

# Midterm #1

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Math 237: Midterm #1

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**Question 1.** Let  $(X, \rho)$  be a compact metric space, and  $f : X \rightarrow X$  a function such that:

$$\rho(f(x), f(y)) < \rho(x, y)$$

for all  $x \neq y$ .

Define  $g : X \rightarrow \mathbb{R}$  via  $g : x \mapsto \rho(x, f(x))$ .

1.1)

Prove that  $g$  is Lipschitz, and that  $g$  has a minimum value, achieved at a point  $x_0 \in X$ . Conclude that there exists  $x \in X$  such that  $g(x) = 0$ .

1.2)

Show that  $f$  has a unique fixed point  $x_0$ .

1.3)

Show that the assumption that  $X$  is compact may not be omitted.

*Solution.* 1.1)

Fix some  $x \in X$ , and let  $y \in X$  be arbitrary. By the triangle inequality, we see that:

$$\begin{cases} \rho(x, f(x)) \leq \rho(x, y) + \rho(y, f(x)) \\ \rho(y, f(x)) \leq \rho(y, f(y)) + \rho(f(x), f(y)) \end{cases}$$

Combining these two equations with the property of  $f$  by hypothesis, we see that:

$$\rho(x, f(x)) - \rho(y, f(y)) \leq \rho(x, y) + \rho(f(x), f(y)) < 2\rho(x, y)$$

However, we notice that we may run the same computation in the triangle inequality, switching the labels of  $x, y$ , as  $\rho(x, y) = \rho(y, x)$ . Thus, we can conclude then that

$$|\rho(x, f(x)) - \rho(y, f(y))| < 2\rho(x, y)$$

and therefore, since the left side is exactly  $d(g(x), g(y))$  with the metric of the real line, we may conclude that  $g$  is Lipschitz with Lipschitz constant at most 2.

Now, since  $g$  is Lipschitz continuous, it is continuous. Hence, since  $X$  is compact,  $g$  achieves its extremas. Hence, we may find  $x_0 \in X$  such that  $g$  achieves its minimum value.

Suppose that  $g(x_0) > 0$ . Then, of course, we would have that  $g(x_0) = \rho(x_0, f(x_0)) > 0$  and hence,  $x_0 \neq f(x_0)$ . Then, we can consider  $g(f(x_0))$ . We have that:

$$g(f(x_0)) = \rho(f(x_0), f(f(x_0))) < \rho(x_0, f(x_0)) = g(x_0)$$

But, this is a contradiction, as we assumed that  $g$  attained a minimum at  $x_0$ . Hence,  $g(x_0) = 0$ .

1.2)

From 1.1, we've shown that there exists  $x_0 \in X$  such that  $g(x_0) = 0$ . Evidently then:

$$g(x_0) = 0 \implies \rho(x_0, f(x_0)) = 0 \implies f(x_0) = x_0$$

Furthermore, this point must be unique, as suppose  $f(x_1) = x_1$  as well. Assuming that  $x_0 \neq x_1$ , we have that:

$$\rho(x_0, x_1) = \rho(f(x_0), f(x_1)) < \rho(x_0, x_1)$$

which is absurd. Hence,  $x_0 = x_1$ .

1.3)

Here are some examples to show that we need  $X$  to be compact. Consider  $X = \mathbb{Z}$ , equipped with the standard metric  $\rho(x, y) = |x - y|$ . Of course, this is not compact, as the sequence  $\{n\}_{n=1}^{\infty}$  cannot admit any convergent subsequence. If we take  $f(x) = \text{round}(x/2)$ , where the round function rounds to the integer closer to 0, then of course, we have that  $\rho(f(x), f(y)) < \rho(x, y)$  for  $x \neq y$ , as it contracts all distances by at least  $1/2$ . On the other hand, it has multiple fixed points,  $-1, 0, 1$ .

Another example is to take the open interval  $(0, 1)$ , equipped with the standard metric  $\rho(x, y)$ , and consider the function  $f(x) = x/2$ . Evidently, in the same fashion, we still have that  $\rho(f(x), f(y)) = |x/2 - y/2| = 1/2|x - y| = 1/2\rho(x, y) < \rho(x, y)$ . However,  $g$  does not attain a minimum and  $f$  does not have a fixed point.

