Math 285
Lecture Notes

1 September 6th

We will start with a review of calculus, recast in into differential forms.

Definition 1.1. *Let $U \in \mathbb{R}^n$ be an open set. For a function $f : U \to \mathbb{R}$, we say $f \in C_p^k$ at a point p if all partial derivatives of f with order $\leq k$ exist and are continuous at k.*

Example: $C^0(\mathbb{R})$ describes functions that are at least continuous over the real numbers. In our setting, we will usually concern ourselves with functions that belong to C^∞, where $C^\infty = \cap_{i=0}^{\infty} C^i$

Definition 1.2. *Let $U \subseteq \mathbb{R}^n$, and let $f : U \to \mathbb{R}$. We call f analytic at a point $p \in U$ if it agrees with its Taylor's series at p in some neighborhood of p.*

We notice that because taking derivatives is linear, that is, we can differentiate term by term, that if f is analytic, then $f \in C^\infty$. However, the converse need not be true:

Consider:

$$f = \begin{cases} e^{-1/x} & \text{when } x > 0 \\ 0 & \text{else} \end{cases}$$

Without too much work, we see that this function is continuous. Moreover, the derivative of $e^{-1/x}$ is equal to $x^{-2}e^{-1/x} = x^{-2}f$. Taking the limit as $x \to 0$, and using L'Hôpital's rule where necessary, we can see this goes to 0. Alternatively, we can look at:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x}$$

with reasonable usage of L'Hôpital's.

The upshot is that, inductively, we may show that $f^{(k)}(0) = 0$, and thus, at the point $x = 0$, the Taylor series for f is identically 0. However, in no neighborhood of 0, is $f(x)$ identically 0. Thus, f is not analytic. However, via computation, we see that $f \in C^\infty$. So, $C^\infty \not\Rightarrow$ analytic.

Another way to see this concept, is if we think about Taylor's Series up to k-th order. This is just a Taylor series truncated at the k-th term, with a remainder term R_{k+1}. Then, in such a view, f is analytic at a point $p \iff \lim_{k\to\infty} R_k = 0$.

Definition 1.3. *Let $U \in \mathbb{R}^n$ be a set, and $p \in U$. We call U star-shaped with respect to p if, for all $q \in U$, that the line segment $\overline{pq} \subset U$.*

This motivates the hypotheses for Taylor's Theorem with a remainder term:

Theorem 1.1. *Let $U \subset \mathbb{R}^n$ be a star-shaped open set with respect to a point $p \in U$. Let $f : U \to \mathbb{R}$. If $f \in C^\infty$, then there exist $g_1, ..., g_n \in C^\infty$ such that*

$$f(x) = f(p) + \sum_{i=1}^{n} g_i(x)(x^i - p^i) \text{ with } g_i(p) = \frac{\partial f}{\partial x^i}(p)$$

Proof: Let $y \in U$, and let $x \in \overline{py} \subseteq U$. Taking a parametrization of $\overline{py} : x(t) = p+t(y-p)$ where the i-th component is given by $x^i(t) = p^i+t(y^i-p^i)$.

Now, consider $f(y) - f(p) = f(x(1)) - f(x(0))$. Using the fundamental theorem:

$$f(x(1)) - f(x(0)) = \int_0^1 \frac{d}{dt} f(x(t))dt = \int_0^1 \sum \frac{\partial f}{\partial x^i}(x(t))\frac{dx^i}{dt}dt =$$

$$\sum_i \int_0^1 \frac{\partial f}{\partial x^i}(p + t(y-p))(y^i - p^i)dt = \sum_i \left(\int_0^1 \frac{\partial f}{\partial x^i}(p + t(y-p))dt \right) (y^i - p^i)$$

Thus, we identify:

$$g_i(y) = \int_0^1 \frac{\partial f}{\partial x^i}(p + t(y-p))dt$$

It should be clear that:

$$g_i(p) = \int_0^1 \frac{\partial f}{\partial x^i}(p + t(p-p))dt = \int_0^1 \frac{\partial f}{\partial x^i}(p)dt = \frac{\partial f}{\partial x^i}(p)$$

as $\frac{\partial f}{\partial x^i}(p)$ is not a function of t.

