Homework #3

Eric Tao
Math 237: Homework #3

March 30, 2024

Question 1. Show that the only function $f \in L^1(\mathbb{R})$ such that $f = f * f$ is $f = 0$ almost everywhere.

Solution. Let $f \in L^1(\mathbb{R})$. Suppose we have that $f = f * f$. We recall that the Fourier transform distributes pointwise over convolution, hence we have that:

$$\hat{f}(\zeta) = \widehat{f * f}(\zeta) = \hat{f}(\zeta)\hat{f}(\zeta)$$

Hence, we already have that at each ζ, $\hat{f} = 0, 1$, by solving the equation $\hat{f}(\zeta)^2 - \hat{f}(\zeta) = 0$.

From Lemma 9.2.3 in Heil, we have that $\hat{f}$ is continuous. Moreover, from the Riemann-Lesbesgue lemma, since $f \in L^1$, we already know that $\hat{f} \in C_0(\mathbb{R})$. Hence, by continuity, since $\hat{f}$ may only take on values $0, 1$, and that it is continuous, $\hat{f} = 0$ everywhere.

Thus, by Corollary 9.2.12 in Heil, since $\hat{f} = 0$ everywhere, it is true almost everywhere of course, and hence $f = 0$ almost everywhere.

□

Question 6. Suppose that $f \in \mathrm{AC}(\mathbb{T})$, that is, 1 periodic and absolutely continuous on $[0, 1]$.

6.1)

Prove that $\hat{f}'(n) = 2\pi i n \hat{f}(n)$ for all $n \in \mathbb{Z}$. Conclude that $\lim_{|n| \to \infty} n\hat{f}(n) = 0$.

6.2)

Show that if $\int_0^1 f(x)dx = 0$, then:

$$\int_0^1 |f(x)|^2 dx \leq \frac{1}{4\pi^2} \int_0^1 |f'(x)|^2 dx$$

Solution. 6.1)

First, we recall that by Corollary 6.1.5 in Heil, that since f is absolutely continuous on $[0, 1]$, we have that f is differentiable almost everywhere, and $f' \in L^1[0, 1]$. Since f' is at least L^1, we may look at its Fourier transform:

$$\hat{f}'(n) = \int_0^1 f'(x) \exp(-2\pi i n x) dx$$

We notice that for a fixed value of $n \in \mathbb{Z}$, that we may compute the derivative of $g = \exp(-2\pi i n x)$ as $g' = -2\pi i n \exp(-2\pi i n x)$. Hence, g is differentiable everywhere on $[0, 1]$, and of course, since $|-2\pi i n \exp(-2\pi i n x)| = 2\pi n$ always, it is bounded. Hence, by Lemma 5.2.5, g is Lipschitz on $[0, 1]$. Thus, since Lipschitz functions are absolutely continuous (6.1.3), we may apply integration by parts on f, g. We have then that:

$$\int_0^1 f'(x) \exp(-2\pi i n x) dx = f(1) \exp(-2\pi i n) - f(0) \exp(0) - \int_0^1 f(x)[-2\pi i n \exp(-2\pi i n x)] dx$$

Now, because f is 1-periodic, we have that $f(0) = f(1)$. Further, $\exp(-2\pi in) = \exp(0)$, of course, since $2\pi n$ is a multiple of 2π for any n. Hence, the first terms vanish.

We are left then with, after using the linearity of the integral:

$$2\pi in \int_0^1 f(x)\exp(-2\pi inx)dx$$

However, we recognize the integral as exactly the Fourier transform of f at n. Hence, we have our result, that:

$$\hat{f}'(n) = 2\pi in\hat{f}(n)$$

Again, by the Riemann-Lebesgue lemma, we have that since $f' \in L^1$, that $\lim_{|n|\to\infty} \hat{f}'(n) = 0$. But, from our result, we have that:

$$\lim_{|n|\to\infty} 2\pi in\hat{f}(n) = \lim_{|n|\to\infty} \hat{f}'(n) = 0 \implies \lim_{|n|\to\infty} n\hat{f}(n) = 0$$

6.2)

First, we use the result from Heil Problem 9.3.24 (b) that the Plancherel equality holds for $f \in L^1(\mathbb{T})$. and prove that later.

