First Assignment

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Math 240: Homework #2

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Problem 2.1. Show that the union of the coordinate axes in $\mathbb{A}^3$ is a closed algebraic set, and determine generators for its ideal.

Solution. Here, we will use x, y, z to denote the coordinates in $\mathbb{A}^3$, and call the union of the coordinate axes C. Suppose a point is in the zero set of the polynomials $Z(\{xy, yz, xz\})$. Since $xy = 0$, suppose we are in the case $x = 0$. Then $xz = 0$ automatically, and just need $yz = 0$. Then the cases would be either $x, y = 0$ or $x, z = 0$, which are the z-axis and the y-axis respectively. Repeating this argument for the other cases, we can also recover the y-axis. Then, we have that a point in the zero set has to be on one of the coordinate axes, and $Z(\{xy, yz, xz\}) \subseteq C$.

Now, suppose we have a point on a coordinate axis. Then, at least 2 of the coordinates must be 0. But then, $xy, yz, xz = 0$ and we have that $C \subseteq Z(\{xy, yz, xz\})$ and thus $C = Z(\{xy, yz, xz\})$. Since the union of the coordinate axes is the zero set of a set of polynomials, it must be a closed algebraic set, and we may take the ideal $< xy, yz, xz >$.

$\square$

Problem 2.2. Consider the curve C in $\mathbb{A}^3$ given in parametric form by (t, t^2, t^3). Show that C is an algebraic set and determine generators for the (radical) ideal of C.

Solution. Consider the zero set of the polynomials $f = y - x^2$ and $g = z - x^3$, $Z(f, g)$. If we have a point on C, we can express it as (t_0, t_0^2, t_0^3) for some $t_0 \in k$. Then, we have that $f(t_0, t_0^2, t_0^3) = t_0^2 - (t_0)^2 = 0$ and $g(t_0, t_0^2, t_0^3) = t_0^3 - (t_0)^3 = 0$, and thus $C \subseteq Z(f, g)$.

Now, suppose we have a point on $Z(f, g)$. Then, we have that $y - x^2 = 0, z - x^3 = 0 \implies y = x^2, z = x^3$. Fix a point $x_0 \in k$, then we have that the points have form (x_0, x_0^2, x_0^3). But these are exactly the points on C. So, $Z(f, g) \subseteq C$, and $C = Z(f, g)$. Since C is the zero set of some polynomials in $k[x, y, z]$, C is an algebraic set, and we can take $< f, g >$ as an ideal of C.

Now, consider the map from $h : k[x, y, z] \to k[x, z]$ via $y \to x^2$. It is clear that this is surjective, so we need only prove that $\ker(h) =< y - x^2 >$. It is clear that $< y - x^2 >\subseteq \ker(h)$. Take $j \in \ker(h)$. We may rewrite the polynomial $j(x, y, z) = \Sigma_{k=0}^n j_k(xz)y^n$. If we do a rescaling, as in class, then we can rewrite this polynomial taking $y \to y - x^2 + x^2$, and we find $j(x, y - x^2 + x^2, z) = \Sigma_{k=0}^n j_k(xz)[(y - x^2) + x^2]^n$. Multiplying out, and regrouping terms, we can group terms as $j = \Sigma_{k=0}^n j'_k(xz)[(y - x^2)]^n$. In particular, since for each power $n \geq 1$, the term goes to 0 when $y \to x^2$, it must have constant term 0. Then, we have that $\ker h =< y - x^2 >$ as $y - x^2$ divides every term of j, and j was a generic polynomial. So, we have that h is an isomorphism.

Without going through every step, we can see that the same argument will work for $m : k[x, z] \to k[x]$ where $z \to x^3$ with $\ker m =< z - x^3 >$. Then, we notice that the kernel of $m \cdot h$ is just the ideal generated by $< \ker h, \ker m >$, which we can write in terms of our generators, $< z - x^3, y - x^2 >$. But, the composition of two isomorphisms is itself an isomorphism, so we have that this ideal is the kernel of a isomorphism to an integral domain, therefore prime, therefore radical. $\square$

Problem 2.3. (a) Consider the set of four points in $X = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \subseteq \mathbb{A}^2$. Find generators for its ideals.

(b) Do the same for the points $Y = \{(0,0),(0,1),(1,1),(2,2)\}$. What is the difference between the two cases, and why?