Further, by an application of the dominated convergence theorem:

$$\frac{\partial}{\partial y^j} g_i(y) = \frac{\partial}{\partial y^j} \left(\int_0^1 \frac{\partial f}{\partial x^i}(p + t(p-p))dt \right) =$$

$$\int_0^1 \frac{\partial}{\partial y^j} \frac{\partial f}{\partial x^i}(p + t(p-p))dt$$

which exists and is continuous because $f \in C^\infty$. Thus, $g_i \in C^\infty$

2 September 11th

We want to reformulate the concept of a tangent vector in a coordinate-free way, because we should not need to immerse our manifold in an ambient Euclidean space.

Note that we will use parentheses for a point, and angle brackets for a vectors.

Recall that for a surface traced out in $\mathbb{R}^3$ by some function $M : f(x^1, x^2, x^3) = 0$, we can say that the tangent space is:

$$T_p(M) = \{v_p \in T_p(\mathbb{R}^3) : \nabla f(p) \cdot v_p = 0\}$$

But this depends on the space we're immersed in.

To move towards a coordinate independent description, we instead look at the directional derivative.

Definition 2.1. *If $v_p \in T_p(U)$ and $f \in C^\infty(U)$, then the directional derivative of f in the direction of v_p at the point p is denoted by $D_{v_p} f$.*

Explicitly, we can describe this as a cross section $f(p + tv)$, and thus:

$$D_{v_p} = \frac{d}{dt}\bigg|_{t=0} = \sum_i \frac{\partial}{\partial x^i}\bigg|_{x=(p+tv_p)} \frac{dx^i}{dt}\bigg|_{t=0} = \sum_i \frac{\partial}{\partial x^i}(p)v^i = \sum_i v_i \frac{\partial}{\partial x^i}\bigg|_p$$

This leads to the concept of the germs of a function:

Definition 2.2. *Let (f, U) denote a C^∞ function and its domain: $f : U \to \mathbb{R}$. Fix a $p \in U$.*

We say that $(f, U) \sim (g, V)$ if there exists $W \subset U \cap V$ such that $p \in W$, and that restricted to W, $f = g$. We denote these equivalence classes as $[(f, U)]$ and call these the germs of functions.

Further, we denote the set of equivalence classes at p as C_p^∞.

With some work, we can show that due to our equivalence being on some neighborhood of p, and the derivative being a local characteristic, that we may apply the directional derivative as:

$$D_{v_p} : C_p^\infty \to \mathbb{R}$$

Without too much trouble, we can see that there is a algebra of germs over $\mathbb{R}$ with the following operations:

$$\begin{cases} [(f, U)] + [(g, V)] = [(f + g, U \cap V)] \\ [(f, U)] * [(g, V)] = [(f * g, U \cap V)] \\ \lambda[(f, U)] = [(\lambda f, U)] \end{cases}$$

Proposition 2.1. *Let $D_{v_p} : C_p^\infty \to \mathbb{R}$.*

(i) D_{v_p} is $\mathbb{R}$-linear.

(ii) D_{v_p} follows a Leibniz rule, that is: $D_{v_p}(fg) = D_{v_p}(f)g(p) + f(p)D_{v_p}(g)$

Definition 2.3. *Let $D : C_p^\infty \to \mathbb{R}$. If D satisfies (i) and (ii) from Proposition 2.1, then we call it a derivation at p, or equivalently, a point-derivation of C_p^∞.*

Definition 2.4. *We denote the set of point-derivations of C_p^∞ as $\mathcal{D}_p(\mathbb{R}^n)$.*

Note that $\mathcal{D}_p(\mathbb{R}^n)$ is closed under addition and scalar multiplication, but not under multiplication. Thus, this forms a vector space, but not an algebra.

So, now we can recast our tangent space.

Theorem 2.1. *The map defined by:*

$$\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R})$$

such that $v_p \mapsto D_{v_p}$
is a linear isomorphism of vector spaces.

Proof. Suppose D_{v_p} is the 0 operator. By definition:

$$\varphi(v_p) = D_{v_p} = \sum_i v^i \frac{\partial}{\partial x^i}\bigg|_p$$

Since this is true for all functions, it is in particular true for the function $f = x^j$. Of course then:

$$D_{v_p}(f) = \sum_i v^i \frac{\partial}{\partial x^i}\bigg|_p (f) = v_p^j = 0$$

Since the choice of x^j was arbitrary, this may be performed for each x^j. Thus, $v_p^i = 0$ for all i, and thus $v_p = 0$.

