Homework #2

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Math 235: Homework #2

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2.1

Problem 2.2.38. Let $E \subseteq \mathbb{R}^d$ with $|E| < \infty$. Prove that the following statements are equivalent:

(a) E is Lebesgue measurable.

(b) For each $\epsilon > 0$, we can write $E = (S \cup A) \setminus B$ where S is a union of finitely many nonoverlapping boxes and $|A|_e, |B|_e < \epsilon$.

Solution. First, suppose E is Lebesgue measurable. Let $\epsilon > 0$ be given. Due to the measurability, we may find an open set U such that $|U \setminus E| < \epsilon$. But also, by Lemma 2.1.5 in Heil, since U is open, there exists countably many nonoverlapping cubes $\{Q_k\}$ such that $U = \cup Q_k$. Since $|E| < \infty$, this implies that our choice of $|U| < \infty$ as well, unless $\epsilon = \infty$. Then, since cubes are measurable, this implies that $| \cup Q_k| = |U| < \infty$. So, since this converges, we may choose k_0 such that $| \cup_{k=0}^{k_0} Q_k| > |U| - \epsilon \implies |U| - | \cup_{k=0}^{k_0} Q_k| < \epsilon$. Set $S = \cup_{k=0}^{k_0} Q_k$, $A = \cup_{k=k_0+1}^{\infty} Q_k$ and $B = U \setminus E$. Then, by our constructions, we have that $(S \cup A) \setminus B = ([\cup_{k=0}^{k_0} Q_k] \cup [\cup_{k=k_0+1}^{\infty} Q_k]) \setminus (U \setminus E) = (\cup_{k=0}^{\infty} Q_k) \setminus (U \setminus E) = U \setminus (U \setminus E) = U \cap [U \cap E^c]^c = U \cap (E \cup U^c) = (U \cap E) \cup (U \cup U^c) = E$, $|B| = |U \setminus E| < \epsilon$, and $|A| = | \cup_{k=k_0+1}^{\infty} Q_k| = |U| - | \cup_{k=0}^{k_0} Q_k| < \epsilon$.

(b)

Firstly, define $E = E_{1,k} = (S_k \cup A_k) \setminus B_k$ for $k \geq 1$ where $|A_k|_e, |B_k|_e < \frac{\epsilon}{2^{k+1}}$. We claim that $E = ((\cup_k S_k) \cup (\cap_k A_k)) \setminus (\cup_k B_k)$.

First, suppose $x \in E$. Then, for every k, we have that either that x is in every A_k, so $x \in \cap_k A_k$. If $x \notin A_{k_0}$ for some k_0, then $x \in S_{k_0} \subseteq \cup_k S_k$ and $x \notin B_k$ for all k, so $x \notin (\cap_k B_k)$. So $E \subseteq ((\cup_k S_k) \cup (\cap_k A_k)) \setminus (\cap_k B_k)$. Now, suppose that we have an $x \in ((\cup_k S_k) \cup (\cap_k A_k)) \setminus (\cup_k B_k)$. In particular then, for each $E_k = (S_k \cup A_k) \setminus B_k$, we have that either $x \in A_k$ or, if not, $x \in S_k$. Further, $x \notin B_k$, as otherwise, $x \in \cup_k B_k$. Thus, we have that $E = E_1 = ((\cup_k S_k) \cup (\cap_k A_k)) \setminus (\cap_k B_k)$. We identify the following: $S_1 = \cup_k S_k, A_1 = \cap_k A_k, B_1 = \cup_k B_k$. We notice then that since $A_1 \subseteq A_k$ for all k, then $|A_k| < \frac{\epsilon}{2^{k+1}}$ then $|A_1|_e = 0$ and thus measurable. Further, we have that S_1 is measurable, as it is a countable union of measurable sets. Finally, we have that $|B_1|_e = | \cup_k B_k|_e \leq \Sigma |B_k| < \Sigma \frac{\epsilon}{2^{k+1}} = \frac{\epsilon}{2}$.

