Homework #1

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Math 235: Homework #1

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2.1

Problem 1.1.20. Assume $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a metric space X, and there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ that converges to $x \in X$. Prove that $x_n \to x$.

Proof. Let $\epsilon > 0$ be given.

Since $\{x_n\}$ is Cauchy, we can choose N such that for all $m, n > N$, $d(x_m, x_n) < \frac{\epsilon}{2}$.

Similarly, since $x_{n_k} \to x$, there exists N_k such that for all $n_k > N_k$, $d(x_{n_k}, x) < \frac{\epsilon}{2}$. In particular, we may choose N_k such that $N_k > N$

Let $m > N_k$.

Then, we have $d(x, x_m) \leq d(x, x_{n_k}) + d(x_m, x_{n_k})$, where $n_k > N_k$, as above. Since $n_k > N_k$, we have that $d(x, x_{n_k}) < \frac{\epsilon}{2}$, by the convergence of the subsequence. Similarly, since the entire sequence is Cauchy, and $m, n_k > N_k > N$, we have that $d(x_m, x_{n_k}) < \frac{\epsilon}{2}$.

Thus, $d(x, x_m) \leq d(x, x_{n_k}) + d(x_m, x_{n_k}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

□

Problem 1.1.21. Given a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space X, prove the following statements:

(a) If $d(x_n, x_{n+1}) < 2^{-n}$ for every $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy.

(b) If $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $d(x_n, x_{n+1}) < 2^{-n}$ for each $k \in \mathbb{N}$

Proof of part (a). Let $\epsilon > 0$ be given.

First, a remark: we notice that due to the condition $d(x_n, x_{n+1}) < 2^{-n}$ for every $n \in \mathbb{N}$, suppose we have $d(x_m, x_n) = d(x_m, x_{m+k})$, for some $m, n \in \mathbb{N}, k > 0$. By iteratively using the triangle equality, we see that:

$$d(x_m, x_{m+k}) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+k}) \leq ... \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + ... + d(x_{m+(k-1)}, x_{m+k})$$

.

However, by our hypothesis, we have that:

$$d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + ... + d(x_{m+(k-1)}, x_{m+k}) < 2^{-m} + 2^{-(m+1)} + ... + 2^{-(m+k)}$$

.

We recognize that last portion as being a geometric series with first term 2^{-m} and ratio 2^{-1}.

Now, choose N such that N is that smallest natural number such that $2^{-N+1} < \epsilon$. This must exist, because ϵ is greater than 0, and $\{2^{-k}\}_{k\in\mathbb{N}}$ converges to 0.

Consider $d(x_m, x_n)$ such that $m, n > N$, and, WLOG, suppose $m > n$ and define $k = m - n$. Then, from our remark, we see that:

$$d(x_m, x_n) = d(x_m, x_{m+k}) < 2^{-m} + 2^{-(m+1)} + ... + 2^{-(m+k)} < 2^{-m} + 2^{-(m+1)} + ...$$

We know that the sum of an infinite geometric series, with first term a and common ratio r is $\frac{a}{1-r}$. So here, we have that:

$$2^{-m} + 2^{-(m+1)} + ... = \frac{2^{-m}}{1-2^{-1}} = 2^{-m+1}$$

. Thus, we have: $d(x_m, x_n) = d(x_m, x_{m+k}) < 2^{-m+1}$. But by our hypothesis, since $m > N$ and N is the smallest natural number such that $2^{-N+1} < \epsilon$, $2^{-m+1} < 2^{-N+1} < \epsilon$. Since the choice of m, n was arbitrary, this works for all m, n. Thus, $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy.

□

Proof of part (b). We will construct our sequence inductively, so as to best see the construction, but the argument need not be inductive in nature.

Firstly, we will find our base element, x_{n_1}. We know that we wish to have $d(x_{n_1}, x_{n_2}) < 2^{-1}$, so we will choose a N_1 such that for all $m_{N_1}, n_{N_1} > N_1$, $d(x_{m_{N_1}}, x_{n_{N_1}}) < 2^{-1}$ due to the fact that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. Choose any $x_{n_{N_1}} > N_1$. In particular, we will choose the smallest such x_{n_1}, but this is not important.

Now, we proceed inductively. Suppose we have constructed a partial subsequence $\{x_{n_k}\}_{k=1}^{k=l}$ of l terms such that $d(x_k, x_{k+1}) < 2^{-k}$ for $k \in \{1, 2, ...l\}$. We wish to show now that we can pick an element x_{l+1} to add to our subsequence such that $d(x_l, x_{l+1}) < 2^l$, but also one such that $d(x_{l+1}, x_{l+2}) < 2^{l+1}$. Thus, we choose N_{l+1} such that for all $m_{N_{l+1}}, n_{N_{l+1}} > N_{l+1}$, $d(x_{m_{N_{l+1}}}, x_{n_{N_{l+1}}}) < 2^{-(l+1)}$. We choose then n_{l+1} such that $n_{l+1} > n_l$ and $n_{l+1} > N_{l+1}$. We note here that, by the construction of n_l, since $n_{l+1} > n_l > N_l$, that $d(x_{n_l}, x_{n_{l+1}}) < 2^{-l}$. But further, by the construction of n_{l+1}, regardless of the choice of n_{l+2}, so long as $n_{l+2} > n_{l+1} > N_{l+1}$, $d(x_{n_{l+1}}, x_{n_{l+2}}) < 2^{-(l+1)}$. This completes our inductive construction.