Now, let $D \in \mathcal{D}_p(\mathbb{R}^n)$ be an arbitrary point-derivation.

Define $D_{v_p} = \sum v^i \frac{\partial}{\partial x^i}\Big|_p$ where $v^j = D(x^j)$.

We claim that for an arbitrary $f \in C_p^\infty$, that $Df = D_{v_p} f$.

Using Theorem 1.1 (Taylor's theorem with Remainder), we expand f as:

$$f(x) = f(p) + \sum g_i(x)(x^i - p^i) : g_i(p) = \frac{\partial f}{\partial x^i}(p)$$

So, computing:

$$D(f) = D(f(p) + \sum g_i(x)(x^i - p^i)) = 0 + \sum D(g_i(x)(x^i - p^i)) =$$

$$\sum D(g_i)(p^i - p^i) + g_i(p) D(x^i - p^i) = \sum \frac{\partial f}{\partial x^i}(p) D(x^i)$$

We notice that this is exactly the same form as $\sum v^i \frac{\partial}{\partial x^i}\Big|_p$ due to our identification of $v^i = D(x^i)$. Thus, $D = D_{v_p}$ on all f. $\square$

Note: we will notate in the future as $e_{i,p} = \frac{\partial}{\partial x^i}\Big|_p$ from now on.

Because we have a bijection, we can establish the following definition:

Definition 2.5. *$T_p(U)$ is the set of point derivations of C_p^∞.*

Definition 2.6. *Let $X : U \to \coprod_{p \in U} T_p(U)$. If $X_p \in T_p(U)$, we call such a function a vector field, where we use $\coprod$ to remind ourselves that the tangent spaces are disjoint.*

Unpacking the definition a little bit, if $X_p \in T_p(U)$ then:

$$X_p = \sum a^i(p) \frac{\partial}{\partial x^i}\Big|_p$$

Denoting $a^i : U \to \mathbb{R}$ as $p \mapsto a^i(p)$, then we have that:

$$X = \sum a^i \frac{\partial}{\partial x^i}$$

Definition 2.7. *Let X be a vector field as above. We say that $X \in C^\infty$ if each $a^i : U \to \mathbb{R}$ is C^∞.*

Notation: The set of C^∞ vector spaces on U is an $\mathbb{R}$-vector space. We denote this by $\mathcal{X}(U)$.

3 September 13th

Multilinear algebra:

Recall some easy examples.

The dot product $\langle u, v\rangle$ is a bilinear function that takes $V \times V \to k$.

Dual spaces:

Note that we like this point of view because we can always multiply functions, but we may not admit a multiplication on vectors. Cross products need not exist.

Definition 3.1. *A covector of a vector space V is a linear function $f : V \to \mathbb{R}$.*

Definition 3.2. *The dual space of V is the set of all covectors of V. We denote this as V^*.*

Theorem 3.1. *Let V be a vector space with basis $\{e_i\}_{I=1}^n$, and we have $\alpha^i \in V^*$, where $\alpha^i(e_j) = \delta_i^j$. Then, $\{\alpha^i\}$ form a basis for V^*. We call this the dual basis to $\{e_j\}$.*

Proof. Suppose $\sum_{I=1}^n c_i \alpha^I = 0$. Since these are functions on V, consider their action on e_j. Then, of course, we find that $c_j = 0$. Repeating this argument, this tells us that $c_i = 0$ for all I. In a similar fashion, if we assume f to be a functional, its action is completely determined on the basis vectors. Then, we may construct g such that $g = \sum_{I=1}^n d_i \alpha^I$. And we notice $f - g = 0$ everywhere, so $f = g$. $\square$

Corollary 3.1. *The dimension of the dual space is the dimension of the vector space.*

