Homework #8

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2.1

Problem 5.2.18. Suppose that $f : [a, b] \to \mathbb{C}$. Show that there exists partitions Γ_k of $[a, b]$ such that Γ_{k+1} is a refinement of Γ_k for each k, and $S_{\Gamma_k} \nearrow V[f; a, b]$ as $k \to \infty$.

Solution. First, we wish to show that for any partition Γ_k and refinement Γ_{k+1}, that $S_{\Gamma_k} \leq S_{\Gamma_{k+1}}$.

Let $S_{\Gamma_k} = \{a = x_0 < ... < x_i = b\}$ and $S_{\Gamma_{k+1}} = \{a = y_0 < ... < y_j = b\}$ be a refinement, where $i < j$ and for every $0 \leq i' \leq i$, there exists a j' such that $x_{i'} = y_{j'}$.

Look at one pair of $x_{i'}, x_{i'+1}$. If, in the refinement, we have that $x_{i'} = y_{j'}$ and $x_{i'+1} = y_{j'+1}$, then we have that $|f(x_{i'+1}) - f(x_{i'})| = |f(y_{j'+1}) - f(y_{j'})|$. Else, suppose not. Then, we have that $x_{i'} = y_{j'}$ and $x_{i'+1} = y_{j'+n}$ for some n. Then, by liberal usage of the triangle inequality, we have that:

$$|f(x_{i'+1}) - f(x_{i'})| = |f(y_{j'+n}) - f(y_{j'})| = \left|f(y_{j'+n}) - f(y_{j'}) + \Sigma_{k=1}^{n-1}(f(y_{j'+k}) - f(y_{j'+k})\right| =$$

$$\left|\Sigma_{k=1}^{n}(f(y_{j'+k}) - f(y_{j'+(k-1)})\right| \leq \Sigma_{k=1}^{n}\left|f(y_{j'+k}) - f(y_{j'+(k-1)})\right|$$

Since we may do this for every $0 \leq i' \leq i$, that means that $S_{\Gamma_k} \leq S_{\Gamma_{k+1}}$.

First, assume $V[f; a, b] < \infty$. Now, since $V[f; a, b]$ is the supremum of S_Γ over every partition Γ, we may construct a sequence Γ_k of partitions such that $V[f; a, b] - S_{\Gamma_k} < 1/k$.

In particular now, define a new sequence of partitions as such. Let $\Gamma'_1 = \Gamma_1$. Then, take $\Gamma'_i = \Gamma'_{i-1} \cup \Gamma_i$, where we understand the union operation as meaning to take every point in Γ'_{i-1}, Γ_i and create a partition with all points. We notice that for each i, Γ'_i is a refinement of both Γ'_{i-1}, Γ_i. Then, we have that $\Gamma'_{i-1} \leq \Gamma'_i$ from the work we did above, and further, we know that $V[f; a, b] - 1/k \leq S_{\Gamma'_i} \leq V[f; a, b]$ by the choice of the Γ_i's. Thus, we have an increasing sequence of refinements that converges to $V[f; a, b]$.

The unbounded case is clear, instead of taking $V[f; a, b] - S_{\Gamma_k} < 1/k$, we simply take $S_{\Gamma_k} > k$ for each $k \geq 1$, and proceed in the same way.

$\square$

Problem 5.2.21. Assume that $E \subseteq \mathbb{R}$ is measurable, and suppose that $f : E \to \mathbb{R}$ is Lipschitz on the set E, that is, there exists a $K \geq 0$ such that:

$$|f(x) - f(y)| \leq K|x - y| \text{ for all } x, y \in E$$

Prove that $|f(A)|_e \leq K|A|_e$, for any $A \subseteq E$.

