# Homework #8

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Math 235: Homework #8

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## 2.1

**Problem 4.6.21.** Assume that $E \subseteq \mathbb{R}^d$ is measurable. Let $f : E \to \overline{F}$ be a measurable function. Define the distribution function of $f$ as follows:

$$\omega(t) = |\{|f| > t\}|, t \geq 0$$

By definition, $\omega$ is a non-negative, extended real-valued function. Prove the following:

(a) $\omega$ is monotone decreasing on $[0, \infty)$.

(b) $\omega$ is right-continuous, that is, $\lim_{s \to t^+} \omega(s) = \omega(t)$ for every $t \geq 0$.

(c) If $f$ is integrable, then $\lim_{s \to t^-} \omega(s) = |\{|f| \geq t\}|$.

(d) $\int_0^\infty \omega(t)dt = \int_E |f(x)|dx$

(e) $f$ is integrable $\iff$ $\omega$ is integrable.

(f) If $f$ is integrable, then $\lim_{n\to\infty} n\omega(n) = 0 = \lim_{n\to\infty} \frac{1}{n}\omega(\frac{1}{n})$.

*Solution.* (a)

We notice that for any $t' \geq t$, that by definition, $\{|f| > t'\} \subseteq \{|f| > t\}$. Then, by the monotonicity of the Lebesgue measure, we have that $|\{|f| > t'\}| \leq |\{|f| > t\}| \implies f(t') \leq f(t)$. Since this is true for all $t' \geq t$, we have that $\omega$ is monotone decreasing.

(b)

Let $\{a_n\}_{n\in\mathbb{N}}$ be any sequence of positive numbers where $a_n \to 0$. Take a monotone subsequence $\{a_{n_k}\}$ such that $a_{n_{k+1}} < a_{n_k}$ for all $k$. Then, consider the sequence of measurable sets $A_{a_{n_k}} = \{|f| > t + a_{n_k}\}$. This is a sequence of measurable sets, with the property that $A_{a_{n_{k+1}}} \subseteq A_{a_{n_k}}$ since we took the monotone subsequence. Then, by convergence from below, we have that $\lim_{n_k \to 0} |A_{a_{n_k}}| = \cup_{n_k} A_{a_{n_k}} = \{|f| > t\}$. Now, we only need to check that this is convergent in the full sequence. First suppose that $|\{|f| > t\}| = \infty$. Then, we have that this must diverge for $a_n$, because since we know $|A_{a_{n_k}}| \to \infty$ monotonically, and $a_n \to 0$, for any $M \in \mathbb{R}$, we take $N_{k_0}$ such that for all $n_k > N_{k_0}, |A_{a_{n_k}}| \geq M$. Then, we pick $N$ such that for all $i > N$, $a_i < a_{N_{k_0}}$, by the convergence of $a_n$. Then, we know that since this is monotone, $|A_{a_{n_i}}| \geq |A_{a_{N_{k_0}}}| \geq M$.

Now, suppose $|\{|f| > t\}| < \infty$. Then, we can choose $N_{k_0}$ such that $|\{|f| > t\}| - |A_{a_{N_{k_0}}}| < \epsilon$. We may choose $N$ such that $a_j < A_{N_{k_0}}$ by the convergence of the $a_n$ for all $j > N$. Then, we have by the monotonicity, that $|\{|f| > t\}| \geq |A_{a_j}| \geq |A_{a_{N_{k_0}}}| \implies |\{|f| > t\}| - |A_{a_j}| < \epsilon$.

(c)

First, we see that if $|E| < \infty$, then we can use continuity from above. Otherwise, suppose $|E| = \infty$. First, for $t = 0$, since we use left-continuous, but $\omega$ only defined for $t \geq 0$, the only permissible sequence is the constant sequence, and of course $|\{|f| > 0\}| = |\{|f| > 0\}|$. Then, for any $t > 0$, we see that $|\{|f| > t\}| < \infty$ as suppose not, then we know that $|f| > t$ the constant function on a set of infinite measure, so $\int_E |f| = \infty$, a contradiction. As such, regardless, we may use continuity from above since we are guaranteed that the sets have finite measure.