□

Problem 1.3.6. Prove that $f : \mathbb{R}^d \to \mathbb{C}$ is uniformly continuous on $\mathbb{R}^d$ if and only if

$$\lim_{a\to 0} \|T_a f - f\|_u = 0,$$

where $T_a f(x) = f(x - a)$ denotes the translation of f by $a \in \mathbb{R}^d$

Proof. We begin by assuming f is uniformly continuous.

Let $\epsilon > 0$ be given. Because f is uniformly continuous, there exists δ such that $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$.

Now, choose $a < \delta$. Then, for all $x \in \mathbb{R}^d$, $\|f(x - a) - f(x)\| = d(f(x - a), f(x)) < \epsilon$. Since this is true for all x, then $\sup_{x\in\mathbb{R}^d} (T_a f - f) = \|T_a f - f\|_u < \epsilon$, and therefore $\lim_{a\to 0} \|T_a f - f\|_u = 0$.

Now assume $\lim_{a\to 0} \|T_a f - f\|_u = 0$.

Let $\epsilon > 0$ be given. Because $\lim_{a\to 0} \|T_a f - f\|_u = 0$, there exists δ such that $d(a, 0) < \delta \Rightarrow \sup_{x\in\mathbb{R}^d} (T_a f - f) < \epsilon$.

Because $\sup_{x\in\mathbb{R}^d} (T_a f - f) < \epsilon$, this implies that for all $x \in \mathbb{R}^d$:

$$|T_a f(x) - f(x)| = |f(x - a) - f(x)| = d(f(x - a), f(x)) < \epsilon$$

. Thus, f is uniformly continuous.

□

Problem 1.3.8. Let $g \in C_0(\mathbb{R})$ be any function that does not belong to $C_c(\mathbb{R})$. For each integer $n > 0$, define a compactly supported approximation to g by setting $g_n(x) = g(x)$ for $|x| \leq n$ and $g_n(x) = 0$ for

$|x| > n+1$, and let g_n be linear on $[n, n+1]$ and $[-n-1, -n]$. Show that $\{g_n\}_{n\in\mathbb{N}}$ is Cauchy in $C_c(\mathbb{R})$ with respect to the uniform norm, but it does not converge uniformly to any function in $C_c(\mathbb{R})$. Conclude that $C_c(\mathbb{R})$ is not complete with respect to $\|\cdot\|_u$ and is not a closed subset of $C_0(\mathbb{R})$

Proof. First, we begin by examining the $\|g_m - g_n\|_u$, where, WLOG, we take $m > n$. We notice then, by the triangle inequality, we have that, for all $x \in \mathbb{R}$:

$$|g_m(x) - g_n(x)| = |g_m(x) + (-g_n(x))| \leq |g_m(x)| + |g_n(x)|$$

.

We also notice that because g_m and g_n only differ on the interval $[n, m+1]$ and $[-m-1, -n]$, we need only consider $x \in [n, m+1] \cup [-m-1, -n]$ or, $n \leq |x| \leq m+1$ as $|g_m - g_n| = 0$ for all x outside of those two intervals.

Then, we proceed as follows. Let $\epsilon > 0$ be given. Since $g \in C_0(\mathbb{R})$, we may choose $k \in \mathbb{N}$ such that for all $x > k$, $|g(x)| < \frac{\epsilon}{6}$.

Now, choose $m > n > k$. From the remarks above, we see the following, where we define $I = [n, m+1] \cup [-m-1, -n]$ for bookkeeping:

$$\|g_m - g_n\|_u = \sup_{x\in I} |g_m(x) - g_n(x)| \leq \sup_{x\in I} (|g_m(x)| + |g_n(x)|)$$

.

However, since $m, n > k$, we have that both $|g_m(x)| < \frac{\epsilon}{6}$ and $|g_n(x)| < \frac{\epsilon}{6}$. Thus:

$$\|g_m - g_n\|_u \leq \sup_{x\in I} (|g_m(x)| + |g_n(x)|) \leq \sup_{x\in I} (\frac{\epsilon}{6} + \frac{\epsilon}{6}) = \frac{\epsilon}{3} < \epsilon$$

. Thus, $\{g_n\}_{n\in\mathbb{N}}$ is Cauchy in $C_c(\mathbb{R})$. However, it cannot be convergent to any function in $C_c(\mathbb{R})$ because it is convergent to g:

Let $\epsilon > 0$ be given. Since $g \in C_0(\mathbb{R})$, we may choose $k \in \mathbb{N}$ such that for all $x > k$, $|g(x)| < \frac{\epsilon}{6}$. Choose any $n > k$. Then, we have:

$$\|g - g_n\|_u = \sup_{x\in\mathbb{R}} |g - g_n| = \sup_{|x|>n} (|g| + |g_n|) \leq \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3} < \epsilon$$

However, by construction, g does not belong to $C_c(\mathbb{R})$, and belongs to the superset $C_0(\mathbb{R})$. Since limits are unique in normed metric spaces, the limit point does not belong in $C_c(\mathbb{R})$, not all Cauchy sequences are complete, and $C_c(\mathbb{R})$ does not contain all of its limit points and is therefore not closed. $\square$

Problem 1.4.3. Define $h : [-1, 1] \to \mathbb{R}$ by $h(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$, and $h(0) = 0$. Prove that h is Lipschitz on $[-1, 1]$.

Proof. Here, we notice that $h(x) = h(-x)$, due to $\sin(x) = \sin(-x)$. We now claim that we need only prove that h is Lipschitz on $[0, 1]$ due to the symmetry here.

