First Assignment

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Math 240: Homework #1

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Problem 1.1. (a) Let $f : R \to S$ be a morphism of rings, with $\text{Ker}(f) = I$. Let J be an ideal in S. Show that $f^{-1}(J) = \{x \in R | f(x) \in J\}$ is an ideal in R such that $I \subseteq f^{-1}(J)$.

(b) Let $f : R \to S$ be a surjective morphism of rings. If I is an ideal in R, show that $f(I)$ is an ideal in S.

(c) Let R be a ring, I an ideal of R. Show that the ideals of R/I are in one-to-one correspondence with the ideals of R that contain I.

(d) Find all ideals of $\mathbb{Z}_{12}$

Solution. (a)

Take J to be a an ideal in S. Because ideals are subrings, we can see that $0_S \in J$, where 0_S is the zero element in S. Then, $f^{-1}(0_S) \in f^{-1}(J)$. In particular, since $I = \text{Ker}(f) = \{r \in R | f(r) = 0_S\}$, we have $I \subseteq f^{-1}(J)$. Now, we confirm that J is a subring. Take $j, j' \in f^{-1}(J)$. Consider the sum $j + (-j')$. $f(j + (-j')) = f(j) + f(-j') = f(j) - f(j')$, which is in J, since $f(j), f(j')$ in J. Therefore, $j + (-j')$ is in $f^{-1}(J)$ and thus it is a subring.

Now, fix a $j \in f^{-1}(J)$, and a $r \in R$. We have the following, by ring morphism properties that $f(rj) = f(r) * f(j)$. Due to $f(j) \in J$ being an ideal, we also have that $f(r) * f(j) \in J$. So, we have that $f(rj) \in J$, which tells us that $rj \in f^{-1}(J)$. Since the choice of r, j was arbitrary, this is true for all $j \in f^{-1}(J)$ and $r \in R$, i.e. $f^{-1}(J)$ is multiplicatively closed. So $f^{-1}(J)$ is a subring closed under multiplication, and thus an ideal.

(b)

Firstly, we will prove that $f(I)$ is a subring of S. Clearly, it is non-empty, as $0_R \in I$, therefore $0_S \in f(I)$. Now, take $s, s' \in f(I)$. Since s, s' in the image of I, we have that there exists i, i' such that $f(i) = s, f(i') = s'$. Now, we have that $f(i + (-i')) = f(i) + f(-i') = f(i) + (-f(i')) = s + (-s')$. Since $i + (-i') \in I$, then this shows that $s + (-s') \in f(I)$, and is a subring.

Now, take any $j \in f(I)$ and any $s \in S$. Since f is surjective, there exists an $r \in R$ such that $f(r) = s$. Because j is in the image of I, there exists $i \in I$ such that $f(i) = j$. Now, because I is an ideal, $ir \in I$. Then, we have that $f(ir) = f(i)f(r) = js$, that is, js is in the image of I. Since the choice of j and s was arbitrary, this works for all such j, s and thus $f(I)$ is closed under multiplication. Therefore, $f(I)$ is an ideal.

(c)

Define a map $f : R \to R/I$ that sends $f(r) = \overline{r} = \{r + I | r \in R\}$, that is, its coset of I. This is a surjective ring morphism, with $\text{Ker}(f) = I$.

From part (b), because f is surjective, we see that for any ideal $V \subseteq R$, that $f(V) = U \in R/I$ is an ideal.

Further, from part (a), for any ideal of $U \subseteq R/I$, there exists an ideal V in R that contains I such that $f(V) = U$.

Now, assume we have two ideals $I \subseteq V_1, V_2 \subseteq R$ and that $f(V_1) = f(V_2)$. Let $v_1 \in V_1$. Then, since $f(V_1) = f(V_2)$, we have that $v_1 + i = v_2 + i'$ for some $i, i' \in I$ and some $v_2 \in V_2$. Then, rearranging, we have that $v_1 = v_2 + i' - i$. But, since $I \subseteq V_2$, and V_2 is an subring, i.e. closed under addition, this implies that $v_1 \in V_2$. Since this choice of $v_1 \in V_1$ was arbitrary, this means that $V_1 \subseteq V_2$. Using the same argument, we see that $V_2 \subseteq V_1$, and thus $V_1 = V_2$.

