Homework #8

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2.1

Problem 5.2.18. Suppose that $f : [a, b] \to \mathbb{C}$. Show that there exists partitions Γ_k of $[a, b]$ such that Γ_{k+1} is a refinement of Γ_k for each k, and $S_{\Gamma_k} \nearrow V[f; a, b]$ as $k \to \infty$.

Solution. First, we wish to show that for any partition Γ_k and refinement Γ_{k+1}, that $S_{\Gamma_k} \leq S_{\Gamma_{k+1}}$.

Let $S_{\Gamma_k} = \{a = x_0 < ... < x_i = b\}$ and $S_{\Gamma_{k+1}} = \{a = y_0 < ... < y_j = b\}$ be a refinement, where $i < j$ and for every $0 \leq i' \leq i$, there exists a j' such that $x_{i'} = y_{j'}$.

Look at one pair of $x_{i'}, x_{i'+1}$. If, in the refinement, we have that $x_{i'} = y_{j'}$ and $x_{i'+1} = y_{j'+1}$, then we have that $|f(x_{i'+1}) - f(x_{i'})| = |f(y_{j'+1}) - f(y_{j'})|$. Else, suppose not. Then, we have that $x_{i'} = y_{j'}$ and $x_{i'+1} = y_{j'+n}$ for some n. Then, by liberal usage of the triangle inequality, we have that:

$$|f(x_{i'+1}) - f(x_{i'})| = |f(y_{j'+n}) - f(y_{j'})| = \left|f(y_{j'+n}) - f(y_{j'}) + \Sigma_{k=1}^{n-1}(f(y_{j'+k}) - f(y_{j'+k})\right| =$$

$$\left|\Sigma_{k=1}^{n}(f(y_{j'+k}) - f(y_{j'+(k-1)})\right| \leq \Sigma_{k=1}^{n} \left|f(y_{j'+k}) - f(y_{j'+(k-1)})\right|$$

Since we may do this for every $0 \leq i' \leq i$, that means that $S_{\Gamma_k} \leq S_{\Gamma_{k+1}}$.

First, assume $V[f; a, b] < \infty$. Now, since $V[f; a, b]$ is the supremum of S_Γ over every partition Γ, we may construct a sequence Γ_k of partitions such that $V[f; a, b] - S_{\Gamma_k} < 1/k$.

In particular now, define a new sequence of partitions as such. Let $\Gamma'_1 = \Gamma_1$. Then, take $\Gamma'_i = \Gamma'_{i-1} \cup \Gamma_i$, where we understand the union operation as meaning to take every point in Γ'_{i-1}, Γ_i and create a partition with all points. We notice that for each i, Γ'_i is a refinement of both Γ'_{i-1}, Γ_i. Then, we have that $\Gamma'_{i-1} \leq \Gamma'_i$ from the work we did above, and further, we know that $V[f; a, b] - 1/k \leq S_{\Gamma'_i} \leq V[f; a, b]$ by the choice of the Γ_i's. Thus, we have an increasing sequence of refinements that converges to $V[f; a, b]$.

The unbounded case is clear, instead of taking $V[f; a, b] - S_{\Gamma_k} < 1/k$, we simply take $S_{\Gamma_k} > k$ for each $k \geq 1$, and proceed in the same way.

□

Problem 5.2.21. Assume that $E \subseteq \mathbb{R}$ is measurable, and suppose that $f : E \to \mathbb{R}$ is Lipschitz on the set E, that is, there exists a $K \geq 0$ such that:

$$|f(x) - f(y)| \leq K|x - y| \text{ for all } x, y \in E$$

Prove that $|f(A)|_e \leq K|A|_e$, for any $A \subseteq E$.