Now, construct E_i in the same way, but instead, force that $|A_k|_e, |B_k|_e < \frac{\epsilon}{2^{k+i}}$. We notice that S_i is still measurable, and A_i is still a set of measure 0, but we have $|B_i| = \Sigma_k \frac{\epsilon}{2^{k+i}} = \frac{\epsilon}{2^i}$, and $E = E_i$ for all i. We now have a countable collection of E_i. Here, we enforce that $S_i \cup A_i = \emptyset$ as if not, we can always just take $\overline{A_i} = A_i \setminus S_i$, which will keep $|\overline{A_i}|_e = 0$ and keep S_i as a union of union of boxes. Further, we enforce that $A_i \cap B_i = \emptyset$ as, if not, we can always find $\overline{A_i} = A_i \setminus B_i$ and same for B_i, which will keep the upper bound on $|B_i|_e < \frac{\epsilon}{2^i}$. Then, we may rewrite $E_i = (S_i \cup A_i) \setminus B_i = (S_i \cup A_i) \cap B_i^c = (S_i \cap B_i^c) \cup (A_i \cap B_i^c) = (S_i \setminus B_i) \cup A_i$.

We claim from here, that we can write $E = (\cap_i S_i \setminus \cap_i B_i) \cup (\cup_i A_i)$. Suppose $x \in E$. Then, for every i we can write $x \in (S_i \setminus B_i) \cup A_i$. If $x \in A_i$ for any i, then we're done. Otherwise, we have that $x \in S_i$ for every i and $x \notin B_i$ for any i. Then $x \in (\cap_i S_i \setminus \cap_i B_i)$ and thus, $E \subseteq (\cap_i S_i \setminus \cap_i B_i) \cup (\cup_i A_i)$. Now, instead, suppose we have an element $x \in (\cap_i S_i \setminus \cap_i B_i) \cup (\cup_i A_i)$. If $x \in (\cup_i A_i)$, then for some i_0, $x \in A_{i_0}$, so then $x \in (S_{i_0} \setminus B_{i_0}) \cup A_{i_0} = E$. Now, suppose not. Then, $x \in (\cap_i S_i \setminus \cap_i B_i)$ which implies that for all i, $x \in S_i$ and $x \notin B_i$. But then, for any i, $x \in S_i \setminus B_i \subseteq (S_i \setminus B_i) \cup A_i = E$. So we have set equality. But, that means that since $\cap_i B_i \subseteq B_i$ for all i, then $| \cap_i B_i|_e \leq |B_i|_e = \frac{\epsilon}{2^i}$ which implies that $| \cap_i B_i|_e = 0$. Then, we have a presentation of E such that $\cap_i S_i$ is measurable, being a countable intersection of measurable sets, $\cap_i B_i$ is a set of outer measure 0 and thus measurable, and $\cup_i A_i$ is a countable union of measurable sets, so thus measurable. Since E lies within the algebra of measurable sets then, E itself is measurable.

$\square$

Problem 2.2.39. Let E be a subset of $\mathbb{R}^d$ such that $0 < |E|_e < \infty$. Given $0 < \alpha < 1$, prove that there exists a cube Q such that $|E \cap Q|_e \geq \alpha|Q|$.

Solution. Fix an $\alpha \in (0,1)$ and an $E \subseteq \mathbb{R}^d$. Choose $\epsilon < \frac{\alpha-1}{\alpha}|E|_e$. We claim that we may relax our conditions to being a box Q such that $|E \cap Q|_e \geq \alpha|Q|$, because if this is true for the box, we may refine our countable cover to find a cube by rearranging our boxes to be cubes. We may find a countable cover of E with boxes $\{Q_k\}$ such that $|E|_e \leq \Sigma|Q_k| \leq |E|_e + \epsilon$. Now, suppose, to the contrary, that $|E \cap Q_k|_e < \alpha|Q_k|$ for all k. Then, we have that since $E = \cup(E \cap Q_k)$, that $|E|_e \leq |\cup(E \cap Q_k)|_e \leq \Sigma|E \cap Q_k|_e \leq \Sigma\alpha|Q_k|$. But, from our original statement, we have $\Sigma|Q_k| \leq |E|_e + \epsilon \implies \Sigma\alpha|Q_k| \leq \alpha|E|_e + \alpha\epsilon$, so then we have that $|E|_e \leq \alpha|E|_e + \alpha\epsilon$. However, from our choice of ϵ, we have that $\alpha|E|_+\alpha\epsilon < \alpha|E|_e + \alpha\frac{1-\alpha}{\alpha}|E|_e = \alpha|E|_e + \alpha - 1|E|_e = |E|_e$. But this implies from our inequality that $|E|_e < |E|_e$, which is a contradiction. Then, for any $\alpha \in (0,1)$, we can find a box Q and thus a cube such that $|E \cap Q|_e \geq \alpha|Q|$.

