Midterm #1

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Math 237: Midterm #1

March 10, 2024

Question 1. Let (X, ρ) be a compact metric space, and $f : X \to X$ a function such that:

$$\rho(f(x), f(y)) < \rho(x, y)$$

for all $x \neq y$.

Define $g : X \to \mathbb{R}$ via $g : x \mapsto \rho(x, f(x))$.

1.1)

Prove that g is Lipschitz, and that g has a minimum value, achieved at a point $x_0 \in X$. Conclude that there exists $x \in X$ such that $g(x) = 0$.

1.2)

Show that f has a unique fixed point x_0.

1.3)

Show that the assumption that X is compact may not be omitted.

Solution. 1.1)

Fix some $x \in X$, and let $y \in X$ be arbitrary. By the triangle inequality, we see that:

$$\begin{cases} \rho(x, f(x)) \leq & \rho(x, y) + \rho(y, f(x)) \\ \rho(y, f(x)) \leq & \rho(y, f(y)) + \rho(f(x), f(y)) \end{cases}$$

Combining these two equations with the property of f by hypothesis, we see that:

$$\rho(x, f(x)) - \rho(y, f(y)) \leq \rho(x, y) + \rho(f(x), f(y)) < 2\rho(x, y)$$

However, we notice that we may run the same computation in the triangle inequality, switching the labels of x, y, as $\rho(x, y) = \rho(y, x)$. Thus, we can conclude then that

$$|\rho(x, f(x)) - \rho(y, f(y))| < 2\rho(x, y)$$

and therefore, since the left side is exactly $d(g(x), g(y))$ with the metric of the real line, we may conclude that g is Lipschitz with Lipschitz constant at most 2.

Now, since g is Lipschitz continuous, it is continuous. Hence, since X is compact, g achieves its extremas. Hence, we may find $x_0 \in X$ such that g achieves its minimum value.

Suppose that $g(x_0) > 0$. Then, of course, we would have that $g(x_0) = \rho(x_0, f(x_0)) > 0$ and hence, $x_0 \neq f(x_0)$. Then, we can consider $g(f(x_0))$. We have that:

$$g(f(x_0)) = \rho(f(x_0), f(f(x_0))) < \rho(x_0, f(x_0)) = g(x_0)$$

But, this is a contradiction, as we assumed that g attained a minimum at x_0. Hence, $g(x_0) = 0$.

1.2)

From 1.1, we've shown that there exists $x_0 \in X$ such that $g(x_0) = 0$. Evidently then:

$$g(x_0) = 0 \implies \rho(x_0, f(x_0)) = 0 \implies f(x_0) = x_0$$

Furthermore, this point must be unique, as suppose $f(x_1) = x_1$ as well. Assuming that $x_0 \neq x_1$, we have that:

$$\rho(x_0, x_1) = \rho(f(x_0), f(x_1)) < \rho(x_0, x_1)$$

which is absurd. Hence, $x_0 = x_1$.

1.3)

Here are some examples to show that we need X to be compact. Consider $X = \mathbb{Z}$, equipped with the standard metric $\rho(x, y) = |x - y|$. Of course, this is not compact, as the sequence $\{n\}_{n=1}^{\infty}$ cannot admit any convergent subsequence. If we take $f(x) = \text{round}(x/2)$, where the round function rounds to the integer closer to 0, then of course, we have that $\rho(f(x), f(y)) < \rho(x, y)$ for $x \neq y$, as it contracts all distances by at least 1/2. On the other hand, it has multiple fixed points, $-1, 0, 1$.

Another example is to take the open interval $(0, 1)$, equipped with the standard metric $\rho(x, y)$, and consider the function $f(x) = x/2$. Evidently, in the same fashion, we still have that $\rho(f(x), f(y)) = |x/2 - y/2| = 1/2|x - y| = 1/2\rho(x, y) < \rho(x, y)$. However, g does not attain a minimum and f does not have a fixed point.

