Homework #4

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Math 233: Homework #4

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Question 1. Let L_1, L_2 be lines in the plane. For which pairs of L_1, L_2 do there exists real functions, harmonic on the entire plane, 0 on $L_1 \cup L_2$, but not vanishing identically?

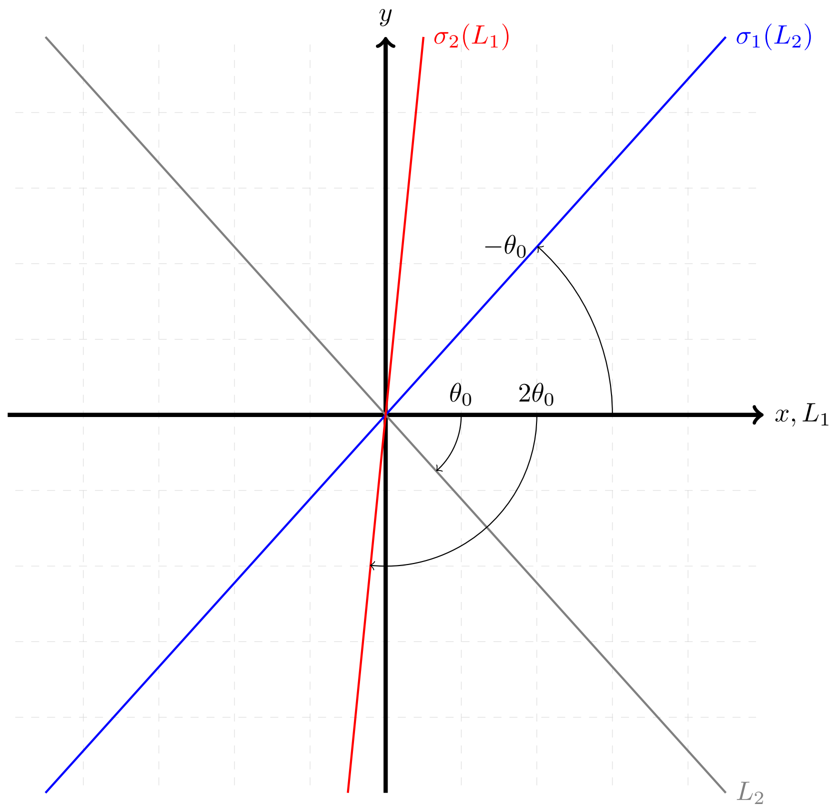
Solution. First, we notice that for any real function v, harmonic on the entire plane, it is the imaginary part of some holomorphic function. First, we know already that by 11.10, every real harmonic function is the real part of a holomorphic function, locally at least. Then, by considering disks around every point $z \in \mathbb{C}$, this can be extended to a holomorphic function f such that $\Re(f) = v$, because on the disks, the local holomorphic functions may only differ by an imaginary constant, and it must align on intersections of disks, thus there may only be a single entire function.

Now, consider if. Since i is a constant, this is clearly holomorphic. Further, by construction $\Im(f) = v$. Thus, we have a holomorphic function such that v is its imaginary part.

Now, suppose v is harmonic, and $v(L_1) = 0, v(L_2) = 0$. Without loss of generality, since we may translate v without affecting the derivatives, we may take $L_1 \cap L_2 = \{(0, 0)\}$. By a further linear change of coordinates, we may assume that L_1 is the real line, which will keep $v_{xx} + v_{yy} = 0$.

Suppose L_1 and L_2 intersect. Suppose that the angle between L_1, L_2 is θ_0.

By the Schwarz reflection principle (11.14), and a relabeling of the two lines as need be, if we call σ_1, σ_2 the reflections of the plane with respect to L_1, L_2, we must have that $f(\sigma_1(z)) = \overline{f}(z), f(\sigma_2(z)) = \overline{f}(z)$. In particular then, on L_1, L_2, we have that $v(\sigma_1(z)) = v(z) = 0, v(\sigma_2(z)) = v(z) = 0$. Pictorially:



where we have that the angle between $L_1, \sigma_1(L_1)$ is $2\theta_0$ because the angle between $\sigma_1(L_1)$ and L_2 is θ_0, due to how reflections work. Further, we also see that $\sigma_1(L_2)$ takes on the angle $-\theta_0$.