$\square$

Problem 2.2.44. Let E be a measurable subset of $\mathbb{R}^d$ such that $|E| < \infty$. Suppose that A and B are disjoint subsets of E such that $E = A \cup B$. Prove that:

A and B are measurable $\iff |E| = |A|_e + |B|_e$

Solution. Clearly, if A, B are measurable, then from our disjoint union and countable additivity, we have that $|E| = |A| + |B|$.

Now, suppose we have $|E| = |A|_e + |B|_e$. Since E is measurable, we may find a $U \supseteq E$ such that $|U \setminus E| < \epsilon/2$. Now, let $\{Q_{k_A}\}$ be a collection of boxes that cover A such that their interiors cover A, and that $\Sigma_{k=1}^{\infty}|Q_{k_A}| = |A|_e + \epsilon/4$. Choose $\{Q_{k_B}\}$ in the same way.

Now, consider the related collection of sets $\{U \cap Q_{k_A}^o\}$ and $\{U \cap Q_{k_B}^o\}$, where $Q_{k_A}^o$ denotes the interior of Q_{k_A} and same for Q_{k_B}. Since the interior of boxes is open, and U is open, each $U \cap Q_{k_A}^o$ and $U \cap Q_{k_B}^o$ is open. Further, since $A \subseteq E$ and $A \subseteq \cup_{k_A=1}^{\infty} Q_{k_A}^o$ by construction, we have that $U_B = \cup_{k_A=1}^{\infty} U \cap Q_{k_A}^o \supseteq A$ and the same occurs for $U_B = \cup_{k_B=1}^{\infty} U \cap Q_{k_B}^o \supseteq B$. Since U_A, U_B are unions of open sets, they are open. Then we consider the following. By Caratheodory's Criterion, we take $U_A \setminus A$ as our set, and E as our measurable set to yield:

$$|U_A \setminus A|_e = |(U_A \setminus A) \cap E|_e + |(U_A \setminus A) \setminus E|_e$$

.

But, we notice that $(U_A \setminus A) \cap E = U_A \cap A^c \cap (A \cup B) = U_A \cap B$, and $(U_A \setminus A) \setminus E = U_A \cap A^c \cap E^c = U_A \cap E^c = U_A \setminus E$. Now, due to our construction, we have that since $U_A \subset U$, that $(U_A \setminus E) \subseteq (U \setminus E)$. Further, because we have $|E| = |A|_e + |B|_e$, then, we can claim from our construction that $|U_A \cup U_B| + |U_A \cap U_B| = |U_A| + |U_B| \leq |\Sigma_{k=1}^{\infty}|Q_{k_A}| + \Sigma_{k=1}^{\infty}|Q_{k_B}| = |A|_e + \epsilon/4 + |B|_e + \epsilon/4 = |E| + \epsilon$, but, by construction, we have $U_A \cup U_B \supseteq E$, so $|U_A \cup U_B|_e \geq |E|$, so $|U_A \cap U_B| \leq \epsilon/2$. But, since $B \subseteq U_B$, we have then that $|U_A \cap B| \leq |U_A \cap U_B| \leq \epsilon/2$.

Thus, we have that:

$$|U_A \setminus A|_e = |U_A \cap B|_e + |U_A \setminus E|_e \leq |U_A \cap U_B|_e + |U \setminus E|_e \leq \epsilon/2 + \epsilon/2 = \epsilon$$

Thus, A is Lebesgue measurable, and B follows with the same argument, switching the labels.

$\square$

Problem 2.2.50. Let X be a set, and let Σ be the collection of all $E \subseteq X$ such that at least one of E or $X \setminus E$ is countable. Prove that Σ is a σ-algebra on X.

Solution. Here, we check that Σ is non-empty, that it is closed under complements in X, and that it is closed under countable unions.

Clearly, Σ is non-empty, as $\emptyset$ is countable, vacuously. Then, $\emptyset, X \in \Sigma$.

Further, by construction, we can see that Σ is closed under complements. Suppose $E \in \Sigma$. Then, consider $E^c = X \setminus E$. Because $E \in \Sigma$, if $X \setminus E$ is countable, then we're done. Otherwise, suppose $X \setminus E$ is not countable, and E is countable. Well, by deMorgan's Laws, we have that: $X \setminus (X \setminus E) = X \cap (X \setminus E)^c = X \cap (X \cap E^c)^c = X \cap (X^c \cup E) = X \cap E = E$, which is countable. Then, E^c is countable.

