Homework #6

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Math 285: Homework #6

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Question 1. Show that the pullback of covectors by a linear map satisfies the two functorial properties:

(i) If $\mathbb{1}_V : V \to V$ is the identity map on V, then $\mathbb{1}_V^* = \mathbb{1}_{A_k(V)}$, the identity map on $A_k(V)$.

(ii) If $K : U \to V$ and $L : V \to W$ are linear maps on vector spaces, then:

$$(L \circ K)^* = K^* \circ L^* : A_k(W) \to A_k(U)$$

Solution. (i)

Let $v_1, ..., v_k \in V, f \in A_k(V)$, and by definition then, we have that:

$$\mathbb{1}_V^*(f)(v_1, ..., v_k) = f(\mathbb{1}_V(v_1), ..., \mathbb{1}_V(v_k)) = f(v_1, ..., v_k)$$

Then, since we have that $\mathbb{1}_V^*(f)$ and f act identically on arbitrary $v_1, ..., v_k \in V$, this implies that $\mathbb{1}_V^*(f) = f$. Since the choice of f was arbitrary, this is true for all $f \in A_k(V)$, and therefore, $\mathbb{1}_V^*$ acts as identity on $A_k(V)$, thus is equal to $\mathbb{1}_{A_k(V)}$.

(ii)

Let $f \in A_k(V)$, and let $v_1, ..., v_k \in V$. We may consider the action of $K^* \circ L^*$ on f:

$$K^* \circ L^*(f)(v_1, ..., v_k) = K^*(L^*(f))(v_1, ..., v_k) = L^*(f)(K(v_1), ..., K(v_k)) =$$
$$f(L(K(v_1)), ..., L(K(v_k))) = f(L \circ K(v_1), ..., L \circ K(v_k)) = (L \circ K)^*(f)(v_1, ..., v_k)$$

Again, since this is true for all $v_1, ..., v_k$, this is an equality of functions $K^* \circ L^*(f) = (L \circ K)^*(f)$. Since this is true for all $f \in A_k(V)$, this is an equality of maps $K^* \circ L^* = (L \circ K)^*$.

□

Question 2. Let $L : V \to V$ be a linear operator on a vector space with dimension n. Show that the pullback $L^* : A_n(V) \to A_n(V)$ acts as multiplication by the determinant of L.

Solution. We recall that from Proposition 3.36, that if $e_1, ..., e_n$ is a basis for V, and $\alpha^1, ..., \alpha^n$ is the dual basis in $V^\vee$, that for a multi-index $I = (i_1 < ... < i_k)$, the alternating k-linear functions have basis α^I. Then, of course, we say that the $A_n(V)$ are scalar multiples of $\alpha^1 \wedge ... \wedge \alpha^n$. Since the pullback is linear, we need only show that L^* acts as multiplication by its determinant on $\alpha^1 \wedge ... \wedge \alpha^n$.

Now, as a 1-linear function, consider the pullback $L * (\alpha^i)$. Considering an arbitrary vector $v = \sum_{j=1}^n v_j e_j \in V$, and writing A as a matrix in the e_j basis, we have that:

$$L^*(\alpha^i)(v) = \alpha^i(L(v)) = \alpha^i \left(\sum_{j=1}^n \left[\sum_{k=1}^n A_{jk} v_k \right] e_j \right) = \sum_{j=1}^n \left[\sum_{k=1}^n A_{jk} v_k \right] \alpha^i e_j = \sum_{j=1}^n \left[\sum_{k=1}^n A_{jk} v_k \right] \delta_i^j = \sum_{k=1}^n A_{ik} v_k$$