We can see  $g$  does not have a minimum as for any  $\epsilon > 0$ , we may choose  $N \geq 1$  such that  $1/N < \epsilon$ . Then,  $g(1/N) = \rho(1/N, f(1/N)) = |1/N - 1/2N| = 1/2N < 1/N < \epsilon$ . Hence,  $g(x)$  can be arbitrarily small. However, we can see that for  $x = 1/2x$ , this is satisfied only at  $x = 0$ , outside of  $(0, 1)$ . Hence, there is no  $x$  such that  $g(x) = 0$  on  $(0, 1)$ , and no fixed point of  $f$  on  $(0, 1)$ .

□

**Question 2.** Let  $X, Y$  be Banach spaces. Let  $T \in L(X, Y)$ . Show that  $T$  is surjective if and only if  $\text{range}(T)$  is not meager in  $Y$ .

*Solution.* One direction is trivial. Suppose  $T$  is surjective. Then,  $Y = \text{range}(T)$ . But, by the Baire Category Theorem (2.21, Heil),  $Y$  is nonmeager in  $Y$ , and we are done.

Now, suppose  $\text{range}(T)$  is not meager. Consider open balls in  $X$  centered on the origin,  $B_n^X(0) = \{x \in X : \|x\| < n\}$ , where we use the superscript to remind ourselves this is in  $X$ . Clearly,  $X = \bigcup_{n=1}^{\infty} B_n^X(0)$ . Therefore, we have that the range of  $T$  can be expressed as:

$$\text{range}(T) = \bigcup_{n=1}^{\infty} T(B_n^X(0))$$

Since  $T$  is non-meager, there exists an  $m$  such that the closure  $\overline{T(B_m^X(0))}$  contains an open ball, as its complement is not dense. We can consider the operator  $mT$ , and the closure  $\overline{mT(B_1^X(0))}$  contains an open ball in  $Y$ , as  $T(B_m^X(0)) = mT(B_1^X(0))$  by linearity. Then, by Lemma 2.26 in Heil, we have that  $mT(B_1^X(0))$  contains an open ball  $B_r^Y(0)$  for some  $r > 0$ . Again, by linearity then, we have that  $T(B_m^X(0))$  contains an open ball  $B_{r/m}^Y(0)$ .

So now, let  $y \in Y$ . In particular, consider  $\frac{y}{\|y\|} * \frac{r}{2m}$ . Evidently, the norm of this vector is  $r/2m$ , and hence is contained within  $B_{r/m}^Y(0)$ . Thus, there exists an  $x \in X$  such that  $T(x) = \frac{y}{\|y\|} * \frac{r}{2m}$ . By linearity then, we have that:

$$T\left(\frac{2m\|y\|}{r}\right) = \frac{2m\|y\|}{r}T(x) = \frac{2m\|y\|}{r} \frac{y}{\|y\|} \frac{r}{2m} = y$$

Hence,  $Y \subseteq \text{range}(T)$ , and therefore,  $Y = \text{range}(T)$ . Thus,  $T$  is surjective. □

**Question 3.** Let  $C_b(\mathbb{R})$  be the space of bounded, continuous, real-valued functions. Let  $C_b^1(\mathbb{R})$  be the space of functions such that  $f, f' \in C_b(\mathbb{R})$ . Equip both of these spaces with the uniform norm.

3.1)

Show that  $C_b$  is complete, and that  $C_b^1$  is not complete.

3.2)

Show that the differentiation operator  $D : C_b^1(\mathbb{R}) \rightarrow C_b(\mathbb{R})$  that sends  $D : f \mapsto f'$  is unbounded, but has a closed graph.

*Solution.* 3.1)

First, consider the family of functions  $f_n(x) = 2^{-n} \cos(13^n \pi x)$  for  $n \geq 1$ , and consider  $g_m(x) = \sum_{n=1}^m f_n(x)$ .

