Assignment

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Math 240: Homework #9

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Problem 9.1. Let X be an irreducible abstract variety. We then have an atlas:

$$X = \cup_i U_i, \phi_i : Y_i \to U_i, \phi_j^{-1}\phi_i : \phi_i^{-1}(U_j \cap U_i) \to \phi_j^{-1}(U_j \cap U_i)$$

with Y_i being affine varieties, ϕ_i homeomorphisms, and $\phi_j^{-1}\phi_i$ regular isomorphisms of varieties.

(a) Show that $\dim Y_i = \dim Y_j$ for all pairs i, j, and therefore, we may define $\dim X = \dim Y_i$.

(b) Let $\alpha : X \to X'$ be a morphism of pre-varieties such that $\alpha(X)$ is dense in X'. Show that $\dim X' \leq \dim X$.

Solution. (a)

First, fix a Y_i, fix a U as an open set in Y_i, and let V be any other open set in Y_i. Consider their image in U_i under ϕ_i, that is, $\phi_i(U), \phi_i(V)$. In particular, by the subspace topology, they come from the intersection of $\phi_i(U) = U_i \cap \overline{U}, \phi_i(V) = U_i \cap \overline{V}$, for some open sets $\overline{U}, \overline{V}$. However, since $U_i, \overline{U}, \overline{V}$ are open sets, $\phi_i(U), \phi_i(V)$ are open sets in X. In particular, they have non-trivial intersection, since X is irreducible, each of these open sets are dense. That is, we can look at $\phi_i(U) \cap \phi_i(V)$. This is another open set, being a finite intersection of open sets, and pulling this back via the homeomorphism ϕ_i, since this is a bijective continuous function, we see that $\phi_i^{-1}(\phi_i(U) \cap \phi_i(V))$ is non-empty, and must be exactly $U \cap V$. Since the choice of the other open set V was arbitrary, this implies that U is dense. Since the choice of U was arbitrary, we can do this procedure with any open set U, and thus every open set in Y_i is dense. Then, we have that every Y_i is irreducible.

Now, take any i, j and consider the map: $\phi_j^{-1} \circ \phi_i : Y_i \supset \phi_i^{-1}(U_i \cap U_j) \to Y_j$. This is a map from a subset of Y_i into a dense subset of Y_j due to the fact that $U_i \cap U_j$ is open, and Y_j is irreducible, therefore, the pullback is an open set, and therefore dense. Then, we have that since the closure of the dense set is the full set, we have that $\dim(Y_j) = \dim(\phi_i^{-1}(U_i \cap U_j)) \leq \dim(Y_i)$. However, because we can run this exact same argument swapping the labels i, j, we also have that $\dim(Y_i) \leq \dim(Y_j)$, which together implies that $\dim(Y_i) = \dim(Y_j)$. Since the choice of i, j is arbitrary, this is true for every pair i, j. Then, because X irreducible, ϕ_i a homeomorphism, $\phi_i(Y_i)$ is an open, dense set in X, and so it follows that $\dim(X) = \dim(Y_i)$ and this is well defined regardless of i.

(b)

Let X' have an atlas

$$X' = \cup_j V_j, \varphi_j : Z_j \to V_j, \varphi_j^{-1}\varphi_i : \varphi^{-1}(V_j \cap V_i) \to \varphi_j^{-1}(V_j \cap V_i)$$

By 5.2.13 in Osserman, we have that for α to be a morphism, it induces morphisms on our quasiaffine sets that look like:

$$\phi_i^{-1}(\alpha^{-1}(V_j)) \xrightarrow{\phi_i} \alpha^{-1}(V_j) \cap U_i \xrightarrow{\alpha} V_j \xrightarrow{\varphi_j^{-1}} Z_j$$

for pair i, j. In particular, we know a few properties. Because $\alpha(X')$ is dense, we know that this is a non-trivial morphism, as the pullback to X, $\phi^{-1}(V_j) \cap U_i$ is a non-empty, and moreover, open set due to the continuity of α. Then, the pullback into Y_i is also a non-empty open set due to the homeomorphism.

From the work done in part (a) then, we have a subset of Y_i that maps into an open, dense set in Z_j, which implies that $\dim(Z_j) \leq \dim(Y_i)$. Then, we have that $\dim(X') \leq \dim(X)$, since we identify $\dim(X') = \dim(Z_j), \dim(X) = \dim(Y_i)$.