We can see g does not have a minimum as for any $\epsilon > 0$, we may choose $N \geq 1$ such that $1/N < \epsilon$. Then, $g(1/N) = \rho(1/N, f(1/N)) = |1/N - 1/2N| = 1/2N < 1/N < \epsilon$. Hence, $g(x)$ can be arbitrarily small. However, we can see that for $x = 1/2x$, this is satisfied only at $x = 0$, outside of $(0, 1)$. Hence, there is no x such that $g(x) = 0$ on $(0, 1)$, and no fixed point of f on $(0, 1)$.

$\square$

Question 2. Let X, Y be Banach spaces. Let $T \in L(X, Y)$. Show that T is surjective if and only if $\text{range}(T)$ is not meager in Y.

Solution. One direction is trivial. Suppose T is surjective. Then, $Y = \text{range}(T)$. But, by the Baire Category Theorem (2.21, Heil), Y is nonmeager in Y, and we are done.

Now, suppose range(T) is not meager. Consider open balls in X centered on the origin, $B_n^X(0) = \{x \in X : \|x\| < n\}$, where we use the superscript to remind ourselves this is in X. Clearly, $X = \cup_{n=1}^{\infty} B_n^X(0)$. Therefore, we have that the range of T can be expressed as:

$$\text{range}(T) = \cup_{n=1}^{\infty} T(B_n^X(0))$$

Since T is non-meager, there exists an m such that the closure $\overline{T(B_m^X(0)\text{ contains an}}$ open ball, as its complement is not dense. We can consider the operator mT, and the closure $\overline{mT(B_1^X(0))}$ contains an open ball in Y, as $T(B_m^X(0) = mT(B_1^X(0))$ by linearity. Then, by Lemma 2.26 in Heil, we have that $mT(B_1^X(0))$ contains an open ball $B_r^Y(0)$ for some $r > 0$. Again, by linearity then, we have that $T(B_m^X(0)$ contains an open ball $B_{r/m}^Y(0)$.

So now, let $y \in Y$. In particular, consider $\frac{y}{\|y\|} * \frac{r}{2m}$. Evidently, the norm of this vector is $r/2m$, and hence is contained within $B_{r/m}^Y(0)$. Thus, there exists an $x \in X$ such that $T(x) = \frac{y}{\|y\|} * \frac{r}{2m}$. By linearity then, we have that:

$$T\left(\frac{2mx\|y\|}{r}\right) = \frac{2m\|y\|}{r} T(x) = \frac{2m\|y\|}{r} \frac{y}{\|y\|} \frac{r}{2m} = y$$

Hence, $Y \subseteq \text{range}(T)$, and therefore, $Y = \text{range}(T)$. Thus, T is surjective.

$\square$

Question 3. Let $C_b(\mathbb{R})$ be the space of bounded, continuous, real-valued functions. Let $C_b^1(\mathbb{R})$ be the space of functions such that $f, f' \in C_b(\mathbb{R})$. Equip both of these spaces with the uniform norm.

3.1)

Show that C_b is complete, and that C_b^1 is not complete.

3.2)

Show that the differentiation operator $D : C_b^1(\mathbb{R}) \to C_b(\mathbb{R})$ that sends $D : f \mapsto f'$ is unbounded, but has a closed graph.

Solution. 3.1)

First, consider the family of functions $f_n(x) = 2^{-n}\cos(7^n \pi x)$ for $n \geq 1$, and consider $g_m(x) = \sum_{n=1}^{m} f_n(x)$.

We have that the sequence of $\{g_m\}$ is uniformly Cauchy, as if we let $\epsilon > 0$, we may choose N such that $2^{-N+1} < \epsilon$, and then for $m, m' > N$ (WLOG, suppose $m > m'$), we have that:

$$|g_m(x) - g_{m'}(x)| = |\sum_{n=1}^{m} f_n(x) - \sum_{n=1}^{m'} f_n(x)| = |\sum_{n=m}^{m'} f_n(x)| \leq |\sum_{n=N}^{\infty} f_n(x)| \leq \sum_{n=N}^{\infty} |f_n(x)| \leq \sum_{n=N}^{\infty} 2^{-N} = 2^{-N+1}$$

Since this is independent of the point x, this is uniformly Cauchy. Since each g_m is continuous, being the finite sum of continuous functions, and the convergence is uniform, the pointwise limit $g(x) = \lim_{m\to\infty} g_m(x)$ is a continuous function. Moreover, we can see easily that g is bounded, as we can see that each of the partial sums are bounded above by $\sum_{n=1}^{\infty} 2^{-n} = 2$. However, this is a Weierstrauss function, famously known for being differentiable nowhere. Since we have demonstrated a sequence of functions in C_b^1, convergent under the uniform norm to a function not in C_b^1, we may conclude that C_b^1 is not complete.