Now, suppose we have $\cup_{k=1}^{\infty} E_k$, with $E_i \in \Sigma$ for all $i \in \mathbb{N}$. If each E_i is countable, then we're done, since a countable union of countable sets is countable. Else, suppose E_n is uncountable for some n. Then, because $E_n \in \Sigma$, we have that E_n^c must be countable. Consider, by deMorgan's Laws, $(\cup_{k=1}^{\infty} E_k)^c = \cap_{k=1}^{\infty} E_k^c$. In particular, we have then that $\cap_{k=1}^{\infty} E_k^c \subseteq E_n^c$, and a subset of a countable set is countable. Thus, Σ is closed under countable unions.

Thus, such a Σ is a σ-algebra. $\square$

2.2

Problem 2.3.17. Assume that $E \subseteq \mathbb{R}^d$ is measurable, with $0 < |E| < \infty$, and $A_n \subseteq E$ are measurable sets such that $|A_n| \to |E|$ as $n \to \infty$. Prove that there exists a subsequence $\{A_{n_k}\}_{k \in \mathbb{N}}$ such that $|\cap A_{n_k}| > 0$. Show by example that this may fail if $|E| = \infty$.

Solution. Arbitrarily, choose A_{n_1} as $A_i : |A_i| > 0.9|E|$. Now, choose A_{n_2} such that $|A_{n_2}| \geq 0.99|E|$ which we can do, because $|A_n| \to |E|$ as $n \to \infty$. We notice that because A_{n_1}, A_{n_2} are measurable, so are their intersection. From 2.2.32 in homework #1, since A_i are measurable for all i, we have that $|A_{n_1}| + |A_{n_2}| = |A_{n_1} \cup A_{n_2}| + |A_{n_1} \cap A_{n_2}|$. But because all of these have finite measure, and because we chose $|A_{n_2}| + |A_{n_1}| \geq |E|$, this implies that $|A_{n_1} \cap A_{n_2}| > 0$. In particular, we have that $|A_{n_1} \cap A_{n_2}| \geq |A_{n_1}| + |A_{n_2}| - |E| \geq (0.9 + 0.99 - 1)|E| = 0.89|E| = (1 - \Sigma_{n=1}^{2} 10^{-n})|E|$. Iteratively, we can continue this process at each step, taking $|A_{n_{m+1}}| \geq (1 - 10^{-n_{m+1}})|E|$, which we can always do due to the convergence. But then, we have that $|\cap_{k=1}^{m} A_{n_k} \cap A_{n_{m+1}}| \geq |\cap_{k=1}^{m} A_{n_k}| + |A_{n_{m+1}}| - |E| \geq (1 - \Sigma_{n=1}^{m} 10^{-n})|E| + (1 - 10^{-(m+1)})|E| - |E| = (1 - \Sigma_{n=1}^{m+1} 10^{-n})|E|$.

Now, we have a sequence of measurable sets $\{\cap_{i=1}^{k} A_{n_k}\}_{k \in \mathbb{N}}$, and further, a decreasing set with $A_{n_1} \supseteq A_{n_1} \cap A_{n_2} \supseteq .... \supseteq \cap_{i=1}^{k} A_{n_k} \supseteq ...$ all with finite measure. Then, we apply continuity from above to find that $|\cap_{i=1}^{\infty} A_{n_i}| = |\cap_{i=1}^{\infty} \cap_{k=1}^{i} A_{n_k}| = \lim_{j \to \infty} |\cap_{k=1}^{j} A_{n_k}|$. But, we have by construction that $|\cap_{k=1}^{j} A_{n_k}| = (1 - \Sigma_{n=1}^{j} 10^{-n})|E|$. Taking the limit of that, we find then that $\lim_{j \to \infty} |\cap_{k=1}^{j} A_{n_k}| = \lim_{j \to \infty} (1 - \Sigma_{n=1}^{j} 10^{-n})|E| = 8/9|E| > 0$.

Now, take $d = 2$, and let $\{p_k\}$ be an enumeration of the prime numbers. Define $E = \cup_k \{(x, y) : p_k < x < p_{k+1}\}$. This is measurable because this is a countable union of open sets, and has infinite measure because it is unbounded in the y-direction. Now, consider $A_i = \{(x, y) : p_i < x < p_{i+1}\}$. This is also measurable, an open set, contained within E, and $|A_i| = |E| = \infty$ for all i. However, by construction, $A_i \cap A_j = \emptyset$ for all $i \neq j$, so there cannot exist a subsequence with intersection with postiive measure.

