Homework #2

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Math 285: Homework #2

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Question 1. Let ω be the 1-form $zdx - dz$ and let X be the vector field $y\partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^3$.
Compute $\omega(X)$ and $d\omega$.

Solution. We recall that $\omega(X)$, in coordinates, is simply $\sum_i a_i b^i$, where $\omega = \sum_i a_i dx^i$, and $X = \sum_j b^j \frac{\partial}{\partial x^j}$
Thus, we have that:

$$\omega(X) = \sum_i a_i b^i = z * y + 0 * x + -1 * 0 = yz$$

In a similar fashion, recall that, by definition:

$$d\omega = \sum_{i,j} \frac{\partial a_i}{\partial x^j} dx^j \wedge dx^i$$

Thus, we have that:

$$d\omega = 1 dz \wedge dx = dz \wedge dx$$

since we notice that the only non-vanishing partial of zdx is $\partial/\partial z$ and none of the partials of $-dz$ survive.
$\square$

Question 2. Suppose the standard coordinates on $\mathbb{R}^3$ are called ρ, ϕ, θ. If we have that:

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho\cos\phi \end{cases}$$

Compute the following quantities in terms of $d\rho, d\phi, d\theta$: $dx, dy, dz, dx \wedge dy \wedge dz$.

Solution. Clearly, x, y, z are C^∞ functions on $\mathbb{R}$. Applying Proposition 4.3, which states that $df = \sum \partial f/\partial x^i dx^i$, we see that:

$$\begin{cases} dx = \sin\phi\cos\theta d\rho + \rho\cos\phi\cos\theta d\phi - \rho\sin\phi\sin\theta d\theta \\ dy = \sin\phi\sin\theta d\rho + \rho\cos\phi\sin\theta d\phi + \rho\sin\phi\cos\theta d\theta \\ dz = \cos\phi d\rho - \rho\sin\phi d\phi \end{cases}$$

Now, we may compute the wedge product $dx \wedge dy \wedge dz$. We recall that odd degree multivectors vanish under the wedge product and that the wedge product distributes over addition, and since our 1-forms are exactly covector fields, hence covectors at each point, we need only consider elements that include some permutation of $d\rho \wedge d\phi \wedge d\theta$:

$$dx\wedge dy\wedge dz = (\sin\phi\cos\theta d\rho)\wedge(\rho\sin\phi\cos\theta d\theta)\wedge(-\rho\sin\phi d\phi)+(\rho\cos\phi\cos\theta d\phi)\wedge(\rho\sin\phi\cos\theta d\theta)\wedge(\cos\phi d\rho)+$$

$$(-\rho\sin\phi\sin\theta d\theta)\wedge(\sin\phi\sin\theta d\rho)\wedge(-\rho\sin\phi d\phi)+(-\rho\sin\phi\sin\theta d\theta)\wedge(\rho\cos\phi\sin\theta d\phi)\wedge(\cos\phi d\rho) =$$

$$-\rho^2\sin^3\phi\cos^2\theta(d\rho\wedge d\theta\wedge d\phi)+\rho^2\sin\phi\cos^2\phi\cos^2\theta(d\phi\wedge d\theta\wedge d\rho)+$$
$$\rho^2\sin^3\phi\sin^2\theta(d\theta\wedge d\rho\wedge d\phi)-\rho^2\sin\phi\cos^2\phi\sin^2\theta(d\theta\wedge d\phi\wedge d\rho)$$

Rewriting everything to be of the form $d\rho\wedge d\phi\wedge d\theta$ using graded commutativity, and pulling out ρ^2:

$$\rho^2(\sin^3\phi\cos^2\theta+\sin\phi\cos^2\phi\cos^2\theta+\sin^3\phi\sin^2\theta+\sin\phi\cos^2\phi\sin^2\theta)(d\rho\wedge d\phi\wedge d\theta)$$