Due to the symmetry of $x, -x$, clearly, if $|f(a) - f(b)| \leq K|a - b|$ for $a, b \in [0, 1]$ and some $K \in \mathbb{R}$, then it must hold true for $-a, -b$ as well. Further, now suppose, that $a, b \in [0, 1]$. Then, we have, since $a, b \geq 0, |a + b| \geq |a - b|$:

$$\frac{|f(a) - f(-b)|}{|a - (-b)|} = \frac{|f(a) - f(b)|}{|a + b|} \leq \frac{|f(a) - f(b)|}{|a - b|} \leq K$$

.

Thus, if h is Lipschitz on $[0, 1]$, by symmetry considerations, it must be Lipschitz on $[-1, 1]$. Here we compute h', the derivative as $2x \sin \frac{1}{x} - \cos \frac{1}{x}$, so h is continuous on $[0, 1]$ (because $-x^2 < x^2 \sin \frac{1}{x} < x^2$, and since $\lim_{x\to 0} x^2 = 0, \lim_{x\to 0} -x^2 = 0$, by the squeeze theorem so does $x^2 \sin \frac{1}{x}$), and differentiable on $(0, 1)$. We notice then, by the MVT, we have that, for any $a, b \in [0, 1]$, that there exists $c \in (0, 1)$ such that:

$$\frac{|f(a) - f(b)|}{|a - b|} = 2c \sin \frac{1}{c} - \cos \frac{1}{c} \leq \max_{x\in[0,1]} (2x \sin \frac{1}{x} - \cos \frac{1}{x}) \leq 3$$

Thus, h is Lipschitz on $[0, 1]$, and by our symmetry concerns above, h is Lipschitz on $[-1, 1]$.

□

Problem 1.4.4. Prove the following statements:

(a) If f is Hölder continuous on an interval I for some exponent $\alpha > 0$, then f is uniformly continuous on I.

(b) If f is Hölder continuous on an interval I for some exponent $\alpha > 1$, then f is constant on I.

(c) The function $f(x) = |x|^{\frac{1}{2}}$ is Hölder continuous on $[-1, 1]$ for exponents $0 < \alpha \leq 1/2$ but not for any exponent $\alpha > 1/2$.

(d) The function g defined by $g(x) = -1/\ln x$ for $x > 0$ and $g(0) = 0$ is uniformly continuous on $[0, 1/2]$, but it is not Hölder continous for any exponent $\alpha > 0$.

Proof. (a)

Let $\epsilon > 0$ be given. Because f is Hölder continuous on I for some $\alpha_0 > 0$, fix a $K_0 > 0$ such that $|f(b) - f(a)| \leq K_0|b - a|^{\alpha_0} = K_0 * d(a, b)^{\alpha_0}$. Choose $\delta > 0$ such that $\delta = (\epsilon/K_0)^{1/\alpha_0}$. Then, for $d(a, b) < \delta$, our Hölder continuity equation becomes:

$$d(f(a), f(b)) = |f(b) - f(a)| \leq K_0|b - a|^{\alpha_0} = K_0 * d(a, b)^{\alpha_0} < K_0 * [(\epsilon/K_0)^{1/\alpha_0}]^{\alpha_0} = K_0 * \epsilon/K_0 = \epsilon$$

.

Thus, f is uniformly continuous on I.

(b)

First, fix some point $x_0 \in I$ and assume f is Hölder continuous with exponent $\alpha_1 > 1$. We may reexpress $\alpha_1 = 1 + \alpha_1'$, with $\alpha_1' > 0$. Then, for any other point $h \in I$, we have that, for some $K_1 > 0$:

$$|f(x_0) - f(h)| \leq K_1|x_0 - h|^{\alpha_1'+1}$$

Rearranging, we have:

$$\frac{|f(x_0) - f(h)|}{|x_0 - h|} \leq K_1|x_0 - h|^{\alpha_1'}$$

Taking the limit as both sides of $h \to x_0$, we find that:

$$\lim_{h \to x_0} \frac{|f(x_0) - f(h)|}{|x_0 - h|} \leq \lim_{h \to x_0} K_1|x_0 - h|^{\alpha_1'} = 0$$

We recognize the left hand side as the derivative of f at x_0, and so we find that at an arbitrary $x_0 \in I$, f' exists and is equal to 0. In particular, since the choice of x_0 was arbitrary, we see that f' exists on I and is identically 0. Then, by the mean value theorem, we have that:

$$\forall a, b \in I, \frac{f(b) - f(a)}{b - a} = f'(c) = 0 \implies f(b) - f(a) = 0 \implies f(a) = f(b)$$

Thus, f is constant.

