First Assignment

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Math 240: Homework #3

September 29, 2022

Problem 3.1. Let $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ be algebraic sets. Consider $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$

(a) Show that $X \times Y$ is an algebraic subset of $\mathbb{A}^{m+n}$.

(b) Show that if either X or Y are reducible, then $X \times Y$ is reducible.

(c) Show that if both X, Y are irreducible, then $X \times Y$ is irreducible.

(d) Compute the dimension of $X \times Y$ in terms of the dimensions of X and Y.

Solution. (a)

Let $I =< f_1, f_2, ..., f_i >$ be the radical ideal in $k[x_1, ..., x_n]$ such that $V(I) = X \subseteq \mathbb{A}^n$, and let $J =< g_1, g_2, ..., g_j >$ be the radical ideal in $k[x_1, ..., x_m]$ such that $V(J) = Y \subseteq \mathbb{A}^m$. Consider, the ideal generated by $\overline{I} =< f_1, f_2, ..., f_i > \subseteq k[x_1, ..., x_{m+n}]$ where we take the f_k to be variables in the first n variables. The zero set of this ideal will have points that look like $V(\overline{I}) = \{(x, y) : x \in X, y \in \mathbb{A}^m\} = X \times \mathbb{A}^m$ due to X being the zero set in the first n variables, and the other variables being free. Analogously, we have the same to be true for Y, that is, we may take $\overline{J} =< g_1, ..., g_j > \subseteq k[x_1, ..., x_{m+n}]$ where we take the polynomials to only be in the last m variables, and free otherwise, and $V(\overline{J}) = \{(x, y) : x \in \mathbb{A}^n, y \in Y\} = \mathbb{A}^n \times Y$. Consider now the ideal generated by $< \overline{I}, \overline{J} > =< f_1, ..., f_i, g_1, ..., g_j >$. We claim that $V(< \overline{I}, \overline{J} >) = X \times Y$, as a point vanishes on any polynomial in the span when it belongs to a point both in X and in Y as a Cartesian product.

Let $x \in V(< \overline{I}, \overline{J} >)$. Then, x is the zero of every polynomial in this ideal. In particular, looking at the generators, this implies that x is 0 on every polynomial in the generators. This implies that $x \in \{(x, y) : x \in X, y \in \mathbb{A}^m\}$ and $x \in \{(x, y) : x \in \mathbb{A}^n, y \in Y\}$, which implies that x is in their intersection, $x \in (X \times \mathbb{A}^m) \cap (\mathbb{A}^n \times Y) = X \times Y$. Now, suppose $z = (x, y) \in X \times Y$, where we associate the first n coordinates with X and the last m with Y. Since $f_k((x, y)) = 0$ for all k, since $X \times Y \subseteq X \times \mathbb{A}^m$ and same for the g_k, we have that z is in the zero set of $(< \overline{I}, \overline{J} >)$. Thus, we have that $X \times Y$ is the zero set of some ideal of polynomials, and is thus algebraic.

(b)

Suppose, without loss of generality, that X is reducible. Then, $X = X_1 \cup X_2$, for X_1, X_2 closed, and $X_1 \neq X$, $X_2 \neq X$. Then, we may find two ideals in $k[x_1, ..., x_n]$, such that $Z(I_1) = X_1$ and $Z(I_2) = X_2$. By the structure we set up in part (a), then, we can see that we can construct $X_1 \times Y$ and $X_2 \times Y$ from the ideals $< \overline{I_1}, \overline{J} >$ and $< \overline{I_2}, \overline{J} >$. Then, we have that $X \times Y = X_1 \times Y \cup X_2 \times Y$ due to our hypothesis that $X = X_1 \cup X_2$, and because $X \neq X_1$ or $X \neq X_2$, $X \times Y \neq X_1 \times Y$ and $X \times Y \neq X_2 \times Y$.

