Homework #7

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Math 285: Homework #7

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Question 1. (a)

Let A, B be disjoint, closed sets on a manifold M. Find a C^∞ function f such that $f|_A = 1$ and $f|_B = 0$.

(b)

Let M be a manifold, and let $A \subset M$ be closed, $U \subset M$ be open, such that $A \subset U$. Show that there exists a C^∞ function f on M such that $f|_A = 1$ and that the support of f is contained within U.

Solution. (a)

Firstly, by Theorem 13.6, we know that relative to the open sets $\{M_A = M \setminus A, M_B = M \setminus B\}$, there exists a C^∞ partition of unity$\{\rho_A, \rho_B\}$ where ρ_A is subordinate to M_A, and analogously for ρ_B.

Clearly then, since the support of ρ_B is a subset of M_B, we have that $\rho_B\big|_B = 0$, as $B \cap M_B = \emptyset$. Furthermore, we can say that on A, we have the analogous result that $\rho_A\big|_A = 0$.

Then, we notice that due to these results, that because this is a partition of unity, we have that:

$$(\rho_A + \rho_B)\big|_A = 1 \implies \rho_A\big|_A + \rho_B\big|_A = 1 \implies \rho_B\big|_A = 1$$

Thus, ρ_B is a C^∞ function that is identically 0 on B and identically 1 on A, as desired.

(b)

Take the disjoint closed sets $M \setminus U$ and A. Then, by part (a), there exists a C^∞ function f such that $f\big|_{M\setminus U} = 0$ and $f\big|_A = 1$.

Furthermore, from construction of f being a partition of unity subordinate to $M \setminus (M \setminus U)$, by deMorgan's laws, we have that

$$M \setminus (M \setminus U) = M \cap (M \setminus U)^c = M \cap (M \cap U^c)^c = M \cap (M^c \cup U^{c^c}) = M \cap (\emptyset \cup U) = M \cap U = U$$

Thus, by definition, since f is subordinate to U, the support of f is contained within U, as requested.

□

Question 2. Let $F : N \to M$ be a C^∞ map of manifolds. Let $h : M \to \mathbb{R}$ be a C^∞ function. Show that the support of $F^*(h)$ is a subset of $F^{-1}(\text{supp}(h))$.

Solution. First, we recall that the pullback is exactly $F^*(h) = h \circ F$.

Now, first, we want to show that $(F^*h)^{-1}(\mathbb{R}^\times) \subset F^{-1}(\text{supp}(h))$.

So, suppose $p \in N$, such that $F^*h(p) \neq 0$. Then, we have that $h \circ F(p) = h(F(p)) \neq 0$, which implies that $F(p) \in \text{supp}(h)$. Thus, $p \in F^{-1}(\text{supp}(h))$. Since the choice of p were arbitrary, this implies that $(F^*h)^{-1}(\mathbb{R}^\times) \subset F^{-1}(\text{supp}(h))$.

Now, by definition, we know that $\text{supp}(F^*h) = \text{cl}[(F^*h)^{-1}(\mathbb{R}^\times)]$, that is, it is the closure of the preimage. Furthermore, we notice that because F is C^∞, F is continuous. Since the $\text{supp}(h)$ is a closed set in M, being a closure, then $F^{-1}(\text{supp}(h))$ is a closed set in N, by continuity.

Hence, since $(F^*h)^{-1}(\mathbb{R}^\times) \subseteq F^{-1}(\text{supp}(h))$ and $F^{-1}(\text{supp}(h))$ is closed, we have that $\text{supp}(F^*h) = \text{cl}[(F^*h)^{-1}(\mathbb{R}^\times)] \subseteq F^{-1}(\text{supp}(h))$, via the characterization of the closure of a set S being the smallest closed set containing S.

□

Question 3. Define $x^1, y^1, ..., x^n, y^n$ as the standard coordinates of $\mathbb{R}^{2n}$. Define the unit sphere S^{2n-1} in an ambient $\mathbb{R}^{2n}$ cut out by the equation $\sum_{i=1}^n (x^i)^2 + (y^i)^2 = 1$.

