Weak Topology Examples

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Abstract

In class, we discussed the weak topology on a set X induced by a collection of subspaces $\{X_\alpha\}_{\alpha \in A}$ for some indexing set A, each equipped with a topology τ_α. In particular, the conditions for compatibility were that (i): for any $\alpha, \beta \in A$, X_α, X_β induce the same topology on $X_\alpha \cap X_\beta$ under the subspace topology and (ii): $X_\alpha \cap X_\beta$ is an open set in both X_α, X_β. We exhibit examples to show that these conditions are independent of each other and, therefore, we need both conditions to be satisifed to conclude that the weak topology is indeed a topology.

Question 1. Let X be a set equipped with a collection of subsets $\{X_\alpha\}_{\alpha \in A}$ for some index set A, possibly uncountably infinite.

Let $\alpha, \beta \in A$, and suppose that X_α, X_β induce the same topology for $X_\alpha \cap X_\beta$ under the subspace topology.

Is it true that $X_\alpha \cap X_\beta$ is an open set of X_α and X_β?

Example. We exhibit an example to answer this question in the negative.

Consider the 3 point set $X = \{a, b, c\}$, and consider the following subsets equipped with their respective topologies:

$$\begin{cases} X_1 = \{a, b\} & \tau_1 = \{\emptyset, \{a\}, \{a, b\} = X_1\} \\ X_2 = \{b, c\} & \tau_2 = \{\emptyset, \{c\}, \{b, c\} = X_2\} \end{cases}$$

Without too much trouble, we can see that τ_1, τ_2 satisfy the properties of a topology, containing the entire set, the empty set, and being closed under both unions and finite intersections.

Now, denote $X_3 = X_1 \cap X_2$. We have that $X_3 = \{b\}$. In particular, we can see that the subspace topologies τ_{13}, τ_{23} induced by X_1, X_2 respectively are exactly:

$$\begin{cases} \tau_{13} = \{\emptyset \cap X_3, \{a\} \cap X_3, \{a, b\} \cap X_3\} = \{\emptyset, \{b\} = X_3\} \\ \tau_{23} = \{\emptyset \cap X_3, \{c\} \cap X_3, \{b, c\} \cap X_3\} = \{\emptyset, \{b\} = X_3\} \end{cases}$$

And thus, $\tau_{13} = \tau_{23}$, that is, they induce the same subspace topology on X_3.

However, looking at the topologies, $X_3 = \{b\}$ is not an element of either τ_1, τ_2, and thus is not open in either X_1, X_2.

$\square$

Question 2. Now, we wish to look at the converse:

With the same setting, suppose that $X_\alpha \cap X_\beta$ is an open set of both X_α, X_β.

Is it true that X_α, X_β induce the same topology for $X_\alpha \cap X_\beta$ under the subspace topology?

Again, we exhibit two examples to answer this question in the negative.

Generically, we may imagine letting the underlying sets $X_\alpha = X_\beta = \mathbb{R}$, but endowing X_α with the standard topology on $\mathbb{R}$, and X_β with the trivial topology. Then, of couse $X_\alpha \cap X_\beta = \mathbb{R}$, which is an open set in both topological spaces, however, it is clear that the intersection inherits different topologies as a subspace of X_α and X_β.

With a concrete example, in the same vein as the first question, we may look at the 4 point set $X = \{a, b, c, d\}$ and consider the following subsets equipped with the respective topologies:

$$\begin{cases} X_1 = \{a, b, c\} & \tau_1 = \{\emptyset, \{b\}, \{b, c\}, \{a, b, c\} = X_1\} \\ X_2 = \{b, c, d\} & \tau_2 = \{\emptyset, \{c\}, \{b, c\}, \{b, c, d\} = X_2\} \end{cases}$$

Again, it is not hard to see that τ_1, τ_2 satisfy the conditions of being a topology on their respective spaces.

Now, denote $X_3 = X_1 \cap X_2 = \{b, c\}$, which, by construction, is an open set in each topological space. However, as a subspace of X_1, X_2, it is clear that $\{b\}$ is an open set in X_3 endowed with the subspace topology of X_1 but not of X_2, and similarly, $\{c\}$ is an open set in the subspace topology of X_2 but not of X_1. Thus, we may have that $X_\alpha \cap X_\beta$ as an open set in both the topologies of X_α, X_β without inducing the same topology as a subspace.