

# Homework #3

Eric Tao  
Math 233: Homework #3

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**Question 1.** Let  $u$  be a harmonic function on a region  $\Omega$ . What can we say about the set of points such that  $\nabla u = 0$ , that is, the set of points where  $u_x = u_y = 0$ ?

*Solution.* Recall that if  $u$  is a real harmonic function, then we may identify it as the real part of a holomorphic function  $f(x, y) = u(x, y) + iv(x, y)$  locally. Suppose  $u_x = u_y = 0$ . Then, by the Cauchy-Riemann equations, we have that at these points,  $v_x = v_y = 0$ . Further, identifying  $f'(z) = \partial f(z)$  for  $\partial = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$ , we have that:

$$f'(z) = \partial f(z) = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (u + iv) = \frac{1}{2} [(u_x + v_y) + i(v_x - u_y)]$$

So, we have that at points where  $u_x = u_y = 0$ , we have that  $f'(z) = 0$ . But, since  $f$  is holomorphic on this neighborhood, so is  $f'$ . Therefore,  $\{(x, y) : \nabla u(x, y) = 0\}$  is either all of the neighborhood, or has no limit points. Since  $\Omega$  is a region, we can always patch our entire region with overlapping neighborhoods, so this extends to all of  $\Omega$ .

Now, if  $u$  is a complex-valued harmonic function, we simply identify it as  $u = w + iv$ , where  $w, v$  are the real and imaginary portions. It should be clear that if  $u$  is harmonic, so must  $w, v$  as:

$$u_{xx} + u_{yy} = w_{xx} + iv_{xx} + w_{yy} + iv_{yy} = (w_{xx} + w_{yy}) + i(v_{xx} + v_{yy}) = 0 \implies w_{xx} + w_{yy} = 0, v_{xx} + v_{yy} = 0$$

Then, suppose  $u_x = u_y = 0$ . At such points, we would have that  $u_x = w_x + iv_x = 0, u_y = w_y + iv_y = 0 \implies w_x = w_y = 0, v_x = v_y = 0$ . But, by the previous work, since  $v, w$  are real harmonic functions, they either have no limit points, or are the full space. It should be clear then, that the set of points where  $\nabla u = 0$  is simply the union of these sets. It too may only be the full space or not have limit points, as if it did, then we could construct a subsequence of points coming from either the set where  $\nabla v = 0$ , or  $\nabla w = 0$ , which would imply that the original set had a limit point, a contradiction.  $\square$

**Question 2.** Let  $u, v$  be real harmonic functions on a plane region  $\Omega$ . Under what conditions is  $uv$  harmonic?

Further, show that  $u^2$  may not be harmonic on  $\Omega$ , unless  $u$  is constant.

Further, for which  $f \in \mathcal{H}(\Omega)$  is  $|f|^2$  harmonic?

*Solution.* We start by proving that if we take the Laplacian of  $uv$ ,  $\Delta(uv)$ , then this is equal to  $2\nabla u \cdot \nabla v$ :

$$\Delta(uv) = (uv)_{xx} + (uv)_{yy} = (u_x v + u v_x)_x + (u_y v + u v_y)_y = u_{xx} v + u_x v_x + u_x v_x + u v_{xx} + u_{yy} v + u_y v_y + u_y v_y + u v_{yy}$$

Because  $u, v$  are harmonic, we know that  $u_{xx} + u_{yy} = 0, v_{xx} + v_{yy} = 0$ , so:

$$= v(u_{xx} + v_{xx}) + 2u_x v_x + u(v_{xx} + v_{yy}) + 2u_y v_y = 2(u_x v_x + u_y v_y) = 2\langle u_x, u_y \rangle \cdot \langle v_x, v_y \rangle = 2\nabla u \cdot \nabla v$$

Here, it should be clear then that if  $u^2$  is not constant, then  $u^2$  is not harmonic. We have that  $\Delta(u^2) = \Delta(uu) = 2\nabla u \cdot \nabla u = 2|\nabla u|^2$ . So, suppose  $u$  is harmonic, then for  $\Delta(u^2) = 0$ , this implies that  $|\nabla u| = 0$  for all  $z \in \Omega$ . However, this implies immediately that  $u$  is constant, and we have the contrapositive.

Now, of course, if  $u$  or  $v$  is constant, suppose  $u = a$  is constant, then of course  $uv = av$  is harmonic, being a scalar multiple of a harmonic function. So, assume  $u, v$  both non-constant.