Now, then, let $\{b_n\}$ be any sequence of non-negative numbers such that $b_n \to t$, where $t > 0$. We use the same construction as (b), where we find a monotone sequence $b_{n_k} \to t$, which extends to a nested sequence of sets $\{|f| > b_{n_0}\} \supseteq \{|f| > b_{n_1}\}$.... We call these sets $B_{b_{n_k}}$. Then, since we are assured that this is eventually constant, since eventually, at least one $b_{n_k} > 0$, we apply continuity from above to find that $\lim_{n_k \to \infty} |B_{b_{n_k}}| = |\cap B_{b_{n_k}}|$. But, we have that $\cap B_{b_{n_k}} = \cap\{|f| > b_{n_k}\} = \{|f| \geq t\}$ because $b_{n_k} \to t$ from the left. Finally, in the same vein as (b), we use the convergence of $b_n \to t$ from the left as well as the monotonicity of $\omega$ to get that the whole thing converges.

(d)

Consider the integral, over $E \times \mathbb{R}^+$, that is, the non-negative real numbers, of $\chi_{\Gamma_{|f|}}$, where $\Gamma_{|f|}$ is the region under the graph of $|f|$ without the boundary defined in previous work $\{(x,t) : x \in E, t < |f(t)|\}$ where $|f(t)|$ can be infinite. Clearly, this is measurable, being a characteristic function of a measurable set, where we know the graph is measurable because of a previous homework 4.2.17, (b) and (a), since the boundary has measure 0. Then, we apply Tonelli's theorem:

$$\int_{\mathbb{R}^+} \left( \int_E \chi_{\Gamma_{|f|}} dx \right) dt = \int_E \left( \int_{\mathbb{R}^+} \chi_{\Gamma_{|f|}} dt \right) dx$$

Here, we notice that fixing $t$, $\int_E \chi_{\Gamma_{|f|}} dx = |\{|f| > t\}| = \omega(t)$, since at any fixed $t_0$, the points in the graph consist of the $(x, t_0)$ such that $|f(x)| > t_0$. On the other side, we have that $\int_{\mathbb{R}^+} \chi_{\Gamma_{|f|}} dt = |f(x)|$ since if we fix an $x_0$, then the points in the graph are just $(x_0, t)$ such that $0 \leq t < |f(x_0)| \implies t \in [0, |f(x_0)|)$, and $|[0, |f(x_0)|)| = |f(x_0)|$ for every $x_0 \in E$.

Then, substituting back into Tonelli's:

$$\int_{\mathbb{R}^+} \left( \int_E \chi_{\Gamma_{|f|}} dx \right) dt = \int_E \left( \int_{\mathbb{R}^+} \chi_{\Gamma_{|f|}} dt \right) dx \implies \int_{\mathbb{R}^+} \omega(t) dt = \int_E |f(x)| dx$$

as desired.

(e)

We have the following:

$$f \text{ integrable} \iff \lim_E f < \infty \iff \int_0^\infty \omega < \infty \iff \omega \text{ integrable}$$

where we use the result from (d).

(f)

Suppose not. Take the sequence $a_n \to \infty$. Then, we have that for all $n \geq N$ for some $N \in \mathbb{N}$, $a_n \omega(a_n) \geq \epsilon$ for some $\epsilon > 0$, where we use the fact that $\omega$ is non-negative to conclude $a_n \omega(a_n) \geq 0$ and monotone decreasing to conclude that it must have a minimum value. But then, we see that $\omega(a_n) \geq \epsilon / a_n$ for all $n \geq N$. This implies then, that on the set $[a_N, \infty]$, that $\omega(a_N)$ is larger than the constant function $\epsilon / a_N$. But then, since $\omega$ is non-negative, we have that $\int_{[0,\infty]} \omega \geq \int_{[a_N,\infty]} \omega \geq \epsilon / a_N |[a_N, \infty]| = \infty$ a contradiction, since we know that by part (d), the integral of $\omega$ aligns with the integral of $|f|$. Then, $a_n \omega(a_n) \to 0$, and since the choice of $a_n$ was arbitrary, this must be true for any sequence $\to \infty$. The same argument works for the second half of the equality, since if we look at $1/b_n \omega(1/b_n) \geq \epsilon \implies \omega(1/b_n) \geq b_n \epsilon$, so we have that on $[1/b_N, \infty]$, $\omega \geq b_N \epsilon$, which has a divergent integral.