Looking at the first two and last two terms, we notice that:

$$\begin{cases}\sin^3\phi\cos^2\theta+\sin\phi\cos^2\phi\cos^2\theta=\sin\phi\cos^2\theta(\sin^2\phi+\cos^2\phi)=\sin\phi\cos^2\theta\\ \sin^3\phi\sin^2\theta+\sin\phi\cos^2\phi\sin^2\theta=\sin\phi\sin^2\theta(\sin^2\phi+\cos^2\phi)=\sin\phi\sin^2\theta\end{cases}$$

So, in the end, we have that:

$$dx\wedge dy\wedge dz=\rho^2(\sin\phi\cos^2\theta+\sin\phi\sin^2\theta)(d\rho\wedge d\phi\wedge d\theta)=\rho^2\sin\phi(d\rho\wedge d\phi\wedge d\theta)$$

□

Question 3. Let V be a vector space of dimension 3 with basis e_1, e_2, e_3 and dual basis $\alpha^1, \alpha^2, \alpha^3$. For a 1-covector $\alpha=\sum_{i=1}^3 a_i\alpha^i$ on V, we associate the vector $v_\alpha=\langle a_1,a_2,a_3\rangle\in\mathbb{R}^3$. For a 2-covector

$$\gamma=c_1\alpha^2\wedge\alpha^3+c_2\alpha^3\wedge\alpha^1+c_3\alpha^1\wedge\alpha^2$$

we assoicate the vector $v_\gamma=\langle c_1,c_2,c_3\rangle\in\mathbb{R}^3$.

Show that under this correspondence, the wedge product of 1-covectors corresponds to the cross product of vectors in $\mathbb{R}^3$, that is:

$$v_{\alpha\wedge\beta}=v_\alpha\times v_\beta$$

Solution. Recall that if we have identifications of 1-covectors: $\alpha=\sum_i a_i dx^i$ and $\beta=\sum_j b_j dx^j$, then we have that:

$$\alpha\wedge\beta=\sum_{i,j}(a_ib_j)dx^i\wedge dx^j$$

Writing this out in terms of coordinates, with respect to the dual basis, i.e. $dx^i=\alpha^i$, we have that:

$$\alpha\wedge\beta=a_1b_2\alpha^1\wedge\alpha^2+a_1b_3\alpha^1\wedge\alpha^3+a_2b_1\alpha^2\wedge\alpha^1+a_2b_3\alpha^2\wedge\alpha^3+a_3b_1\alpha^3\wedge\alpha^1+a_3b_2\alpha^3\wedge\alpha^2=$$

$$(a_2b_3-a_3b_2)\alpha^2\wedge\alpha^3+(a_3b_1-a_1b_3)\alpha^3\wedge\alpha^1+(a_1b_2-b_1a_2)\alpha^1\wedge\alpha^2$$

where we've used the fact that since α^i are covectors, $\alpha^i\wedge\alpha^i=0$.

So, we have that:

$$v_{\alpha\wedge\beta} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - b_1a_2\rangle$$

In contrast, let's consider the cross product of $v_\alpha \times v_\beta = \langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle$. Using matrix notation:

$$v_\alpha \times v_\beta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k =$$

$$\langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

We notice these are the same, and we conclude that $v_{\alpha\wedge\beta} = v_\alpha \times v_\beta$.

□

Question 4. Let $A = \oplus_{k=-\infty}^{\infty} A^k$ be a graded algebra over a field K, with $A^k = 0$ for $k < 0$. Let $m \in \mathbb{Z}$.

Define a superderviation of A with degree m as a K-linear map $D : A \to A$ such that for all $k \in \mathbb{Z}$, we have that $D(A^k) \subset A^{k+m}$ and that for all $a \in A^k, b \in A^l$:

$$D(ab) = (Da)b + (-1)^{km} aDb$$

Let D_1, D_2 be superderivations of A with degrees m_1, m_2 respectively. Define their commutator as:

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{m_1m_2} D_2 \circ D_1$$

Show that the commutator $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$.

Solution. Fix a $k \in \mathbb{Z}$, and suppose $x \in A^k$.