We notice that we may iterate this process, and in fact generate lines of $k\theta_0$ via successive reflections. However, we know that if θ_0 is not a rational multiple of π, then $\{e^{im\theta_0} : m \in \mathbb{Z}\}$ is dense in T. And since $v = 0$ on all of these lines, if it is 0 on a dense set, then it is 0 everywhere by continuity. Thus, this implies that we must have that θ_0 is a rational multiple of π.

Now, suppose instead that L_1, L_2 are parallel. In such a case, applying the Schwarz reflection principle on successive lines, we note that then we must have that v is periodic, 0 at each interval $d = \text{dist}(L_1, L_2)$, since we can keep translating and applying reflections to find a line on the opposite side. For example, assuming $L_1 : x = 1, L_2 : x = 5$ one such v could be $v(x, y) = e^y \sin(\pi(x - 1)/\pi)$. This is generalizable with a suitable linear transformation on x, y to match our parallel 1-D lattice.

$\square$

Question 2. Suppose Δ is a closed equilateral triangle in the plane, with vertices a, b, c. Find $\max\{|z - a||z - b||z - c|\}$ for $z \in \Delta$.

Solution. First, fix some $a, b, c \in \mathbb{C}$. We notice that the function:

$$f(z) = (z - a)(z - b)(z - c)$$

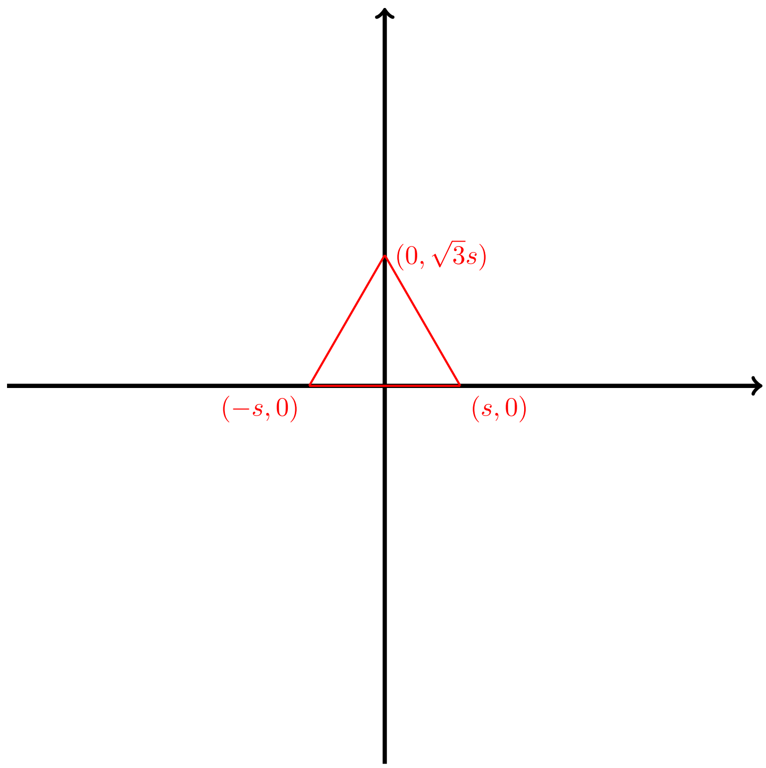
is a polynomial, thus entire. Then, we may apply the maximum modulus principle to our closed triangle which then says that:

$$|z - a||z - b||z - c| \leq \|(z - a)(z - b)(z - c)\|_{\partial\Delta}$$

Thus, it is sufficient to consider the value of $(z - a)(z - b)(z - c)$ on the boundary of our equilateral triangle. Further, since the quantity we are concerned about is

$$\|(z-a)(z-b)(z-c)\|_{\partial\Delta} = \max\{|z-a||z-b||z-c| : z \in \partial\Delta\}$$

where we use max instead of sup due to being compact, this is simply the product of the distances from z to a, b, c. Thus, under any isometries, this product is preserved. Therefore, we may take rotations and translations such that the following picture holds, for $2s$ the side length of our triangle:



Due to symmetries, we can also restrict ourselves to analyzing z on the real line, as a simple rotation will find us the value on the other sides. It should also be clear that due to reflectional symmetries, we can restrict ourselves to the non-negative reals as well.