□

**Problem 4.6.27.** Let $f \in L^1(\mathbb{R}), g \in L^\infty(\mathbb{R})$. Prove the following:

(a) The integral that defines $(f * g)(x)$ exists for every $x \in \mathbb{R}$.

(b) $f * g$ is continuous on $\mathbb{R}$.

(c) $f * g$ is bounded on $\mathbb{R}$, and $\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty$.

*Solution.* (a)

Recall that for any $x$, we define $(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$. Since we know that $f^+, f^- \leq |f|$ by definition, it suffices then to show that $\int_{\mathbb{R}} |f(y)g(x-y)|dy < \infty$. Since $g \in L^\infty(\mathbb{R})$, we can say that $|g| \leq \|g\|_\infty$ a.e. But then, we have that $|f(y)g(x-y)| \leq \|g\|_\infty |f(y)|$ for almost every $y \in \mathbb{R}$. So, we have that:

$$\int_{\mathbb{R}} |f(y)g(x-y)|dy \leq \int_{\mathbb{R}} \|g\|_\infty |f(y)|dy \leq \|g\|_\infty \|f\|_1 < \infty$$

Thus, $(f * g)(x)$ exists for all $x \in \mathbb{R}$.

(b)

By theorem 4.5.8, we can find a function $h \in L^1(\mathbb{R})$ such that $h \in C_c(\mathbb{R})$ and $\|f - h\|_1 < \epsilon$, for any $\epsilon > 0$. We also notice that $\int_{\mathbb{R}} |h(y)g(x-y)|dy \leq \int_{\mathbb{R}} |h(y)|\|g\|_\infty dy = \|g\|_\infty \|h\|_1$ where we don't know if $h$ is $L^1$ yet, so this could be infinite. But, by the reverse triangle inequality, we have that $|\|f\|_1 - \|h\|_1| \leq \|f - h\|_1 < \epsilon < \infty$, and since $\|f\|_1 < \infty$, so too must be $\|h\|_1$. Thus, $h(y)g(x-y)$ is integrable.

Further, we notice that the convolution is commutative, since we can take the translation $z = x - y \implies y = x - z$, as then $(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy = \int_{\mathbb{R}} f(x-z)g(z)dz = (g * f)(x)$.

Now $h$ be an arbitrary continuous function with compact support, we can say that $h$ is uniformly continuous. Further, since $h$ is compactly support, let $S$ be the support of $h$, then we can say that $S + S \subseteq [-n, n]$ for some $n$, since compact sets are bounded in $\mathbb{R}$. Then, we can say $|S+S| < \infty$, where $|S+S|$ is the same as 4.6.28. Let $\eta > 0$. Then, we may choose $\delta(x) > 0$ such that $d(x, y) < \delta \implies d(h(x), h(y)) < \eta/|S+S|\|g\|_\infty$. Now, let $x, x' \in \mathbb{R}$ such that $d(x, x') < \delta$. Then, we have that, by the commutativity of the convolution, that

$$|(h*g)(x) - (h*g)(x')| = \left| \int_{\mathbb{R}} h(x-y)g(y)dy - \int_{\mathbb{R}} h(x'-y)g(y) \right| = \left| \int_{\mathbb{R}} g(y)[h(x-y) - h(x'-y)] \right| \leq$$

$$\int_{\mathbb{R}} |g(y)||h(x-y) - h(x'-y)| \leq \|g\|_\infty \int_{\mathbb{R}} |h(x-y) - h(x'-y)|$$

However, we have that $d(x-y, x'-y) = |(x-y) - (x'-y)| = |x - x'| = d(x, x') < \delta$, so we have that:

$$\|g\|_\infty \int_{\mathbb{R}} |h(x-y) - h(x'-y)| \leq \|g\|_\infty \int_{\mathbb{S}+\mathbb{S}} |h(x-y) - h(x'-y)| \leq \|g\|_\infty \frac{\eta}{|S+S|\|g\|_\infty} |S+S| = \eta$$