First, we wish to show that $[D_1, D_2](x) \in A^{k+m_1+m_2}$.

It is enough to show that $D_1 \circ D_2(x), D_2 \circ D_1(x) \in A^{k+m_1+m_2}$, because if that is true, then the sum and multiplication by scalars remains in $A^{k+m_1+m_2}$ due to it being an algebra.

Well, because D_1, D_2 are superderivations, we have that $D_2(x) \in A^{k+m_2}$, so $D_1(D_2(x)) \in A^{k+m_2+m_1}$. Similarly, $D_1(x) \in A^{k+m_1}$, so $D_2(D_1(x)) \in A^{k+m_1+m_2}$ for the same reason.

Since algebras are closed under addition and scalar multiplication, this implies that $D_1 \circ D_2 - (-1)^{m_1m_2} D_2 \circ D_1(x) \in A^{k+m_1+m_2}$.

Now, we need to check the condition on products. Fix $k, l \in \mathbb{Z}$, and suppose that $x \in A^k, y \in A^l$.

Consider $[D_1, D_2](xy)$. We have that:

$$[D_1, D_2](xy) = D_1 \circ D_2 - (-1)^{m_1m_2} D_2 \circ D_1(xy) = D_1(D_2(xy)) - (-1)^{m_1m_2} D_2(D_1(xy))$$

Because D_1, D_2 are superderivations, we have that:

$$D_i(xy) = (D_i(x))y + (-1)^{km_i} x(D_iy)$$

Therefore:

$$[D_1, D_2](xy) = D_1((D_2(x))y + (-1)^{km_2} x(D_2y)) - (-1)^{m_1m_2} D_2((D_1(x))y + (-1)^{km_1} x(D_1y)) =$$

$$D_1(D_2(x)y) + (-1)^{km_2} D_1(xD_2(y)) - (-1)^{m_1m_2}[D_2(D_1(x)y) + (-1)^{km_1} D_2(xD_1(y))]$$

Recalling that $D_2(x) \in A^{k+m_2}, D_1(x) \in A^{k+m_1}$, applying the fact that D_1, D_2 are superderivations again, we reduce to:

$$D_1 \circ D_2(x)y + (-1)^{(k+m_2)m_1} D_2(x)D_1(y) + (-1)^{km_2}[D_1(x)D_2(y) + (-1)^{km_1} xD_1 \circ D_2(y)] -$$

$$(-1)^{m_1m_2}[D_2 \circ D_1(x)y + (-1)^{(k+m_1)m_2}D_1(x)D_2(y) + (-1)^{km_1}[D_2(x)D_1(y) + (-1)^{km_2}xD_2 \circ D_1(y)]]$$

First, we look at terms of the form $D_2(x)D_1(y)$. These are:

$$(-1)^{(k+m_2)m_1}D_2(x)D_1(y) - (-1)^{m_1m_2}[(-1)^{km_1}D_2(x)D_1(y)] = (-1)^{km_1+m_1m_2}[D_2(x)D_1(y) - D_2(x)D_1(y)] = 0$$

Similarly, for $D_1(x)D_2(y)$, since $(-1)^{2p} = 1$ for $p \in \mathbb{Z}$:

$$(-1)^{km_2}D_1(x)D_2(y) - (-1)^{m_1m_2}(-1)^{(k+m_1)m_2}D_1(x)D_2(y) = (-1)^{km_2}[D_1(x)D_2(y) - D_1(x)D_2(y)] = 0$$

For terms with a y:

$$D_1 \circ D_2(x)y - (-1)^{m_1m_2}D_2 \circ D_1(x)y = [D_1, D_2](x)y$$

And similarly, for terms with an x:

$$(-1)^{km_2}(-1)^{km_1}xD_1 \circ D_2(y) - (-1)^{m_1m_2}(-1)^{km_1}(-1)^{km_2}xD_2 \circ D_1(y) =$$

$$(-1)^{k(m_1+m_2)}[xD_1 \circ D_2(y) - (-1)^{m_1m_2}xD_2 \circ D_1(y)] = (-1)^{k(m_1+m_2)}x[D_1, D_2](y)$$

Thus, this horrible mess shows us that:

$$[D_1, D_2](xy) = [D_1, D_2](x)y + (-1)^{k(m_1+m_2)}x[D_1, D_2](y)$$

Therefore, $[D_1, D_2]$ satsifies the conditions to be a superderivation of degree $m_1 + m_2$.

