Homework #2

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Math 285: Homework #2

September 21, 2023

Question 1. Let ω be the 1-form $zdx - dz$ and let X be the vector field $y\partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^3$.

Compute $\omega(X)$ and $d\omega$.

Solution. We recall that $\omega(X)$, in coordinates, is simply $\sum_i a_i b^i$, where $\omega = \sum_i a_i dx^i$, and $X = \sum_j b^j \frac{\partial}{\partial x^j}$

Thus, we have that:

$$\omega(X) = \sum_i a_i b^i = z * y + 0 * x + -1 * 0 = yz$$

In a similar fashion, recall that, by definition:

$$d\omega = \sum_{i,j} \frac{\partial a_i}{\partial x^j} dx^j \wedge dx^i$$

Thus, we have that:

$$d\omega = 1dz \wedge dx = dz \wedge dx$$

since we notice that the only non-vanishing partial of zdx is $\partial/\partial z$ and none of the partials of $-dz$ survive.

□

Question 2. Suppose the standard coordinates on $\mathbb{R}^3$ are called ρ, ϕ, θ. If we have that:

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ x = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

Compute the following quantities in terms of $d\rho, d\phi, d\theta$: $dx, dy, dz, dx \wedge dy \wedge dz$.

Solution.

□

Question 3. Let V be a vector space of dimension 3 with basis e_1, e_2, e_3 and dual basis $\alpha^1, \alpha^2, \alpha^3$. For a 1-covector $\alpha = \sum_{i=1}^3 a_i \alpha^i$ on V, we associate the vector $v_\alpha = \langle a_1, a_2, a_3 \rangle \in \mathbb{R}^3$. For a 2-covector

$$\gamma = c_1 \alpha^2 \wedge \alpha^3 + c_2 \alpha^3 \wedge \alpha^1 + c_3 \alpha^1 \wedge \alpha^2$$

we assoicate the vector $v_\gamma = \langle c_1, c_2, c_3 \rangle \in \mathbb{R}^3$.

Show that under this correspondence, the wedge product of 1-covectors corresponds to the cross product of vectors in $\mathbb{R}^3$, that is:

$$v_{\alpha \wedge \beta} = v_\alpha \times v_\beta$$

Solution. Recall that if we have identifications of 1-covectors: $\alpha = \sum_i a_i dx^i$ and $\beta = \sum_j b_j dx^j$, then we have that:

$$\alpha \wedge \beta = \sum_{i,j} (a_i b_j) dx^i \wedge dx^j$$

Writing this out in terms of coordinates, with respect to the dual basis, i.e. $dx^i = \alpha^i$, we have that:

$$\alpha \wedge \beta = a_1 b_2 \alpha^1 \wedge \alpha^2 + a_1 b_3 \alpha^1 \wedge \alpha^3 + a_2 b_1 \alpha^2 \wedge \alpha^1 + a_2 b_3 \alpha^2 \wedge \alpha^3 + a_3 b_1 \alpha^3 \wedge \alpha^1 + a_3 b_2 \alpha^3 \wedge \alpha^2 =$$

$$(a_2 b_3 - a_3 b_2) \alpha^2 \wedge \alpha^3 + (a_3 b_1 - a_1 b_3) \alpha^3 \wedge \alpha^1 + (a_1 b_2 - b_1 a_2) \alpha^1 \wedge \alpha^2$$

where we've used the fact that since α^i are covectors, $\alpha^i \wedge \alpha^i = 0$.

So, we have that:

$$v_{\alpha \wedge \beta} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - b_1 a_2 \rangle$$

In contrast, let's consider the cross product of $v_\alpha \times v_\beta = \langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle$. Using matrix notation:

$$v_\alpha \times v_\beta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) j + (a_1 b_2 - a_2 b_1) k =$$

$$\langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

We notice these are the same, and we conclude that $v_{\alpha \wedge \beta} = v_\alpha \times v_\beta$.

□

Question 4. Let $A = \oplus_{k=-\infty}^{\infty} A^k$ be a graded algebra over a field K, with $A^k = 0$ for $k < 0$. Let $m \in \mathbb{Z}$.

Define a superderviation of A with degree m as a K-linear map $D : A \to A$ such that for all $k \in \mathbb{Z}$, we have that $D(A^k) \subset A^{k+m}$ and that for all $a \in A^k, b \in A^l$:

$$D(ab) = (Da)b + (-1)^{km} a(Db)$$

Let D_1, D_2 be superderivations of A with degrees m_1, m_2 respectively. Define their commutator as:

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{m_1 m_2} D_2 \circ D_1$$

Show that the commutator $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$.