We have that the sequence of  $\{g_m\}$  is uniformly Cauchy, as if we let  $\epsilon > 0$ , we may choose  $N$  such that  $2^{-N+1} < \epsilon$ , and then for  $m, m' > N$  (WLOG, suppose  $m > m'$ ), we have that:

$$|g_m(x) - g_{m'}(x)| = \left| \sum_{n=1}^m f_n(x) - \sum_{n=1}^{m'} f_n(x) \right| = \left| \sum_{n=m'+1}^m f_n(x) \right| \leq \sum_{n=m'+1}^m |f_n(x)| \leq \sum_{n=N}^{\infty} |f_n(x)| \leq \sum_{n=N}^{\infty} 2^{-n} = 2^{-N+1}$$

Since this is independent of the point  $x$ , this is uniformly Cauchy. Since each  $g_m$  is continuous, being the finite sum of continuous functions, and the convergence is uniform, the pointwise limit  $g(x) = \lim_{m \rightarrow \infty} g_m(x)$  is a continuous function. Moreover, we can see easily that  $g$  is bounded, as we can see that each of the partial sums are bounded above by  $\sum_{n=1}^{\infty} 2^{-n} = 2$ . However, this is a Weierstrauss function, famously known for being differentiable nowhere, as we satisfy the condition that  $13/2 = 7.5 > 1 + 3/2\pi$ . Since we have demonstrated a sequence of functions in  $C_b^1$ , convergent under the uniform norm to a function not in  $C_b^1$ , we may conclude that  $C_b^1$  is not complete.

On the other hand, let  $\{f_n\}_{n=1}^{\infty} \subseteq C_b$ , with  $\sum_{n=1}^{\infty} \|f_n\|_u < \infty$ . Consider  $f = \sum_{n=1}^{\infty} f_n$ , and we will show that  $f$  is both bounded, and the uniform limit of the partial sums.

Evidently,  $f$  is bounded, as we can look at the partial sums  $\sum_{n=1}^N f_n$ . We have that  $\|\sum_{n=1}^N f_n\|_u \leq \sum_{n=1}^N \|f_n\|_u < \sum_{n=1}^{\infty} \|f_n\|_u < \infty$ , where the first inequality comes from the triangle inequality, and the second is simply our hypothesis of being absolutely convergent. Since this bound holds for all  $N > 0$ , it must hold in the limit as well. Hence,  $\|f\|_u < \sum_{n=1}^{\infty} \|f_n\|_u < \infty$ .

Now, we wish to show that  $\sum_{n=1}^N f_n \rightarrow f$  uniformly. Since  $\sum_{n=1}^{\infty} \|f_n\|_u < \infty$ , for  $\epsilon > 0$ , we may find a  $M > 0$  such that for all  $m > M$ ,  $\sum_{n=m}^{\infty} \|f_n\|_u < \epsilon$ . Now, let  $m > M$ , and consider  $\|f - \sum_{n=1}^m f_n\|_u$ . We see that:

$$\|f - \sum_{n=1}^m f_n\|_u = \left\| \sum_{n=m+1}^{\infty} f_n \right\|_u$$

Now, due to the positivity of the norm, since we have for each finite sum:  $\|\sum_{n=m+1}^p f_n\|_u \leq \sum_{n=m+1}^p \|f_n\|_u \leq \sum_{n=m+1}^{\infty} \|f_n\|_u$ , we may conclude that this holds in the limit as well.

Hence, we have that:

$$\left\| \sum_{n=m+1}^{\infty} f_n \right\|_u \leq \sum_{n=m+1}^{\infty} \|f_n\|_u < \epsilon$$

Thus,  $f_n \rightarrow f$  uniformly, and hence,  $f$  is continuous. Therefore,  $f \in C_b$ , as desired, and  $f_n \rightarrow f$  under the norm. Since the choice of absolutely convergent sequence was arbitrary, by 5.1 in Folland, since every absolutely convergent sequence converges,  $C_b$  must be complete.

3.2)

Evidently,  $D$  is unbounded. For example, take the family of functions  $f_k = \sin(kx)$ , for  $k \in \mathbb{N}$ . Clearly, this is a continuous function, bounded above by 1, and so  $\|f_k\|_u = 1$ . Furthermore, its derivative is  $k \cos(kx)$ , continuous, and for each  $k$ , bounded above by  $k$ . However,  $\|D(f_k)\|_u = \|k \cos(kx)\|_u = k$ . Since we may choose  $k$  arbitrarily large without affecting the norm of  $f_k$ ,  $D$  is unbounded.

