Homework #2

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Math 237: Homework #2

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Question 4. Let X be the space of C^1 functions on $[0,1]$ such that $f(0) = 0$. Define the bilinear function:

$$\langle f, x \rangle = \int_0^1 f'(x)\overline{g'(x)}dx$$

4.1

Prove that H, the completion of X, is a reproducing kernel Hilbert space.

4.2

Prove that $K(x,y) = \min(x,y)$.

Solution. □

Question 6. Let $Y = l^1(\mathbb{N})$, and define $X = \{f \in Y : \sum n|f(n)| < \infty\}$, equipped with the l^1 norm.

6.1

Prove that X is a proper dense subspace of Y, and hence X is not complete.

6.2

Define $T : X \to Y$ that sends $f(n) \mapsto nf(n)$. Show that T is a closed map, but not bounded.

6.3

Let $S = T^{-1}$. Prove that $S : Y \to X$ is bounded, surjective, but not open.

Solution. 6.1)

Without too much trouble, we can see X is a vector subspace of Y. Of course, the sequence of all 0s satisfies this condition. Further, if $f, g \in X$, then, looking at partial sums, we have that:

$$\sum_{i=1}^k i|f+g(i)| = \sum_{i=1}^k i|f(i)+g(i)| \leq \sum_{i=1}^k i|f(i)| + i|g(i)| = \sum_{i=1}^k i|f(i)| + \sum_{j=1}^\infty j|g(j)|$$

Since the right side converges as $k \to \infty$, due to $f, g \in X$, and all quantities being positive, so too must $\sum i|f+g(i)| < \infty$.

Similarly, we can see scalar multiplication as:

$$\sum_{i=1}^k i|af(i)| = \sum_{i=1}^k i|a||f(i)| = |a|\sum_{i=1}^k i|f(i)|$$

and again, since the right side converges, so too must the left.

Furthermore, this inclusion is proper. Consider the sequence $\{1/n^2\} \in Y$. Clearly, via the integral test, the series $\sum_{n=1}^\infty 1/n^2 < \infty$. However, on the other hand, $\sum_{n=1}^\infty n1/n^2 = \sum_{n=1}^\infty 1/n$ diverges, and hence this sequence is not in X.

Lastly, we want to show this subspace is dense. Let $g(n)$ be a sequence in $Y \setminus X$. Because we have that $\sum_{n=1}^{\infty} |g(n)| < \infty$, for every $\epsilon > 0$, there exists M such that for all $m > M$, $\sum_{n=m}^{\infty} |g(n)| < \epsilon$. Thus, construct the sequence f such that $f(n) = g(n)$ for all $n \leq M$, and 0 for all $n > M$. Clearly, f resides in X, as only finitely many terms are non-0, hence $\sum_{n=1}^{\infty} n|f(n)|$ is a finite sum, and therefore finite.

Without too much trouble, by definition, we have that:

$$\|f - g\|_1 = \sum_{n=1}^{\infty} |f(n) - g(n)| = \sum_{n=M+1} | - g(n)| < \epsilon$$

Thus, since ϵ can be as small as we want, for every open ball around our $g \in Y \setminus X$, we may find an f in such an open ball. Hence, X is dense.

However, X cannot be complete then. As above, take a sequence of $f_n \to g$, where $\|f_n - g\|_1 < 1/n$. Evidently, this is a Cauchy sequence from the argument above, and it converges to an element in $Y \setminus X$.