(c)

By a similar argument to the proof of 1.4.3 above, we will look at $f(x) = |x|^{\frac{1}{2}}$ on $[0, 1]$ and claim by symmetry, that this extends to $[-1, 1]$. First, let's consider the following quantity, for $\alpha \in (0, 1]$ and $a, b \in [0, 1]$, with, wlog, $b > a$:

$$\frac{|f(b) - f(a)|}{|b - a|^\alpha} = \frac{|\sqrt{b} - \sqrt{a}|}{|\sqrt{b} - \sqrt{a}|^\alpha |\sqrt{a} + \sqrt{b}|^\alpha} = \frac{|\sqrt{b} - \sqrt{a}|^\alpha}{|\sqrt{a} + \sqrt{b}|^\alpha} * |\sqrt{b} - \sqrt{a}|^{1-2\alpha} = \left|1 - \frac{2\sqrt{a}}{\sqrt{a} + \sqrt{b}}\right|^\alpha * |\sqrt{b} - \sqrt{a}|^{1-2\alpha}$$

We notice that for the quantity, $\frac{2\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, it is non-negative, and since $b > a$, $\sqrt{b} + \sqrt{a} > 2\sqrt{a}$ we have that $0 \leq \frac{2\sqrt{a}}{\sqrt{a}+\sqrt{b}} \leq \frac{2\sqrt{a}}{2\sqrt{a}} = 1$. Then:

$$\left|1 - \frac{2\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right|^{\alpha} \leq |1|^{\alpha} = 1$$

Now, we look at the second quantity $K = |\sqrt{b} - \sqrt{a}|^{1-2\alpha}$. Here, we split into three cases: $1 - 2\alpha < 0$,$1 - 2\alpha > 0$,$1 - 2\alpha = 0$.

For $1 - 2\alpha < 0$, or, $\alpha > 1/2$, we have that this quantity is unbounded, as it looks like $1/|\sqrt{b} - \sqrt{a}|^{2\alpha-1}$ where $2\alpha - 1$ is positive. Suppose, for example, we fix $a = 0$. Then, for the quantity $1/\sqrt{b}^{2\alpha-1}$, as $b \to 0$, this value is unbounded. Then, there can not exist any constant to make f Hölder continuous.

Now, suppose $1 - 2\alpha = 0 \implies \alpha = 1/2$. Then we have $K = 1$, and, in particular, 1 is a bound.

Finally, suppose $1 - 2\alpha > 0 \implies 0 < \alpha < 1/2$. We notice that $\sqrt{b} - \sqrt{a}$ is bounded above by 1, so we have 1 to a positive exponent, which is itself 1. Then, as in the second case, 1 is a bound.

From these cases, we conclude that f is Hölder continuous on $[0, 1]$ and thus $[-1, 1]$ with exponent α when $\alpha \in (0, 1/2]$ and not for any $\alpha > 1/2$.

(d)

Firstly, we see that because $-\ln x$ is continuous on $(0, 0.5]$, we have that $-1/\ln x$ is continuous on $(0, 0.5]$. Further, we see that because as $x \to 0$, $\ln x \to \infty$, so $-1/\ln x \to 0$, so g is continuous all of $[0, 0.5]$. Then, because f is continous, $[0, 0.5]$ a closed, bounded subset of $\mathbb{R}$ and thus compact, and we live in a metric space, we have that f is uniformly continuous by the Heine-Cantor Theorem.

However, consider the quantity for $\alpha \in (0, 1]$ and $b \in [0, 1]$:

$$\frac{|f(b) - f(0)|}{|b - 0|^{\alpha}} = \frac{|-1/\ln b|}{(b)^{\alpha}} = \frac{1/b^{\alpha}}{-\ln b}$$

We notice that as $b \to 0$, $1/b^{\alpha} \to \infty$ and $-\ln b \to \infty$, $1/x, -\ln x$ differentiable on $(0, 0.5)$, we satisfy the conditions of L'Hôpital's rule.

Then, we have that:

$$\lim_{b\to 0} \frac{1/b^{\alpha}}{-\ln b} = \lim_{b\to 0} \frac{-\alpha/b^{\alpha+1}}{-1/b} = \lim_{b\to 0} \frac{\alpha}{b^{\alpha}} \to \infty$$

Then, by L'Hôpital's rule, the original quantity is unbounded. Since the choice of α did not affect the quantity, we conclude that g cannot be Hölder continuous for any $\alpha \in (0, 1]$ on $[0, 0.5]$.

□

2.2

Problem 2.1.29. Prove that a countable union of sets that each have exterior measure zero has exterior measure zero. That is, if $Z_k \subseteq \mathbb{R}^d$ and $|Z_k|_e = 0$ for each $k \in \mathbb{N}$, then $|\cup_k Z_k|_e = 0$

Proof. Here, we apply Theorem 2.1.13 from Heil, the countable subadditivity of exterior measures, by that, we have:

$$|\cup_k Z_k|_e \leq \Sigma_{k=1}^{\infty} |Z_k|_e$$

Since $|Z_k|_e = 0$ for all k, then:

$$|\cup_k Z_k|_e \leq \Sigma_{k=1}^{\infty} |Z_k|_e = \Sigma_{k=1}^{\infty} 0 = 0$$

. Since by the definition of exterior measure, the measure of a set is non-negative as the volume of boxes are non-negative, we have that $|\cup_k Z_k|_e \leq 0 \to |\cup_k Z_k|_e = 0$ $\square$

Problem 2.1.30. Show that if $Z \subseteq \mathbb{R}^d$ and $|Z|_e = 0$, then $\mathbb{R}^d/Z$ is dense in $\mathbb{R}^d$.

Proof. Suppose not. Then, there exists a non-empty open set $U \subseteq \mathbb{R}^d$ such that for all u in U, u is in the compliment of $\mathbb{R}^d/Z$, i.e, $U \subseteq Z$. Choose some $u_0 \in U$. By the definition of an open set, we have some open ball with $r_0 > 0$, $B_{r_0}(u_0)$ that is completely contained within U.