□

Question 5. Consider the set $S = \mathbb{R} \setminus \{0\} \cup \{A, B\}$, the bug-eyed line or the line with two origins.

For $c, d \in \mathbb{R}$, define the following notation:

$$\begin{cases} I_A(-c, d) = (-c, 0) \cup \{A\} \cup (0, d) \\ I_B(-c, d) = (-c, 0) \cup \{B\} \cup (0, d) \end{cases}$$

Define a topology on S as follows: On $\mathbb{R} \setminus \{0\}$, use the subspace topology from $\mathbb{R}$ with open intervals as a basis. At the point A, use the collection of sets $\{I_A(-c, d) : c, d > 0\}$ as a basis, and analogously at B.

(a) Prove that the map $h : I_A(-c, d) \to (-c, d) \subseteq \mathbb{R}$ defined by:

$$\begin{cases} h(x) = x & \text{when } x \neq A \\ h(A) = 0 & \text{else} \end{cases}$$

is a homeomorphism.

(b) Show that S is locally Euclidean, second countable, but not Hausdorff.

Solution. (a)

Take the map h as defined in the statement above. We need only show that h is a continous bijection that admits a continuous inverse.

We will show that $g : (-c, d) \to I_A(-c, d)$ defined by:

$$g(a) = \begin{cases} A & \text{if } a = 0 \\ a & \text{else} \end{cases}$$

is a left inverse and a right inverse.

First, consider the map $h \circ g : (-c, d) \to (-c, d)$.

Fix an $a \in (-c, d)$. If $a = 0$, then we have that:

$$h \circ g(0) = h(g(0)) = h(A) = 0$$

Else, suppose $a \neq 0$. Then, by definition, we have that:

$$h \circ g(a) = h(g(a)) = h(a) = a$$

Thus, g is a right inverse.

Similarly, looking at $g \circ h : I_A(-c, d) \to I_A(-c, d)$, fixing a $b \in I_A(-c, d)$, if $b = A$, then we have that:

$$g \circ h(A) = g(h(A)) = g(0) = A$$

otherwise, for $b \neq A$, we have that:

$$g \circ h(b) = g(h(b)) = g(b) = b$$

Thus, we have that g acts as a left and right inverse, and thus h is bijective, and g is an inverse to h.

Now, we wish to show that h, g is continuous. To do so, we need only show that pre-images of basis elements are taken to basis elements. This is because, working in our image space, suppose $U = \cup_{B \in \mathcal{B}} B$ for a collection of basis elements $\mathcal{B}$. If we have that $h^{-1}(B)$ is a basis element in our codomain for every B, then of course, $\cup_{B \in \mathcal{B}} h^{-1}(B)$, being a union of basis elements is an open set, and thus $h^{-1}(U)$ is open.

Then, it is enough to consider an open interval $(a, b) \subseteq (-c, d)$. If $0 \notin (a, b)$, then $(a, b) \subseteq \mathbb{R} \setminus \{0\}$. Since S inherits the subspace topology on this set, then of course (a, b) is a basis element of the topology on S. Furthermore, since h acts via identity on $\mathbb{R} \setminus \{0\}$, $h^{-1}((a, b)) = (a, b)$.

Now, suppose $0 \in (a, b)$. In the notation we have established then, write this interval as $(-a, b)$. Then, from the action of h, we see that $h^{-1}((-a, b)) = (-a, 0) \cup A \cup (0, b)$. But, from the definiton of I_A, this is exactly $I_A(-a, b)$, and from the definition of the topology on S, this is exactly a basis element for neighborhoods of A.

