Homework #9

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Math 285: Homework #9

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Question 1. Consider S^1 as a subset of the unit circle. We see that it admits multiplication, given by:

$$e^{it} \cdot e^{iu} = e^{i(t+u)}$$

for $u, t \in \mathbb{R}$.

Viewed as complex numbers, we can also express complex multiplication by an element $\cos t + i \sin t \in S^1$ as:

$$(\cos t + i \sin t)(x + iy) = ((\cos t)x - (\sin t)y) + i((\sin t)x + (\cos t)y)$$

Then for $g = (\cos t, \sin t) \in S^1 \subset \mathbb{R}^2$, the left multiplication is given by:

$$l_g(x, y) = ((\cos t)x - (\sin t)y, (\sin t)x + (\cos t)y)$$

Let $\omega = -ydx + xdy$ be a 1-form on S^1. Prove that $l_g^* \omega = \omega$ for all $g \in S^1$.

Solution. We recall that for any pullback, by 17.10 and 17.9:

$$l_g^*(-ydx + xdy) = -l_g^*(y)d(l_g^*(x)) + l_g^*(x)d(l_g^*(y))$$

Fixing a $g = (\cos t, \sin t)$, we see that l_g^* has the action of sending $x \mapsto (\cos t)x - (\sin t)y, y \mapsto (\sin t)x + (\cos t)y$.

Thus, we have that:

$$-l_g^*(y)d(l_g^*(x)) + l_g^*(x)d(l_g^*(y)) = -[(\sin t)x + (\cos t)y]d((\cos t)x - (\sin t)y) + [(\cos t)x - (\sin t)y]d((\sin t)x + (\cos t)y)$$

Doing some more algebra, we see that this is equal to:

$$-[(\sin t)x + (\cos t)y][\cos t dx - \sin t dy] + [(\cos t)x - (\sin t)y][(\sin t)dx + (\cos t)dy] =$$

$$-\sin t \cos t(xdx) + \sin^2 t(xdy) - \cos^2 t(ydx) + \sin t \cos t(ydy) + \cos t \sin t(xdx) + \cos^2 t(xdy) - \sin^2 t(ydx) - \sin t \cos t(ydy) =$$

$$(-\sin t \cos t + \cos t \sin t)xdx + \sin^2 t + \cos^2 t xdy + (\sin t \cos t - \sin t \cos t)ydy - (\sin^2 t + \cos^2 t)ydx =$$

$$-ydx + xdy = \omega$$

Since the choice of g were arbitrary, as we used no properties of t other than trigonometric identities, we have that $l_g^*\omega = \omega$ for all $g \in S^1$.

$\square$

Question 2. We say that a sum $\sum \omega_\alpha$ of differential k-forms on a manifold M is locally finite if the collection of supports $\{\operatorname{supp} \omega_\alpha\}$ is locally finite. Suppose that we have two locally finite sums $\sum \omega_\alpha, \sum \tau_\beta$, and $f \in C^\infty(M)$.

(a) Show that for every $p \in M$, there exists a neighborhood $p \in U$ such that $\sum \omega_\alpha$ is finite.

(b) Show that $\sum(\omega_\alpha + \tau_\alpha)$ is a locally finite sum, and

$$\sum(\omega_\alpha + \tau_\alpha) = \sum \omega_\alpha + \sum \tau_\alpha$$

(c) Show that $\sum f\omega_\alpha$ is a locally finite sum, and that:

$$\sum f \cdot \omega_\alpha = f \cdot \left(\sum \omega_\alpha\right)$$

Solution. (a)

Fix some $p \in M$. Since the supports of ω_α are locally finite, we know that there exists a neighborhood U of p such that only finitely many of $\operatorname{supp}(\omega_\alpha) \cap U \neq \emptyset$. Choose such a neighborhood U, and index the supports with non-empty intersection $\operatorname{supp}(\omega_{\alpha_i})$ for $i \in (1, ..., n)$.

Then, consider the value of $\sum \omega_\alpha$. Of course, if the support has trivial intersection with U, then $\omega_\alpha = 0$ on U, as the support of a k-form is defined as the closure of the complement of the zero set, hence the complement of the support is contained within the zero set.