Definition 3.3. *Let f is a function, k-linear on V. That is, a function from $V \times V ... \times V \to \mathbb{R}$, linear in each argument. We call this object a k-tensor on V. Further, we call f symmetric if, for any permutation of the arguments, f is constant.*

$$f(v_{\sigma(1)}, ..., v_{\sigma(n)}) = f(v_1, ..., v_n)$$

for any $\sigma \in S_n$. Call f alternative if, instead:

$$f(v_{\sigma(1)}, ..., v_{\sigma(n)}) = sgn(\sigma) f(v_1, ..., v_n)$$

Definition 3.4. *Let $\sigma \in S_n$. We say that:*

$$sgn(\sigma) = \begin{cases} 1 \text{ if } \sigma \text{ can be expressed as a product of an even number of transpositions} \\ -1 \text{ if } \sigma \text{ can be expressed as a product of an odd number of transpositions} \end{cases}$$

Equivalently, we can count the inversions.

It turns out, that the symmetric tensors will not be terribly interesting for this course, but the alternating ones will be.

Definition 3.5. *Denote the k-tensors on V as the set $L_k(V)$. Denote the alternating k-tensors on V as the set $A_k(V)$*

Definition 3.6. *Let $f \in L_k(V), g \in L_l(V)$. Denote the tensor product of f and g as $f \otimes g \in L_{k+k}(V)$, where:*

$$f \otimes g(v_1, ..., v_{k+l}) = f(v_1, ..., v_k) g(v_{k+1}, ..., v_{k+l})$$

Note that this is not commutative, but it is associative.

Definition 3.7. *Let $f \in A_k(V), g \in A_l(V)$. We denote the wedge product of f and g as $f \wedge g$, computed as:*

$$f \wedge g(v_1, ...v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} sgn(\sigma) f(v_{\sigma(1)}, ..., v_{\sigma(k)}) g(v_{\sigma(k+1)}, ..., v_{\sigma(k+l)})$$

Definition 3.8. *Let $\sigma \in S_{k+l}$. If $\sigma(1) < ... < \sigma(k)$ and $\sigma(k+1) < ... < \sigma(k+l)$, we call σ a (k,l) shuffle. Note that we have $\binom{k+l}{k}$*

We notice that instead of summing over all permutations of S_n, we may simply define the wedge product over (k,l) shuffles, and drop the fraction in front.

Definition 3.9. *Let $\sigma \in S_k$, and let $f \in L_k(V)$. Then, we denote the permutation of the arguments as $\sigma f(v_1, .., v_k) = f(v_{\sigma(1)}, ..., v_{\sigma(k)})$.*

Definition 3.10. *Let $f \in L_k(V)$. We call the function:*

$$A(f) = \sum_{\sigma \in S_k} sgn(\sigma)(\sigma f)$$

the alternator of f.

Theorem 3.2. *For $f \in L_k(V)$, $A(k)$ is alternating.*

Proof. Let $\tau \in S_n$. Then, we have that:

$$\tau A(f) = \tau \sum_{\sigma} \text{sgn}(\sigma)(\sigma f) = \sum_{\sigma} \text{sgn}(\sigma)(\tau\sigma f) = \sum_{\sigma} \text{sgn}(\sigma)(\tau\sigma f) = \text{sgn}(\tau) \sum_{\sigma} \text{sgn}(\tau\sigma)(\tau\sigma) f$$

But, we see that summing over σ is equivalent to summing over $\tau\sigma$ because it's just a translation of the group. So, this is equal to $\text{sgn}(\tau)A(f)$, and we're done. $\square$

Thus, we notice that we can also write the wedge product:

$$f \otimes g = \frac{1}{k!l!} A(f \otimes g)$$

Remark: The wedge product has the following properties: (i) $f \wedge g = (-1)^{d_f d_g} g \wedge f$ where d_f is the degree of f and same for g. (ii) It is associative, $(f \wedge g) \wedge h = f \wedge (g \wedge h)$. (iii) $f \wedge g \wedge h = \frac{1}{k!l!m!} A(f \otimes g \otimes h)$. (iv) For $\alpha^i \in V^*$, $(\alpha^1 \wedge ... \wedge \alpha^k)(v_1, ..., v_k) = \det[\alpha^i(v_j)]$

4 September 18th

Differential forms on $\mathbb{R}^n$

Recall that a covector is a linear map from $V \to \mathbb{R}$, 1-tensor, alternating 1-tensor.

