Homework #1

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Math 233: Homework #1

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Question 1. The following fact was tacitly used in this chapter: if A, B are disjoint subsets of the plane, A is compact, B is closed, then there exists a $\delta > 0$ such that, for all $\alpha \in A$, $\beta \in B$, $|\alpha - \beta| \geq \delta > 0$. Prove this for $A, B \subset X$ for X an arbitrary metric space.

Solution. Let X be a metric space, $A \subseteq X$ compact, $B \subseteq X$ closed, $A \cap B = \emptyset$

Suppose not. Then, there exist pairs of points (α_n, β_n) such that $d(\alpha_n, \beta_n) < \frac{1}{n}$. Now, consider the sequence of points $\{\alpha_n\}_{n=1}^{\infty}$. Since A is compact, we know that there exists a subsequence $\{\alpha_{n_k}\}_{k=1}^{\infty}$, convergent to α.

Let $\epsilon > 0$ be given. Since $\alpha_{n_k} \to \alpha$, choose N_k such that $d(\alpha, \alpha_{n_k}) < \frac{\epsilon}{2}$ for all $n_k > N_k$. Choose N such that $\frac{1}{n} < \frac{\epsilon}{2}$ for all $n > N$. Choose M_k such that $M = \max(N, N_k)$. Assume $m > M, m \in \{n_k\}_{k=1}^{\infty}$. Consider the sequence of $\{\beta_{n_k}\}_{k=1}^{\infty}$, and in particular, consider:

$$d(\alpha, \beta_m) \leq d(\alpha, \alpha_m) + d(\alpha_m, \beta_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, we have that $\beta_{n_k} \to \alpha$. Since $\{\beta_{n_k}\}_{k=1}^{\infty} \subset B$, a closed set, $\alpha \subset B$, because closed sets contain its limit points. But, this is a contradiction. Thus, $\delta > 0$ exists. □

Question 3. Suppose f, g are entire functions, and suppose that for all $z \in \mathbb{C}$, that $|f(z)| \leq |g(z)|$. What conclusion can you draw?

Solution. Claim: for some $m \in \mathbb{C}$, $f = mg$.

First suppose $g = 0$. Then, since $|f| \leq |g| = 0$, this implies that $f = 0$ everywhere. Then, of course $f = mg$, for actually any m.

Now, suppose not. Then, define $Z(g) = \{z \in \mathbb{C} : g(z) = 0\}$, that is, the zero set of g, and consider the function $h = \frac{f}{g}$. By the algebra of holomorphic functions, we have that h is holomorphic on at least $\mathbb{C} \setminus Z(g)$.

Because $\mathbb{C}$ is of course a connected open set, we have the result that $Z(g)$ has no limit points in $\mathbb{C}$. Then, let $a \in Z(g)$. Because a is not a limit point, there exists $r > 0$ such that $D(a, r) \cap Z(g) = \emptyset$. We have then that h is holomorphic on $D(a, r) \setminus \{a\}$, a region. Further, on $\mathbb{C} \setminus Z(g)$, we have that $|h| = \frac{|f|}{|g|} \leq 1$. So, in particular, on $D'(a, \frac{r}{2}) = \{z \in \mathbb{C} : 0 < |z - a| < \frac{r}{2}\} \subseteq \mathbb{C} \setminus Z(g)$, we have that h is bounded. Then, by Theorem 10.20 from Rudin, we have that f has a removable singularity at a.

Now, we recall from Theorem 10.18, that $Z(g)$ is at most countable. So, we may patch h countably many times at each point in $Z(g)$ to produce a holomorphic function everywhere, which we call $\tilde{h}$. Further, since $\tilde{h}$ is holomorphic, it must be continuous everywhere. Thus, since $|\tilde{h}(z)| \leq 1$ at every point other than $z \in Z(g)$, we must have that $|\tilde{h}(z)| \leq 1$ everywhere by continuity. Thus, we have that $\tilde{h}$ is a bounded, entire function, and by Liouville's Theorem, it must be constant, that is, $\tilde{h} = k$ for some $k \in \mathbb{C}$. Then, we have that at least on $\mathbb{C} \setminus Z(g)$, that $f(z) = kg(z)$.

However, $kg(z)$ is certainly holomorphic, and it agrees with $f(z)$ almost everywhere, which of course is a set with limit points in Ω. Thus, $f = kg$ everywhere.

□

Question 4. Suppose that f is an entire function, and

$$|f(z)| \leq A + B|z|^k$$

for all z, where A, B, k are positive real numbers. Prove that f must be polynomial.