Then, we see that $\sum \omega_\alpha = \sum_{i=1}^n \omega_{\alpha_i}$. But, at each point $q \in U$, each ω_{α_i} is a alternating k-tensor, hence a k-linear real function. Then, at q, this is a finite sum of finite values, hence finite.

Since this is true for all $q \in U$, since the choice of q was arbitrary, this is finite for all of U, and we may say that $\sum \omega_\alpha = \sum_{i=1}^n \omega_{\alpha_i} < \infty$ on all of U.

(b)

We start by proving a lemma:

Lemma. *Let ω, τ be k-forms on a manifold M.*

Then, we have that:

$$supp(\omega + \tau) \subseteq supp(\omega) \cup supp(\tau)$$

Proof. Clearly, we know that as sets:

$$\{p \in M : \omega_p + \tau_p \neq 0\} \subseteq \{p \in M : \omega_p \neq 0\} \cup \{q \in M : \tau \not\emptyset\}$$

as of course, if $\omega_p + \tau_p \neq 0$, then at least one of ω_p, τ_p is non-0.

Since we have that $\{p \in M : \omega_p \neq 0\} \subseteq \operatorname{supp}(\omega)$, being the closure, and same with τ, we see that:

$$\{p \in M : \omega_p + \tau_p \neq 0\} \subseteq \operatorname{supp}(\omega) \cup \operatorname{supp}(\tau)$$

Now, since $\operatorname{supp}(\omega)$ is closed, and same with $\operatorname{supp}(\tau)$, we see that the right hand side is a closed set, being the finite union of closed sets.

Then, since the closure of the left-hand side, is the smallest closed set that contains the left-hand side, and that the right hand side is a closed set, we have that:

$$\operatorname{supp}(\omega + \tau) = \operatorname{cl}(\{p \in M : \omega_p + \tau_p \neq 0\}) \subseteq \operatorname{supp}(\omega) \cup \operatorname{supp}(\tau)$$

as desired.

$\square$

We start by proving the equality at a fixed point. Let p be an arbitrary point in M. Since $\sum \omega_\alpha$ is locally finite, we may find a neighborhood U such that the intersection with $\text{supp}(\omega_\alpha)$ is trivial for all but α_i, $i \in (1, ..., n)$. Similarly, there exists V such that the interesection with $\text{supp}(\tau_\beta)$ is trivial for all but β_j, $j \in (1, ..., m)$.

Then, on $U \cap V$, we have that:

$$\sum \omega_\alpha + \sum \tau_\beta = \sum_{i=1}^{n} \omega_{\alpha_i} + \sum_{j=1}^{m} \tau_{\beta_j} = \sum_{i=1}^{\max(m,n)} \omega_{\alpha_i} + \tau_{\beta_j}$$

where WLOG if $m < n$, we pad with $\beta_{m+1}, ..., \beta_n = 0$.

Thus, on $p \in U \cap V$, we have that:

$$\sum \omega_\alpha + \sum \tau_\beta = \sum_{i=1}^{n} \omega_{\alpha_i} + \sum_{j=1}^{m} \tau_{\beta_j} = \sum_{i=1}^{l} \omega_{\alpha_i} + \tau_{\beta_i} = \sum \omega_\alpha + \tau_\beta$$

Moreover, we see that on $U \cap V$, only $\omega_{\alpha_i}, \tau_{\beta_j}$ potentially non-0 for $i \in (1, ..., n), j \in (1, ..., m)$. Hence, for whatever numbering $\sum_\alpha \omega_\alpha + \tau_\alpha$ takes on, only these can have supports with non-trivial interesection. Since by our lemma, the support of $\omega + \tau$ is contained within the union of the supports, only up to $m + n$ distinct $\omega_\alpha + \tau_\alpha$ may have supports with non-trivial intersection. Thus, $\sum_\alpha \omega_\alpha + \tau_\alpha$ is locally finite at p.

However, the choice of p was trivial. Thus, we can repeat this argument for every point $p \in M$, and thus the sum is locally finite on all of M, and the equality holds on all of M.

(c)

In the same vein as (b), we first look at the support. It should be clear that for a single k-form ω, that we have set-wise:

$$\{p \in M : \omega_p = 0\} \subseteq \{p \in M : f \cdot \omega_p = 0\}$$

as if $\omega_p = 0$, of course $f \cdot \omega_p = 0$.

