Homework #3

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Math 235: Homework #3

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2.1

Problem 2.4.8. (a) Prove that continuity from below holds for exterior Lebesgue measure. That is, if $E_1 \subseteq E_2 \subseteq ...$ is any nested increasing sequence of subsets of $\mathbb{R}^d$, then $|\cup E_k|_e = \lim_{k\to\infty} |E_k|_e$.

(b) Show that there exists sets $E_1 \supseteq E_2 \supseteq ...$ in $\mathbb{R}$ such that $|E_k|_e < \infty$ for every k, and that:

$$|\cap_{k=1}^{\infty} E_k|_e < \lim_{k\to\infty} |E_k|_e$$

Solution. (a)

First, we note that due to the monotonicity of the outer measure, we have that $E_i \subseteq \cup_{k=1}^{\infty} E_i \implies |E_i|_e \leq |\cup_{k=1}^{\infty} E_i|_e$. Now, assume that $\lim k \to \infty |E_k|_e = \infty$. Then, due to monotonicity, we have that $|\cup_{k=1}^{\infty} E_i|_e = \infty$, as otherwise, if it were bounded, we could find a $|E_i|_e \geq |\cup_{k=1}^{\infty} E_i|_e$. More generally, due to monotonicity, we already will have that $\lim_{k\to\infty} |E_k|_e \leq |\cup E_k|_e$.

Now, suppose $\lim k \to \infty |E_k|_e$ is finite. Let $\epsilon > 0$ be given. For each E_i, we may find an open set $U_i \supseteq E_i$ such that $|E_i| \leq |U_i| \leq |E_i| + \epsilon$. Construct the related sequence of sets $V_k \cup_{i=k}^{\infty} U_k$. By construction, we have that these sets are nested $V_1 \subseteq V_2 \subseteq ....$ Consider the union over all k $\cup_k V_k$. By the construction of the U_i, since $E_i \subseteq E_j$ for $j > i$, then, $E_i \subseteq U_i$ for all $j > i$, and thus $E_i \subseteq V_i \subseteq U_i$ for all i, it follows that $\cup E_k \subseteq \cup V_k$. Then, we have that, via continuity from below, that $|\cup E_k| \leq |\cup V_k| = \lim_{k\to\infty} |V_k|$. But, from our construction, we also have that for each i, $|E_i| \leq |V_i| \leq |U_i| \leq |E_i| + \epsilon$, and thus $\lim_{k\to\infty} |V_k| \leq \lim_{k\to\infty} |E_k| + \epsilon$. Then, we have that:

$$|\cup E_k| \leq |\cup V_k| = \lim_{k\to\infty} |V_k| \leq \lim_{k\to\infty} |E_k| + \epsilon$$

Since ϵ can be taken to be arbitrarily small, this now implies that $|\cup E_k| = \lim_{k\to\infty} |E_k|$, as desired.

(b)

Take the set constructed in Heil for the proof of Theorem 2.4.5. That is, define the set M as such: Start with the interval $[0,1]$ and define the equivalence relation $x \sim y = \{x = y + q : q \in \mathbb{Q}\}$. Consider the equivalence classes of $[0,1]/\sim$. Construct M by applying the axiom of choice, and selecting one element from each equivalence class. Continue and construct the collection $\{M_k\}$ where we take $\{q_k\}$ as an enumeration of the rationals, and we define $M_k = (M + q_k)/[0,1]$, that is, modulo the interval $[0,1]$, so that each $M_k \subseteq [0,1]$. We notice that M is in our collection, because 0 is rational. Because equivalence classes partition a set, we are guaranteed that each M_k is disjoint, and that $\cup M_k = [0,1]$.

Now, consider the following sequence of sets. Define $E_1 = [0,1]$, and define $E_i = E_1 \setminus \cup_{k=2}^{i} M_{k-1}$ for $i \geq 2$, where we just assume the enumeration of the M_k starts at $k = 1$.