However, now consider the box of the form $Q = \Pi_{j=1}^d [u_{0_j} - r_0/3, u_{0_j} + r_0/3]$, where u_{0_j} denotes the j-th component of the vector u. We have the following inclusion: $Q \subseteq B_{r_0}(u_0) \subseteq U$. But, since Q is a box, we have:

$$|Q|_e = \text{vol}(Q) = \Pi_{j=1}^d (u_{0_j} + r_0/3 - [u_{0_j} - r_0/3]) = (2r_0/3)^d$$

.

However, by Lemma 2.1.11 from Heil, we have from our inclusion that since $Q \subseteq Z$, that $|Q|_e = (2r_0/3)^d \leq |Z|_e = 0$, a contradiction.

Thus, there cannot exist a non-empty open set $U \subseteq Z$, and thus $\mathbb{R}^d/Z$ must be dense in $\mathbb{R}^d$. $\square$

Problem 2.1.31. Let Z be a subset of $\mathbb{R}$ such that $|Z|_e = 0$. Set $Z^2 = \{x^2 : x \in Z\}$, and prove that $|Z^2|_e = 0$.

Proof. First, we note a few properties.

Let $\{Q_i\}$ be a countable collection of boxes that covers Z. Then, $\{Q_i^2\}$ is a countable collection that covers Z^2, where we define, in the same way, $Q_i^2 = \{q_i^2 | q_i \in Q_i\}$. To see this, let $z \in Z$. Then, since $\{Q_i\}$ covers Z, there exists $q_j \in Q_j$ for some j such that $z = q_j$. Then $z^2 \in Q_j^2$, and since the choice of z was arbitrary, this means that this is true for all z in Z.

Now, consider the quantity $\text{vol}(Q_j^2)$. If $Q_j = (a_j, b_j)$, then $Q_j^2 = (\min(a_j^2, b_j^2), \max(a_j^2, b_j^2))$. Then, we would have that $\text{vol}(Q_j^2) = |a_j^2 - b_j^2| = |a_j + b_j||a_j - b_j| = |a_j + b_j|\text{vol}(Q_j)$.

Now, let $\epsilon > 0$ be given. Choose a collection of boxes that covers Z, $\{Q_i\}$, with the condition that $\text{vol}(Q_i) < \epsilon/2^{-(i+1)}|a_i + b_i|$, where a_i, b_i are the endpoints of the box $Q_i = (a_i, b_i)$. We may do this because $|Z|_e$, so the sum of the volumes of the boxes must be arbitrarily small, which imply that the volume of each individual box can be arbitrarily small.

Then, we have that $\text{vol}(Q_i^2) = |a_i + b_i|\text{vol}(Q_i) < \epsilon|a_i + b_i/2^{-(i+1)}|a_i + b_i| = \epsilon/2^{-(i+1)}$.

From the remarks above, this is a covering of Z^2, and in particular, the sum of the volumes then is:

$$\Sigma_{j=1}^{\infty} \text{vol}(Q_j^2) \leq \epsilon/2^{-(j+1)} = \epsilon[\frac{1/4}{1 - 1/2}] = \epsilon/2 < \epsilon$$

Then, we can construct a covering of Z^2 with arbitrarily small volume, which means $|Z^2|_e = 0$. $\square$

Problem 2.1.32. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then its graph

$$\Gamma_f = \{(x, f(x)) | x \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

has measure zero, i.e., $|\Gamma_f|_e = 0$.

Proof. Let $\epsilon > 0$ be given. Let $\{q_n | n \in \mathbb{N}\}$ be an enumeration of the rationals.

Because f is continous, in particular, continuous at each point q_n, there exists a $\delta > 0$ such that $|x - q_n| < \delta \to |f(x) - f(q_n)| < \epsilon/2^{(n+1)}$. Set $0 < w_n < \max(\delta, 1/2)$ and define the interval $I_n = [q_n - w_n, q_n + w_n]$. Then, construct the box:

$$Q_n = I_n \times [f(q_n) - \epsilon/2^{n+1}, f(q_n) + \epsilon/2^{n+1}]$$

Since the rationals are dense, these intervals I_n cover $\mathbb{R}$. Also, because of the continuity of f as above, we see that Γ_f in the interval I_n is completely contained within Q_n. We see that for any x_n in I_n, $|q_n - x_n| \le w_n$, so that:

$$|f(q_n) - f(x_n)| < \epsilon/2^{n+1} \to f(q_n) - \epsilon/2^{n+1} < f(x_n) < f(q_n) + \epsilon/2^{n+1}$$

Further, due to our condition on w_n, we have that:

$$\text{vol}(Q_n) = ([q_n + w_n] - [q_n - w_n]) * ([f(q_n) + \epsilon/2^{n+1}] - [f(q_n) - \epsilon/2^{n+1}]) = 2w_n * 2\epsilon/2^{n+1} < \epsilon/2^n$$

Thus, we have that the collection of $\{Q_n\}$ for $n \in \mathbb{N}$ has volume $\Sigma_{n=1}^{\infty} \epsilon/2^n = \epsilon$ and since we can construct a collection of boxes with the sum of their volumes as arbitrarily small, the outer measure $|\Gamma_f|_e = 0$.

□

Problem 2.1.34. Given $E \subseteq \mathbb{R}^d$, prove that $|E|_e = \inf\{\Sigma \text{vol}(Q_k)\}$, where the infimum is taken over all countable collections of boxes $\{Q_k\}$ such that $E \subseteq \cup Q_k^o$

Proof. □

Problem 2.1.35. Find the exterior measures of the following sets.

(a) $L = \{(x, x) | 0 \le x \le 1\}$ the diagonal of the unit square in $\mathbb{R}^2$.