Now, because complements reverse the inclusion, this implies that:

$$\{p \in M : f \cdot \omega_p \neq 0\}^c \subseteq \{p \in M : \omega_p \neq 0\}$$

Then, since the support of ω is the closure, we have that:

$$\{p \in M : f \cdot \omega_p \neq 0\}^c \subseteq \text{supp}(\omega)$$

And finally, since the right hand side is a closed set, the closure of the left hand set must be contained within the right hand side, and so we have that:

$$\text{supp}(f \cdot \omega) \subseteq \text{supp}(\omega)$$

Now, look at the sum $f \cdot \sum \omega_\alpha$. Fix some $p \in M$. Since $\sum \omega_\alpha$ is locally finite, we may find a neighborhood such that only finitely many ω_α have supports which intersect U non-trivially. Label these forms ω_{α_i} for $i \in (1, ..., n)$.

Then, on U, we have that

$$f \cdot \sum \omega_\alpha = f \cdot \sum_{i=1}^{n} \omega_{\alpha_i}$$

as on U, since the intersection with the support is trivial, if $\alpha \notin \{\alpha_i\}_{i=1}^n$, then $\omega_\alpha = 0$ identically on U.

Thus, since this is now a finite sum, we may distribute into the sum, and we find that we have a neighborhood with p such that:

$$f \cdot \sum \omega_\alpha = f \cdot \sum_{i=1}^n \omega_{\alpha_i} = \sum_{i=1}^n f \cdot \omega_{\alpha_i} = \sum_\alpha f \cdot \omega_\alpha$$

where the last equality comes from the fact that $\omega_\alpha = 0 \implies f \cdot \omega_\alpha = 0$ for all $\alpha \notin \{\alpha_i\}_{i=1}^n$ because, by the statement proved about the containment of the supports, $f \cdot \omega_\alpha$ must be locally finite, as only the $\text{supp}(\omega_{\alpha_i})$ have non-trivial intersection with U, and thus only the $\text{supp}(f \cdot \omega_{\alpha_i})$ could have non-trivial intersection.

Finally, since this procedure may be done for every point p, this sum becomes an equality of functions across all $p \in M$ and is locally finite everywhere, and so we conclude that:

$$f \cdot \sum \omega_\alpha = \sum f \cdot \omega_\alpha$$

with $\sum f\omega_\alpha$ as locally finite.

$\square$

Question 3. Let U be the open set $(0, \infty) \times (0, \pi) \times (0, 2\pi) \subseteq \mathbb{R}^3$, with the spherical coordinates (ρ, ϕ, θ). Define $F : U \to \mathbb{R}^3$ via:

$$F(\rho, \phi, \theta) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$$

Let x, y, z be the standard coordinates on $\mathbb{R}^3$. Show that:

$$F^*(dx \wedge dy \wedge dz) = \rho^2 \sin \phi \, d\rho \wedge d\phi \wedge d\theta$$

Solution. First, we recall that by proposition 18.10, the pullback commutes with the wedge product. Thus, we have that:

$$F^*(dx \wedge dy \wedge dz) = F^*(dx) \wedge F^*(dy) \wedge F^*(dz)$$

Now, by proposition 17.9, since F is clearly C^∞, each coordinate being the product of C^∞ functions, we see that:

$$F^*(dx) \wedge F^*(dy) \wedge F^*(dz) = dF^*(x) \wedge dF^*(y) \wedge dF^*(z) = d(x \circ F) \wedge d(y \circ F) \wedge d(z \circ F) =$$

$$d(\rho \sin \phi \cos \theta) \wedge d(\rho \sin \phi \sin \theta) \wedge d(\rho \cos \phi)$$

Expanding, we compute:

$$d(\rho \sin \phi \cos \theta) \wedge d(\rho \sin \phi \sin \theta) \wedge (\rho \cos \phi) = (\sin \phi \cos \theta d\rho + \rho \cos \phi \cos \theta d\phi - \rho \sin \phi \sin \theta d\theta) \wedge$$

$$(\sin \phi \sin \theta d\rho + \rho \cos \phi \sin \theta d\phi + \rho \sin \phi \cos \theta d\theta) \wedge (\cos \phi d\rho - \rho \sin \phi d\phi)$$

