Homework #3

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Math 285: Homework #3

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Question 1. Let σ be the permutation $(1\ 3\ 2)$ and let τ be the permutation $(1\ 2)$. If f is a 3-linear function on the vector space V, compute using the definition of the permutation action $(\sigma(\tau f))(v_1, v_2, v_3)$ and $((\sigma\tau)f)(v_1, v_2, v_3)$, for $v_i \in V$.

Solution. We recall, that by definition, we have that $(\sigma f)(v_1, v_2, v_3) = f(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)})$. Thus, we have that:

$$(\sigma(\tau f))(v_1, v_2, v_3) = (\sigma f)(v_2, v_1, v_3) = f(v_1, v_3, v_2)$$

On the other hand, we can see that by computation, that $(\sigma\tau) = (2\ 3)$. Therefore, we have that:

$$((\sigma\tau)f)(v_1, v_2, v_3) = f(v_1, v_3, v_2)$$

And we see that we may compute the composition in S_3 of $\sigma\tau$ first or apply them onto f one by one, and the result aligns.

$\square$

Question 2. Prove Proposition 5.14:

Let $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ be C^∞ atlases for manifolds M, N of dimension m, n respectively. Prove that the collection

$$\{U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n\}$$

of charts is a C^∞ atlas on $M \times N$, making this a C^∞ manifold of dimension $m + n$.

Solution. Denote $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$, $\mathfrak{V} = \{(V_\beta, \psi_\beta)\}, \mathfrak{U} \times \mathfrak{V} = \{U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n\}$.

First, we wish to show that for any $(p, q) \in M \times N$, there exists a $U_\alpha \times V_\beta$ in the charts of $\mathfrak{U} \times \mathfrak{V}$ such that $(p, q) \in U_\alpha \times V_\beta$. But this is clear. Since $\mathfrak{U}$ is a C^∞ atlas on M, there exists a U_p such that $m \in U_p$. Similarly, we may make the same argument to find a $V_q \subseteq N$ such that $q \in V_q$. Then, of course, $(p, q) \in U_p \times V_q$, and by the construction of $\mathfrak{U} \times \mathfrak{V}$, $U_p \times V_q$ is an open set for one of the charts in this atlas. Thus, this collection of open sets and maps covers all of $M \times N$.

Now, we wish to show that these pairs are truly charts for $M \times N$. From the above, for a point (p, q), we already know that we may find a $(U_p \times V_q, \phi_p \times \psi_q)$ such that $(p, q) \in U_p \times V_q$. We need only show then that $\phi_p \times \psi_q$ is a homeomorphism from $U_p \times V_q$ to some open subset of $\mathbb{R}^{m+n}$. Since $\mathfrak{U}$ is an atlas, (U_p, ϕ_p) is a chart, and we may consider the open set $\phi_p(U_p) \subseteq \mathbb{R}^m$. Similarly for $\mathfrak{V}$, we may consider the open set $\psi_q(V_q) \subseteq \mathbb{R}^n$. Then, we may consider the open set $\phi_p(U_p) \times \psi_q(V_q) \subseteq \mathbb{R}^m \times \mathbb{R}^n$.

We claim that $\phi_p \times \psi_q$ is a homeomorphism from $U_p \times V_q$ to $\phi_p(U_p) \times \psi_q(V_q)$. Here, we use without proof that $f : X \to X'$ and $g : Y \to Y'$ if and only if $f \times g : X \times Y \to X' \times Y'$ is continuous. Since ϕ_p is continous, and ψ_q is contiuous, so must be $\phi_p \times \psi_q$. Further, we will show that this is bijective by showing that $\phi_p^{-1} \times \psi_q^{-1}$ acts as a left and a right inverse.

Let $x \in U_p, y \in V_p$.

$$(\phi_p^{-1} \times \psi_q^{-1}) \circ (\phi_p \times \psi_q)(x, y) = (\phi_p^{-1} \times \psi_q^{-1})(\phi_p(x), \psi_q(y)) = (\phi_p^{-1}(\phi_p(x)), \psi_q^{-1}(\psi_q(y))) = (x, y)$$

where we have used the fact that ϕ_p, ψ_q are homemorphic, and thus bijective.

Let $a \in \phi_p(U_p), b \in \psi_q(V_q)$:

$$(\phi_p \times \psi_q) \circ (\phi_p^{-1} \times \psi_q^{-1})(a, b) = (\phi_p \times \psi_q)(\phi_p^{-1}(a), \psi_q^{-1}(b)) = (\phi_p(\phi_p^{-1}(a)), \psi_q(\psi_q^{-1}(b))) = (a, b)$$

where, again, we've used the bijectivity of ϕ_p, ψ_q.

