Midterm #1

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Math 237: Midterm #1

March 8, 2024

Question 1. Let (X, ρ) be a compact metric space, and $f : X \to X$ a function such that:

$$\rho(f(x), f(y)) < \rho(x, y)$$

for all $x \neq y$.

Define $g : X \to \mathbb{R}$ via $g : x \mapsto \rho(x, f(x))$.

1.1)

Prove that g is Lipschitz, and that g has a minimum value, achieved at a point $x_0 \in X$. Conclude that there exists $x \in X$ such that $g(x) = 0$.

1.2)

Show that f has a unique fixed point x_0.

1.3)

Show that the assumption that X is compact may not be omitted.

Solution. 1.1)

Fix some $x \in X$, and let $y \in X$ be arbitrary. By the triangle inequality, we see that:

$$\begin{cases} \rho(x, f(x)) \leq & \rho(x, y) + \rho(y, f(x)) \\ \rho(y, f(x)) \leq & \rho(y, f(y)) + \rho(f(x), f(y)) \end{cases}$$

Combining these two equations with the property of f by hypothesis, we see that:

$$\rho(x, f(x)) - \rho(y, f(y)) \leq \rho(x, y) + \rho(f(x), f(y)) < 2\rho(x, y)$$

However, we notice that we may run the same computation in the triangle inequality, switching the labels of x, y, as $\rho(x, y) = \rho(y, x)$. Thus, we can conclude then that

$$|\rho(x, f(x)) - \rho(y, f(y))| < 2\rho(x, y)$$

and therefore, since the left side is exactly $d(g(x), g(y))$ with the metric of the real line, we may conclude that g is Lipschitz with Lipschitz constant at most 2.

Now, since g is Lipschitz continuous, it is continuous. Hence, since X is compact, g achieves its extremas. Hence, we may find $x_0 \in X$ such that g achieves its minimum value.

Suppose that $g(x_0) > 0$. Then, of course, we would have that $g(x_0) = \rho(x_0, f(x_0)) > 0$ and hence, $x_0 \neq f(x_0)$. Then, we can consider $g(f(x_0))$. We have that:

$$g(f(x_0)) = \rho(f(x_0), f(f(x_0))) < \rho(x_0, f(x_0)) = g(x_0)$$

But, this is a contradiction, as we assumed that g attained a minimum at x_0. Hence, $g(x_0) = 0$.

1.2)

From 1.1, we've shown that there exists $x_0 \in X$ such that $g(x_0) = 0$. Evidently then:

$$g(x_0) = 0 \implies \rho(x_0, f(x_0)) = 0 \implies f(x_0) = x_0$$

Furthermore, this point must be unique, as suppose $f(x_1) = x_1$ as well. Assuming that $x_0 \neq x_1$, we have that:

$$\rho(x_0, x_1) = \rho(f(x_0), f(x_1)) < \rho(x_0, x_1)$$

which is absurd. Hence, $x_0 = x_1$.

1.3)

Here are some examples to show that we need X to be compact. Consider $X = \mathbb{Z}$, equipped with the standard metric $\rho(x, y) = |x - y|$. Of course, this is not compact, as the sequence $\{n\}_{n=1}^{\infty}$ cannot admit any convergent subsequence. If we take $f(x) = \text{round}(x/2)$, where the round function rounds to the integer closer to 0, then of course, we have that $\rho(f(x), f(y)) < \rho(x, y)$ for $x \neq y$, as it contracts all distances by at least $1/2$. On the other hand, it has multiple fixed points, $-1, 0, 1$.

Another example is to take the open interval $(0, 1)$, equipped with the standard metric $\rho(x, y)$, and consider the function $f(x) = x/2$. Evidently, in the same fashion, we still have that $\rho(f(x), f(y)) = |x/2 - y/2| = 1/2|x - y| = 1/2\rho(x, y) < \rho(x, y)$. However, g does not attain a minimum and f does not have a fixed point.

We can see g does not have a minimum as for any $\epsilon > 0$, we may choose $N \geq 1$ such that $1/N < \epsilon$. Then, $g(1/N) = \rho(1/N, f(1/N)) = |1/N - 1/2N| = 1/2N < 1/N < \epsilon$. Hence, $g(x)$ can be arbitrarily small. However, we can see that for $x = 1/2x$, this is satisfied only at $x = 0$, outside of $(0, 1)$. Hence, there is no x such that $g(x) = 0$ on $(0, 1)$, and no fixed point of f on $(0, 1)$.

□

Question 2. Let X, Y be Banach spaces. Let $T \in L(X, Y)$. Show that T is surjective if and only if range(T) is not meager in Y.