$\square$

Problem 8.2. Let $C_1 = \mathbb{P}^1, C_2 = V(X_0 X_1^2 - X_2^3) \subseteq \mathbb{P}^2, C_3 = V(X_0 X_1^2 - X_0 X_2^2 + X_2^3) \subseteq \mathbb{P}^2$.

(a) Find a birational map $\alpha_1 : C_1 \to C_2$ and its birational inverse.

(b) Find a birational map $\alpha_2 : C_1 \to C_3$ and its birational inverse.

(c) Show that C_1 is not isomorphic to either C_2, C_3.

(d) Show that C_2 is not isomorphic to C_3.

(e) Explain how (c),(d) do not contradict the statement that we proved in class that projective, non-singular curves are birational to each other if and only if the curves are isomorphic.

Solution. (a)

Consider the map from $\phi : C_1 \to C_2$ that sends $[X_0, X_1] \to [X_0^3, X_1^3, X_0 X_1^2]$, and the map $\psi : C_2 \to C_1$ that takes $[X_0, X_1, X_2] \to [X_2/X_1, 1]$, where we exclude $V(X_0 X_1)$ from C_1, as well as $V(X_1 X_2)$ from C_2.

Consider $\phi \circ \psi(X_0, X_1)$ and $\psi \circ \phi(X_0, X_1, X_2)$. We have that:

$$\psi \circ \phi(X_0, X_1) = \psi([X_0^3, X_1^3, X_0 X_1^2]) = [X_0 X_1^2/X_1^3, 1] = [X_0/X_1, 1] = [X_0, X_1]$$

where we multiply by a factor of X_1 since it is non-0

$$\phi \circ \psi(X_0, X_1, X_2) = \phi([X_2/X_1, 1]) = \phi([X_2, X_1]) = [X_2^3, X_1^3, X_2 X_1^2] = [X_0 X_1^2, X_1^3, X_2 X_1^2] = [X_0, X_1, X_2]$$

where we use the fact that we are on the curve and apply $X_0 X_1^2 = X_2^3$ and pulling out a scale factor of X_1^2.

We see that $\psi \circ \phi$ acts as identity on $C_1 \setminus V(X_0 X_1)$, and $\phi \circ \psi$ acts as identity on $C_2 \setminus V(X_1 X_2)$, so we have a pair of rational maps from open sets on C_1, C_2 that are inverses of each other, thus, birational.

(b)

Consider the map from $\phi : C_1 \to C_2$ that sends $[X_0, X_1] \to [X_0^3, X_1(X_1^2 - X_0^2), -X_0(X_1^2 - X_0^2)]$, and the map $\psi : C_2 \to C_1$ that takes $[X_0, X_1, X_2] \to [X_2, -X_1]$, where we exclude $V(X_0 X_1 [X_1^2 - X_0^2])$ from C_1, as well as $V(X_1 X_2)$ from C_2, so that at least one is always non-0.

First, we validate that ϕ actually sends us into the variety. Well, substituting into the curve:

$$X_0^3(X_1^2(X_1^2 - X_0^2)^2) - X_0^3(X_0^2(X_1^2 - X_0^2)^2 - X_0^3(X_1^2 - X_0^2)^3 =$$
$$X_0^3[(X_1^6 - 2X_1^4 X_0^2 + X_1^2 X_0^4) - (X_1^4 X_0^2 - 2X_1^2 X_0^4 + X_0^6) - (X_1^6 - 3X_1^4 X_0^2 + 3X_1^4 X_0^2 - X_0^6)] =$$
$$X_0^3[(X_1^6 - 3X_1^4 X_0^2 + 3X_1^2 X_0^4 - X_0^6) - (X_1^6 - 3X_1^4 X_0^2 + 3X_1^4 X_0^2 - X_0^6)] = 0$$

So, great.