(c)

Suppose we have closed sets $Z_1, Z_2 \subseteq Z_1 \cup Z_2$ such that $Z_1 \cup Z_2 = X \times Y$. Consider the subset $S_y = X \times \{y\}$ for some fixed element $y \in Y$. This must be contained within one of Z_1 or Z_2 as, suppose not, then we would have $X_1 \times \{y\} \subseteq Z_1$ and $X_2 \times \{y\} \subseteq Z_2$, and we've found two closed sets X_1, X_2 that union to X, but neither are the full space. WLOG, suppose $X \times \{y\} \subseteq Z_1$. Now, consider the set of $\{y \in Y : X \times \{y\} \subseteq Z_1\}$. If this is all of Y, then we are done, otherwise, suppose not. Then, we have that due to the irreducibility of Y, then $\{y \in Y : X \times \{y\} \subseteq Z_2\} = Y$, as otherwise we've found two closed sets that join up to Y and neither are all of Y. But if that's true, then by construction, $Z_2 = X \times Y$ and we are done.

(d)

Let I be the ideal in $k[x_1, ..., x_n]$ such that $V(I) = X \subseteq \mathbb{A}^n$, and let $J =< g_1, g_2, ..., g_j >$ be the radical ideal in $k[x_1, ..., x_m]$ such that $V(J) = Y \subseteq \mathbb{A}^m$. Consider the ideal in $k[x_1, x_2, ..., x_{m+n}]$ generated by the image $\overline{I}$ under inclusion where we associate the variables in I with the first m variables. In particular, the generators are exactly the same, since we just include them into a larger space. This must also be true for the image of I_y, where we associate those n variables with the last n variables. Now, consider the degree of $k[x_1, x_2, ..., x_{m+n}]/ < \overline{I}, \overline{J} >$. In particular, we notice here that because the generators of $\overline{I}$ are polynomials only in the first m variables, and $\overline{J}$ in the last n variables, then modding out by $\overline{I}$ will not affect the last m variables, and vice versa for $\overline{J}$. In particular, if we call $\dim(X) = a, \dim(Y) = b$, construct the map that sends $k[x_1, x_2, ..., x_{m+n}] \to k[y_1, y_2, ..., y_{a+b}]$ that sends a polynomial $f(x_1..., x_n) \to [f]$ modulo I identified with the first a variables, a polynomial $f(x_{n+1}, ..., x_{m+n}) \to [g]$ modulo J identified with the last b variables, and extend this linearly. This must be surjective, as we know that $k[x_1, ..., x_n]/I \cong k[x_1, .., x_a]$ as k-algebras by definition of the degree, and same with the last b variables. Then, for any monomial in $g \in k[y_1, y_2, ..., y_{a+b}]$, we can idenitfy a coset $[a] \in k[x_1, ..., x_n]/I$ and one in $[b] \in k[x_1, ..., x_m]/J$ such that $[a] * [b] = g$ from our identification, which we can lift to a polynomial in $k[x_1, x_2, ..., x_{m+n}]$. Further, the kernel is exactly those things that go to the zero coset in either, which is exactly the ideal generated by $< \overline{I}, \overline{J} >$. So, then, the dimensionality of $X \times Y$ is the trascendental degree of $k[x_1, x_2, ..., x_{m+n}]/ < \overline{I}, \overline{J} >$ which by our isomorphism is $\dim(X) + \dim(Y)$.

□

Problem 3.2. Let X be a closed set in $\mathbb{A}^n$. Consider $\mathbb{A}^n$ as a linear subspace of $\mathbb{A}^m$, for $m > n$ by taking the last $m - n$ coordinates equal to 0. Let $P = (0, ..., 1) \in \mathbb{A}^m$. Define the set $Y \subseteq \mathbb{A}^m$ via:

$$Y = \{Q \in \mathbb{A}^m \setminus \{P\} : \text{ the line } \overline{PQ} \cap X \neq \emptyset\} \cup \{P\}$$

(a) Show that Y is an algebraic subset of $\mathbb{A}^m$.

(b) Compute the dimension of Y.

Solution. (a)

Let $a = (a_1, ..., a_m)$ be a point in Y, and consider the line $\overline{ap}$ that we can express parametrically as $h(t) = at + p(1-t)$ for $t \in k$. Consider the intersection of the line with a point in x, which must exist due to construction. By our embedding, we have that the points in X have form $(x_1, ..., x_n, 0, ...0)$. Then, we know that this intersects X when $a_n * t + (1 - t) = 0$, which implies that $t = \frac{1}{1-a_m}$. Then, we have the following set of coordinates on the line: $h_i = \frac{a_i}{1-a_m}$ for $i \leq n$, $h_j = 0$ for $n < j < m$, and h_m is free.