Thus, we have that $(h*g)$ is continuous, actually, uniformly continuous. Now, consider:

$$|(f*g)(x) - (f*g)(x')| = \left| \int_{\mathbb{R}} f(x-y)g(y)dy - \int_{\mathbb{R}} f(x'-y)g(y)dy \right| =$$

$$\left| \int_{\mathbb{R}} f(x-y)g(y)dy - \int_{\mathbb{R}} h(x-y)g(y) + \int_{\mathbb{R}} h(x-y)g(y) + \int_{\mathbb{R}} h(x'-y)g(y) - \int_{\mathbb{R}} h(x'-y)g(y) - \int_{\mathbb{R}} f(x'-y)g(y)dy \right|$$

Using the triangle inequality, we break the sum up into:

$$\left| \int_{\mathbb{R}} f(x-y)g(y)dy - \int_{\mathbb{R}} h(x-y)g(y) \right| \leq \|g\|_\infty \|f - h\|_1$$

$$\left| \int_{\mathbb{R}} h(x-y)g(y)dy - \int_{\mathbb{R}} h'(x-y)g(y) \right| \leq \|g\|_\infty \int_{\mathbb{R}} |h(x-y) - h'(x-y)|$$

$$\left| \int_{\mathbb{R}} h(x'-y)g(y)dy - \int_{\mathbb{R}} f(x'-y)g(y) \right| \leq \|g\|_\infty \|h - f\|_1$$

Then, since we can control $h$ such that $\|f-h\|_1 = \|h-f\|_1 < \epsilon/3\|g\|_\infty$ due to the statement of the theorem, and we can control $d(x,x')$ such that $d(x,x') < \delta \implies \int_{\mathbb{R}} |h(x-y) - h'(x-y)| < \epsilon/3\|g\|_\infty$ due to the continuity on $h$ if $h \in C_c(\mathbb{R})$, we can control the whole sum to be less than $\epsilon$.

(c)

Well, I somehow did this to show (a), because we know that

$$|f*g| = \left|\int_{\mathbb{R}} f(y)g(x-y)dy\right| \le \int_{\mathbb{R}} |f(y)g(x-y)|dy \le \int_{\mathbb{R}} \|g\|_\infty |f(y)|dy \le \|g\|_\infty \|f\|_1 < \infty$$

□

**Problem 4.6.28.** (a) Show that if $f, g \in C_c(\mathbb{R})$, then $f*g \in C_c(\mathbb{R})$ and

$$\mathrm{supp}(f*g) \subseteq \mathrm{supp}(f) + \mathrm{supp}(g) = \{f+g : x \in \mathrm{supp}(f), y \in \mathrm{supp}(g)\}$$

Conclude that $C_c(\mathbb{R})$ is closed under convolution.

(b) Is $C_c^1(\mathbb{R})$ closed under convolution?

*Solution.* (a)

Suppose $f, g \in C_c(\mathbb{R})$. Then, since compact sets are closed and bounded on $\mathbb{R}$, we can say that $\mathrm{supp}(f) \subseteq [-m,m], \mathrm{supp}(g) \subseteq [-n,n]$ for $m,n \ge 0$. Consider the shape of $f*g(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$. Suppose $y \in \mathrm{supp}(f)$. Then, by necessity, $y \in [-m,m]$. Then, for $y-x \in \mathrm{supp}(g)$, we must have that $y-x \in [-n,n]$. This implies then that $x \in [-m-n, m+n]$ since we can see if $y=m$, then $x \in [m+n, m-n]$ and if $y=-m$, $x \in [-m+n, -m-n]$, so we can take these to be the max and min of the allowable $x$. But, by definition, since $\mathrm{supp}(f*g)$ is defined as the closure, it is a closed set. Further, we can see that for any $x : (f*g)(x) \neq 0$ we have that $|x| \le m+n$, then so too must be the closure, since no sequence of points that satisfy those bounds can have limit above $m+n$ or below $-m-n$. Then, we have that $\mathrm{supp}(f*g)$ as a bounded, closed set, and therefore, it must be compact as well.