Solution. Let $\{Q_k\}_k$ be a collection of boxes such that $A \subseteq \cup_k Q_k$. Let's look at one specific box, Q_i. Since $A \subseteq E$, we can take $d_i = \sup(\{x - y : x, y \in E \cap Q_i\})$, where we notice $d_i \leq \mathrm{Vol}(Q_i)$ Consider the image of $f(E \cap Q_i)$. Since f is Lipschitz, and $Q_i \cap E$ an intersection of measurable sets, the image is measurable. In particular, we notice that, for $x, y \in E \cap Q_i$, we have:

$$|f(x) - f(y)| \leq K|x - y| \leq Kd_i$$

Then, if we fix an x, that means $f(E \cap Q_i)$ can be contained within an interval of length Kd_i. We may repeat this process for each Q_i. We notice, since Q_k covers A, then so must $E \cap Q_k$. So, we have that

$$| \cup_k f(E \cap Q_k)|_e \leq \Sigma_k(Kd_k) \leq K\Sigma_k(d_k) \leq K\Sigma_k \text{Vol}(Q_k)$$

Since we can do this for any cover by boxes Q_k of A, $f(A) \subseteq \cup_k f(E \cap Q_k)$ for every collection of boxes, and via the properties of the infimum, we have that:

$$|f(A)|_e \leq K|A|_e$$

□

Problem 5.2.22. Fix $a, b > 0$ and define:

$$f(x) = \begin{cases} |x|^a \sin(|x|^{-b}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Prove the following:

(a) $f \in \text{BV}[-1, 1] \iff a > b$

(b) If $a = b$ then $f \in C^\alpha[-1, 1]$ with exponent $\alpha = \frac{b}{b+1}$.

(c) $C^\alpha[-1, 1]$ is not contained in $\text{BV}[-1, 1]$ for any $0 < \alpha < 1$.

Solution. (a)

First, we notice that f is symmetric across $x = 0$, and so we restrict ourselves to looking on $[0, 1]$, and we may drop the absolute values. Computing f' on $(0, 1]$, we find that

$$f' = ax^{a-1} \sin(x^{-b}) + x^a \cos(x^{-b}) - bx^{-b-1} = ax^{a-1} \sin(x^{-b}) - bx^{a-b-1} \cos(x^{-b})$$

□

Problem 5.2.23. (a) Suppose that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of complex-valued functions $f_n : [a, b] \to \mathbb{C}$ and that $f_n \to f$ pointwise on $[a, b]$. Prove that:

$$V[f; a, b] \leq \liminf_{n \to \infty} V[f_n; a, b]$$

(b) Exhibit functions f_n, f such that $f_n \in \text{BV}[a, b]$ for each $n \in \mathbb{N}$ and $f_n \to f$ pointwise, but $f \notin \text{BV}[a, b]$.

Solution. □

Problem 5.2.26. Prove the following:

(a) $\|f\| = V[f; a, b]$ defines a seminorm on $\text{BV}[a, b]$ and

$$\|f\|_{\text{BV}} = V[f; a, b] + \|f\|_u : f \in \text{BV}[a, b]$$

is a norm on $\text{BV}[a, b]$.

(b) $\text{BV}[a, b]$ is a Banach space with respect to $\| \cdot \|_{\text{BV}}$.

(c) $\|f\|_{\text{BV}'} = V[f; a, b] + |f(a)|$ defines an equivalent norm for $\text{BV}[a, b]$. That is, it is a norm, and there exists $C_1, C_2 > 0$ such that:

$$C_1\|f\|_{\text{BV}} \leq \|f\|_{\text{BV}'} \leq C_2\|f\|_{\text{BV}} : f \in \text{BV}[a, b]$$

Solution. (a)

Clearly, we have that $V[f;a,b] \geq 0$ for any $f \in \mathrm{BV}[a,b]$, because it is the supremum of non-negative numbers. Then, we need only check for the triangle inequality, and factoring scalars.