Analogously then, a covector field on an open set $U \subseteq \mathbb{R}$ assigns a covector to each point $u \in U$.

An alternative name for a covector field is a 1-form.

Differentials of f:

Definition 4.1. *For $f \in C^\infty(U)$, define the 1-form df on U via, for $p \in U$:*

$$(df)_p(X_p) = D_{X_p} f := X_p f$$

Example 4.1.

$$(dx^i)_p \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) = \left. \frac{\partial}{\partial x^j} \right|_p x^i = \delta^i_j$$

We notice that $\left\{\frac{\partial}{\partial x^i}\right\}$ is a basis, and the standard basis for $T_p(\mathbb{R}^n)$.

Therefore, $\{dx^i\}$ is the dual basis for $(T_p\mathbb{R}^n)^v := T_p^*(\mathbb{R}^n)$, the cotangent space at p.

Then, of course, every covector w_p at p may be expressed in this basis as:

$$w_p = \sum b_i(p) dx_p^i$$

Viewing b_i as a function over U, then:

$$w = \sum b_i dx^i$$

where $b_i : U \to \mathbb{R}$.

Definition 4.2. *A 1-form $w = \sum b_i dx^i$ is C^∞ if all $b_i \in C^\infty$.*

Example 4.2. *Consider $\{x, y, z\} \subset \mathbb{R}^3$ as the standard coordinates.*

dx is a 1-form on $\mathbb{R}^3$ such that

$$dx\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\right) = a$$

that is, it extracts the x-coordinate of a vector.

If we consider extracting coordinates via the dual basis, it's not hard to see that, for $f \in C^\infty$:

$$df = \sum \frac{\partial f}{\partial x^i} dx^i$$

Extending to k arguments, we have the following:

Definition 4.3. *A k-form on U is an alternating k-tensor w_p defined at each point $p \in U$.*

Recall the following theorem:

Theorem 4.1. *If $\alpha^1, ..., \alpha^n$ is a basis for $A_1(V)$, then a basis for $A_k(V)$ is $\alpha^{i_1} \wedge ... \wedge \alpha^{i_k}$ where $i_1 < .. < i_k$.*

Example 4.3. *In $\mathbb{R}^3$:*

A 0-form associates to each point a number, so is a linear function $f : U \to \mathbb{R}$.

A 1-form looks like:

$$w = f dx + g dy + h dz$$

A 2-form looks like:

$$w = P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy$$

A 3-form looks like:

$$w = f dx \wedge dy \wedge dz$$

In general, we define a k-form on $U \subset \mathbb{R}^n$ as:

$$w = \sum_I b_I dx^I$$

where I is a multi-index such that $1 \leq i_1 < ... < i_k \leq n$.

We can see that there exists a 1-1 correspondence between 1-forms/2-forms and vector fields.

We can also see that these exists a 1-1 correspondence between 0-forms/3-forms and functions.

Not too hard to see, we would just look at these as abstract vector fields.

Exterior Derivative:

Notationally, we will denote $\Omega^k(U)$ as the C^∞ k-forms on U.

Definition 4.4. *We define the exterior derivative as a map:*

$$d : \Omega^k(U) \to \Omega^{k+1}(U)$$

where $d\left(\sum_I b_I dx^I\right) = \sum_I db_I \wedge dx^I = \sum_{I,j} \frac{\partial b_I}{\partial x^j} dx^j \wedge dx^I$

Proposition 4.1. *(i) d is an antiderivation of degree 1:*

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau$$

(ii) $d \cdot c = 0$.

(iii) If f is a 0-form, then:

$$df(X) = Xf$$

See proof in book.

Recall some basics from vector calculus.