Define the set  $A = \{z \in \Omega : \nabla u(z) = 0 \text{ or } \nabla v(z) = 0\}$ . By the first problem, we know that neither of those sets have limit points in  $\Omega$ . Since both of those are closed conditions,  $A$  is the union of two closed sets, and thus closed. Thus, consider  $\Omega' = \Omega \setminus A$ .

This is an open set, of course, being open minus closed, or equivalently, open intersect open. Further, it must be connected, since the points of  $A$  have no limit points, and are at most countable. Suppose  $x, y \in \Omega'$ , and consider a path between them in  $\Omega$ . This may have at most countably many disconnections when we move to  $\Omega'$ . Since  $A$  has no limit points, we may restrict down into a small enough punctured disk around any connection and take a path there - this punctured disk must be completely contained within  $\Omega'$  due to  $A$  having no limit points. Since we have merely countably many of these issues, we are assured that we can patch this. Finally, this must be dense because let  $U$  be any open set in  $\Omega$ . Choose any  $a \in U$ . There exists a disk  $D(a, r) \subset U$ , with uncountable cardinality. But,  $A$  is merely countable, thus  $D(a, r) \setminus A \neq \emptyset$ . Thus, since  $A \cup \Omega' = \Omega$ , we must have that  $D(a, r) \cap \Omega' \neq \emptyset$ . Thus, we have that  $\Omega'$  is a region.

Now, we have that since  $\Delta(uv) = 0$ , we must have that  $u_x v_x + u_y v_y = 0 \implies u_x v_x = -u_y v_y$ . Since we wish  $uv$  to be harmonic, this must hold for all  $z \in \Omega'$ , which leads us to two cases, since  $u_x, u_y, v_x, v_y \neq 0$  on  $\Omega'$ :

Case 1:

$$\begin{cases} v_x = -\lambda u_y \\ v_y = \lambda u_x \end{cases}$$

It should be clear that due to the definition of  $\Omega'$ , that  $\lambda \neq 0$ . In particular, since  $u, v$  are harmonic on  $\Omega$ , they are continuous on all of  $\Omega$ , with continuous first derivatives. Thus, these must actually hold for all of  $\Omega$ , since  $u_x, u_y, v_x, v_y$ . Thus, we can say that the function

$$f = \lambda u + iv$$

is holomorphic, since these are exactly the Cauchy-Riemann equations for  $u' = \lambda u, v' = v$ . Thus, in this case,  $uv$  is harmonic if we may find a  $\lambda$  such that  $u, v$  are real and imaginary parts of a holomorphic function.

Case 2:

$$\begin{cases} u_x = -\lambda u_y \\ v_y = \lambda v_x \end{cases}$$

Consider the first equation. This implies that  $u_{xx} = -\lambda u_{yx}$  and  $u_{yy} = -\frac{1}{\lambda} u_{xy}$ . Thus, in such a case, since  $u$  is harmonic, we must have that:

$$u_{xx} + u_{yy} = 0 \implies -\lambda u_{yx} - \frac{1}{\lambda} u_{xy} = 0 \implies u_{xy} = 0$$

Similarly:

$$v_{xx} + v_{yy} = 0 \implies \lambda v_{yx} + \frac{1}{\lambda} v_{xy} = 0 \implies v_{xy} = 0$$

However, since  $u_x, u_y \neq 0$  on  $\Omega'$ , this implies that  $u_x = f(x)$  since  $u_{xy} = 0$  and  $u_y = g(y)$  since  $u_{yx} = 0$ . Then, we must have that  $u = F(x) + G(y)$  for  $F' = f, G' = g$ , and due to harmonicity, we further have that  $f'(x) + g'(y) = 0$ . This can only be true on all of  $\Omega'$  if  $f', g'$  are constant, which implies that  $F, G$  are at most quadratics. However, since we started with  $u_x = -\lambda u_y$ , this implies that  $F'(x) = -\lambda G'(y)$ , and if  $F, G$  are polynomials, this implies then that  $F', G'$  are constants and thus  $F, G$  are linear. Thus, we have that:

$$u = -\lambda ax + ay + b$$

Running through the same logic with  $v$ , we see that:

$$v = cx + \lambda cy + d$$

However, here, we notice that:

$$\begin{cases} u_x = -\lambda a \\ u_y = a \\ v_x = c \\ v_y = \lambda c \end{cases}$$

Choosing  $\lambda' = -\frac{c}{a}$ , we see that:

$$\begin{cases} -\lambda' u_y = \frac{c}{a} a = c = v_x \\ \lambda' u_x = -\frac{c}{a} \cdot -\lambda a = \lambda c = v_y \end{cases}$$

and thus we are back in case 1. Thus, in either case, we see that  $uv$  is harmonic for  $u, v$  non-constant if there exists a  $\lambda \neq 0$  such that  $\lambda u + iv$  is holomorphic.