Solution. Let $\{Q_k\}_k$ be a collection of boxes such that $A \subseteq \cup_k Q_k$. Let's look at one specific box, Q_i. Since $A \subseteq E$, we can take $d_i = \sup(\{x - y : x, y \in E \cap Q_i\})$, where we notice $d_i \leq \text{Vol}(Q_i)$ Consider the image of $f(E \cap Q_i)$. Since f is Lipschitz, and $Q_i \cap E$ an intersection of measurable sets, the image is measurable. In particular, we notice that, for $x, y \in E \cap Q_i$, we have:

$$|f(x) - f(y)| \le K|x-y| \le Kd_i$$

Then, if we fix an x, that means $f(E \cap Q_i)$ can be contained within an interval of length Kd_i. We may repeat this process for each Q_i. We notice, since Q_k covers A, then so must $E \cap Q_k$. So, we have that

$$| \cup_k f(E \cap Q_k)|_e \le \Sigma_k(Kd_k) \le K\Sigma_k(d_k) \le K\Sigma_k \mathrm{Vol}(Q_k)$$

Since we can do this for any cover by boxes Q_k of A, $f(A) \subseteq \cup_k f(E \cap Q_k)$ for every collection of boxes, and via the properties of the infimum, we have that:

$$|f(A)|_e \le K|A|_e$$

□

Problem 5.2.22. Fix $a, b > 0$ and define:

$$f(x) = \begin{cases} |x|^a \sin(|x|^{-b}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Prove the following:

(a) $f \in \mathrm{BV}[-1,1] \iff a > b$

(b) If $a = b$ then $f \in C^\alpha[-1,1]$ with exponent $\alpha = \frac{b}{b+1}$.

(c) $C^\alpha[-1,1]$ is not contained in $\mathrm{BV}[-1,1]$ for any $0 < \alpha < 1$.

Solution. (a)

First, we notice that f is symmetric across $x = 0$, and so we restrict ourselves to looking on $[0,1]$, and we may drop the absolute values. Computing f' on $(0,1]$, we find that

$$f' = ax^{a-1}\sin(x^{-b}) + x^a \cos(x^{-b}) - bx^{-b-1} = ax^{a-1}\sin(x^{-b}) - bx^{a-b-1}\cos(x^{-b})$$

Now, we wish to check if this function is in $L^1[0,1]$. We see that, via the triangle inequality, and the fact that $|\sin(y)|, |\cos(y)| \le 1$ for all y:

$$\int_0^1 |ax^{a-1}\sin(x^{-b}) - bx^{a-b-1}\cos(x^{-b})| \le \int_0^1 |ax^{a-1}\sin(x^{-b})| + \int_0^1 |bx^{a-b-1}\cos(x^{-b})| \le$$
$$\int_0^1 |ax^{a-1}| + \int_0^1 |bx^{a-b-1}| = \int_0^1 ax^{a-1} + \int_0^1 bx^{a-b-1} =$$

We notice that if $a = b$, then the integral on the right diverges, since the integral becomes $\int_0^1 bx^{-1} = b\ln(x)\Big|_0^1$ which diverges. So, here, we take the case $a \neq b$:

$$x^a\Big|_0^1 + \frac{b}{a-b}x^{a-b}\Big|_0^1 = 1 + \frac{b}{a-b}x^{a-b}\Big|_0^1$$

Here, we notice that if $a < b$, that the remaining integrand goes to infinity at 0, but if we have that $a > b$, then:

$$1 + \frac{b}{a-b}x^{a-b} = 1 + \frac{b}{a-b} = \frac{a}{a-b} < \infty$$

.

So, we have then that if $a > b$, then $\|f'\|_1 < \infty$, and thus, by 5.2.9, $f \in \mathrm{BV}[0,1]$.

Now, we consider a partition with form $\Gamma_k = \{1 > (2/\pi)^{1/b} > ... > (2/k\pi)^{1/b} > 0\}$, where we take $k \geq 4$. Let's compute S_{Γ_k}.

$$S_{\Gamma_k} = |\sin(1) - (2/\pi)^{a/b}\sin(\pi/2)| + |(2/\pi)^{a/b}\sin(\pi/2) - (2/2\pi)^{a/b}\sin(2\pi/2)| + ... + |(2/k\pi)^{a/b}\sin(k\pi/2) - 0| \leq$$
$$\Sigma_{i=1}^{k-1}|(2/i\pi)^{a/b}\sin(i\pi/2) - (2/(i+1)\pi)^{a/b}\sin((i+1)\pi/2)|$$

where we've omitted the first and last term. We notice, that $\sin(i\pi/2)$ is 0 whenever i is even. Then, we can rewrite this as:

$$\Sigma_{i=1}^{k-1}|(2/i\pi)^{a/b}\sin(i\pi/2) - (2/(i+1)\pi)^{a/b}\sin((i+1)\pi/2)| = 2\Sigma_{i=1}^{\lfloor k/2\rfloor-1}(2/(2i+1)\pi)^{a/b}$$

Because we count each odd $(2/i\pi)$ twice, once with $i-1$, and once with $i+1$, and we drop the sin and absolute values, because sin takes on ± 1. We also ignore $2i-1=1$, because it's only counted once, due to the $\sin(1)$ term.

