Homework #11

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2.1

Problem 6.3.6. Assume that $g : [a, b] \to [c, d]$ and $f : [c, d] \to \mathbb{C}$ are continuous. Prove the following statements:

(a) If f is Lipschitz and $g \in \mathrm{AC}[a, b]$, then $f \circ g \in \mathrm{AC}[a, b]$

(b) If $f \in \mathrm{AC}[c, d], g \in \mathrm{AC}[a, b]$ and g monotone increasing on $[a, b]$, then $f \circ g \in \mathrm{AC}[a, b]$

(c) If $f \in \mathrm{AC}[c, d], g \in \mathrm{AC}[a, b]$, then

$$f \circ g \in \mathrm{AC}[a, b] \iff f \circ g \in \mathrm{BV}[a, b]$$

Solution. (a)

Let $\epsilon > 0$ be given.

Since f is Lipschitz, we may find $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$.

Since g is absolutely continuous, we may find a $\delta > 0$ such that for collections of nonoverlapping subintervals of $[a, b]$, that

$$\Sigma_j(b_j - a_j) < \delta \implies \Sigma_j|g(b_j) - g(a_j)| < \frac{\epsilon}{K}$$

Well, take $[a_j, b_j]_j$ as a collection of countable, nonoverlapping subintervals of $[a, b]$ such that $\Sigma_j(b_j - a_j) < \delta$, and consider

$$\Sigma|f \circ g(b_j) - f \circ g(a_j)| \leq \Sigma K|g(b_j) - g(a_j) = K\Sigma|g(b_j) - g(a_j)| < K\frac{\epsilon}{K} = \epsilon$$

Thus, $f \circ g \in \mathrm{AC}[a, b]$.

(b)

Let $\epsilon > 0$ be given.

Because f is absolutely continuous, we may find $\delta > 0$ such that, for $\{[c_j, d_j]\}_j$ intervals in $[c, d]$, we have that

$$\Sigma_j(d_j - c_j) < \delta \implies \Sigma_j|f(d_j) - f(c_j)| < \epsilon$$

Further, since g is absolutely continuous, we may find a $\delta' > 0$ such that for $\{[a_i, b_i]\}_i$ intervals in $[a, b]$, we have that

$$\Sigma_i(b_i - a_i) < \delta' \implies \Sigma_i|g(b_i) - g(a_i)| < \delta$$

Now, take $\{[a_i, b_i]\}_i$ intervals in $[a, b]$ such that $\Sigma_i(b_i - a_i) < \delta'$. Since g is monotone increasing, we notice that $\{[g(a_i), g(b_i)]\}_i$ are actually intervals, non-overlapping since, due to the monotone increasing nature of g, they may only overlap on their endpoints. Further, from the δ' condition, we have that $\Sigma_i |g(b_i) - g(a_i)| < \delta$, which implies then that $\Sigma_i |f(g(b_i)) - f(g(a_i))| < \epsilon$.

(c)

By Lemma 6.1.3, we know already that $h \in \mathrm{AC}[a,b] \implies h \in \mathrm{BV}[a,b]$. So, we need only prove that $f \circ g \in \mathrm{BV}[a,b] \implies f \circ g \in \mathrm{AC}[a,b]$. However, this is easy.

Let $Z \subseteq [a,b]$ be a set of measure 0. By corollary 6.3.2, $g(Z) \subseteq [c,d]$ is a set of measure 0. However, now we use the absolute continuity of f as well, to see that $f(g(Z))$ is also a set of measure 0. Since the choice of Z was arbitrary, we have that $|Z| = 0 \implies |f \circ g(Z)| = |f(g(Z))| = 0$. Then, by Banach-Zaretsky again, we have that $f \circ g \in \mathrm{AC}[a,b]$.

□

Problem 6.3.10. Suppose that $f : [a,b] \to \mathbb{C}$ is differentiable everywhere on $[a,b]$. Prove the following:

(a) $f \in \mathrm{AC}[a,b]$ if and only if $f \in \mathrm{BV}[a,b]$

(b) $f' = 0$ a.e. if and only if f is constant on $[a,b]$.

Solution. (a)

We already have that $f \in \mathrm{AC}[a,b] \implies f \in \mathrm{BV}[a,b]$ by Lemma 6.1.3. So, now assume $f \in \mathrm{BV}[a,b]$.

