Homework #1

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Math 285: Homework #1

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Question 1. Define $f : \mathbb{R} \to \mathbb{R}$ via:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Using the definition of the derivative, prove that $f \in C^\infty(\mathbb{R})$, and that the derivatives $f^{(k)}(0)$ vanish for all $k \in \mathbb{N}$.

Solution. Well, first, we restrict ourselves to $\mathbb{R} \setminus \{0\}$. On this domain, $f(x) = e^{-1/x^2}$, and, familiarly, we can see that

$$f'(x) = e^{-x^{-2}} \frac{d}{dx}(-x^{-2}) = e^{-x^{-2}} 2x^{-3} = 2x^{-3} f(x)$$

then, we identify:

$$f^{(2)}(x) = f'(x) 2x^{-3} - 6x^{-4} f(x) = 2x^{-3} f(x) 2x^{-3} - 6x^{-4} f(x) = f(x)(4x^{-6} - 6x^{-4})$$

More generally, we can see that:

$$f^{(n)}(x) = f(x) p_n\left(\frac{1}{x}\right)$$

for $p_n(1/x)$ a polynomial in $1/x$ because the derivative of a polynomial in $1/x$ is simply a polynomial, and the derivative of f is $f'(x) = 2x^{-3} f(x)$. Explicitly:

$$f^{(n+1)}(x) = \frac{d}{dx}\left[f(x) p_n\left(\frac{1}{x}\right)\right] = f'(x) p_n\left(\frac{1}{x}\right) + f(x) p_n'\left(\frac{1}{x}\right)\frac{-1}{x^2} = f(x)\left[2x^{-3} p_n\left(\frac{1}{x}\right) - p_n'\left(\frac{1}{x}\right) x^{-2}\right] = f(x) p_{n+1}\left(\frac{1}{x}\right)$$

Further, we notice that by our identification:

$$p_{n+1}\left(\frac{1}{x}\right) = 2x^{-3} p_n\left(\frac{1}{x}\right) - p_n'\left(\frac{1}{x}\right) x^{-2}$$

and that because $p_0(1/x) = 1$, that $\deg(p_n(y)) = \deg(p_{n-1}(y)) + 3 = 3n$, where we can say this because the degree of the first term in the recurrence is $\deg(p_n) + 3$ and the degree of the second term is $\deg(p_n') + 2 = \deg(p_n) - 1 + 2 = \deg(p_n) + 1$, so the leading term in the first term cannot vanish, and we have that the overall polynomial has degree $\deg(p_n) + 3$.

Now, let's examine the limit as $x \to 0$ of $f^{(n)}(x)$. Well:

$$\lim_{x \to 0} f^{(n)}(x) = \lim_{x \to 0} f(x) p_n\left(\frac{1}{x}\right)$$

Here, we rewrite this as examining two easier limits, by letting $y = 1/x$:

$$\begin{cases} \lim_{y\to\infty} e^{-y^2} p_n(y) \\ \lim_{y\to-\infty} e^{-y^2} p_n(y) \end{cases}$$

where of course, $p_n(y)$ is a polynomial in y, with non-negative powers. Well:

$$\lim_{y\to\infty} e^{-y^2} p_n(y) = \lim_{y\to\infty} \frac{p_n(y)}{e^{y^2}}$$

Making the assumption that the leading coefficient of $p_n(y)$ is positive (if not, we pull out a negative sign and examine $-p_n(y)$), we see that the conditions for L'Hôpital's is met, as both numerator and denominator diverge to positive infinity.

Thus, we may take a derivative to find:

$$\lim_{y\to\infty} \frac{p_n(y)}{e^{y^2}} = \lim_{y\to\infty} \frac{p_n'(y)}{2ye^{y^2}} = \lim_{y\to\infty} \frac{p_n'(y)/2y}{e^{y^2}}$$

Well, if $p_n'(y)/2y$ has no positive powers, then we're done, as in that case, $\lim_{y\to\infty} p_n'(y)/2y = c$, at most a constant. Otherwise, if it does have a positive power, its limit remains ∞, so we may iteratively keep taking derivatives since p_n has finite degree until this is true. Thus, because $\lim_{y\to\ infty} e^{y^2} = \infty$, we have that:

$$\lim_{y\to\infty} \frac{p_n(y)}{e^{y^2}} = 0$$

The same argument works for $y \to -\infty$, with potentially examining $-p_n(y)$ if the leading term has odd degree. Thus, in terms of our original limit, we have that:

$$\lim_{x\to 0} f^{(n)}(x) = \lim_{x\to 0} f(x) p_n\left(\frac{1}{x}\right) = 0$$