On the other hand, let $\{f_n\}_{n=1}^{\infty} \subseteq C_b$, with $\sum_{n=1}^{\infty} \|f_n\|_u < \infty$. Consider $f = \sum_{n=1}^{\infty} f_n$, and we will show that f is both bounded, and the uniform limit of the partial sums.

Evidently, f is bounded, as we can look at the partial sums $\sum_{n=1}^{N} f_n$. We have that $\|\sum_{n=1}^{N} f_n\|_u \leq \sum_{n=1}^{N} \|f_n\|_u < \sum_{n=1}^{\infty} \|f_n\|_u < \infty$, where the first inequality comes from the triangle inequality, and the second is simply our hypothesis of being absolutely convergent. Since this bound holds for all $N > 0$, it must hold in the limit as well. Hence, $\|f\|_u < \sum_{n=1}^{\infty} \|f_n\|_u < \infty$.

Now, we wish to show that $\sum_{n=1}^{N} f_n \to f$ uniformly. Since $\sum_{n=1}^{\infty} \|f_n\|_u < \infty$, for $\epsilon > 0$, we may find a $M > 0$ such that for all $m > M$, $\sum_{n=M}^{\infty} \|f_n\|_u < \epsilon$. Now, let $m > M$, and consider $\|f - \sum_{n=1}^{m} f_n\|_u$. We see that:

$$\|f - \sum_{n=1}^{m} f_n\|_u = \|\sum_{n=m+1}^{\infty} f_n\|_u$$

Now, due to the positivitiy of the norm, since we have for each finite sum: $\|\sum_{n=m+1}^{p} f_n\|_u \leq \sum_{n=m+1}^{p} \|f_n\|_u \leq \sum_{n=m+1}^{\infty} \|f_n\|_u$, we may conclude that this holds in the limit as well.

Hence, we have that:

$$\|\sum_{n=m+1}^{\infty} f_n\|_u \leq \sum_{n=m+1}^{\infty} \|f_n\|_u < \epsilon$$

Thus, $f_n \to f$ uniformly, and hence, f is continuous. Therefore, $f \in C_b$, as desired, and $f_n \to f$ under the norm. Since the choice of absolutely convergent sequence was arbitrary, by 5.1 in Folland, since every absolutely convergent sequence converges, C_b must be complete.

3.2)

Evidently, D is unbounded. For example, take the family of functions $f_k = \sin(kx)$, for $k \in \mathbb{N}$. Clearly, this is a continuous function, bounded above by 1, and so $\|f_k\|_u = 1$. Furthermore, its derivative is $k\cos(kx)$, continuous, and for each k, bounded above by k. However, $\|D(f_k)\|_u = \|k\cos(kx)\|_u = k$. Since we may choose k arbitrarily large without affecting the norm of f_k, D is unbounded.

Now, suppose that we have $f_n \to f \in C_b^1$, and $Df_n = f_n' \to g \in C^1$, uniformly in both cases. Fix an arbitrary point $a \in \mathbb{R}$, and consider, for $x > a$, the closed interval $[a, x]$. Since we have that $f_n' \to g$ uniformly, evidently, $\|f_n'\|_u$ is bounded. Then, we can take $\sup_n \|f_n'\|_u < \infty$ as an upper bound for all $|f_n'(y)|, y \in [a, x]$. Of course also, if $f_n' \to g$ uniformly, it does so pointwise as well. Therefore, by the Lesbesgue Dominated Convergence Theorem, we have that:

$$\lim_{n\to\infty} \int_a^x f_n'(y)dy = \int_a^x g(y)dy$$

However, we know that f_n is differentiable on $[a, x]$, and f_n', its derivative is continuous. Thus, we may transform the left hand side via the Fundamental Theorem of Calculus to obtain:

$$\lim_{n\to\infty} f_n(x) - f_n(a) = \int_a^x g(y)dy$$

Now, since $f_n \to f$ uniformly, it does so pointwise as well, so we have that:

$$f(x) - f(a) = \int_a^x g(y)dy$$

and finally, we can apply D to both sides of this equation, and since g is continuous, we can apply the other statement of the FTC to obtain:

$$D(f(x) - f(a)) = D\left(\int_a^x g(y)dy\right) \implies D(f)(x) = g(x)$$

Since the choice of a were arbitrary, we may repeat this argument for every x. Hence, varying across all $x \in \mathbb{R}$, we obtain an equality of functions, and conclude that $Df = g$.

Since this is true for an arbitrary $f_n \to f, f_n' \to g$, this is true for all cases where both sequences simultaneously converge, and hence D has a closed graph.

□

Question 4. Let $\mathcal{H} = L^2[0, 1]$, the Lebesgue measurable and square-integrable functions defined on $[0, 1]$. Let K be a non-empty, closed, convex subset of $\mathcal{H}$. Define $P = P_K$ as the orthogonal projection of H onto K.

4.1)

Let $x \in \mathcal{H}$. Prove that the following are equivalent:

i) There exists a unique $z \in K$ such that $\|x - z\| = \min_{y\in K} \|x - y\|$.

ii) $z \in K$ and $\langle x - z, y - z\rangle \leq 0$ for all $y \in K$.

4.2)

Let A be a continuous bilinear mapping from $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$ such that, for some $\alpha > 0$, we have:

$$A(f, f) \geq \alpha \|f\|_2^2$$

for every $f \in \mathcal{H}$. We will prove the following statement in parts:

For every $f \in \mathcal{H}$, there exists a unique $u \in K$ such that:

$$A(u, v - u) \geq \langle f, v - u\rangle$$

for all $v \in K$.

4.2.1)

Fix a $u \in \mathcal{H}$, and prove that there exists a unique $Tu \in \mathcal{H}$ such that $A(u, v) = \langle Tu, v\rangle$ for every $v \in \mathcal{H}$. Prove that T is a bounded linear mapping on $\mathcal{H}$.

4.2.2)

Fix a $\rho > 0$, $f \in \mathcal{H}$, and define a map $S_\rho : K \to K$ that sends $v \mapsto P(\rho f - \rho Tv + v)$. Prove that we may choose ρ such that there exists a $0 < k < 1$ with the property that:

$$\|S_\rho(v_1) - S_\rho(v_2)\| \leq k\|v_1 - v_2\|$$

for all $v_1, v_2 \in K$.

4.2.3)

Conclude that for the value of $\rho > 0$ chosen in 4.2.2, that S_ρ is a contraction, and therefore has a unique fixed point $u \in K$.

4.2.4)

Note that we can rewrite $\rho f - \rho Tu = \rho f - \rho Tu + u - u$. Then, use 4.1 to show that:

$$\langle \rho f - \rho Tu, v - u\rangle \leq 0$$

for every $v \in K$.

4.2.5)

Conclude that, for every $f \in \mathcal{H}$, there exists a unique $u \in K$ such that:

$$A(u, v - u) \geq \langle f, v - u\rangle$$

Solution. 4.1)

First, we show that if $\langle x - z, y - z\rangle \leq 0$, then we get that $\|x - z\| = \min \|x - y\|$.

We have the following sequence of equalities, for arbitrary y:

$$\langle x - z, y - z\rangle = \langle x - z, y + (x - x) - z\rangle = \langle x - z, x - z\rangle + \langle x - z, y - x\rangle = \|x - z\|^2 + \langle x - z, y - x\rangle$$

Then, we have that:

$$\langle x - z, y - z\rangle \leq 0 \implies \|x - z\|^2 + \langle x - z, y - x\rangle \leq 0 \implies \|x - z\|^2 \leq -\langle x - z, y - x\rangle$$

Since the norm is positive, we may harmlessly replace $\langle x - z, y - x\rangle$ with its absolute value. Then, by the Cauchy-Schwarz inequality, we retrieve:

$$\|x - z\|^2 \leq \|x - z\|\|y - x\| \implies \|x - z\| \leq \|y - x\| = \|x - y\|$$

Since this is true for all $y \in K$, including z itself, we conclude that $\|x - z\| = \min_{y \in K} \|x - y\|$.