We notice that since 3-forms are spanned exactly by $d\rho \wedge d\phi \wedge d\theta$, we can look at only the terms that have one of each of $d\rho, d\phi, d\theta$ in some order, as otherwise, because $df \wedge df = 0$, and we can alternate over the wedge product at the cost of a factor of (-1) the other terms that repeat any single form vanish:

$$= \sin \phi \cos \theta d\rho \wedge \rho \sin \phi \cos \theta d\theta \wedge (-\rho \sin \phi d\phi) + \rho \cos \phi \cos \theta d\phi \wedge \rho \sin \phi \cos \theta d\theta \wedge \cos \phi d\rho +$$

$$(-\rho\sin\phi\sin\theta d\theta)\wedge\sin\phi\sin\theta d\rho\wedge(-\rho\sin\phi d\phi)+(-\rho\sin\phi\sin\theta d\theta)\wedge\rho\cos\phi\sin\theta d\phi\wedge\cos\phi d\rho$$

Cleaning up with some algebra, this is equal to:

$$-\rho^2\sin^3\phi\cos^2\theta(d\rho\wedge d\theta\wedge d\phi)+\rho^2\sin\phi\cos^2\phi\cos^2\theta(d\phi\wedge d\theta\wedge d\rho)+$$
$$\rho^2\sin^3\phi\sin^2\theta(d\theta\wedge d\rho\wedge d\phi)-\rho^2\sin\phi\cos^2\phi\sin^2\theta(d\theta\wedge d\phi\wedge d\rho)$$

Factoring out a $\rho^2\sin\phi$ from the entire sum, and using the alternating nature to swap 1 forms to rewrite everything in terms of $d\rho\wedge d\phi\wedge d\theta$:

$$\rho^2\sin\phi(\sin^2\phi\cos^2\theta+\cos^2\phi\cos^2\theta+\sin^2\phi\sin^2\theta+\cos^2\phi\sin^2\theta)(d\rho\wedge d\phi\wedge d\theta)$$

Regrouping, we use the fact that $\sin^2+\cos^2=1$ twice to find this to be equal to:

$$\rho^2\sin\phi(\sin^2\phi(\cos^2\theta+\sin^2\theta)+\cos^2\phi(\cos^2\theta+\sin^2\theta)(d\rho\wedge d\phi\wedge d\theta)=$$

$$\rho^2\sin\phi(\sin^2\phi+\cos^2\phi)(d\rho\wedge d\phi\wedge d\theta)=\rho^2\sin\phi\, d\rho\wedge d\phi\wedge d\theta$$

as desired.

□

Question 4. Recall that the electrical and magnetic fields obey the following equations (Maxwell's equations):

$$\begin{cases}\nabla\times E=-\frac{\partial B}{\partial t} & \nabla\times B=\frac{\partial E}{\partial t}\\ \nabla\cdot E=0 & \nabla\cdot B=0\end{cases}$$

Corresponding to the vector fields E, B, we have the following forms:

$$E=E_1dx+E_2dy+E_3dz$$

$$B=B_1dy\wedge dz+B_2dz\wedge dx+B_3dx\wedge dy$$

Let $\mathbb{R}^4$ be flat space-time with coordinates (x,y,z,t). Define F to be the 2-form:

$$F=E\wedge dt+B$$

Decide which of Maxwell's equations are equivalent to the condition $dF=0$, and prove this fact.

Solution. Before we claim anything, we compute F in terms of the wedge product of 1-forms:

$$F=E\wedge dt+B=(E_1dx+E_2dy+E_3dz)\wedge dt+(B_1dy\wedge dz+B_2dz\wedge dx+B_3dx\wedge dy)=$$
$$E_1dx\wedge dt+E_2dy\wedge dt+E_3dz\wedge dt+B_1dy\wedge dz+B_2dz\wedge dx+B_3dx\wedge dy$$