Now, let $\{z_n\}$ be any sequence of points, such that $(f*g)(z_n) \neq 0$ for each $n$. We notice, a necessary condition for an integral to not be 0 is that it must be non-0 at least somewhere, since of course the integral of 0 everywhere is 0. Then, looking at $\int_{\mathbb{R}} f(y)g(z_n-y)dy$, we must have that $y \in \mathrm{supp}(f), z_n - y \in \mathrm{supp}(g)$. Then, suppose $z_n - y = x \in \mathrm{supp}(g)$, we then have that $z_n = x+y \implies z_n \in \mathrm{supp}(f)+\mathrm{supp}(g)$. Then, we have that $\lim_n z_n \in \mathrm{supp}(f)+\mathrm{supp}(g)$, since we see that this extends to families of sequences of points $z_n = x_n + y_n$, and taking the limit of both sides, and using the fact that $\mathrm{supp}(f), \mathrm{supp}(g)$ are closures, we find that $z = x+y$, where $z_n \to z, x_n \to x, y_n \to y$, where $x \in \mathrm{supp}(g), y \in \mathrm{supp}(f)$. Therefore, $\mathrm{supp}(f*g) \subseteq \mathrm{supp}(f)+\mathrm{supp}(g)$.

(b)

Let $f, g \in C_c^1(\mathbb{R})$. Firstly, by part (a), $f*g \in C_c(\mathbb{R})$. Then, we need only make sure that this has bounded derivative. Consider, for any $x \in \mathbb{R}$, :

$$\lim_{h\to 0} \frac{1}{h}\left[\int_{\mathbb{R}} f(y)g(x-y) - \int_{\mathbb{R}} f(y)g(x+h-y)\right] = \lim_{h\to 0}\int_{\mathbb{R}} \frac{f(y)(g(x-y)-g(x-y+h))}{h}$$

Define $f_n(y) = f(y)(g(x-y)-g(x-y+h_n))/h_n$, as $h_n \to 0$. By the differentiability of $g$, we see that $\lim_{n\to\infty} f(y)(g(x-y)-g(x-y+h_n))/h_n = f(y)\lim_n (g(x-y)-g(x-y+h_n))/h_n = f(y)g'(x-y)$, so these must converge pointwise. Further, because $f, g \in C_c^1$, we notice that $f, g$ must be bounded on its support. Then, if $|g| \le M$, $|f| \le N$, $|g(x-y)-g(x-y+h_n)| \le 2M$. Then, we have that $|f(y)(g(x-y)-g(x-y+h_n))/h_n| \le 2MN/h_n$, which, since we're on compact support, is integrable over the compact set.

Then, we may apply the DCT, and we get that:

$$\lim_{n\to 0}\int_{\mathbb{R}} \frac{f(y)(g(x-y)-g(x-y+h_n))}{h_n} = \int_{\mathbb{R}} f(y)g'(x-y)$$

But here, we use the fact that $g'$ is bounded, to bound this by $\|f\|_1\|g'\|_\infty$, where we use $\|f\|_1$ because $f \in C_c \implies f \in L^1$. Explicitly:

$$\left|\int_{\mathbb{R}} f(y)g'(x-y)\right| \leq \int_{\mathbb{R}} |f(y)g'(x-y)| \leq \int_{\mathbb{R}} |f(y)|\|g\|_\infty = \|f\|_1\|g\|_\infty$$

Since the choice of sequence of $h_n \to 0$ was arbitrary, this works for all such $h_n \to 0$, so it holds for the limit as $h \to 0$ more generally.

Thus, $f * g \in C_c$, and its derivative exists, and is bounded. Therefore, $f * g \in C_c^1(\mathbb{R})$.

□

**Problem 4.6.29.** Let $E \subseteq \mathbb{R}$ be a measurable subset with $0 < |E| < \infty$.

(a) Prove that the convolution $\chi_E * \chi_{-E}$ is continuous.

(b) Prove the Steinhaus Theorem: The set $E - E = \{x - y : x, y \in E\}$ contains an open interval centered at the origin.

(c) Show that $\lim_{t\to 0} |E \cap (E+t)| = |E|, \lim_{t\to\pm\infty} |E \cap (E+t)| = 0$.

*Solution.* (a)

By 4.6.27 (b), since $\chi_{-E}$ is an indicator function, it is bounded on $\mathbb{R}$, in particular, by 1, so $\chi_{-E} \in L^\infty(\mathbb{R})$ and $\chi_E \in L^1(\mathbb{R})$ since $\int_{\mathbb{R}} \chi_E = |E| < \infty$, their convolution is continous.