So, we have that to remove the discontinuity, we would enforce that $f^{(n)}(0) = 0$ for all n. Thus, derivatives exist on all of $\mathbb{R}$ continuously and smoothly for all n, as we may patch the discontinuity at $x = 0$. Thus, $f \in C^\infty(\mathbb{R})$.

□

Question 2. Let $\overline{0} = (0, 0)$ be the origin in $\mathbb{R}^2$, and let $B(\overline{0}, 1)$ be the open unit disk centered at the origin. To find a diffeomorphism between $B(\overline{0}, 1)$ and $\mathbb{R}^2$, we identify $\mathbb{R}^2$ as $\{(x, y, 0) : x, y, \in \mathbb{R}\} \subseteq \mathbb{R}^3$, and introduce the lower open hemisphere:

$$S = \{(x, y, z) : x^2 + y^2 + (z-1)^2 = 1; z < 1\} \subseteq \mathbb{R}^3$$

as an intermediate space. Notice that the map:

$$f : B(\overline{0}, 1) \to S \text{ via } (a, b) \mapsto (a, b, 1 - \sqrt{1 - a^2 - b^2})$$

is a bijection.

(a) The stereographic projection $g : S \to \mathbb{R}^2$ through $(0, 0, 1)$ is the map that sends a point $(a, b, c) \in S$ to the point in the xy plane given by the line through $(0, 0, 1)$ and (a, b, c). Show that it is given by:

$$(a, b, c) \mapsto (u, v) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right); c = 1 - \sqrt{1 - a^2 - b^2}$$

and its inverse is given by:

$$(u,v) \mapsto \left(\frac{u}{w}, \frac{v}{w}, 1 - \frac{1}{w}\right); w = \sqrt{1+u^2+v^2}$$

(b)

Define

$$h = g \circ f : B(\overline{0}, 1) \to \mathbb{R}^2$$

Show that:

$$h(a,b) = \left(\frac{a}{\sqrt{1-a^2-b^2}}, \frac{b}{\sqrt{1-a^2-b^2}}\right)$$

Further, find a formula for $h^{-1}(u,v) = (f^{-1} \circ g^{-1})(u,v)$ and conclude that h is a diffeomorphism.

(c)

Generalize part (b) to $\mathbb{R}^n$.

Solution. □

Question 3. Let A be an algebra over a field k. Let D_1, D_2 be derivations of A. Show that $D_1 \circ D_2$ need not be a derivation, but $D_1 \circ D_2 - D_2 \circ D_1$ is.

Solution. Counterexample: Let $A = k[x]$, the polynomial ring over the field k. Let $D_1 = \frac{d}{dx}$, the formal derivative with respect to x, and let $D_2 = x\frac{d}{dx}$.

First, we check that these are actual derivations. We can see this by the action on x^n for arbitrary n, m that these are k linear maps:

$$D_1 : \begin{cases} \frac{d}{dx}(cx^n) = cnx^{n-1} = c\frac{d}{dx}(x^n) \\ \frac{d}{dx}(x^n + x^m) = nx^{n-1} + mx^{m-1} = \frac{d}{dx}x^n + \frac{d}{dx}x^m \end{cases}$$

$$D_2 : \begin{cases} x\frac{d}{dx}(cx^n) = xcnx^{n-1} = cxnx^{n-1} = cx\frac{d}{dx}(x^n) \\ x\frac{d}{dx}(x^n + x^m) = xnx^{n-1} + xmx^{m-1} = x\frac{d}{dx}x^n + x\frac{d}{dx}x^m \end{cases}$$

Further, we can see that this follows the Leibniz rule:

$$D_1(fg) = \frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f\frac{d}{dx}(g) = D_1(f)g + fD_1(g)$$

$$D_2(fg) = x\frac{d}{dx}(fg) = x\left[\frac{d}{dx}(f)g + f\frac{d}{dx}(g)\right] = x\frac{d}{dx}(f)g + fx\frac{d}{dx}(g) = D_2(f)g + fD_2(g)$$