We notice, that because $v_k = \alpha^k(v)$, that we may rewrite this as:

$$L^*(\alpha^i)(v) = \sum_{k=1}^{n} A_{ik}\alpha^k(v)$$

Since the choice of v were arbitrary, this is an equality of covectors:

$$L^*(\alpha^i) = \sum_{k=1}^{n} A_{ik}\alpha^k$$

Then, we consider $L^*(\alpha^1 \wedge ... \wedge \alpha^n)(v_1, ..., v_n)$, for arbitrary vectors $v_1, ..., v_n \in V$. We see that:

$$L^*(\alpha^1 \wedge ... \wedge \alpha^n)(v_1, ..., v_n) = (\alpha^1 \wedge ... \wedge \alpha^n)(L(v_1), ..., L(v_n)) = A(\alpha^1 \otimes ... \otimes \alpha^n)(L(v_1), ..., L(v_n)) =$$

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\alpha^1(L(v_{\sigma(1)}))...\alpha^n(L(v_{\sigma(n)})) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)L^*(\alpha^1)(v_{\sigma(1)})...L^*(\alpha^n)(v_{\sigma(n)}) =$$

$$A(L^*(\alpha^1) \otimes ... \otimes L^*(\alpha^n))(v_1, ..., v_n) = L^*(\alpha^1) \wedge ... \wedge L^*(\alpha^n)(v_1, ..., v_n)$$

Again, varying over all $v_1, ..., v_n$, we see an equality of covectors:

$$L^*(\alpha^1 \wedge ... \wedge \alpha^n) = L^*(\alpha^1) \wedge ... \wedge L^*(\alpha^n)$$

Now, by homework 1, question 7, we have that if $\beta^i = \sum_{j=1}^{k} a^i_j \gamma^j$, for two sets of covectors $\{\beta^i\}, \{\gamma^j\}$, $1 \leq i, j \leq k$, we have that:

$$\beta^1 \wedge ... \wedge \beta^k = \det(A)\gamma^1 \wedge ... \wedge \gamma^k$$

Taking $\beta^i = L^*(\alpha^i)$, and $\gamma^i = \alpha^i$, we see that because $L^*(\alpha^i) = \sum_{k=1}^{n} A_{ik}\alpha^k$, we have that:

$$L^*(\alpha^1 \wedge ... \wedge \alpha^n) = L^*(\alpha^1) \wedge ... \wedge L^*(\alpha^n) = \det(A)\alpha^1 \wedge ... \wedge \alpha^n$$

Thus, L^* acts on $A_n(V)$ by multiplication by $\det(A)$, from the linearity and basis considerations before.

□

Question 3. (a) Let $i : S^1 \to \mathbb{R}^2$ be the inclusion map of the unit circle. Denote the standard coordinates on $\mathbb{R}^2$ as (x, y) and denote the restriction of these coordinates to S^1 as $(\overline{x}, \overline{y})$. Clearly, we have that $\overline{x} = i^*(x), \overline{y} = i^*(y)$.

On the upper semicircle $U = \{(a, b) \in S^1 : b > 0\}$, $\overline{x}$ is a local coordinate, so $\partial/\partial\overline{x}$ is well-defined. Prove that for $p \in U$, we have that:

$$i_*\left(\left.\frac{\partial}{\partial\overline{x}}\right|_p\right) = \left.\left(\frac{\partial}{\partial x} + \frac{\partial\overline{y}}{\partial\overline{x}}\frac{\partial}{\partial y}\right)\right|_p$$

(b) Let C be a smooth curve in $\mathbb{R}^2$. Let U be a chart on C such that $\overline{x}$, the restriction of the coordinate x on $\mathbb{R}^2$ is a local coordinate. Generalize the result of part (a) to C.

Solution. (a)

Let $p \in U$. First, we want to show that the inclusion is smooth on $(U, \overline{x})$. Let $\overline{x}(p) = x_0 \in V \subseteq \mathbb{R}$

Clearly, following the chart then, we have that:

$$i \circ \overline{x}^{-1} = \left(x_0, \sqrt{1 - x_0^2}\right)$$

With respect to x_0, the coordinate value for $\overline{x}$, this is smooth in both of the standard coordinate functions of $i \circ \overline{x}^{-1}$ as x_0 is polynomial, being identity, and $\sqrt{1-x_0^2}$ being C^∞ since on U, $\overline{x}(p) \geq 0$. Since the choice of $p \in U$ were arbitrary, i is smooth on every point in U.

