# Homework #11

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## 2.1

**Problem 6.3.6.** Assume that $g : [a, b] \to [c, d]$ and $f : [c, d] \to \mathbb{C}$ are continuous. Prove the following statements:

(a) If $f$ is Lipschitz and $g \in \mathrm{AC}[a, b]$, then $f \circ g \in \mathrm{AC}[a, b]$

(b) If $f \in \mathrm{AC}[c, d], g \in \mathrm{AC}[a, b]$ and $g$ monotone increasing on $[a, b]$, then $f \circ g \in \mathrm{AC}[a, b]$

(c) If $f \in \mathrm{AC}[c, d], g \in \mathrm{AC}[a, b]$, then

$$f \circ g \in \mathrm{AC}[a, b] \iff f \circ g \in \mathrm{BV}[a, b]$$

*Solution.* (a)

Let $\epsilon > 0$ be given.

Since $f$ is Lipschitz, we may find $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$.

Since $g$ is absolutely continuous, we may find a $\delta > 0$ such that for collections of nonoverlapping subintervals of $[a, b]$, that

$$\Sigma_j (b_j - a_j) < \delta \implies \Sigma_j |g(b_j) - g(a_j)| < \frac{\epsilon}{K}$$

Well, take $[a_j, b_j]_j$ as a collection of countable, nonoverlapping subintervals of $[a, b]$ such that $\Sigma_j (b_j - a_j) < \delta$, and consider

$$\Sigma |f \circ g(b_j) - f \circ g(a_j)| \leq \Sigma K|g(b_j) - g(a_j) = K\Sigma |g(b_j) - g(a_j)| < K\frac{\epsilon}{K} = \epsilon$$

Thus, $f \circ g \in \mathrm{AC}[a, b]$.

(b)

Let $\epsilon > 0$ be given.

Because $f$ is absolutely continuous, we may find $\delta > 0$ such that, for $\{[c_j, d_j]\}_j$ intervals in $[c, d]$, we have that

$$\Sigma_j (d_j - c_j) < \delta \implies \Sigma_j |f(d_j) - f(c_j)| < \epsilon$$

Further, since $g$ is absolutely continuous, we may find a $\delta' > 0$ such that for $\{[a_i, b_i]\}_i$ intervals in $[a, b]$, we have that

$$\Sigma_i (b_i - a_i) < \delta' \implies \Sigma_i |g(b_i) - g(a_i)| < \delta$$

Now, take $\{[a_i, b_i]\}_i$ intervals in $[a, b]$ such that $\Sigma_i (b_i - a_i) < \delta'$. Since $g$ is monotone increasing, we notice that $\{[g(a_i), g(b_i)]\}_i$ are actually intervals, non-overlapping since, due to the monotone increasing nature of $g$, they may only overlap on their endpoints. Further, from the $\delta'$ condition, we have that $\Sigma_i|g(b_i)-g(a_i)|<\delta$, which implies then that $\Sigma_i|f(g(b_i))-f(g(a_i))|<\epsilon$.

(c)

By Lemma 6.1.3, we know already that $h \in \mathrm{AC}[a,b] \implies h \in \mathrm{BV}[a,b]$. So, we need only prove that $f \circ g \in \mathrm{BV}[a,b] \implies f \circ g \in \mathrm{AC}[a,b]$. However, this is easy.

Let $Z \subseteq [a,b]$ be a set of measure 0. By corollary 6.3.2, $g(Z) \subseteq [c,d]$ is a set of measure 0. However, now we use the absolute continuity of $f$ as well, to see that $f(g(Z))$ is also a set of measure 0. Since the choice of $Z$ was arbitrary, we have that $|Z|=0 \implies |f \circ g(Z)| = |f(g(Z))| = 0$. Then, by Banach-Zaretsky again, we have that $f \circ g \in \mathrm{AC}[a,b]$.

□

**Problem 6.3.10.** Suppose that $f:[a,b] \to \mathbb{C}$ is differentiable everywhere on $[a,b]$. Prove the following:

(a) $f \in \mathrm{AC}[a,b]$ if and only if $f \in \mathrm{BV}[a,b]$

(b) $f'=0$ a.e. if and only if $f$ is constant on $[a,b]$.

*Solution.* (a)

We already have that $f \in \mathrm{AC}[a,b] \implies f \in \mathrm{BV}[a,b]$ by Lemma 6.1.3. So, now assume $f \in \mathrm{BV}[a,b]$.

By Corollary 5.4.3, since $f \in \mathrm{BV}[a,b]$, we have that $f' \in L^1[a,b]$. Then, by Corollary 6.3.3, since $f$ differentiable everywhere by hypothesis, we have that $f \in \mathrm{AC}[a,b]$.

(b)

Clearly, if $f$ is constant on $[a,b]$, then $f'=0$ everywhere, stronger than almost everywhere.