Here, we consider the sum $2\Sigma_{i=1}^{\lfloor k/2\rfloor-1}(2/(2i+1)\pi)^{a/b} = 2(2/\pi)^{a/b}\Sigma_{i=1}^{\lfloor k/2\rfloor-1}(1/(2i+1))^{a/b}$. We recognize this as some constant times the sum of odd reciprocals. In particular, we know that as $k \to \infty$, this sum diverges so long as $a/b \leq 1$. Thus, since we have found a partition that diverges, $V[f;a,b]$ must diverge as well, since we can always union this sequence of partitions into any other partition. Therefore, for $f \in bv[a,b]$, $a/b > 1$, and thus, $a > b$.

Therefore, we have a biconditional.

(b)

Suppose $a = b$, then $f = x^b sin(x^{-b})$ on $(0,1]$. Again, we restrict ourselves to looking on $(0,1]$ due to symmetry, as if it is true here, then it is true on all of $[0,1]$

First, suppose $0 < x < y \leq 1$, define $h = y - x$, and then consider the case where $h \leq x^b + 1$.

We have, via the Mean Value Theorem, that because f is differentiable on $(0,1]$, that $|f(x) - f(y)| = |f'(t)|h$ for some $x < t < b$. From part (a), we computed the derivative as:

$$f'(x) = bx^{b-1}\sin(x^{-b}) - bx^{-1}\cos(x^{-b}) = bx^{-1}(x^b\sin(x^{-b}) - \cos(x^{-b}))$$

.

We notice, that because $\sin, \cos$ are bounded by ± 1, and since $t \in [0,1]$, we have that $t^b \in (0,1)$, we may take the estimate:

$$|f'(t)| = |bt^{-1}||t^b\sin(t^{-b}) - \cos(t^{-b})| \leq |b/t|(|t^b\sin(t^{-b})| + |\cos(t^{-b})|) \leq 2b/t$$

.

Then, we have that $|f(x) - f(y)| = |f'(t)|h \leq 2bh/t$. Now, we have that $x < t < y$, so therefore, since $b+1 > 0$, we have that $x^{b+1} \geq t^{b+1}$, and by our case, this implies that $t^{b+1} > h \implies t > h^{1/(b+1)}$. Since this is a lower bound for t, this is an upper bound for the fraction $2b/t$, so we have that:

$$|f(x) - f(y)| \leq 2bh/t \leq 2bh/h^{1/(b+1)} = 2bh^{1-1/(b+1)} = 2bh^{b/b+1}$$

Since $2b$ is a constant, we have $b/b+1$ as a Hölder exponent in this case.

Now, suppose $h > x^{b+1}$.

If we look at $|f(y) - f(x)|$, we have that:

$$|f(y) - f(x)| \leq |f(y)| + |f(x)| \leq |y^b\sin(y^{-b})| + |x^b\sin(x^{-b})| \leq y^b + x^b$$

Now, from our case, we already have that because $x^{b+1} < h$, since $b > 0 \implies b/b+1 > 0$, we may take both sides to the $b/b+1$-th power, and obtain that $(x^{b+1})^{b/b+1} < h^{b/b+1} \implies x^b < h^\alpha$.

On the other hand, we look at y^b/h^α. In particular, since $0 < b/b+1 < 1, 0 < h < 1$, we have that $y^b/h^\alpha \le y^b/h^b = (y/y-x)^b$. We notice here that because $h > x^{b+1}$, that instead, if we view this as fixing a y, h can be no less than some constant multiple of y, Cy, as otherwise, x cannot get too close to y without making $h \le x^{b+1}$. Then, we have that:

$$|f(y) - f(x)| \le y^b + x^b \le h^\alpha + C^b h^\alpha = (1 + C^b)h^\alpha$$

Now, to finish, we just take our Hölder constant to be the max of $(1 + C^b), 2b$ and we are done.