(b) An arbitrary line segment, ray, or line in $\mathbb{R}^2$

Proof. First, we will prove part (a).

Define a parametrization of the line L via $f : [0, 1] \to L$ where $f(t) = t(1, 1) + (0, 0)$. We claim this has exterior measure 0. Before we prove that, let's first show that this line L is differentiable and thus continuous.

$$f'(t_0) = \lim_{h\to 0} \frac{f(t_0 + h) - f(t_0)}{h} = \lim_{h\to 0} \frac{(t_0 + h)(1, 1) + (0, 0) - [t_0(1, 1) + (0, 0)]}{h} = \lim_{h\to 0} \frac{h(1, 1)}{h} = (1, 1)$$

.

Since this is true irrespective of the point t_0, the derivative exists for all points on the domain. Therefore, f is also continuous on its domain.

Let $\epsilon > 0$ be given.

Let $\{q_n\}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$. Then, use the same construction and argument as 2.1.32 above.

Part (b) follows in a similar fashion. Here, we note that for a line segment S, we have a parametrization that looks like $f : [0, 1] \to \mathbb{R}^2$ where $f(t) = t(x_1, y_1) + (x_0, y_0)$, where $(x_0, y_0), (x_1, y_1)$ are the endpoints of the line segment. For a ray R, we have a parametrization of form $g : [0, \infty) \to \mathbb{R}^2$ where $g(t) = t(x_1', y_1') + (x_0', y_0')$ for (x_0', y_0') the fixed starting point and (x_1', y_1') any other point on the ray. Finally, for a line L, we have a parametrization of form $h : \mathbb{R} \to \mathbb{R}^2$ where $h(t) = t(x_1'', y_1'') + (x_0'', y_0'')$ for any two fixed points $(x_1'', y_1''), (x_0'', y_0'')$.

The same argument applies. The functions f, g, h are differentiable via the same calculation, therefore continuous. Then, we may apply a construction as 2.1.32 above, and therefore the exterior measure is 0. □

Problem 2.1.39. Given a set $E \subseteq \mathbb{R}^d$, show that $|E|_e = 0$ if and only if there exist countably many boxes Q_k such that $\Sigma \text{vol}(Q_k) < \infty$ and each point $x \in E$ belongs to infinitely many Q_k.

Proof. First, suppose $|E|_e = 0$. Then, for each $k \in \mathbb{N}$, there must exists some collection of countably many boxes $Q_k = \{Q_{k,j}\}$ such that $\Sigma_j \text{vol}(Q_{k,j}) < 2^{-k}$ due to the properties of the infimum. Now, consider the collection of collection of boxes $\{Q_k\} = \{Q_{k,j}\}$ with $k, j \in \mathbb{N}$. This collection is countably infinite, because there exists a bijection from $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Further, $\Sigma_k \Sigma_j \text{vol}(Q_{k,j}) \le \Sigma_k 2^{-k} = 1$. Lastly, since each Q_k is a cover of E, for any point $e \in E$, and for each k, there exists a box $\{Q_{k,j}\}$ such that $e \in \{Q_{k,j}\}$.

Now, suppose we have a set $E \subseteq \mathbb{R}^d$, and that there exists countably many boxes Q_k, $\Sigma\text{vol}(Q_k) < \infty$, and, for each point $x \in E$, x belongs to infinitely many Q_k. Let $\epsilon > 0$ be given. Construct a collection of boxes $\{Q_j\}$ as follows. First, choose any Q_1 such that $\text{vol}(Q_1) < \epsilon/2$. Then, choose any Q_2 such that $\text{vol}(Q_2) < \epsilon/2^2$ and $Q_2 \cap (E \setminus Q_1)$ is non-empty. More generally, construct Q_i such that $\text{vol}(Q_i) < \epsilon/2^k$ and $Q_i \cap (E \setminus \cup_{n=1}^{i-1} Q_n)$ is non-empty, where this iterative process may or may not terminate.

Let's justify our construction first. By hypothesis, $x \in E$ belongs to infinitely many Q_k, i.e. we have a collection of countably infinitely many Q_k. However, we also have that $\Sigma\text{vol}(Q_k) < \infty$, so then this implies that for some $k_0 > K$ and $\epsilon > 0$, that $\text{vol}(Q_k) < \epsilon$ for all $k > k_0$ as otherwise, we would have that the infimum is non-0, and then our lower bound on the sum of the volumes would be $\inf_k Q_k \times \#\{Q_k\} = \infty$. Further, we can refine this and say that for any point x, that we may find a small enough Q_k that contains x for approximately the same reason. Suppose not, then for every Q_j that contains some x_0, we have that $\inf_j\{Q_j\} > 0$. But, there are infinitely many of such Q_j, so then we have that $\Sigma_j(Q_j) \geq \inf_j\{Q_j\} \times \#\{Q_j\} = \infty$, a contradiction. Finally, since we know that we have only countably many boxes, we are certain that this sequence either terminates, or becomes a countably infinite subset of Q_k.

Now, we have that $\Sigma_i \text{vol}(Q_i) \leq \Sigma_i \epsilon/2^i = \epsilon$. Since we can find a sub-cover of arbitrarily small volume of E, it follows that $|E|_e = 0$. □

Problem 2.1.42. Let C be the Cantor set, and let $D = \{\Sigma_{n=1}^{\infty} 3^{-n} c_n | c_n = 0, 1\}$. Show that $D + D = [0,1]$, and use this to show that $C + C = [0,2]$. Therefore, $|C + C|_e = 2$ even though $|C|_e = 0$.

