Homework #1

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Math 285: Homework #1

August 13, 2023

Question 1. Define $f : \mathbb{R} \to \mathbb{R}$ via:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Using the definition of the derivative, prove that $f \in C^\infty(\mathbb{R})$, and that the derivatives $f^{(k)}(0)$ vanish for all $k \in \mathbb{N}$.

Solution. Well, first, we restrict ourselves to $\mathbb{R} \setminus \{0\}$. On this domain, $f(x) = e^{-1/x^2}$, and, familiarly, we can see that

$$f'(x) = e^{-x^{-2}} \frac{d}{dx}(-x^{-2}) = e^{-x^{-2}} 2x^{-3} = 2x^{-3} f(x)$$

then, we identify:

$$f^{(2)}(x) = f'(x) 2x^{-3} - 6x^{-4} f(x) = 2x^{-3} f(x) 2x^{-3} - 6x^{-4} f(x) = f(x)(4x^{-6} - 6x^{-4})$$

More generally, we can see that:

$$f^{(n)}(x) = f(x) p_n \left(\frac{1}{x} \right)$$

for $p_n(1/x)$ a polynomial in $1/x$ because the derivative of a polynomial in $1/x$ is simply a polynomial, and the derivative of f is $f'(x) = 2x^{-3} f(x)$. Explicitly:

$$f^{(n+1)}(x) = \frac{d}{dx} \left[f(x) p_n \left(\frac{1}{x} \right) \right] = f'(x) p_n \left(\frac{1}{x} \right) + f(x) p_n' \left(\frac{1}{x} \right) \frac{-1}{x^2} = f(x) \left[2x^{-3} p_n \left(\frac{1}{x} \right) - p_n' \left(\frac{1}{x} \right) x^{-2} \right] = f(x) p_{n+1} \left(\frac{1}{x} \right)$$

Further, we notice that by our identification:

$$p_{n+1} \left(\frac{1}{x} \right) = 2x^{-3} p_n \left(\frac{1}{x} \right) - p_n' \left(\frac{1}{x} \right) x^{-2}$$

and that because $p_0(1/x) = 1$, that $\deg(p_n(y)) = \deg(p_{n-1}(y)) + 3 = 3n$, where we can say this because the degree of the first term in the recurrence is $\deg(p_n) + 3$ and the degree of the second term is $\deg(p_n') + 2 = \deg(p_n) - 1 + 2 = \deg(p_n) + 1$, so the leading term in the first term cannot vanish, and we have that the overall polynomial has degree $\deg(p_n) + 3$.

Now, let's examine the limit as $x \to 0$ of $f^{(n)}(x)$. Well:

$$\lim_{x \to 0} f^{(n)}(x) = \lim_{x \to 0} f(x) p_n \left(\frac{1}{x} \right)$$

Here, we rewrite this as examining two easier limits, by letting $y = 1/x$:

$$\begin{cases} \lim_{y\to\infty} e^{-y^2} p_n(y) \\ \lim_{y\to-\infty} e^{-y^2} p_n(y) \end{cases}$$

where of course, $p_n(y)$ is a polynomial in y, with non-negative powers. Well:

$$\lim_{y\to\infty} e^{-y^2} p_n(y) = \lim_{y\to\infty} \frac{p_n(y)}{e^{y^2}}$$

Making the assumption that the leading coefficient of $p_n(y)$ is positive (if not, we pull out a negative sign and examine $-p_n(y)$), we see that the conditions for L'Hôpital's is met, as both numerator and denominator diverge to positive infinity.

Thus, we may take a derivative to find:

$$\lim_{y\to\infty} \frac{p_n(y)}{e^{y^2}} = \lim_{y\to\infty} \frac{p_n'(y)}{2ye^{y^2}} = \lim_{y\to\infty} \frac{p_n'(y)/2y}{e^{y^2}}$$

Well, if $p_n'(y)/2y$ has no positive powers, then we're done, as in that case, $\lim_{y\to\infty} p_n'(y)/2y = c$, at most a constant. Otherwise, if it does have a positive power, its limit remains ∞, so we may iteratively keep taking derivatives since p_n has finite degree until this is true. Thus, because $\lim_{y\to\ infty} e^{y^2} = \infty$, we have that:

$$\lim_{y\to\infty} \frac{p_n(y)}{e^{y^2}} = 0$$

The same argument works for $y \to -\infty$, with potentially examining $-p_n(y)$ if the leading term has odd degree. Thus, in terms of our original limit, we have that:

$$\lim_{x\to 0} f^{(n)}(x) = \lim_{x\to 0} f(x) p_n\left(\frac{1}{x}\right) = 0$$

So, we have that to remove the discontinuity, we would enforce that $f^{(n)}(0) = 0$ for all n. Thus, derivatives exist on all of $\mathbb{R}$ continuously and smoothly for all n, as we may patch the discontinuity at $x = 0$. Thus, $f \in C^\infty(\mathbb{R})$.