Now, suppose that we have  $f_n \rightarrow f \in C_b^1$ , and  $Df_n = f'_n \rightarrow g \in C^1$ , uniformly in both cases. Fix an arbitrary point  $a \in \mathbb{R}$ , and consider, for  $x > a$ , the closed interval  $[a, x]$ . Since we have that  $f'_n \rightarrow g$  uniformly, evidently,  $\|f'_n\|_u$  is bounded. Then, we can take  $\sup_n \|f'_n\|_u < \infty$  as an upper bound for all  $|f'_n(y)|$ ,  $y \in [a, x]$ . Of course also, if  $f'_n \rightarrow g$  uniformly, it does so pointwise as well. Therefore, by the Lebesgue Dominated Convergence Theorem, we have that:

$$\lim_{n \rightarrow \infty} \int_a^x f'_n(y) dy = \int_a^x g(y) dy$$

However, we know that  $f_n$  is differentiable on  $[a, x]$ , and  $f'_n$ , its derivative is continuous. Thus, we may transform the left hand side via the Fundamental Theorem of Calculus to obtain:

$$\lim_{n \rightarrow \infty} f_n(x) - f_n(a) = \int_a^x g(y) dy$$

Now, since  $f_n \rightarrow f$  uniformly, it does so pointwise as well, so we have that:

$$f(x) - f(a) = \int_a^x g(y) dy$$

and finally, we can apply  $D$  to both sides of this equation, and since  $g$  is continuous, we can apply the other statement of the FTC to obtain:

$$D(f(x) - f(a)) = D\left(\int_a^x g(y) dy\right) \implies D(f)(x) = g(x)$$

Since the choice of  $a$  were arbitrary, we may repeat this argument for every  $x$ . Hence, varying across all  $x \in \mathbb{R}$ , we obtain an equality of functions, and conclude that  $Df$  exists, and is equal to  $g$  everywhere.

Since this is true for an arbitrary  $f_n \rightarrow f, f'_n \rightarrow g$ , this is true for all cases where both sequences simultaneously converge, and hence  $D$  has a closed graph.

□

**Question 4.** Let  $\mathcal{H} = L^2[0, 1]$ , the Lebesgue measurable and square-integrable functions defined on  $[0, 1]$ . Let  $K$  be a non-empty, closed, convex subset of  $\mathcal{H}$ . Define  $P = P_K$  as the orthogonal projection of  $H$  onto  $K$ .

4.1)

Let  $x \in \mathcal{H}$ . Prove that the following are equivalent:

- i) There exists a unique  $z \in K$  such that  $\|x - z\| = \min_{y \in K} \|x - y\|$ .
- ii)  $z \in K$  and  $\langle x - z, y - z \rangle \leq 0$  for all  $y \in K$ .

4.2)

Let  $A$  be a continuous bilinear mapping from  $\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$  such that, for some  $\alpha > 0$ , we have:

$$A(f, f) \geq \alpha \|f\|_2^2$$

for every  $f \in \mathcal{H}$ . We will prove the following statement in parts:

For every  $f \in \mathcal{H}$ , there exists a unique  $u \in K$  such that:

$$A(u, v - u) \geq \langle f, v - u \rangle$$

for all  $v \in K$ .

4.2.1)

Fix a  $u \in \mathcal{H}$ , and prove that there exists a unique  $Tu \in \mathcal{H}$  such that  $A(u, v) = \langle Tu, v \rangle$  for every  $v \in \mathcal{H}$ . Prove that  $T$  is a bounded linear mapping on  $\mathcal{H}$ .

4.2.2)

Fix a  $\rho > 0$ ,  $f \in \mathcal{H}$ , and define a map  $S_\rho : K \rightarrow K$  that sends  $v \mapsto P(\rho f - \rho Tv + v)$ . Prove that we may choose  $\rho$  such that there exists a  $0 < k < 1$  with the property that:

$$\|S_\rho(v_1) - S_\rho(v_2)\| \leq k \|v_1 - v_2\|$$

for all  $v_1, v_2 \in K$ .

4.2.3)

Conclude that for the value of  $\rho > 0$  chosen in 4.2.2, that  $S_\rho$  is a contraction, and therefore has a unique fixed point  $u \in K$ .

4.2.4)

Note that we can rewrite  $\rho f - \rho Tu = \rho f - \rho Tu + u - u$ . Then, use 4.1 to show that:

$$\langle \rho f - \rho Tu, v - u \rangle \leq 0$$

for every  $v \in K$ .

4.2.5)

Conclude that, for every  $f \in \mathcal{H}$ , there exists a unique  $u \in K$  such that:

$$A(u, v - u) \geq \langle f, v - u \rangle$$

*Solution.* 4.1)

First, we show that if  $\langle x - z, y - z \rangle \leq 0$ , then we get that  $\|x - z\| = \min \|x - y\|$ .