Thus, f is an surjective and injective map that brings ideals of R that contain I to ideals of R/I and the sets are in one-to-one correspondence.

(d)

The ideals of $\mathbb{Z}/12\mathbb{Z}$ are: $\mathbb{Z}/12\mathbb{Z}$, (2), (3), (4), (6), $\{0\}$.

We can see this because of course we have the trivial ideals, the entire space and just 0.

Then, we notice that if $\gcd(n, 12) = 1$, by Bézout's identity, there exists $an + 12b = 1$ in $\mathbb{Z}$, which, under modulo 12, becomes $an = 1$, that is, n is invertible. But this implies then if $I \subseteq \mathbb{Z}/12\mathbb{Z}$ an ideal, and there exists $n \in I$ with $\gcd(n, 12) = 1$, then $1 \in I$ and thus $I = \mathbb{Z}/12\mathbb{Z}$.

Then, the other non-trivial ideals can only be the subrings additively generated by the elements of $\mathbb{Z}/12\mathbb{Z}$ non-coprime to 12, and we can see quickly that (2), (3), (4), (6) are multiplicatively closed under multiplication by elements of $\mathbb{Z}/12\mathbb{Z}$.

□

Problem 1.2. Let R be a ring. Call an element $a \in R$ nilpotent, if there exists $n > 0$ such that $a^n = 0$.

(a) Show that the set of nilpotent elements N is an ideal in R.

(b) Show that R/N has no non-zero nilpotent elements.

Solution. (a) Firstly, since $0^n = 0$ for all $n \in \mathbb{N}$, $0 \in N$. Now, let r, s be non-zero elements of N. Since they are nilpotent, take n_r, n_s such that $r^{n_r} = 0, s^{n_s} = 0$. Consider

$$(r - s)^{n_r+n_s} = \Sigma_{k=0}^{n_r+n_s} c_k r^k (-s)^{n_r+n_s-k}$$

for some coefficient c_k, where we understand $2r = r + r$. We notice that if $k < n_r$, then $n_r + n_s - k \geq n_s$. Similarly,if $n_r + n_s - k < n_s$, then $k \geq n_r$. Thus:

$$(r - s)^{n_r+n_s} = \Sigma_{k=0}^{n_r+n_s} c_k r^k (-s)^{n_r+n_s-k} = \Sigma_{k=0}^{n_r-1} c_k r^k * 0 + \Sigma_{k=n_r}^{n_r+n_s} c_k 0 * (-s)^{n_r+n_s-k} = 0$$

Therefore, for any $r, s \in N$, $r - s \in N$, and therefore, N is a subring of R.

Now, let $t \in R$, and $r \in N$ with $r^{n_r} = 0$. Then, consider tr. $(tr)^{n_0} = t^{n_0} r^{n_0} = 0$, and $tr \in N$. Since this can be applied for any $r \in N$ and any element of R, N is multiplicatively closed in R, and thus an ideal.

(b)

Suppose R/N has a nilpotent element, that is, $\overline{x}^{n_x} = 0 \in R/N$. Then, this implies that for a representative of $\overline{x}$, $x \in R$, $x^{n_x} \in N$. But, that implies that (x^{n_x}) is nilpotent as being a member of N. Since (x^{n_x}) is nilpotent, there exists m_x such that $(x^{n_x})^{m_x} = 0$. But that implies that $x^{n_x * m_x} = 0$, which implies that $x \in N$ itself. Therefore, any nilpotent element of $\overline{x} = 0 \in R/N$.

□

Problem 1.3. Let K be an algebraically closed field, $R = K[x_1, x_n, ..., x_n]$, the ring of polynomials in n variables. Recall that for a set of polynomials $f_n \in R$,

$$V(f_1, ..., f_k) = \{(a_1, ..., a_n) \in K^n | f_i(a_1, ..., a_n) = 0 \text{ for each } i = 1, ...k\}$$

with a similar definition when one replaces the set of polynomials by an ideal $I \subseteq R$.

