Homework #4

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Math 233: Homework #4

April 6, 2023

Question 1. Let L_1, L_2 be lines in the plane. For which pairs of L_1, L_2 do there exists real functions, harmonic on the entire plane, 0 on $L_1 \cup L_2$, but not vanishing identically?

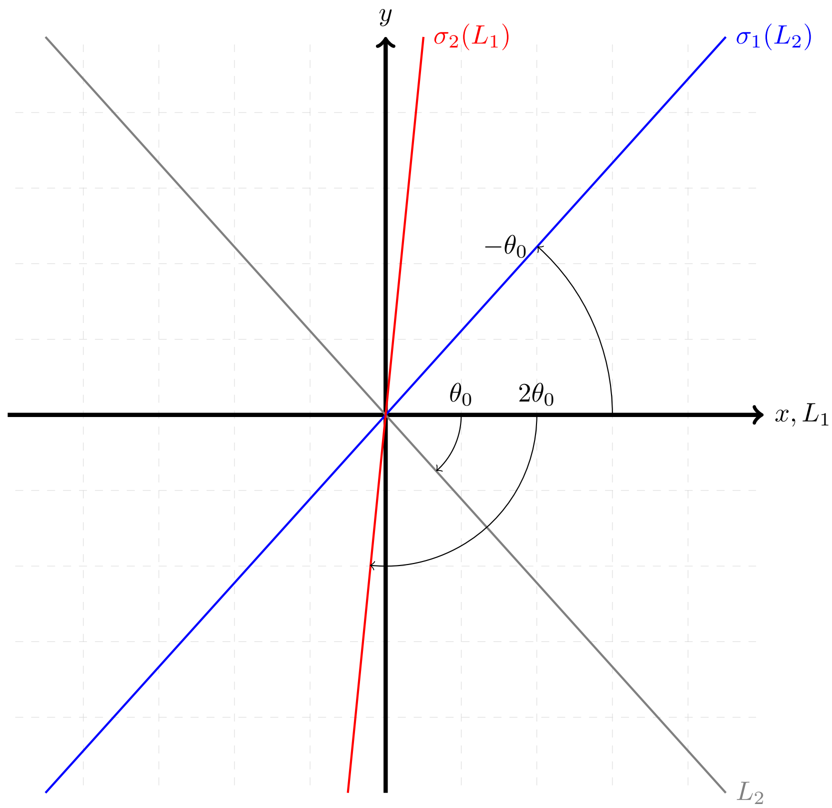
Solution. First, we notice that for any real function v, harmonic on the entire plane, it is the imaginary part of some holomorphic function. First, we know already that by 11.10, every real harmonic function is the real part of a holomorphic function, locally at least. Then, by considering disks around every point $z \in \mathbb{C}$, this can be extended to a holomorphic function f such that $\Re(f) = v$, because on the disks, the local holomorphic functions may only differ by an imaginary constant, and it must align on intersections of disks, thus there may only be a single entire function.

Now, consider if. Since i is a constant, this is clearly holomorphic. Further, by construction $\Im(f) = v$. Thus, we have a holomorphic function such that v is its imaginary part.

Now, suppose v is harmonic, and $v(L_1) = 0, v(L_2) = 0$. Without loss of generality, since we may translate v without affecting the derivatives, we may take $L_1 \cap L_2 = \{(0,0)\}$. By a further linear change of coordinates, we may assume that L_1 is the real line, which will keep $v_{xx} + v_{yy} = 0$.

Suppose L_1 and L_2 intersect. Suppose that the angle between L_1, L_2 is θ_0.

By the Schwarz reflection principle (11.14), and a relabeling of the two lines as need be, if we call σ_1, σ_2 the reflections of the plane with respect to L_1, L_2, we must have that $f(\sigma_1(z)) = \overline{f}(z), f(\sigma_2(z)) = \overline{f}(z)$. In particular then, on L_1, L_2, we have that $v(\sigma_1(z)) = v(z) = 0, v(\sigma_2(z)) = v(z) = 0$. Pictorially:



where we have that the angle between $L_1, \sigma_1(L_1)$ is $2\theta_0$ because the angle between $\sigma_1(L_1)$ and L_2 is θ_0, due to how reflections work. Further, we also see that $\sigma_1(L_2)$ takes on the angle $-\theta_0$.

We notice that we may iterate this process, and in fact generate lines of $k\theta_0$ via successive reflections. However, we know that if θ_0 is not a rational multiple of π, then $\{e^{im\theta_0} : m \in \mathbb{Z}\}$ is dense in T. And since $v = 0$ on all of these lines, if it is 0 on a dense set, then it is 0 everywhere by continuity. Thus, this implies that we must have that θ_0 is a rational multiple of π.