Then, since $f' \in L^1(\mathbb{T})$, we have that:

$$\sum_{n\in\mathbb{Z}} |\hat{f}'(n)|^2 = \|f'\|_2^2 = \int_0^1 |f'(x)|^2 dx$$

Now, from 6.1, we have that $\hat{f}'(n) = 2\pi in\hat{f}(n)$ for each n. Hence, we have that:

$$\sum_{n\in\mathbb{Z}} |\hat{f}'(n)|^2 = \sum_{n\in\mathbb{Z}} 4\pi^2 n^2 |\hat{f}(n)|^2 = 4\pi^2 \sum_{n\in\mathbb{Z}} n^2 |\hat{f}(n)|^2$$

Now, we notice that since $\int_0^1 f(x)dx = 0$, that evidently, $\hat{f}(0) = 0$, as $\hat{f}(0) = \int_0^1 f(x)\exp(-2\pi i0x)dx = \int_0^1 f(x)dx = 0$.

Because this is 0, and for all $n \in \mathbb{Z}, n \neq 0$, we have that $|\hat{f}(n)| \leq |n\hat{f}(n)|$, we can conclude that:

$$4\pi^2 \sum_{n\in\mathbb{Z}} n^2 |\hat{f}(n)|^2 \geq 4\pi^2 \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$$

Finally, applying Plancherel's equaity again for f, as $f \in L^1(\mathbb{T})$, we see that:

$$4\pi^2 \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = 4\pi^2 \|f\|_2^2 = 4\pi^2 \int_0^1 |f(x)|^2 dx$$

And rewriting all of these inequalities together, we have that:

$$4\pi^2 \int_0^1 |f(x)|^2 \leq \int_0^1 |f'(x)|^2 dx \implies \int_0^1 |f(x)|^2 \leq \frac{1}{4\pi^2} \int_0^1 |f'(x)|^2 dx$$

as desired.

Now, we return to proving 9.3.24.

First, we wish to show that if $f \in L^1(\mathbb{T})$, and $\hat{f} \in L^2(\mathbb{Z})$, then $f \in L^2(\mathbb{T})$.

Next, we want to show that the Plancherel equality holds, either both sides being finite or infinite. If $f \in L^2 \cap L^1$, then we're fine, by Corollary 9.3.14.

Then, we assume $f \notin L^2(\mathbb{T})$ but $f \in L^1(\mathbb{T})$. Then, by the previous result, we have that $\hat{f} \notin L^2(\mathbb{Z})$. Hence, we have that both sides of the Plancherel equality are infinite, and we are done.

$\square$

Question 12. Fix a $g \in L^2(\mathbb{R})$. Let $k \in \mathbb{Z}$, and define the operator T_k on g that sends $T_k(g(x)) \mapsto g(x-k)$. Prove that the family $\{T_k g\}_{k\in\mathbb{Z}}$ is an orthonormal sequence if and only if $\sum_{k\in\mathbb{Z}} |\hat{g}(\zeta - k)| = 1$ almost everywhere.

Solution. $\square$

Question 14. Let $p(x) = \chi_{[0,1)}(x)$, $h(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x)$. Let $j, k \in \mathbb{Z}$, and define $I_{jk} = [2^{-j}k, 2^{-j}(k+1))$. Further define the following functions:

$$\begin{cases} p_{jk} = 2^{j/2} p(2^j x - k) \\ h_{jk} = 2^{j/2} h(2^j x - k) \end{cases}$$

14.1)

Prove that $\{h_{jk}\}$ is an orthonormal sequence in L^2.

14.2)

For each fixed $j \in \mathbb{Z}$, prove that $\{p_{jk}\}_{k\in\mathbb{Z}}$ is an orthonormal sequence in L^2.

14.3)

Fix a $j \in \mathbb{Z}$. Let g_j be any step function, constant on each interval I_{jk} for $k \in \mathbb{Z}$. Show that we may express $g_j(x) = g_{j-1}(x) + r_{j-1}(x)$, where

$$r_{j-1}(x) = \sum_{k\in\mathbb{Z}} a_{j-1}(k) h_{j-1,k}(x)$$

for some coefficients $a_{j-1}(k)$ and some step function $g_{j-1}(x)$, constant on intervals $I_{j-1,k}$.