Let $z = (x, 0)$ with $x \in [0, s]$. Computing the value of $|f|$, we find:

$$|f(z)| = |(x,0)-(s,0)|+|(x,0)-(-s,0)|+|(x,0)-(0,\sqrt{3}s)| = (s-x)(x+s)\sqrt{x^2+3s^2} = (s^2-x^2)\sqrt{x^2+3s^2}$$

But now, this is a real function, so we may take a derivative and check endpoints to find the maximum. We see pretty clearly that:

$$f'(x) = -2x\sqrt{x^2+3s^2} + x(s^2-x^2)\frac{1}{\sqrt{x^2+3s^2}} = \frac{1}{\sqrt{x^2+3s^2}}\left(-2x(x^2+3s^2)+x(s^2-x^2)\right) =$$

$$\frac{1}{\sqrt{x^2+3s^2}}(-3x^3-5xs^2) = \frac{1}{\sqrt{x^2+3s^2}}x(-3x^2-5s^2)$$

Clearly, since $x^2 \geq 0, s^2 > 0$, we have that x^2+3s^2 and $-3x^2-5s^2$ never vanish. Thus, we have only the critical point $x = 0$. Since at $x = s$, f vanishes, because this is also a boundary of our domain, this must be the maximum. Thus, we have that the maximum of f is equal to:

$$f(0) = s^2\sqrt{3s^2} = \sqrt{3}s^3$$

where $2s = |a-b| = |b-c| = |c-a|$

$\square$

Question 3. Suppose $f \in \mathcal{H}(\Pi^+)$, where $\Pi^+ = \{z = x + yi : y > 0\}$, and $|f| \leq 1$. How large can $|f'(i)|$ be? Find the extremal functions.

Solution. First, for $U = \{z : |z| < 1\}$, we consider the map $\psi : U \to \Pi^+$ via:

$$\psi(z) = i\frac{1-z}{1+z}$$

.

On U, this map is holomorphic. Further, this is injective. Suppose we have that $\psi(z) = \psi(w)$. Then, since on U, $z, w \neq -1$:

$$i\frac{1-z}{1+z} = i\frac{1-w}{1+w} \implies (1+w)(1-z) = (1+z)(1-w) \implies 1+w-z-wz = 1+z-w-wz \implies 2w = 2z \implies w = z$$

Further, we have that this map is surjective onto Π^+. Let $\zeta = a + bi \in \Pi^+$. Then, we have that, for $z = x + yi$:

$$f(z) = \zeta \iff i\frac{1-x-yi}{1+x+yi} = a + bi \iff 1 - x - yi = -ai - axi + ay + b + bx + byi$$
$$\iff \begin{cases} 1 - x = ay + b + bx \\ -y = -a - ax + by \end{cases} \iff x = \frac{1-ay-b}{1+b}$$

where we've used the fact that $z \in U$ so $1 + x + yi \neq 0$ and $\zeta = \Pi^+$, so $b \neq -1$. Now, substituting into the second equation, this would enforce that:

$$-y = -a - a\frac{1-ay-b}{1+b} + by \iff -y\left(1 + b + \frac{a^2}{b+1}\right) = -a - \frac{a-ab}{1+b} = \frac{-2a}{1+b} \iff$$
$$y = \frac{2a}{1+b} \cdot \frac{b+1}{a^2+(b+1)^2} = \frac{2a}{a^2+(b+1)^2}$$

Now, substituting back in for x, we find that:

$$x = \frac{1 - a\frac{2a}{a^2+(b+1)^2} - b}{1+b} = \frac{1}{1+b} \cdot \frac{a^2 + (b+1)^2 - 2a^2 - a^2b - b(b+1)^2}{a^2+(b+1)^2} = \frac{1}{b+1}\frac{-a^2(b+1) + (b+1)^2(1-b)}{a^2+(b+1)^2} =$$
$$\frac{1-a^2-b^2}{a^2+(b+1)^2}$$