$\square$

Question 2. Let $\overline{0} = (0,0)$ be the origin in $\mathbb{R}^2$, and let $B(\overline{0}, 1)$ be the open unit disk centered at the origin. To find a diffeomorphism between $B(\overline{0}, 1)$ and $\mathbb{R}^2$, we identify $\mathbb{R}^2$ as $\{(x, y, 0) : x, y, \in \mathbb{R}\} \subseteq \mathbb{R}^3$, and introduce the lower open hemisphere:

$$S = \{(x, y, z) : x^2 + y^2 + (z-1)^2 = 1; z < 1\} \subseteq \mathbb{R}^3$$

as an intermediate space. Notice that the map:

$$f : B(\overline{0}, 1) \to S \text{ via } (a, b) \mapsto (a, b, 1 - \sqrt{1 - a^2 - b^2})$$

is a bijection.

(a) The stereographic projection $g : S \to \mathbb{R}^2$ through $(0, 0, 1)$ is the map that sends a point $(a, b, c) \in S$ to the point in the xy plane given by the line through $(0, 0, 1)$ and (a, b, c). Show that it is given by:

$$(a, b, c) \mapsto (u, v) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right); c = 1 - \sqrt{1 - a^2 - b^2}$$

and its inverse is given by:

$$(u,v) \mapsto \left(\frac{u}{w}, \frac{v}{w}, 1-\frac{1}{w}\right); w = \sqrt{1+u^2+v^2}$$

(b)

Define

$$h = g \circ f : B(\overline{0}, 1) \to \mathbb{R}^2$$

Show that:

$$h(a,b) = \left(\frac{a}{\sqrt{1-a^2-b^2}}, \frac{b}{\sqrt{1-a^2-b^2}}\right)$$

Further, find a formula for $h^{-1}(u,v) = (f^{-1} \circ g^{-1})(u,v)$ and conclude that h is a diffeomorphism.

(c)

Generalize part (b) to $\mathbb{R}^n$.

Solution. (a)

Let $(a,b,c) \in S$, and consider the line through $(0,0,1)$ and (a,b,c). In vector form, we may write this as:

$$(x,y,z) = (0,0,1)t + (a,b,c)(1-t) \text{ s.t. } t \in \mathbb{R}$$

We wish to find the point at which $z = 0$. We find that for $z = 0$, then the value of t is:

$$0 = t_{z=0} + c(1-t_{z=0}) \implies t_{z=0} = \frac{c}{c-1}$$

Now, we wish to see the values of (x,y) at $t_{z=0}$. Substituting, we find that:

$$\begin{cases} x_{z=0} = a - at = a - a\frac{c}{c-1} = \frac{ac-a-ac}{c-1} = \frac{-a}{c-1} = \frac{a}{1-c} \\ y_{z=0} = b - bt = b - b\frac{c}{c-1} = \frac{bc-b-bc}{c-1} = \frac{-b}{c-1} = \frac{b}{1-c} \end{cases}$$

Here, we notice that since we live on the lower open hemisphere, that we have that:

$$a^2+b^2+(c-1)^2 = 1 \implies c = 1-\sqrt{1-a^2-b^2}$$

where we choose the negative sign for the square root as we live on the lower hemisphere.

Thus, we have that the map from the hemisphere to the disk in the plane as being from:

$$(a,b,c) \mapsto (u,v) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right) \text{ where } c = 1-\sqrt{1-a^2-b^2}$$

Now, we can examine the inverse. Let (u,v) be a point on the plane identified with $z = 0$, and consider the line that joins it and $(0,0,1)$.

We can parametrize the line with the form:

$$(x,y,z) = (0,0,1)(1-t) + (u,v,0)t$$

Now, we need to intersect this with the sphere. We have that:

$$\begin{cases} x = ut \\ y = vt \\ z = (1-t) \end{cases}$$

So, substituting, we have that:

$$u^2t^2 + v^2t^2 + (1-t-1)^2 = 1 \implies t^2(u^2+v^2+1) = 1 \implies t = \frac{1}{\sqrt{1+u^2+v^2}}$$