By Corollary 5.4.3, since $f \in \mathrm{BV}[a,b]$, we have that $f' \in L^1[a,b]$. Then, by Corollary 6.3.3, since f differentiable everywhere by hypothesis, we have that $f \in \mathrm{AC}[a,b]$.

(b)

Clearly, if f is constant on $[a,b]$, then $f' = 0$ everywhere, stronger than almost everywhere.

Now, suppose $f' = 0$ almost everywhere. Clearly then, $f' \in L^1[a,b]$, because in particular, $\int_{[a,b]} f' = 0$. Therefore, we have that $f \in \mathrm{AC}[a,b]$ by 6.3.3 again. Further, by definition, since $f' = 0$ almost everywhere, f is singular. Then, by 6.3.4, since f is both singular and absolutely continuous, f must actually be constant.

□

2.2

Problem 6.4.10. Show that $f : [a,b] \to \mathbb{C}$ is Lipschitz if and only if $f \in \mathrm{AC}[a,b]$ and $f' \in L^\infty[a,b]$.

Solution. Firstly, suppose f is Lipschitz. We have already that Lipschitz implies absolutely continuous by 6.1.3, which implies that f' exists almost everywhere, by 6.1.5. Now, let x be somewhere the dervative exists at. Then, we have that, for any $y \in [a,b], y \neq x$, by the definition of Lipschitz, there exists an $M > 0$ such that:

$$|f(y) - f(x)| \leq M|x - y| \implies \frac{|f(y) - f(x)|}{|y - x|} \leq M$$

Now, if we view $y = x + h$, and then take the limit as $h \to 0$, this implies that $|f'(x)| \leq M$ as well. Since the existence of M is independent of the point x, coming from the Lipschitz condition, we have then that on the $[a,b] \setminus Z, |Z| = 0$ where f' is defined, that $|f'| \leq M \implies f' \in L^\infty[a,b]$.

Now, instead, suppose $f \in \mathrm{AC}[a,b]$ with $f' \in L^\infty[a,b]$. By the fundamental theorem of calculus (6.4.2), we have that $f' \in L^1$, and:

$$f(x) - f(a) = \int_a^x f'(t)dt \implies f(x) = f(a) + \int_a^x f'(t)dt$$

Now, consider the difference $|f(y) - f(x)|$ for $x, y \in [a,b]$. We have that:

$$|f(y) - f(x)| = |f(a) + \int_a^y f'(t)dt - f(a) - \int_a^x f'(t)dt| = |\int_x^y f'(t)dt|$$

Now, since we have that f' is essentially bounded, suppose that $f' \leq \|f'\|_\infty$ almost everywhere. Then, we can say that on $[x, y]$, $f' \leq \|f'\|_\infty$ almost everywhere, so we have that:

$$|f(y) - f(x)| = |\int_x^y f'(t)dt| \leq |\int_x^y \|f'\|_\infty dt| = |x - y|\|f'\|_\infty$$

Thus, f is Lipschitz, as we just take the Lipschitz constant as the uniform norm of f'.

□

Problem 6.4.13. Suppose that $f \in L^1(\mathbb{R})$ is such that $f' \in L^1(\mathbb{R})$ and $f \in \mathrm{AC}[a, b]$ for every finite interval $[a, b]$. Show that $\lim_{|x|\to\infty} f(x) = 0 = \int_{-\infty}^{\infty} f'$.

Solution. First, we wish to prove that f is actually uniformly continuous. Let $\epsilon > 0$ be given. Because f' is integrable, we have that there exists $\delta > 0$ such that for all measurable $E \subseteq \mathbb{R}$:

$$|E| < \delta \implies \int_E |f| < \epsilon$$

Now, let $x, y \in \mathbb{R}$ such that $|x - y| < \delta$. WLOG, suppose $x < y$. Since $f \in \mathrm{AC}[x, y]$, by the fundamental theorem, we have that

$$|f(y) - f(x)| = \left| \int_x^y f'(t)dt \right| \leq \int_x^y |f'(t)|dt < \epsilon$$

Thus, f is uniformly continuous.

Now, suppose that $\lim_{x\to\infty} \neq 0$, where we tackle the positive infinity first. Then, fix any $\epsilon > 0$. We may find $\{x_n\}_n \to \infty$ such that $|f(x_i)| > \epsilon$. In order to space these out, since we take $x_n \to \infty$, we may assume that $x_{n+1} - x_n > k$ for all n, as if it is not, we may always take a subsequence such that our points are sufficiently spaced out.

