Homework #8

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Math 235: Homework #8

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2.1

Problem 4.6.21. Assume that $E \subseteq \mathbb{R}^d$ is measurable. Let $f : E \to \overline{F}$ be a measurable function. Define the distribution function of f as follows:

$$\omega(t) = |\{|f| > t\}|, t \geq 0$$

By definition, ω is a non-negative, extended real-valued function. Prove the following:

(a) ω is monotone decreasing on $[0, \infty)$.

(b) ω is right-continuous, that is, $\lim_{s\to t^+} \omega(s) = \omega(t)$ for every $t \geq 0$.

(c) If f is integrable, then $\lim_{s\to t^-} \omega(s) = |\{|f| \geq t\}|$.

(d) $\int_0^\infty \omega(t)dt = \int_E |f(x)|dx$

(e) f is integrable $\iff$ ω is integrable.

(f) If f is integrable, then $\lim_{n\to\infty} n\omega(n) = 0 = \lim_{n\to\infty} \frac{1}{n}\omega(\frac{1}{n})$.

Solution. (a)

We notice that for any $t' \geq t$, that by definition, $\{|f| > t'\} \subseteq \{|f| > t\}$. Then, by the monotonicity of the Lebesgue measure, we have that $|\{|f| > t'\}| \leq |\{|f| > t\}| \implies f(t') \leq f(t)$. Since this is true for all $t' \geq t$, we have that ω is monotone decreasing.

(b)

Let $\{a_n\}_{n\in\mathbb{N}}$ be any sequence of positive numbers where $a_n \to 0$. Take a monotone subsequence $\{a_{n_k}\}$ such that $a_{n_{k+1}} < a_{n_k}$ for all k. Then, consider the sequence of measurable sets $A_{a_{n_k}} = \{|f| > t + a_{n_k}\}$. This is a sequence of measurable sets, with the property that $A_{a_{n_{k+1}}} \subseteq A_{a_{n_k}}$ since we took the monotone subsequence. Then, by convergence from below, we have that $\lim_{n_k\to 0} |A_{a_{n_k}}| = \cup_{n_k} A_{a_{n_k}} = \{|f| > t\}$. Now, we only need to check that this is convergent in the full sequence. First suppose that $|\{|f| > t\}| = \infty$. Then, we have that this must diverge for a_n, because since we know $|A_{a_{n_k}}| \to \infty$ monotonically, and $a_n \to 0$, for any $M \in \mathbb{R}$, we take N_{k_0} such that for all $n_k > N_{k_0}$, $|A_{a_{n_k}}| \geq M$. Then, we pick N such that for all $i > N$, $a_i < a_{N_{k_0}}$, by the convergence of a_n. Then, we know that since this is monotone, $|A_{a_{n_i}}| \geq |A_{a_{N_{k_0}}}| \geq M$.

Now, suppose $|\{|f| > t\}| < \infty$. Then, we can choose N_{k_0} such that $|\{|f| > t\}| - |A_{a_{N_{k_0}}}| < \epsilon$. We may choose N such that $a_j < A_{N_{k_0}}$ by the convergence of the a_n for all $j > N$. Then, we have by the monotonicity, that $|\{|f| > t\}| \geq |A_{a_j}| \geq |A_{a_{N_{k_0}}}| \implies |\{|f| > t\}| - |A_{a_j}| < \epsilon$.

(c)

First, we see that if $|E| < \infty$, then we can use continuity from above. Otherwise, suppose $|E| = \infty$. First, for $t = 0$, since we use left-continuous, but ω only defined for $t \geq 0$, the only permissible sequence is the constant sequence, and of course $|\{|f| > 0\}| = |\{|f| > 0\}|$. Then, for any $t > 0$, we see that $|\{|f| > t\}| < \infty$ as suppose not, then we know that $|f| > t$ the constant function on a set of infinite measure, so $\int_E |f| = \infty$, a contradiction. As such, regardless, we may use continuity from above since we are guaranteed that the sets have finite measure.

