Homework #10

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2.1

Problem 5.4.6. Assume that $f : [a, b] to \mathbb{R}$ is continuous, and $D^+ f \geq 0$ on (a, b). Prove that f is monotone increasing on $[a, b]$.

Solution. First, suppose $D^+ f \geq \delta > 0$, and we have $x, y \in (a, b)$ such that $x < y$. Then, since f is a continuous function on a closed and bounded interval $[x, y]$ it attains a maximum on that interval. Suppose x_0 be a point on (x, y) such that $f(x_0)$ is a maximum. Then, we have, for $t > x_0$:

$$\frac{f(t) - f(x_0)}{t - x_0} \leq 0$$

due to being a maximum. Then, since this is true for any $t > x_0$, this implies that:

$$D^+ f(x_0) = \limsup_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0$$

But, by hypothesis, $D^+ f \geq \delta$ on $(a, b) \supset [x, y]$, and we have a contradiction. Thus, this means that $x_0 \notin (x, y)$, and therefore, we may only have a maximum at x or y itself. But, because of the rightwards limit on D^+, we may make the same argument for x. Therefore, $f(y)$ is a maximum on $[x, y]$, and thus $f(x) \leq f(y)$. Since the choice of $x, y \in (a, b)$ was arbitrary, this means that we are monotone increasing on all of (a, b), and due to continuity, this remains true on $[a, b]$.

Now, suppose we have $D^+ f \geq 0$. Fix some $\delta < 0$, and define the function $g(x) = f(x) + \delta x$. This is a continuous function on $[a, b]$, being the sum of two continuous functions, and further, we have that:

$$D^+ g(x) = \limsup_{h \to 0^+} \frac{g(x + h) - g(x)}{h} = \limsup_{h \to 0^+} \frac{f(x + h) + \delta(x + h) - f(x) - \delta x}{h} = \limsup_{h \to 0^+} \frac{f(x + h) - f(x)}{h} + \delta = D^+ f + \delta$$

Since we have that $D^+ f \geq 0, \delta > 0$, we have that $D^+ g > 0$. Then, we have that g is monotone increasing, by above. Then, take $x, y \in [a, b]$ such that $x < y$. We have that:

$$g(x) \leq g(y) \implies f(x) + \delta x \leq f(y) + \delta y \implies f(y) - f(x) \geq \delta(x - y)$$

However, the choice of $\delta > 0$ was arbitrary. So, we take a sequence of $\delta \to 0$ and retrive that $f(y) - f(x) \geq 0$. Thus, f is monotone increasing on $[a, b]$.

□

Problem 5.4.8. Let ϕ be the Cantor-Lebesgue function on $[0, 1]$. Extend ϕ onto all of $\mathbb{R}$ by setting $\phi(x) = \phi(0) = 0$ for $x < 0$ and $\phi(x) = \phi(1) = 1$ for $x > 1$. Let $\{[a_n, b_n]\}_n$ be an enumeration of all subintervals of $[0, 1]$ such that a_n, b_n are rational endpoints in $[0, 1]$ with $a_n < b_n$. For each $n \in \mathbb{N}$, set:

$$f_n(x) = 2^{-n}\phi\left(\frac{x-a_n}{b_n-a_n}\right)$$

Observe that f_n is monotone increasing on $\mathbb{R}$ and has uniform norm $\|f_n\|_u = 2^{-n}$. Prove the following:

(a) The series $f = \Sigma f_n$ converges uniformly on $[0,1]$.

(b) f is continuous and monotone increasing on $[0,1]$.

(c) f is strictly increasing on $[0,1]$.

(d) f is singular on $[0,1]$, that is, $f'(x)$ exists for almost every $x \in [0,1]$ and $f' = 0$ almost everywhere.