Thus, we have, as expected that:

$$(u,v) \mapsto uw, vw, 1-w \text{ where } w = \frac{1}{\sqrt{1+u^2+v^2}}$$

Now, we need only show that these are inverses. Calling the first map f_1 and the second map f_2, we see that:

$$f_1 \circ f_2(u,v) = f_1((uw, vw, 1-w)) = \left(\frac{uw}{1-(1-w)}, \frac{vw}{1-(1-w)}\right) = (u,v)$$

and that:

$$f_2 \circ f_1((a,b,c)) = f_2\left(\frac{a}{1-c}, \frac{b}{1-c}\right)$$

Computing first the value of w, we find that:

$$w = \frac{1}{\sqrt{1+\frac{a^2}{(1-c)^2}+\frac{b^2}{(1-c)^2}}} = \frac{(1-c)}{\sqrt{(1-c)^2+a^2+b^2}} = (1-c)$$

where we use the fact that (a,b,c) are points on the sphere and replace $(1-c)^2+a^2+b^2 = 1$.
Thus:

$$f_2\left(\frac{a}{1-c}, \frac{b}{1-c}\right) = \left(\frac{a}{1-c}(1-c), \frac{b}{1-c}(1-c), 1-(1-c)\right) = (a,b,c)$$

Thus, we have that these maps act as inverses as well.
(b)
As instructed, we define $h = g \circ f : B(\overline{0},1) \to \mathbb{R}^2$.
Let's consider the action of h on a point $(a,b) \in B(\overline{0},1)$:

$$h((a,b)) = g \circ f((a,b)) = g(f(a,b)) = g((a,b,1-\sqrt{1-a^2-b^2})) = \left(\frac{a}{\sqrt{1-a^2-b^2}}, \frac{b}{\sqrt{1-a^2-b^2}}\right)$$

Next, let's consider such an inverse. Firstly, we notice that the inverse to f acts like:

$$f^{-1}(a,b,c) = (a,b)$$

Then, we have that, for a point $(u,v) \in \mathbb{R}^2$:

$$h^{-1}(u,v) = (f^{-1} \circ g^{-1})(u,v) = f^{-1}(uw,vw,1-w) = (uw,vw) = \left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v}{\sqrt{1+u^2+v^2}}\right)$$

without writing it out, it should be clear that composing these maps will show that these are inverses. Then, we should just check that the inverse actually brings points to the disk. Indeed:

$$\left|\left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v}{\sqrt{1+u^2+v^2}}\right)\right|^2 = \frac{u^2}{1+u^2+v^2} + \frac{v^2}{1+u^2+v^2} = \frac{u^2+v^2}{1+u^2+v^2} < \frac{1+u^2+v^2}{1+u^2+v^2} = 1$$

Then, indeed, h^{-1} is a map to the open disk.

(c)

Generalizing this process, take $\overline{0}_n$ to be the origin in $\mathbb{R}^n$. We may take the open disk $B(\overline{0}_n, 1)$ to be the open unit disk centered on the origin, and take a diffeomorphism to the sphere in $n+1$ dimensions defined via:

$$(a_1, ..., a_n) \mapsto (a_1, ..., a_n, 1 - \sqrt{1 - a^2 - b^2}$$

Then, in a similar fashion, we pass to $\mathbb{R}^n$ via a stereographic projection, taking:

$$(a_1, ..., a_{n+1}) \mapsto (u, v) = \left(\frac{a_1}{1-c}, \frac{a_2}{1-c}, ..., \frac{a_n}{1-c}\right) \text{ where } c = 1 - \sqrt{1 - \sum_{i=1}^{n} a_i^2}$$

Composing these maps, analogously to parts a,b, we find that these act as smooth bijections between the open disk in $\mathbb{R}^n$ and all of $\mathbb{R}^n$ and thus there exists a diffeomorphism relating them.

□

Question 3. Let A be an algebra over a field k. Let D_1, D_2 be derivations of A. Show that $D_1 \circ D_2$ need not be a derivation, but $D_1 \circ D_2 - D_2 \circ D_1$ is.

Solution. Counterexample: Let $A = k[x]$, the polynomial ring over the field k. Let $D_1 = \frac{d}{dx}$, the formal derivative with respect to x, and let $D_2 = x\frac{d}{dx}$.

First, we check that these are actual derivations. We can see this by the action on x^n for arbitrary n, m that these are k linear maps:

$$D_1 : \begin{cases} \frac{d}{dx}(cx^n) = cnx^{n-1} = c\frac{d}{dx}(x^n) \\ \frac{d}{dx}(x^n + x^m) = nx^{n-1} + mx^{m-1} = \frac{d}{dx}x^n + \frac{d}{dx}x^m \end{cases}$$

$$D_2 : \begin{cases} x\frac{d}{dx}(cx^n) = xcnx^{n-1} = cxnx^{n-1} = cx\frac{d}{dx}(x^n) \\ x\frac{d}{dx}(x^n + x^m) = xnx^{n-1} + xmx^{m-1} = x\frac{d}{dx}x^n + x\frac{d}{dx}x^m \end{cases}$$