(c)

Fix an $0 < \alpha < 1$. Then, since $\alpha = b/b + 1 = 1 - 1/b + 1$, we have that $1/(1 + b) = 1 - \alpha \implies b + 1 = 1/(1 - \alpha) \implies b = \alpha/(1 - \alpha)$. By our choice of α, $b > 0$. Then, from part (a), (b), we may find a function f_b defined as above with this choice of b such that it belongs to $C^\alpha[-1, 1]$ but does not belong to $\mathrm{BV}[-1, 1]$. □

Problem 5.2.23. (a) Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of complex-valued functions $f_n : [a, b] \to \mathbb{C}$ and that $f_n \to f$ pointwise on $[a, b]$. Prove that:

$$V[f; a, b] \le \liminf_{n\to\infty} V[f_n; a, b]$$

(b) Exhibit functions f_n, f such that $f_n \in \mathrm{BV}[a, b]$ for each $n \in \mathbb{N}$ and $f_n \to f$ pointwise, but $f \notin \mathrm{BV}[a, b]$.

Solution. (a)

Let Γ be a partition on $[a, b]$. Then, by 4.2.18, Fatou's lemma for series, we can say that

$$S_\Gamma = \Sigma_{j=1}^n |f(x_j) - f(x_{j-1})| = \Sigma_{j=1}^n \liminf_{n\to\infty} |f_n(x_j) - f_n(x_{j-1})| \le$$

$$\liminf_{n\to\infty} \Sigma_{j=1}^n |f_n(x_j) - f_n(x_{j-1})| = \liminf_{n\to\infty} S_\Gamma[f_n; a, b]$$

Since this is true for an arbitrary partition, this is true for every partition. Then, since V is the sup over all Γ of S_Γ, this implies that:

$$V[f; a, b] \le \liminf_{n\to\infty} V[f_n; a, b]$$

(b)

Consider the sequence of functions

$$f_n = \begin{cases} 0, & \text{if } x \in [a, a + 1/n) \\ 1/(x - a), & \text{if } x \in [a + 1/n, b] \end{cases}$$

It is clear that this function has bounded variation, because for any f_n, it is monotone increasing on $[a, a+1/n]$ and monotone decreasing on $[a+1/n, b]$, so it has total variation exactly equal to $n+(n-1/(b-a)) = 2n - 1/(b - a)$, thus $f_n \in \mathrm{BV}[a, b]$ for all $n \ge 1$. However, this converges to $1/(x - a)$, which is not bounded, and thus is not in $\mathrm{BV}[a, b]$.

□

Problem 5.2.26. Prove the following:

(a) $\|f\| = V[f; a, b]$ defines a seminorm on $\mathrm{BV}[a, b]$ and

$$\|f\|_{\mathrm{BV}} = V[f; a, b] + \|f\|_u : f \in \mathrm{BV}[a, b]$$

is a norm on $\mathrm{BV}[a, b]$.

(b) BV$[a,b]$ is a Banach space with respect to $\|\cdot\|_{\text{BV}}$.

(c) $\|f\|_{\text{BV}'} = V[f;a,b] + |f(a)|$ defines an equivalent norm for BV$[a,b]$. That is, it is a norm, and there exists $C_1, C_2 > 0$ such that:

$$C_1\|f\|_{\text{BV}} \leq \|f\|_{\text{BV}'} \leq C_2\|f\|_{\text{BV}} : f \in \text{BV}[a,b]$$

Solution. (a)

Clearly, we have that $V[f;a,b] \geq 0$ for any $f \in \text{BV}[a,b]$, because it is the supremum of non-negative numbers. Then, we need only check for the triangle inequality, and factoring scalars.

Let $f, g \in \text{BV}[a,b]$, and fix a partition $\Gamma = \{a = x_0 < ... < x_n = b\}$. We notice, by the triangle inequality on the complex numbers, we have that, for each (x_i, x_{i+1}):

$$|f+g(x_{i+1}) - f+g(x_i)| = |f(x_{i+1}) + g(x_{i+1}) - f(x_i) - g(x_0)| \leq |f(x_{i+1}) - f(x_i)| + |g(x_{i+1}) - g(x_i)|$$

Since this is true for every interval in the partition, this implies then that $S_\Gamma^{f+g} \leq S_\Gamma^f + S_\Gamma^g$, where we use S_Γ^f to denote the sum for the function f. Then, since the variation is simply the supremum over all partitions, and this holds for every partition, we have that:

$$\|f+g\| = V[f+g;a,b] \leq V[f;a,b] + V[g;a,b] = \|f\| + \|g\|$$

.