Thus, we have shown that $\phi_p \times \psi_q$ is a continuous bijection, with $\phi_p^{-1} \times \psi_q^{-1}$ acting as $(\phi_p \times \psi_q)^{-1}$, and by the same argument on ϕ_p^{-1}, ψ_q^{-1} being continous implying that $\phi_p^{-1} \times \psi_q^{-1}$ being continuous, we can conclude that $\phi_p \times \psi_q$ is a homeomorphism.

Further, if we map $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{m+n}$ by sending $[(x^1, ..., x^m), (y^1, ..., y^n)] \mapsto (x^1, ..., x^m, y^1, ..., y^n)$, we can see that these are isomorphic as topological spaces, and we can identify $\phi_p(U_p) \times \psi_q(V_q)$ as an open set in $\mathbb{R}^{m+n}$. Since we may do this procedure for each $(p, q) \in M \times N$, we can conclude that each $(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)$ forms a chart on $M \times N$, with dimension $m + n$. Thus, our pairs of open neighborhoods and maps form charts on $M \times N$.

We have already shown that these are charts that cover $M \times N$, and that $M \times N$ is a locally Euclidean space of dimension $m + n$. We need only show then that these charts are pairwise C^∞ compatible.

Let $U_a \times V_b, U_{a'} \times V_{b'}$ be open sets of charts such that their intersection is not empty.

Let $(u, v) \in U_a \times V_b \cap U_{a'} \times V_{b'}$ such that $u \in U_a \cap U_{a'}, v \in V_b \cap V_{b'}$.

Consider:

$$(\phi_a \times \psi_b) \circ (\phi_{a'}^{-1} \times \psi_{b'}^{-1})((\phi_{a'} \times \psi_{b'})(u, v) = (\phi_a \circ \phi_{a'}^{-1}(\phi_{a'}(u)), \psi_b \circ \psi_{b'}^{-1}(\psi_{b'}(v)))$$

Because $\mathfrak{U}$ is a set of C^∞ compatible maps, we must have that $\phi_a \circ \phi_{a'}^{-1}$ is C^∞ at $\phi_{a'}(u)$. Similarly, since $\mathfrak{V}$ is also C^∞ pairwise compatible, $\psi_b \circ \psi_{b'}^{-1}$ must be C^∞ at $\psi_{b'}(v)$. Thus, since the components are C^∞, so must be $(\phi_a \times \psi_b) \circ (\phi_{a'}^{-1} \times \psi_{b'}^{-1})$ at $(\phi_{a'} \times \psi_{b'})(u, v) = (\phi_{a'}(u), \psi_{b'}(v))$.

Without too much trouble, we can see the same will occur with:

$$(\phi_{a'} \times \psi_{b'}) \circ (\phi_a^{-1} \times \psi_b^{-1})((\phi_a \times \psi_b)(u, v)) = (\phi_{a'} \circ \phi_a^{-1}(\phi_a(u)), \psi_{b'} \circ \psi_b^{-1}(\psi_b(v)))$$

and with the same argument, because each coordinate transition map $\phi_{a'} \circ \phi_a^{-1}, \psi_{b'} \circ \psi_b^{-1}$ is C^∞ at $\phi_a(u), \psi_b(v)$ respectively, due to the C^∞ compatibility of $\mathfrak{U}, \mathfrak{V}$, so too must be $(\phi_{a'} \times \psi_{b'}) \circ (\phi_a^{-1} \times \psi_b^{-1})$.

Thus, $(U_a \times V_b, \phi_a \times \psi_b)$ and $(U_{a'} \times V_{b'}, \phi_{a'} \times \psi_{b'})$ are C^∞ compatible maps, and since the choice of charts were arbitrary, other than having non-empty intersection, this is true for all charts with non-empty intersection. Since we say that if the intersection is empty, that the charts are automatically compatible, this implies that every pair of charts is C^∞ compatible. Thus, this collection named $\mathfrak{U} \times \mathfrak{V}$ is a collection of C^∞ compatible charts on $M \times N$, a locally Euclidean space of dimension $m + n$, such that the charts cover $M \times N$. Therefore, this collection is a C^∞ atlas on $M \times N$, and we can find a maximal atlas, compatible with $\mathfrak{U} \times \mathfrak{V}$ endowing $M \times N$ with the structure of a C^∞ manifold with dimension $m + n$.

