Homework #5

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Math 123: Homework #5

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Question 1. Let $\{x_i\}_{i=1}^n \subset \mathbb{R}^D$ be a discrete set on unique points. Recall that the DBSCAN algorithm depends on two parameters: ϵ and MinPts.

(a) Describe the behavior of DBSCAN as $\epsilon \to \infty$ and $\epsilon \to 0^+$.

(b) Describe the behavior of DBSCAN as MinPts $\to \infty$ and as MinPts $\to 0^+$.

Solution. (a)

We recall that ϵ represents the distance under which a point is considered "close enough". Fix some $1 <$ MinPts $\leq n$ (for sanity, as of course if MinPts ≤ 1, then everything is automatically a core point, due to itself being within ϵ automatically, and if MinPts $\geq n+1$, every point is automatically an outlier, as there are only n total points), and fix some data point x_i. Since there are a finite number of points n, we may look at the set $\{\|x_j - x_i\| : 1 \leq j \leq n\}$. This is a set of real numbers, finite. Therefore, it achieves a maximum, call that value M_{x_i}. Consider any $\epsilon > M_{x_i}$. For such an ϵ and for any other point x_j, we have that:

$$\|x_j - x_i\| \leq M_{x_i} < \epsilon$$

Thus, so long as MinPts is a reasonable parameter for our set, that is, no larger than the size of the set, we have that every point is within ϵ of x_i, and thus x_i is a core point as $\epsilon \to \infty$. In particular, we may choose ϵ as it grows to ∞ to be larger than all such M_{x_i}, since across the index i, this too is a finite set of real numbers, which achives a maximum. Thus, repeating this argument for any arbitrary point x_i, DBSCAN will indicate that every point is a core point of a single cluster as $\epsilon \to \infty$.

Conversely, suppose $\epsilon \to 0$. In a similar fashion then, we fix a MinPts, x_i, and consider the set $\{\|x_i - x_j\| : i \neq j\}$. Because we have that the data points are unique, this must be a set of non-negative numbers, finite. Thus, the set achieves a minimum, call this m_{x_i}. If we choose an $0 < \epsilon < m_{x_i}$, then we have that, for any other point x_j:

$$\|x_j - x_i\| \geq m_{x_i} > \epsilon$$

and thus, x_i is an outlier point. Again, since the set of m_{x_i} is a finite set of non-negative real numbers, this too attains a minimum. We then can choose ϵ to be greater than 0, but less than every m_{x_i} and repeat the argument to say that DBSCAN will indicate that every point is an outlier as $\epsilon \to 0^+$.

(b)

In a similar fashion, we may look at MinPts, while keeping ϵ fixed. Clearly, if MinPts is larger than n, the number of data points in a set, every point must be an outlier, since the set of points within ϵ can only be as large as the underlying data set.

The reverse is clear as well, as regardless of the $\epsilon \geq 0$, when MinPts is equal to 1, every point x_i is a core point, since x_i is a point within $0 \leq \epsilon$ distance from x_i. However, we do note the edge case for if $\epsilon < 0$, then because distance functions are non-negative, every point remains an outlier regardless, unless MinPts is exactly equal to 0, as cardinalities of sets are non-negative.

□

Question 2. Let $L = D - W \in \mathbb{R}^{n\times n}$ be the graph Laplacian for data with an associated symmetric weight matrix W, and $w_{ij} \in [0,1]$ for all $i, j = 1, ..., n$.

(a) Show L is positive semidefinite.

(b) Show L is not positive definite by proving 0 is an eigenvalue of L.

Solution. (a)

Let $y \in \mathbb{R}^n$, realized as a column vector.