Now, suppose $f'=0$ almost everywhere. Clearly then, $f' \in L^1[a,b]$, because in particular, $\int_{[a,b]} f' = 0$. Therefore, we have that $f \in \mathrm{AC}[a,b]$ by 6.3.3 again. Further, by definition, since $f'=0$ almost everywhere, $f$ is singular. Then, by 6.3.4, since $f$ is both singular and absolutely continuous, $f$ must actually be constant.

□

# 2.2

**Problem 6.4.10.** Show that $f:[a,b] \to \mathbb{C}$ is Lipschitz if and only if $f \in \mathrm{AC}[a,b]$ and $f' \in L^\infty[a,b]$.

*Solution.* Firstly, suppose $f$ is Lipschitz. We have already that Lipschitz implies absolutely continuous by 6.1.3, which implies that $f'$ exists almost everywhere, by 6.1.5. Now, let $x$ be somewhere the dervative exists at. Then, we have that, for any $y \in [a,b], y \neq x$, by the definition of Lipschitz, there exists an $M>0$ such that:

$$|f(y)-f(x)| \leq M|x-y| \implies \frac{|f(y)-f(x)|}{|y-x|} \leq M$$

Now, if we view $y=x+h$, and then take the limit as $h \to 0$, this implies that $|f'(x)| \leq M$ as well. Since the existence of $M$ is independent of the point $x$, coming from the Lipschitz condition, we have then that on the $[a,b] \setminus Z, |Z|=0$ where $f'$ is defined, that $|f'| \leq M \implies f' \in L^\infty[a,b]$.

Now, instead, suppose $f \in \mathrm{AC}[a,b]$ with $f' \in L^\infty[a,b]$. By the fundamental theorem of calculus (6.4.2), we have that $f' \in L^1$, and:

$$f(x)-f(a) = \int_a^x f'(t)dt \implies f(x) = f(a) + \int_a^x f'(t)dt$$

Now, consider the difference $|f(y)-f(x)|$ for $x,y \in [a,b]$. We have that:

$$|f(y)-f(x)| = |f(a) + \int_a^y f'(t)dt - f(a) - \int_a^x f'(t)dt| = |\int_x^y f'(t)dt|$$

Now, since we have that $f'$ is essentially bounded, suppose that $f' \leq \|f'\|_\infty$ almost everywhere. Then, we can say that on $[x, y]$, $f' \leq \|f'\|_\infty$ almost everywhere, so we have that:

$$|f(y) - f(x)| = |\int_x^y f'(t)dt| \leq |\int_x^y \|f'\|_\infty dt| = |x - y|\|f'\|_\infty$$

Thus, $f$ is Lipschitz, as we just take the Lipschitz constant as the uniform norm of $f'$.

□

**Problem 6.4.13.** Suppose that $f \in L^1(\mathbb{R})$ is such that $f' \in L^1(\mathbb{R})$ and $f \in \mathrm{AC}[a, b]$ for every finite interval $[a, b]$. Show that $\lim_{|x| \to \infty} f(x) = 0 = \int_{-\infty}^{\infty} f'$.

*Solution.* □

## 2.3

**Problem 7.3.22.** Let $E$ be a measurable subset of $\mathbb{R}^d$, and fix a $1 \leq p < \infty$

(a) Suppose that $\Sigma f_n$ is absolutely convergent in $L^p(E)$, that is, $f_n \in L^p(E)$ for all $n$ and $\Sigma\|f_n\|_p < \infty$. Prove the following:

- the series $f(x) = \Sigma_{n=1}^{\infty} f_n(x)$ converges for almost every $x \in E$
- $f \in L^p(E)$
- the series $f = \Sigma f_n$ converges in the $L^p$ norm, that is, $\lim_{N \to \infty} \|f - \Sigma_n^N f_n\|_p = 0$

(b) Use part (a) and theorem 1.2.8 to give another proof that $L^P(E)$ is complete with respect to $\|\cdot\|_p$.

(c) Show that if $\Sigma f_n$ is an absolutely convergent series in $L^1(E)$, then

$$\int_E \Sigma_{n=1}^{\infty} f_n = \Sigma_{n=1}^{\infty} \int_E f_n$$

*Solution.* □

**Problem 7.3.23.** Fix a $1 \leq p < \infty$. Given $f_n \in L^p(\mathbb{R}^d)$, prove that $f_n \to f$ in $L^p(\mathbb{R}^d$ if and only if the following three conditions hold.

(a) $f_n \xrightarrow{m} f$

(b) For each $\epsilon > 0$, there exists a $\delta > 0$ such that for every measurable set $E \subseteq \mathbb{R}^d$ with $|E| < \delta$, we have that $\int_E |f_n|^p < \epsilon$ for every $n$.

(c) For each $\epsilon > 0$, there exists a measurable set $E \subseteq \mathbb{R}^d$ such that $|E| < \infty$ and $\int_{E^c} |f_n|^p < \epsilon$ for every $n$.

*Solution.* □