6.2)

It should be clear that T must be a closed map. If $f_k \to f$, and Tf_k is convergent, we show that $Tf_k \to Tf$. Indeed, we see that:

$$\|Tf - Tf_k\|_1 = \sum_{n=1}^{\infty} |Tf(n) - Tf_k(n)| = \sum_{n=1}^{\infty} |nf(n) - nf_k(n)| = \sum_{n=1}^{\infty} |n||f(n) - f_k(n)|$$

Now, since $f_k \to f$, we have that $\sum_{n=1}^{\infty} |f(n) - f_k(n)| \to 0$. In particular, that means we can choose k such that for each n, $|f(n) - f_k(n)| < \frac{1}{n}\frac{\epsilon}{2^n}$. Under this choice, the above inequality becomes:

$$\sum_{n=1}^{\infty} |n||f(n) - f_k(n)| \leq \sum_{n=1}^{\infty} |n|\frac{1}{n}\frac{\epsilon}{2^n} = \epsilon$$

Thus, we have that $Tf_k \to Tf$, and thus T is closed.

However, it is clear that T is not bounded. Let f_i be the family of sequences with 0 everywhere, except for at the i-th position. Evidently, $\|f_i\|_1 = 1$. However, $\|Tf_i\|_1 = i$, and since in this family, we can choose i arbitrarily large, T cannot be bounded.

6.3)

Clearly, S is surjective, since as we saw, T is a map that takes $f \in X$ to Tf, and evidently, $S \circ T(f) = f \in X$. Hence, we can always find a sequence in Y that maps to any sequence in X.

Moreover, S must be bounded. If we look at the action of S, S sends $f(n) \in Y$ to $\frac{1}{n}f(n) \in X$. Evidently then, we have that:

$$\|Sf\|_1 = \sum_{k=1}^{\infty} |1/kf(k)| \leq \sum_{k=1}^{\infty} |f(k)| = \|f\|_1$$

and so S is bounded, at least by 1.

However, S is certainly not open. By Folland, we may say that S is open if and only if, for B the open ball of radius 1 around 0 in Y, $S(B)$ contains a ball centered around 0 in X.

Fix an $i \in \mathbb{N}$. We can then consider the family of sequences g_i such that $g(i) = 2/i$, and 0 otherwise. Evidently, we may find a sequence as close to 0 as we want, since $\|g_i\|_1 = 2/i$. However, for no i does there exists a $y \in B$ such that $Sy = g_i$, since under the map T, we see that $Tg_i(n)$ is 2 when $n = i$ and 0 otherwise, outside of B. Hence, S is not open.

$\square$

Question 16. Define $\text{Lip}[0,1] = \{f \in C[0,1] : f \text{ is Lipschitz }\}$. For each $n \geq 1$, define $F_n = \{f \in C[0,1] : |f(x) - f(y)| \leq n|x - y| \text{ for all } x, y \in [0,1]\}$.

16.1

Prove that for each $n \geq 1$, F_n is a closed, nowhere dense subset of $\mathrm{Lip}[0,1]$.

16.2

Conclude that $\mathrm{Lip}[0,1]$ is a countable union of nowhere dense subsets of $C[0,1]$.

Solution. Obviously, $F_n \subset \mathrm{Lip}[0,1]$, as we may just take n to be a Lipschitz constant.

Consider $U_n = (F_n)^c$, the compliment of F_n. We wish to show that this set is open.

Let $f \in U_n$. Evidently then, there exists $x, y \in [0,1]$ such that $|f(x) - f(y)| > n|x - y|$. Fix a choice x_0, y_0 (where, evidently, $x_0 \neq y_0$ as the inequality always holds then), without loss of generality, we may assume $f(x_0) > f(y_0)$ and otherwise swap labels, and call $d = f(x_0) - f(y_0) - n|x_0 - y_0|$.

Now, let g be in the open ball around f with radius at most $d/2$. From the condition of being in the ball of radius at most $d/2$, we must have that $g(y_0) > g(x_0)$, since $d < |f(x_0) - f(y_0)|$, and hence, from the supremum norm, $g(y_0) \geq f(y_0) - d/2 > [f(x_0) + f(y_0)]/2$ and similarly, $g(x_0) \leq f(x_0) + d/2 < [f(x_0) + f(y_0)]/2$. Thus, we have that at x_0, y_0 for g, that:

$$g(y_0) - g(x_0) \geq f(y_0) - d/2 - f(x_0) - d/2 = f(y_0) - f(x_0) - d = n|x_0 - y_0|$$

Since we can arbitrarily shrink the ball to have radius smaller than $d/2$, say $d/3$, this shows that we may find an open ball around any $f \in U_n$, hence U_n is open, and thus F_n is closed.