Consider the quantity $y^T Ly$. Specifically, since $L = D - W$, we have that:

$$(Ly)_i = (Dy)_i - (Wy)_i = \sum_{j=1}^{n} w_{ij} y_i - \sum_{j=1}^{n} w_{ij} y_j = \sum_{j=1}^{n} w_{ij}(y_i - y_j)$$

Thus, we have that:

$$y^T Ly = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(y_i - y_j)y_i$$

Here, we look at a trick: Consider relabeling $i \to j, j \to i$ and then interchanging the summation. This does not change our sum, but we would have:

$$y^T Ly = \sum_{j=1}^{n}\sum_{i=1}^{n} w_{ji}(y_j - y_i)y_j = -\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ji}(y_i - y_j)y_j$$

Summing these two equations, and noticing that because W is symmetric, we have that $w_{ij} = w_{ji}$:

$$2(y^T Ly) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(y_i - y_j)y_i - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ji}(y_i - y_j)y_j = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(y_i - y_j)^2$$

which, of course, implies that:

$$y^T Ly = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(y_i - y_j)^2$$

Now, by hypothesis, $w_{ij} \geq 0$. And, since $y \in \mathbb{R}^n$, $(y_i - y_j)^2 \geq 0$. Thus, this is a (double) sum of non-negative numbers, and hence must be non-negative, that is, $y^T Ly \geq 0$

(b)

We notice from our computation, that we arrived at

$$y^T Ly = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(y_i - y_j)^2$$

Thus, choose any y such that $y = a(1, ..., 1)$. Then, for all i, j, we have that $y_i = y_j$, and thus $y^T Ly = 0$. Equivalently, looking at $(Ly)_i$, we have:

$$(Ly)_i = \sum_{j=1}^{n} w_{ij}(y_i - y_j) = 0$$

for all i.

Thus, for this choice of y, we have that:

$$Ly = 0$$

and thus, 0 is an eigenvalue, and thus L is not positive definite.

$\square$

Question 3. Compute the graph Laplacian with $W_{ij} = \exp(-\|x_i - x_j\|_2^2/\sigma^2)$ on the image in Ncut_Data.mat using a range of σ values. For each of these σ, use the second eigenvector (the one with second smallest eigenvalue) to segment the image by thresholding at 0. Discuss the results. Do they make sense, and how do the results depend on σ?

Solution. Here follows some charts with two clusters from Ncut, with clusters coloured differently:

