Math 235: Groupwork #9

Eric

December 13, 2022

Problem 1. Fix a $1 \leq p < \infty$. Prove that the set S of simple functions is dense in $L^p(\mathbb{R})$.

Solution. Let $f \in L^p(R)$. By Corollary 3.2.15, there exists simple functions ϕ_n such that $\phi_n(x) \to f(x)$ for each $x \in \mathbb{R}$ and $|\phi_n(x)| \leq |f(x)|$ for all $x \in \mathbb{R}, n \in \mathbb{N}$.

Then, we have that $f - \phi_n \to 0$ pointwise everywhere. Further, by an application of the triangle inequality, we have that, for all x:

$$|f(x) - \phi_n(x)| \leq |f(x)| + |\phi_n(x)| \leq 2|f(x)|$$

Then, since these are non-negative real numbers, we use the monotonicity of x^p on $[0, \infty)$ to conclude that:

$$|f(x) - \phi_n(x)|^p \leq 2^p|f(x)|^p$$

We notice that, because $f \in L^p(R)$, we have that $2^p|f|^p$ is integrable, as we notice $\int_{\mathbb{R}} 2^p|f|^p = 2^p \left(\left(\int_{\mathbb{R}} |f|^p\right)^{1/p}\right)^p = 2^p\|f\|_p^p < \infty$

Thus, by the Dominated Convergence Theorem, we have that:

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f - \phi_n|^p = 0 \implies \lim_{n\to\infty} \left(\int_{\mathbb{R}} |f - \phi_n|^p\right)^{1/p} = 0$$

Since if the integral goes to 0, then taking the $1/p$ power will also go to 0.

Thus, we have shown that for an arbitrary function $f \in L^p(\mathbb{R})$, that we can find simple functions ϕ_n such that $\phi_n \to f$ in the L^p-norm. Thus, the set of simple functions is dense in $L^p(\mathbb{R})$.

$\square$