14.4)

Fix a $J \geq 0$. Consider the set:

$$\{p_{Jk} : 0 \leq k \leq 2^J - 1\} \cup \{h_{j,k} : j \geq J, 0 \leq k \leq 2^j - 1\}$$

Prove that this set is an orthonormal sequence in $L^2[0,1]$.

14.5)

For $f \in L^2[0,1]$, and a fixed $J \geq 0$, show that we may find g_j step functions for $j \geq J$, such that they are constant on each $I_{j,k}$, and that g_j approximates f in the L^2 norm.

Use this result and the result of 14.4 to show that the set in 14.4 is an orthonormal basis for $L^2[0,1]$.

Solution. 14.1)

Fix a $j \in \mathbb{Z}$, and consider the family of $\{h_{jk}\}$ iterating across k.

First, we wish to show that $\|h_{jk}\|_2 = 1$, regardless of the choice of j, k. Fix a choice of j, k. We see that, by the definition of h, that $h(2^j x - k)$ takes on 1 on $I_{j+1,2k}$, as we can see at the endpoints, $h(2^j 2^{-j-1} 2k - k) = h(k-k) = h(0) = 1$. Further, $I_{j+1,2k}$ has a length 2^{-j-1}, and we can see that if $h(x) = 1$ on $[0, 1/2)$, then $h(2^j x)$ takes on 1 on an interval of length 2^{-j-1}, from $[0, 2^{-j-1})$. Translating this interval over by k, we see that this is exactly $I_{j+1,2k}$. In a similar argument, we see that h takes on -1 on $I_{j+1,2k+1}$. Hence, we may rewrite h_{jk} as:

$$h_{jk} = 2^{j/2}(\chi_{I_{j+1,2k}} - \chi_{I_{j+1,2k+1}})$$

Thus, we can look at the L^2 norm of this function. We have that:

$$\|h_{jk}\|_2^2 = \int_{\mathbb{R}} |h_{jk}|^2 = \int_{I_{j+1,2k}} 2^j \chi_{I_{j+1,2k}} + \int_{I_{j+1,2k+1}} 2^j \chi_{I_{j+1,2k}} = 2^j(2^{-j-1} + 2^{-j-1}) = 1$$

where we've used the fact that the square of a characteristic function is itself, the measure of $I_{j+1,2k}$ is equal to 2^{-j-1}.

Now, we want to take $h_{jk}, h_{jk'}$, and look at $\langle h_{jk}, h_{jk'}\rangle$. Computing directly, and dropping the complex conjugate as this family is strictly real-valued, we see that:

$$\langle h_{jk}, h_{jk'}\rangle = \int_{\mathbb{R}} 2^{j/2}(\chi_{I_{j+1,2k}} - \chi_{I_{j+1,2k+1}})2^{j/2}(\chi_{I_{j+1,2k'}} - \chi_{I_{j+1,2k'+1}}) =$$

$$2^j \int_{\mathbb{R}} \chi_{I_{j+1,2k}}\chi_{I_{j+1,2k'}} - \chi_{I_{j+1,2k}}\chi_{I_{j+1,2k'+1}} - \chi_{I_{j+1,2k+1}}\chi_{I_{j+1,2k'}} + \chi_{I_{j+1,2k+1}}\chi_{I_{j+1,2k'+1}}$$

By definition though, $I_{j+1,2k} \cap I_{j+1,2k'} = [2^{-j-1}2k, 2^{-j-1}(2k+1)) \cap [2^{-j-1}k', 2^{-j-1}(2k'+1))$, which is non-empty if and only if $k = k'$, as j, k are integers, and so we have endpoints exactly at multiples of 2^{-j-1}. We may disregard this case, as this means $h_{jk} = h_{jk'}$, and we wish to look at distinct elements of this family. Hence, if we have distinct elements, we have that $\chi_{I_{j+1,2k}}\chi_{I_{j+1,2k'}}, \chi_{I_{j+1,2k+1}}\chi_{I_{j+1,2k'+1}}$ are identically 0.

Then, we need only look at the terms $-\chi_{I_{j+1,2k}}\chi_{I_{j+1,2k'+1}} - \chi_{I_{j+1,2k+1}}\chi_{I_{j+1,2k'}}$. However, in a similar fashion to the previous terms, these can be non-0 if and only if either $2k = 2k'+1$, or $2k' = 2k+1$. However, $k, k' \in \mathbb{Z}$, and hence there are no values of k, k' such that these are non-0.