Here, we recall Propostion 8.13, that the differential i_* is represented by the following matrix on U:

$$\begin{bmatrix} \left.\frac{\partial(x \circ i)}{\partial \overline{x}}\right|_p & \left.\frac{\partial(y \circ i)}{\partial \overline{x}}\right|_p \end{bmatrix}$$

Then, by the setting of the problem, we have that $\overline{x} = i^*(x) = x \circ i$ and similarly for $\overline{y}$. Thus, we may rewrite this as:

$$\begin{bmatrix} \left.\frac{\partial \overline{x}}{\partial \overline{x}}\right|_p & \left.\frac{\partial \overline{y}}{\partial \overline{x}}\right|_p \end{bmatrix} = \begin{bmatrix} 1\Big|_p & \left.\frac{\partial \overline{y}}{\partial \overline{x}}\right|_p \end{bmatrix}$$

Unpacking this statement relative to the basis $\{\frac{\partial}{\partial x}_{i(p)}, \frac{\partial}{\partial y}_{i(p)}\}$, and identifying $i(p)$ with p as i acts via inclusion, we see that:

$$i_* \left(\left.\frac{\partial}{\partial \overline{x}}\right|_p \right) = \left. \left(\frac{\partial}{\partial x} + \frac{\partial \overline{y}}{\partial \overline{x}} \frac{\partial}{\partial y} \right) \right|_p$$

as desired.

(b)

Generically then, let $C : (a, b) \to \mathbb{R}^2$ be a smooth map sending $t \mapsto (C_1(t), C_2(t))$. Fix a chart such that $\overline{x}$, the restriction of $\mathbb{R}^2$ to the image of C_1 is a local coordinate.

Clearly, on U, C_1 is smooth with respect to itself, being the identity. So it remains to show that C_2 is smooth with respect to C_1 on U.

However, this must be true because U is a C^∞ chart, and thus, looking at any chart that admits $\overline{y}$ as a local coordinate, $\overline{y} = y \circ \overline{x}^{-1}$ must be C^∞ with respect to $\overline{x}$, and thus, since we may identify C_1, C_2 with $\overline{x}, \overline{y}$ on a suitable chart, this implies that C_2 is C^∞ with respect to C_1 on U.

Therefore, we may go through the same procedure as part (a) since we do not use any properties of the circle there, and use Proposition 8.13 again to obtain:

$$i_* \left(\left.\frac{\partial}{\partial \overline{x}}\right|_p \right) = \left. \left(\frac{\partial}{\partial x} + \frac{\partial \overline{y}}{\partial \overline{x}} \frac{\partial}{\partial y} \right) \right|_p$$

We note because we are a smooth curve, by the chain rule, we may use $\frac{\partial \overline{y}}{\partial \overline{x}} = \frac{\partial C_2}{\partial t} \left(\frac{\partial C_1}{\partial t} \right)^{-1}$ to help compute this.

□

Question 4. Let $f : GL_n(\mathbb{R}) \to \mathbb{R}$ be the determinant map $A \mapsto \det(A)$. Consider a matrix $B \in SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det(A) = 1\}$. By example 9.10 (note: numbering based off of Chapter 3, v1-1 in Canvas), for $A = [a_{kl}]$, there exists a (k, l) such that the partial dervative $\frac{\partial f}{\partial a_{kl}}(A) \neq 0$.

Use Lemma 9.9 and the implicit function theorem (9.8) to prove the following:

(a) There exists a neighborhood of A in $SL_n(\mathbb{R})$ such that a_{ij}, $(i, j) \neq (k, l)$ forms a coordinate system, and a_{kl} is a C^∞ function of the other entries.

(b) The group multiplication map:

$$\overline{\mu} : SL_n(\mathbb{R}) \times SL_n(\mathbb{R}) \to SL_n(\mathbb{R})$$

is C^∞.