We have the following sequence of equalities, for arbitrary  $y$ :

$$\langle x - z, y - z \rangle = \langle x - z, y + (x - x) - z \rangle = \langle x - z, x - z \rangle + \langle x - z, y - x \rangle = \|x - z\|^2 + \langle x - z, y - x \rangle$$

Then, we have that:

$$\langle x - z, y - z \rangle \leq 0 \implies \|x - z\|^2 + \langle x - z, y - x \rangle \leq 0 \implies \|x - z\|^2 \leq -\langle x - z, y - x \rangle$$

Since the norm is positive, we may harmlessly replace  $\langle x - z, y - x \rangle$  with its absolute value. Then, by the Cauchy-Schwarz inequality, we retrieve:

$$\|x - z\|^2 \leq \|x - z\| \|y - x\| \implies \|x - z\| \leq \|y - x\| = \|x - y\|$$

Since this is true for all  $y \in K$ , including  $z$  itself, we conclude that  $\|x - z\| = \min_{y \in K} \|x - y\|$ .

Now, suppose that  $z \in K$  is such that  $\|x - z\| = \min_{y \in K} \|x - y\|$ . By convexity, for any  $y \in K$ , we may reexpress  $y = (1 - t)z + tw$  for at least some fixed  $w \in K, t \in [0, 1]$ , hence, we have that:

$$\|x - z\| \leq \|x - (1 - t)z + tw\| = \|x - z - t(w - z)\|$$

We may safely square both sides and examine the inner product instead. Thus, we have that:

$$\langle x - z, x - z \rangle \leq \langle x - z - t(w - z), x - z - t(w - z) \rangle$$

Using the linearity and conjugate linearity of the inner product, we see that the RHS can be rewritten as:

$$\langle x - z - t(w - z), x - z - t(w - z) \rangle = \langle x - z, x - z \rangle - t\langle x - z, w - z \rangle - t\langle w - z, x - z \rangle + t^2\langle w - z, w - z \rangle$$

Hence, we have that:

$$\langle x - z, x - z \rangle \leq \langle x - z - t(w - z), x - z - t(w - z) \rangle \implies \langle x - z, w - z \rangle + \langle w - z, x - z \rangle \leq t\langle w - z, w - z \rangle$$

If we live in a complex Hilbert space, at most, I can conclude that the real component is non-positive.

On the other hand, assuming that  $\langle x - z, w - z \rangle$  is purely real as we live in a real Hilbert space, then as we vary  $t$ , since the inner products are constants, this must hold for all  $t \in (0, 1]$ , and hence, we have that:

$$2\langle x - z, w - z \rangle \leq 0 \implies \langle x - z, w - z \rangle \leq 0$$

as desired.

4.2.1)

Let  $u \in \mathcal{H}$ . By the bilinearity of  $A$ , we have that:

$$A_u : \mathcal{H} \rightarrow \mathbb{R} \quad A_u : v \mapsto A(u, v)$$

is a linear functional on  $\mathcal{H}$ . Moreover, since  $A$  is continuous, it is continuous in each variable, and hence  $A_u$  is a continuous linear functional. Thus, since  $\mathcal{H}, \mathbb{R}$  are normed linear spaces, and  $A_u$  is a continuous linear operator,  $A_u$  is bounded (1.63, Heil).

Since  $\mathcal{H}$  is a Hilbert space, we can identify a  $w_u$  such that  $A_u(v) = \langle v, w_u \rangle$  by the Riesz Representation Theorem (Folland, 5.25). Since  $A$  is real-valued, we can freely pick  $w_u$  to be in the first or second argument due to conjugate symmetry - we will from now on use  $A_u(v) = \langle w_u, v \rangle$ .

So now, we may define  $T : \mathcal{H} \rightarrow \mathcal{H}$  that sends  $u \mapsto w_u$ . Evidently, due to the bilinearity of  $A$ ,  $T$  is linear:

$$\begin{cases} \langle T(u + u'), v \rangle = A(u + u', v) = A(u, v) + A(u', v) = \langle T(u), v \rangle + \langle T(u'), v \rangle = \langle T(u) + T(u'), v \rangle \\ \langle T(ku), v \rangle = A(ku, v) = kA(u, v) = k\langle T(u), v \rangle \end{cases}$$



Now, we wish to show that  $T$  is bounded. First, restrict ourselves to  $u \in \mathcal{H}$  such that  $\|u\| = 1$ , and consider the related family of operators as defined above  $A_u$ .