Therefore, $h^{-1}(a, b)$ for any $a, b \in \mathbb{R}$ is taken to a basis element of S, and therefore h is continuous.

In a similar fashion, we may do the same for $g : (-c, d) \to I_A(-c, d)$.

Take a basis element from $I_A(-c, d)$, and call it C. If $A \notin C$, then of course C comes from an open interval on $\mathbb{R} \setminus \{0\}$, and thus $g^{-1}(C) = C$, as it acts via identity on $\mathbb{R} \setminus \{0\}$.

Now, suppose $A \in C$. Then, being a basis element, $C = I_A(-a, b)$ for $-c \leq -a < b \leq d$. Looking at the action of $g^{-1}(I_A(-a, b))$, we see that this is exactly:

$$g^{-1}(I_A(-a, b)) = g^{-1}((-a, 0) \cup \{A\} \cup (0, b)) = g^{-1}((-a, 0)) \cup g^{-1}(A) \cup g^{-1}(A)((0, b)) =$$

$$(-a, 0) \cup \{0\} \cup (0, b) = (-a, b)$$

Thus, for every basis element in $I_A(-c, d)$, the inverse image under g is a basis element of $(-c, d)$. Thus, g is continuous.

Therefore, since h is a continuous bijection that admits a continuous inverse, h is a homeomorphism.

(b)

Without too much trouble, it should be clear that S is locally Euclidean. Fix a $c, d \in \mathbb{R} : c, d > 0$. From part (a), we already have a chart from $I_A(-c, d)$ to a neighborhood of $\mathbb{R}$, an open interval, via h. It should be easy to see that swapping B for A everywhere, this also extends to a similar chart for $I_B(-c, d)$. Furthermore, on $S \setminus \{A, B\}$, we see that we may take $f : S \setminus \{A, B\} \to \mathbb{R}$ via $f(x) = x$, the identity, and the image is exactly $\mathbb{R}\setminus\{0\}$ an open set. It should be clear that the identity is continuous. Thus, between these three charts, S is locally Euclidean (of dimension 1).

Furthermore, S is second countable. We may take our basis to be the union of:

1) Open intervals with rational endpoints in $\mathbb{R}$ such that either both endpoints are positive or both are negative:

$$\{(a,b) : a,b \in \mathbb{Q}, a \neq 0, ab > 0\}$$

2) Open intervals of the form $I_A(-c,d)$ where $c,d > 0, c,d \in \mathbb{Q}$.

3) Open intervals of the form $I_B(-c,d)$ where $c,d > 0, c,d \in \mathbb{Q}$.

Using the fact that open intervals with rational endpoints are a countable basis for $\mathbb{R}$, we see that (1) generates the open sets for $\mathbb{R}\setminus\{0\}$. Further, by the definition of the topology for S, (2) and (3) generate the neighborhoods for A, B respectively, since for any $c,d \in \mathbb{R}$, we may take a sequence of rational numbers approaching c,d from above and below, respectively.

Since each of these sets are countable, being at most $\mathbb{Q}\times\mathbb{Q}$, their union is also countable. Thus S is second countable.

However, it should be clear that S is not Hausdorff. Take the points A, B. From the definition of our topology, we already know that the neighborhoods of A can be generated by $I_A(-c,d)$ and analogously for B, $I_B(-e,f)$.

Fix any two neighborhoods $I_A(-c,d), I_B(-e,f)$. Pick any point:

$$p \in (\max\{-c,-e\}, \min\{d,f\}) \setminus \{0\} \subseteq \mathbb{R}$$

Clearly, since $\max\{-c,-e\} < p < \min\{d,f\}$, we have that $p \in (-c,d)\setminus\{0\}$ and that $p \in (-e,f)\setminus\{0\}$. Thus, $p \in I_A(-c,d)$ and $p \in I_B(-e,f)$. Since this procedure may be done regardless of the choice of c,d,e,f, we can never find disjoint neighborhoods of A, B, and therefore S is not Hausdorff.