Solution. Because f is entire, it is analytic, specifcally at $a = 0$, with infinite radius of convergence. Then, we may rewrite f as:

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

Now, we apply Theorem 10.22. We have that:

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta$$

Here, we use our hypothesis. Since we have that $|f(z)| \leq A + B|z|^k$, we must have that:

$$|f(re^{i\theta})| \leq A + B|re^{i\theta}|^k = A + Br^k$$

Thus, using our first equation then, we have a bound:

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} (A + Br^k)^2 d\theta = (A + Br^k)^2$$

Now, suppose we have that $c_n \neq 0$ for some $n > k$. Then, we would have that:

$$\frac{|c_n| r^{2n}}{(A + Br^k)^2} = \frac{|c_n| r^{2(n-k)}}{(\frac{A}{r^k} + B)^2}$$

Now, since f is entire and thus the radius of convergence is infinite, we may take the limit as $r \to \infty$. But, since $n > k$, we have that:

$$\lim_{r \to \infty} \frac{|c_n| r^{2(n-k)}}{(\frac{A}{r^k} + B)^2} = \infty$$

Then, $c_n = 0$ for every $n > k$. Then, this implies that we have that

$$f(z) = \sum_{n=0}^{\lfloor k \rfloor} c_n z^n$$

and since this holds everywhere, with finite degree, f is polynomial.

□

Question 6. There is a region Ω such that $\exp(\Omega) = D(1, 1)$. Show that the exponential function is one-to-one on Ω, but that there are many such Ω. Fix one, and define $\log(z)$, for $|z - 1| < 1$ to be $w \in \Omega$ such that $e^w = z$. Prove that $\log'(z) = \frac{1}{z}$. Further, find the coefficients a_n in

$$\frac{1}{z} = \sum_{n=0}^{\infty} a_n (z - 1)^n$$

and hence, find the coefficients c_n in the expansion

$$\log z = \sum_{n=0}^{\infty} c_n (z-1)^n$$

.

In what other discs can this be done?

Solution. First, we find the shape of one such Ω. We notice that $D(1,1) = \{z : |z-1| < 1\}$. Thus, we would have that, for $x, y \in \mathbb{R}, z = x + yi$:

$$|e^z - 1| < 1 \implies |e^x(\cos(y) + i\sin(y)) - 1| < 1 \implies |e^x \cos(y) - 1 + ie^x \sin y| < 1 \implies \sqrt{e^{2x} - 2e^x \cos(y) + 1} < 1$$

However, we make one more observation, that first of all e^z is cyclic in the imaginary component y, with a period of 2π. Further, we have that in terms of the radial component, $D(1,1)$ is completely contained within $(-\pi/2, \pi/2)$, and that other regions may be found, but they are separated by integer multiples of 2π and therefore disconnected from this one. So, then, we have that we may describe our region as $\Omega = \{x + yi : \sqrt{e^{2x} - 2e^x \cos(y) + 1} < 1 . y \in (-\pi/2, \pi/2)\}$. We notice that this must be 1:1 because the exponential $e^z = e^x(\cos(y) + i\sin(y))$ must be 1:1 on $y \in (-\pi/2, \pi/2)$:

$$e^x(\cos(y) + i\sin(y)) = e^{x'}(\cos(y') + i\sin(y')) \implies \begin{cases} e^x \cos(y) = e^{x'} \cos(y') \\ e^x \sin(y) = e^{x'} \sin(y') \end{cases}$$

$$\implies \begin{cases} e^{2x} \cos^2(y) = e^{2x'} \cos^2(y') \\ e^{2x} \sin^2(y) = e^{2x'} \sin^2(y') \end{cases} \implies e^{2x} = e^{2x}(\cos^2(y) + \sin^2(y)) = e^{2x'}(\cos^2(y') + \sin^2(y')) = e^{2x'}$$

Thus, we have that $x = x'$. Now, we note that on $(-\pi/2, \pi/2)$, sin is 1:1, so therefore

$$e^x \sin(y) = e^{x'} \sin(y') \implies \sin(y) = \sin(y') \implies y = y'$$

Thus, we have that e^z is one-to-one on this region.

As noted earlier though, we notice that we can find another region easily - $\Omega' = \{x + yi : \sqrt{e^{2x} - 2e^x \cos(y) + 1} < 1 . y \in (3\pi/2, 5\pi/2)\}$ is certainly another valid region, and there are actually infinitely many, separated by $2\pi n, n \in \mathbb{N}$.