The gradient of f may be viewed as:

$$\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_x \end{bmatrix}$$

We say this corresponds to the following 1-form:

$$df = f_x dx + f_y dy + f_z dz$$

Similarly, for the curl:

$$\nabla \times f = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}$$

This corresponds to the following 2-form:

$$(R_y - Q_z) dy \wedge dz + (P_z - R_x) dz \wedge dx + (Q_x - P_y) dx \wedge dy = d(Pdx + Qdy + Rdz)$$

Without writing it out, we can look at the divergence of a vector field, and we notice that it corresponds to the following 2-form:

$$d(Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy)$$

In particular then, we see that these are special cases of the exterior derivative. More generally, if we let $\mathfrak{X}(U)$ denote C^∞ vector fields on U, we see that we have a sequence:

$$C^\infty(U) \to_{\text{grad}} \mathfrak{X}(U) \to_{\text{curl}} \mathfrak{X}(U) \to_{\text{div}} C^\infty(U)$$

and we have a parallel sequence of isomorphic structures:

$$\Omega^0(U) \to_d \Omega^1(U) \to_d \Omega^2(U) \to_d \Omega^3(U)$$

Theorem 4.2.

$$d^2 = 0 \iff \mathit{curl}(\mathit{grad}\ f) = 0, \mathit{div}(\mathit{curl}\ F) = 0$$

Similar, we can look at Green's theorem as a statement on differential forms:

Theorem 4.3. *The Generalized Stokes' Theorem*

$$\int_{\partial D} \omega = \iint_D d\omega$$

5 September 20

Topological Manifolds:

Definition 5.1. *We call a topological space M locally Euclidean if, for every $p \in M$, there exists a neighborhood $p \in U$ such that U is homeomorphic to a neighborhood of a Euclidean space $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$, for some n. For such an n, we say that M is locally Euclidean of dimension n. We call a pair (U, ϕ) a chart, or a coordinate neighborhood of M.*

Definition 5.2. *If a topological space is locally Euclidean, Hausdorff, and second countable, we call it a topological manifold.*

Note that the idea here is that hope that topological manifolds can be embedded in some $\mathbb{R}^n$, and $\mathbb{R}^n$ is Hausdorff, second countable. Since topological subspaces remain Hausdorff, second countable, we restrict ourselves to the study of this class of manifolds.

Example 5.1. *We want to show that the set $\{(x, y) : xy = 0\}$ is not locally Euclidean. Suppose it were around the origin $p = (0, 0)$. Then, we note that we have a homeomorphism from $U \setminus p \to \phi(U \setminus p) \subseteq \mathbb{R}^n$. However, we notice that $U \setminus p$ has 4 connected components, but removing a single point from any Euclidean neighborhood gives us either 2 components in $\mathbb{R}$ and 1 else. Thus, such a homeomorphism cannot exist.*

Definition 5.3. *Let U, V be open sets, locally Euclidean with charts $\varphi : U \to \mathbb{R}^n, \psi : V \to \mathbb{R}^n$. We call ψ, φ C^∞ compatible if both:*

$$\begin{cases} \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V) \\ \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V) \end{cases}$$

are C^∞ maps.

Definition 5.4. *Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ be a collection of C^∞ charts, such that $M = \cup_\alpha U_\alpha$. We call this collection an atlas.*

Definition 5.5. *Let $\mathfrak{U}$ be a C^∞ atlas. We call it maximal if it is not contained in any other C^∞ atlas, that is, if $\mathfrak{U} \subseteq \mathfrak{M}$, then $\mathfrak{U} = \mathfrak{M}$.*

Definition 5.6. *We call a topological manifold equipped with a maximal C^∞ atlas, a C^∞ manifold.*

Theorem 5.1. *Every C^∞ atlas on a locally Euclidean space is contained within a unique, maximal C^∞ atlas.*

Theorem 5.2. *If M, N are C^∞ manifolds, so too is $M \times N$.*

6 September 25

We want to define smooth maps, the morphisms in the category of topological manifolds.

First, let's talk about smooth functions, that is, for M a topological manifold, $f : M \to \mathbb{R}$.

Definition 6.1. *Let M be a topological manifold. We say that a map $f : M \to \mathbb{R}$ is C^∞ at a point $p \in M$ if there exists a chart (U, ϕ) of M about p such that:*

$$f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$$

is C^∞ at $\phi(p)$.

First, is this well-defined? Is this independent of the choice of chart?

Suppose that we have another chart about p: (V, ψ). Of course, $p \in U \cap V$, so we may use the transition maps:

$$f \circ \psi^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1})$$

We have a priori that $f \circ \phi^{-1}$, is C^∞. And, because these charts belong to the same maximal atlas, they must be C^∞ compatible, hence $\phi \circ \psi^{-1}$ is C^∞. And the composition of two C^∞ functions is C^∞.