$\square$

Question 6. Define $S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\} \subseteq \mathbb{R}^3$, the unit sphere in 3-D.

Define the following charts:

$$\begin{cases} U_1 = \{(x,y,z) \in S^2 : x > 0\}, & \phi_1(x,y,z) = (y,z) \\ U_2 = \{(x,y,z) \in S^2 : x < 0\}, & \phi_2(x,y,z) = (y,z) \\ U_3 = \{(x,y,z) \in S^2 : y > 0\}, & \phi_3(x,y,z) = (x,z) \\ U_4 = \{(x,y,z) \in S^2 : y < 0\}, & \phi_4(x,y,z) = (x,z) \\ U_5 = \{(x,y,z) \in S^2 : z > 0\}, & \phi_5(x,y,z) = (x,y) \\ U_6 = \{(x,y,z) \in S^2 : z < 0\}, & \phi_6(x,y,z) = (x,y) \end{cases}$$

Describe the domain of $\phi_1 \circ \phi_4^{-1}$, and show that $\phi_1 \circ \phi_4^{-1}$ is a C^∞ function on its domain. Do the same for $\phi_6 \circ \phi_1^{-1}$.

Solution. We recall that $\phi_1 \circ \phi_4^{-1}$ is a map from $\phi_4(U_1 \cap U_4)$. Since we know that ϕ_4 acting on U_4 takes $(x,y,z) \mapsto (x,z)$, if we include the condition on U_1, where $x > 0$, this restricts us to the open half disk $\{(x,z) \subseteq \mathbb{R}^2 : x^2+z^2 < 1, x > 0\}$. (Alternatively, relabelling the axes, this is the right half of the open disk).

On this domain, ϕ_4^{-1} takes $(x,z) \mapsto (x, -\sqrt{1-x^2-z^2}, z)$ since $y < 0$, and ϕ_1 takes $(x,y,z) \mapsto (y,z)$. Therefore, we have that:

$$\phi_1 \circ \phi_4^{-1} : \phi_4(U_1 \cap U_4) \to \phi_1(U_1 \cap U_4) \text{ via } (x,z) \mapsto (-\sqrt{1-x^2-z^2}, z)$$

Clearly, the coordinate function z is the identity, thus polynomial and C^∞. Meanwhile, we notice that via power rule on $-(1-x^2-z^2)^{1/2}$, and the product rule on higher order derivatives, since we have that

$y < 0, y = -\sqrt{1 - x^2 - z^2}$ via rearrangment of the equation for S^2, we see that $1 - x^2 - z^2$ does not vanish on this domain, and therefore is also C^∞.

Doing the same thing for $\phi_6 \circ \phi_1^{-1}$, we see that the domain is the set $\phi_1(U_1 \cap U_6)$. Of course, since U_6 restricts U_1 to additionally have $z < 0$, we see that this is exactly $\{(y, z) \subseteq \mathbb{R} : y^2 + z^2 < 1, z < 0\}$, a different half open disk. Relabelling axes again, we can visualize this as the lower half of the open disk.

On this domain, we see that ϕ_1^{-1} has the action of taking $(y, z) \mapsto (\sqrt{1 - y^2 - z^2}, y, z)$ and ϕ_6 takes $(x, y, z) \mapsto (x, y)$.

Thus, we have that:

$$\phi_6 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_6) \to \phi_6(U_1 \cap U_6) \text{ via } (y, z) \mapsto (\sqrt{1 - y^2 - z^2}, y)$$

Using the same argument, since on $U_1 \cap U_6$, $x > 0$, we must have that $1 - y^2 - z^2$ cannot vanish by rearranging $x = \sqrt{1 - y^2 - z^2}$, and therefore $\sqrt{1 - y^2 - z^2}$ is C^∞ via the power rule and product rule. And, of course y is polynomial, thus C^∞.

$\square$