Show that

$$X = \sum_{i=1}^n -y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i}$$

is a smooth vector field that does not vanish anywhere on S^{2n-1}.

Solution. First, we prove part of problem 11.1, that is:

Lemma 1. *Define the unit sphere $S^n \subset \mathbb{R}^{n+1}$ as being cut out by the equation $\sum_{i=1}^{n+1}(x^i)^2 = 1$. For a point $p = (p^1, ..., p^{n+1}) \in S^n$, and a tangent vector at p,*

$$X_p = \sum a^i \frac{\partial}{\partial x^i}\bigg|_p \in T_p(\mathbb{R}^{n+1})$$

show that $\sum a^i p^i = 0$ if and only if X_p is tangent to S^n at p.

Proof. We follow the discussion in Section 11.5.

Since we have the equation for an n-sphere, we have then an equation for the tangent space for S^n at a point p as being, in terms of a tangent vector $a = \langle a^1, ..., a^{n+1}\rangle$ for TS^n:

$$\sum_{j=1}^{n+1} \frac{\partial}{\partial x^j}\left(\sum_{i=1}^{n+1}(x^i)^2\right)(p)a^j = 0 \iff$$

$$\sum_{j=1}^{n+1}\sum_{i=1}^{n+1} \delta_j^i 2x^i(p)a^j = 0 \iff$$

$$\sum_{j=1}^{n+1} 2p^j a^j = 0 \iff \sum_{j=1}^{n+1} p^j a^j = 0$$

So, we have that if $\sum_{j=1}^{n+1} p^j a^j = 0$, then $a = \langle a^1, ..., a^{n+1}\rangle$ is the image of some tangent vector in TS^n under the differential of the inclusion, and thus X_p is tangent to S^n at p as desired. Note that to be clear, the tangent vector a should be thought of the image of the differential of the inclusion of the n sphere into $\mathbb{R}^{n+1}$, and so we may identify it with its image in $T\mathbb{R}^{n+1}$.

□

Thus, we fix a point $p = (p^1, ..., p^{2n}) \in S^{2n-1}$, and we wish to show that the vector field

$$X = \sum_{i=1}^n -y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i}$$

is tangent to S^{n-1}.

By the lemma above, we need only check if $\sum_{j=1}^n -y^i\big|_p p^{2i-1} + x^i\big|_p p^{2i} = 0$ at a point p. However, this is clear:

$$\sum_{j=1}^{n} -y^i\big|_p p^{2i-1} + x^i\big|_p p^{2i} = \sum_{j=1}^{n} -p^{2i}p^{2i-1} + p^{2i-1}p^{2i} = \sum_{j=1}^{n} 0 = 0$$

Since the choice of p was arbitrary, this works for all p, and since X is tangent at every point $p \in S^{2n-1}$, it defines a vector field on S^{2n-1}, where we see smoothness as the lift from $\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$ is smooth by being polynomial in each coordinate, and taking an appropriate restriction.

However, it is clear that it is nowhere vanishing, as at any point p, we have that:

$$X_p = \sum_{i=1}^{n} -y^i \frac{\partial}{\partial x^i}\bigg|_p + x^i \frac{\partial}{\partial y^i}\bigg|_p = \sum_{i=1}^{n} -p^{2n} \frac{\partial}{\partial x^i}\bigg|_p + p^{2n-1} \frac{\partial}{\partial y^i}\bigg|_p$$

Therefore, $X_p = 0$ if and only if $p^i = 0$ for all i. However, the point $(0, ..., 0) \notin S^{2n-1}$, and thus X does not vanish on S^{2n-1}.

□

Question 4. Let $M = \mathbb{R} \setminus \{0\}$. Let $X = \frac{d}{dx}$ on M. Find the maximal integral curve of X with initial point $x = 1$.

Solution. Clearly, if we have that $c(t) = x(t)$, in order to be an integral curve, we must have that:

$$x'(t) = 1 \implies x(t) = t + a$$

for indeterminant $a \in \mathbb{R}$.