Now, computing dF, and notating $\frac{\partial E_i}{\partial x}=(E_i)_x$, and same for y,z, and similarly for B_i we see that:

$$dF=dE_1dx\wedge dt+dE_2dy\wedge dt+dE_3dz\wedge dt+dB_1dy\wedge dz+dB_2dz\wedge dx+dB_3dx\wedge dy=$$

$$(E_1)_ydy\wedge dx\wedge dt+(E_1)_zdz\wedge dx\wedge dt+(E_2)_xdx\wedge dy\wedge dt+(E_2)_zdz\wedge dy\wedge dt+(E_3)_xdx\wedge dz\wedge dt+(E_3)_ydy\wedge dz\wedge dt+$$

$$(B_1)_x dx\wedge dy\wedge dz+(B_1)_t dt\wedge dy\wedge dz+(B_2)_y dy\wedge dz\wedge dx+(B_2)_t dt\wedge dz\wedge dx+(B_3)_z dz\wedge dx\wedge dy+(B_3)_t dt\wedge dx\wedge dy$$

where we have omitted writing any cases where we would have two of the same 1-forms in the wedge product, because they vanish due to $df \wedge df = 0$.

Collecting like terms by the antcommutativity of the wedge product, we see this as equal to:

$$[(B_1)_x + (B_2)_y + (B_3)_z](dx \wedge dy \wedge dz) + [-(E_1)_y + (E_2)_x dx + (B_3)_t)](dx \wedge dy \wedge dt) +$$
$$[-(E_2)_z + (E_3)_y + (B_1)_t](dy \wedge dz \wedge dt) + [(E_1)_z - (E_3)_x + (B_2)_t](dz \wedge dx \wedge dt]$$

For this to vanish, we have the following equations then:

$$\begin{cases} (B_1)_x + (B_2)_y + (B_3)_z = 0 \\ (B_1)_t = (E_2)_z - (E_3)_y \\ (B_2)_t = -(E_1)_z + (E_3)_x \\ (B_3)_t = (E_1)_y - (E_2)_x \end{cases}$$

It should be clear, that the first equation corresponds to $\nabla \cdot B = 0$.

To see the final 3 equations, we compute $\nabla \times E$:

$$\begin{vmatrix} x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_1 & E_2 & E_3 \end{vmatrix} = [(E_3)_y - (E_2)_z]x + [(E_1)_z - (E_3)_x]y + [(E_2)_x - (E_1)_y]z$$

Thus, we see that the last three equations correspond exactly to:

$$\frac{\partial}{\partial t} B = -\nabla \times E$$

Thus, we have that $dF = 0$ is equivalent to the following of Maxwell's equations:

$$\begin{cases} \nabla \times E = -\frac{\partial B}{\partial t} \\ \nabla \cdot B = 0 \end{cases}$$

□

Question 5. Let $\omega = dx^1 \wedge ... \wedge dx^n$ be the volume form and $X = \sum x^i \partial/\partial x^i$ be the radial vector field on $\mathbb{R}^n$. Compute the contraction $i_X \omega$.

Solution. First, we compute $i_X(dx^j)$:

$$i_X(dx^j) = dx^j(X) = dx^j \left(\sum_k x^k \frac{\partial}{\partial x^k} \right) = x^j$$

Now, using the fact that i_X is an antiderivation of degree -1, we proceed inductively. Denote $dx^1 \wedge ... \wedge dx^n = \bigwedge_i dx^i$, that is, in order.

We wish to show that for $n \geq 2$:

$$i_X(\omega) = \sum_{k=1}^{n} (-1)^{k-1} x^k \left(\bigwedge_{j \neq k} dx^j \right)$$

Technically, we see that for the special case $n = 1$, then this sum still makes sense, as the wedge product is over nothing, and we have that as we proved for $\omega = x^1$ as a 1-form, $i_X(\omega) = x^1$.

Suppose $n = 2$.

Then, of course, we have by the action on the 1-form:

$$i_X(\omega) = i_X(dx^1 \wedge dx^2) = i_X(dx^1) \wedge dx^2 + (-1)dx^1 \wedge i_X(dx^2) = x^1 dx^2 - x^2 dx^1$$

which is exactly what we want in the case $n = 2$.

Now, suppose that we have that:

$$i_X(\omega) = i_X(dx^1 \wedge ... \wedge dx^l) = \sum_k^l (-1)^{k-1} x^k \left(\bigwedge_{j \neq k} dx^j \right)$$

for all $l < n$.