$\square$

Problem 2.3.19. Let E be a measurable subset of $\mathbb{R}^d$, and set $f(t) = |E \cap B_t(0)|$ for $t > 0$. Prove the following statements:

(a) f is monotone increasing and continuous on $(0, \infty)$

(b) $\lim_{t \to 0^+} f(t) = 0$

(c) $\lim_{t \to \infty} f(t) = |E|$

(d) If $|E| < \infty$, then f is uniformly continuous on $(0, \infty)$.

Solution. (a)

Firstly, we claim that f is monotone increasing. Take real numbers $t_1 > t_0$ where $t_0, t_1 > 0$. Suppose $x \in E \cap B_{t_0}(0)$. Then, $x \in E$ and $x \in B_{t_0}(0)$. Since $t_1 > t_0$, $B_{t_0}(0) \subseteq B_{t_1}(0)$. Then, $x \in B_{t_1}(0)$ and thus, $x \in E \cap B_{t_1}(0)$. Since the choice of x was arbitrary, we have that $E \cap B_{t_0}(0) \subseteq E \cap B_{t_1}(0)$, and by the monotonicity of the Lebesgue measure, this implies that $|E \cap B_{t_0}(0)| \leq |E \cap B_{t_1}(0)|$. Since the choice of t_0, t_1 was arbitrary other than $t_1 > t_0$, this is true for every $t_1 > t_0$, and thus f is monotone increasing.

Now, claim that this is continuous. From Theorem 2.3.15, we have that if L is a linear transformation $L : \mathbb{R}^d \to \mathbb{R}^d$, and E a measurable set, then $|L(E)| = |\det(L)||E|$. Then, since rI for r a real postiive number, I the identity matrix maps the unit ball centered on the origin to the ball with radius r centered on the origin, and $\det(rI) = r^d$, we have that $|B_{t_0}(0)| = t_0^n|B_1)(0)|$. Consider $f(x) - f(y)$ and, wlog, choose x ¿ y. Then we have $f(x) - f(y) = |E \cap B_x(0)| - |E \cap B_y(0)|$. Due to measures being positive, the worst this could be would be if $|E \cap B_y(0)| = 0$. But, in that case, since $B_y(0) \subseteq B_x(0)$, we would have $|E \cap B_x(0)| - |E \cap B_y(0)| \leq |E \cap (B_x(0) \setminus B_y(0))| \leq |B_x(0) \setminus B_y(0)| = |B_x(0)| - |B_y(0)| = x^d|B_1(0)| - y^d|B_1(0)|$. But, we already know that polynomials are continuous, so we know that for any $\epsilon > 0$, we can choose $\delta > 0$ such that $d(x, y) \implies |x^d - y^d||B_1(0)| < \epsilon$. Thus, f is continuous.

(b)

To prove this, we want to prove a related claim: $\lim_{t \to 0} B_t(0) = \{0\}$.

Let $\epsilon > 0$ be given. Then, we choose $\delta = \epsilon$. Here, we recall quickly that the defintion of $B_t(0) = \{x \in \mathbb{R}^d : \|x\| = d(x, 0) \leq t\}$, where we use $\|x\|$ to denote the norm of x. Let $x_t \in B_t(0)$ such that $0 < t < \delta$. Well, $d(x_t, 0) = \|x_t\| \leq t < \delta = \epsilon$. Then, by problem 1.1.23's definition, we have that $x_t \to 0$ as $t \to 0$. Since the choice of x_t was arbitrary, this implies that $\lim_{t \to 0} B_t(0) = \{0\}$.

Then, we have that, since E is constant with respect to t, $\lim_{t \to 0^+} f(t) = \lim_{t \to 0^+} |E \cap B_t(0)| = |E \cap (\lim_{t \to 0^+} B_t(0)) = |E \cap \{0\}| \leq |\{0\}|$. This is a countable set, so $|\{0\}| = 0$, so $\lim_{t \to 0^+} f(t) = 0$.

(c)

In a similar vein to (b), we will prove the related claim: $\lim_{t \to \infty} B_t(0) = \mathbb{R}^d$. Here, we just need to show that for any $x \in \mathbb{R}^d$, there exists a t_0 such that $x \in B_{t_1}(0)$ when $t_1 > t_0$. But this is easy - just choose $t_0 = \|x\|$. Then, by construction, $x \in B_{t_1}(0)$ for $t_1 > t_0$.