Fix any  $v \in \mathcal{H}$ , and consider  $\sup_{\|u\|=1} \|A_u(v)\|$ . By considering the related bounded linear function  $\tilde{A}_v : \tilde{A}_v(u) = A(u, v)$ , bounded and linear for the same reasons as  $A_u$  due to the bilinearity and continuity of  $A$ , we can see that:

$$\sup_{\|u\|=1} \|A_u(v)\| = \sup_{\|u\|=1} \|\tilde{A}_v(u)\| = \|\tilde{A}_v\| < \infty$$

Since we may repeat this argument for each fixed  $v$ , we satisfy the conditions for the uniform boundedness principle. Hence, we have that:

$$\sup \|A_u\| < \infty$$

Call this supremum  $N_A$ .

Now, for each  $u$  then, we may consider the value of  $A_u(Tu)$ . We have the following sequence of inequalities:

$$\|Tu\|^2 = \langle Tu, Tu \rangle = A_u(Tu) \leq \|A_u\| \|Tu\| \leq N_A \|Tu\|$$

which of course, implies that:

$$\|Tu\| \leq N_A < \infty$$

Since this is true for arbitrary unit norm  $u$ , this is true for all, and hence,  $T$  is bounded.

4.2.2)

First of all, using the equivalent statement in 4.1.ii, we see that:

$$\langle \rho f - \rho T v + v - S_\rho(v), y - S_\rho(v) \rangle \leq 0$$

for every  $y \in K$ .

Then, letting  $v_1, v_2 \in K$ , we have the following statements:

$$\begin{cases} \langle \rho f - \rho T v_1 + v_1 - S_\rho(v_1), S_\rho(v_2) - S_\rho(v_1) \rangle \leq 0 \\ \langle \rho f - \rho T v_2 + v_2 - S_\rho(v_2), S_\rho(v_1) - S_\rho(v_2) \rangle \leq 0 \end{cases}$$

Summing these equations then, and pulling out a factor of  $-1$  from the second argument in the second equation, we find that:

$$\langle \rho f - \rho T v_1 + v_1 - S_\rho(v_1) - \rho f + \rho T v_2 - v_2 + S_\rho(v_2), S_\rho(v_2) - S_\rho(v_1) \rangle \leq 0 \implies$$

$$\langle -\rho T(v_1 - v_2) + v_1 - v_2 + S_\rho(v_2) - S_\rho(v_1), S_\rho(v_2) - S_\rho(v_1) \rangle \leq 0 \implies$$

$$\langle -\rho T(v_1 - v_2) + (v_1 - v_2) + S_\rho(v_2) - S_\rho(v_1), S_\rho(v_1) - S_\rho(v_2) \rangle \geq 0$$

where we've used the linearity of  $T$ , and then multiplied through by  $-1$ , bringing it into the second argument.

We examine the square of the norm, to leverage the inner product.

We have that:

$$\langle S_\rho(v_1) - S_\rho(v_2), S_\rho(v_1) - S_\rho(v_2) \rangle \leq$$

$$\langle S_\rho(v_1) - S_\rho(v_2), S_\rho(v_1) - S_\rho(v_2) \rangle + \langle -\rho T(v_1 - v_2) + (v_1 - v_2) + S_\rho(v_2) - S_\rho(v_1), S_\rho(v_1) - S_\rho(v_2) \rangle =$$

$$\langle S_\rho(v_1) - S_\rho(v_2) - \rho T(v_1 - v_2) + (v_1 - v_2) + S_\rho(v_2) - S_\rho(v_1), S_\rho(v_1) - S_\rho(v_2) \rangle =$$

$$\langle -\rho T(v_1 - v_2) + (v_1 - v_2), S_\rho(v_1) - S_\rho(v_2) \rangle \leq \| -\rho T(v_1 - v_2) + (v_1 - v_2) \| \| S_\rho(v_1) - S_\rho(v_2) \|$$

where we add the positive quantity determined above in line 2, and the final inequality comes from the Cauchy-Schwarz inequality.