Then, consider $\omega = dx^1 \wedge ... \wedge dx^n$. We have that:

$$i_X(\omega) = i_X(dx^1 \wedge ... \wedge dx^n) = i_X(dx^1) \wedge \bigwedge_{j=2}^n dx^j + (-1)dx^1 \wedge i_X \left(\bigwedge_{j=2}^n dx^j \right)$$

By the inductive step, since $i_X \left(\bigwedge_{j=2}^n dx^j \right)$ is the action on a $n-1$ form, this is equal to:

$$x^1 \bigwedge_{j=2}^n dx^j - dx^1 \wedge \sum_{l=2}^n (-1)^{l-2} x^l \left(\bigwedge_{i \neq l, i=2}^n dx^i \right)$$

where we notice that we have $(-1)^{l-2}$ because the first term is at $l = 2$, so we need to do a mild bit of reindexing.

Bringing the (-1) into the sum and the dx^1 into the wedge product on the left, and rewriting the first term a bit, we see:

$$x^1(-1)^{1-1} \bigwedge_{j=1, j \neq 1}^n dx^j + \sum_{l=2}^n (-1)^{l-1} x^l \left(\bigwedge_{i \neq l, i=1}^n dx^i \right)$$

We identify the first term as the $l = 1$ term of the sum on the right, so we can combine this into a single sum:

$$\sum_{l=1}^n (-1)^{l-1} x^l \left(\bigwedge_{j=1, j \neq l}^n dx^j \right)$$

as desired.

Thus, we see that

$$i_X(\omega) = \sum_{l=1}^n (-1)^{l-1} x^l \left(\bigwedge_{j=1, j \neq l}^n dx^j \right)$$

□

Question 6. Let $U = (0, \infty) \times (0, 2\pi) \subseteq \mathbb{R}^2$. Define $F : U \to \mathbb{R}^2$ via $F(r, \theta) = (r \cos \theta, r \sin \theta)$. Decide whether F is orientation-preserving or reversing as a diffeomorphism onto its image.

Solution. As per the definition, we take the standard orientations (r, θ) for U and (x, y) for $\mathbb{R}^2$. By definition then, we need to see what orientation equivalence class $F^*(dx^1 \wedge dx^2)$ lands in, where we see $dx^1 \wedge dx^2$ as a representative of the orientation class of counterclockwise orientation.

Well, by the commutativity with the wedge product and the commutativity with the differential, the pullback acts via the following:

$$F^*(dx^1 \wedge dx^2) = F^*(dx) \wedge F^*(dy) = d(F^*(x)) \wedge d(F^*(y)) = d(r\cos\theta) \wedge d(r\sin\theta) =$$

$$[\cos\theta dr - r\sin\theta d\theta] \wedge [\sin\theta dr + r\cos\theta d\theta]$$

Again, using the fact that $dr \wedge dr = 0 = d\theta \wedge d\theta$, we look only at terms that contain $dr \wedge d\theta, d\theta \wedge dr$ as those will be the only surviving terms:

$$r\cos^2\theta(dr \wedge d\theta) - r\sin^2\theta d\theta \wedge dr = [r\cos^2\theta + r\sin^2\theta]dr \wedge d\theta = rdr \wedge d\theta$$

Since $r \in (0, \infty)$ we see that F^* takes $dx \wedge dy$ to the orientation class of $[dr \wedge d\theta]$ and hence is orientation preserving relative to the choice of orientations for the manifolds $(U, [dr \wedge d\theta]), (\mathbb{R}^2, [dx \wedge dy])$.

I realize at this point, I could've used Proposition 21.11 and just computed the Jacobian determinant. Take the ordering of the coordinates as $x^1 = r, x^2 = \theta$, and the ordering in the codomain as $y^1 = x = r\cos\theta, y^2 = y = r\sin\theta$.

$$\det\left[\frac{\partial F^i}{\partial x^j}\right] = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos\theta^2 + r\sin^2\theta = r$$

But again, since $r > 0$, we have that this is always positive, hence orientation preserving with respect to $[dr \wedge d\theta]$ and $[dx \wedge dy]$.

$\square$