Now, then, let $\{b_n\}$ be any sequence of non-negative numbers such that $b_n \to t$, where $t > 0$. We use the same construction as (b), where we find a monotone sequence $b_{n_k} \to t$, which extends to a nested sequence of sets $\{|f| > b_{n_0}\} \supseteq \{|f| > b_{n_1}\}$.... We call these sets $B_{b_{n_k}}$. Then, since we are assured that this is eventually constant, since eventually, at least one $b_{n_k} > 0$, we apply continuity from above to find that $\lim_{n_k \to \infty} |B_{b_{n_k}}| = |\cap B_{b_{n_k}}|$. But, we have that $\cap B_{b_{n_k}} = \cap\{|f| > b_{n_k}\} = \{|f| \geq t\}$ because $b_{n_k} \to t$ from the left. Finally, in the same vein as (b), we use the convergence of $b_n \to t$ from the left as well as the monotonicity of ω to get that the whole thing converges.

(d)

Consider the integral, over $E \times \mathbb{R}^+$, that is, the non-negative real numbers, of $\chi_{\Gamma_{|f|}}$, where $\Gamma_{|f|}$ is the region under the graph of $|f|$ without the boundary defined in previous work $\{(x,t) : x \in E, t < |f(t)|\}$ where $|f(t)|$ can be infinite. Clearly, this is measurable, being a characteristic function of a measurable set, where we know the graph is measurable because of a previous homework 4.2.17, (b) and (a), since the boundary has measure 0. Then, we apply Tonelli's theorem:

$$\int_{\mathbb{R}^+} \left(\int_E \chi_{\Gamma_{|f|}} dx \right) dt = \int_E \left(\int_{\mathbb{R}^+} \chi_{\Gamma_{|f|}} dt \right) dx$$

Here, we notice that fixing t, $\int_E \chi_{\Gamma_{|f|}} dx = |\{|f| > t\}| = \omega(t)$, since at any fixed t_0, the points in the graph consist of the (x, t_0) such that $|f(x)| > t_0$. On the other side, we have that $\int_{\mathbb{R}^+} \chi_{\Gamma_{|f|}} dt = |f(x)|$ since if we fix an x_0, then the points in the graph are just (x_0, t) such that $0 \leq t < |f(x_0)| \implies t \in [0, |f(x_0)|)$, and $|[0, |f(x_0)|)| = |f(x_0)|$ for every $x_0 \in E$.

Then, substituting back into Tonelli's:

$$\int_{\mathbb{R}^+} \left(\int_E \chi_{\Gamma_{|f|}} dx \right) dt = \int_E \left(\int_{\mathbb{R}^+} \chi_{\Gamma_{|f|}} dt \right) dx \implies \int_{\mathbb{R}^+} \omega(t) dt = \int_E |f(x)| dx$$

as desired.

(e)

We have the following:

$$f \text{ integrable} \iff \lim_E f < \infty \iff \int_0^\infty \omega < \infty \iff \omega \text{ integrable}$$

where we use the result from (d).

(f)

Suppose not. Take the sequence $a_n \to \infty$. Then, we have that for all $n \geq N$ for some $N \in \mathbb{N}$, $a_n \omega(a_n) \geq \epsilon$ for some $\epsilon > 0$, where we use the fact that ω is non-negative to conclude $a_n \omega(a_n) \geq 0$ and monotone decreasing to conclude that it must have a minimum value. But then, we see that $\omega(a_n) \geq \epsilon / a_n$ for all $n \geq N$. This implies then, that on the set $[a_N, \infty]$, that $\omega(a_N)$ is larger than the constant function ϵ / a_N. But then, since ω is non-negative, we have that $\int_{[0,\infty]} \omega \geq \int_{[a_N, \infty]} \omega \geq \epsilon / a_N |[a_N, \infty]| = \infty$ a contradiction, since we know that by part (d), the integral of ω aligns with the integral of $|f|$. Then, $a_n \omega(a_n) \to 0$, and since the choice of a_n was arbitrary, this must be true for any sequence $\to \infty$. The same argument works for the second half of the equality, since if we look at $1/b_n \omega(1/b_n) \geq \epsilon \implies \omega(1/b_n) \geq b_n \epsilon$, so we have that on $[1/b_N, \infty]$, $\omega \geq b_N \epsilon$, which has a divergent integral.

$\square$

Problem 4.6.27. Let $f \in L^1(\mathbb{R}), g \in L^\infty(\mathbb{R})$. Prove the following:

(a) The integral that defines $(f * g)(x)$ exists for every $x \in \mathbb{R}$.

(b) $f * g$ is continuous on $\mathbb{R}$.

(c) $f * g$ is bounded on $\mathbb{R}$, and $\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty$.