Now, let  $f \in \mathcal{H}(\Omega)$ , and consider  $|f|^2$ . Explicitly taking derivatives:

$$\frac{\partial^2}{\partial x^2} |f|^2 = \frac{\partial^2}{\partial x^2} (u^2 + v^2) = \frac{\partial}{\partial x} (2uu_x + 2vv_x) = 2(u_x^2 + uu_{xx} + v_x^2 + vv_{xx})$$

Of course then, the same equation will hold for the  $y$ , just switching the labels. Thus:

$$2(u_x^2 + uu_{xx} + v_x^2 + vv_{xx}) + 2(u_y^2 + uu_{yy} + v_y^2 + vv_{yy}) = 2(u(u_{xx} + u_{yy}) + u_x^2 + u_y^2 + v(v_{xx} + v_{yy}) + v_x^2 + v_y^2) = 2(u_x^2 + u_y^2 + v_x^2 + v_y^2)$$

where we've used the fact that because  $u, v$  come from the real, imaginary parts of a holomorphic function,  $u, v$  are harmonic.

Now, applying the Cauchy-Riemann equations, we obtain:

$$2(u_x^2 + u_y^2 + v_x^2 + v_y^2) = 2(2v_x^2 + 2v_y^2) = 4(v_x^2 + v_y^2) = 4(u_x^2 + u_y^2)$$

However, since  $u$  is a real-valued function, so must be  $u_x, u_y$ . Then, since  $u_x^2, u_y^2 \geq 0$ , for this to be harmonic, we must have  $u_x, u_y = 0$ . But that implies that  $u$  and thus  $v$ , are constants. Thus, we have that  $|f|^2$  is harmonic iff  $f$  is constant.  $\square$

**Question 3.** Suppose  $f$  is a complex function on a region  $\Omega$ , and both  $f, f^2$  are harmonic on  $\Omega$ . Prove that either  $f, \bar{f}$  must be holomorphic on  $\Omega$ .

*Solution.* It is clear that if  $f = a \in \mathbb{C}$ , that is, constant, then  $f, f^2$  are harmonic and  $f, \bar{f}$  are both holomorphic. Thus, we restrict ourselves to  $f$  non-constant.

Now, we see that:

$$\Delta(f^2) = (2ff_x)_x + (2ff_y)_y = 2[f_x^2 + f_y^2] = 2[(f_x + if_y)(f_x - if_y)] = 2\bar{\partial}f\partial f$$

where, as in the text, we identify:

$$\partial = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \bar{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

Where we've used the fact that  $f$  is harmonic to say that  $f(f_{xx} + f_{yy}) = 0$ . Now, since  $f^2$  is harmonic, we have that  $\Delta(f^2) = 0$ , which implies that at every point in  $\Omega$ , either  $\partial f = 0$  or  $\bar{\partial} f = 0$ . Now, consider  $\partial f, \bar{\partial} f$ . In particular, consider the quantity  $\bar{\partial}(\partial f)$ :

$$\bar{\partial}(\partial f) = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \frac{1}{2} (f_x - i f_y) = \frac{1}{4} (f_{xx} + i f_{xy} - i f_{yx} + f_{yy}) = 0$$

That is, for  $f$  harmonic,  $\partial f$  is holomorphic on  $\Omega$ , because the Cauchy-Riemann equations hold. In particular, its zero set is either all of  $\Omega$ , or a countable subset without limit points. If its zero set is all of  $\Omega$ , we are done, since if we expand out  $\partial f$ , we find that:

$$\partial f = \frac{1}{2} (f_x - i f_y) = \frac{1}{2} (u_x + i v_x - i u_y + v_y) = 0 \implies \begin{cases} u_x = -v_y \\ u_y = v_x \end{cases}$$

and thus we have that:

$$\bar{\partial}(\bar{f}) = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u - i v) = \frac{1}{2} (u_x - i v_x + i u_y + v_y) = 0$$