Note that without too much extra trouble, it is clear that this is Hausdorff and second countable:

Let $(p, q), (p', q') \in M \times N$ such that at least one of $p \neq p', q \neq q'$ is true. WLOG, suppose $p \neq p'$. Since M is a manifold, M is Hausdorff, so we may find neighborhoods $U_p, U_{p'}$ such that $p \in U_p, p' \in U_{p'}$ and $U_p \cap U_{p'} = \emptyset$. Then, consider the sets $U_p \times N, U_{p'} \times N$. Of course, this is an open set in the product topology, and further, due to construction, it should be clear that $(p, q) \in U_p \times N, (p', q') \in U_{p'} \times N$ and the intersection being trivial implies that $(U_p \times N) \cap (U_{p'} \times N) = \emptyset$. Since we can see that this works similar if $p = p'$ and $q \neq q'$ we can do this for every point, and thus $M \times N$ is Hausdorff.

Similarly, due to the construction of the product topology, we can find second countable to be true because take a countable basis of M as U_i and a countable basis of N as V_j. Then, $\{U_i \times V_j\}_{i,j\in\mathbb{N}}$ is a basis for $M \times N$, and $\mathbb{N} \times \mathbb{N}$ is still countable.

□

Question 3. Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas of the chart $\{\mathbb{R}, \phi = \mathbb{1} : \mathbb{R} \to \mathbb{R}\}$. Let $\mathbb{R}'$ be the real line with the differentiable structure given by the maximal atlas of the chart $\{\mathbb{R}, \psi : \mathbb{R} \to \mathbb{R}\}$ via $\psi(x) = x^{1/3}$.

(a) Show that these two differentiable structures are distinct.

(b) Show that there exists a diffeomorphism from $\mathbb{R} \to \mathbb{R}'$.

Solution. (a)

If these charts are not C^∞ compatible, then the maximal atlas for $\mathbb{R}, \mathbb{R}'$ must differ. So, we need only show that $(\mathbb{R}, \mathbb{1})$ is not compatible with $(\mathbb{R}, \psi)$ with $\psi(x) = x^{1/3}$.

In particular, for $x \in \mathbb{R}$, consider the function $\psi \circ \mathbb{1}^{-1} : \mathbb{R} \to \mathbb{R}$. We have that:

$$\psi \circ \mathbb{1}^{-1}(x) = \psi((\mathbb{1}^{-1})(x)) = \psi(x) = x^{1/3}$$

However, this map is not C^∞ at $x = 0$. Taking the familiar derivative via the power rule, we see that $\frac{d}{dx}x^{1/3} = 1/3x^{-2/3}$, which is undefined at $x = 0$. Thus, these charts are not compatible, and must belong to different maximal atlases, and thus have distinct differentiable structures.

(b)

Consider the map $F : \mathbb{R} \to \mathbb{R}'$ that sends $x \to x^3$. We recall that we say F is C^∞ at a point $y \in \mathbb{R}$ if for generic charts (U, ϕ) of $p \in M$, and (V, ψ) of $F(p) \in N$, that $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

Applying this for $(U, \phi) = (\mathbb{R}, \mathbb{1})$, $(V, \psi) = (\mathbb{R}, \psi)$, we consider the following for an arbitrary $p \in \mathbb{R}$, since $\mathbb{1}(\mathbb{R}) = \mathbb{R}$:

$$\psi \circ F \circ \mathbb{1}^{-1}(p) = \psi \circ F(\mathbb{1}^{-1}(p)) = \psi(F(p)) = \psi(p^3) = p$$

Since this acts as identity on p on $\mathbb{R} \to \mathbb{R}$, certainly this is C^∞, so we call F a C^∞ at p. Since p was an arbitrary point, we can say that F is a C^∞ map.

Then, we consider the map $F^{-1} : \mathbb{R}' \to \mathbb{R}$ that sends $y \to y^{1/3}$. Without too much trouble, we can see that this function acts as a left and right inverse:

$$F \circ F^{-1} : \mathbb{R}' \to \mathbb{R}' \text{ has the action of } F \circ F^{-1}(y) = F(y^{1/3}) = (y^{1/3})^3 = y$$

and

$$F^{-1} \circ F : \mathbb{R} \to \mathbb{R} \text{ has the action of } F^{-1} \circ F(x) = F^{-1}(x^3) = (x^3)^{1/3} = x$$

where we use the fact that the cube root and cubing a real number are bijective functions and inverses of each other on $\mathbb{R} \to \mathbb{R}$.