Solution. (a)

Without too much trouble, it is easy to see f is a C^∞ map of manifolds, as we can view it as a subset of $\mathbb{R}^{n^2}$, so we may take charts compatible with standard coordinates being each matrix entry. Since the determinant is a degree n homogeneous polynomial in the matrix entries, it is C^∞ on this chart. Since we may take the open set of this chart to be all of $GL_n(\mathbb{R})$, we see f as a C^∞ map.

We notice that we can view $SL_n(\mathbb{R}) = f^{-1}(1)$. Thus, by Theorem 9.8, $SL_n(\mathbb{R})$ is a regular submanifold with dimension $n^2 - 1$.

Fix some $A \in SL_n(\mathbb{R})$.

Now, following example 9.12 with the special linear group, defining m_{ij} as the determinant of the submatrix obtained by deleting the i-th row and the j-th column, we may rewrite the map f as, for a selected row $1 \leq i \leq n$:

$$f(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} m_{ij}$$

Since m_{ij}, varying across j, is obtained by deleting the i-th row, m_{ij} is not a function of a_{il} for any $1 \leq j, l \leq n$. Further, since the determinant of matrices in $SL_n(\mathbb{R})$ is exactly 1, by the determinental rank being n, there must exist (k, l) such that $m_{kl} \neq 0$

Then, for such a (k, l) we have that:

$$\frac{\partial f}{\partial a_{kl}} = \sum_{j=1}^{n} \frac{\partial}{\partial a_{kl}} (-1)^{k+j} a_{kj} m_{kj} = \sum_{j=1}^{n} (-1)^{k+j} \delta_j^l m_{kj} = (-1)^{k+j} m_{kl}$$

Since this is itself a C^∞, being a homogeneous polynomial of degree $n-1$ in $GL_n(\mathbb{R})$, it is in particular, continuous. Thus, we may find some neighborhood U such that $A \in U$ and $\frac{\partial f}{\partial a_{kl}} \neq 0 \implies m_{kl} \neq 0$, by considering the open set $\mathbb{R} \setminus \{0\}$, and looking at the inverse image of the derivative.

Then, by Lemma 9.9, with a change of coordinates $F = f - 1$ and therefore $f^{-1}(1) = F^{-1}(0)$, but $\partial F \partial a_{kl} = \frac{\partial f}{\partial a_{kl}}$, we see that since on U, the Jacobian $J(F) = [\partial F \partial a_{kl}] \neq 0$, and therefore, we may replace the coordinate a_{kl} with $F = \det(A) - 1$ to obtain an adapted chart for $GL_n(\mathbb{R})$ relative to $SL_n(\mathbb{R})$.

Then, we have the chart $(U, a_{ij}, \det(A) - 1)$ with $1 \leq i, j \leq n, (i, j) \neq (k, l)$. Of course, due to the definition of $SL_n(\mathbb{R})$, we can see that $U \cap SL_n(\mathbb{R})$ is defined by the vanishing of $\det(A) - 1$, and therefore, the other a_{ij} coordinates form a coordinate system on this neighborhood.

Now, we wish to just see that we may define a_{kl} as a C^∞ function of the other entries on U. With some algebraic manipulation:

$$f(A) = \det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} m_{kj} \implies \det(A) - \sum_{j=1, j\neq l}^{n} (-1)^{k+j} a_{kj} m_{kj} = (-1)^{k+l} a_{kl} m_{kl} \implies$$

$$a_{kl} = (-1)^{k+l} \frac{1}{m_{kl}} \left(\det(A) - \sum_{j=1, j\neq l}^{n} (-1)^{k+j} a_{kj} m_{kj} \right)$$

Since $\det(A) = 1$ on $U \cap SL_n(\mathbb{R})$, we have that:

$$a_{kl} = (-1)^{k+l} \frac{1}{m_{kl}} \left(1 - \sum_{j=1, j\neq l}^{n} (-1)^{k+j} a_{kj} m_{kj} \right)$$

Of course, m_{ij} is a polynomial without a_{kl}, hence C^∞ in the other coordinates. Further, m_{kl} is non-0 on U here, and polynomial, hence $\frac{1}{m_{kl}}$ is C^∞. Thus, this is a sum and product of C^∞ functions, hence C^∞ on this domain.