Thus, if we have two distinct elements $h_{jk}, h_{jk'}$, we have that $\langle h_{jk}, h_{jk'}\rangle = 0$. Hence, the family $\{h_{jk}\}_{k\in\mathbb{Z}}$ for fixed j is a orthonormal sequence in L^2.

14.2)

In a similar fashion to 14.1, but maybe slightly cleaner, we do the same procedure. Fix a choice of $j \in \mathbb{Z}$. First, we reexpress p_{jk} in terms of characteristic functions. We see that, in analogy to 14.1, that

$$p_{jk} = 2^{j/2}\chi_{I_{jk}}$$

as we can see that $2^j x - k$ takes $2^{-j}k$ to 0 and $2^{-j}k + 1$ to 1, hence takes $[2^{-j}k, 2^{-j}k+1)$ to $[0,1)$ as this is linear in x.

Then, considering $\langle p_{jk}, p_{jk'}\rangle$, dropping the complex conjugate again, we see that:

$$\langle p_{jk}, p_{jk'}\rangle = \int_{\mathbb{R}} 2^{j/2}\chi_{I_{jk}} 2^{j/2}\chi_{I_{jk'}} = 2^j \int_{\mathbb{R}} \chi_{I_{jk}}\chi_{I_{jk'}}$$

By definition, we may look at $I_{jk} \cap I_{jk'}$. We have that:

$$I_{jk} \cap I_{jk'} = [2^{-j}k, 2^{-j}(k+1)) \cap [2^{-j}k', 2^{-j}(k'+1))$$

Since $j, k \in \mathbb{Z}$, these intervals have endpoints at multiple of 2^{-j}, and hence these have overlap if and only if $k = k'$. Hence, we can say that:

$$2^j \int_{\mathbb{R}} \chi_{I_{jk}}\chi_{I_{jk'}} = 2^j \int_{\mathbb{R}} \chi_{I_{jk}}\delta_k^{k'} = 2^j|2^{-j}(k+1) - 2^{-j}k|\delta_k^{k'} = 2^j 2^{-k}\delta_k^{k'} = \delta_k^{k'}$$

Therefore, $\{p_{jk}\}_{k\in\mathbb{Z}}$ is an orthonormal sequence in L^2 for fixed $j \in \mathbb{Z}$.

$\square$

Question 16. Let ϕ be a non-0 function in $L^2(\mathbb{R})$. For any $f \in L^2(\mathbb{R})$, define $V_\phi f$ via:

$$V_\phi(f)(x,\zeta) = \int_{\mathbb{R}} f(t)\overline{\phi(t-x)}\exp(-2\pi i t\zeta)dt$$

For $a, b \in \mathbb{R}$, let T_a be the translation operator that sends $T_a(f(x)) \mapsto f(x-a)$ and let M_b the modulation operator that sends $M_b(f(x)) \mapsto \exp(2\pi bx)f(x)$.

16.1)

Prove that for each $f \in L^2$, $V_\phi f$ is uniformly continuous on $\mathbb{R}^2$, and that $\lim_{|(x,\zeta)|\to\infty} V_\phi f = 0$.

16.2)

Recall that the Schwarz space $\mathcal{S}(\mathbb{R})$ is defined as:

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \in L^\infty(\mathbb{R}) \text{ for all } m, n \geq 0\}$$

Prove that if $f \in \mathcal{S}(\mathbb{R})$, then $V_\phi \in S(\mathbb{R}^2)$.

16.3)

Prove that V_ϕ acts as an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^2)$, and that $\|V_\phi f\|_{L^2(\mathbb{R}^2)} = \|\phi\|_{L^2(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}$ for every $f \in L^2$.

16.4)

Show that the operator V_ϕ^* defined by:

$$V_\phi^* F(t) - \|\phi\|_2^{-2} \iint_{\mathbb{R}^2} F(x,\zeta)\exp(2\pi i\zeta t)\phi(t-x)dxd\zeta$$

takes $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R})$, and that for each $f \in L^2(\mathbb{R})$, we can make sense of the following inversion formula:

$$f(t) = \|\phi\|_2^{-2} \iint_{\mathbb{R}^2} V_\phi f(x,\zeta)\exp(2\pi i\zeta t)\phi(t-x)dxd\zeta$$

Solution. □