Proof. First,we begin by proving that $D + D \subseteq [0,1]$.

We do this by examining the bounds of D. Clearly, the smallest value $d_0 \in D$ is $d_0 = \Sigma_{n=1}^{\infty} 3^{-n} * 0 = 0$. So the minimum value that $D + D$ can attain is $d_0 + d_0 = 0$. Next, we consider the maximum value. In a similar fashion, we have $d_1 = \Sigma_{n=1}^{\infty} 3^{-n} * 1 = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = 1/2$. Then, the maximum that $D + D$ can attain is $d_1 + d_1 = 1$. Thus, for all $d, d' \in D$, $0 \leq d + d' \leq 1$, so $d + d' \in [0,1]$ and $D + D \subseteq [0,1]$.

Next, we wish to prove that $[0,1] \subseteq D + D$.

Let $x \in [0,1]$. x has a ternary expansion $x = \Sigma_{n=1}^{\infty} c_n 3^{-n} | c_n \in \{0,1,2\}$, where, if a number has multiple ternary expansions, we choose the one that ends in infinitely many 2s; we do this so that such a ternary expansion always exists, as otherwise, 1 would not be representable. Construct d, d' as follows, where a_n denotes the ternary expansion coefficients of d and b_n the coefficients of d'.

$$a_n = \begin{cases} 0 & c_n = 0 \\ 1 & c_n = 1,2 \end{cases} \quad b_n = \begin{cases} 0 & c_n = 0,1 \\ 1 & c_n = 2 \end{cases}$$

□

By construction, $d, d' \in D$, and also by construction, $d + d' = x$. We remark that there may be multiple d, d' that satisfy x if there are multiple ternary expansions, but this does not create issues with showing that $x \in D + D$ and thus $[0,1] \subseteq D + D$.

Now, here, we notice that by exercise 2.1.24 in Heil, that any element of the Cantor set C has ternary expansion with $c_n = 0, 2$. However, here, we note this is exactly $2D = \{2 * d | d \in D\}$. Then, consider the following:

$$C + C = 2D + 2D = \{2d + 2d' | d, d' \in D\} = \{2(d + d') | d, d' \in D\} = 2(D + D) = 2 * [0,1] = [0,2]$$

.

And we have $|C + C|_e = 2$, as claimed.

2.3

Problem 2.2.32. Show that if A and B are any measureable subsets of $\mathbb{R}^d$, then

$$|A \cup B| + |A \cap B| = |A| + |B|$$

.

Proof. We proceed here by using Carathéodory's criterion:

Since A is measurable, we have that $|B| = |A \cap B| + |A \setminus B|$, where we drop the exterior measure since we know that B is measurable, $A \cap B$ is an intersection of measurable sets, thus measurable, and $A \setminus B$ is measurable because $A \setminus B = A \cap B^c$, and A, B^c are measurable sets.

Similarly, we also have that $|A| = |B \cap A| + |B \setminus A|$ due to the measurability of B.

So, we have that $|A|+|B| = 2*|A\cap B|+|A\setminus B|+|B\setminus A|$. Now, we consider the sum $|A\cap B|+|A\setminus B|+|B\setminus A|$. These are measurable sets, but moreover, they must be disjoint. Then, via countable additivity, we have that:

$$|A|+|B| = |A\cap B|+[|A\cap B|+|A\setminus B|+|B\setminus A|] = |A\cap B|+|[(A\cap B)\cup(A\setminus B)\cup(B\setminus A)]| = |A\cup B|+|A\cap B|$$

. $\square$

Problem 2.2.33. Assume that $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of measurable subsets of $\mathbb{R}^d$ such that $|E_m \cap E_n| = 0$ whenever $m \neq n$. Prove that $|\cup E_n| = \Sigma|E_n|$

Proof. Argue by induction on n the number of distinct sets in the sequence.

From 2.2.32, we have that $|E_1| + |E_2| = |E_1 \cup E_2| + |E_1 \cap E_2| = |E_1 \cup E_2|$.

Now, suppose we have that $|\cup_k E_k| = \Sigma_k|E_k|$ for $k = 1, ...m$. By 2.2.32 again, we have then that: $|E_{m+1}| + \Sigma_{k=1}^m|E_k| = |E_{m+1}| + |\cup_{k=1}^m E_k| = |\cup_{k=1}^{m+1} E_k| + |E_{m+1} \cap (\cup_{k=1}^m E_k)|$.

But we see that $E_{m+1}\cap(\cup_{k=1}^m E_k) = \cup_{k=1}^m(E_{m+1}\cap E_k)$. But, by countable subadditivity and by hypothesis, we have that:

$$|\cup_{k=1}^m (E_{m+1} \cap E_k)| \leq \Sigma_{k=1}^m(E_{m+1} \cap E_k) = 0 \implies |\cup_{k=1}^m (E_{m+1} \cap E_k)| = 0$$

.

So, we have that $|E_{m+1}| + |\cup_{k=1}^m E_k| = |\cup_{k=1}^{m+1} E_k| + |E_{m+1} \cap (\cup_{k=1}^m E_k)| = |\cup_{k=1}^{m+1} E_k|$, completing our inductive hypothesis.

$\square$

Problem 2.2.34. Let $S_r = \{x \in \mathbb{R}^d | \|x\| = r\}$ be the sphere of radius r in $\mathbb{R}^d$ centered at the origin. Prove that $|S_r| = 0$.