Now, we do the same check on $\phi \circ \psi(X_0, X_1)$ and $\psi \circ \phi(X_0, X_1, X_2)$. We have that:

$$\psi \circ \phi(X_0, X_1) = \psi([X_0^3, X_1(X_1^2 - X_0^2), -X_0(X_1^2 - X_0^2)]) = [(-X_0(X_1^2 - X_0^2, -X_1(X_1^2 - X_0^2)]] = [X_0, X_1]$$

$$\phi \circ \psi(X_0, X_1, X_2) = \phi([X_2, -X_1]) = [X_2^3, -X_1(X_1^2 - X_2^2), -X_2(X_1^2 - X_2^2)]$$

Substituting for the curve with $-X_0 X_1^2 + X_0 X_2^2 = X_2^3$, we get:

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$$[-X_0X_1^2 + X_0X_2^2, X_1(X_1^2 - X_2^2), -X_2(X_1^2 - X_2^2)] = [-X_0(X_1^2 - X_2^2), -X_1(X_1^2 - X_2^2), -X_2(X_1^2 - X_2^2)]$$

Factoring out $-(X_1^2 - X_2^2)$ because we live in a projective space, we find:

$$[-X_0(X_1^2 - X_2^2), -X_1(X_1^2 - X_2^2), -X_2(X_1^2 - X_2^2)] = [X_0, X_1, X_2]$$

Then, in the same vein as part (a), we have found regular functions on open sets such that we go from an open set $\mathbb{P}^1 \setminus V(X_0X_1[X_1^2 - X_0^2])$ to and from the open set $C_2 \setminus V(X_1X_2)$.

(c)

Consider the Jacobian of C_2. This has form:

$$\mathcal{J}(C_2) = [X_1^2, 2X_0X_1, 3X_2^2]$$

In particular, we notice that this vanishes whenever $X_1, X_2 = 0$, but X_0 may be kept free, that is, the point of form $(1, 0, 0)$. We also quickly see that this is a point on the curve, as $X_0X_1^2 - X_2^3$ vanishes as well, thus, $(1, 0, 0)$ is a singular point on C_2

Similarly, we may look at the Jacobian of C_3, in a similar fashion. This has form:

$$\mathcal{J}(C_3) = [X_1^2 - X_2^2, 2X_0X_1, 3X_2 - 2X_0X_2]$$

From the second entry, we know that either $X_0 = 0$ or $X_1 = 0$. Suppose $X_0 = 0$, then from the last equation, we get that $3X_2 = 0 \implies X_2 = 0$, which implies $X_1 = 0$ from the first equation. That is not allowed, so we assume $X_1 = 0, X_0 \neq 0$. Then, from the first equation, we have that $-X_2^2 = 0 \implies X_2 = 0$, so we have that the singular points are of form $(1, 0, 0)$.

Then, we may not permit an isomorphism from C_1 to either C_2, C_3, as a morphism of varieties must send singular points to singular points, but $\mathbb{P}^1$ is not singular.

(d)

Consider the tangent cones for C_2, C_3 on the copy of the affine planes, where we delete $X_0 = 0$. In that case, we may take our copy to be of the form $(1, X_1, X_2)$, and we notice that this copy contains our singular point in both cases.

In the case for C_2, we see that the polynomial in this copy has form $X_1^2 - X_2^3$, which, if we take the lowest term, we find that $X_1 = 0$ is a tangent line with multiplicity 2.

However, in the case of C_3, we notice that in the version of the affine plane, we have $X_1^2 - X_2^2 + X_2^3$, which, if we take the lowest term again, this time of degree 2, we see that we have two tangent lines in the cone, with multiplicity 1: $X_1 - X_2 = 0, X_1 + X_2 = 0$. Since the tangent cone is a local invariant of our point, we see that the characterization of C_2 having a line with multiplicity 2, and C_3 having no double points. Thus, there cannot be an isomorphism between C_2, C_3.

(e)

This is fine because C_1, C_2, as seen in part (c) are both singular. Thus, they do not fulfill the condition of the statement we proved in class. We also notice that the birational maps cut out the bad point, so that we have isomorphisms between quasiprojective varieties that do not contain singularities. As such, those isomorphisms on the open sets are allowable.

$\square$

Problem 8.3. Let $C_1 = \mathbb{P}^1, C_2 = V(X_0X_2^2 - X_1(X_1 - X_0)(X_1 - aX_0)) \subseteq \mathbb{P}^2$ for a in our base field such that $a \neq 0, a \neq 1$.

(a) Show that C_2 is non-singular.

(b) Show that there does not exist a birational map $\alpha : C_1 \to C_2$. Hint: if a rational map were to exist, it would be an isomorphism that gives rise to an isomorphism $C_2 \setminus \{p\} \to \mathbb{A}^1$ for p some point, defined on some open set of $\mathbb{A}^1 \subseteq \mathbb{P}^1$.