Here, we go off to the side and prove a result from Heil: 2.2.43(d). Define the inner Lebesgue measure of a set $A \subseteq \mathbb{R}^d$ to be $|A|_i = \sup\{|F| : F \text{ is closed and } F \subseteq A\}$. If E is Lebesgue measurable, and $A \subseteq E$, then $|E| = |A|_i + |E \setminus A|_e$. Because E is Lebesgue measurable, we may take a $U \supseteq E$ such that $|U \setminus E| < \epsilon$. Take any closed set $F \subseteq A$. We notice that $U \setminus F$ is open, because $U \cap F^c$ is an intersection of open sets. Moreover, it is a cover of $E \setminus A$ by construction. So, we have that $|U| = |E| + |U \setminus E| = |F| + |U \setminus F|$, where we have equality because $F, U \setminus F$ are measurable. Now, take any sequence of F_k such that $|A|_i leq |F_k| + 1/k$, which we may do because the inner measure is a supremum. Then, we note for each F_k, $(U \setminus F_k)$ is a sequence of sets such that these are open, and converge to $|U \setminus A|_e$. Then, we have that $|E| + |U \setminus E| = |A|_i + |U \setminus A|_e$. Now, since the choice of U is arbitrary, we can actually shrink U such that $|U \setminus E| \to 0$, and $|U \setminus A|_e \to |E \setminus A|_e$, because $|U \setminus A| \leq |E \setminus A| + |(U \setminus E) \setminus A| \leq |E \setminus A| + |U \setminus E|$, and $|U \setminus E| < \epsilon$. Then, we find that $|E| = |A|_i + |E \setminus A|_e$.

Now, consider the inner Lebesgue measure. Clearly, we have that it is translation invariant, as the Lebesgue measure of a closed set is translation invariant. Further, we also have monotonicity from the monotonicity of the Lebesgue measure, as well as subadditivity. (that is, suppose we have $A \cup B$, and $F_A \subseteq A, F_B \subseteq B$ with F_A, F_B closed. Then, $F_A \cup F_B$ is closed, and $|F_A \cup F_B| \leq |F_A| + |F_B|$ since they need not be disjoint. Since this is true for any F_A, F_B, this implies that $|F_A \cup F_B|_i \leq |F_A|_i + |F_B|_i$.)

Then, by the same argument that shows M as non-measurable in Heil, we can claim that because $[0,1] = \overline{[0,1]}$, the closure, that $[0,1]$ has inner measure 1, and that $|M|_i = 0$ because otherwise, we have a countable sum of inner measures of M as M_k are just translations.

Now, consider the outer measure of each E_i. $E_1 = [0,1]$. From what we proved about the inner measure, we have that $|[0,1]| = |\cup_{k=2}^{i} M_{k-1}|_i + |[0,1] \setminus \cup_{k=2}^{i} M_{k-1}|_e \implies |E_i|_e = |[0,1] \setminus \cup_{k=2}^{i} M_{k-1}|_e = |[0,1]| = 1$. So, we have a sequence of sets, with outer measure identically 1, so then we have that $\lim_{k\to\infty} |E_k|_e = 1$. However, we know that $\cap E_k = \emptyset$ because since the M_k partition $[0,1]$, for every $x \in [0,1]$, $x \in M_{k_0}$ for exactly one k_0. But then, by construction, this means that $x \in \cup_{k=2}^{i} M_{k-1}$ for $i > k_0$, so $x \notin E_i$ for any $i > k_0$. Since the choice of x was arbitrary, this is true for all x, and thus $\cap E_k = \emptyset \implies |\cap E_k|_e = 0 < \lim_{k\to\infty} |E_k|_e = 1$ □

Problem 2.4.10. Given any integer $d > 0$, show that there exists a set $N \subseteq \mathbb{R}^d$ that is not Lebesgue measurable.

Solution. We use the same construction and argument in Heil, and extend to multiple dimensions.

Fix a dimension d. Consider the rationals in the unit box $\Pi_{i=1}^{d}[0,1] \cap \mathbb{Q}^d$. We define the equivalence relationship $x \sim y \iff x - y \in \mathbb{Q}^d$. This is an equivalence relation because it is reflexive ($x - x = 0 \in \mathbb{Q}^d$), symmetric (if $x - y \in \mathbb{Q}^d$, then $-(x-y) = y - x \in \mathbb{Q}^d$ by being a ring) and transitive (if $x - y \in \mathbb{Q}^d$ and $y - z \in \mathbb{Q}^d$, then $x - z = x - y + y - z = (x-y) + (y-z) \in \mathbb{Q}^d$ due to $\mathbb{Q}^d$ being a ring). Then, the equivalence classes partition $[0,1]^d$ by virtue of being an equivalence relationship. Using the axiom of choice, construct a set M such that M has one representative from each (uncountably many) equivalence class.

Suppose M, and actually, all sets are measurable, under the Lebesgue measure μ, which we note to have the following properties for measurable sets:

(a) $\mu([0,1]^d) = 1$

(b) If $\{E_i\}$ is a countable collection of disjoint measurable subsets of $\mathbb{R}^d$, then $\mu(\cup E_i) = \Sigma \mu(E_i)$

(c) $\mu(E + h) = \mu(E)$ for every $E \subseteq \mathbb{R}^d$ and for any $h \in \mathbb{R}^d$.