Now, we need to show that F_n is nowhere dense. Let $f \in F_n$. Let $\epsilon > 0$, and suppose without loss of generality that $f(0) \geq f(\epsilon/n + 1)$. Consider the continuous function g_ϵ such that it takes on ϵ at 0, joined linearly to 0 at $\epsilon/n + 1$, and constantly 0 otherwise. Of course, this is continuous, with $\|g_\epsilon\|_u = \epsilon$. Furthermore, we can consider the function $f + g_\epsilon$, and in particular, compute $f + g_\epsilon(0) - [f + g_\epsilon(\epsilon/n + 1)]$. We have that:

$$f + g_\epsilon(0) - [f + g_\epsilon(\epsilon/n+1)] = f(0) - f(\epsilon/n+1) + g_n(0) - g_n(\epsilon/n+1) \geq g_n(0) - g_n(\epsilon/n+1) = \epsilon > \epsilon/n + 1$$

where we use the fact that $f(0) \geq f(\epsilon/n + 1)$ for one inequality, and the fact that $n \geq 1$ for the other. Hence, $f + g_n$ is not Lipschitz. If $f(0) < f(\epsilon/n + 1)$, the same argument holds by taking $-g_\epsilon$.

Since the choice of f was arbitrary, this implies that F_n contains no non-trivial open subsets, and thus, as it is its own closure, F_n is nowhere dense.

Clearly, $\mathrm{Lip}[0,1] = \cup F_n$. We have that $F_n \subseteq \mathrm{Lip}[0,1]$ for each n, since we may take n as a Lipschitz constant. Hence, $\cup F_n \subseteq \mathrm{Lip}[0,1]$.

On the other hand, let $g \in \mathrm{Lip}[0,1]$. Then, there exists a constant $K \geq 0$ such that $|g(x) - g(y)| \leq K|x - y|$ for all $x, y \in [0,1]$. Then, since this inequality clearly holds for $K' \geq K$, taking any integer $k \geq K$, we see that $g \in F_k$. Hence, $\mathrm{Lip}[0,1] \subseteq \cup F_n$, and we are done.

□

Question 17. Let f be a nonnegative Lebesgue measurable function defined on $\mathbb{R}$. Assume that for all $g \in L^2(\mathbb{R})$, $fg \in L^1(\mathbb{R})$. Prove that $T_f(g) = \int_{\mathbb{R}} gf$ is a bounded linear functional on L^2, and conclude that $f \in L^2$.

Solution. First, we use the σ-finiteness of $\mathbb{R}$, and consider the compact sets $E_i = [-i, i]$. Additionally, for each E_i, we define f_n in the following fashion:

$$f_n(x) = \begin{cases} 0 & x \notin E_n \\ \min(f(x), n) & x \in E_n \end{cases}$$

Certainly, if f is measurable, so too are each f_n, as if we look at, for $a > 0$, $\{f_n > a\}$, if $a \geq n$, then this is exactly $\{f \geq n\}$, if $a < n$, then it aligns exactly with $\{f > a\}$. And lastly, if $a \leq 0$, then the preimage is the entire codomain, hence still measurable.

Moreover, each f_n is clearly in L^2, as it has compact support, contained within E_n, and is bounded above by n. Hence, we can say that:

$$\|f_n\|_2^2 = \int_{\mathbb{R}} f_n = \int_{E_n} f_n \leq \int_{E_n} n = 2n^2 < \infty$$

Now, consider the family of linear functionals determined as:

$$\begin{aligned} T_n :& L^2(\mathbb{R}) \to \mathbb{R} \\ T_n :& g \mapsto \int f_n g \end{aligned}$$

where this is clearly linear in terms of g, due to the linearity of the integral and distributivity of mutliplication.