Now, since X is an algebraically closed set in $\mathbb{A}^n$, take $X = V(I)$ for some ideal $I \subseteq k[x_1, ..., x_n]$. In particular, for each $f \in I$ we may then write $f(\frac{x_1}{1-x_m}, \frac{x_2}{1-x_m}, ..., \frac{x_n}{1-x_m}) \in k[x_1, ..., x_n]$. We may clear the denominators by multiplying through via $(1-x_m)^k$ for some k to recover $(1-x_m)^k f(\frac{x_1}{1-x_m}, \frac{x_2}{1-x_m}, ..., \frac{x_n}{1-x_m}) \in k[x_1, ..., x_n, x_m]$. Because this vanishes on X and we've merely introduced a parametrization on a line intersecting X, f vanishes on any copy of X scaled along our line through $\overline{ap}$. Therefore, the cone Y is algebraic.

(b)

Since the dimension of X is well defined, take $X = V(I)$ as above, we know that $k[x_1, ..., x_n]/I$ has trascendental degree $\dim(X)$ over k as a k-algebra. In particular, since the point P is linearly independent of every point in X, we claim that $\dim(Y) > \dim(X)$. However, we also see from part (a) that we can express the zero set of Y as a polynomial in $k[x_1, ..., x_n, x_m]$, and thus, one extra variable, therefore the transcendental degree can be at most one more than the transcendental degree of X because we add one new variable. Thus, $\dim(Y) = \dim(X) + 1$.

$k[x_1, ..., x_n, x_m]/I$

□

Problem 3.3. Fix a polynomial $f_0 \in k[x_1, ..., x_n]$ without multiple irreducible factors. Define $A = \{\frac{g}{f_0^k}\}$ for $k \in \mathbb{N}$, where we identify $\frac{g}{f_0^k} \sim \frac{g'}{f_0^{k'}}$ if $gf_0^{k'} = g'f_0^k$.

(a) Show that A is a ring with a natural addition and multiplication.

(b) Show that there exists a natural injective morphism $k[x_1, ..., x_n] \to A$.

(c) Show that the prime ideals of A are in bijection with the prime ideals of $k[x_1, ..., x_n]$ that do not contain f_0.

(d) Let U be the open set of $\mathbb{A}^n$ given by $f_0 \neq 0$. Show that the ring of regular functions in U is identified with A.

Solution. (a)

Well, we claim that we can define an addition via:

$$\frac{g}{f_0^k} + \frac{g'}{f_0^{k'}} = \frac{f_0^{k'}g + f_0^k g'}{f_0^{k+k'}}$$

and a multiplication via:

$$\frac{g}{f_0^k} * \frac{g'}{f_0^{k'}} = \frac{gg'}{f_0^{k+k'}}$$

Firstly, we wish to check that this is well-defined. Suppose we have that $\frac{g}{f_0^k} \sim \frac{g''}{f_0^{k''}} \implies gf_0^{k''} = g''f_0^k$. Consider $\frac{f_0^{k'}g+f_0^k g'}{f_0^{k+k'}}$ and $\frac{f_0^{k'}g''+f_0^{k''}g'}{f_0^{k''+k'}}$. In particular, consider $(f_0^{k'}g + f_0^k g) * (f_0^{k''+k'})$ and $(f_0^{k'}g'' + f_0^{k''}g') * (f_0^{k+k'})$. We have that:

$$(f_0^{k'}g + f_0^k g') * (f_0^{k''+k'}) = (f_0^{2k'+k''}g + f_0^{k+k'+k''}g') = (f_0^{2k'}g''f_0^k + f_0^{k+k'+k''}g') = (f_0^{2k'+k}g'' + f_0^{k+k'+k''}g')$$

and

$$(f_0^{k'}g'' + f_0^{k''}g') * (f_0^{k+k'}) = (f_0^{k+2k'}g'' + f_0^{k+k'+k''}g')$$

So addition is well-defined.