Now, suppose that $z \in K$ is such that $\|x - z\| = \min_{y \in K} \|x - y\|$. By convexity, for any $y \in K$, we may reexpress $y = (1 - t)z + tw$ for at least some $w \in K, t \in [0, 1]$, hence, we have that:

$$\|x - z\| \leq \|x - (1 - t)z + tw\| = \|x - z - t(w - z)\|$$

We may safely square both sides and examine the inner product instead. Thus, we have that:

$$\langle x - z, x - z\rangle \leq \langle x - z - t(w - z), x - z - t(w - z)\rangle$$

Using the linearity and conjugate linearity of the inner product, we see that the RHS can be rewritten as:

$$\langle x - z - t(w - z), x - z - t(w - z)\rangle = \langle x - z, x - z\rangle - t\langle x - z, w - z\rangle - t\langle w - z, x - z\rangle + t^2\langle w - z, w - z\rangle$$

Hence, we have that:

$$\langle x - z, x - z\rangle \leq \langle x - z - t(w - z), x - z - t(w - z)\rangle \implies \langle x - z, w - z\rangle + \langle w - z, x - z\rangle \leq t\langle w - z, w - z\rangle$$

Wait, uhhhh.

4.2.1)

Let $u \in \mathcal{H}$. By the bilinearity of A, we have that:

$$A_u : \mathcal{H} \to \mathbb{R} \quad A_u : v \mapsto A(u, v)$$

is a linear functional on $\mathcal{H}$. Moreover, since A is continuous, it is continuous in each variable, and hence A_u is a continuous linear functional. Thus, since $\mathcal{H}, \mathbb{R}$ are normed linear spaces, and A_u is a continuous linear operators, A_u is bounded (1.63, Heil).

Since $\mathcal{H}$ is a Hilbert space, we can identify a w_u such that $A_u(v) = \langle v, w_u\rangle$ by the Riesz Representation Theorem (Folland, 5.25). Since A is real-valued, we can freely pick w_u to be in the first or second argument due to conjugate symmetry - we will from now on use $A_u(v) = \langle w_u, v\rangle$.

So now, we may define $T : \mathcal{H} \to \mathcal{H}$ that sends $u \mapsto w_u$. Evidently, due to the bilinearity of A, T is linear:

$$\begin{cases} \langle T(u + u'), v\rangle = A(u + u', v) = A(u, v) + A(u', v) = \langle T(u), v\rangle + \langle T(u'), v\rangle = \langle T(u) + T(u'), v\rangle \\ \langle T(ku), v\rangle = A(ku, v) = kA(u, v) = k\langle T(u), v\rangle \end{cases}$$

By the Cauchy-Schwarz inequality:

$$\alpha\|f\|^2 \leq A(f, f) = \langle Tf, f\rangle \leq \|Tf\|\|f\| \implies \alpha\|f\| \leq \|Tf\|$$

....that's not the right way for this inequality hrm.

4.2.2)

First of all, using the equivalent statement in 4.1, we see that:

$$\langle \rho f - \rho Tv + v - S_\rho(v), y - S_\rho(v)\rangle \leq 0$$

for every $y \in K$.

Then, letting $v_1, v_2 \in \mathcal{H}$, we have the following statements:

$$\begin{cases} \langle \rho f - \rho Tv_1 + v_1 - S_\rho(v_1), S_\rho(v_2) - S_\rho(v_1)\rangle \leq 0 \\ \langle \rho f - \rho Tv_2 + v_2 - S_\rho(v_2), S_\rho(v_1) - S_\rho(v_2)\rangle \leq 0 \end{cases}$$

