Homework #6

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Math 123: Homework #6

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Question 1. Let $L = D - W$ be the unnormalized graph Laplacian associated to a graph $\mathcal{G} = (V, W)$ on points $V = \{x_i\}_{i=1}^n$ with symmetric weight matrix W and diagonal degree matrix D. Let $\{C, \overline{C}\}$ be any partition of V, and let:

$$f_i^C = \begin{cases} -\sqrt{\mathrm{Vol}(\overline{C})/\mathrm{Vol}(C)} & \text{if } x_i \in C \\ -\sqrt{\mathrm{Vol}(C)/\mathrm{Vol}(\overline{C})} & \text{if } x_i \in \overline{C} \end{cases}$$

(a) Prove that $\langle Df^C, \mathbb{1} \rangle = 0$.

(b) Prove that $(f^C)^T D f^C = \mathrm{Vol}(V)$.

(c) Prove that $(f^C)^T L f^C = \mathrm{Vol}(V) \cdot \mathrm{Ncut}(C, \overline{C})$.

Solution. □

Question 2. Recall that one construction of the weight matrix for a graph on data points $\{x_i\}_{i=1}^n$ is to use the Gaussian kernel

$$W_{ij} = \begin{cases} \exp(-\|x_i - x_j\|_2^2/\sigma^2) & i \neq j \\ 0 & i = j \end{cases}$$

for some choice of $\sigma > 0$.

(a) What happens to the resulting Laplacian matrix as $\sigma \to 0$?

(b) What happens to the resulting Laplacian matrix as $\sigma \to \infty$?

Solution. □

Question 3.

Solution. □