Doing the same for multiplication, we have that:

$$gg'f_0^{k''+k'} = (gf_0^{k''})g'f_0^{k'} = g''f_0^k g'f_0^{k'} = g''g'f_0^{k'+k}$$

and

$$g''g'f_0^{k+k'}$$

Now, we have that these operations are associative, commutative, etc from the inheritance on polynomial addition/multiplication. So, we need only check that we have a group under addition, a multiplicative identity exists, and that multiplication distributes over addition.

We identify $\frac{0}{1}$ as an additive identity, where we denote $0, 1$ as the zero and one from $k[x_1, ..., x_n]$, because $\frac{0}{1} + \frac{g}{f_0^k} = \frac{0+g}{f_0^k} = \frac{g}{f_0^k}$

It should be clear that every element has an additive inverse. For any $\frac{g}{f_0^k}$, take $\frac{-g}{f_0^k}$, where $-g$ is the additive inverse of g in $k[x_1, ..., x_n]$. Then, we have:

$$\frac{g}{f_0^k} + \frac{-g}{f_0^k} = \frac{f_0^k g + f_0^k(-g)}{f_0^{2k}} = \frac{f_0^k g + (-g)}{f_0^{2k}} = \frac{0}{f_0^{2k}}$$

which we see as equivalent to $\frac{0}{f_0^0} = \frac{0}{1}$, as $1 * 0 = f_0^{2k} * 0$.

We identify $\frac{1}{1}$ as a multiplicative identity, as $\frac{g}{f_0^k} * \frac{1}{1} = \frac{g*1}{f_0^k*1} = \frac{g}{f_0^k}$.

Now, we just need to show that this multiplication distributes over addition. Here, we notice that $\frac{f_0^k}{f_0^k} \sim \frac{1}{1}$.

$$\frac{g}{f_0^k}(\frac{g'}{f_0^{k'}} + \frac{g''}{f_0^{k''}}) = \frac{g}{f_0^k} * \frac{f_0^{k'}g'' + f_0^{k''}g'}{f_0^{k''+k'}} = \frac{gg''f_0^{k'} + gg'f_0^{k''}}{f_0^{k+k'+k''}} * \frac{f_0^k}{f_0^k} = \frac{gg''f_0^{k+k'} + gg'f_0^{k+k''}}{f_0^{(k+k')}f_0^{(k+k'')}} = \frac{gg''}{f_0^{k+k''}} + \frac{gg'}{f_0^{k+k'}}$$

Thus, A has a ring structure.

(b)

Take the morphism $i : k[x_1, ..., x_n] \to A$ that sends a polynomial $f \to \frac{f}{1}$.

We may verify that this is actually a ring hom:

$$i(f) + i(g) = \frac{f}{1} + \frac{g}{1} = \frac{f*1 + g*1}{1*1} = \frac{f+g}{1} = i(f+g)$$

and

$$i(f)i(g) = \frac{f}{1} * \frac{g}{1} = \frac{f*g}{1*1} = \frac{fg}{1} = i(fg)$$

.

Further, this must be injective. Suppose $i(f) = i(f')$. Then, we have that $\frac{f}{1} = \frac{f'}{1}$. This implies that $\frac{f}{1} + \frac{-f'}{1} = \frac{0}{1} \implies \frac{f-f'}{1} = \frac{0}{1}$. Then, we have that $f - f' = 0 \in k[x_1, ..., x_n]$, and thus that $f = f'$.

(c)

This is exactly the same as what we did in class with localizations of rings. However, let's write it out.