Take an enumeration of $\mathbb{Q}^d \cap [-1,1]^d$, and call it $\{q_k\}$. This should exist because d is finite, countable, and $\mathbb{Q}$ is countable, so has cardinality of at most $\mathbb{N} \times \mathbb{N}$, which is countable. Consider the sets $M_k = M + q_k$. These sets must be disjoint, because, suppose not, that is $x \in M_i \cap M_j$. Then, $x = [x] + q_i = [x'] + q_j$, for some equivalence classes $[x], [x']$. But then, we have that $[x] = [x'] + (q_j - q_i)$, with $q_j - q_i$ rational. But, then $[x], [x']$ differ by a rational, they are the same equivalence class then, which implies $x = x'$ as we only pick one element from each equivalence class, which implies that $q_j = q_i$.

Consider the union of all such M_k, $\cup_{k=1}^{\infty} M_k$. This is a countable union of disjoint subsets of $\mathbb{R}^d$. Further, since $q_k \in [-1,1]^d$, we have that each $M_k \subseteq [-1,2]^d$. But also, because M contains one element from every equivalence relation, we hit with any rational in $\mathbb{Q}^d \cap [-1,1]^d$, and every element of $[0,1]^d$ belongs to some equivalence class, $[0,1]^d \subseteq \cup M_k$.

We notice by (a), that we have $\mu([0,1]^d) = 1$. By using the translations and countable additivity, we also have that $\mu([-1,2]^d) = 3^d$.

Then, using the monotonicity of the Lebesgue measure with our set inclusions, we have that:

$$1 = \mu([0,1]^d) \leq \mu(\cup_{k=1}^{\infty} M_k) \leq \mu([-1,2]^d) = 3^d$$

However, by the definition of M_k, (b), and (c), we have that:

$$\mu(\cup_{k=1}^{\infty} M_k) = \Sigma_{k=1}^{\infty} \mu(M_k) = \Sigma_{k=1}^{\infty} \mu(M)$$

Then, we have that $1 \leq \Sigma_{k=1}^{\infty} \mu(M) \leq 3^d$. But, μ can only take on values in $[0, \infty]$, and in particular then, $\Sigma_{k=1}^{\infty} \mu(M)$ is either 0 if $\mu(M) = 0$ and infinite otherwise. But that is a contradiction with our inequality.

Thus, M may not be a Lebesgue measurable set.

□

2.2

Problem 3.1.15. Let $E \subseteq \mathbb{R}^d$. Prove that $f : \to [-\infty, \infty]$ is measurable if and only if $\{f > r\}$ is measurable for every rational number r.

Solution. First, suppose f is measurable. Then, we have that $\{f > r\}$ is measurable for all $r \in \mathbb{R}$. In particular, since rational nubmers are real, this means that this is true for rational numbers as well.

Now, suppose $\{f > r\}$ is measurable for every rational number r. Let $x \in \mathbb{R} \setminus \mathbb{Q}$, that is, an irrational number. Take a sequence of points $\{x_k\}_k$ such that $x_k \in (x, x + 1/k)$, which we can do, since the rationals are dense. Clearly, the sequence converges to x. We claim that $\cup_k \{f > x_k\} = \{f > x\}$. First, suppose $y \in \cup_k \{f > x_k\}$. Then, for at least some x_{k_0}, $y \in \{f > x_{k_0}\}$. But, we have that $x_{k_0} > x$, by definition, so $f(y) \in (x_{k_0}, \infty] \subseteq (x, \infty]$, and $y \in \{f > x\}$, so $\cup_k \{f > x_k\} \subseteq \{f > x\}$. Now, suppose $y \in \{f > x\}$. Then, $f(y) \in (x, \infty]$. Consider the interval $(x, f(y))$. In particular, for some $1/k_1$, we must have that $(x, x + 1/k_1) \subseteq (x, f(y))$ as otherwise, we would need $f(y) \leq x$, a contradiction. Then, $f(y) \in \{f > x_{k_1}\}$, and thus $\{f > x\} \subseteq \cup_k \{f > x_k\}$. Then, $\cup_k \{f > x_k\} = \{f > x\}$. But, this is a countable union of measurable sets, so $\{f > x\}$ is measurable. Since the choice of x was arbitrary, this is true for every irrational, so for every real number, $\{f > x\}$ is measurable so f is measurable. □

Problem 3.1.16. Let E be a subset of $\mathbb{R}^d$. Prove that if $f : E \to [-\infty, \infty]$ is a measurable function, and $\{f = -\infty\}$ is a measurable set, then E is measurable.

