# Math 233
# Lecture Notes

## Jan 18th

First, settle some notation:

Disks:

Let $a \in \mathbb{C}$, and $r > 0$. Denote the open disk of radius $r$, centered at $a$ as:

$$D(a,r) = \{z \in \mathbb{C} : |z - a| < r\}$$

Similarly, denote the closed disk as:

$$\overline{D}(a,r) = \{z \in \mathbb{C} : |z - a| \leq r\}$$

Connected sets and components:

Let $X$ be a topological space. Call $E \subseteq X$ disconnected when there exist non-empty subsets $A, B \subset E$ such that $A \cup B = E$ and $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. We say that $A, B$ is said to separate $E$.

We call $E \subseteq X$ if it does not admit a separation into subsets.

Now, suppose $E \subseteq X, x_0 \in E$. Then, if we have $A \subset E$ such that $x_0 \in A$, and $A$ connected, then $\cup A$ is connected. Moreover, since we have the (potentially uncountable) union over all connected sets, this must be the largest such connected sets that includes $x_0$. Call this maximal connected set the component of $E$ that contains $x_0$. Call the collection of such connected subsets of $E$ over all $x_0$ the connected components of $E$. It should be clear that the set $E$ must be the disjoint union of the connected components.

Now, let $E \subseteq \mathbb{C}$ be an open set. Let $a \in E$. Because $E$ is open, there exists $r > 0$ such that $D(a,r) \subseteq E$. Then, $D(a,r)$ is in the connected component containing $a$. Thus, the components of $E$ are open.

Note: we will use $\Omega$ to denote open sets in $\mathbb{C}$.

Call an open, connected subset of $\mathbb{C}$ a region.

Derivatives:

Suppose that $\Omega \subseteq \mathbb{C}$ is an open set, and we have $f : \Omega \to \mathbb{C}$. Let $z \in \Omega$. Then, if it exists, we define the derivative of $f$ at $z$ as:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

That is, for the limit to exists, we must have that:

$$\lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| = 0$$

We note that in the $\mathbb{R}^2$ sense of derivatives, complex differentiable implies real differentiable, but not the converse, because the extra structure from the Cauchy-Riemann equations.

Theorem:

If $f'(z_0)$ exists, then $f$ is continuous at $z_0$.

Let $\Omega \subset \mathbb{C}$ be an open set, $f : \Omega \to \mathbb{C}$. If $f$ is differentiable on all of $\Omega$, then we call $f$ holomorphic (or analytic).

We denote the set of all holomorphic functions on $\Omega$ by $\mathcal{H}(\Omega)$.

Let $f, g \in \mathcal{H}(\Omega)$. Then $f + g, fg \in \mathcal{H}(\Omega)$ and $f/g \in \mathcal{H}(\Omega)$ if $0 \notin g(\Omega)$. The normal rules hold: product rule, quotient rule, etc.

Chain rule:

Suppose that $g \in \mathcal{H}(\Omega)$, $g(\Omega) \subset \Omega_1$, and $f \in \mathcal{H}(\Omega_1)$. Then we claim that $h = f \circ g \in \mathcal{H}(\Omega)$ and $h' = f'(g(z_0)) * g'(z_0)$

Proof:

Let $z_0 \in \Omega, w_0 = g(z_0) \in \Omega_1$. Since $f'(w_0)$ exists, because $f$ is holomorphic, define $\phi : \Omega_1 \to \mathbb{C}$ as:

$$\phi(w) = \begin{cases} \frac{f(w) - f(w_0)}{w - w_0} & \text{if } w \in \Omega \setminus \{w_0\} \\ f'(w_0) & \text{if } w = w_0 \end{cases}$$

We see that by the definition of the derivative, that $\phi$ is continuous on $\Omega_1$, and we have that $f(w) - f(w_0) = \phi(w)(w - w_0)$. However, we have that $w_0 = g(z_0), w = g(z)$, so we have that:

$$\frac{f(g(z)) - f(g(z_0)}{z - z_0} = \phi(g(z)) \frac{g(z) - g(z_0)}{z - z_0}$$

If we take the limit of both sides as $z \to z_0$, we see that the right hand side is exactly $\phi(g(z_0))g'(z_0)$. Further, the left hand side is exactly the definition of the derivative of $h$ at $z_0$. Hence, $h'(z_0)$ exists, and equals $f'(g(z_0))g'(z_0)$.

Power rule:

By direct computation, we have that $\frac{d}{dz}(z^n) = nz^{n-1}$. This holds for all $n \in \mathbb{Z}$.

Definition:

Let $f : \mathbb{C} \to \mathbb{C}$. If $f$ is holomorphic, then we call $f$ entire. Example: $1, z, z^2$ are entire.

Functions representable by Power Series:

Consider a power series $\Sigma_{n=0}^{\infty} c_n (z - a_n)^n$.

By the root test, we can say this series has radius of convergence $R$, where $\frac{1}{R} = \limsup_{n \to \infty} |c_n|^{1/n}$. In particular, the series converges absolutely for all $z \in D(a, R)$. Further, the series diverges for all $z \notin \overline{D}(a, R)$. Now, for $0 \leq r < R$, the power series converges uniformly via the Weierstrauss M-test.

Now, let $f : \Omega \to \mathbb{C}$. We say that $f$ is representable by power series on $\Omega$ provided that for any disk $D(a, r) \subseteq \Omega$, $f$ may be represented by a power series $f(z) = \Sigma_{n=0}^{\infty} c_n (z - a)^n, z \in D(a, r)$.

Theorem:

Suppose that $f$ is representable by a power series on $\Omega$. Then $f$ is holomorphic on $\Omega$, that is, $f \in \mathcal{H}(\Omega)$. Moreover, if $D(a, r) \subset \Omega$, and $f(z) = \Sigma_{n=0}^{\infty} c_n (z-a)^n$ on this disk, then for any $z \in D(a, r)$, $f'(z) = \Sigma_{n=1}^{\infty} n c_n (z-a)^n$. We notice that this is also a power series; as such, $f'$ is also representable by a power series (and, thus, holomorphic).

Proof omitted, long computation, refer to lecture notes or Rudin.

If $f$ is representable by power series on $\Omega$, then $f'$ is also representable, and thus holomorphic. Moreover, $f^{(n)}$ is holomorphic. By term-by-term differentiation, we have that:

$$f^{(k)} = \Sigma_{n=k}^{\infty} n(n-1)...(n-k+1) c_n (z-a)^{n-k}$$

Further, we have that at $z = a$, that $f^{(k)}(a) = k! c_k \implies c_k = f^{(k)}(a)/k!$.