Firstly, via our injective morphism $\phi : k[x_1, ..., x_n] \to A$, we can bring any prime ideal P to $\phi(P)$ via $f \to \frac{f}{1}$. Clearly, if $f_0^k \in P$, then $\phi(P)$ is trivial, since $\frac{f_0^k}{1} \in \phi(P)$ and $\frac{f_0^k}{1} * \frac{1}{f_0^k} = \frac{f_0^k}{f_0^k} \sim \frac{1}{f_0^0} = \frac{1}{1}$ since $f_0^k = f_0^0 f_0^k = f_0^k * 1 = f_0^k$. Then, a unit is in $\phi(P)$ and $\phi(P) = A$. So we can assume first that $f_0^k \notin P$ for any $k \geq 0$. Now, consider the prime ideal spanned by $\phi(P)$, $S = \{f_0^k : k \in \mathbb{N}\}$ called $S^{-1}P$. From class, we showed that this must have form $\{[\frac{p}{f_0^k}] : p \in P, k \in \mathbb{N}\}$. These must be distinct, because suppose not, then, two images have the same span, but because our original morphism was injective, they must come from the same elements in $k[x_1, ..., x_n]$. Further, these clearly pull back to the same preimage under ϕ^{-1}, since for any $p \in P$, we havethat $[\frac{p}{1}] \in S^{-1}P$, so $p \in \phi^{-1}(S^{-1}P)$, and if $p \in \phi^{-1}(S^{-1}P)$, then we can say $p = \phi^{-1}(a)$ for some $a \in S^{-1}P$. But by the construction, $a = \frac{p}{1}$, which implies that $p \in P$.

Now, instead, suppose we have a prime ideal in A called P_A. We can see that $\phi^{-1}(P_A)$ is a prime ideal. Suppose $x, y \in \phi^{-1}(P_A)$, then $\phi(x - y) = \phi(x) - \phi(y) \in P_A$, so $x - y \in \phi^{-1}(P_A)$. and we have that it is a subring. Further, let $f \in k[x_1, ..., x_n]$, and take $x \in \phi^{-1}(P_A)$. Then, $\phi(xf) = \phi(x)\phi(f) \in P_A$, because $\phi(x) \in P_A$ and P_A an ideal, so $xf \in \phi^{-1}(P_A)$. So this has the property of multiplicative absorption. Further, f_0^k cannot be in this because if it were, then $\phi(f_0^k) \in P_A$, which we've shown to be a unit, and then P_A is trivial. Now, consider $S^{-1}\phi^{-1}(P_A)$. It is clear that $S^{-1}\phi^{-1}(P_A) \subseteq P_A$. Now suppose we have a $\frac{a}{b} \in P_A$. Then, we notice that since $\frac{b}{1} \in A$, then $\frac{ab}{b} \sim \frac{a}{1} \in P_A$. Then, we have that $a \in \phi^{-1}(P_A)$, and then of course, $\frac{a}{b} \in S^{-1}\phi^{-1}(P_A)$, because $\frac{a}{1} \in \phi(\phi^{-1}(P_A))$. Further, this must be injective because if $\phi^{-1}(P) = \phi^{-1}(P')$, then traveling back by ϕ into A, they would generate the same span, and if the generating set for two ideals are equal, the original ideals were equal. So, by these two constructions, we have a one to one association that takes a prime ideal in A to $\phi^{-1}(A)$ and a prime ideal in $k[x_1, ..., x_n]$ to the prime ideal spanned by its image in A.

(d)

Recall that the ring of regular functions on $U \subseteq \mathbb{A}^n$ is defined as functions $\phi : V \to \mathbb{A}^1$ for $V \subseteq U$ an open neighborhood and acting via $\phi : x \to \frac{f}{g}(x)$, that is, via some rational functions of polynomials such that $g(x) \neq 0$ on V. Here, we notice something: since f_0 has only a single irreducible factor, call it h, then $h \in \text{rad}(< f_0 >)$. In particular, since h is the only irreducible polynomial that divides f_0, then

$< h >= \mathrm{rad}(< f_0 >)$ and $f_0 = ch^j$ for some field element c and some number j. Then, for any regular function, we may take a g that's defined on the whole space that vanishes on the zeros of f_0.Now, let f/g be a regular function defined on all of U, which we can do, because if it were defined piecewise, we may find a smaller open set that it agrees on piece by piece.. Then, if g is non-constant, g may only be 0 on the zeroes of f_0. Then, we have that g is exactly a power of h. In particular, since $k[x_1, ..., x_n]$ is a UFD, we may identify a power i such that $g = c'h^i$ and for c' a field element. Then, we may identify a regular function f/g as m/f_0^k, where$k$ is the smallest natural such that $jk - i > 0$, and $m = c'^{-1}c^k fh^{jk-i}$. We may confirm this: $gm = c'h^i * c'^{-1}c^k fh^{jk-i} = c^k fh^{jk} = f_0^k f$.

□