Hence, we conclude that:

$$\| S_\rho(v_1) - S_\rho(v_2) \|^2 \leq \| -\rho T(v_1 - v_2) + (v_1 - v_2) \| \| S_\rho(v_1) - S_\rho(v_2) \| \implies \| S_\rho(v_1) - S_\rho(v_2) \| \leq \| -\rho T(v_1 - v_2) + (v_1 - v_2) \|$$

Now, since norms are non-negative, we may examine the square of this quantity on the right hand side and apply a polarization identity. We have that:

$$\begin{aligned} \|(v_1 - v_2) - \rho T(v_1 - v_2)\|^2 &= \|v_1 - v_2\|^2 + \rho \|T(v_1 - v_2)\|^2 + 2\rho \langle v_1 - v_2, -T(v_1 - v_2) \rangle = \\ &= \|v_1 - v_2\|^2 + \rho \|T(v_1 - v_2)\|^2 - 2\rho A(v_1 - v_2, v_1 - v_2) \end{aligned}$$

Using the definition of the operator norm as well as the hypothesis on  $A$ , we may conclude that this quantity is at most:

$$\|v_1 - v_2\|^2 + \rho^2 \|T\|^2 \|v_1 - v_2\|^2 - 2\rho\alpha \|v_1 - v_2\|^2 \leq \|v_1 - v_2\|^2 (1 + \rho^2 \|T\|^2 - 2\rho\alpha)$$

Looking at that coefficient, we see that to have  $1 + \rho^2 \|T\|^2 - 2\rho\alpha < 1$ , this implies that, since this is a parabola in  $\rho$  opening upwards, with vertex under the  $y$ -axis due to  $\|T\| > 0, \alpha > 0$ :

$$\rho^2 \|T\|^2 - 2\rho\alpha < 0 \implies \rho(\rho \|T\|^2 - 2\alpha) < 0 \implies \rho \in \left(0, \frac{2\alpha}{\|T\|^2}\right)$$

Hence, for any  $\rho$  in this range, we have that:

$$\|S_\rho(v_1) - S_\rho(v_2)\|^2 \leq \|v_1 - v_2\|^2 (1 + \rho^2 \|T\|^2 - 2\rho\alpha) < \|v_1 - v_2\|^2 k$$

for some  $k \in (0, 1)$ . Hence, taking the square root of both sides, for this choice of  $\rho$ ,  $\sqrt{k}$  remains a bound for  $\|S_\rho(v_1) - S_\rho(v_2)\| \leq \sqrt{k} \|v_1 - v_2\|$ , and if  $k \in (0, 1)$ , then  $\sqrt{k} \in (0, 1)$ .

4.2.3)

By definition then, since the  $\rho$  in 4.2.2 gives rise to a  $k \in (0, 1)$  such that  $\|S_\rho(v_1) - S_\rho(v_2)\| \leq k \|v_1 - v_2\|$ , we see that  $S_\rho$  is a contraction on the metric. Hence, by the Banach fixed-point Theorem, there exists a unique fixed point  $u \in K$  such that  $S_\rho(u) = u$ .

Alternatively, if we do not wish to appeal to the Banach fixed point Theorem for Metric spaces, we notice that that condition for  $\rho$  implies that we have satisfied the conditions for problem 1. Hence, by 1.2, there exists a unique fixed point, as we simply consider the metric induced by the norm.

4.2.4)

Identifying  $\rho f - \rho T u + u$  as  $x$ ,  $P(\rho f - \rho T u + u) = z = S_\rho(u) = u$ , and renaming  $y$  to  $v$ , we see that:

$$\langle \rho f - \rho T u + u - u, v - u \rangle \leq 0 \implies \langle \rho f - \rho T u, v - u \rangle \leq 0$$

4.2.5)

Ok, from here, consider  $\rho A(u, v - u)$ , where  $\rho$  is small enough such that we may find  $u$ , the unique fixed point associated to  $S_\rho$  determined by  $f$ . From 4.2.1, we have that:

$$\begin{aligned}\rho A(u, v - u) &= \rho \langle Tu, v - u \rangle = \langle \rho f - \rho f + \rho Tu, v - u \rangle = \\ &= \rho \langle f, v - u \rangle + \langle -\rho f + \rho Tu, v - u \rangle\end{aligned}$$

But, from 4.2.4, we see that:

$$\langle -\rho f + \rho Tu, v - u \rangle = -\langle \rho f - \rho Tu, v - u \rangle \geq 0$$

Hence, we conclude that:

$$\rho \langle f, v - u \rangle \leq \rho A(u, v - u) \implies \langle f, v - u \rangle \leq A(u, v - u)$$

□

## References

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