Now, we need only check that this lives within U. Well:

$$x^2 + y^2 = \frac{1}{(a^2+(b+1)^2)^2}[(1-a^2-b^2)^2 + 4a^2]$$

It should be clear that this is always less than the denominator. If we expand everyyhing out, we see that we have the numerator as:

$$1 + a^4 + b^4 + 2a^2 - 2b^2 + 2a^2b^2$$

and the denominator as:

$$a^4 + 2a^2(b+1)^2 + (b+1)^4 = a^4 + 2a^2b^2 + 4a^2b + 2a^2 + b^4 + 4b^3 + 8b^2 + 4b + 1$$

Subtracting the numerator from the denominator, we see:

$$(a^4 + 2a^2b^2 + 4a^2b + 2a^2 + b^4 + 4b^3 + 8b^2 + 4b + 1) - (1 + a^4 + b^4 + 2a^2 - 2b^2 + 2a^2b^2) =$$
$$4a^2b + 4b^3 + 10b^2 + 4b$$

Now, because (a, b) are chosen from the upper half plane, we have that this number must be positive, since $a^2 \geq 0$, and $b > 0$. Thus, we have that $x^2 + y^2 < 1$, and therefore $z \in U$. Thus, ψ is surjective.

Lastly, we consider the action of ψ on $T = \{z : |z| = 1\}$, or really, $T \setminus \{-1\}$. Well, if $|z| = 1$, we may write it as $z = e^{i\varphi}$. First, we notice that:

$$\begin{cases} \sin(x) = \frac{e^{ix}-e^{-ix}}{2i} \\ \cos(x) = \frac{e^{ix}+e^{-ix}}{2} \end{cases} \implies \tan(x) = i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = i\frac{e^{2ix} - 1}{e^{2ix} + 1}$$

Then, we have that:

$$\psi(e^{i\varphi}) = i\frac{1 - e^{i\varphi}}{1 + e^{i\varphi}} = -i\frac{e^{i\varphi} - 1}{1 + e^{i\varphi}} = -i \cdot i \tan\left(\frac{\varphi}{2}\right) = \tan\left(\frac{\varphi}{2}\right)$$

Since on $T \setminus \{-1\}$, $\varphi \in (-\pi, \pi)$, and on $x \in (-\pi/2, \pi/2)$, $\tan(x) \in (-\infty, \infty)$, $\tan(\varphi/2)$ covers the real line.

Now, let f be as given, and consider the map $g = f \circ \psi : U \to \mathbb{C}$. Because $|f| \leq 1$ on the upper half plane, and the work we've done above, we have that $g \in \mathcal{H}^\infty(U)$, $\|g\|_\infty \leq 1$, and since g is defined on U, we have that, as stated in 12.5, we may take $\alpha = 0 < 1$. Further, if $g(0) = \beta$, then we may assume that $|\beta| < 1$, as otherwise, by the maximum modulus principle, since $|g| \leq 1$ on U, this extends to the boundary by continuity. So, if $|g(0)| = 1$, then g is constant everywhere and the derivative is 0.

Then, by the discussion in 12.5, we have that:

$$|g'(0)| \leq 1 - |\beta|^2$$

However, here, we notice that because $g = f \circ \psi$, $\psi(0) = i\frac{1-0}{1+0} = i$, so $g'(0) = f'(i), g(0) = \beta = f(i)$.

Thus, restated in terms of f, we have that:

$$|f'(i)| \leq 1 - |f(i)|^2$$

Thus, we have two conditions to realize the maximum value here across all functions f. Firstly, we require $f(i) = 0$, and secondly, by Theorem 12.2, if $f(i) = \beta = 0$, then we have that $|g'(0)| = 1$ occurs if and only if $g = \lambda z$, for some $\lambda \in \mathbb{C} : |\lambda| = 1$, that is, f composed with ψ acts as a rotation by some λ on the unit disk U.

This means that, we need only take an inverse to ψ, with some scale factor for the rotation, and a translation such that $f(i) = 0$. Well, I claim that $f(z) = \frac{iz+1}{-iz+1}$ acts as a left inverse to ψ:

$$f\left(i\frac{1 - z}{1 + z}\right) = \frac{-\frac{1-z}{1+z} + 1}{\frac{1-z}{1+z} + 1} = \frac{-1 + z + z + 1}{1 - z + 1 + z} = \frac{2z}{2} = z$$

Further, we see that $f(i) = \frac{i^2+1}{-i^2+1} = \frac{0}{2} = 0$. So that part is all set.