Now, from the uniform continuity, we have that there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon/2$. Enforce that $\overline{\delta} = \min(\delta, 1)$.

Consider the intervals of form $I_n = (x_n, x_n + \overline{\delta})$. By continuity, this implies that $|f| \geq \epsilon/2$ on these intervals. Then, we would have that

$$\int_0^\infty |f| \geq \Sigma \int_{I_n} |f| \geq \Sigma \int_{I_n} \epsilon/2 = \Sigma\delta\epsilon/2 = \infty$$

since there are countably many of these intervals, and $\delta, \epsilon > 0$.

But, this is a contradiction, thus $\lim_{x\to\infty} f(x) = 0$. The same argument works for $x \to -\infty$. So, we have that $\lim_{|x|\to\infty} f(x) = 0$.

Now, consider $f_n = f\chi_{[-n,n]}$. We notice, since $f \in \mathrm{AC}[-n, n]$ for each n, then $f_n \in \mathrm{AC}[-n, n]$. Then, we may apply the fundamental theorem to see that:

$$f_n(n) - f_n(-n) = \int_{[a,b]} f_n'(t)dt = \int_{\mathbb{R}} f_n'(t)dt$$

where we use the construction of f_n to argue that the integral of f_n' is the same on $[-n, n]$ as it is on $\mathbb{R}$ since f_n' is identically 0 outside of $[-n.n]$.

We have that, by construction, $f_n' \to f'$ pointwise a.e., and $|f_n'(x)| \leq f'$ a.e.

Then, we have that

$$\int_{\mathbb{R}} f' = \lim_{n\to\infty} \int_{\mathbb{R}} f_n'(t) = \lim_{n\to\infty} f(n) - f(-n) = 0$$

□

2.3

Problem 7.3.22. Let E be a measurable subset of $\mathbb{R}^d$, and fix a $1 \leq p < \infty$

(a) Suppose that Σf_n is absolutely convergent in $L^p(E)$, that is, $f_n \in L^p(E)$ for all n and $\Sigma\|f_n\|_p < \infty$. Prove the following:

- the series $f(x) = \Sigma_{n=1}^{\infty} f_n(x)$ converges for almost every $x \in E$
- $f \in L^p(E)$
- the series $f = \Sigma f_n$ converges in the L^p norm, that is, $\lim_{N\to\infty} \|f - \Sigma_n^N f_n\|_p = 0$

(b) Use part (a) and theorem 1.2.8 to give another proof that $L^P(E)$ is complete with respect to $\|\cdot\|_p$.

(c) Show that if Σf_n is an absolutely convergent series in $L^1(E)$, then

$$\int_E \Sigma_{n=1}^{\infty} f_n = \Sigma_{n=1}^{\infty} \int_E f_n$$

Solution. (a)

First, we check the convergence of $\Sigma_{n=1}^{\infty}|f_n|$. Looking at the partial sums, we have that:

$$\|\Sigma_{n=1}^{N}|f_n|\|_p \leq \Sigma_{n=1}^{N}\|f_n\|_p \leq \Sigma\|f_n\|_p < \infty$$

Now, because $\|\Sigma_{n=1}^{N}|f_n|\|_p$, varying over N, is a monotone increasing set of numbers by the triangle inequality, bounded above by $\Sigma\|f_n\|_p < \infty$, we may apply the Monotone Convergence Theorem to state that $\|\Sigma_{n=1}^{\infty}|f_n|\|_p < \infty$. Then, $\Sigma_{n=1}^{\infty}|f_n| < \infty$ almost everywhere by Lemma 4.1.8, since we would have that:

$$\int_E (\Sigma_{n=1}^{\infty}|f_n|)^p < \infty \implies \int_E \Sigma_{n=1}^{\infty}|f_n| < \infty$$

Then, this implies that almost everywhere, $\Sigma_{n=1}^{\infty}|f_n(x)| < \infty$, and an absolutely convergent series of complex or real numbers is itself convergent. Thus, $f(x) = \Sigma_{n=1}^{\infty} f_n(x)$ converges for almost every $x \in E$.

