|f(z)| \geq \delta_0 > 0$ for all $z \in \delta\overline{D}$.

Now, because $f_n \to f$ uniformly on compact subsets, we may find a $N > 0$ such that for all $n \geq N$, $|f(z) - f_n(z)| \leq |f - f_n|_\infty < \delta_0 \leq |f(z)|$. Then, by Rouche's Theorem (10.43(b)), we have that the zeros of f_n, N_{f_n}, counted with multiplicity, are the same as the zeros of f, N_f, also counted with multiplicity, on $D(z_0, r)$. However, we have that by hypothesis, that $N_{f_n} = 0$ and $N_f = 1$, a contradiction.

Therefore, either f is identically 0, or f has no zeros on Ω.

□