Now, choose an arbitrary one of these Ω. Define $\log z = w \in \Omega$ such that $e^w = z$, where $z \in D(1,1)$. It should be clear that because the exponential in injective on Ω, log must be injective on $D(1,1)$.

Fix some $z_0 \in D(1,1)$. Since the exponential is one-to-one on Ω, this corresponds to $\log(z_0) = w_0$. Then, for arbitrary $w \in \Omega$ and $z \in D(1,1)$, we have that:

$$\frac{\log(z) - \log(z_0)}{z - z_0} = \frac{w - w_0}{e^w - e^{w_0}}$$

just by the injective nature of these functions and using the fact that $z = e^w, w = \log(z)$.

Then, by the continuity of the exponential, we have that as $w \to w_0$, that $z \to z_0$. Thus, we have that, by taking the limit of both sides as $w \to w_0, z \to z_0$:

$$\frac{\log(z) - \log(z_0)}{z - z_0} = \frac{w - w_0}{e^w - e^{w_0}} \implies \lim_{z \to z_0} \frac{\log(z) - \log(z_0)}{z - z_0} = \lim_{w \to w_0} \frac{w - w_0}{e^w - e^{w_0}} \implies \log'(z_0) = \frac{1}{e^{w_0}}$$

But, by our definition of log, e^{w_0} is exactly z_0. So $\log'(z_0) = \frac{1}{z_0}$ as desired.

Well, now we notice that $\frac{1}{z}$ is holomorphic on regions that exclude the origin, thus we can use the corollary to 10.6 to compute coefficients to our power series. Since

$$\frac{d^n}{dz^n}\frac{1}{z} = \frac{(-1)^n n!}{z^{n+1}}$$

we have that

$$n!c_n = f^{(n)}(1) \implies c_n = \frac{1}{n!}\frac{(-1)^n n!}{1^{n+1}} = -1^n$$

So, we have that

$$\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

Doing term by term integration, then, we have that the n-th term becomes:

$$\int (-1)^n(z-1)^n dz = \frac{(-1)^n}{n+1}(z-1)^{n+1}$$

up to a constant. So, then, we have that:

$$\log(z) = c_0 + \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(z-1)^{n+1}$$

where c_0 is some constant connected to the choice of Ω.

We remark that this procedure is not special to the disk $D(1,1)$, but rather, is permissable on any disk that does not include the origin, as if it does, there is no z such that $e^z = 0$. In such a case, the inverse function defined on a region missing the origin would have a pole at 0, and discs that include 0 may come from a single region, and may be hard to restrict to a disconnected domain. For example, the disk $D(0,1)$ has, as a preimage under the exponential, $\{x + yi : x \leq 0\}$, which has a many to 1 relation with $D(0,1)$.

□

Question 7. Let $f \in \mathcal{H}(\Omega)$. Under certain conditions on z, Γ, we have that:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta$$

for $n \in \mathbb{N}$. State these, and prove the formula.

Solution. Before we start, we will prove a necessary result. We will prove that for a closed cycle Γ, an open set Ω, and $z \in \Omega \setminus \Gamma^*$, that $\int_\Gamma (z-\zeta)^{-m}d\zeta = 0$ for $m > 1$.

Well, since $z \notin \Gamma^*$, we have that $(z-\zeta)^{-m}$ is continuous, and thus integrable on Γ^*. In particular, since we have that $m > 1$, it has exactly anti-derivative $F(\zeta) = \frac{(z-\zeta)^{-m+1}}{-m+1}$. Then, if $\Gamma = \gamma_1 + \gamma_2 + ... + \gamma_n$, and if the endpoints of γ_i are α_i, β_i, we can rewrite this as:

$$\int_\Gamma (z-\zeta)^{-m}d\zeta = \sum_{i=1}^{n}\int_{\gamma_i}(z-\zeta)^{-m}d\zeta = \sum_{i=1}^{n}F(\beta_i) - F(\alpha_i) = 0$$

because since Γ is a closed cycle, we must have that $\beta_i = \alpha_{i+1}$ with the understanding that $\beta_n = \alpha_1$. So, this is a telescoping sum and vanishes, at least when $m > 1$. Note that although we may prove this for really, more generally, $m \neq 1$ in the same manner, this is all we need here.

We will need that $z \in \Omega \setminus \Gamma^*$, so that we can take a contour integral over Γ, as well as $\mathrm{Ind}_\Gamma(z) = 1$ and $\mathrm{Ind}_\Gamma(\alpha) = 0$ when $\alpha \notin \Omega$ for use in Cauchy's theorem.