Clearly, to satisfy the initial condition, we then must have:

$$x(0) = 0 + a = 1 \implies a = 1$$

And thus, our integral curve desired has equation $c(t) = x(t) = t + 1$.

Now, here, we want to determine what the domain should be.

We notice that since c is a smooth curve, in particular, it is continuous. Thus, because the domain of the curve is an interval in $\mathbb{R}$, hence connected, because the continous image of a connected set is also connected (Proposition A.42), we must have that the image of the curve is connected.

Thus, since we notice that we may find connected components on M under the subspace topology as $(-\infty, 0), (0, \infty)$, the natural maximal integral curve is simply the preimage of the connected component containing our initial point.

Thus, we see that our desired domain is:

$$c^{-1}((0, \infty)) = (-1, \infty)$$

Hence, our maximal curve is:

$$c : (-1, \infty) \to M \text{ via } t \mapsto t + 1$$

A remark, if we consider curves to only have finite endpoints, we may not find a maximal curve. Since, suppose $c_0 : (a, b) \to \mathbb{R}$ were a maximal candidate, that sends $t \mapsto t + 1$.

The problem here is that we can always construct $c_1 : (a, b + 1) \to \mathbb{R}$ with the same map. Since we can do this for any finite b, there is no maximal integral curve with finite endpoints as its domain for such a manifold, initial point, and vector field.

□

Question 5. Find the integral curves of the vector field:

$$X_{(x,y)} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

in an ambient $\mathbb{R}^2$.

Proof. Define the initial condition as the point x_0, y_0.

For $c(t)$ to be an integral curve, if we define $c(t) = (x(t), y(t))$, and denoting a derivative with respect to only time with a $'$ we have the following system of equations:

$$\begin{cases} x'(t) = x \\ y'(t) = -y \end{cases}$$

We recognize this as first order differential equations, with the following solution set:

$$\begin{cases} x(t) = Ae^t \\ y(t) = Be^{-t} \end{cases}$$

for indeterminants $A, B \in \mathbb{R}$.

Using our initial conditions, we see that:

$$c(0) = (x(0), y(0)) = (x_0, y_0) \implies A = x_0, B = y_0$$

Thus, relative to the initial point $p = (x_0, y_0)$, we see that integral curves for the vector field $X = \langle x, -y \rangle$ take on the form:

$$c(t) = c_t(p) = (x_0 e^t, y_0 e^{-t})$$

Here, we note that the interval curve can take on arbitrary intervals of $\mathbb{R}$ as its domain.

$\square$

Question 6. Let f, g be C^∞ functions and X, Y be C^∞ vector fields, all on a manifold M. Show that:

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

Solution. Fix some $h \in C^\infty(M)$, and consider the action of $[fX, gY]h$. We have that:

$$[fX, gY]h = fX(gY(h)) - gY(fX(h))$$

Now, using the fact that a vector field is a derivation, we can use the Leibniz rule on $X(g * [Y(h)])$ and $Y(f * X(h))$:

$$fX(gY(h)) - gY(fX(h)) = f(Xg)(Yh) + fg(XY(h)) - g(Yf)(Xh) - gf(YX(h))$$

Regrouping a little, we can see that because $[X, Y] := XY - YX$, and $fg = gf$ being scalar functions:

$$f(Xg)(Yh)+fg(XY(h))-g(Yf)(Xh)-gf(YX(h)) = (fg(XY(h))-gf(YX(h)))+f(Xg)(Yh)-g(Yf)(Xh) =$$

$$fg(XY - YX)h + f(Xg)(Yh) - g(Yf)(Xh) = fg[X, Y]h + f(Xg)(Yh) - g(Yf)(Xh)$$

Thus, we see that:

$$[fX, gY]h = fg[X, Y]h + f(Xg)(Yh) - g(Yf)(Xh)$$

Since the choice of h was arbitrary, varying over h, we obtain an equality of vector fields:

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

as desired.

$\square$