Solution. (a) We notice that we can think of this zero set as the intersection of the zero sets of the lines $x, x-1$ and $y, y-1$. We can also think of, for example, $x, x-1$ as the union of the zero sets of x, $x-1$. Then, from proposition 2.1.6 from Osserman, we have:

$$Z(\{x\}) \cup Z(\{x-1\}) = Z(\{x(x-1)\})$$

and

$$Z(\{x(x-1)\}) \cap Z(\{y(y-1)\}) = Z(\{x(x-1), y(y-1)\})$$

Now, we notice that $Z(\{x\}) = \{(0,y)\}$ and $Z(\{x-1\}) = \{(1,y)\}$, so we have that $Z(\{x(x-1)\}) = \{(x,y) : x = 0,1\}$ and analogously for y.

Then, $Z(\{x(x-1)\}) \cap Z(\{y(y-1)\}) = \{(x,y) : x = 0,1\} \cap \{(x,y) : y = 0,1\} = \{(x,y) : x, y = 0,1\} = X$. Thus we have that $Z(\{x(x-1), y(y-1)\}) = X$, and thus we can take our ideal as $< x(x-1), y(y-1) > \subseteq k[x,y]$.

(b) Working analgously, we can see that we can take this as the intersection of the union of sets of lines again. In particular, we will take the following zero sets:

$$Z_1 = Z(\{x(y-1)(1/2x+1)\})$$

and

$$Z_2 = Z(\{(y-x)(y-x-1)\})$$

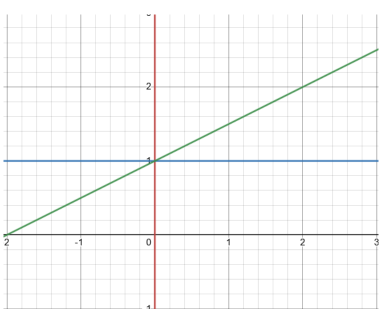

Figure 1: The lines $x = 0$, $y = 1$, and $y = \frac{x}{2} + 1$.

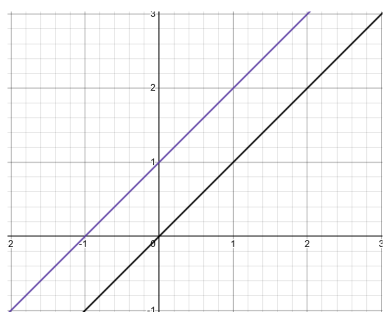

Figure 2: The lines $y = x$ and $y = x + 1$.

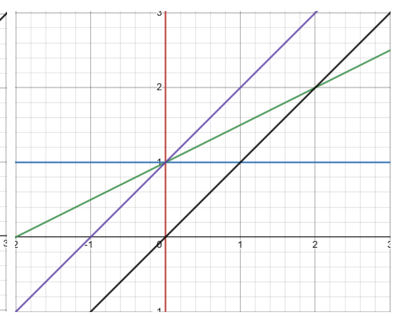

Figure 3: All lines in the previous 2 figures.

Let's justify this a little bit. We note that because lines can only meet in $0, 1$ or infinitely many points, we are guaranteed that these are the only points of intersection for our zero set, and we claim by picture that $Z_1 \cap Z_2 = Z(\{x(y-1)(1/2x+1), (y-x)(y-x-1)\}) = Y$.

The difference here is because we notice that in part (a), we never have 3 collinear points, so we may find 2 pairs of parallel lines such that they intersect in those points. On the other hand, once we have collinear points, we need to be a little more clever. In this case, one solution is to take a pair of parallel lines, one of which being the collinear points, and then lines that meet at the lone isolated point.

□

Problem 2.4. (a) Let Y be the plane curve $y = x^2$ or, in the language we used in class, the zero set of the polynomial $y - x^2$. Show that the coordinate ring $A(Y)$ is isomorphic to a polynomial ring in one variable over k

(b) Let Z be the plane curve $xy = 1$. Show that the coordinate ring $A(Z)$ is not isomorphic to a polynomial ring in one variable over k

(c) Let f be an irreducible quadratic polynomial in $k[x, y]$, and W the conic defined by f. Show that $A(W)$ is isomorphic either to $A(Y)$ or to $A(Z)$. Could you say which one it is and when?

Solution. (a) Consider the morphism of rings $f : k[x, y] \to k[x]$ via $f(g(x, y)) = f(g(x, x^2))$. Clearly, we can see this is a surjective map, because if we have any polynomial in $k[x]$, it is a member of $k[x, y]$, where it just doesn't contain any y terms. Further, we see that if g has no dependency on y, then $f(g) = g$, where we abuse notation somewhat to indicate that even though we're talking about polynomials in different spaces, they have variables only in x and with the same coefficients.