Solution. First, we notice that we can write $[-\infty, \infty] = \cup_{N=1}^{\infty} (-N, \infty] \cup \{-\infty\}$. We see that by hypothesis, $f^{-1}(\{-\infty\})$ is measurable. Further, because f is measurable, we have that $f^{-1}((-N, \infty])$ is measurable. Then, since this is a countable union, we have that the whole thing is measurable. So, $f^{-1}([-\infty, \infty])$ is measurable. Since the entirety of the codomain is measurable, that means our domain is measurable. Hence, E is measurable.

□

Problem 3.1.18. (a) Prove that $f : \mathbb{R}^d \to \mathbb{R}$ is a measurable function if and only if $f^{-1}(U)$ is a measurable set for every open set $U \subseteq \mathbb{R}$.

(b) Prove that $f : \mathbb{R}^d \to \mathbb{C}$ is a measurable function if and only if $f^{-1}(U)$ is a measurable set for every open set $U \subseteq \mathbb{C}$.

Solution. (a)

Clearly, if $f^{-1}(U)$ is measurable for every open set, then f must be a measurable function because, let $x \in \mathbb{R}$. We may consider $\cup_{N=1}^{\infty} (x, x + N)$. Each of these are open sets, and this union is equal to (x, ∞), since for any $y \in (x, \infty)$, I can find an $N \in \mathbb{N} : N > y - x \implies x + N > y \implies y \in (x, x + N)$, and we clearly have that $(x, x + N) \subseteq (x, \infty)$ for every N. But, the inverse image of each $(x, x + N)$ is measurable, and taking the countable union over N, we have that $\{f > x\}$ is measurable. Since the choice of x was arbitrary, this works for any x, and hence f is measurable.

Now, suppose f is a measurable function. Since we are working in $\mathbb{R}$, we may express any open set $U \subseteq \mathbb{C}$ as the union of at most countably many bounded open intervals (a_k, b_k), that is, $U = \cup_{k=1}^{\infty}(a_k, b_k)$. However, we may rewrite $(a_k, b_k) = (-\infty, b_k) \cap (a_k, \infty)$. Since f is measurable, we know that $f^{-1}((a_k, \infty))$ is measurable, and by Lemma 3.1.5 in Heil, we know that $f^{-1}((-\infty, b_k))$ is measurable too, for each k. Then, since measurable sets are closed under both countable unions and intersections, we have that $f^{-1}((a_k, b_k)) = f^{-1}((-\infty, b_k) \cap (a_k, \infty))$ is measurable, and $f^{-1}(\cup_{k=1}^{\infty}(a_k, b_k)) = f^{-1}(U)$ is measurable. Since the choice of U was arbitrary, the inverse image for every open set is measurable.

(b)

First, suppose $f^{-1}(U)$ is measurable for every open set in $\mathbb{C}$. Fix a real number $a \in \mathbb{R}$, and consider the open sets $U_n = \{z \in \mathbb{C} : a < \Re(z) < a + n\}$ for $n > 0$, that is, such that the real part is between $a, a+n \in \mathbb{R}$. This is open because we identify this set with the following set in $\mathbb{R}^2$: $\cup_{m=1}^{\infty}\{(a, a+n) \times (-m, m)\}$, a countable union of open boxes, which is open. Consider the union of such sets, $\cup_n U_n$. In the same argument as part (a), this would be exactly the set $\{z \in \mathbb{C} : a < \Re(z)\}$. But, consider $f^{-1}(U_n)$. We know that this is measurable, but if we break into components $f = f_r + i f_i$, we have that $f^{-1}(U_n) = f_r^{-1}((a, a+n)) \cap f_i^{-1}(\mathbb{R})$, as it is exactly the points such that the image under f_r lands within $(a, a+n)$ and f_i is any imaginary part. But, $f_i^{-1}(\mathbb{R}) = \mathbb{R}^d$, so we have that $f^{-1}(U_n) = f_r^{-1}((a, a+n))$, and thus $f_r^{-1}((a, a+n))$ is measurable. Since each individual one is measurable, taking the union over U_n, we have that $f^{-1}(\cup_n U_n) = f_r^{-1}(\cup_{n=1}^{\infty}(a, a+n)) = f_r^{-1}((a, \infty])$ is measurable. Thus, f_r is measurable. Repeating the argument above for the open sets $V_n = \{z \in \mathbb{C} : a < \Im(z) < a + n\}$ and identifying them in $\mathbb{R}^2$ as $\cup_{m=1}^{\infty}\{(-m, m) \times (a, a+n)\}$, we see that f_i must be measurable. Then, f is measurable.