(b)

Fix a $(A, B) \in SL_n(\mathbb{R}) \times SL_n(\mathbb{R})$.

By part (a), we may find $U \in SL_n(\mathbb{R})$ such that $A \in U$, and is defined as a submanifold chart with coordinates $a_{ij}, (i, j) \neq (k, l)$. Similarly, we may find V with $B \in V$ and with coordinates $b_{ij}, (i, j) \neq (k', l')$.

We may look at the neighborhood $(A, B) \in U \times V \subseteq SL_n(\mathbb{R}) \times SL_n(\mathbb{R})$. Further, for $C = A * B$, we may find a neighborhood $W \subseteq SL_n(\mathbb{R})$, such that $C \in W$ and C takes on the coordinates $c_{i''j''}, (i'', j'') \neq (k'', l'')$. Using this to look at the components $\overline{\mu}$, on $U \times V$, the natural matrix multiplication has the form, for $A * B = C = [c_{ij}]$ and $(m, n) \neq (k'', l'')$:

$$\overline{\mu}^{mn}(A, B) = c_{mn} = \sum_{p=1}^{n} a_{mp} b_{pn}$$

For $m \neq k, n \neq l'$, we can see that c_{mn} is familiarly a homogeneous polynomial of the coordinates, hence C^∞. On the other hand, when either $m = k$ or $n = l'$ we have a sum of degree 2 polynomials, as well as a term of the form $a_{kl} b_{ln}$ or $a_{mk'} b_{k'l'}$. By the considerations of part (a), a_{kl} is a C^∞ function of the other $n^2 - 1$ coordinates, and so is $b_{k'l'}$. Thus, in these cases, the overall sum is a sum and product of C^∞ functions, hence C^∞.

Thus, by this argument, each of the entries of C is a C^∞ function on the $a_{ij}, b_{i'j'}, (i, j) \neq (k, l), (i', j') \neq (k', l')$. Hence, each component of $\overline{\mu}$ is a C^∞ function on these coordinates. Since these are a set of coordinates for $U \times V$, this implies that $\overline{\mu}$ is a C^∞ function at (A, B). Since we may repeat this procedure for any $(A, B) \in SL_n(\mathbb{R}) \times SL_n(\mathbb{R})$, this implies that $\overline{\mu}$ is C^∞ on the entire set.

Note that technically, we don't need to exclude k'', l'' from the components (m, n) but since we merely need to verify from a coordinate neighborhood of $SL_n(\mathbb{R}) \times SL_n(\mathbb{R})$ to a suitable coordinate neighborhood of $SL_n(\mathbb{R})$, it is enough to look at the relevant coordinate functions.

□

Question 5. Let M be a manifold, and let $(U, \phi) = (U, x^1, ..., x^m), (V, \psi) = (V, y^1, ..., y^m)$ be charts such that $U \cap V \neq \emptyset$.

Consider the induced charts $(TU, \tilde{\phi}), (TV, \tilde{\psi})$ on TM, the total space of the tangent bundle, with transition map $\tilde{\psi} \circ \tilde{\phi}^{-1}$ that sends:

$$(x^1, ..., x^m, a^1, ..., a^m) \mapsto (y^1, ..., y^m, b^1, ..., b^m)$$

(a) Compute the Jacobian matrix of the transition map at $\phi(p)$.

(b) Show that the determinant of the transition map at $\phi(p)$ takes on the value:

$$\left(\det\left[\frac{\partial y^i}{\partial x^j}\right]\right)^2$$

Proof. (a)

By definition, the Jacobian matrix of a map $F : N \to M$ relative to a chart $(x^1, ..., x^n)$ of N is simply $J(F) = \left[\frac{\partial F^i}{\partial x^j}\right]$, where F^i is the i-th component of F in a chart of M.