By Hölder's inequality then, since $f_n, g \in L^2$, we have that:

$$\|f_n g\|_1 \leq \|f_n\|_2 \|g\|_2$$

However, we've already computed $\|f_n\|_2 \leq 2n^2$. Hence, we have that:

$$|T_n(g)| = \left| \int f_n g \right| \leq \int |f_n g| = \|f_n g\|_1 \leq \|f_n\|_2 \|g\|_2 \leq 2n^2 \|g\|_2$$

Hence, for each n, T_n is a bounded linear functional, with norm at most $2n^2$.

Now, fix any $g \in L^2$, and consider $|T_n(g)|$ again. For every n, we have that due to the fact that $f_n \leq f$ everywhere:

$$|T_n(g)| = \left| \int f_n g \right| \leq \int |f_n g| \leq \int |f||g| = \|fg\|_1 < \infty$$

Since this is true for all n, we may conclude that for all $g \in L^2$, that $\sup_n |T_n(g)| < \infty$. Since L^2 is a Banach space, we may now apply the Uniform Boundedness Principle and conclude that

$$\sup_n \|T_n\| < \infty$$

Thus, we need only show that $\|T_f\| = \sup_n \|T_n\|$. Of course, $f_n g \to fg$ almost everywhere, by the definition of f_n, and further, fg itself is integrable and then so is $|fg|$, and $|f_n g| \leq |fg|$ by the definition of f_n. Hence, we may use an application of the Dominated Convergence Theorem, as well as picking an arbitrary $g \in L^2$ with $\|g\| = 1$, to see that:

$$|T_f(g)| = \left| \int fg \right| = \left| \lim_{n \to \infty} \int f_n g \right| = \left| \lim_{n \to \infty} T_n(g) \right| = \lim_{n \to \infty} |T_n(g)| \leq \lim_{n \to \infty} \|T_n\| \leq \sup_n \|T_n\|$$

Since this is true for all $g \in L^2$ with unit norm, we may conclude then that $\|T_f\| \leq \sup_n \|T_n\|$. However, earlier, we already showed that $|T_n(g)| \leq |T_f(g)|$ due to the fact that $f_n \leq f$ everywhere. Hence, we have that $\|T_f\| = \sup_n \|T_n\| < \infty$, and hence T_f is a bounded linear functional.

Now, by the Riesz Representation Theorem, since L^2 is a Hilbert space, this implies that we may find an $h \in L^2$ such that $T_n(g) = \langle g, h \rangle$. In particular then, we have that:

$$T_n(g) = \int fg = \langle g, h \rangle = \int gh \implies \int fg - \int gh = 0 \implies \int g(f-h) = 0 \implies f = h \; a.e.$$

Where because this works for all $g \in L^2$, we are free to pick g to be non-negative, and $f = h$ follows because this single function h must work regardless of the choice of g. Hence, f may be identified as a member of an equivalence class in L^2. $\square$

Question 19. Suppose $\mathcal{B}$ is a Banach space, and let S be a closed proper subspace. Fix some $f_0 \notin S$. Show that there exists a continuous linear functional γ such that $\gamma(f) = 0$ for all $f \in S$, and $\gamma(f_0) = 1$. Moreover, show that we may choose the linear functional such that $\|\gamma\| = 1/d$, where d is the distance from f_0 to S.

Solution. First, we notice that d, the distance between f_0, S must be positive as otherwise, we could find a descending series of s_n such that $\|f_0 - s_n\| \to 0$. In such a case, S being closed would imply that $f_0 \in S$, a contradiction.

Now, consider the vector subspace $T = \text{span}\{S, f_0\}$. We claim that for every $t \in T$, there exists a unique representation $t = s_t + a_t f_0$, for $s_t \in S$ and a_t a scalar.