Solution. (a)

We look again at the Jacobian criterion. The Jacobian here has form:

$$\mathcal{J}(C_2) = [X_2^2 - 2aX_0X_1 + (a+1)X_1^2, -3X_1^2 + 2(a+1)X_0X_1 - aX_0^2, 2X_0X_2]$$

First, we look at $2X_0X_2$. This implies either $X_0 = 0$ or $X_2 = 0$.

If $X_0 = 0$, then we have from $3X_1^2 + 2(a+1)X_0X_1 - aX_0^2$, that $3X_1^2 = 0 \implies X_1 = 0$. Then, from the first equation, we have that $X_2^2 - 2aX_0X_1 + (a+1)X_1^2 \implies X_2^2 = 0 \implies X_2 = 0$, and $(0,0,0) \notin \mathbb{P}^2$

Now, suppose $X_2 = 0$. Then, we have that $-2aX_0X_1 + (a+1)X_1^2 = 0 \implies X_1[(a+1)X_1 - 2aX_0)] = 0 \implies X_1 = 0$ or $(a+1)X_1 - 2aX_0) = 0$. If $X_1 = 0$, then $X_0 = 0$ follows easily, so we assume the latter. Substituting $X_1 = 2a(a+1)^{-1}X_0$ into $-3X_1^2 + 2(a+1)X_0X_1 - aX_0^2$, we find that

$$-3(2a(a+1)^{-1}X_0)^2 + 2X_0(2a(a+1)^{-1}X_0) + (a+1)(2a(a+1)^{-1}X_0)^2 - aX_0^2 =$$

$$(a+1)^{-1}[-6aX_0^2 + 4aX_0^2 + 4a^2X_0^2 - (a^2+1)X_0^2] = (a+1)^{-1}X_0^2(-3a^2 + 4a - 1) =$$

$$-(a+1)^{-1}X_0^2(3a-1)(a-1)$$

Since $a \neq 1$, we see that the choices are $X_0 = 0$ or $a = 1/3$. If $X_0 = 0$, then $X_1 = 0$, and the point does not exist, so we assume $a = 1/3$. Then, we have that $4/3X_1 - 2/3X_0 = 0 \implies 2X_1 = X_0$. However, let's look at points of the form $(2X_1, X_1, 0)$, and when they lie on the curve. Indeed, if we let $f(X_0, X_1, X_2) = X_0X_2^2 - X_1(X_1 - X_0)(X_1 - 1/3X_0)$:

$$f(2X_1, X_1, 0) = 0 - X_1(X_1 - 2X_1)\left(X_1 - \frac{1}{3}2X_1\right) = -X_1(-X_1)\left(\frac{1}{3}X_1\right) = \frac{X_1^3}{3}$$

And this vanishes only if $X_1 = 0$, which, again, via $2X_1 = X_0$, implies that $X_0 = 0$, which is still not a point in $\mathbb{P}^2$. Thus, regardless of the choice of a as long as $a \neq 1$, C_2 may not be singular.

(b)

Suppose a birational map exists. Then, we have an isomorphism between open sets in $\mathbb{P}^1, C_2$. Since this is a curve, we notice that the form of an open set in C_2 is $U = C_2 \setminus \{p\}$, that is, the curve missing a finite number of points. Similarly, we an argue the same for $\mathbb{P}^1$, but, in particular, we enforce that it is the point at infinity, that is, $\mathbb{P}^1 \setminus \{(1,0)\}$, which we can do, since we can take a linear transformation as a rotation of the form $X_0' = \cos(\theta)X_0 + \sin(\theta)X_1, X_1' = -\sin(\theta)X_0 + \cos(\theta)X_1$ and/or switch the labels on X_0, X_1 such that we delete this specific point, $(1,0)$. Well, we recall that this is a copy of $\mathbb{A}^1$ sitting inside $\mathbb{P}^1$, so this is an isomorphism between $U \to \mathbb{A}^1$.

However, this would induce an isomorphism between the coordinate rings. However, the coordinate ring of $\mathbb{A}$ is all of $k[x]$, whereas the coordinate ring of U is the same as the coordinate ring of C_2, being $k[X_0, X_1, X_2]/(X_0X_2^2 - X_1(X_1 - X_0)(X_1 - aX_0))$. In particular, we notice that since the ideal is of degree 3, irreducible, then we would need element from $k[x]$ that map to X_0, X_1, X_2 individually. Further, by a degree argument, this may only be satisfied by elements of degree 1. However, this is impossible in an isomorphism, as we would then have relations between elements in $k[x]$. Thus, these curves may not be birational.

$\square$