Solution.

□

Question 5. Consider the set $S = \mathbb{R} \setminus \{0\} \cup \{A, B\}$, the bug-eyed line or the line with two origins.

For $c, d \in \mathbb{R}$, define the following notation:

$$\begin{cases} I_A(-c, d) = (-c, 0) \cup \{A\} \cup (0, d) \\ I_B(-c, d) = (-c, 0) \cup \{B\} \cup (0, d) \end{cases}$$

Define a topology on S as follows: On $\mathbb{R} \setminus \{0\}$, use the subspace topology from $\mathbb{R}$ with open intervals as a basis. At the point A, use the collection of sets $\{I_A(-c, d) : c, d > 0\}$ as a basis, and analogously at B.

(a) Prove that the map $h : I_A(-c, d) \to (-c, d) \subseteq \mathbb{R}$ defined by:

$$\begin{cases} h(x) = x & \text{when } x \neq A \\ h(A) = 0 & \text{else} \end{cases}$$

is a homeomorphism.

(b) Show that S is locally Euclidean, second countable, but not Hausdorff.

Solution. (a)

Take the map h as defined in the statement above. We need only show that h is a continous bijection that admits a continuous inverse.

We will show that $g : (-c, d) \to I_A(-c, d)$ defined by:

$$g(a) = \begin{cases} A & \text{if } a = 0 \\ a & \text{else} \end{cases}$$

is a left inverse and a right inverse.

First, consider the map $h \circ g : (-c, d) \to (-c, d)$.

Fix an $a \in (-c, d)$. If $a = 0$, then we have that:

$$h \circ g(0) = h(g(0)) = h(A) = 0$$

Else, suppose $a \neq 0$. Then, by definition, we have that:

$$h \circ g(a) = h(g(a)) = h(a) = a$$

Thus, g is a right inverse.

Similarly, looking at $g \circ h : I_A(-c, d) \to I_A(-c, d)$, fixing a $b \in I_A(-c, d)$, if $b = A$, then we have that:

$$g \circ h(A) = g(h(A)) = g(0) = A$$

otherwise, for $b \neq A$, we have that:

$$g \circ h(b) = g(h(b)) = g(b) = b$$

Thus, we have that g acts as a left and right inverse, and thus h is bijective, and g is an inverse to h.

Now, we wish to show that h, g is continuous. To do so, we need only show that pre-images of basis elements are taken to basis elements. This is because, working in our image space, suppose $U = \cup_{B \in \mathcal{B}} B$ for a collection of basis elements $\mathcal{B}$. If we have that $h^{-1}(B)$ is a basis element in our codomain for every B, then of course, $\cup_{B \in \mathcal{B}} h^{-1}(B)$, being a union of basis elements is an open set, and thus $h^{-1}(U)$ is open.

Then, it is enough to consider an open interval $(a, b) \subseteq (-c, d)$. If $0 \notin (a, b)$, then $(a, b) \subseteq \mathbb{R} \setminus \{0\}$. Since S inherits the subspace topology on this set, then of course (a, b) is a basis element of the topology on S. Furthermore, since h acts via identity on $\mathbb{R} \setminus \{0\}$, $h^{-1}((a, b)) = (a, b)$.

Now, suppose $0 \in (a, b)$. In the notation we have established then, write this interval as $(-a, b)$. Then, from the action of h, we see that $h^{-1}((-a, b)) = (-a, 0) \cup A \cup (0, b)$. But, from the definiton of I_A, this is exactly $I_A(-a, b)$, and from the definition of the topology on S, this is exactly a basis element for neighborhoods of A.

Therefore, $h^{-1}(a, b)$ for any $a, b \in \mathbb{R}$ is taken to a basis element of S, and therefore h is continuous.

In a similar fashion, we may do the same for $g : (-c, d) \to I_A(-c, d)$.

Take a basis element from $I_A(-c, d)$, and call it C. If $A \notin C$, then of course C comes from an open interval on $\mathbb{R} \setminus \{0\}$, and thus $g^{-1}(C) = C$, as it acts via identity on $\mathbb{R} \setminus \{0\}$.