(a) Show that if $f \in R$, $f \neq 0$, then $V(f) \neq \mathbb{A}^n$.

(b) Show that if $f \in R$, f non-constant, then $V(f) \neq \emptyset$.

(c) Use part (a) to show that $\mathbb{A}^n$ is irreducible.

Solution. (a)

Construct the family of functions $g_{x_{2_i}, ...x_{n_i}}(x_1)$ for $i \in I$, i not necessarily countable, such that $g_{x_{2_i}, ...x_{n_i}}(x_1) = f(x_1, x_{2_i}, ..., x_{n_i})$. We claim that at least one such g is not identically 0. Suppose not. Then, for every $(x_{2_i}, ..., x_{n_i}), g(x_1) = 0$ for all x_1. Then, this implies that for all $(x_1, ..., x_n) \in A_k^n$, we have that $f(x_1, ..., x_n) = 0$, a contradiction.

Then, we have a $g_j = g_{x_{2_j},...x_{n_j}}(x_1)$ for some j such that g_j is not identically 0. Since g_j is not identically 0, there exists some x_{1_j} such that $g_j(x_{1_j}) \neq 0$. Then, the point $(x_{1_j}, ..., x_{n_j}) \notin V(f)$, and thus $V(f) \neq \mathbb{A}^n$.

(b)

In a similar argument as above, construct the family now over all x_i, fixing one x_i at a time. That is, denote $g_{\{x_{m_i}\}}(x_j)$ for $i \in I$ to be the level curves of constant x_i where $i \neq j$ and consider the collections of $g_{\{x_{m_i}\}}(x_j)$ for $j = 1, ..n$. Claim that there is at least one level curve in this set that is not constant.

Suppose not. Then, let $(a_1, ...a_n)$ and $(b_1, ..., b_n)$ be two arbitrary points in $\mathbb{A}^n$. Then, they are connected by the level curves $g_{(a_2,a_3,...a_n)}(x_1), g_{(b_1,a_3,...a_n)}(x_2), ...g_{(b_1,...b_{k-1},a_{k+1},...,a_n)}(x_k), g_{(b_1,...b_{n-1})}(x_n)$ via the line segments that have the form $f_m : [0,1] \to \mathbb{R}^n$ with $f_m(t) = t(b_1, ...b_{m-1}, b_m, a_{m+1}, ..., a_n) + (b_1, ..., b_{m-1}, a_m, a_{m+1}, ...a_n)$. But, because f is constant on all of these level curves, f is constant on all of these line segments, therefore $f(a_1, ...a_n) = (b_1, ..., b_n)$. Since the choice of points was arbitrary, this is true for all points in $\mathbb{A}^n$, and then f is constant, a contradiction.

Choose a level curve that is not constant $g_{\{x_{m_k}\}}(x_j)$. This is a non-constant polynomial in one variable, over an algebraically closed field. This implies that there exists at least some x_{j_0} such that $g_{\{x_{m_k}\}}(x_{j_0}) = 0$. Then, $g_{\{x_{m_k}\}}(x_{j_0}) = f(x_{1_k}, ...x_{j_0}, ...x_{n_k}) = 0$, and thus $(x_{1_k}, ...x_{j_0}, ...x_{n_k}) \in V(f)$ i.e. $V(f) \neq \emptyset$.