that is,  $\bar{f}$  is holomorphic. Otherwise, suppose  $Z = \{z \in \Omega : \partial f(z) = 0\}$  has no limit points. Since  $f$  harmonic, at least one of  $\partial f, \bar{\partial} f = 0$  so on  $\Omega \setminus Z$ ,  $\bar{\partial} f = 0$ . But, because  $Z$  has no limit points, by continuity,  $\bar{\partial} f = 0$  actually on all of  $\Omega$ , where we know this must be continuous, because it is the linear combination of continuous functions. Then, we would have that:

$$\bar{\partial} f = \frac{1}{2} (f_x + i f_y) = \frac{1}{2} (u_x + i v_x + i u_y - v_y) \implies \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

and thus

$$\bar{\partial} f = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u + i v) = \frac{1}{2} (u_x + i v_x + i u_y - v_y) = 0$$

that is,  $f$  is holomorphic. □

**Question 4.** Let  $\Omega$  be a region, and  $f_n \in \mathcal{H}(\Omega)$  for all  $n$ . Set  $u_n = \Re(f_n)$ , and suppose  $u_n$  converges uniformly on compact subsets of  $\Omega$  and that there exists  $z \in \Omega$  such that  $f_n(z)$  converges. Prove that  $f_n$  converges uniformly on compact subsets of  $\Omega$ .

*Solution.* By hypothesis, there exists a  $z_0 \in \Omega$  such that  $f_n(z_0)$  converges. Since  $\Omega$  is open, we may choose an  $R > 0$  such that  $\bar{D}(z_0, R) \subset \Omega$ , since if the disk  $D(a, r)$  is contained in  $\Omega$ , the closed disk  $\bar{D}(a, r/2)$  is as well.

Since this is a compact set, and  $u_n$  converges uniformly on compact sets, if we set  $u = \lim_{n \rightarrow \infty} u_n(z)$  for  $z \in \bar{D}(z_0, R)$ , by theorem 11.11, we have that  $u$  is harmonic. Since  $u$  is harmonic on  $D(z_0, R)$  and continuous on the boundary, we have that by 11.9 that  $u$  is the real part of a holomorphic function defined by:

$$f(z_0 + z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{Re^{it} + z}{Re^{it} - z} u(z_0 + Re^{it}) dt$$

for  $|z| < R$ .

In the same way, we see that since each  $u_n$  is harmonic on the same disk, we may find a sequence of holomorphic functions  $g_n$  such that:

$$g_n(z_0 + z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{Re^{it} + z}{Re^{it} - z} u_n(z_0 + Re^{it}) dt$$

But,  $u_n$  is also the real part of  $f_n$ , also a holomorphic function. Thus, by 11.10, these holomorphic functions may only differ by an imaginary additive constant, and we may say that there exists  $c_n \in \mathbb{R}$  such that  $f_n = g_n + ic_n$ .

First, we wish to show that  $g_n \rightarrow f$  uniformly for any  $r < R$ , the closed disk  $\overline{D}(z_0, r)$ . Let  $\epsilon > 0$  be given. Well, by definition, we have that for any point  $|z| < r$ :

$$\begin{aligned} |f - g_n| &= \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{Re^{it} + z}{Re^{it} - z} u(z_0 + Re^{it}) dt - \frac{1}{2\pi} \int_0^{2\pi} \frac{Re^{it} + z}{Re^{it} - z} u_n(z_0 + Re^{it}) dt \right| = \\ &= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{Re^{it} + z}{Re^{it} - z} u(z_0 + Re^{it}) dt - \int_0^{2\pi} \frac{Re^{it} + z}{Re^{it} - z} u_n(z_0 + Re^{it}) dt \right| \leq \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{Re^{it} + z}{Re^{it} - z} \right| |u(z_0 + Re^{it}) - u_n(z_0 + Re^{it})| dt \end{aligned}$$