Then, the maximal functions take on exactly the form $f_\lambda(z) = \lambda\frac{iz+1}{-iz+1}$ for $\lambda \in \mathbb{C} : |\lambda| = 1$.

□

Question 4. Suppose $f \in \mathcal{H}(\Omega)$. Under what conditions can $|f|$ have a local minimum in Ω?

Solution. We see that for f non-constant, $|f|$ may have a local minimum on a connected component of Ω if and only if f never attain 0 on that component. Equivalently, we prove that $|f|$ has a non-0 local minimum on a connected component if and only if f is constant on that component.

For what follows, let Ω be a single connected component. Clearly, if f is a non-0 constant, then $|f|$ has a non-0 local minimum, as at any point $\zeta \in \Omega$, $f(\zeta) = f(z) \implies f(\zeta) = f(z)$ for all $z \in \Omega$. So we need only prove the other direction.

Suppose $|f|$ has a non-0 local minimum at $\zeta \in \Omega$, that is, $0 < |f(\zeta)| \leq |f(z)|$ for all $z \in \Omega$. Then, f has no 0s on Ω, but is holomorphic. Thus, $g = 1/f$ is a holomorphic function on Ω. In particular, at ζ we have that $g(\zeta) = \frac{1}{f(\zeta)}$. Since $|f|$ is at a minimum at ζ, $|g| = |1/f|$ is at a maximum at ζ. Thus, we would have that $|g|$ has a local maximum on Ω. However, by the maximum modulus principle, since $|g|$ has a local maximum, we have that g is constant. But, if g is constant, then since $g = 1/f \implies f = 1/g$, f too must be constant.

Thus, $|f|$ having a positive local minimum is equivalent to f being constant on the component containing a local minimum.

$\square$

Question 5. (a) Suppose that Ω is a region, D is a disc, $\overline{D} \subset \Omega, f \in \mathcal{H}(\Omega)$, non-constant, and $|f|$ is constant on ∂D. Prove that f has at least one zero in D.

(b) Find all entire functions f such that $|f(z)| = 1$ for all $|z| = 1$.

Solution. (a)

Since $\overline{D}$ is compact and $|f|$ is a continuous function, $|f|$ achieves a minimum somewhere on $\overline{D}$. First, suppose $|f| \geq \delta > 0$ on $\overline{D}$.

If this occurs on the interior, then by the last problem, f must be constant on D, a contradiction. Thus, suppose this occurs on the boundary, that is $|f|$ attains a minimum on $\overline{D}$ on ∂D. Since $|f|$ is constant, the minimum is attained at all points on the boundary, call it a. However, by the maximum modulus principle, we have that $|f(z)| \leq \|f\|_{\partial D} = a$ for all $z \in D$. Thus, we have that for all $z \in D$, $a = \|f\|_{\partial D} \leq |f(z)| \leq \|f\|_{\partial D} = a \implies |f(z)| = a$. Since f is holomorphic, for this to satisfy the Cauchy-Riemann equations on a disk, we must have that f is constant. Thus, we cannot have that we achieve a positive minimum anywhere.

Now, suppose that $|f|$ achieves 0 somewhere. If it is on the boundary, because $|f|$ is constant on the boundary, then $|f| = 0$ on all of ∂D. However, by the maximum modulus principle then, $f = 0$ everywhere, a contradiction. Then, we must have that $|f| = 0$ somewhere on D, and thus $f = 0$ somewhere on D.

(b)

We claim that under these hypotheses, this is satisfied only if $f = cz^m$ for some $m \geq 0$ and $c \in \mathbb{C}$ such that $|c| = 1$. Clearly, if f is constant, then any function that satisfies this is just $f = e^{i\theta}$ for $\theta \in [0, 2\pi)$ and this satisfies the above conditions for $m = 0$.

So, assume f is non-constant. Let m be the multiplicity of the zero of f at $z = 0$, with $m = 0$ if $f(0) \neq 0$. Define:

$$f_1(z) = \frac{f(z)}{z^m}$$

We notice the following. When $|z| = 1$, we have that $|f_1(z)| = \frac{|f(z)|}{|z|^m} = \frac{1}{1} = 1$, and that by definition, for $m \neq 0$, we can find a $h(z)$ such that $f(z) = z^m h(z)$ on Ω with $h(0) \neq 0$, that

$$f_1(z) = \frac{f(z)}{z^m} = \frac{z^m h(z)}{z^m} = h(z)$$

and thus $f_1(0) \neq 0$.