Solution. (a)

Let $\epsilon > 0$ be given. We notice, by the shape of the f_n, that because ϕ is bounded between 0 and 1, that f_n is bounded between 0 and 2^{-n}. Then, take any point $x \in [0,1]$, and choose k such that $2^{-k} < \epsilon$. If we look at partial sums, then we notice:

$$f(x) - \Sigma_{i=1}^{M} f_i(x) = \Sigma_{i=M+1}^{\infty} f_i(x) \leq \Sigma_{i=M+1}^{\infty} 2^{-i} = 2^{-M}$$

Thus, if we choose $M = k$, then we have that the difference from f to the partial sum $\Sigma_i^k f_i$ can be no more than $2^{-k} < \epsilon$. Since this is true regardless of the point x, this implies that this is uniform convergence.

(b)

We recall that ϕ is continuous, therefore, since f_n merely multiplies it by a constant, and shifts the window on where f_n is increasing, f_n is continuous as well. Then, since we've proved in part (a) that the convergence to f is uniform, we must have that f is continuous, since the uniform convergence of continuous functions is continuous. Further, because each f_n is monotone increasing, the sum of monotone increasing, non-negative functions must also be monotone.

(c)

Let $0 \leq x < y \leq 1$. We may find two rational points p, q such that $0 \leq x < p < q < y \leq 1$. Since these are rational numbers, it has some enumeration in the subintervals with rational endpoints $\{[a_i, b_i]\}$ and corresponds with a $f_i = 2^{-i}\phi\left(\frac{x-a_i}{b_i-a_i}\right)$. In particular, we notice that $f_i(p) = f_i(a_i) = 0, f_i(q) = f_i(b_i) = 2^{-i}$. Then, if we consider the series $\Sigma f_n(y), \Sigma f_n(x)$, looking term by term, because each of the f_n are monotone, non-negative, and because at least $f_i(y) = f_i(q) > f_i(p) > f_i(x)$, we have that $\Sigma f_n(y) > \Sigma f_n(x)$. Since this can be done with any choice of x, y, we have then that f is actually strictly increasing.

(d)

Fixing an $x \in [0,1]$, due to the fact that we are bounded above on each f_n by $\|f_n\|_u = 2^{-n}$, we are actually bounded above on f by $\Sigma_{n=1}^{\infty} 2^{-n} = 1$. Further, because of the fact that the f_n are non-negative, we have that the partial sums are monotone increasing. Thus, by the monotone convergence theorem, we have that the series $f = \Sigma f_n$ converges for every $x \in [0,1]$. Then, by lemma 5.4.4, we have that f is differentiable almost everywhere, and:

$$f'(x) = \Sigma f_n'(x)$$

almost everywhere. However, we know from working with the Cantor-Lebesgue function, that this function has 0 derviative almost everywhere on $[0,1]$, and on the extension to the full real line, it still has 0 derivative almost everywhere. Then, we can see that, for each f_n, there is a Z_n such that $|Z_n| = 0$, and that f_n has non-0 derivative. Then, if we look at $[0,1] \setminus \cup_n Z_n$, on this set, by definition, $f_n' = 0$ for all n. Then, on that set, we have that:

$$f'(x) = \Sigma f_n'(x) = \Sigma 0 = 0$$

and because $|\cup_n Z_n| = 0$, this is almost everywhere.

□

2.2

Problem 5.5.17. Given a locally integrable function f on $\mathbb{R}^d$, define a non-centered maximal function by:

$$M^*f(x) = \sup\left\{\frac{1}{|B|}\int_B |f| : B \text{ is any open ball that contains } x\right\}$$

Prove that $Mf \leq M^*f \leq 2^d Mf$.