Then, recalling section 12.2, we have that the action of the transition map $\tilde{\psi} \circ \tilde{\phi}^{-1}$ has the following form, where we recall that $x^i = r^i \circ \phi$ is the i-th component of ϕ and similar for y^j and ψ:

$$(\phi(p), a^1, ..., a^m) = (x^1(p), ..., x^m(p), a^1, ..., a^m) \mapsto (y^1(p), ..., y^m(p), b^1, ..., b^m) = (\psi \circ \phi^{-1}(\phi(p)), b^1, ..., b^n)$$

To compute the transformation that takes the a^i to a specified b^j, we recall that at a point $p \in U \cap V$, we may describe a fixed tangent vector $v \in T_pM$ by the bases $\left\{\frac{\partial}{\partial x^i}\right\}$ or equivalently by $\left\{\frac{\partial}{\partial y^j}\right\}$. Thus, we have the equality:

$$\sum_i a^i \frac{\partial}{\partial x^i} = \sum_j b^j \frac{\partial}{\partial y^j}$$

Using the standard trick and applying both sides onto y^k, we see that:

$$\sum_i a^i \frac{\partial y^k}{\partial x^i} = \sum_j b^j \frac{\partial y^k}{\partial y^j} = \sum_j b^j \delta^k_j = b^k$$

Thus, we have that:

$$(x^1(p), ..., x^m(p), a^1, ..., a^m) \mapsto \left(y^1(p), ..., y^m(p), \sum_i a^i \frac{\partial y^1}{\partial x^i}, ..., \sum_i a^i \frac{\partial y^m}{\partial x^i} \right)$$

Now, we are equipped to describe the Jacobian of this map. We see that for $1 \leq i \leq m$, $F^i = y^i$, and so, for $1 \leq j \leq m$, the derivatives correspond to the x^i, and so we have that via this numbering and denoting $\frac{\partial}{\partial x^i} y^j = y^j_i$, the upper left $m \times m$ submatrix A has the form:

$$A = \begin{bmatrix} y^1_1 & \dots & y^1_m \\ \vdots & \ddots & \vdots \\ y^m_1 & \dots & y^m_m \end{bmatrix}$$

On the other hand, the coordinates from $m+1$ to $2m$ represent the a^i. Since the transition map $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is induced via the transition map $\psi \circ \phi^{-1}$, we must have that the y^i are independent of the a^j. Thus, the Jacobian matrix has a $m \times m$ zero matrix in the top right.

Now, using the explicit description of the b^j, we may compute each block matrix of the lower $m+1, ..., 2m$ rows. In the first m coordinates, we see that for b^j, and denoting $\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} y^k = y^k_{ji}$:

$$\frac{\partial}{\partial x^k} \sum_i a^i y^j_i = \sum_i a^i \frac{\partial}{\partial x^k} y^j_i = \sum_i a^i y^j_{ik}$$

Thus, the lower left $m \times m$ block matrix has the form:

$$C = \begin{bmatrix} \sum_i a^i y^1_{i1} & \dots & \sum_i a^i y^1_{im} \\ \vdots & \ddots & \vdots \\ \sum_i a^i y^m_{i1} & \dots & \sum_i a^i y^m_{im} \end{bmatrix}$$

Lastly, with the same argument that y^j is independent of a^i for all i, j the lower right block matrix has the form, for b^j:

$$\frac{\partial}{\partial a^k} b^j = \frac{\partial}{\partial a^k} \sum_i a^i y^j_i = \sum_i \frac{\partial a^i}{\partial a^k} y^j_i = \sum_i \delta^i_k y^j_i = y^j_k$$