Further, we can see that this follows the Leibniz rule:

$$D_1(fg) = \frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f\frac{d}{dx}(g) = D_1(f)g + fD_1(g)$$

$$D_2(fg) = x\frac{d}{dx}(fg) = x\left[\frac{d}{dx}(f)g + f\frac{d}{dx}(g)\right] = x\frac{d}{dx}(f)g + fx\frac{d}{dx}(g) = D_2(f)g + fD_2(g)$$

However, consider $D_2 \circ D_1$ acting on $x^3 = x^2 * x$:

$$D_2 \circ D_1(x^3) = D_2(\frac{d}{dx}x^3) = D_2(3x^2) = 6x$$

On the other hand:

$$D_2 \circ D_1(x^2) * x + x^2 D_2 \circ D_1(x) = D_2(2x) * x + x^2 D_2(1) = 2x^2 + 0 = 2x^2$$

where we've used the fact that $\frac{d}{dx}(1) = 0$ to conclude $D_2(1) = 0$.

Thus, even though D_1, D_2 are derivations, $D_2 \circ D_1$ need not be a derivation.

However, now consider $D_1 \circ D_2 - D_2 \circ D_1$ for generic derivations D_1, D_2.

Well:

$$D_1 \circ D_2 - D_2 \circ D_1(fg) = D_1 \circ D_2(fg) - D_2 \circ D_1(fg) = D_1(D_2(f)g + fD_2(g)) - D_2(D_1(f)g + fD_1(g)) =$$

$$D_1{\circ}D_2(f)g{+}D_2(f)D_1(g){+}D_1(f)D_2(g){+}fD_1{\circ}D_2(g){-}[D_2{\circ}D_1(f)g{+}D_1(f)D_2(g){+}D_2(f)D_1(g){+}fD_2{\circ}D_1(g)] =$$

$$D_1{\circ}D_2(f)g{+}fD_1{\circ}D_2(g){-}D_2{\circ}D_1(f)g{-}fD_2{\circ}D_1(g) = [D_1{\circ}D_2(f){-}D_2{\circ}D_1(f)]g{+}f[D_1{\circ}D_2(g){-}D_2{\circ}D_1(g)] =$$

$$D_1 \circ D_2 - D_2 \circ D_1(f)g + fD_1 \circ D_2 - D_2 \circ D_1(g)$$

□

Question 4. Let U be a neighborhood of a point $p \in \mathbb{R}^n$, and let X, Y be smooth vector fields on U. Define a function $Z_p : C_p^\infty \to \mathbb{R}$ via:

$$Z_p(f) = X_p(Yf)$$

(a)

Show that Z_p does not satisfy the Leibniz rule, and is therefore not a derivation at p.

(b)

Show that $X_pY - Y_pX$ satsifies the Leibniz rule on C_p^∞. Hence, $[X, Y]_p := X_pY - Y_pX$ is a tangent vector at p, and $[X, Y]$ is a vector field on U. Call this the Lie bracket of X, Y.

(c)

Let $X = \sum_i a^i \frac{\partial}{\partial x^i}, Y = \sum_j b^j \frac{\partial}{\partial x^j}$. /Find the coefficient c_k in $[X, Y] = \sum_k c^k \frac{\partial}{\partial x^k}$.

(d)

Show that if the vector fields X, Y are smooth on U, then their Lie brakcet is also a smooth vector field on U.

Solution. □

Question 5. Let V be a vector space of dimension n, with basis $e_1, ..., e_n$. Let $\alpha^1, ..., \alpha^n$ be the dual basis for V^*. Show that a basis for the space of k-linear functions of V, $L_k(V)$ is $\{\alpha^{i_1} \otimes ... \otimes \alpha^{i_k}$ for any multi-index $(i_1, ..., i_k)$. In particular, conclude that $\dim(L_k(V)) = n^k$.

Solution. □

Question 6. Let V be a vector space. For $a, b \in \mathbb{R}, f \in A_k(V), g \in A_l(V)$, show that $af \wedge bg = (ab)f \wedge g$

Solution. □

Question 7. Suppose we have two sets of covectors on a vector space V: $\{\beta^1, ..., \beta^k\}, \{\gamma^1, ..., \gamma^k\}$. Further, suppose we have that they are related by:

$$\beta^i = \sum_{j=1}^{k} a^i_j \gamma^j$$

for each $1 \leq i \leq k$, such that the a^i_j form the entries of a $k \times k$ matrix A. Show that:

$$\beta^1 \wedge ... \wedge \beta^k = \det(A)\gamma^1 \wedge ... \wedge \gamma^k$$

Solution. □