Homework #7

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Math 123: Homework #7

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Question 1. Consider the cube $C_r^D = [-r/2, r/2]^D \subset \mathbb{R}^D$ in D dimensions. Let $\text{Vol}_D(A)$ denote the volume of a set A in $\mathbb{R}^D$, that is:

$$\text{Vol}_D(A) = \int_A dx_1 ... dx_D$$

(a) Prove using integration, that $\text{Vol}_D(C_r^D) = r^D$.

(b) For an $\epsilon > 0$, define $A_{\epsilon,r}^D = \{x \in C_r^D : x \notin C_{r-\epsilon}^D\}$. Compute

$$\frac{\text{Vol}_D(A_{\epsilon,r}^D)}{\text{Vol}_D(C_r^D)}$$

(c) Using (b), argue that most of the volume for a high-dimensional cube is near the boundary. Can you make this precise?

Solution. (a)

Without too much difficulty, we see that:

$$\int_{C_r^D} dx_1 ... dx_D = \underbrace{\int \cdots \int_{-r/2}^{r/2}}_{D \text{ times}} dx_1 \cdots dx_D = \left(\frac{r}{2} - \frac{-r}{2} \right)^D = r^D$$

(b)(c)

Similarly, here, we compute $\text{Vol}_D(A_{\epsilon,r}^D) = \text{Vol}_D(C_r^D) - \text{Vol}_D(C_{r-\epsilon}^D)$. But, from part (a), we have that:

$$\text{Vol}_D(A_{\epsilon,r}^D) = \text{Vol}_D(C_r^D) - \text{Vol}_D(C_{r-\epsilon}^D) = r^D - (r - \epsilon)^D$$

Then, we have that:

$$\frac{\text{Vol}_D(A_{\epsilon,r}^D)}{\text{Vol}_D(C_r^D)} = \frac{r^D - (r-\epsilon)^D}{r^D} = 1 - \left(1 - \frac{\epsilon}{r}\right)^D$$

If we do a second order approximation, assuming $\frac{\epsilon}{r}$ is small, then we can say:

$$\left(1 - \frac{\epsilon}{r}\right)^D \approx 1 - \frac{D\epsilon}{r} + \frac{D(D-1)}{2}\frac{\epsilon^2}{r^2}$$

Then we have that:

$$\frac{\text{Vol}_D(A_{\epsilon,r}^D)}{\text{Vol}_D(C_r^D)} \approx \frac{D\epsilon}{r} - \frac{D(D-1)}{2}\frac{\epsilon^2}{r^2}$$

Setting this equal to one half, we can solve for $z = \frac{\epsilon}{r}$:

$$2Dz - D(D-1)z^2 = 1 \implies D(D-1)z^2 - 2Dz + 1 = 0 \implies z = \frac{2D \pm \sqrt{4D^2 - 4(D^2 - D)}}{2D(D-1)} = \frac{D \pm \sqrt{D}}{D(D-1)}$$

□

Question 2. Fix some $w \in \mathbb{R}^{D \times 1}$.

(a) Show that $\{x \in \mathbb{R}^{D\times 1} : w^T x = 0\}$ is a $(D-1)$-dimensional linear subspace of $\mathbb{R}^D$, if $w \neq 0$.

(b) Fix some $b \in \mathbb{R}$. Is the set $\{x \in \mathbb{R}^{D\times 1} : w^T x = b\}$ a $(D-1)$-dimensional linear subspace of $\mathbb{R}^D$? Prove or show a counterexample.

Solution. □

Question 3. Using the dataset "kNN_ClassifierSyntheticData.mat", randomly select 100 different testing points in the dataset, and run a kNN-classifier for kNN $= \{1, 10, 50, 100, 500, 900\}$ using the remaining points as training points. How does performance change with the change in kNN?

Solution. □

Question 4. Using the Salina A dataset, randomly select 100 different testing points in the dataset, and run a kNN-classifier for kNN $= \{1, 10, 50, 100, 500, 900\}$ using the remaining points as training points. How does performance change with the change in kNN?

Solution. □