Now, let $k \in \mathbb{R}$. Consider now $\|kf\|$. Again, looking at any partition Γ, we see that:

$$|kf(x_{i+1}) - kf(x_i)| = |k||f(x_{i+1}) - f(x_i)|$$

Since this is true for each interval in our partition, it implies that $S_\Gamma^{kf} = |k|S_\Gamma^f$. Again, via the properties of the supremum, this implies then that $\|kf\| = |k|\|f\|$.

Now, we look at $\|f\|_{\text{BV}} = V[f;a,b] + \|f\|_u : f \in \text{BV}[a,b]$. Because of the fact that we have shown that $V[f;a,b]$ is a seminorm on BV$[a,b]$ and that we already know that $\|f\|_u$ is a norm, we know that this is already a seminorm. Then, it suffices to show that $\|f\|_{\text{BV}} = 0 \implies f = 0$. Since both portions are non-negative, this implies, in particular, $\|f\|_u = 0$. But, because this is a norm, this implies that $f = 0$, and we are done. Thus, this is a norm.

(b)

Suppose we have a Cauchy sequence of functions $f_n \in \text{BV}[a,b]$, that is, such that $\|f_m - f_n\|_{\text{BV}} \to 0$ as $m, n \to \infty$. By the definition of $\|\cdot\|_u$, for this to go to 0, we must have that $\|f_m - f_n\|_u \to 0$ as well, that is, it must be Cauchy with respect to the uniform norm. Then, fix any $x \in [a,b]$, and look at $|f_m(x) - f_n(x)|$. In particular, we have that, for an $\epsilon > 0$ given, there must be N such that for all $m, n > N$, $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_u < \epsilon$, by the properties of the supremum. Then, this means that $f_n(x)$ is a sequence of Cauchy real numbers, and thus convergent. Then, define $f(x) = \lim_{n\to\infty} f_n(x)$, that is, the point-wise convergence of the sequence.

Now, we claim that if f_n is Cauchy, then it is convergent to f, and that $f \in \text{BV}[a,b]$. Firstly, we see that f must be bounded, because from the fact that $f_n \to f$ in the uniform norm, let $\epsilon > 0$, we can see that $\|f - f_n\|_u < \epsilon$ for at least some n. Then, by the reverse triangle inequality, we have that $|\|f\|_u - \|f_n\|_u| < \epsilon \implies -\epsilon < \|f\|_u - \|f_n\|_u < \epsilon \implies -\epsilon < \|f_n\|_u - \epsilon < \|f\|_u < \|f_n\|_u + \epsilon \implies \|f\|_u < \|f_n\|_u + \epsilon < \infty$.

Now, we wish that f to be of bounded variation. Because the f_n are Cauchy in $\|\cdot\|_{\text{BV}}$, we have that they must be Cauchy as well in $\|\cdot\|$, that is, in their variation, since both the uniform norm and the seminorm must go to 0. But, this then implies that the sequence of $\|f_n\|$ under the seminorm is bounded. Then, if that's bounded, we have from problem 5.2.23 part (a), that:

$$V[f;a,b] \leq \liminf_{n\to\infty} V[f_n;a,b] < \infty$$

Thus, $f \in \mathrm{BV}[a,b]$. Then, it is clear from the triangle inequality and from the seminorm properties that $f_n \to f$ in the seminorm as well, and thus $f_n \to f$ in the full norm.

(c)

First, we look at the case $f(a) \geq 0$. Then, using the Jordan decomposition on $f = g - h$ for g, h monotone increasing, and the seminorm properties to see that $V[f;a,b] \leq V[g;a,b] + V[h;a,b]$, we conclude that $f(a) \leq \|f\|_u \leq f(a) + V[f;a,b]$, since to maximize $|f|$, we would need $V[h;a,b] = 0$. We can actually see that this argument works for $f(a) < 0$, where instead of taking the positive distance, we take $V[g;a,b] = 0$ to maximize $|f|$. So, we actually have that $|f(a)| \leq \|f\|_u \leq |f(a)| + V[f;a,b]$.

Then, we take $C_1 = 1, C_2 = 2$.

