Homework #8

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Math 235: Homework #8

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2.1

Problem 4.6.21. Assume that $E \subseteq \mathbb{R}^d$ is measurable. Let $f : E \to \overline{F}$ be a measurable function. Define the distribution function of f as follows:

$$\omega(t) = |\{|f| > t\}|, t \geq 0$$

By definition, ω is a non-negative, extended real-valued function. Prove the following:

(a) ω is monotone decreasing on $[0, \infty)$.

(b) ω is right-continuous, that is, $\lim_{s \to t^+} \omega(s) = \omega(t)$ for every $t \geq 0$.

(c) If f is integrable, then $\lim_{s \to t^-} \omega(s) = |\{|f| \geq t\}|$.

(d) $\int_0^\infty \omega(t)dt = \int_E |f(x)|dx$

(e) f is integrable $\iff$ ω is integrable.

(f) If f is integrable, then $\lim_{n\to\infty} n\omega(n) = 0 = \lim_{n\to\infty} \frac{1}{n}\omega(\frac{1}{n})$.

Solution. (a)

We notice that for any $t' \geq t$, that by definition, $\{|f| > t'\} \subseteq \{|f| > t\}$. Then, by the monotonicity of the Lebesgue measure, we have that $|\{|f| > t'\}| \leq |\{|f| > t\}| \implies f(t') \leq f(t)$. Since this is true for all $t' \geq t$, we have that ω is monotone decreasing.

(b)

Let $\{a_n\}_{n\in\mathbb{N}}$ be any sequence of positive numbers where $a_n \to 0$. Take a monotone subsequence $\{a_{n_k}\}$ such that $a_{n_{k+1}} < a_{n_k}$ for all k.

$\square$

Problem 4.6.27. Let $f \in L^1(\mathbb{R}), g \in L^\infty(\mathbb{R})$. Prove the following:

(a) The integral that defines $(f * g)(x)$ exists for every $x \in \mathbb{R}$.

(b) $f * g$ is continuous on $\mathbb{R}$.

(c) $f * g$ is bounded on $\mathbb{R}$, and $\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty$.

Solution. $\square$

Problem 4.6.28. (a) Show that if $f, g \in C_c(\mathbb{R})$, then $f * g \in C_c(\mathbb{R})$ and

$$\text{supp}(f * g) \subseteq \text{supp}(f) + \text{supp}(g) = \{f + g : x \in \text{supp}(f), y \in \text{supp}(g)\}$$

Conclude that $C_c(\mathbb{R})$ is closed under convolution.

(b) Is $C_c^1(\mathbb{R})$ closed under convolution?

Solution. ☐

Problem 4.6.29. Let $E \subseteq \mathbb{R}$ be a measurable subset with $0 < |E| < \infty$.

(a) Prove that the convolution $\chi_E * \chi_{-E}$ is continuous.

(b) Prove the Steinhaus Theorem: The set $E - E = \{x - y : x, y \in E\}$ contains an open interval centered at the origin.

(c) Show that $\lim_{t \to 0} |E \cap (E + t)| = |E|, \lim_{t \to \pm\infty} |E \cap (E + t)| = 0$.

Solution. ☐

2.2

Problem 5.1.5. Prove that the Cantor-Lebesgue function is Hölder continuous for $0 < \alpha \leq \log_3 2$. In particular, notice that it is not Lipschitz.

Solution. ☐

Problem 5.1.7. Let C be the Cantor set, let ϕ be the Cantor-Lebesgue function, and define $g(x) = \phi(x) + x$ for $x \in [0, 1]$.

(a) Prove that $g : [0, 1] \to [0, 2]$ is continous, strictly increasing, and a bijection. Further, its inverse $h = g^{-1} : [0, 2] \to [0, 1]$ is also a continuous, strictly increasing, bijection.

(b) Show that $g(C)$ is a closed subset of $[0, 2]$ and that $|g(C)| = 1$.

(c) Since $g(C)$ has positive measure, it follows that there exists $N \subseteq g(C)$ such that N is not Lebesgue measurable. Show that $A = h(N)$ is a Lebesgue measurable subset of $[0, 1]$.

(d) Set $f = \chi_A$. Prove that $f \circ h$ is not a Lebesgue measurable function.

Solution. ☐