However, consider $D_2 \circ D_1$ acting on $x^3 = x^2 * x$:

$$D_2 \circ D_1(x^3) = D_2(\frac{d}{dx}x^3) = D_2(3x^2) = 6x$$

On the other hand:

$$D_2 \circ D_1(x^2) * x + x^2 D_2 \circ D_1(x) = D_2(2x) * x + x^2 D_2(1) = 2x^2 + 0 = 2x^2$$

where we've used the fact that $\frac{d}{dx}(1) = 0$ to conclude $D_2(1) = 0$.

Thus, even though D_1, D_2 are derivations, $D_2 \circ D_1$ need not be a derivation.

However, now consider $D_1 \circ D_2 - D_2 \circ D_1$ for generic derivations D_1, D_2.

Well:

$$D_1 \circ D_2 - D_2 \circ D_1(fg) = D_1 \circ D_2(fg) - D_2 \circ D_1(fg) = D_1(D_2(f)g + fD_2(g)) - D_2(D_1(f)g + fD_1(g)) =$$

$$D_1 \circ D_2(f)g + D_2(f)D_1(g) + D_1(f)D_2(g) + fD_1 \circ D_2(g) - [D_2 \circ D_1(f)g + D_1(f)D_2(g) + D_2(f)D_1(g) + fD_2 \circ D_1(g)] =$$

$$D_1 \circ D_2(f)g + fD_1 \circ D_2(g) - D_2 \circ D_1(f)g - fD_2 \circ D_1(g) = [D_1 \circ D_2(f) - D_2 \circ D_1(f)]g + f[D_1 \circ D_2(g) - D_2 \circ D_1(g)] =$$

$$D_1 \circ D_2 - D_2 \circ D_1(f)g + fD_1 \circ D_2 - D_2 \circ D_1(g)$$

□

Question 4. Let U be a neighborhood of a point $p \in \mathbb{R}^n$, and let X, Y be smooth vector fields on U. Define a function $Z_p : C_p^\infty \to \mathbb{R}$ via:

$$Z_p(f) = X_p(Yf)$$

(a)

Show that Z_p does not satisfy the Leibniz rule, and is therefore not a derivation at p.

(b)

Show that $X_pY - Y_pX$ satsifies the Leibniz rule on C_p^∞. Hence, $[X, Y]_p := X_pY - Y_pX$ is a tangent vector at p, and $[X, Y]$ is a vector field on U. Call this the Lie bracket of X, Y.

(c)

Let $X = \sum_i a^i \frac{\partial}{\partial x^i}, Y = \sum_j b^j \frac{\partial}{\partial x^j}$. /Find the coefficient c_k in $[X, Y] = \sum_k c^k \frac{\partial}{\partial x^k}$.

(d)

Show that if the vector fields X, Y are smooth on U, then their Lie brakcet is also a smooth vector field on U.

Solution. □

Question 5. Let V be a vector space of dimension n, with basis $e_1, ..., e_n$. Let $\alpha^1, ..., \alpha^n$ be the dual basis for V^*. Show that a basis for the space of k-linear functions of V, $L_k(V)$ is $\{\alpha^{i_1} \otimes ... \otimes \alpha^{i_k}$ for any multi-index $(i_1, ..., i_k)$. In particular, conclude that $\dim(L_k(V)) = n^k$.

Solution. □

Question 6. Let V be a vector space. For $a, b \in \mathbb{R}, f \in A_k(V), g \in A_l(V)$, show that $af \wedge bg = (ab)f \wedge g$

Solution. □

Question 7. Suppose we have two sets of covectors on a vector space V: $\{\beta^1, ..., \beta^k\}, \{\gamma^1, ..., \gamma^k\}$. Further, suppose we have that they are related by:

$$\beta^i = \sum_{j=1}^{k} a_j^i \gamma^j$$

for each $1 \leq i \leq k$, such that the a_j^i form the entries of a $k \times k$ matrix A. Show that:

$$\beta^1 \wedge ... \wedge \beta^k = \det(A)\gamma^1 \wedge ... \wedge \gamma^k$$

Solution. □