Homework #3

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Math 233: Homework #3

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Question 1. Let u be a harmonic function on a region Ω. What can we say about the set of points such that $\nabla u = 0$, that is, the set of points where $u_x = u_y = 0$?

Solution. Recall that if u is a real harmonic function, then we may identify it as the real part of a holomorphic function $f(x,y) = u(x,y) + iv(x,y)$ locally. Suppose $u_x = u_y = 0$. Then, by the Cauchy-Riemann equations, we have that at these points, $v_x = v_y = 0$. Further, identifying $f'(z) = \partial f(z)$ for $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, we have that:

$$f'(z) = \partial f(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}\left[(u_x + v_y) + i(v_x - u_y)\right]$$

So, we have that at points where $u_x = u_y = 0$, we have that $f'(z) = 0$. But, since f is holomorphic on this neighborhood, so is f'. Therefore, $\{(x,y) : \nabla u(x,y) = 0\}$ is either all of the neighborhood, or has no limit points. Since Ω is a region, we can always patch our entire region with overlapping neighborhoods, so this extends to all of Ω.

Now, if u is a complex-valued harmonic function, we simply identify it as $u = w + iv$, where w, v are the real and imaginary portions. It should be clear that if u is harmonic, so must w, v as:

$$u_{xx} + u_{yy} = w_{xx} + iv_{xx} + w_{yy} + v_{yy} = (w_{xx} + w_{yy}) + i(v_{xx} + v_{yy}) = 0 \implies w_{xx} + w_{yy} = 0, v_{xx} + v_{yy} = 0$$

Then, suppose $u_x = u_y = 0$. At such points, we would have that $u_x = w_x + iv_x = 0, u_y = w_y + iv_y = 0 \implies w_x = w_y = 0, v_x = v_y = 0$. But, by the previous work, since v, w are real harmonic functions, they either have no limit points, or are the full space. It should be clear then, that the set of points where $\nabla u = 0$ is simply the union of these sets. It too may only be the full space or not have limit points, as if it did, then we could construct a subsequence of points coming from either the set where $\nabla v = 0$, or $\nabla w = 0$, which would imply that the original set had a limit point, a contradiction. □

Question 2. Let u, v be real harmonic functions on a plane region Ω. Under what conditions is uv harmonic?

Further, show that u^2 may not be harmonic on Ω, unless u is constant.

Further, for which $f \in \mathcal{H}(\Omega)$ is $|f|^2$ harmonic?

Solution. We start by proving that if we take the Laplacian of uv, $\Delta(uv)$, then this is equal to $2\nabla u \cdot \nabla v$:

$$\Delta(uv) = (uv)_{xx} + (uv)_{yy} = (u_x v + u v_x)_x + (u_y v + u v_y)_y = u_{xx}v + u_x v_x + u_x v_x + u v_{xx} + u_{yy} v + u_y v_y + u_y v_y + u v_{yy}$$

Because u, v are harmonic, we know that $u_{xx} + u_{yy} = 0, v_{xx} + v_{yy} = 0$, so:

$$= v(u_{xx} + v_{xx}) + 2u_x v_x + u(v_{xx} + v_{yy}) + 2u_y v_y = 2(u_x v_x + u_y v_y) = 2\langle u_x, u_y\rangle \cdot \langle v_x, v_y\rangle = 2\nabla u \cdot \nabla v$$

Here, it should be clear then that if u^2 is not constant, then u^2 is not harmonic. We have that $\Delta(u^2) = \Delta(uu) = 2\nabla u \cdot \nabla u = 2|\nabla u|^2$. So, suppose u is harmonic, then for $\Delta(u^2) = 0$, this implies that $|\nabla u| = 0$ for all $z \in \Omega$. However, this implies immediately that u is constant, and we have the contrapositive.

Now, of course, if u or v is constant, suppose $u = a$ is constant, then of course $uv = av$ is harmonic, being a scalar multiple of a harmonic function. So, assume u, v both non-constant.

Define the set $A = \{z \in \Omega : \nabla u(z) = 0 \text{ or } \nabla v(z) = 0\}$. By the first problem, we know that neither of those sets have limit points in Ω. Since both of those are closed conditions, A is the union of two closed sets, and thus closed. Thus, consider $\Omega' = \Omega \setminus A$.

This is an open set, of course, being open minus closed, or equivalently, open intersect open. Further, it must be connected, since the points of A have no limit points, and are at most countable. Suppose $x, y \in \Omega'$, and consider a path between them in Ω. This may have at most countably many disconnections when we move to Ω'. Since A has no limit points, we may restrict down into a small enough punctured disk around any connection and take a path there - this punctured disk must be completely contained within Ω' due to A having no limit points. Since we have merely countably many of these issues, we are assured that we can patch this. Finally, this must be dense because let U be any open set in Ω. Choose any $a \in U$. There exists a disk $D(a, r) \subset U$, with uncountable cardinality. But, A is merely countable, thus $D(a, r) \setminus A \neq \emptyset$. Thus, since $A \cup \Omega' = \Omega$, we must have that $D(a, r) \cap \Omega' \neq \emptyset$. Thus, we have that Ω' is a region.