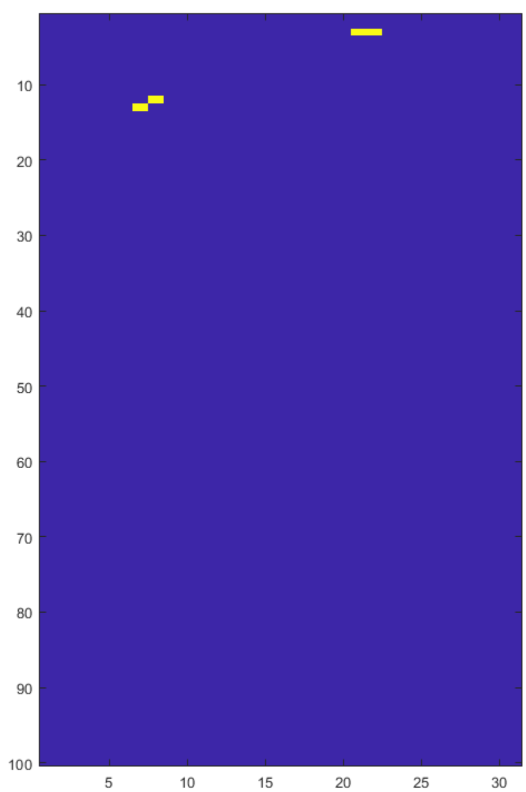


Figure 1: $\sigma = 0.01$

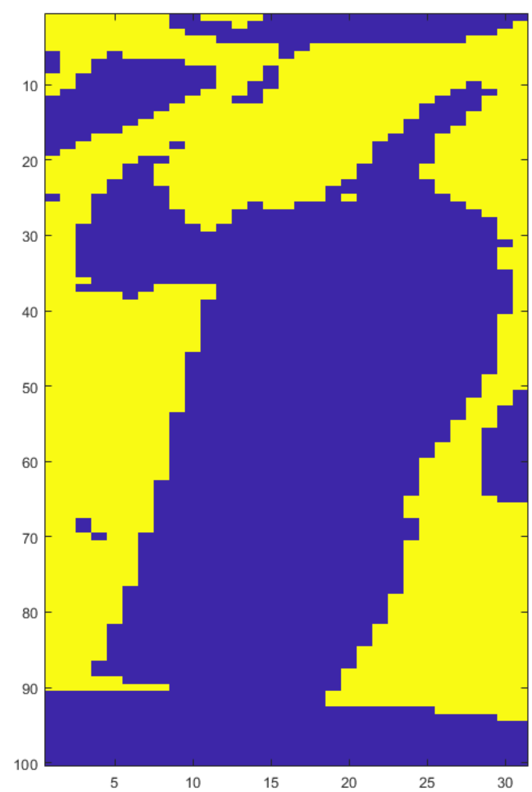
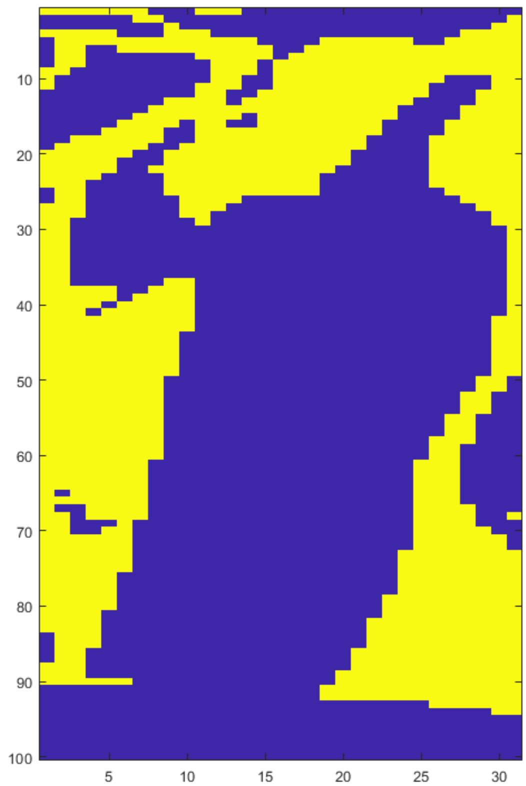


Figure 2: $\sigma = 0.1$



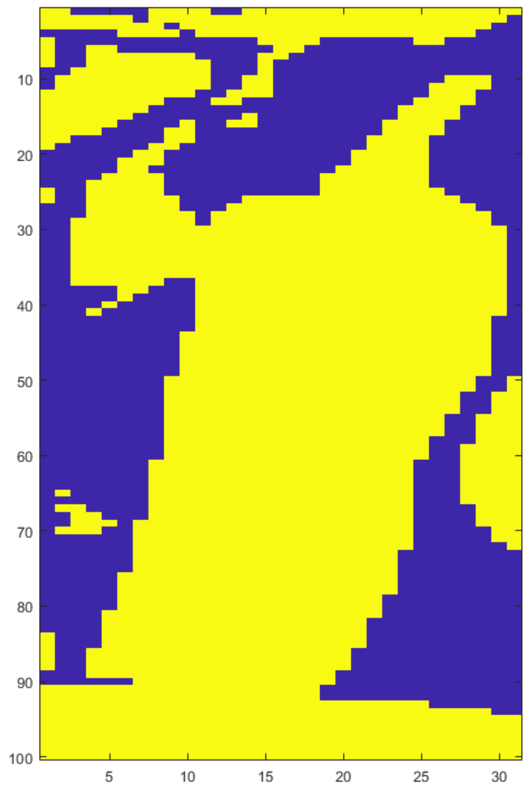


Figure 3: $\sigma = 1$

Figure 4: $\sigma = 10$

Generally speaking, these preserve the pepper strucure pretty well, in the sense that we do distinctly see a shadow of a figure in the main image for all but the first figure.

We notice that there is a sweet spot for σ values that tries to make distinctions between the background and the pepper, but also have a sharp boundary. In the first figure, of course, σ is too small to distinguish between background and foreground objects. On the other hand, as we increase σ to 100, we notice that finer structures, especially those near the bottom left, become washed out a bit - we can see a gap that closes from figure 2, 3, and 4 that might be a gap between the pepper and its base, which may represent the background colors. In a nutshell, as σ is very large or very small, we start to lose granularity in our clustering.

Lastly, we notice that clustering wise, as we increase σ, there is a switch from the pepper as being the blue color (i.e., below our threshold) to the yellow color (above the threshold). This isn't a very interesting thing actually, since it merely indicates which cluster has more points.

$\square$