(c)

Let X_1, X_2 be closed sets such that $\mathbb{A}^n = X_1 \cup X_2$. Then, we have $\mathbb{A}^n = V(f_1, ...f_m) \cup V(g_1, ...g_n)$ for some indices $m \in I, n \in J$. Then, from what we proved in class, we can take $V(f_1, ...f_m) \cup V(g_1, ...g_n) = V(f_1g_1, f_2g_1, ..., f_ig_1, f_1, g_2, ..., f_mg_n)$. But, also from class, $V(f_1g_1, f_2g_1, ..., f_mg_1, f_1, g_2, ..., f_ng_m) = \cap_{i\in I, j\in J} V(f_ig_j)$. Since this is an intersection of sets, it follows then that $V(f_ig_j) = \mathbb{A}^n$ for all i, j. But, by part (a) then, $f_ig_j = 0$. We claim that either $V(f_1, ...f_m)$ or $V(g_1, ...g_n)$. Suppose $f_1, ...f_m = 0$ for all i. Then we're done, as $V(f_1, ...f_i) = V(0) = \mathbb{A}^n$. Else, there exists $f_{i_0} \neq 0$ for some i_0. However, for all $n \in J$, $f_{i_0}g_n = 0$, thus $g_n = 0$ and $V(g_1, ...g_n) = \mathbb{A}^n$.

□

Problem 1.4. Let X be a topological space, and let $Y \subseteq X$. Call Y dense in X if for every non-empty open set $U \subseteq X$, $U \cap Y$ is non-empty.

Call a topological space X irreducible if for any closed sets X_1, X_2, $X = X_1 \cup X_2 \implies X_1 = X$ or $X_2 = X$.

Let X be a topological space in the following:

(a) If $Y \subseteq X$, show that Y is dense in X if and only if the closure $\overline{Y}$ of Y in X satisfies $\overline{Y} = X$.

(b) Show that X is irreducible if and only if every non-empty open subset $U \subseteq X$ is dense in X.

(c) If $Y \subseteq X$ such that Y is irreducible and Y is dense in X, then X is also irreducible.

(d) If $Y \subseteq X$ such that Y is dense in X and X is irreducible, then Y is also irreducible.

(e) If $f : A \to B$ is a continuous map of topological spaces and $X \subseteq A$ is an irreducible set, show that $f(X) \subseteq B$ is an irreducible set.

(f) If $Y \subseteq X$ satisfies that, for every $P \in X \setminus Y$, there exists a topological space Z and a continuous map $f : Z \to X$ with $P \in f(Z)$ and $f^{-1}(Y)$ dense in Z, then Y is dense in X.

Solution. (a)

By construction, $\overline{Y} \subseteq X$ as it is the closure with respect to X. Now, suppose Y is dense in X. Let $x \in X \setminus Y$. Let V be any neighborhood of x. Then, since V contains an open set U such that $x \in U$, and Y is dense, then there exists $y \in Y$ such that $y \in U$. Since the choice of neighborhood was arbitrary, this is true for every neighborhood, and every $x \in X \setminus Y$ is a limit point of Y. Then, that implies that $X \subseteq \overline{Y}$. Thus, $X = \overline{Y}$.

Now, suppose $\overline{Y} = X$. Let $U \subseteq X$ be an open set. Let $u \in U$. If $u \in Y$, then we are done. Otherwise, suppose $u \in X \setminus Y$. But then, by hypothesis, $u \in \overline{Y}$, so take a small enough neighborhood V of u such that $V \subseteq U$. Since u is in the closure of Y and not in Y, u must be a limit point of Y, so there exists $y \in Y$ such that $y \in V$. But, by construction, $y \in U$. Thus, for any arbitrary open set $U \subseteq X$, there exists $u \in U$ such that $u \in Y$ and therefore Y is dense in X.

(b)

Suppose X is irreducible. Let U be a non-empty open subset of X. Consider the quantity $U^c \cup \overline{U}$, where U^c is the compliment of U in X. It should be clear that $U^c \cup \overline{U} = X$. $U^c \cup \overline{U} \subseteq X$ follows from construction, and $X \subseteq U^c \cup \overline{U}$ as for $x \in X$, $x \in U \subset \overline{U}$ or $x \in U^c$. Further, $\overline{U}$ is closed by construction, and since U is open, its complement is closed. Thus, we have X as a union of closed sets. Further, since U is non-empty, U^c cannot be X, therefore $\overline{U} = X$. But, by part (a), then U is dense in X.