Solution. (a)

Recall that for any x, we define $(f*g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$. Since we know that $f^+, f^- \leq |f|$ by definition, it suffices then to show that $\int_{\mathbb{R}} |f(y)g(x-y)|dy < \infty$. Since $g \in L^\infty(\mathbb{R})$, we can say that $|g| \leq \|g\|_\infty$ a.e. But then, we have that $|f(y)g(x-y)| \leq \|g\|_\infty |f(y)|$ for almost every $y \in \mathbb{R}$. So, we have that:

$$\int_{\mathbb{R}} |f(y)g(x-y)|dy \leq \int_{\mathbb{R}} \|g\|_\infty |f(y)|dy \leq \|g\|_\infty \|f\|_1 < \infty$$

Thus, $(f*g)(x)$ exists for all $x \in \mathbb{R}$.

(b)

By theorem 4.5.8, we can find a function $h \in L^1(\mathbb{R})$ such that $h \in C_c(\mathbb{R})$ and $\|f-h\|_1 < \epsilon$, for any $\epsilon > 0$. We also notice that $\int_{\mathbb{R}} |h(y)g(x-y)|dy \leq \int_{\mathbb{R}} |h(y)|\|g\|_\infty dy = \|g\|_\infty \|h\|_1$ where we don't know if h is L^1 yet, so this could be infinite. But, by the reverse triangle inequality, we have that $|\|f\|_1 - \|h\|_1| \leq \|f-h\|_1 < \epsilon < \infty$, and since $\|f\|_1 < \infty$, so too must be $\|h\|_1$. Thus, $h(y)g(x-y)$ is integrable.

Further, we notice that the convolution is commutative, since we can take the translation $z = x-y \implies y = x-z$, as then $(f*g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy = \int_{\mathbb{R}} f(x-z)g(z)dz = (g*f)(x)$.

Now h be an arbitrary continuous function with compact support, we can say that h is uniformly continuous. Further, since h is compactly support, let S be the support of h, then we can say that $S+S \subseteq [-n,n]$ for some n, since compact sets are bounded in $\mathbb{R}$. Then, we can say $|S+S| < \infty$, where $|S+S|$ is the same as 4.6.28. Let $\eta > 0$. Then, we may choose $\delta(x) > 0$ such that $d(x,y) < \delta \implies d(h(x), h(y)) < \eta/|S+S|\|g\|_\infty$. Now, let $x, x' \in \mathbb{R}$ such that $d(x,x') < \delta$. Then, we have that, by the commutativity of the convolution, that

$$|(h*g)(x) - (h*g)(x')| = \left| \int_{\mathbb{R}} h(x-y)g(y)dy - \int_{\mathbb{R}} h(x'-y)g(y) \right| = \left| \int_{\mathbb{R}} g(y)[h(x-y) - h(x'-y)] \right| \leq$$

$$\int_{\mathbb{R}} |g(y)||h(x-y) - h(x'-y)| \leq \|g\|_\infty \int_{\mathbb{R}} |h(x-y) - h(x'-y)|$$

However, we have that $d(x-y, x'-y) = |(x-y) - (x'-y)| = |x-x'| = d(x,x') < \delta$, so we have that:

$$\|g\|_\infty \int_{\mathbb{R}} |h(x-y) - h(x'-y)| \leq \|g\|_\infty \int_{\mathbb{S}+\mathbb{S}} |h(x-y) - h(x'-y)| \leq \|g\|_\infty \frac{\eta}{|S+S|\|g\|_\infty} |S+S| = \eta$$

Thus, we have that $(h*g)$ is continuous, actually, uniformly continuous. Now, consider:

$$|(f*g)(x) - (f*g)(x')| = \left| \int_{\mathbb{R}} f(x-y)g(y)dy - \int_{\mathbb{R}} f(x'-y)g(y)dy \right| =$$

$$\left| \int_{\mathbb{R}} f(x-y)g(y)dy - \int_{\mathbb{R}} h(x-y)g(y) + \int_{\mathbb{R}} h(x-y)g(y) + \int_{\mathbb{R}} h(x'-y)g(y) - \int_{\mathbb{R}} h(x'-y)g(y) - \int_{\mathbb{R}} f(x'-y)g(y)dy \right|$$

Using the triangle inequality, we break the sum up into:

$$\left| \int_{\mathbb{R}} f(x-y)g(y)dy - \int_{\mathbb{R}} h(x-y)g(y) \right| \leq \|g\|_\infty \|f-h\|_1$$

$$\left| \int_{\mathbb{R}} h(x-y)g(y)dy - \int_{\mathbb{R}} h'(x-y)g(y) \right| \leq \|g\|_\infty \int_{\mathbb{R}} |h(x-y) - h'(x-y)|$$

$$\left| \int_{\mathbb{R}} h(x'-y)g(y)dy - \int_{\mathbb{R}} f(x'-y)g(y) \right| \leq \|g\|_\infty \|h-f\|_1$$