(b)

First, consider $\chi_E * \chi_{-E}(0)$. By definition, this is exactly $\int_{\mathbb{R}} \chi_E(y)\chi_{-E}(-y)dy$, which we notice is exactly 1 on $y \in E$, and 0 otherwise. Then, this integral evaluates to $|E| > 0$. Now, consider the interval $(|E|/2, 3|E|/2)$. This is an open set, and from part (a), we know that the inverse image of this interval is open by continuity. Further, we just showed that $\chi_E * \chi_{-E}(0) = |E| \in (|E|/2, 3|E|/2)$, so $0 \in (\chi_E * \chi_{-E})^{-1}(|E|/2, 3|E|/2)$. Then, because this is open, there exists an open ball, i.e. an open interval around 0.

(c)

Let $\{t_n\}$ be any sequence with $t_n \to 0$. Consider $\chi_E * \chi_{-E}(t_n)$. In integral form, this looks like $\int_{\mathbb{R}} \chi_E(y)\chi_{-E}(t_n - y)$. For the integrand to be non-0, we notice that $y \in E$, and $t_n - y \in -E \implies y - t_n \in E \implies y \in E + t_n$. In reverse, we see that if $y \in E \cap E + t_n$, then $y \in E$, and by definition, $y - t_n \in E$, so $t_n - y \in -E$. Thus, we have that $\chi_E(y)\chi_{-E}(t_n - y) = \chi_{E\cap E+t_n} y$. Now, we apply this to our sequence, using the fact that the convolution is contininuous. We have that:

$$\chi_E * \chi_{-E}(t_n) = \int_{\mathbb{R}} \chi_{E\cap E+t_n} y dy = |E \cap E + t_n|$$

But also, we know that:

$$\chi_E * \chi_{-E}(0) = \int_{\mathbb{R}} \chi_E(y)\chi_{-E}(-y) = |E|$$

Since continuous functions preserve convergence, and that the choice of $t_n$ was arbitrary, we have that $\lim_{t\to 0} |E \cap (E+t)| = |E|$.

Let $\epsilon > 0$ be given. Since $0 < |E| < \infty$, we have that $\chi_E, \chi_{-E} \in L^1(\mathbb{R})$. Since we know the convolution is closed with respect to integrable functions, $\chi_E * \chi_{-E} \in L^1$. Then, there exists a function $f \in C_c(\mathbb{R})$ such that $\|f - \chi_E * \chi_{-E}\|_1 < \epsilon$. In particular, since $f$ has compact support, take its support to be contained within $[-M, M]$ and look at $\int_{\mathbb{R}\setminus[-M,M]} |f(x) - \chi_E * \chi_{-E}(x)|dx$. In particular, here, $f = 0$, so we have that $\int_{\mathbb{R}\setminus[-M,M]} \chi_E * \chi_{-E}(x)dx < \epsilon$. Since indicator functions are non-negative, their integral is non-negative, so we have that for $\int_{\mathbb{R}\setminus[-M,M]} \chi_E * \chi_{-E}(x)dx < \epsilon$, then there exists a $K$ such that for almost every $x \geq K$, $\chi_E * \chi_{-E}(x) < \epsilon$. But, earlier, we found that $\chi_E * \chi_{-E}(x)$ can be viewed as $|E \cap E + x|$, which we've bounded by $\epsilon$. Since we can always find a continuous function with compact support small enough, we can find a lower bound for $x$ such that $|E \cap E + x| < \epsilon$

□

## 2.2

**Problem 5.1.5.** Prove that the Cantor-Lebesgue function is Hölder continuous for $0 < \alpha \leq \log_3 2$. In particular, notice that it is not Lipschitz.

*Solution.* First, we notice that for $\phi_k$ in the construction of the Cantor Lebesgue function, that the iteration of the Cantor set, $C_k$ at that point has measure $(2/3)^k$, and therefore, has only sections of slope $(3/2)^k$ and constants, where we get $(3/2)^k$ because the function must traverse from $\phi_k(0) = 0$ to $\phi_k(1) = 1$. Then, we have that $|\phi_k(x) - \phi_k(y)| \leq (3/2)^k|x - y|$, because wlog, suppose $x < y$. Because $\phi_k$ is composed of sections of slope $(3/2)^k$ and slope 0, the line joining $x, y$ must have slope between $0 < m < (3/2)^k$.