Now, suppose $A \in C$. Then, being a basis element, $C = I_A(-a,b)$ for $-c \leq -a < b \leq d$. Looking at the action of $g^{-1}(I_A(-a,b))$, we see that this is exactly:

$$g^{-1}(I_A(-a,b)) = g^{-1}((-a,0) \cup \{A\} \cup (0,b)) = g^{-1}((-a,0)) \cup g^{-1}(A) \cup g^{-1}(A)((0,b)) =$$

$$(-a,0) \cup \{0\} \cup (0,b) = (-a,b)$$

Thus, for every basis element in $I_A(-c,d)$, the inverse image under g is a basis element of $(-c,d)$. Thus, g is continuous.

Therefore, since h is a continuous bijection that admits a continuous inverse, h is a homeomorphism.

(b)

Without too much trouble, it should be clear that S is locally Euclidean. Fix a $c,d \in \mathbb{R} : c,d > 0$. From part (a), we already have a chart from $I_A(-c,d)$ to a neighborhood of $\mathbb{R}$, an open interval, via h. It should be easy to see that swapping B for A everywhere, this also extends to a similar chart for $I_B(-c,d)$. Furthermore, on $S \setminus \{A,B\}$, we see that we may take $f : S \setminus \{A,B\} \to \mathbb{R}$ via $f(x) = x$, the identity, and the image is exactly $\mathbb{R} \setminus \{0\}$ an open set. It should be clear that the identity is continuous. Thus, between these three charts, S is locally Euclidean (of dimension 1).

Furthermore, S is second countable. We may take our basis to be the union of:

1) Open intervals with rational endpoints in $\mathbb{R}$ such that either both endpoints are positive or both are negative:

$$\{(a,b) : a,b \in \mathbb{Q}, a \neq 0, ab > 0\}$$

2) Open intervals of the form $I_A(-c,d)$ where $c,d > 0, c,d \in \mathbb{Q}$.

3) Open intervals of the form $I_B(-c,d)$ where $c,d > 0, c,d \in \mathbb{Q}$.

Using the fact that open intervals with rational endpoints are a countable basis for $\mathbb{R}$, we see that (1) generates the open sets for $\mathbb{R} \setminus \{0\}$. Further, by the definition of the topology for S, (2) and (3) generate the neighborhoods for A,B respectively, since for any $c,d \in \mathbb{R}$, we may take a sequence of rational numbers approaching c,d from above and below, respectively.

Since each of these sets are countable, being at most $\mathbb{Q} \times \mathbb{Q}$, their union is also countable. Thus S is second countable.

However, it should be clear that S is not Hausdorff. Take the points A,B. From the definition of our topology, we already know that the neighborhoods of A can be generated by $I_A(-c,d)$ and analogously for $B, I_B(-e,f)$.

Fix any two neighborhoods $I_A(-c,d), I_B(-e,f)$. Pick any point:

$$p \in (\max\{-c,-e\}, \min\{d,f\}) \setminus \{0\} \subseteq \mathbb{R}$$

Clearly, since $\max\{-c,-e\} < p < \min\{d,f\}$, we have that $p \in (-c,d) \setminus \{0\}$ and that $p \in (-e,f) \setminus \{0\}$. Thus, $p \in I_A(-c,d)$ and $p \in I_B(-e,f)$. Since this procedure may be done regardless of the choice of c,d,e,f, we can never find disjoint neighborhoods of A,B, and therefore S is not Hausdorff.

$\square$

Question 6. Define $S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\} \subseteq \mathbb{R}^3$, the unit sphere in 3-D.

Define the following charts:

$$\begin{cases} U_1 = \{(x,y,z) \in S^2 : x > 0\}, & \phi_1(x,y,z) = (y,z) \\ U_2 = \{(x,y,z) \in S^2 : x < 0\}, & \phi_2(x,y,z) = (y,z) \\ U_3 = \{(x,y,z) \in S^2 : y > 0\}, & \phi_3(x,y,z) = (x,z) \\ U_4 = \{(x,y,z) \in S^2 : y < 0\}, & \phi_1(x,y,z) = (x,z) \\ U_5 = \{(x,y,z) \in S^2 : z > 0\}, & \phi_5(x,y,z) = (x,y) \\ U_6 = \{(x,y,z) \in S^2 : z < 0\}, & \phi_6(x,y,z) = (x,y) \end{cases}$$

Describe the domain of $\phi_1 \circ \phi_4^{-1}$, and show that $\phi_1 \circ \phi_4^{-1}$ is a C^∞ function on its domain. Do the same for $\phi_6 \circ \phi_1^{-1}$.

Solution. □