First, fix some $z \in \Omega \setminus \Gamma^*$, and choose some $n \in \mathbb{N}$. Define the related function $P(\zeta)$ on Ω via:

$$P(\zeta) = f(z) + f'(z)(\zeta - z) + ... + \frac{f^{(n-1)}(z)}{(n-1)!}(\zeta - z)^{n-1}$$

First, we show that $f(\zeta) - P(\zeta) = (\zeta - n)^n h(\zeta)$:

Since f is holomorphic and thus analytic, we can write a power series for f around z:

$$f(\zeta) = f(z) + f'(z)(\zeta - z) + ... = \sum_{n=0}^{\infty} \frac{f^{(i)}(z)}{i!}(\zeta - z)^i$$

where the coefficients come from the corollary to theorem 10.6.

Then, we can compute:

$$f(\zeta) - P(\zeta) = \sum_{i=0}^{\infty} \frac{f^{(i)}(z)}{i!}(\zeta - z)^i - \sum_{i=0}^{n} \frac{f^{(i)}(z)}{i!}(\zeta - z)^i = \sum_{i=n}^{\infty} \frac{f^{(i)}(z)}{i!}(\zeta - z)^i = (\zeta - z)^n \sum_{i=n}^{\infty} \frac{f^{(i)}(z)}{i!}(\zeta - z)^{i-n}$$

Identifying $h(\zeta) = \sum_{i=n}^{\infty} \frac{f^{(i)}(z)}{i!}(\zeta - z)^{i-n}$, we notice that:

$$h(z) = \sum_{i=n}^{\infty} \frac{f^{(i)}(z)}{i!}(z - z)^{i-n} = \frac{f^{(n)}(z)}{n!}$$

.

Next, we claim that:

$$\int_{\Gamma} \frac{P(\zeta)}{(\zeta - z)^{n+1}} d\zeta = 0$$

.

Well:

$$\int_{\Gamma} \frac{P(\zeta)}{(\zeta - z)^{n+1}} d\zeta = \int_{\Gamma} \sum_{i=0}^{n-1} \frac{f^{(i)}(z)(\zeta - z)^i}{i!(\zeta - z)^{n+1}} d\zeta = \sum_{i=0}^{n-1} \int_{\Gamma} \frac{f^{(i)}(z)}{i!(\zeta - z)^{n+1-i}} d\zeta = \sum_{i=0}^{n-1} \frac{f^{(i)}(z)}{i!} \int_{\Gamma} (\zeta - z)^{i-n-1} d\zeta$$

Since for all $0 \leq i \leq n-1$, we have that $-n-1 \leq i-n-1 \leq -2$ and since $z \notin \Gamma^*$, we apply the lemma we proved at the beginning to show that $\int_{\Gamma} (\zeta - z)^{i-n-1} d\zeta = 0$ for all i. Thus, the entire integral vanishes, and we have that:

$$\int_{\Gamma} \frac{P(\zeta)}{(\zeta - z)^{n+1}} d\zeta = 0$$

Now, consider the following quantity:

$$\int_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta = \int_{\Gamma} \frac{P(\zeta) + (\zeta - n)^n h(\zeta)}{(\zeta - z)^{n+1}} d\zeta = \int_{\Gamma} \frac{P(\zeta))}{(\zeta - z)^{n+1}} d\zeta + \int_{\Gamma} \frac{h(\zeta)(\zeta - n)^n}{(\zeta - z)^{n+1}} d\zeta = \int_{\Gamma} \frac{h(\zeta)}{(\zeta - z)} d\zeta$$

where we applied the facts that $\int_{\Gamma} \frac{P(\zeta)}{(\zeta - z)^{n+1}} d\zeta = 0$ and $f(\zeta) - P(\zeta) = (\zeta - n)^n h(\zeta) \implies f(\zeta) = P(\zeta) + (\zeta - n)^n h(\zeta)$.

Now, since h is analytic, it must be holomorphic on Ω. Further, due to our conditions on z and Γ, we may apply Cauchy's theorem to claim that:

$$h(z) * \mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{h(\zeta)}{\zeta - z} = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

Now, we use the condition that $\mathrm{Ind}_\Gamma(z) = 1$, and the result that $h(z) = \frac{f^{(n)}(z)}{n!}$ to get that:

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \implies f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}$$

Since the choice of n was arbitrary, we can use this procedure for any natural number n, the desired result.

$\square$