Here, we notice that in terms of moduli,  $|Re^{it} + z| \leq |Re^{it}| + |z| \leq R + r$ , and similarly,  $|Re^{it} - z| \geq R - r$ . Thus, we have the estimate:

$$\left| \frac{Re^{it} + z}{Re^{it} - z} \right| \leq \frac{R + r}{R - r}$$

for all  $|z| < r$ . Further, since  $\overline{D}(a, r)$  is compact, we have that  $u_n \rightarrow u$  uniformly. Then, choose  $N$  such that for all  $n > N$ ,  $|u(z) - u_n(z)| < \epsilon \frac{R-r}{R+r}$ . Then, for any  $n > N$  and for every  $z \in \overline{D}(a, r)$ , we have that:

$$\begin{aligned} |f(z) - g_n(z)| &\leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{Re^{it} + z}{Re^{it} - z} \right| |u(z_0 + Re^{it}) - u_n(z_0 + Re^{it})| dt \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{R + r}{R - r} \epsilon \frac{R - r}{R + r} dt = \\ &= \frac{1}{2\pi} \int_0^{2\pi} \epsilon dt = \frac{1}{2\pi} \epsilon 2\pi = \epsilon \end{aligned}$$

Thus, we have that  $g_n \rightarrow f$  uniformly for every  $\overline{D}(a, r)$ ,  $r < R$ .

Next, we restrict our focus to  $z_0$ . We have that  $f_n = g_n + ic_n$ . Thus, at  $z_0$ , since  $g_n(z_0) \rightarrow f(z_0)$  because of what we showed above, we have that:

$$f(z_0) = \lim_{n \rightarrow \infty} f_n(z_0) = \lim_{n \rightarrow \infty} (g_n(z_0) + ic_n) = \lim_{n \rightarrow \infty} g_n(z_0) + i \lim_{n \rightarrow \infty} c_n = f(z_0) + i \lim_{n \rightarrow \infty} c_n$$

Thus, we have that  $\lim_{n \rightarrow \infty} c_n$  exists and is equal to 0.

Now, we look at any closed disk  $\overline{D}(z_0, r)$ ,  $r < R$  again, and look at  $f_n$  this time. Let  $\epsilon > 0$  be given. We have that:

$$\|f - f_n\| = \|f - g_n - ic_n\| \leq \|f - g_n\| + \|c_n\|$$

Since we have that  $g_n \rightarrow f$  uniformly, there exists  $N_g$  such that for all  $n > N_g$ ,  $\|f - g_n\| < \epsilon/2$ . Since  $c_n \rightarrow 0$  is a sequence of constant numbers, there exists  $N_c$  such that for all  $n > N_c$ ,  $\|c_n\| < \epsilon/2$ . Then, for any  $n > \max(N_g, N_c)$ :

$$\|f - f_n\| \leq \|f - g_n\| + \|c_n\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, we have that  $f_n \rightarrow f$  uniformly for any  $\overline{D}(z_0, r)$ ,  $r < R$ .

Define  $\Omega_1, \Omega_2$  via the following:

$$\Omega_1 = \{z \in \Omega : \{f_n(z)\} \text{ converges} \}$$

$$\Omega_2 = \{z \in \Omega : \{f_n(z)\} \text{ does not converge} \}$$

From what we've shown above,  $\Omega_1$  must be open, since we've shown that there exists a disk around a convergent point  $z_0$  such that for any concentric, closed disk contained within this neighborhood,  $f_n$  converges uniformly on the closed disk.

However, we notice that  $\Omega_2$  must also be open, because we chose a closed disk  $\overline{D}(z_0, R)$  to analyze and the complement of such in  $\Omega$  is open. Thus, the union of such complements is also open. Further, by definition,  $\Omega_1 \cap \Omega_2 = \emptyset$ .

Thus, because  $\Omega$  is a region, we must have that either  $\Omega = \Omega_1$ , or  $\Omega = \Omega_2$ , and by hypothesis, we see that  $z_0 \in \Omega_1 \implies \Omega = \Omega_1$ , and thus  $f_n \rightarrow f$  on all of  $\Omega$ .

Then, the result is clear. Let  $K$  be any compact subset of  $\Omega$ . For each  $k \in K$ , we may find  $r_k > 0$  such that  $D(k, r_k) \subset \Omega$ . Consider  $\cup_k D(k, r_k)$ . Clearly, this is an open cover of  $K$ , so by the compactness of  $K$ , there exists a finite subcover

$$K \subset \cup_{i=1}^n D(k_i, r_{k_i}) \subset \Omega$$

Then, let  $\epsilon > 0$  be given. For each  $i$ , choose  $r_i$  such that  $r_i < r_{k_i}$ , but that  $\cup_{i=1}^n D(k_i, r_i)$  remains a cover of  $K$ . We may do this because of homework 1, finding the minimum distance between  $K$  and the complement of  $\cup_{i=1}^n D(k_i, r_{k_i})$ , a closed set. Then, by the work above, we have that on each  $\overline{D}(k_i, r_i)$ , that since  $f_n \rightarrow f$  uniformly, there exists  $N_i$  such that for all  $n > N_i$ ,  $\|f - f_n\| < \epsilon$  on  $D(k_i, r_i)$ . We notice, that there are only finitely many  $N_i$  and thus it achieves a maximum. Thus, of course, we have that for  $N = \max_i N_i$ , for any  $n > N$ :