Suppose we had that $s_t + a_t f_0 = t = s_t' + a_t' f_0$. Then, we must have that $(s_t - s_t') + (a_t - a_t')f_0 = 0$. Since $f_0 \notin S$, we must have that $(s_t - s_t') = 0, (a_t - a_t')f_0 = 0$. Hence, we must have that $s_t = s_t'$, and that $a_t - a_t' = 0 \implies a_t = a_t'$.

Now, define a functional $\lambda : T \to F$ that sends $t \mapsto a_t$, where a_t comes from the representation above. Clearly, we see from the representation that this must be linear:

$$\lambda(t + t') = \lambda(s_t + a_t f_0 + s_{t'} + a_{t'} f_0) = \lambda((s_t + s_{t'}) + (a_t + a_{t'})f_0) = a_t + a_{t'} = \lambda(t) + \lambda(t')$$

where we've used the fact that S is a subspace to conclude $s_t + s_{t'} \in S$, and:

$$\lambda(bt) = \lambda(b(s_t + a_t f_0)) = \lambda(bs_t + ba_t f_0) = ba_t = b\lambda(t)$$

and we omit $\lambda(0) = 0$, as of course, $0 = 0 + 0f_0$.

Also, certainly, $\lambda|_S = 0$, since for any $s \in S$, we have the representation in T, $s = s + 0f_0$, and of course then, $\lambda(s) = 0$. Finally, in a similar fashion, we have that $f_0 = 0 + f_0$, and hence $\lambda(f_0) = 1$.

Thus, we need only show that λ is bounded, with norm $\frac{1}{d}$.

Let t be an arbitrary non-0 vector in T, with non-0 component of f_0, as of course, if that were true, $\lambda(t) = 0$. We notice that, since $t = s_t + a_t f_0 = a_t(a_t^{-1} s_t + f_0)$, that:

$$\|t\| = \|a_t(a_t^{-1} s_t + f_0)\| = |a_t| \|(a_t^{-1} s_t + f_0\|$$

and since $a_t^{-1} s_t \in S$ due to being a scalar multiple, we must have that $\|a_t^{-1} s_t + f_0\| \geq d$, as d is the infimum of $\|f_0 - s\|$, and we can always replace s with $-s_t$, as necessary. Hence, using the fact that $a_t = \lambda(t)$, we have that:

$$\|t\| \geq |a_t| d = |\lambda(t)| d \implies |\lambda(t)| \leq \frac{1}{d} \|t\|$$

Since this is true for all t, including when $a_t = 0$ due to the observation earlier, we have that λ is bounded, and $\|\lambda\|_{T^*} \leq \frac{1}{d}$ by definition.

Now, again, since d is the infimum of $\|f_0 - s\|$ for $s \in S$, we may find $\{s_n\} \subset S$ such that $\|f_0 - s_n\| \to d$. Then, considering the action of lambda on each of these, we have that:

$$1 = |\lambda(f_0)| = |\lambda(f_0 - s_n)| \leq \|\lambda\|_{T^*} \|f_0 - s_n\|_T$$

Taking the limit as $n \to \infty$ then, we retrieve the inequality

$$1 \leq \|\lambda\|_{T^*} d \implies \|\lambda\|_{T^*} \geq \frac{1}{d}$$

Hence, $\|\lambda\|_{T^*} = \frac{1}{d}$.

Now, by Corollary 2.2 in Heil, since $\mathcal{B}$ is a Banach space, T a subspace of $\mathcal{B}$, $\lambda \in T^*$, there exists a $\gamma \in \mathcal{B}^*$ such that $\gamma(f_0) = \lambda(f_0) = 1$, $\gamma(s) = \lambda(s) = 0$ for all $s \in S$, and $\|\gamma\|_{\mathcal{B}^*} = \|\lambda\|_{T^*} = \frac{1}{d}$ as desired. $\square$