Now, we have that since $\Delta(uv) = 0$, we must have that $u_x v_x + u_y v_y = 0 \implies u_x v_x = -u_y v_y$. Since we wish uv to be harmonic, this must hold for all $z \in \Omega'$, which leads us to two cases, since $u_x, u_y, v_x, v_y \neq 0$ on Ω':

Case 1:

$$\begin{cases} v_x = -\lambda u_y \\ v_y = \lambda u_x \end{cases}$$

It should be clear that due to the definition of Ω', that $\lambda \neq 0$. In particular, since u, v are harmonic on Ω, they are continuous on all of Ω, with continuous first derivatives. Thus, these must actually hold for all of Ω, since u_x, u_y, v_x, v_y. Thus, we can say that the function

$$f = \lambda u + iv$$

is holomorphic, since these are exactly the Cauchy-Riemann equations for $u' = \lambda u, v' = v$. Thus, in this case, uv is harmonic if we may find a λ such that u, v are real and imaginary parts of a holomorphic function.

Case 2:

$$\begin{cases} u_x = -\lambda u_y \\ v_y = \lambda v_x \end{cases}$$

Consider the first equation. This implies that $u_{xx} = -\lambda u_{yx}$ and $u_{yy} = -\frac{1}{\lambda} u_{xy}$. Thus, in such a case, since u is harmonic, we must have that:

$$u_{xx} + u_{yy} = 0 \implies -\lambda u_{yx} - \frac{1}{\lambda} u_{xy} = 0 \implies u_{xy} = 0$$

Similarly:

$$v_{xx} + v_{yy} = 0 \implies \lambda v_{yx} + \frac{1}{\lambda} v_{xy} = 0 \implies v_{xy} = 0$$

However, since $u_x, u_y \neq 0$ on Ω', this implies that $u_x = f(x)$ since $u_{xy} = 0$ and $u_y = g(y)$ since $u_{yx} = 0$. Then, we must have that $u = F(x) + G(y)$ for $F' = f, G' = g$, and due to harmonicity, we further have that $f'(x) + g'(y) = 0$. This can only be true on all of Ω' if f', g' are constant, which implies that F, G are at most quadratics. However, since we started with $u_x = -\lambda u_y$, this implies that $F'(x) = -\lambda G'(y)$, and if F, G are polynomials, this implies then that F', G' are constants and thus F, G are linear. Thus, we have that:

$$u = -\lambda a x + a y + b$$

Running through the same logic with v, we see that:

$$v = cx + \lambda c y + d$$

However, here, we notice that:

$$\begin{cases} u_x = -\lambda a \\ u_y = a \\ v_x = c \\ v_y = \lambda c \end{cases}$$

Choosing $\lambda' = -\frac{c}{a}$, we see that:

$$\begin{cases} -\lambda' u_y = \frac{c}{a} a = c = v_x \\ \lambda' u_x = -\frac{c}{a} \cdot -\lambda a = \lambda c = v_y \end{cases}$$

and thus we are back in case 1. Thus, in either case, we see that uv is harmonic for u, v non-constant if there exists a $\lambda \neq 0$ such that $\lambda u + iv$ is holomorphic.

Now, let $f \in \mathcal{H}(\Omega)$, and consider $|f|^2$. Explicitly taking derivatives:

$$\frac{\partial^2}{\partial x^2}|f|^2 = \frac{\partial^2}{\partial x^2}(u^2 + v^2) = \frac{\partial}{\partial x}(2uu_x + 2vv_x) = 2(u_x^2 + uu_{xx} + v_x^2 + vv_{xx})$$

Of course then, the same equation will hold for the y, just switching the labels. Thus:

$$2(u_x^2+uu_{xx}+v_x^2+vv_{xx})+2(u_y^2+uu_{yy}+v_y^2+vv_{yy}) = 2(u(u_{xx}+u_{yy})+u_x^2+u_y^2+v(v_{xx}+v_{yy})+v_x^2+v_y^2) = 2(u_x^2+u_y^2+v_x^2+v_y^2)$$

where we've used the fact that because u, v come from the real, imaginary parts of a holomorphic function, u, v are harmonic.

Now, applying the Cauchy-Riemann equations, we obtain:

$$2(u_x^2 + u_y^2 + v_x^2 + v_y^2) = 2(2v_x^2 + 2v_y^2) = 4(v_x^2 + v_y^2) = 4(u_x^2 + u_y^2)$$

However, since u is a real-valued function, so must be u_x, u_y. Then, since $u_x^2, u_y^2 \geq 0$, for this to be harmonic, we must have $u_x, u_y = 0$. But that implies that u and thus v, are constants. Thus, we have that $|f|^2$ is harmonic iff f is constant. □

Question 3. Suppose f is a complex function on a region Ω, and both f, f^2 are harmonic on Ω. Prove that either $f, \overline{f}$ must be holomorphic on Ω.

Solution. It is clear that if $f = a \in \mathbb{C}$, that is, constant, then f, f^2 are harmonic and $f, \overline{f}$ are both holomorphic. Thus, we restrict ourselves to f non-constant.

□

Question 4. Let Ω be a region, and $f_n \in \mathcal{H}(\Omega)$ for all n. Set $u_n = \Re(f_n)$, and suppose u_n converges uniformly on compact subsets of Ω and that there exists $z \in \Omega$ such that $f_n(z)$ converges. Prove that f_n converges uniformly on compact subsets of Ω.

Solution. □

Question 5. Let Ω be a region, K a compact subset of Ω, and fix some $z_0 \in \Omega$. Let u be any positive harmonic function. Prove that there exists $\alpha, \beta > 0$ such that

$$\alpha u(z_0) \leq u(z) \leq \beta u(z_0)$$

for all $z \in K$.

Solution. $\square$