Then, since we can control h such that $\|f-h\|_1 = \|h-f\|_1 < \epsilon/3\|g\|_\infty$ due to the statement of the theorem, and we can control $d(x,x')$ such that $d(x,x') < \delta \implies \int_{\mathbb{R}} |h(x-y) - h'(x-y)| < \epsilon/3\|g\|_\infty$ due to the continuity on h if $h \in C_c(\mathbb{R})$, we can control the whole sum to be less than ϵ.

(c)

Well, I somehow did this to show (a), because we know that

$$|f*g| = \left|\int_{\mathbb{R}} f(y)g(x-y)dy\right| \leq \int_{\mathbb{R}} |f(y)g(x-y)|dy \leq \int_{\mathbb{R}} \|g\|_\infty |f(y)|dy \leq \|g\|_\infty \|f\|_1 < \infty$$

□

Problem 4.6.28. (a) Show that if $f, g \in C_c(\mathbb{R})$, then $f * g \in C_c(\mathbb{R})$ and

$$\text{supp}(f*g) \subseteq \text{supp}(f) + \text{supp}(g) = \{f + g : x \in \text{supp}(f), y \in \text{supp}(g)\}$$

Conclude that $C_c(\mathbb{R})$ is closed under convolution.

(b) Is $C_c^1(\mathbb{R})$ closed under convolution?

Solution. □

Problem 4.6.29. Let $E \subseteq \mathbb{R}$ be a measurable subset with $0 < |E| < \infty$.

(a) Prove that the convolution $\chi_E * \chi_{-E}$ is continuous.

(b) Prove the Steinhaus Theorem: The set $E - E = \{x - y : x, y \in E\}$ contains an open interval centered at the origin.

(c) Show that $\lim_{t\to 0} |E \cap (E+t)| = |E|, \lim_{t\to\pm\infty} |E \cap (E+t)| = 0$.

Solution. (a)

By 4.6.27 (b), since χ_{-E} is an indicator function, it is bounded on $\mathbb{R}$, in particular, by 1, so $\chi_{-E} \in L^\infty(\mathbb{R})$ and $\chi_E \in L^1(\mathbb{R})$ since $\int_{\mathbb{R}} \chi_E = |E| < \infty$, their convolution is continous.

(b)

First, consider $\chi_E * \chi_{-E}(0)$. By definition, this is exactly $\int_{\mathbb{R}} \chi_E(y)\chi_{-E}(-y)dy$, which we notice is exactly 1 on $y \in E$, and 0 otherwise. Then, this integral evaluates to $|E| > 0$. Now, consider the interval $(|E|/2, 3|E|/2)$. This is an open set, and from part (a), we know that the inverse image of this interval is open by continuity. Further, we just showed that $\chi_E * \chi_{-E}(0) = |E| \in (|E|/2, 3|E|/2)$, so $0 \in (\chi_E * \chi_{-E})^{-1}(|E|/2, 3|E|/2)$. Then, because this is open, there exists an open ball, i.e. an open interval around 0.

(c)

□

2.2

Problem 5.1.5. Prove that the Cantor-Lebesgue function is Hölder continuous for $0 < \alpha \leq \log_3 2$. In particular, notice that it is not Lipschitz.

Solution. □

Problem 5.1.7. Let C be the Cantor set, let ϕ be the Cantor-Lebesgue function, and define $g(x) = \phi(x) + x$ for $x \in [0,1]$.

(a) Prove that $g : [0,1] \to [0,2]$ is continous, strictly increasing, and a bijection. Further, its inverse $h = g^{-1} : [0,2] \to [0,1]$ is also a continuous, strictly increasing, bijection.

(b) Show that $g(C)$ is a closed subset of $[0,2]$ and that $|g(C)| = 1$.

(c) Since $g(C)$ has positive measure, it follows that there exists $N \subseteq g(C)$ such that N is not Lebesgue measurable. Show that $A = h(N)$ is a Lebesgue measurable subset of $[0, 1]$.

(d) Set $f = \chi_A$. Prove that $f \circ h$ is not a Lebesgue measurable function.

Solution. $\square$