Now, suppose we have every non-empty subset $U \subseteq X$ dense in X, and suppose $X = X_1 \cup X_2$ for X_1, X_2 closed. If $X_1 = X$, then we are done, else, consider the open set X_1^c, non-empty. We have then that $X_1^c \subseteq X_2$. By the properties of the closure of U being the smallest such closed set that is a superset of U, we have that $\overline{X_1^c} \subseteq X_2$. But, by part (a), we have that $\overline{X_1^c} = X$, so we have that $X \subseteq X_2$. We also have $X_2 \subseteq X$ from the original union. Thus, if $X_1 \neq X$, $X_2 = X$.

(c)

Let U be a non-empty open subset of X. Then, we can consider the closed sets $\overline{U}, U^c$. In particular, consider the closed sets in Y, $\overline{U} \cap Y, U^c \cap Y$ and, $Y = (\overline{U} \cap Y) \cup (U^c \cap Y)$

Since Y is irreducible, we have either that $U^c \cap Y = Y$ or $\overline{U} \cap Y = Y$.

Suppose $U^c \cap Y = Y$. However, since Y is dense in X, and U is open, there exists $y_u \in Y$ such that $y_u \in U$. But then, $y_u \notin U^c$, therefore $y_u \notin U^c \cap Y$, a contradiction.

Then, we must have $\overline{U} \cap Y = Y$. But then we have that $Y \subseteq U$ in X. And, in particular, since Y is dense, so must be U. Thus, every non-empty open subset of X is dense, and X is irreducible.

(d)

Suppose there exists closed sets in Y, $V_1, V_2 \subseteq Y$ such that $V_1 \cup V_2 = Y$. From the subspace topology, we have closed sets in X, $V_1', V_2' \subseteq X$ such that $V_1' \cap Y = V_1, V_2' \cap Y = V_2$. Since Y is dense in X, $\overline{Y} = X$. But $Y \subseteq V_1' \cup V_2'$, which is a closed set, and the closure is the smallest such closed set that contains Y, so we have $\overline{Y} = X \subseteq V_1' \cup V_2'$. Then, $X = V_1' \cup V_2'$. Since X is irreducible, this means that either $V_1' = X$ or $V_2' = X$. Suppose $V_1' = X$. Then, $V_1 = V_1' \cap Y = Y$, and a similar calculation follows for V_2'. Because the choice of closed sets V_1, V_2 in Y is arbitrary, this means this is true for any such closed sets that cover Y, and thus Y is irreducible.

(e)

Let $U \subseteq f(X)$ be a non-empty open set in $f(X)$. Because f is continuous, $f^{-1}(U)$ is an open set in A, and because it is wholly contained within X, is also an open set in X. Now, since X is irreducible, this implies that $\overline{f^{-1}(U)} = X$. But also, we have, due to the continuity of f, that $f(X) = f(\overline{f^{-1}(U)}) \subseteq \overline{U}$. However, by definition, the closure of U in $f(X)$ is a subset of $f(X)$ itself. Therefore $\overline{U} = f(X)$, and we have that U is dense in $f(X)$ by part (a). Since the choice of U was arbitrary, this is true for all $U \subseteq f(X)$, and thus by part (b), $f(X)$ is irreducible.

(f)

For each $P \in X \setminus Y$, because there exists Z_p with $f_p^{-1}(Y)$ dense in Z_p, we have that the closure$\overline{f_p^{-1}(Y)} = Z_p$. Then, due to the continuity of f_p, we have that $f(Z_p) = f(\overline{f_p^{-1}(Y)}) \subseteq \overline{Y}$. In particular, this tells us that $P \in \overline{Y}$. Since we can do this for all $P \in X \setminus Y$, this implies that $X \setminus Y \subseteq \overline{Y}$. Further, by definition, $Y \subseteq \overline{Y}$. Therefore, we have that $X = Y \cup (X \setminus Y) \subseteq \overline{Y}$, so $X \subseteq \overline{Y}$ and, because the closure in X is a clear subset of X, we have $X = \overline{Y}$. Then, by part (a), we have that Y is dense in X. $\square$