6.1 Smooth maps

First, a technical note:

Theorem 6.1. *If $F : M \to N$ and $G : N \to P$ are C^∞ maps, then $G \circ F$ is a C^∞ map*

Definition 6.2. *Let M, N be topological manifolds, and let $F : M \to N$.*

We say that F is C^∞ at a point $p \in M$ if there exists charts (U, ϕ) of $p \in M$ and (V, ψ) of $F(p) \in N$ such that $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

Without too much trouble, you can see that every coordinate map of a chart is C^∞ by following the definition, and picking ϕ^{-1} to pull from $\mathbb{R}^n$, and taking the identity on the target $\mathbb{R}^n$. This also applies to the inverse of the coordinate map.

Definition 6.3. *Let $F : M \to N$ be an bijective, C^∞ map. If F^{-1} is also C^∞, then we call F a diffeomorphism.*

Example 6.1. *Let (U, ϕ) be a chart of M. Then, ϕ, ϕ^{-1} are diffeomorphisms between $U, \phi(U) \subseteq \mathbb{R}^n$*

Theorem 6.2. *Let $F : U \to F(U)$ be a diffeomorphism from an open subset $U \subseteq M$ to a subset of $\mathbb{R}^n$ for some n. Then, (U, F) defines a chart of M. In particular, this chart is compatible with the maximal atlas fixed by the choice of M.*

Proof. First, we want to see that F is a homeomorphism. Without too much trouble, we look at $\psi \circ F \circ \phi^{-1}$, where ψ is a chart for $\mathbb{R}^n$, and ϕ is a chart for U. Since this is a C^∞ map, it is continuous, and since ψ, ϕ^{-1} are continuous, so too must be F. Therefore, F is a homeomorphism, since we can play the same game with F^{-1}.

Next, suppose (U_α, ϕ_α) is a member of the maximal atlas on M. Well, since F, ϕ_α are C^∞ maps, we have that $F \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ F^{-1}$ are C^∞. Thus, (U, F) is C^∞ compatible with (U_α, ϕ_α). Since the choice of (U_α, ϕ_α) were arbitrary, (U, F) is compatible with every chart in your maximal atlas, and by the definition of the maximal atlas, it must belong to the maximal atlas. $\square$

Therefore, since we have an association with diffeomorphisms and charts, we can say that each diffeomorphism gives rise to a choice of coordinate system.

6.2 Partial Derivatives on a Chart

Let (U, ϕ) be a chart of M. We define the partial derivatives on a chart as follows, where we define $x^i = r^i \circ \phi$.

Definition 6.4.
$$\frac{\partial f}{\partial x^i}(p) = \left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f \circ \phi^{-1}}{\partial r^i}(\phi(p))$$

Theorem 6.3.

$$\frac{\partial x^i}{\partial x^j}(p) = \delta^i_j$$

Proof. Clear, by using the definition of x^i as combined with the definition of partial derivatives.

□

6.3 Inverse Function Theorem

Definition 6.5. *We call a map $F : M \to N$ locally invertible at $p \in M$ (equivalently, a local diffeomorphism at p) if there exists U, a neighborhood of p and V, a neighborhood of $F(p) \in N$ such that $F : U \to V$ is a diffeomorphism.*

Theorem 6.4. *Let $F : W \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be C^∞ on an open set W. Then, F is locally invertible at p if and only if, for r^i the standard coordinates on $\mathbb{R}^n$, the determinant of the matrix with i, j-th entry: $\partial F^i/\partial r^j$ does not vanish at p.*

Theorem 6.5. *Let $(U, \phi) = (U, (x^1 ..., x^n))$ be a chart of M around a point p, and let $(V, \psi) = (V, (y^1, ..., y^n))$ be a chart of N around $F(p)$, where $F : U \to V$ is a C^∞ map.*

Define $F^i = y^i \circ F$, that is, the i-th component function of F. Then, we have that F is locally invertible at $p \in U$ if and only if the determinant of the matrix with i, j-th entry: $\partial F^i/\partial x^j$ does not vanish at p.

