Some but not all of the questions will be graded. Usually model answers will be provided for all of them.

1. Abstract varieties, Curves

Question 1. Let X be an irreducible abstract variety. We then have an atlas

$$X = \cup U_i,\ \varphi_i : Y_i \to U_i,\ \varphi_j^{-1}\varphi_i : \varphi_i^{-1}(U_j \cap U_i) \to \varphi_j^{-1}(U_j \cap U_i)$$

with Y_i affine varieties , φ_i homeomorphisms and $\varphi_j^{-1}\varphi_i$ regular isomorphisms of quasiaffine varieties.

(a) Show that $\dim Y_i = \dim Y_j$ for all pairs i, j and therefore, we can define $\dim X = \dim Y_i$.
(b) Let $\alpha : X \to X'$ be a morphism of pre-varieties so that $\alpha(X)$ is dense in X'. Show that $\dim X' \leq \dim X$.

Question 2. Let $C_1 = \mathbb{P}^1, C_2 = V(X_0X_1^2 - X_2^3) \subseteq \mathbb{P}^2, C_3 = V(X_0X_1^2 - X_0X_2^2 + X_2^3) \subseteq \mathbb{P}^2$.

(a) Find a birational map $\alpha_1 : C_1 \dashrightarrow C_2$ and its birational inverse (that is a regular map defined only in one open subset of C_1 with an inverse defined in an open set of C_2)
(b) Find a birational map $\alpha_2 : C_1 \dashrightarrow C_3$ and its birational inverse
(c) Show that C_1 is not isomorphic to either C_2 or C_3 Hint: C_2, C_3 are singular.
(d) Show that C_2 is not isomorphic to C_3. Hint: the tangent cones at the singular point are different.
(e) Explain why (c) and (d) do not contradict the statement that we proved in class that projective non-singular curves are birational to each other if and only if they are isomorphic.

Question 3. Let $C_1 = \mathbb{P}^1, C_2 = V(X_0X_2^2 - X_1(X_1 - X_0)(X_1 - aX_0)) \subseteq \mathbb{P}^2$ for some a in the base field $a \neq 0, a \neq 1$.

(a) Show that C_2 is non-singular.
(b) Show that there is no birational map $\alpha_1 : C_1 \dashrightarrow C_2$. Hint: if such rational map were to exist, it would be an isomorphism that would give rise to an isomorphism $C_2 - point \to \mathbb{A}^1$ it would be defined in some opens set of $\mathbb{A}^1 \subseteq \mathbb{P}^1$.