Now, we wish only to show that F^{-1} is also C^∞. Since $\psi(\mathbb{R}) = \mathbb{R}$, consider, for some arbitrary $q \in \mathbb{R}$, the function:

$$\mathbb{1} \circ F^{-1} \circ \psi^{-1}(q) = \mathbb{1} \circ F^{-1}(q^3) = \mathbb{1}(q^3)^{1/3} = \mathbb{1}(q) = q$$

Thus, since $\mathbb{1} \circ F^{-1} \circ \psi^{-1}$ acts as identity on $\mathbb{R}$, we can also claim that F^{-1} is C^∞ at q. Since the choice of q were arbitrary, we can actually say that F^{-1} is C^∞.

Thus, F is a bijective C^∞ map such that its inverse is also C^∞. Therefore, F is a diffeomorphism from $\mathbb{R} \to \mathbb{R}'$.

□

Question 4. Let M, N be manifolds, and fix some $q_0 \in N$. Prove that the inclusion map $i_{q_0} : M \to M \times N$ via $p \mapsto (p, q_0)$ is C^∞.

Solution. Fix an arbitrary point $p \in M$. Let $(U, \phi) = (U, x^1, ..., x^m)$ be a chart of M about p. Let $(V, \psi) = (V, y^1, ..., y^n)$ be any chart about $q_0 \in N$.

By Proposition 5.14, or problem 2, we have that $(U \times V, \phi \times \psi) = (U \times V, x^1, ..., x^m, y^1, ..., y^n)$ is a chart of $M \times N$ about the point (p, q_0). Then, we have that:

$$((\phi \times \psi) \circ i_{q_0} \circ \phi^{-1})(a^1, ..., a^m) = (a^1, ..., a^m, b^1, ..., b^n)$$

for $\psi(q_0) = (b^1, ..., b^n), \phi(p) = (a^1, ..., a^m)$.

We can see that each coordinate map is C^∞, as it acts as identity on the x^i coordinates and is constant on the y^j coordinates. Thus, the composite map is C^∞ at the point $(a^1, ..., a^m)$, which by definition, says that i_{q_0} is C^∞ at the point $p = \phi^{-1}(a^1, ..., a^m)$. Since the choice of p were arbitrary, this implies that i_{q_0} is C^∞ on all of M, and thus this choice of inclusion map is C^∞. Since of course, $q_0 \in N$ was arbitrary, we can more generally conclude that inclusion maps for any fixed q_0 are C^∞.

□

Question 5. Let $f : X \to Y$ be a map of sets, and let $B \subseteq Y$. Prove that $f(f^{-1}(B)) = B \cap f(X)$. Conclude that if f is surjective, then $f(f^{-1}(B)) = B$.

Solution. First, we wish to show that $f(f^{-1}(B)) \subseteq B \cap f(X)$. We recall that by definition, $f^{-1}(B) = \{x \in X : f(x) \in B\}$. Then, of course, $f^{-1}(B) \subseteq X$, because for any $y \in f(f^{-1}(B))$, by definition, there exists an $x_b \in f^{-1}(B)$ such that $f(x_b) = y$. But, since $f^{-1}(B) \subseteq X$, $x_b \in X$, and thus $y \in f(X)$. Since y was arbitrary, we must have that $f(f^{-1}(B)) \subseteq f(X)$. Further, let $x \in f^{-1}(B)$. Then, by definition, we have that $f(x) \in B$. Since this is true for an arbitrary element of $x \in f^{-1}(B)$, this is true for the entire set, and we have that $f(f^{-1}(B)) \subseteq B$. Therefore, we have the desired conclusion $f(f^{-1}(B)) \subseteq B \cap f(X)$.

Next, we show that $B \cap f(X) \subseteq f(f^{-1}(B))$. Let $y \in B \cap f(X) \subseteq Y$. Since $y \in f(X)$, there exists a $x \in X$ such that $f(x) = y$. Furthermore, since $y \in B$, by the definition of $f^{-1}(B) = \{x \in X : f(x) \in B\}$, $x \in f^{-1}(B)$. Then, of course, $y \in f(f^{-1}(B))$, since we have found an $x \in f^{-1}(B)$, such that $f(x) = y$.

Since the choice of y was arbitrary, this applies to all of the elements of the intersection, and thus we conclude that $B \cap f(X) \subseteq f(f^{-1}(B))$.

Thus, since we have subsets in both directions, we have equality, and we conclude that $f(f^{-1}(B)) = B \cap f(X)$.

Of course, then, if f is surjective, then $f(X) = Y$, and since $B \subseteq Y$, of course $B \cap Y = B$. In such a case then, we have that $f(f^{-1}(B)) = B \cap f(X) = B \cap Y = B$.

□