Solution. One direction is trivial. Suppose T is surjective. Then, $Y = \text{range}(T)$. But, by the Baire Category Theorem (2.21, Heil), Y is nonmeager in Y, and we are done.

□

Question 3. Let $C_b(\mathbb{R})$ be the space of bounded, continuous, real-valued functions. Let $C_b^1(\mathbb{R})$ be the space of functions such that $f, f' \in C_b(\mathbb{R})$. Equip both of these spaces with the uniform norm.

3.1)

Show that C_b is complete, and that C_b^1 is not complete.

3.2)

Show that the differentiation operator $D : C_b^1(\mathbb{R}) \to C_b(\mathbb{R})$ that sends $D : f \mapsto f'$ is unbounded, but has a closed graph.

Solution. 3.1)

First, consider the family of functions $f_n(x) = 2^{-n} \cos(7^n \pi x)$ for $n \geq 1$, and consider $g_m(x) = \sum_{n=1}^{m} f_n(x)$.

We have that the sequence of $\{g_m\}$ is uniformly Cauchy, as if we let $\epsilon > 0$, we may choose N such that $2^{-N+1} < \epsilon$, and then for $m, m' > N$ (WLOG, suppose $m > m'$), we have that:

$$|g_m(x) - g_{m'}(x)| = |\sum_{n=1}^{m} f_n(x) - \sum_{n=1}^{m'} f_n(x)| = |\sum_{n=m}^{m'} f_n(x)| \leq |\sum_{n=N}^{\infty} f_n(x)| \leq \sum_{n=N}^{\infty} |f_n(x)| \leq \sum_{n=N}^{\infty} 2^{-N} = 2^{-N+1}$$

Since this is independent of the point x, this is uniformly Cauchy. Since each g_m is continuous, being the finite sum of continuous functions, and the convergence is uniform, the pointwise limit $g(x) = \lim_{m\to\infty} g_m(x)$ is a continuous function. Moreover, we can see easily that g is bounded, as we can see that each of the partial sums are bounded above by $\sum_{n=1}^{\infty} 2^{-n} = 2$. However, this is a Weierstrauss function, famously known for being differentiable nowhere. Since we have demonstrated a sequence of functions in C_b^1, convergent under the uniform norm to a function not in C_b^1, we may conclude that C_b^1 is not complete.

On the other hand, let $\{f_n\}_{n=1}^{\infty} \subseteq C_b$, with $\sum_{n=1}^{\infty} \|f_n\|_u < \infty$. Consider $f = \sum_{n=1}^{\infty} f_n$, and we will show that f is both bounded, and the uniform limit of the partial sums.

Evidently, f is bounded, as we can look at the partial sums $\sum_{n=1}^{N} f_n$. We have that $\|\sum_{n=1}^{N} f_n\|_u \leq \sum_{n=1}^{N} \|f_n\|_u < \sum_{n=1}^{\infty} \|f_n\|_u < \infty$, where the first inequality comes from the triangle inequality, and the second is simply our hypothesis of being absolutely convergent. Since this bound holds for all $N > 0$, it must hold in the limit as well. Hence, $\|f\|_u < \sum_{n=1}^{\infty} \|f_n\|_u < \infty$.

Now, we wish to show that $\sum_{n=1}^{N} f_n \to f$ uniformly. Since $\sum_{n=1}^{\infty} \|f_n\|_u < \infty$, for $\epsilon > 0$, we may find a $M > 0$ such that for all $m > M$, $\sum_{n=M}^{\infty} \|f_n\|_u < \epsilon$. Now, let $m > M$, and consider $\|f - \sum_{n=1}^{m} f_n\|_u$. We see that:

$$\|f - \sum_{n=1}^{m} f_n\|_u = \|\sum_{n=m+1}^{\infty} f_n\|_u$$

Now, due to the positivitiy of the norm, since we have for each finite sum: $\|\sum_{n=m+1}^{p} f_n\|_u \leq \sum_{n=m+1}^{p} \|f_n\|_u \leq \sum_{n=m+1}^{\infty} \|f_n\|_u$, we may conclude that this holds in the limit as well.

Hence, we have that:

$$\|\sum_{n=m+1}^{\infty} f_n\|_u \leq \sum_{n=m+1}^{\infty} \|f_n\|_u < \epsilon$$

Thus, $f_n \to f$ uniformly, and hence, f is continuous. Therefore, $f \in C_b$, as desired, and $f_n \to f$ under the norm. Since the choice of absolutely convergent sequence was arbitrary, by 5.1 in Folland, since every absolutely convergent sequence converges, C_b must be complete.