Thus, we have the lower right matrix takes on the form:

$$D = \begin{bmatrix} y^1_1 & \dots & y^1_m \\ \vdots & \ddots & \vdots \\ y^m_1 & \dots & y^m_m \end{bmatrix}$$

Thus, the Jacobian has the following block form:

$$J(F) = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$$

with 0 denoting a $m \times m$ matrix with entries identically 0, and A, C, D as computed above. In particular, we notice that $A = D$, and thus, we may rewrite this as:

$$J(F) = \begin{bmatrix} A & 0 \\ C & A \end{bmatrix}$$

with:

$$\begin{cases} A = \left[\frac{\partial y^i}{\partial x^j}\right]_{i,j} \\ C = \left[\sum_l a^l y^i_{lj}\right]_{i,j} \end{cases}$$

(b)

First, we will prove the following lemma:

Lemma. *Let M be a $n \times n$ matrix, $n \geq 2$. Suppose that in block matrix form, we have that:*

$$M = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$$

where A, D are square submatrices, $i \times i, j \times j$ respectively, $i + j = n$ and C is a $j \times i$ submatrix, and 0 *a $i \times j$ submatrix with entries identically 0.*

Then, $\det(M) = \det(A)\det(D)$.

Proof. Proceed by induction on n.

In the base case, $n = 2$. Then, the only case is that A, C, D are exactly scalar values, and we have that:

$$M = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$$

for $a, c, d \in k$, our base field.

Then, by direct computation, $\det(M) = ad - c0 = ad = \det(A)\det(D)$.

Now, suppose this is true for all $n \leq k - 1$, and consider M a $k \times k$ matrix. Let A be square of shape $i \times i$ and D be square of shape $j \times j$.

First, suppose $i = 1$. Then, $A = [a]$, and we have that:

$$M = \begin{bmatrix} a & 0 \\ C & D \end{bmatrix}$$

Computing the determinant by expanding along the first row and denoting the determinant of the submatrix obtained by deleting the i-th row and j-th column by m_{ij}, we find that since the only non-0 term in the first row is a, that:

$$\det(M) = am_{11} = a\det(D) = \det(A)\det(D)$$

Now, suppose $i > 1$.

Computing the determinant by expanding along the first row and denoting the determinant of the submatrix obtained by deleting the i-th row and j-th column by m_{ij}, we find that since the $i + 1, ..., k$ entries in the first row are 0:

$$\det(M) = \sum_{l=1}^{i} (-1)^{l+1} a_{1l} m_{1l}$$

However, we notice, the submatrix S_{1l} obtained by deleting the first row and l-th column of M is a matrix of dimension $k-1 \times k-1$, and has the shape

$$S_{1l} = \begin{bmatrix} A_{1l} & 0 \\ C_l & D \end{bmatrix}$$

where we denote A_{1l} as the submatrix of A obtained by deleting the first row, and l-th column, and C_l from deleting the l-th column. In particular, by the induction hypothesis, we have that:

$$m_{1l} = \det(S_{1l}) = \det(A_{1l}) \det(D)$$

Therefore, we may rewrite $\det(M)$ as

$$\det(M) = \sum_{l=1}^{i} (-1)^{l+1} a_{1l} m_{1l} = \sum_{l=1}^{i} (-1)^{l+1} a_{1l} \det(A_{1l}) \det(D) = \det(D) \left(\sum_{l=1}^{i} (-1)^{l+1} a_{1l} \det(A_{1l}) \right)$$

However, we recognize the sum as exactly the expansion computation for $\det(A)$, viewed as an $i \times i$ square matrix and expanded along its first row. Thus, we have that:

$$\det(M) = \det(D) \left(\sum_{l=1}^{i} (-1)^{l+1} a_{1l} \det(A_{1l}) \right) = \det(D) \det(A)$$

as desired.

$\square$

Now, using this lemma and the results from part (a), we have that

$$\det(J(F)) = \det(A) \det(A) = \left(\det \left[\frac{\partial y^i}{\partial x^j} \right] \right)^2$$

as desired. $\square$