Solution. Clearly, since Mf is defined as

$$Mf(x) = \sup_{h>0} \frac{1}{|B_h(x)|}\int_{B_h(x)} |f(t)|dt$$

that is, the supremum over only balls centered on x and we are defining M^*f over every ball containing x, which includes balls centered on x, this implies, by the properties of the supremum, that $Mf \leq M^*f$. So, we need only prove that $M^*f \leq 2^d Mf$

Suppose we have a ball B with center c, radius r, such that $x \in B$. Let $z \in B$. We claim that $|z-x| < 2r$. We can see this via the triangle inequality:

$$|z-x| \leq |z-c| + |c-x| \leq r + r = 2r$$

Therefore, z is contained within a ball of radius $2r$ around x. Since the choice of z was arbitrary, this implies that all of B is contained within this ball, which we will call B'. We also recall, that from 2.3.15, about linear changes of variable, since this is merely a translation composed with a dilation by 2 of B, that we have that $|B'| = |L(B)| = |2I \cdot T(B)| = |\det(2I \cdot T)||B|$, where we use the trick about looking at the ball in a $\mathbb{R}^{d+1}$ space to view a translation as a linear transformation.

Here, we notice that the determinant of a translation is 1, and the determinant of a dilation by 2 in every coordinate is 2^d. Thus, we have that $|B'| = 2^d|B|$.

Then, looking at the integrand of the maximal functions, we have that:

$$\frac{1}{|B|}\int_B |f| \leq \frac{2^d}{|B'|}\int_{B'} |f|dt$$

because the fractions are equal, but $B \subseteq B'$ and $|f|$ is non-negative, so $\int_B |f| \leq \int_{B'} |f|$.

But, then we have that, by the definition of Mf, that since B' is a ball centered on x:

$$\frac{2^d}{|B'|}\int_{B'} |f|dt = 2^d \frac{1}{|B'|}\int_{B'} |f|dt \leq 2^d Mf$$

Since we may do this for every ball B that contains x, this extends to the supremum. Thus, we have that $M^*f \leq 2^d Mf$ □

Problem 5.5.19. Let A be any subset of $\mathbb{R}^d$ with $|A|_e > 0$. Define the density of A at a point $x \in \mathbb{R}^d$ to be:

$$D_A(x) = \lim_{r\to 0} \frac{|A \cap B_r(x)|_e}{|B_r(x)|}$$

whenever this limits exists. Prove the following:

(a) $D_A(x) = 1$ for almost every $x \in A$.

(b) A is measurable if and only if $D_A(x) = 0$ for almost every $x \in A$.

Additionally, exhibit a measurable set E and a point x such that $D_E(x)$ does not exist, and given $0 < \alpha < 1$, exhibit a measurable set E and a point x such that $D_E(x) = \alpha$.

Solution. (a)

I'm really not quite sure how to prove this in the general case. This is clear for a measurable set E, since then we may take $f = \chi_E$, locally integrable, so by applying the Lebesgue Differentiation Theorem, we find that:

$$\lim_{h \to 0} \frac{1}{|B_h(x)|} \int_{B_h(x)} f(t)dt = f(x)$$

However, we notice that $\int_{B_h(x)} f(t)dt = \int_{B_h(x)} \chi_E(t)dt = |B_h(x) \cap \chi_E|$, so we get that:

$$\lim_{h \to 0} \frac{|B_h(x) \cap \chi_E|}{|B_h(x)|} = \chi_E(x)$$

for almost every $x \in \mathbb{R}^d$. In particular then, this means that restricting to A, this is 1 for almost every $x \in A$.

(b)

The forward direction is clear, from another application of the LDT, and noticing that $\chi_A(x) = 0$ for $x \notin A$. I'm not sure how to attack the reverse direction.

An easy example for $\alpha \in (0, 1)$ in $\mathbb{R}^2$. Take a point $x = (0, 0)$, and take E to be defined in radial coordinates, as $E = \{(r, \theta) : 0 \leq \theta < 2\pi\alpha\}$. Clearly, for any ball centered on the origin, we cut out exactly $2\pi\alpha/2\pi$ of the ball.

I do not see an easy example for when $D_E(x)$ does not exist.