Let $f, g \in \mathrm{BV}[a,b]$, and fix a partition $\Gamma = \{a = x_0 < ... < x_n = b\}$. We notice, by the triangle inequality on the complex numbers, we have that, for each (x_i, x_{i+1}):

$$|f + g(x_{i+1}) - f + g(x_i)| = |f(x_{i+1}) + g(x_{i+1}) - f(x_i) - g(x_0)| \leq |f(x_{i+1}) - f(x_i)| + |g(x_{i+1}) - g(x_i)|$$

Since this is true for every interval in the partition, this implies then that $S_\Gamma^{f+g} \leq S_\Gamma^f + S_\Gamma^g$, where we use S_Γ^f to denote the sum for the function f. Then, since the variation is simply the supremum over all partitions, and this holds for every partition, we have that:

$$\|f + g\| = V[f + g; a, b] \leq V[f; a, b] + V[g; a, b] = \|f\| + \|g\|$$

.

Now, let $k \in \mathbb{R}$. Consider now $\|kf\|$. Again, looking at any partition Γ, we see that:

$$|kf(x_{i+1}) - kf(x_i)| = |k||f(x_{i+1}) - f(x_i)|$$

Since this is true for each interval in our partition, it implies that $S_\Gamma^{kf} = |k|S_\Gamma^f$. Again, via the properties of the supremum, this implies then that $\|kf\| = |k|\|f\|$.

Now, we look at $\|f\|_{\mathrm{BV}} = V[f;a,b] + \|f\|_u : f \in \mathrm{BV}[a,b]$. Because of the fact that we have shown that $V[f;a,b]$ is a seminorm on $\mathrm{BV}[a,b]$ and that we already know that $\|f\|_u$ is a norm, we know that this is already a seminorm. Then, it suffices to show that $\|f\|_{\mathrm{BV}} = 0 \implies f = 0$. Since both portions are non-negative, this implies, in particular, $\|f\|_u = 0$. But, because this is a norm, this implies that $f = 0$, and we are done. Thus, this is a norm.

(b)

(c)

First, we look at the case $f(a) \geq 0$. Then, using the Jordan decomposition on $f = g - h$ for g, h monotone increasing, and the seminorm properties to see that $V[f;a,b] \leq V[g;a,b] + V[h;a,b]$, we conclude that $f(a) \leq \|f\|_u \leq f(a) + V[f;a,b]$, since to maximize $|f|$, we would need $V[h;a,b] = 0$. We can actually see that this argument works for $f(a) < 0$, where instead of taking the positive distance, we take $V[g;a,b] = 0$ to maximize $|f|$. So, we actually have that $|f(a)| \leq \|f\|_u \leq |f(a)| + V[f;a,b]$.

Then, we take $C_1 = 1, C_2 = 2$.

From $|f(a)| \leq \|f\|_u$, we can add $V[f;a,b]$ to both sides to obtain:

$$\|f\|_{\mathrm{BV}'} = V[f;a,b] + |f(a)| \leq V[f;a,b] + \|f\|_u = \|f\|_{\mathrm{BV}}$$

, so we have that $C_1\|f\|_{\mathrm{BV}'} = \|f\|_{\mathrm{BV}'} \leq \|f\|_{\mathrm{BV}}$

Further, we have that from the other side, we obtain:

$$\|f\|_u \leq |f(a)| + V[f;a,b] \implies V[f;a,b] + \|f\|_u \leq |f(a)| + 2V[f;a,b]$$

so we can see that:

$$C_2\|f\|_{\mathrm{BV}'} = 2|f(a)| + 2V[f;a,b] \geq |f(a)| + 2V[f;a,b] \geq V[f;a,b] + \|f\|_u = \|f\|_{\mathrm{BV}}$$

.

Thus, these norms are equivalent. If you really want the other inclusion, we can reverse the inclusions by dividing via the constants. $\square$

2.2

Problem 5.3.5. Assume that $E \subseteq \mathbb{R}^d$ satisfies that $0 < |E|_e < \infty$, and let $\mathcal{B}$ be a Vitali covering of E. Given an $\epsilon > 0$, prove that there exist a countable collection of balls $B_k \in \mathcal{B}$ such that

$$|E \setminus \cup_k B_k|_e = 0 \text{ and } \Sigma_k |B_k| < |E|_e + \epsilon$$

Solution. □