3.2)

Evidently, D is unbounded. For example, take the family of functions $f_k = \sin(kx)$, for $k \in \mathbb{N}$. Clearly, this is a continuous function, bounded above by 1, and so $\|f_k\|_u = 1$. Furthermore, its derivative is $k\cos(kx)$, continuous, and for each k, bounded above by k. However, $\|D(f_k)\|_u = \|k\cos(kx)\|_u = k$. Since we may choose k arbitrarily large without affecting the norm of f_k, D is unbounded.

Now, suppose that we have $f_n \to f \in C_b^1$, and $Df_n = f_n' \to g \in C^1$, uniformly in both cases. Fix an arbitrary point $a \in \mathbb{R}$, and consider, for $x > a$, the closed interval $[a, x]$. Since we have that $f_n' \to g$ uniformly, evidently, $\|f_n'\|_u$ is bounded. Then, we can take $\sup_n \|f_n'\|_u < \infty$ as an upper bound for all $|f_n'(y)|, y \in [a, x]$. Of course also, if $f_n' \to g$ uniformly, it does so pointwise as well. Therefore, by the Lesbesgue Dominated Convergence Theorem, we have that:

$$\lim_{n\to\infty} \int_a^x f_n'(y)dy = \int_a^x g(y)dy$$

However, we know that f_n is differentiable on $[a, x]$, and f_n', its derivative is continuous. Thus, we may transform the left hand side via the Fundamental Theorem of Calculus to obtain:

$$\lim_{n\to\infty} f_n(x) - f_n(a) = \int_a^x g(y)dy$$

Now, since $f_n \to f$ uniformly, it does so pointwise as well, so we have that:

$$f(x) - f(a) = \int_a^x g(y)dy$$

and finally, we can apply D to both sides of this equation, and since g is continuous, we can apply the other statement of the FTC to obtain:

$$D(f(x) - f(a)) = D\left(\int_a^x g(y)dy\right) \implies D(f)(x) = g(x)$$

Since the choice of a were arbitrary, we may repeat this argument for every x. Hence, varying across all $x \in \mathbb{R}$, we obtain an equality of functions, and conclude that $Df = g$.

Since this is true for an arbitrary $f_n \to f, f_n' \to g$, this is true for all cases where both sequences simultaneously converge, and hence D has a closed graph.

$\square$

Question 4. Let $\mathcal{H} = L^2[0,1]$, the Lebesgue measurable and square-integrable functions defined on $[0,1]$. Let K be a non-empty, closed, convex subset of $\mathcal{H}$. Define $P = P_K$ as the orthogonal projection of H onto K.

4.1)

Let $x \in \mathcal{H}$. Prove that the following are equivalent:

i) There exists a unique $z \in K$ such that $\|x - z\| = \min_{y \in K} \|x - y\|$.

ii) $z \in K$ and $\langle x - z, y - z\rangle \leq 0$ for all $y \in K$.

4.2)

Let A be a continuous bilinear mapping from $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$ such that, for some $\alpha > 0$, we have:

$$A(f, f) \geq \alpha \|f\|_2^2$$

for every $f \in \mathcal{H}$. We will prove the following statement in parts:

For every $f \in \mathcal{H}$, there exists a unique $u \in K$ such that:

$$A(u, v - u) \geq \langle f, v - u\rangle$$

for all $v \in K$.

4.2.1)

Fix a $u \in \mathcal{H}$, and prove that there exists a unique $Tu \in \mathcal{H}$ such that $A(u, v) = \langle Tu, v\rangle$ for every $v \in \mathcal{H}$. Prove that T is a bounded linear mapping on $\mathcal{H}$.

4.2.2)

Fix a $\rho > 0$, $f \in \mathcal{H}$, and define a map $S_\rho : K \to K$ that sends $v \mapsto P(\rho f - \rho T v + v)$. Prove that we may choose ρ such that there exists a $0 < k < 1$ with the property that:

$$\|S_\rho(v_1) - S_\rho(v_2)\| \leq k\|v_1 - v_2\|$$

for all $v_1, v_2 \in K$.

4.2.3)

Conclude that for the value of $\rho > 0$ chosen in 4.2.2, that S_ρ is a contraction, and therefore has a unique fixed point $u \in K$.

4.2.4)

Note that we can rewrite $\rho f - \rho Tu = \rho f - \rho Tu + u - u$. Then, use 4.1 to show that:

$$\langle \rho f - \rho T u, v - u\rangle \leq 0$$

for every $v \in K$.

4.2.5)

Conclude that, for every $f \in \mathcal{H}$, there exists a unique $u \in K$ such that:

$$A(u, v - u) \geq \langle f, v - u \rangle$$

Solution. $\square$