□

2.3

Problem 6.1.9. Prove that $f \in \mathrm{AC}[a, b]$ if and only if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every finite collection of nonoverlapping subintervals $\{[a_j, b_j]\}_j$ of $[a, b]$, we have that:

$$\Sigma_{j=1}^{N}(b_j - a_j) < \delta \implies \Sigma_{j=1}^{N}|f(b_j) - f(a_j)| < \epsilon$$

Solution. It is clear that if $f \in \mathrm{AC}[a, b]$, then the statement holds, because we recall that we define absolutely continuous as, for every $\epsilon > 0$, there exists $\delta > 0$ such that for either finite or countably infinite non overlapping collections of subintervals,

$$\Sigma_{j=1}^{N}(b_j - a_j) < \delta \implies \Sigma_{j=1}^{N}|f(b_j) - f(a_j)| < \epsilon$$

So, it is already true by definition.

Now, instead, suppose we only know that the $\epsilon - \delta$ criteria holds for finitely many collections of subintervals. Then, we wish that this holds for countably infinite collections of subintervals.

Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that, for every finite collection of intervals, we have that

$$\Sigma_{j=1}^{N}(b_j - a_j) < \delta \implies \Sigma_{j=1}^{N}|f(b_j) - f(a_j)| < \epsilon/2$$

Let $\{[x_j, y_j]\}_j$ be a countably infinite collection of nonoverlapping subintervals such that $[a_j, b_j] \subseteq [a, b]$ for all j, and such that

$$\Sigma_{j=1}^{\infty}(y_j - x_j) < \delta$$

Then, we look at a sequence of finite collection of subintervals, that is, $\{x_j, y_j\}_j^M$. In particular, we have that:

$$\Sigma_{k=1}^{M}(y_j - x_j) \leq \Sigma_{j=1}^{\infty}(y_j - x_j) < \delta$$

because of the fact $y_j - x_j \geq 0$. Then, we have that, by hypothesis:

$$\Sigma_{j=1}^{M}|f(y_j) - f(x_j)| < \epsilon/2$$

But, this is true for every N, since they are all finite. Then, taking the limit as $N \to \infty$, we have that:

$$\Sigma_{j=1}^{\infty} = \lim_{M\to\infty} \Sigma_{j=1}^{M}|f(y_j) - f(x_j)| < \epsilon/2 < \epsilon$$

Thus, $f \in \mathrm{AC}[a, b]$.

□

Problem 6.1.10. (a) Prove that $\mathrm{AC}[a, b]$ is a closed subspace of $\mathrm{BV}[a, b]$ with respect to the norm $\|f\|_{\mathrm{BV}}$ defined by 5.2.26. That is, show that if $f_n \in \mathrm{AC}[a, b]$, $f \in \mathrm{BV}[a, b]$, and $\|f - f_n\|_{\mathrm{BV}} \to 0$, then $f \in \mathrm{AC}[a, b]$.

(b) Exhibit functions f_n, f such that $f_n \in \mathrm{AC}[a, b]$ and f_n converges uniformly to $f \in \mathrm{BV}[a, b]$, but $f \notin \mathrm{AC}[a, b]$. Thus the uniform limit of absolutely continuous functions need not be absolutely continuous.

Solution. (a)

Let $\epsilon > 0$ be given. First, since $\|f_n - f\|_{\mathrm{BV}} \to 0$, we may pick N such that $\|f_m - f\|_{\mathrm{BV}} < \epsilon/2$ for every $m > N$. Choose n such that n is the smallest such m that works. Since $f_n \in \mathrm{AC}[a, b]$, we may choose δ such that for $\{[a_j, b_j]\}_{j=1}^{M}$:

$$\Sigma_j^M b_j - a_j < \delta \implies \Sigma_j^M |f_n(b_j) - f_n(a_j)| < \epsilon/2$$

Then, consider the sum:

$$\Sigma_j^M|f(b_j) - f(a_j)| = \Sigma_j^M|f(b_j) - f_n(b_j) + f_n(b_j) - f_n(a_j) + f_n(a_j) - f(a_j)| \leq$$

$$\Sigma_j^M|[f(b_j)-f_n(b_j)]-[f(a_j)-f_n(a_j)]|+\Sigma^M|f_n(b_j)-f_n(a_j)| = \Sigma_j^M|(f-f_n)(b_j)-(f-f_n)(a_j)]|+\Sigma_j^M|f_n(b_j)-f_n(a_j)|$$