Now, if $f_1(z)$ is constant, we are done, since of course, we must have that $|\lambda| = 1$, as:

$$f_1(z) = \lambda \implies \frac{f(z)}{z^m} = \lambda \implies f(z) = \lambda z^m$$

and since by hypothesis, when $|z| = 1$, $|f(z)| = 1$, therefore $|\lambda| = 1$.

Then, suppose $f_1(z)$ is not constant. Then, by part (a), we must have at least one zero on U, away from $z = 0$. Let $\alpha_1, ..., \alpha_m$ be the zeros of f_1 in U with multiplicity according to the order of the zero.Recall the function defined in Rudin (12.4) as:

$$\varphi_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}$$

Then, consider the function:

$$g(z) = \frac{f_1(z)}{\varphi_{\alpha_1}(z)...\varphi_{\alpha_m}(z)} = f_1(z) \cdot \prod_{j=i}^{m} \frac{1 - \overline{\alpha}_j z}{z - \alpha_j}$$

We notice that by construction, g must be holomorphic on U, because suppose f_1 has a zero of order m at α_i. Then we may write $f_1 = (z - \alpha_i)^m h(z)$ for some holomorphic function h_{α_i} that is non-0 at α_i. Further, by construction, since we counted α_i as zeros of f with multiplicity according to the order of the 0, there are exactly m copies of $\frac{1}{z-\alpha_i}$ in the product. Thus, g has a removable singularity at each 0 of $f_1(z)$.

Further, suppose $g = 0$ on U. This cannot happen at any of the zeros of f_1, due to the analysis presented above, as the order of the zero is cancelled out by the denominators, and we are left with a non-0 function, as $(1 - \overline{\alpha_j}\alpha_i) = 0 \iff \overline{\alpha_j}\alpha_i = 1$, which implies that $|\overline{\alpha_j}| = |\alpha_j| = \frac{1}{|\alpha_i} > 1$. But since α_j is a 0 of f_1 in U, $|\alpha_j| < 1$, so this cannot happen. Further, by the same logic, none of the $1 - \overline{\alpha_j} z$ disappear on U. Thus, g is non-0 on U.

Further, g must be at least meromorphic everywhere, since f_1 is at least meromorphic, being the ratio of an entire function with a polynomial, and it is being multiplied by a finite product of rational functions.

Lastly, suppose $|z| = 1$. We recall from the text, that φ_α is a map that takes $T \to T$. Thus, we notice that when $|z| = 1$, we had earlier that $|f_1| = 1$, and that for each $\frac{1}{\varphi_{\alpha_j}}$, that for any $|z| = 1$, $\left|\frac{1}{\varphi_{\alpha_j}(z)}\right| = \frac{1}{|\varphi_{\alpha_j}(z)|} = 1$. Therefore, when $|z| = 1$, so too does $|g(z)| = 1$.

Now, we have that g is holomorphic on the disk, with $|g|$ constant on the boundary, with no 0. By the discussion above in part (a), this implies then that g is constant on D, and by continuity, $g = \lambda$ such that $|\lambda| = 1$.

However, let's consider what this means for f. Since g is constant, and $f_1(z) = \frac{f(z)}{z^m}$, we have then that at least on U:

$$\lambda = \frac{f(z)}{z^m} \cdot \prod_{j=i}^{m} \frac{1 - \overline{\alpha}_j z}{z - \alpha_j} \implies f(z) = \lambda z^m \prod_{j=i}^{m} \frac{z - \alpha_j}{1 - \overline{\alpha}_j z}$$

However, since g is meromorphic, since it is constant on U, this must hold everywhere. But this is a contradiction, as this expression has poles at each $\frac{1}{\overline{\alpha}_j}$. Therefore, this case cannot satisfy f as an entire function, and we have that f may only be constant (that is, exponent of 0), or of the form $f(z) = \lambda z^m$ for some $|\lambda| = 1$.

$\square$