Homework #2

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Math 237: Homework #2

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Question 4. Let X be the space of C^1 functions on $[0,1]$ such that $f(0) = 0$. Define the bilinear function:

$$\langle f, x \rangle = \int_0^1 f'(x)\overline{g'(x)}dx$$

4.1

Prove that H, the completion of X, is a reproducing kernel Hilbert space.

4.2

Prove that $K(x,y) = \min(x,y)$.

Solution. ☐

Question 6. Let $Y = l^1(\mathbb{N})$, and define $X = \{f \in Y : \sum n|f(n)| < \infty\}$, equipped with the l^1 norm.

6.1

Prove that X is a proper dense subspace of Y, and hence X is not complete.

6.2

Define $T : X \to Y$ that sends $f(n) \mapsto nf(n)$. Show that T is a closed map, but not bounded.

6.3

Let $S = T^{-1}$. Prove that $S : Y \to X$ is bounded, surjective, but not open.

Solution. 6.1)

Without too much trouble, we can see X is a vector subspace of Y. Of course, the sequence of all 0s satisfies this condition. Further, if $f, g \in X$, then, looking at partial sums, we have that:

$$\sum_{i=1}^{k} i|f+g(i)| = \sum_{i=1}^{k} i|f(i)+g(i)| \leq \sum_{i=1}^{k} i|f(i)| + i|g(i)| = \sum_{i=1}^{k} i|f(i)| + \sum_{j=1}^{\infty} j|g(j)|$$

Since the right side converges as $k \to \infty$, due to $f, g \in X$, and all quantities being positive, so too must $\sum i|f+g(i)| < \infty$.

Similarly, we can see scalar multiplication as:

$$\sum_{i=1}^{k} i|af(i)| = \sum_{i=1}^{k} i|a||f(i)| = |a|\sum_{i=1}^{k} i|f(i)|$$

and again, since the right side converges, so too must the left.

Furthermore, this inclusion is proper. Consider the sequence $\{1/n^2\} \in Y$. Clearly, via the integral test, the series $\sum_{n=1}^{\infty} 1/n^2 < \infty$. However, on the other hand, $\sum_{n=1}^{\infty} n1/n^2 = \sum_{n=1}^{\infty} 1/n$ diverges, and hence this sequence is not in X.

Lastly, we want to show this subspace is dense. Let $g(n)$ be a sequence in $Y \setminus X$. Because we have that $\sum_{n=1}^{\infty} |g(n)| < \infty$, for every $\epsilon > 0$, there exists M such that for all $m > M$, $\sum_{n=m}^{\infty} |g(n)| < \epsilon$. Thus, construct the sequence f such that $f(n) = g(n)$ for all $n \leq M$, and 0 for all $n > M$. Clearly, f resides in X, as only finitely many terms are non-0, hence $\sum_{n=1}^{\infty} n|f(n)|$ is a finite sum, and therefore finite.

Without too much trouble, by definition, we have that:

$$\|f - g\|_1 = \sum_{n=1}^{\infty} |f(n) - g(n)| = \sum_{n=M+1} |-g(n)| < \epsilon$$

Thus, since ϵ can be as small as we want, for every open ball around our $g \in Y \setminus X$, we may find an f in such an open ball. Hence, X is dense.

However, X cannot be complete then. As above, take a sequence of $f_n \to g$, where $\|f_n - g\|_1 < 1/n$. Evidently, this is a Cauchy sequence from the argument above, and it converges to an element in $Y \setminus X$.

6.2)

It should be clear that T must be a closed map. If $f_k \to f$, and Tf_k is convergent, we show that $Tf_k \to Tf$. Indeed, we see that:

$$\|Tf - Tf_k\|_1 = \sum_{n=1}^{\infty} |Tf(n) - Tf_k(n)| = \sum_{n=1}^{\infty} |nf(n) - nf_k(n)| = \sum_{n=1}^{\infty} |n||f(n) - f_k(n)|$$

Now, since $f_k \to f$, we have that $\sum_{n=1}^{\infty} |f(n) - f_k(n)| \to 0$. In particular, that means we can choose k such that for each n, $|f(n) - f_k(n)| < \frac{1}{n}\frac{\epsilon}{2^n}$. Under this choice, the above inequality becomes:

$$\sum_{n=1}^{\infty} |n||f(n) - f_k(n)| \leq \sum_{n=1}^{\infty} |n|\frac{1}{n}\frac{\epsilon}{2^n} = \epsilon$$

Thus, we have that $Tf_k \to Tf$, and thus T is closed.

However, it is clear that T is not bounded. Let f_i be the family of sequences with 0 everywhere, except for at the i-th position. Evidently, $\|f_i\|_1 = 1$. However, $\|Tf_i\|_1 = i$, and since in this family, we can choose i arbitrarily large, T cannot be bounded.

6.3)

Clearly, S is surjective, since as we saw, T is a map that takes $f \in X$ to Tf, and evidently, $S \circ T(f) = f \in X$. Hence, we can always find a sequence in Y that maps to any sequence in X.

Moreover, S must be bounded. If we look at the action of S, S sends $f(n) \in Y$ to $\frac{1}{n}f(n) \in X$. Evidently then, we have that:

$$\|Sf\|_1 = \sum_{k=1}^{\infty} |1/kf(k)| \leq \sum_{k=1}^{\infty} |f(k)| = \|f\|_1$$

and so S is bounded, at least by 1.