Now, consider $\|\phi - \phi_k\|_u$. This can differ at most on $C_k$, since the agree and are equal on the complement of $C_k$. In particular, we have that $\|\phi - \phi_k\|_u \leq 2^{-k}$, since the distance between constant lines on the complement of $C_k$ is $2^{-k}$. Then, we look at:

$$|\phi(x)-\phi(y)| \leq |\phi(x)-\phi_k(x)+\phi_k(x)-\phi_k(y)+\phi_k(y)-\phi(y)| \leq |\phi(x)-\phi_k(x)|+|\phi_k(x)-\phi_k(y)|+|\phi_k(y)-\phi(y)| \leq$$

$$2 * 2^{-k} + |\phi_k(x) - \phi_k(y)| \leq 2 * 2^{-k} + \left(\frac{3}{2}\right)^k |x - y| = 2^{-k}(2 + 3^k|x - y|)$$

However, the choice of $k$ was arbitrary. In particular, we may choose $k, \phi_k$ dependent on $|x - y|$ such that $3^{k-1} \leq |x - y| < 3^k$. Then, we notice that $2 + 3^k|x - y| \leq 7/3$, and that $2^{-k} = 3^{-k\log_3 2} = (3^{-k})^{\log_3 2}$ so $|x - y| \geq 3^{-k}, \log_3 2 > 0 \implies |x - y|^{\log_3 2} \geq (3^{-k})^{\log_3 2} = 2^{-k}$. Then:

$$2^{-k}(2 + 3^k|x - y|) \leq 2^{-k}\frac{7}{3} \leq \frac{7}{3}|x - y|^{\log_3 2} \implies |\phi(x) - \phi(y)| \leq \frac{7}{3}|x - y|^{\log_3 2}$$

Further, since $|x - y| \leq 1$, and for any $0 < x \leq 1$, $a < b \implies x^a \geq x^b$, for any $0 < \alpha \leq \log_3 2$,

$$|\phi(x) - \phi(y)| \leq \frac{7}{3}|x - y|^{\log_3 2} \leq \frac{7}{3}|x - y|^{\alpha}$$

Thus, $\phi$ is Hölder continuous for $0 < \alpha \leq \log_3 2$.

However, we see that $\phi$ cannot be Lipschitz. In partiular, take, on the construction of $\phi_k$, the first interval in $C_k$ of the form $(1/3)^k$. We look at points that look like $\phi(0), \phi((1/3)^k)$. We notice that although $(1/3)^k$ belongs to the Cantor set, because we construct $\phi_k$ by making it continuous, the value of $\phi((1/3)^k)$ aligns with that of the middle third removed on $((1/3)^k, 2(1/3)^k)$. Then, we have that $\phi(0) = 0, \phi((1/3)^k) = 2^{-k}$. Then, we notice that:

$$\frac{\phi((1/3)^k) - \phi(0)}{(1/3)^k} = \frac{2^{-k}}{3^{-k}} = \frac{3^k}{2^k} = \left(\frac{3}{2}\right)^k$$

.

That is, since $3/2 > 1$, unbounded with respect to $k$. Then, suppose we claimed that $K$ were a Lipschitz constant such that $|\phi(x) - \phi(y)| \leq K|x - y|$ for all $x, y \in [0, 1]$. Then, for $x \neq y$, we could look at $|\phi(x) - \phi(y)|/|x - y| \leq K$, and simply choose $k_0$ such that $(3/2)^{k_0} > K$. Then, we would have:

$$\frac{\phi((1/3)^{k_0})}{(1/3)^{k_0}} = \left(\frac{3}{2}\right)^{k_0} > K$$

a contradiction. Thus, no such constant $K$ may exist.

$\square$

**Problem 5.1.7.** Let $C$ be the Cantor set, let $\phi$ be the Cantor-Lebesgue function, and define $g(x) = \phi(x)+x$ for $x \in [0, 1]$.

(a) Prove that $g : [0,1] \to [0,2]$ is continous, strictly increasing, and a bijection. Further, its inverse $h = g^{-1} : [0,2] \to [0,1]$ is also a continuous, strictly increasing, bijection.

(b) Show that $g(C)$ is a closed subset of $[0,2]$ and that $|g(C)| = 1$.

(c) Since $g(C)$ has positive measure, it follows that there exists $N \subseteq g(C)$ such that $N$ is not Lebesgue measurable. Show that $A = h(N)$ is a Lebesgue measurable subset of $[0,1]$.