Proof. First, we remark that S_r is measurable because S_r is closed. This can be relatively easily seen by the fact that S_r^c is the union of the open ball centered on the origin of radius r, and the compliment of the closed ball with the same radius and center.

Now, let $A_r = \{x \in \mathbb{R}^d : \|x\| \leq r\}$, that is, the closed ball with radius r. Here, we apply Carathéodory's Criterion on S_r and find that $|A_r|_e = |A_r \cap S_r|_e + |A_r \setminus S_r|_e$. But, we notice $|A_r \cap S_r|_e = |S_r|_e$ and $|A_r \setminus S_r|_e = |B_r|_e$, where $B_r = \{x \in \mathbb{R}^d : \|x\| < r\}$ the open ball with radius r. So we have that $|A_r|_e = |S_r|_e + |B_r|_e$. In particular, we see that $|A_r|_e \leq |B_r|_e$.

However, we also have that the exterior Lebesgue measure is monotonic, and, by definition, we see that $B_r \subseteq A_r$, so $|B_r|_e \leq |A_r|_e$. Then, we have $|B_r|_e = |A_r|_e$. Further, we also know that this must be finite, as they are both contained within the box $Q = [-r, r]^d$, with $\text{vol}(Q) = (2r)^d$. Then, we have that $|A_r|_e = |S_r|_e + |B_r|_e \implies |S_r|_e = |A_r|_e - |B_r|_e = 0$ and since S_r is measurable, then $|S_r|_e = |S_r| = 0$.

$\square$

Problem 2.2.36. Let $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ be measurable sets. Assume that $P(x, y)$ is a statement that is either true or false for each pair $(x, y) \in E \times F$ Suppose that for every $x \in E$, $P(x, y)$ is true for almost every $y \in F$.

Must it then be true that for almost every $y \in F$, $P(x, y)$ is true for every $x \in E$?

Proof. Consider $E = \mathbb{R}$, $F = \mathbb{R}$, and say that $P(x, y)$ is false only when $x = y$ and true otherwise.

Then, it is true that for every $x \in E$, $P(x, y)$ is true for almost every $y \in F$. In particular, it is true for all but $x \in F$, a finite and therefore set of measure zero.

However, for no $y \in F$ is $P(x, y)$ true for every $y \in E$. In particular, for any $y \in F$, $P(y, y)$ is false. □

Problem 2.2.37. Given a set $E \subseteq \mathbb{R}^d$, prove that the following statements are equivalent:

(a) E is Lebesgue measurable.

(b) For every $\epsilon > 0$, there exists an open set U and a closed set F such that $F \subseteq E \subseteq U$ and $|U \setminus F| < \epsilon$.

(c) There exists a G_δ-set G and a F_σ-set H such that $H \subseteq E \subseteq G$ and $|G \setminus H| = 0$.

Proof. □

Problem 2.2.38. Given a set $E \subseteq \mathbb{R}^d$ with $|E|_e < \infty$, show that the following two statements are equivalent:

(a) E is Lebesgue measurable.

(b) For each $\epsilon > 0$, we can write $E = (S \cup A) \setminus B$ where S is a union of finitely many nonoverlapping boxes and $|A|_e, |B|_e < \epsilon$.

Proof. □

Problem 2.2.42. This problem will show that there exist closed sets with positive measure that have empty interior.

The Cantor set construction removes 2^{n-1} intervals from F_n, each of length 3^{-n} to obtain F_{n+1}. Modify this construction by removing 2^{n-1} intervals from F_n that each have length a_n and set $P = \cap F_n$.

(a) Show that P is closed, P contains no open intervals, $P^o = \emptyset$, $P = \partial P$, and $U = [0, 1] \setminus P$ is dense in $[0, 1]$.

(b) Show that if $a_n \to 0$ quickly enough, then $|P| > 0$. In fact, given $0 < \epsilon < 1$, exhibit a_n such that $|P| = 1 - \epsilon$.

Proof. □

Problem 2.2.43. Define the inner Lebesgue measure of a set $A \subseteq \mathbb{R}^d$ to be:

$$|A|_i = \sup\{|F| : F \text{ is closed and } F \subseteq A\}$$

Prove the following statements:

(a) If A is Lebesgue measurable, then $|A|_e = |A|_i$.

(b) If $|A|_e < \infty$ and $|A|_e = |A|_i$, then A is Lebesgue measurable.

(c) Assuming that nonmeasurable sets exist, there exists a nonmeasurable set A that satisfies $|A|_e = |A|_i = \infty$.

(d) If $E \subseteq \mathbb{R}^d$ is Lebesgue measurable and $A \subseteq E$, then:

$$|E| = |A|_i + |E \setminus A|_e$$

Proof. □

Problem 2.2.47. Given a function $f : \mathbb{R}^d \to \mathbb{C}$, define the oscillation of f at the point x to be:

$$\operatorname{osc}_f(x) = \inf_{\delta>0} \sup\{|f(y) - f(z)| : y, x \in B_\delta(x)\}$$

Prove the following statements:

(a) f is continuous at x if and only if $\operatorname{osc}_f(x) = 0$.

(b) For each $\epsilon > 0$, the set $\{x \in \mathbb{R}^d : \operatorname{osc}_f(x) \geq \epsilon\}$ is closed.

(c) $D = \{x \in \mathbb{R}^d : f \text{ is discontinuous at } x\}$ is an F_σ-set, and therefore the set of continuities of f is a G_δ-set.

Proof. $\square$