From $|f(a)| \leq \|f\|_u$, we can add $V[f;a,b]$ to both sides to obtain:

$$\|f\|_{\mathrm{BV}'} = V[f;a,b] + |f(a)| \leq V[f;a,b] + \|f\|_u = \|f\|_{\mathrm{BV}}$$

so we have that $C_1\|f\|_{\mathrm{BV}'} = \|f\|_{\mathrm{BV}'} \leq \|f\|_{\mathrm{BV}}$

Further, we have that from the other side, we obtain:

$$\|f\|_u \leq |f(a)| + V[f;a,b] \implies V[f;a,b] + \|f\|_u \leq |f(a)| + 2V[f;a,b]$$

so we can see that:

$$C_2\|f\|_{\mathrm{BV}'} = 2|f(a)| + 2V[f;a,b] \geq |f(a)| + 2V[f;a,b] \geq V[f;a,b] + \|f\|_u = \|f\|_{\mathrm{BV}}$$

.

Thus, these norms are equivalent. If you really want the other inclusion, we can reverse the inclusions by dividing via the constants. $\square$

2.2

Problem 5.3.5. Assume that $E \subseteq \mathbb{R}^d$ satisfies that $0 < |E|_e < \infty$, and let $\mathcal{B}$ be a Vitali covering of E. Given an $\epsilon > 0$, prove that there exist a countable collection of balls $B_k \in \mathcal{B}$ such that

$$|E \setminus \cup_k B_k|_e = 0 \text{ and } \Sigma_k |B_k| < |E|_e + \epsilon$$

Solution. We first proceed in the same way as the proof of 5.3.3.

Let $U \supseteq E$ be an open set such that $|U| < |E|_e + \epsilon$. Call $\mathcal{B}'$ the restriction of $\mathcal{B}$ such that for all $B \in \mathcal{B}'$, $B \subseteq U$. Since these were closed sets, and we live in an open set surrounding U, we must still have a Vitali cover, as we just shrink ourselves to the case where the ball has radius less than the open ball around each point.

Fix any $B_1 \in \mathcal{B}$ and proceed inductively, picking disjoint balls as follows. Suppose we have picked n balls. Then, if $|E \setminus B_1 \cup ... \cup B_n|_e = 0$ we are done. Otherwise, pick a point in $E \setminus B_1 \cup ... \cup B_n$. Since this has positive measure, we can find an open set around it $U' \setminus B_1 \cup ... \cup B_n$, with set difference of measure less than ϵ. Then, we pick B_{n+1} such that it contains x, disjoint from the other $B_1, ...B_n$, and, defining

$$s_n = \sup\{\mathrm{radius}(B) : B \in \mathcal{B}, B \cap B_i, 1 \leq i < n\}$$

such that $\mathrm{radius}(B_{n+1})$. We continue this process, stopping only if $E \setminus B_1 \cup ... \cup B_N|_e = 0$, otherwise obtaining a countable collection of disjoint balls. From the argument of 5.3.3, we have that:

$$\Sigma_{k=1}^{\infty} |B_k| = |\cup_k B_k| \leq |U| < |E|_e + \epsilon$$

Now, take a point $x \in E \setminus \cup_k^\infty B_k$. Fix a m. By necessity, x must also be in $x \in E \setminus \cup_{k=1}^m B_k$. Then, by the argument in 5.3.3, for some $i > m$, it belongs to some B_i^*, where this is a closed ball with the same center as B_i but radius$(B_i^*) = 5$radius(B_i). Then, we have that:

$$|E \setminus \cup_k^\infty B_k|_e \leq |E \setminus \cup_k^m B_k|_e \leq \Sigma_{k=m+1}^\infty |B_k^*| = 5^d \Sigma_{k=m+1}^\infty |B_k|$$

However, since $\Sigma_{k=1}^\infty |B_k| < \infty$, we must have that $\Sigma_{k=m+1}^\infty |B_k|$ can be picked arbitrarily small, i.e. we can find m_n such that $\Sigma_{k=m_k+1}^\infty |B_k| < 1/k$. Since the choice of m was arbitrary, we can pick this sequence of m_n's which implies then that:

$$|E \setminus \cup_k^\infty B_k|_e \leq \Sigma_{k=m_k+1}^\infty |B_k| < 1/k$$

for every k. Then, we must have that $|E \setminus \cup_k B_k|_e = 0$.

□