$$\|f - f_n\|_K = \|f - f_n\|_{\overline{D}(k_j, r_j)} < \epsilon$$

for some  $k_j, r_j$  since they cover  $K$ . Thus,  $f_n \rightarrow f$  uniformly on compact subsets of  $\Omega$ . □

**Question 5.** Let  $\Omega$  be a region,  $K$  a compact subset of  $\Omega$ , and fix some  $z_0 \in \Omega$ . Let  $u$  be any positive harmonic function. Prove that there exists  $\alpha, \beta > 0$  such that

$$\alpha u(z_0) \leq u(z) \leq \beta u(z_0)$$

for all  $z \in K$ .

If  $\{u_n\}$  is a sequence of positive harmonic functions in  $\Omega$ , and  $u_n(z_0) \rightarrow 0$ , describe the behavior of  $\{u_n\}$  on the rest of  $\Omega$ . Repeat this process for if  $u_n(z_0) \rightarrow \infty$ . Show that  $\{u_n\}$  must be positive.

*Solution.* First, fix some  $z_0 \in \Omega$ . Let  $u$  be any positive harmonic function on  $\Omega$ . Let  $z \in \Omega$  be any other point, and let  $\gamma$  be a path,  $\gamma^* \subset \Omega$  such that  $\gamma(0) = z_0, \gamma(1) = z$ , which exists since  $\Omega$  is connected. Further, assume that  $\gamma$  has a finite length. Such a path must exist. Since the path is a compact set, and the complement of  $\Omega$  is closed, this implies that we may find a  $R > 0$  such that  $D(\zeta, R) \subset \Omega$  for all  $\zeta \in \gamma^*$ .

Now, consider  $\overline{D}(z_0, R/2) \subset \Omega$ . If  $z \in D(z_0, R/3)$ , then we can say that, for  $r = |z - z_0|$ , that:

$$\frac{R/2 - r}{R/2 + r} u(z_0) \leq u(z) \leq \frac{R/2 + r}{R/2 - r} u(z_0) \implies \frac{R/2 - r}{R/2 + r} \leq \frac{u(z)}{u(z_0)} \leq \frac{R/2 + r}{R/2 - r} \implies \frac{1}{5} \leq \frac{u(z)}{u(z_0)} \leq 5$$

Where we use the fact that  $\frac{R/2-r}{R/2+r}$  is a decreasing function, since as  $r \rightarrow R/2$ ,  $R/2 - r$  decreases, and  $R/2 + r$  increases, so the fraction decreases, so it takes on its minimum value at  $r = R/3$ . The same logic applies for the upper bound, as an increasing function.

Otherwise, take the boundary  $\partial D(z_0, R/3)$ . Since  $z$  is not contained within  $D(z_0, R/2) \supset D(z_0, R/3)$ , there exists at least some point  $\zeta \in \gamma^*$  such that  $\zeta \in \gamma^* \cap \partial D(z_0, R/3)$ . If there are multiple such  $\zeta$ , we choose the one corresponding to the largest parameter in  $\gamma(t)$ . Calling this point  $\zeta_1$ , we have that:

$$\frac{R/2 - R/3}{R/2 + R/3} u(z_0) \leq u(\zeta_1) \leq \frac{R/2 + R/3}{R/2 - R/3} u(z_0) \implies \frac{1}{5} u(z_0) \leq u(\zeta_1) \leq 5u(z_0)$$

Now, we repeat this process for  $\zeta_1$  taking the role of  $z_0$ . If  $z$  is contained within  $D(\zeta_1, R/3)$ , then we have that, for  $r = |z - \zeta_1|$  again:

$$\frac{R/2 - r}{R/2 + r} u(\zeta_1) \leq u(z) \leq \frac{R/2 + r}{R/2 - r} u(\zeta_1) \implies \frac{1}{5} u(z_0) \leq u(z) \leq 5^2 u(z_0) \implies \frac{1}{5} \leq \frac{u(z)}{u(z_0)} \leq 5^2$$

Otherwise, choose  $\zeta_2$  in the same fashion as  $\zeta_1$ , by looking at the boundary  $\partial D(\zeta_1, R/3)$ . Importantly, this process must terminate in a finite amount of steps - in particular, it must terminate in at most  $\lceil L/(R/3) \rceil = \lceil \frac{3L}{R} \rceil$  steps, where  $L$  is the path length of  $\gamma$ . Thus, letting  $n = \lceil \frac{3L}{R} \rceil$ , we have for our estimate, that:

$$\frac{1}{5} \leq \frac{u(z)}{u(z_0)} \leq 5^n$$

We notice that this is actually independent of  $u$ , and is strictly a function of the geometry.