Now, since $\|f - f_n\|_{\mathrm{BV}} < \epsilon/2$, we have that, in particular, $V[f - f_n; a, b] < \epsilon/2$. Then, since $\{[a_j, b_j]\}_{j=1}^{M}$ are non-overlapping, we may extend this to a partition on $[a, b]$ by including every a_j, b_j with a, b, that is, if we have that $a_1 < b_1 < a_2 < b_2 < ... < a_M < b_M$, then we can take the partition:

$$\Gamma = \{a = x_0 < a_1 = x_1 < b_1 = x_2 < ... < b_M = x_{2M} < b = x_{2M+1}\}$$

This is a proper partition on $[a, b]$, and we have that:

$$\Sigma_j^M|(f - f_n)(b_j) - (f - f_n)(a_j)| \leq \Sigma_{i=0}^{2M}|(f - f_n)(x_{i+1}) - (f - f_n)(x_i)| \leq \|f - f_n\|_{\mathrm{BV}} < \epsilon/2$$

because every subinterval $\{[a_j, b_j]\}_{j=1}^{M}$ is contained within the parition, and because $\|f - f_n\|_{\mathrm{BV}} = V[f - f_n; a, b] + \|f - f_n\|_u$, then since they are all non-negative, we have that $\Sigma_{i=0}^{2M}|(f - f_n)(x_{i+1}) - (f - f_n)(x_i)| \leq V[f - f_n; a, b] \leq \|f - f_n\|_{\mathrm{BV}} < \epsilon/2$.

Further, by the choice of δ, we have that $\Sigma_j^M|f_n(b_j) - f_n(a_j)| < \epsilon/2$

Thus, we have that with this choice of δ, that:

$$\Sigma_j^M|f(b_j) - f(a_j)| \leq \Sigma_j^M|(f - f_n)(b_j) - (f - f_n)(a_j)]| + \Sigma_j^M|f_n(b_j) - f_n(a_j)| < \epsilon/2 + \epsilon/2 = \epsilon$$

By the last problem, 6.1.9, we have that showing this works for finite subintervals is enough to show that $f \in \mathrm{AC}[a, b]$.

(b)

Consier the functions that iterate to the Cantor-Lebesgue function, ϕ. That is, suppose $C_1 = [0,1] \setminus (1/3, 2/3)$, and C_n defined iteratively by removing middle thirds, and ϕ_1 being linear on C_1 and constantly 2^{-1} on $(1/3, 2/3)$, and defining ϕ_n iteratively.

From the book, we know that $\phi_n \to \phi$ uniformly, since it can only differ at most by 2^{-n} regardless of the choice of x. Further, we see that $f_n \in \mathrm{AC}[a,b]$ for all n. This is because fix an n. Then, the measure of the construction of the Cantor set on which ϕ_n is linear is exactly $(2/3)^n$. Then, we know that the slope on those segments is exactly $(3/2)^n$, since it must range from 0 to 1. Then, let $\epsilon > 0$ be given. Choose δ such that $\delta < (2/3)^n \epsilon$. Then, consider any set of countable non-overlapping intervals $\{[a_j, b_j]\}_{j=1}$ such that $[a_j, b_j] \subseteq [a,b]$ and $\Sigma_j b_j - a_j < \delta$. Then, since ϕ_n is linear only on the complement of the iterations of the Cantor set, we have that:

$$\Sigma f(b_j) - f(a_j) \leq \Sigma (3/2)^n (b_j - a_j) \leq (3/2)^n \delta < \epsilon$$

Thus, for each n, $\phi_n \in \mathrm{AC}[a,b]$. Further, since ϕ is monotone, we know that $V[\phi; a, b] = 1$ and thus $\phi \in \mathrm{BV}[a,b]$. However, from example 6.1.2 in the book, ϕ is not in $\mathrm{AC}[a,b]$. Thus, the uniform limit of absolutely continuous functions need not be absolutely continuous.

$\square$