Summing these equations then, and pulling out a factor of -1 from the second argument in the second equation, we find that:

$$\langle \rho f - \rho Tv_1 + v_1 - S_\rho(v_1) - \rho f + \rho Tv_2 - v_2 + S_\rho(v_2), S_\rho(v_2) - S_\rho(v_1)\rangle \leq 0 \implies$$

$$\langle \rho T(v_2 - v_1) + v_2 - v_1 + S_\rho(v_2) - S_\rho(v_1), S_\rho(v_2) - S_\rho(v_1)\rangle \leq 0 \implies \langle \rho T(v_2 - v_1) + v_2 - v_1 + S_\rho(v_2) - S_\rho(v_1), S_\rho(v_1) - S_\rho(v_2)\rangle \geq 0$$

where we've used the linearity of T, and then multiplied through by -1, bringing it into the second argument.

We examine the square of the norm, to leverage the inner product.

We have that:

$$\langle S_\rho(v_1) - S_\rho(v_2), S_\rho(v_1) - S_\rho(v_2) \rangle \leq$$

$$\langle S_\rho(v_1) - S_\rho(v_2), S_\rho(v_1) - S_\rho(v_2) \rangle + \langle \rho T(v_2 - v_1) + v_2 - v_1 + S_\rho(v_2) - S_\rho(v_1), S_\rho(v_1) - S_\rho(v_2) \rangle =$$

$$\langle S_\rho(v_1) - S_\rho(v_2) + \rho T(v_2 - v_1) + v_2 - v_1 + S_\rho(v_2) - S_\rho(v_1), S_\rho(v_1) - S_\rho(v_2) \rangle =$$

$$\langle \rho T(v_2 - v_1) + v_2 - v_1, S_\rho(v_1) - S_\rho(v_2) \rangle \leq \|\rho T(v_2 - v_1) + v_2 - v_1\| \|S_\rho(v_1) - S_\rho(v_2)\|$$

where we add the positive quantity determined above in line 2, and the final inequality comes from the Cauchy-Schwarz inequality.

Hence, we conclude that:

$$\|S_\rho(v_1) - S_\rho(v_2)\|^2 \leq \|\rho T(v_2 - v_1) + v_2 - v_1\| \|S_\rho(v_1) - S_\rho(v_2)\| \implies \|S_\rho(v_1) - S_\rho(v_2)\| \leq \|\rho T(v_2 - v_1) + v_2 - v_1\|$$

But now, let's examine the right side a bit more. By the triangle inequality and the definition of the operator norm, we find that:

$$\|\rho T(v_2 - v_1) + v_2 - v_1\| \leq \|\rho T(v_2 - v_1)\| + \|v_2 - v_1\| \leq (\rho \|T\| + 1) \|v_2 - v_1\|$$

Clearly something has gone horribly wrong.

4.2.3)

By definition then, since the ρ in 4.2.2 gives rise to a $k \in (0, 1)$ such that $\|S_\rho(v_1) - S_\rho(v_2)\| \leq k\|v_1 - v_2\|$, we see that S_ρ is a contraction on the metric. Hence, by the Banach fixed-point Theorem, there exists a unique fixed point $u \in K$ such that $S_\rho(u) = u$.

4.2.4)

Identifying $\rho f - \rho T u + u$ as x, $P(\rho f - \rho T u + u) = z = S_\rho(u) = u$, and renaming y to v, we see that:

$$\langle \rho f - \rho T u + u - u, v - u \rangle \leq 0 \implies \langle \rho f - \rho T u, v - u \rangle \leq 0$$

4.2.5)

Ok, from here, consider $\rho A(u, v - u)$, where ρ is small enough such that we may find u, the unique fixed point associated to S_ρ determined by f. From 4.2.1, we have that:

$$\rho A(u, v - u) = \rho \langle T u, v - u \rangle = \langle \rho f - \rho f + \rho T u, v - u \rangle =$$

$$\rho \langle f, v - u \rangle + \langle -\rho f + \rho T u, v - u \rangle$$

But, from 4.2.4, we see that:

$$\langle -\rho f + \rho T u, v - u \rangle = -\langle \rho f - \rho T u, v - u \rangle \geq 0$$

Hence, we conclude that:

$$\rho \langle f, v - u \rangle \leq \rho A(u, v - u) \implies \langle f, v - u \rangle \leq A(u, v - u)$$

□