Now, suppose f is a measurable function. Then, for $f = f_r + i f_i$, for $f : \mathbb{R}^d \to \mathbb{R}$, we have that f_r and f_i are measurable. Let $U \subseteq \mathbb{C}$ be an open set. Identifying the open sets in $\mathbb{C}$ as the open sets in $\mathbb{R}^2$, we see from Lemma 2.1.5, that we may find countably many (non-overlapping) cubes such that $U = \cup Q_k = \cup[x_k, x_k'] \times [y_k, y_k']$. However, here, we notice that we may rewrite, for each k, $[x_k, x_k'] \times [y_k, y_k'] = \cup_{n=1}^{\infty}(x_k+1/n, x_k'-1/n) \times (y_k+1/n, y_k'-1/n)$. Now, from part (a), since f_r, f_i measurable, and $(x_k+1/n, x_k'-1/n), (y_k+1/n, y_k'-1/n)$ open sets for each n, then, the inverse image of those are measurable. Then, since we have that $f^{-1} = f_r^{-1} \cap f_i^{-1}$ on boxes, we have that the inverse image $f^{-1}([x_k, x_k'] \times [y_k, y_k'])$ is measurable. Finally, since each individual inverse image is measurable, we have that $f^{-1}(U) = f^{-1}(\cup[x_k, x_k'] \times [y_k, y_k']) = \cup f^{-1}([x_k, x_k'] \times [y_k, y_k'])$, a countable union of measurable sets, thus measurable. Since the choice of U was arbitrary, we have that $f^{-1}(U)$ is a measurable set for every open set $U \subseteq \mathbb{C}$.

□

Problem 3.1.19. Let $E \subseteq \mathbb{R}^d$ be a measurable set with $|E| > 0$, and assume that $f : E \to \overline{F}$ is a measurable function.

(a) Show that if f is finite almost everywhere, then there exists a measurable set $A \subseteq E$ such that $|A| > 0$ and f is bounded on A.

(b) Suppose that it is not the case $f = 0$ almost everywhere, that is, f is non-zero on a set of positive measure. Prove that there exists a measurable set $A \subseteq E$ and a number $\delta > 0$ such that $|A| > 0$ and $|f| \geq \delta$ on A.

Solution. (a)

Suppose such an A does not exist. Then, we have, for any $x \in \mathbb{R} \setminus \{0\}$, that the measure of $\{f > |x|\} \cup \{f < |x|\}$ must be equal to that of $|E|$, as otherwise, the complement is exactly what we're looking for, A. In particular, this must be true for every x. Then, take a sequence $x_k = N$ where $N = 1, 2....$ We would have that $|\{f > |x_k|\} \cup \{f < |x_k|\}| = |E|$ for all k, and thus f is infinity almost everywhere, a contradiction. Thus, such an A must exist. Note that we can run this same argument when $\overline{F} = \mathbb{C}$, simply by breaking things up into components, and considering the measure of $(\{f_r > |x|\} \cup \{f_r < |x|\}) \cap (\{f_i > |x|\} \cup \{f_i < |x|\})$.

(b)

First, we notice we may rewrite $E = A \cup B$, where $A = \{x \in E : f(x) = 0\}$ and $B = E \setminus A$, $|B| > 0$. Suppose that, contrary to hypothesis, there does not exist a C, δ such that $|C| > 0$ and $|f| \geq \delta$ on C. In a similar fashion to part (a), for any real number $a \in \mathbb{R} \setminus \{0\}$, and consider the set $\{f < -a\} \cup \{f > a\}$. This is measurable since it is the union of measurable sets. Further, take a sequence of rational numbers to 0, for example, take the sets $\{f < -1/k\} \cup \{f > 1/k\}$ for $k \geq 1$. Then, we have that $B = \cup_{k=1}^{\infty}(\{f < -1/k\} \cup \{f > 1/k\})$, so B is measurable, and $|B| \leq |\cup_{k=1}^{\infty}(\{f < -1/k\} \cup \{f > 1/k\})|$. By our hypothesis, since no C exists with positive measure, then $|(\{f < -1/k\} \cup \{f > 1/k\})| = 0$ for all k. Then, we have that $|B| = 0$. But this is a contradiction. Then, that means, for some k_0, we have that $|(\{f < -1/k_0\} \cup \{f > 1/k_0\})| > 0$, and we may take that to be our C, and $\delta = 1/k$. We may run the same argument for $F = \mathbb{C}$, where we need just break things up into components again, and run over $(\{f_r < -1/k\} \cup \{f_r > 1/k\}) \cap (\{f_i < -1/k\} \cup \{f_i > 1/k\})$. $\square$