Then, in the same fashion, because E is independent of t, we have that $\lim_{t \to \infty} f(t) = \lim_{t \to \infty} |E \cap B_t(0)| = |E \cap (\lim_{t \to \infty} B_t(0)) = |E \cap \mathbb{R}^d| = |E|$.

(d)

Essentially, we have a monotone, continous, bounded function. So, let $\epsilon > 0$ be given. If $|E| = 0$, then we're done, as $f(t)$ is identically zero. Else, because the function is monotone, there exists t_0 such that $|E| > f(t_1) > (|E| - \epsilon)$ for all $t_1 > t_0$. This implies then, that for $t_2, t_3 > t_0$, that $|f(t_3) - f(t_2)| \leq (|E| - (|E| - \epsilon)) = \epsilon$, irrespective of the point on the interval (t_0, ∞). Now, consider the related function on $[0, t_0]$, $\overline{f}$ which agrees with f on $(0, t_0]$ and attains 0 at $t = 0$. By part (a) combined with part (b), this function is continous. Moreover, because this is a continous function on a compact set, it is uniformly continuous on the entirety of the compact set. Then, this implies that for our ϵ, there exists a δ such that $d(x, y) < \delta \implies d(\overline{f}(x), \overline{f}(y)) < \epsilon$. But, because $\overline{f} = f$ on $(0, t_0)$, this δ works for f on $(0, t_0]$ as well. Since δ exists, and is independent of a point in $(0, \infty)$ that we are checking continuity at, f is uniformly continuous. □

2.3

Problem 2.4.8. (a) Prove that continuity from below holds for exterior Lebesgue measure. That is, if $E_1 \subseteq E_2 \subseteq ...$ is any nested increasing sequence of subsets of $\mathbb{R}^d$, then $|\cup E_k|_e = \lim_{k \to \infty} |E_k|_e$.

(b) Show that there exists sets $E_1 \supseteq E_2 \supseteq ...$ in $\mathbb{R}$ such that $|E_k|_e < \infty$ for every k, and that:

$$|\cap_{k=1}^{\infty} E_k|_e < \lim_{k \to \infty} |E_k|_e$$

Solution. (a)

First, we note that due to the monotonicity of the outer measure, we have that $E_i \subseteq \cup_{k=1}^{\infty} E_i \implies |E_i|_e \leq |\cup_{k=1}^{\infty} E_i|_e$. Now, assume that $\lim k \to \infty |E_k|_e = \infty$. Then, due to monotonicity, we have that $|\cup_{k=1}^{\infty} E_i|_e = \infty$, as otherwise, if it were bounded, we could find a $|E_i|_e \geq |\cup_{k=1}^{\infty} E_i|_e$. More generally, due to monotonicity, we already will have that $\lim_{k\to\infty} |E_k|_e \leq |\cup E_k|_e$.

Now, suppose $\lim k \to \infty |E_k|_e$ is finite. Let $\epsilon > 0$ be given. For each E_i, we may find an open set $U_i \supseteq E_i$ such that $|E_i| \leq |U_i| \leq |E_i| + \epsilon$. Construct the related sequence of sets $V_k \cup_{i=k}^{\infty} U_k$. By construction, we have that these sets are nested $V_1 \subseteq V_2 \subseteq ....$ Consider the union over all k $\cup_k V_k$. By the construction of the U_i, since $E_i \subseteq E_j$ for $j > i$, then, $E_i \subseteq U_i$ for all $j > i$, and thus $E_i \subseteq V_i \subseteq U_i$ for all i, it follows that $\cup E_k \subseteq \cup V_k$. Then, we have that, via continuity from below, that $|\cup E_k| \leq |\cup V_k| = \lim_{k\to\infty} |V_k|$. But, from our construction, we also have that for each i, $|E_i| \leq |V_i| \leq |U_i| \leq |E_i| + \epsilon$, and thus $\lim_{k\to\infty} |V_k| \leq \lim_{k\to\infty} |E_k| + \epsilon$. Then, we have that:

$$|\cup E_k| \leq |\cup V_k| = \lim_{k\to\infty} |V_k| \leq \lim_{k\to\infty} |E_k| + \epsilon$$

Since ϵ can be taken to be arbitrarily small, this now implies that $|\cup E_k| = \lim_{k\to\infty} |E_k|$, as desired.