Now, consider the function:

$$f(z) = \inf_{\gamma} \int_0^1 |\gamma'(t)| dt, \gamma(0) = z_0, \gamma(1) = z$$

that is, the length of the shortest path from  $z_0 \rightarrow z$ . This function is continuous. Let  $\epsilon > 0$  be given. Consider the disk  $D(z, \epsilon) \subset \Omega$  and fix some  $z'$  in the disk. For any path  $\gamma$  from  $z_0 \rightarrow z$ , consider the new path  $\gamma_1$  defined by:

$$\gamma_1 = \begin{cases} \gamma(2t) & \text{for } t \in [0, 1/2] \\ z'(2t - 1) + z(2 - 2t) & \text{for } t \in [1/2, 1] \end{cases}$$

that is, the path that traverses  $\gamma$  twice as fast, concatenated with the straight line  $z \rightarrow z'$ . Then, we have that:

$$f(z') \leq \int_0^1 |\gamma_1'(t)| dt = \int_0^{1/2} |\gamma'(t)| dt + \int_{1/2}^1 |2z' - 2z| dt < \int_0^{1/2} |\gamma'(t)| dt + \epsilon$$

Since  $f$  is defined as the infimum, and we have that for any point  $z'$ , that the length from  $z_0 \rightarrow z'$  is at most the length from  $z_0 \rightarrow z$  plus  $\epsilon$ , since the infimum is linear, we have that  $f(z') \leq f(z) + \epsilon$ . Further, we see that this function is positive if  $z_0$  is not in our set, being the integral of a real or complex modulus.

Now, let  $K$  be any compact set on  $\Omega$ . Since  $f$  is continuous,  $f$  attains a maximum on  $K$ . Then, we can find a maximum distance  $N = \max\{f(z) : z \in K\}$ . But, applying our analysis on the path length, we have then that the largest value of  $\lceil \frac{3L}{R} \rceil$  happens when  $L = N$ , at  $\lceil \frac{3N}{R} \rceil$ . Thus, we have that on  $K$ , that:

$$\frac{1}{5} \leq \frac{u(z)}{u(z_0)} \leq 5^{\lceil \frac{3N}{R} \rceil}$$

Now, suppose  $u_n$  is a sequence of positive harmonic functions, and  $u_n(z_0) \rightarrow 0$ . Then, to maintain the ratio  $u_n(z)/u_n(z_0)$ , we must have that  $u_n(z) \rightarrow 0$  on all of  $\Omega$ , since we can always find a compact subset

(i.e., a closed disk) around any point in  $\Omega$ . Similarly, it should be clear that if  $u_n(z_0) \rightarrow \infty$ , we must have that  $u_n(z) \rightarrow \infty$ .

Easy issue if  $u$  is not necessarily positive. Consider  $u_n((x, y)) = x + \frac{1}{n}$ , which is clearly a harmonic function, and take the region  $\Omega = \{(x, y) : x^2 + y^2 < 1\}$ . Clearly, we see that  $u_n((0, 0)) = \frac{1}{n}$ , and as  $n \rightarrow \infty$ , of course we have that  $1/n \rightarrow 0$ . However, we see that for any point,  $(x, y)$ , that  $u_n((x, y)) = x + \frac{1}{n} \rightarrow x$ , which does not align with what we've said.

In a similar fashion, we can think of the function  $u_n((x, y)) = u_n(x) = nx$ . Clearly, for any  $x > 0$ , we have that  $\lim_{n \rightarrow \infty} u_n(x) = \lim_{n \rightarrow \infty} nx = \infty$ . However, consider any  $(0, y)$ . On this line, we have that  $u_n(x) = n \cdot 0 = 0$ . Thus, when we do not have positivity on the region, we need not have the inequalities stated above.  $\square$