Further, via this process, it should be clear that $f \in L^p(E)$, as $|f| \leq \Sigma_{n=1}^{\infty}|f_n| < \infty$, so we can take both sides to the $p - th$ power, and, looking at the integrals, this remains true.

Lastly, we have that f must converge in the L^p norm, because we have the following chain of inequalities:

$$\left|f - \Sigma_{n=1}^{N} f_n\right|^p \leq \left(|f| + \left|\Sigma_{n=1}^{N} f_n\right|\right)^p \leq \left(|f| + \Sigma_{n=1}^{N}|f_n|\right)^p \leq$$
$$(\Sigma_{n=1}^{\infty}|f_n| + \Sigma_{n=1}^{\infty}|f_n|)^p = (2\Sigma_{n=1}^{\infty}|f_n|)^p < \infty$$

Therefore, we have that $|f - \Sigma_{n=1}^{N} f_n|^p \to 0$ almost everywhere, and that $\left|f - \Sigma_{n=1}^{N} f_n\right|^p \leq (2\Sigma_{n=1}^{\infty}|f_n|)^p < \infty$, so by the Dominated Convergence Theorem, we have that

$$\lim_{N\to\infty} \|f - \Sigma_n^N f_n\|_p = \lim_{N\to\infty} \int_E \left|f - \Sigma_{n=1}^{N} f_n\right|^p = \int_E 0 = 0$$

(b)

If we believe in part (a), theorem 1.2.8 states that if every absolutely convergence series in a metric space X converges in X, then X is complete. Part (a) just said that, for an arbitrary absolutely convergence series, it converges in $L^p(E)$, so we are done.

(c)

Without repeating the argument in (a), we see that if Σf_n is absolutely convergent in $L^1(E)$, then we have that $f = \Sigma f_n$ converges a.e., $f \in L^1(E)$, and that it converges in the L^1 norm. Then, by convergence in L^1, we have that:

$$\lim_{N\to\infty} \int_E \Sigma_{n=1}^N f_n = \int_E f = \int_E \Sigma f_n$$

However, we know that because the f_n are absolutely convergent, this means that they are in L^1 individually, so by the linearity of the integral, we have that:

$$\lim_{N\to\infty} \int_E \Sigma_{n=1}^N f_n = \lim_{N\to\infty} \Sigma_{n=1}^N \int_E f_n$$

Lastly, since they are absolutely convergent, we have that:

$$\Sigma \|f_n\|_1 = \Sigma_n \int_E |f_n| < \infty$$

which implies that $\int_E f_n$ is an absolutely convergent series, and thus convergent. Therefore, we may replace this as:

$$\Sigma_n \int_E f_n = \lim_{N\to\infty} \Sigma_n^N \int_E f_n = \lim_{N\to\infty} \int_E \Sigma_{n=1}^N f_n = \int_E f = \int_E \Sigma f_n$$

□

Problem 7.3.23. Fix a $1 \leq p < \infty$. Given $f_n \in L^p(\mathbb{R}^d)$, prove that $f_n \to f$ in $L^p(\mathbb{R}^d)$ if and only if the following three conditions hold.

(a) $f_n \xrightarrow{m} f$

(b) For each $\epsilon > 0$, there exists a $\delta > 0$ such that for every measurable set $E \subseteq \mathbb{R}^d$ with $|E| < \delta$, we have that $\int_E |f_n|^p < \epsilon$ for every n.

(c) For each $\epsilon > 0$, there exists a measurable set $E \subseteq \mathbb{R}^d$ such that $|E| < \infty$ and $\int_{E^c} |f_n|^p < \epsilon$ for every n.

Solution. First, assume (a)-(c) are true.

Let $\epsilon > 0$ be given.

It is fairly clear that if $f_n \to f$ in $L^p(\mathbb{R}^d)$ is true, (a)-(c) is true:

(a)

By Theorem 7.3.4, if they converge in the L^p norm, we automatically have that $f_n \xrightarrow{m} f$.

(b)

First, we assume that the f_n are simple functions with compact support. Then, because $f_n \in L^p(\mathbb{R}^d$, we can look at $max(\{f_n(x) : x \in \text{Supp}(f_n)\}) < \infty$. This must be finite because, if not, $\|f_n\|_p = \infty$. Further, we may talk in particular about the max over all such f_n. This must be finite, because if not,

Ok, you know, I don't know how this works.

□