(b)

Take the set constructed in Heil for the proof of Theorem 2.4.5. That is, define the set M as such: Start with the interval $[0, 1]$ and define the equivalence relation $x \sim y = \{x = y + q : q \in \mathbb{Q}\}$. Consider the equivalence classes of $[0, 1]/\sim$. Construct M by applying the axiom of choice, and selecting one element from each equivalence class. Continue and construct the collection $\{M_k\}$ where we take $\{q_k\}$ as an enumeration of the rationals, and we define $M_k = (M + q_k)/[0, 1]$, that is, modulo the interval $[0, 1]$, so that each $M_k \subseteq [0, 1]$. We notice that M is in our collection, because 0 is rational. Because equivalence classes partition a set, we are guaranteed that each M_k is disjoint, and that $\cup M_k = [0, 1]$.

Now, consider the following sequence of sets. Define $E_1 = [0, 1]$, and define $E_i = E_1 \setminus \cup_{k=2}^{i} M_{k-1}$ for $i \geq 2$, where we just assume the enumeration of the M_k starts at $k = 1$.

Here, we go off to the side and prove a result from Heil: 2.2.43(d). Define the inner Lebesgue measure of a set $A \subseteq \mathbb{R}^d$ to be $|A|_i = \sup\{|F| : F$ is closed and $F \subseteq A\}$. If E is Lebesgue measurable, and $A \subseteq E$, then $|E| = |A|_i + |E \setminus A|_e$. Because E is Lebesgue measurable, we may take a $U \supseteq E$ such that $|U \setminus E| < \epsilon$. Take any closed set $F \subseteq A$. We notice that $U \setminus F$ is open, because $U \cap F^c$ is an intersection of open sets. Moreover, it is a cover of $E \setminus A$ by construction. So, we have that $|U| = |E| + |U \setminus E| = |F| + |U \setminus F|$, where we have equality because $F, U \setminus F$ are measurable. Now, take any sequence of F_k such that $|A|_i leq |F_k| + 1/k$, which we may do because the inner measure is a supremum. Then, we note for each F_k, $(U \setminus F_k)$ is a sequence of sets such that these are open, and converge to $|U \setminus A|_e$. Then, we have that $|E| + |U \setminus E| = |A|_i + |U \setminus A|_e$. Now, since the choice of U is arbitrary, we can actually shrink U such that $|U \setminus E| \to 0$, and $|U \setminus A|_e \to |E \setminus A|_e$, because $|U \setminus A| \leq |E \setminus A| + |(U \setminus E) \setminus A| \leq |E \setminus A| + |U \setminus E|$, and $|U \setminus E| < \epsilon$. Then, we find that $|E| = |A|_i + |E \setminus A|_e$.

Now, consider the inner Lebesgue measure. Clearly, we have that it is translation invariant, as the Lebesgue measure of a closed set is translation invariant. Further, we also have monotonicity from the monotonicity of the Lebesgue measure, as well as subadditivity. (that is, suppose we have $A \cup B$, and $F_A \subseteq A, F_B \subseteq B$ with F_A, F_B closed. Then, $F_A \cup F_B$ is closed, and $|F_A \cup F_B| \leq |F_A| + |F_B|$ since they need not be disjoint. Since this is true for any F_A, F_B, this implies that $|F_A \cup F_B|_i \leq |F_A|_i + |F_B|_i$.)

Then, by the same argument that shows M as non-measurable in Heil, we can claim that because $[0, 1] = \overline{[0, 1]}$, the closure, that $[0, 1]$ has inner measure 1, and that $|M|_i = 0$ because otherwise, we have a countable sum of inner measures of M as M_k are just translations.

Now, consider the outer measure of each E_i. $E_1 = [0, 1]$. From what we proved about the inner measure, we have that $|[0, 1]| = |\cup_{k=2}^{i} M_{k-1}|_i + |[0, 1] \setminus \cup_{k=2}^{i} M_{k-1}|_e \implies |E_i|_e = |[0, 1] \setminus \cup_{k=2}^{i} M_{k-1}|_e = |[0, 1]| = 1$. So, we have a sequence of sets, with outer measure identically 1, so then we have that $\lim_{k\to\infty} |E_k|_e = 1$. However, we know that $\cap E_k = \emptyset$ because since the M_k partition $[0, 1]$, for every $x \in [0, 1]$, $x \in M_{k_0}$ for exactly one k_0. But then, by construction, this means that $x \in \cup_{k=2}^{i} M_{k-1}$ for $i > k_0$, so $x \notin E_i$ for any $i > k_0$. Since the choice of x was arbitrary, this is true for all x, and thus $\cap E_k = \emptyset \implies |\cap E_k|_e = 0 < \lim_{k\to\infty} |E_k|_e = 1$ $\square$

Problem 2.4.10. Given any integer $d > 0$, show that there exists a set $N \subseteq \mathbb{R}^d$ that is not Lebesgue measurable.