However, S is certainly not open. By Folland, we may say that S is open if and only if, for B the open ball of radius 1 around 0 in Y, $S(B)$ contains a ball centered around 0 in X.

Fix an $i \in \mathbb{N}$. We can then consider the family of sequences g_i such that $g(i) = 2/i$, and 0 otherwise. Evidently, we may find a sequence as close to 0 as we want, since $\|g_i\|_1 = 2/i$. However, for no i does there exists a $y \in B$ such that $Sy = g_i$, since under the map T, we see that $Tg_i(n)$ is 2 when $n = i$ and 0 otherwise, outside of B. Hence, S is not open.

$\square$

Question 16. Define $\text{Lip}[0,1] = \{f \in C[0,1] : f \text{ is Lipschitz }\}$. For each $n \geq 1$, define $F_n = \{f \in C[0,1] : |f(x) - f(y)| \leq n|x - y| \text{ for all } x, y \in [0,1]\}$.

16.1

Prove that for each $n \geq 1$, F_n is a closed, nowhere dense subset of $\text{Lip}[0,1]$.

16.2

Conclude that $\text{Lip}[0,1]$ is a countable union of nowhere dense subsets of $C[0,1]$.

Solution. Obviously, $F_n \subset \text{Lip}[0,1]$, as we may just take n to be a Lipschitz constant.

Consider $U_n = (F_n)^c$, the compliment of F_n. We wish to show that this set is open.

Let $f \in U_n$. Evidently then, there exists $x, y \in [0,1]$ such that $|f(x) - f(y)| > n|x-y|$. Fix a choice x_0, y_0 (where, evidently, $x_0 \neq y_0$ as the inequality always holds then), without loss of generality, we may assume $f(x_0) > f(y_0)$ and otherwise swap labels, and call $d = f(x_0) - f(y_0) - n|x_0 - y_0|$.

Now, let g be in the open ball around f with radius at most $d/2$. From the condition of being in the ball of radius at most $d/2$, we must have that $g(y_0) > g(x_0)$, since $d < |f(x_0) - f(y_0)|$, and hence, from the supremum norm, $g(y_0) \geq f(y_0) - d/2 > [f(x_0) + f(y_0)]/2$ and similarly, $g(x_0) \leq f(x_0) + d/2 < [f(x_0) + f(y_0)]/2$. Thus, we have that at x_0, y_0 for g, that:

$$g(y_0) - g(x_0) \geq f(y_0) - d/2 - f(x_0) - d/2 = f(y_0) - f(x_0) - d = n|x_0 - y_0|$$

Since we can arbitrarily shrink the ball to have radius smaller than $d/2$, say $d/3$, this shows that we may find an open ball around any $f \in U_n$, hence U_n is open, and thus F_n is closed.

Now, we need to show that F_n is nowhere dense. Let $f \in F_n$. Let $\epsilon > 0$, and suppose without loss of generality that $f(0) \geq f(\epsilon/n+1)$. Consider the continuous function g_ϵ such that it takes on ϵ at 0, joined linearly to 0 at $\epsilon/n+1$, and constantly 0 otherwise. Of course, this is continuous, with $\|g_\epsilon\|_u = \epsilon$. Furthermore, we can consider the function $f + g_\epsilon$, and in particular, compute $f + g_\epsilon(0) - [f + g_\epsilon(\epsilon/n+1)]$. We have that:

$$f + g_\epsilon(0) - [f + g_\epsilon(\epsilon/n+1)] = f(0) - f(\epsilon/n+1) + g_n(0) - g_n(\epsilon/n+1) \geq g_n(0) - g_n(\epsilon/n+1) = \epsilon > \epsilon/n+1$$

where we use the fact that $f(0) \geq f(\epsilon/n+1)$ for one inequality, and the fact that $n \geq 1$ for the other. Hence, $f + g_n$ is not Lipschitz. If $f(0) < f(\epsilon/n+1)$, the same argument holds by taking $-g_\epsilon$.

Since the choice of f was arbitrary, this implies that F_n contains no non-trivial open subsets, and thus, as it is its own closure, F_n is nowhere dense.

Clearly, $\text{Lip}[0,1] = \cup F_n$. We have that $F_n \subseteq \text{Lip}[0,1]$ for each n, since we may take n as a Lipschitz constant. Hence, $\cup F_n \subseteq \text{Lip}[0,1]$.

On the other hand, let $g \in \text{Lip}[0,1]$. Then, there exists a constant $K \geq 0$ such that $|g(x) - g(y)| \leq K|x-y|$ for all $x, y \in [0,1]$. Then, since this inequality clearly holds for $K' \geq K$, taking any integer $k \geq K$, we see that $g \in F_k$. Hence, $\text{Lip}[0,1] \subseteq \cup F_n$, and we are done.

□

Question 17. Let f be a nonnegative Lebesgue measurable function defined on $\mathbb{R}$. Assume that for all $g \in L^2(\mathbb{R})$, $fg \in L^1(\mathbb{R})$. Prove that $T_f(g) = \int_{\mathbb{R}} gf$ is a bounded linear functional on L^2, and conclude that $f \in L^2$.

Solution. □

Question 19. Suppose $\mathcal{B}$ is a Banach space, and let S be a closed proper subspace. Fix some $f_0 \notin S$. Show that there exists a continuous linear functional γ such that $\gamma(f) = 0$ for all $f \in S$, and $\gamma(f_0) = 1$. Moreover, show that we may choose the linear functional such that $\|\gamma\| = 1/d$, where d is the distance from f_0 to S.

Solution. □