Now, consider an element of $< y - x^2 >$, the ideal generated by that polynomial in $k[x, y]$. Then, we can express this element as $h(x, y)(y - x^2)$, for some $h(x, y) \in k[x]$. Then, $f(h(x, y)(y - x^2)) = f(h)f(x^2 - x^2) = f(h) * 0 = 0$. Thus, we have that $< y - x^2 >\subseteq \ker(f)$.

Now, suppose we have that $f(g) = 0$. Now, rewrite $g(x, y) = \Sigma_{n=0}^m h_n(x)y^n$, for $h_n(x)$ some polynomial in one variable. Now, consider the rescale

$$g(x, y) = g(x, (y - x^2) + x^2) = \Sigma_{n=0}^m h_n(x)[(y - x^2) + x^2]^n = \Sigma_{n=0}^m h'_n(x)[(y - x^2)]^n$$

for some new polynomials h'_n after multiplying terms out. However, this is the same polynomial, and, in particular, it still belongs to the kernel of f. Then, it must have constant term 0, as for every $1 \leq n \leq m$, under the map f, that term becomes $f(h'_n(x))f([(y - x^2)]^n) = h'_n(x) * 0 = 0$, so all terms with positive degree are 0 under this map. Then, for the entire sum to be 0, $h'_0(x) = 0$.

So, then we have that we can reexpress $g(x, y) = \Sigma_{n=1}^m h'_n(x)(y-x^2)^n$. But now, we see that $y-x^2$ divides each term of the sum, therefore $y-x^2$ divides g, and $g \in< y-x^2 >$. Thus, we have that $\ker(f) =< y-x^2 >$. So, we have by the first isomorphism theorem, that $k[x, y]/ < y - x^2 >\cong k[x]$. But, by definition, the left side is $A(Y)$, so $A(Y) = k[x, y]/ < y - x^2 >\cong k[x]$

(b) Let $f : k[x, y] \to k[x]$ be any morphism of rings. Suppose $< 1 - xy >\subseteq \ker(f)$. In particular then, $1 - xy \in \ker(f)$. Then, we have that $0 = f(1 - xy) = f(1) - f(x)f(y) = 1 - f(x)f(y)$. However, this means that $f(x)f(y) = 1$, and thus $f(x), f(y)$ are in the units of $k[x]$, which implies that $f(x), f(y) \in k$, by the degree function.

Then, $f(g) \in k$ for all $g \in k[x, y]$, and thus, no morphism f where $1 - xy \in \ker(f)$ can be surjective. Since no surjective map can exist from $k[x, y]$ to $k[x]$ with $1 - xy$ in the kernel, $k[x, y]/ < 1 - xy >\ncong k[x]$, and $A(Z) \ncong k[x]$.

(c) Let $f = ax^2 + by^2 + cx + dy + exy + g$. First, suppose at least one of a, b non-zero. Without loss of generality, assume a is non-0. Then, we can rewrite $f = x^2 + by^2 + cx + dy + exy + g$, where it's understood we multiplied through by a^{-1}. Rearranging, we have that $f = x^2 + x(c + ey) + by^2 + dy + g$. Here, we can take the transformation $x \to x - \frac{1}{2}(c + ey)$, completing the square:

$$f = (x-1/2(c+ey))^2+x(c+ey)+by^2+dy+g = x^2-x(c+ey)+1/4(c+ey)^2+x(c+ey)-\frac{1}{2}(c+ey)^2+by^2+dy+g$$

$$= x^2 + b'y^2 + d'y + g'$$

Where we denote b', d', g' as the resultant coefficients after expansion. Now, we can see that if $b' = 0$, then we have $f = x^2 + d'y + g'$, which is in the form of Y, and then $A(W) = A(Y)$.

Now, suppose not. Then, we may complete the square again by sending $y \to y - d'/2b'$ to find $f = x^2 + b'(y^2 - d'/b'y + (d'/2b')^2) + d'y + g' = x^2 + b'y^2 + g''$ where again, we collect the constants into g'', non-0 to be irreducible. Since we live in a algebraically closed field, $x^2 + by^2$ factors into linear factors of form $(x + cy), (x - cy)$ such that $c^2 = -b$. Finally,we notice that we can take a change of variables $\alpha = x + cy, \beta = x - cy$ and find $f = \alpha\beta + g''$, which is of form Z, and in this case we find $A(W) = A(Z)$.

Now, suppose both a, b both 0. Then, we have $f = cx + dy + exy + g$. Using a similar rescaling, take the transformation $x = \alpha+\beta, y = \alpha-\beta$ and we recover something of the form $f = e\alpha^2+(c+d)\alpha-e\beta^2+(c-d)\beta+g$. From here, we may go through the same transformations as above, identifying α as x and β as y.

□