Solution. We use the same construction and argument in Heil, and extend to multiple dimensions.

Fix a dimension d. Consider the rationals in the unit box $\Pi_{i=1}^d[0,1] \cap \mathbb{Q}^d$. We define the equivalence relationship $x \sim y \iff x - y \in \mathbb{Q}^d$. This is an equivalence relation because it is reflexive ($x - x = 0 \in \mathbb{Q}^d$), symmetric (if $x - y \in \mathbb{Q}^d$, then $-(x-y) = y - x \in \mathbb{Q}^d$ by being a ring) and transitive (if $x - y \in \mathbb{Q}^d$ and $y - z \in \mathbb{Q}^d$, then $x - z = x - y + y - z = (x-y) + (y-z) \in \mathbb{Q}^d$ due to $\mathbb{Q}^d$ being a ring). Then, the equivalence classes partition $[0,1]^d$ by virtue of being an equivalence relationship. Using the axiom of choice, construct a set M such that M has one representative from each (uncountably many) equivalence class.

Suppose M, and actually, all sets are measurable, under the Lebesgue measure μ, which we note to have the following properties for measurable sets:

(a) $\mu([0,1]^d) = 1$

(b) If $\{E_i\}$ is a countable collection of disjoint measurable subsets of $\mathbb{R}^d$, then $\mu(\cup E_i) = \Sigma\mu(E_i)$

(c) $\mu(E + h) = \mu(E)$ for every $E \subseteq \mathbb{R}^d$ and for any $h \in \mathbb{R}^d$.

Take an enumeration of $\mathbb{Q}^d \cap [-1,1]^d$, and call it $\{q_k\}$. This should exist because d is finite, countable, and $\mathbb{Q}$ is countable, so has cardinality of at most $\mathbb{N} \times \mathbb{N}$, which is countable. Consider the sets $M_k = M + q_k$. These sets must be disjoint, because, suppose not, that is $x \in M_i \cap M_j$. Then, $x = [x] + q_i = [x'] + q_j$, for some equivalence classes $[x], [x']$. But then, we have that $[x] = [x'] + (q_j - q_i)$, with $q_j - q_i$ rational. But, then $[x], [x']$ differ by a rational, they are the same equivalence class then, which implies $x = x'$ as we only pick one element from each equivalence class, which implies that $q_j = q_i$.

Consider the union of all such M_k, $\cup_{k=1}^\infty M_k$. This is a countable union of disjoint subsets of $\mathbb{R}^d$. Further, since $q_k \in [-1,1]^d$, we have that each $M_k \subseteq [-1,2]^d$. But also, because M contains one element from every equivalence relation, we hit with any rational in $\mathbb{Q}^d \cap [-1,1]^d$, and every element of $[0,1]^d$ belongs to some equivalence class, $[0,1]^d \subseteq \cup M_k$.

We notice by (a), that we have $\mu([0,1]^d) = 1$. By using the translations and countable additivity, we also have that $\mu([-1,2]^d) = 3^d$.

Then, using the monotonicity of the Lebesgue measure with our set inclusions, we have that:

$$1 = \mu([0,1]^d) \leq \mu(\cup_{k=1}^\infty M_k) \leq \mu([-1,2]^d) = 3^d$$

However, by the definition of M_k, (b), and (c), we have that:

$$\mu(\cup_{k=1}^\infty M_k) = \Sigma_{k=1}^\infty \mu(M_k) = \Sigma_{k=1}^\infty \mu(M)$$

Then, we have that $1 \leq \Sigma_{k=1}^\infty \mu(M) \leq 3^d$. But, μ can only take on values in $[0,\infty]$, and in particular then, $\Sigma_{k=1}^\infty \mu(M)$ is either 0 if $\mu(M) = 0$ and infinite otherwise. But that is a contradiction with our inequality.

Thus, M may not be a Lebesgue measurable set.

□