(d) Set $f = \chi_A$. Prove that $f \circ h$ is not a Lebesgue measurable function.

*Solution.* (a)

Firstly, we already have that $\phi$ is continuous, and $f = x$ is continuous, therefore $g$ is a sum of continuous functions, thus continuous. Further, it must be strictly increasing, since $\phi$ is monotone increasing, and $f = x$ is strictly increasing, so for $x' > x$, we have that $\phi(x') \geq \phi(x)$, so $\phi(x') + x' > \phi(x) + x$. Because it is strictly increasing, it must be injective, as otherwise, suppose $x \neq x'$, but $g(x) = g(x')$. Well, wlog, $x > x'$, which then implies $g(x) > g(x')$, a contradiction. Lastly, we see that since $g(0) = \phi(0) + 0 = 0$, and $g(1) = \phi(1) + 1 = 2$, that we may apply the intermediate value theorem to see that $g$ is surjective.

Now, we look at $h = g^{-1}$. Firstly, $h$ must be a bijection because $g$ is a bijection. We see that because for every $y \in [0,1]$, we just take the unique $g(y)$ to be the element that maps via $h(g(y)) = y$. Now, because $[0,1]$ is compact, we notice that every closed subset $F \subseteq [0,1]$ is compact as well. Consider $h^{-1}(F)$. Because $g$ is injective, we can say that this is exactly the set $g(F)$, since we see that $h \circ g(F) = F$, where we have set equality due to the bijection. But, because $g$ is continous, and $F$ is compact, $g(F)$ is compact in $[0,2]$, and closed. Then, $h$ is continous. Moreover, $h$ must be strictly increasing, since take $x, y \in [0,2] : x > y$, and consider $g(h(x)), g(h(y))$. Since this acts via identity due to being inverse functions, we see that $g(h(x)) = x > y = g(h(y))$. But, because $g$ is strictly increasing, this implies that $h(x) > h(y)$.

(b)

Because the Cantor set is closed, and $[0,1]$ bounded, the Cantor set is in fact compact. Then, since $g$ continous, $g(C)$ must be compact, and thus closed. Now, we look at the complement of the Cantor set in $[0,1]$. Without writing it explicitly, we notice that this looks like intervals $(1/3, 2/3)$, etc. Consider $g((1/3, 2/3))$. Breaking this up into the identity part and the $\phi$ part, we notice that since $\phi$ is constant on the complement of the Cantor set, we can fix a $\phi(1/2)$, and say that $g((1/3,2/3)) = (1/3,2/3) + \phi(1/2)$. However, we notice then that $|g((1/3,2/3))| = |(1/3,2/3) + \phi(1/2)| = |(1/3,2/3)|$, by the translation invariance of the measure. Since this is actually true for each interval in the complement of $C$ in $[0,1]$, we find that $|g([0,1] \setminus C)| = |[0,1] \setminus C| = 1$, and further, being open intervals, it is a measurable subset. Then, since $C, [0,1] \setminus C$ partition $[0,1]$, and because $g$ is a bijection, then $g(C), g([0,1] \setminus C)$ partition $[0,2]$, so we have that $2 = |[0,2]| = |g(C)| + |g([0,1] \setminus C)| = |g(C)| + 1$. Thus, $|g(C)| = 1$.

(c)

This should be clear. Because $g, h$ bijective inverses, since $N \subseteq g(C)$, then we notice that $g(h(N)) =\subseteq g(C)$. Due to the injectivity of $g$ then, $h(N) \subseteq C$. But, $|C| = 0$, so by the monotonicity of the outer measure, $|h(N)| \leq |C| = 0 \implies |h(N)| = 0$. Then, since $h(N)$ has outer measure 0, it is Lebesgue measurable.

(d)

Consider $\{f \circ h \geq 1/2\}$, or really, any real value in $(0,1]$. Because of the shape of $f = \chi_A$, this takes on non-0 values only when $h(x) \in A$. But, due to the definition of $A = h(N)$, and $h$ being injective, $h(x) \implies x \in N$. Then, we have that $\{f \circ h \geq 1/2\} = N$. But $N$ is not measurable, therefore $f \circ h$ is not a measurable function.

$\square$