Proof. Using the definition of partial derivatives on a chart, we can instead say then that F is a local diffeomorphism at $p \in U \subseteq M$ if and only if $\psi \circ F \circ \phi^{-1}$ is a local diffeomorphism at $\phi(p) \in \mathbb{R}^n$ if and only if the matrix with entries:

$$\left[\frac{\partial(\psi \circ F \circ \phi^{-1})^i}{\partial r^j}\right]_{i,j}$$

has non vanishing determinant at $\phi(p)$

However, we notice that:

$$(\psi \circ F \circ \phi^{-1})^i = \psi^i \circ F \circ \phi^{-1} = y^i \circ F \circ \phi^{-1} = F^i \circ \phi^{-1}$$

Thus, we recover the statement in the theorem. □

7 September 27th

So we can discuss some examples of C^∞ manifolds by exhibiting an atlas explicitly. However, we have other ways to construct manifolds.

We think of this via gluing edges. The classic example is identifying opposite edges of a square to make a torus. But, how do we formalize this?

Start with the unit square as an example. First, define $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$. From here, we wish to generate an equivalence relationship by enforcing the reflexivity, symmetric, and transitive properties.

In general, suppose $\sim$ is an equivalence relationship on a topological space S. Denote $[x]$ as the equivalence class of x. Define the quotient of S by $\sim$ as the set of equivalence classes $[x]$. We denote this quotient as $S/\sim$.

We may naturally endow this space with the quotient topology from S, that is, we say $U \subseteq S/\sim$ is open if, for the projection $\pi : S \to S/\sim$ which sends $x \mapsto [x]$, $\pi^{-1}(U)$ is open. Equivalently, we take the finest topology such that the projection is continuous.

Another example: Take $S = \mathbb{R}$, and take the equivalence relationship that only associates $0 \sim 1$. Then, of course, this is not a manifold because it is not locally Euclidean.

7.1 A necessary condition for a quotient to be Hausdorff

Proposition 7.1. *Let X be a Hausdorff topological space. Let $\{x\} \subset X$ be any singleton set. Then, $\{x\}$ is closed.*

Theorem 7.1. *If $S/\sim$ is Hausdorff, then for all $x \in S$, the equivalence class $[x] \subseteq S$ is a closed set in S.*

So of course, now we can say when a quotient is not Hausdorff. However, we would like a sufficient condition as well.

7.2 Open maps

Definition 7.1. *Let $f : X \to Y$ be a map between topological spaces. We call f open if, for any open set $U \subset X$, $f(U)$ is open in Y.*

Non-example: $f : \mathbb{R} \to \mathbb{R}$ that sends $x \to x^2$ is not an open map.

Definition 7.2. *Suppose $\pi : S \to S/\sim$, the projection, is an open map. Then, we call $\sim$ an open equivalence relation.*

Definition 7.3. *If $\sim$ is an equivalence relation on S, define $R = \{(x, y) \in S \times S : x \sim y\}$.*

Theorem 7.2. *If $\pi : S \to S/\sim$ is an open map, and R is closed in $S \times S$, then the quotient space $S/\sim$ is Hausdorff.*

Proof. Let $[x] \neq [y] \in S/\sim$. Then, we must have that $x \not\sim y$ and therefore $(x, y) \notin R$. Then, since R is closed, we may find an open rectangle $U \times V$ that contains (x, y) such that $(U \times V) \cap R$ is trivial.

Of course then, we have that $x \in U$, $y \in V$. By the condition that the intersection is trivial, we must have that for all $u \in U, v \in V$, $u \not\sim v$.

We may look then at $\pi(U), \pi(V)$. Of course, since U, V are open sets in S, we must have that these are open sets in the quotient. Furthermore, of course $[x] \in \pi(U), [y] \in \pi(V)$. Lastly, their intersection must be trivial as if we may find a $[z] \in \pi(U) \cap \pi(V)$, then, we would have a $(z, z) \in R$. But that is a contradiction. Therefore, we can find disjoint neighborhoods of $[x], [y]$. Since $[x], [y]$ were arbitrary, this can be done for any $[x], [y]$, and therefore, $S/\sim$ is Hausdorff. $\square$

Corollary 7.1. *If the diagonal Δ is a closed set in $S \times S$, then S is Hausdorff.*

In fact, the converse is also true.