Now, suppose instead that L_1, L_2 are parallel. In such a case, applying the Schwarz reflection principle on successive lines, we note that then we must have that v is periodic, 0 at each interval $d = \text{dist}(L_1, L_2)$, since we can keep translating and applying reflections to find a line on the opposite side. For example, assuming $L_1 : x = 1, L_2 : x = 5$ one such v could be $v(x, y) = e^y \sin(\pi(x-1)/\pi)$. This is generalizable with a suitable linear transformation on x, y to match our parallel 1-D lattice.

$\square$

Question 2. Suppose Δ is a closed equilateral triangle in the plane, with vertices a, b, c. Find $\max\{|z - a||z-b||z-c|\}$ for $z \in \Delta$.

Solution. First, fix some $a, b, c \in \mathbb{C}$. We notice that the function:

$$f(z) = (z-a)(z-b)(z-c)$$

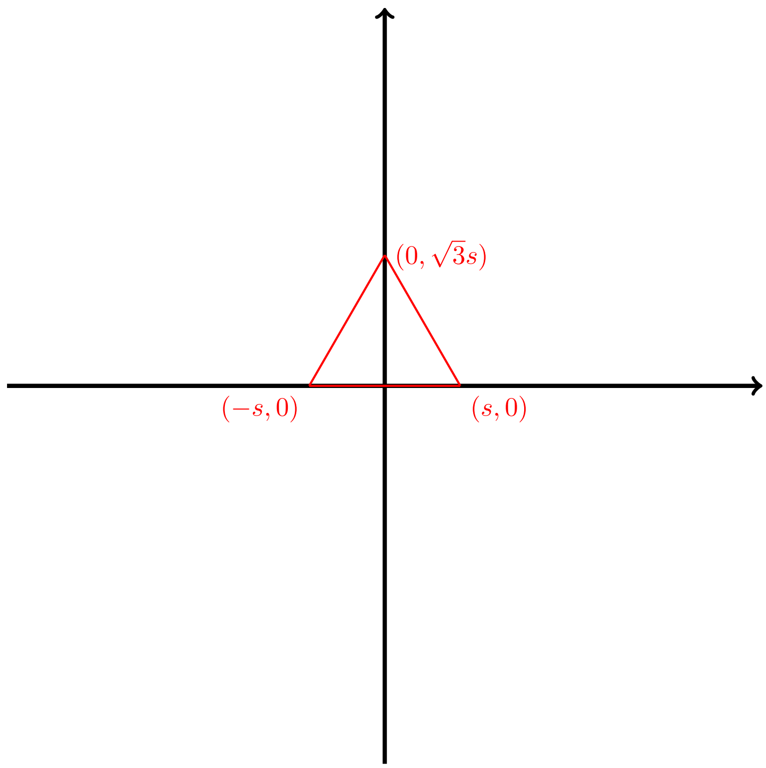
is a polynomial, thus entire. Then, we may apply the maximum modulus principle to our closed triangle which then says that:

$$|z-a||z-b||z-c| \leq \|(z-a)(z-b)(z-c)\|_{\partial\Delta}$$

Thus, it is sufficient to consider the value of $(z-a)(z-b)(z-c)$ on the boundary of our equilateral triangle. Further, since the quantity we are concerned about is

$$\|(z-a)(z-b)(z-c)\|_{\partial\Delta} = \max\{|z-a||z-b||z-c| : z \in \partial\Delta\}$$

where we use max instead of sup due to being compact, this is simply the product of the distances from z to a, b, c. Thus, under any isometries, this product is preserved. Therefore, we may take rotations and translations such that the following picture holds, for $2s$ the side length of our triangle:



Due to symmetries, we can also restrict ourselves to analyzing z on the real line, as a simple rotation will find us the value on the other sides. It should also be clear that due to reflectional symmetries, we can restrict ourselves to the non-negative reals as well.

Let $z = (x, 0)$ with $x \in [0, s]$. Computing the value of $|f|$, we find:

$$|f(z)| = |(x,0)-(s,0)|+|(x,0)-(-s,0)|+|(x,0)-(0,\sqrt{3}s)| = (s-x)(x+s)\sqrt{x^2+3s^2} = (s^2-x^2)\sqrt{x^2+3s^2}$$

But now, this is a real function, so we may take a derivative and check endpoints to find the maximum. We see pretty clearly that:

$$f'(x) = -2x\sqrt{x^2+3s^2} + x(s^2-x^2)\frac{1}{\sqrt{x^2+3s^2}} = \frac{1}{\sqrt{x^2+3s^2}}\left(-2x(x^2+3s^2) + x(s^2-x^2)\right) =$$

$$\frac{1}{\sqrt{x^2+3s^2}}(-3x^3 - 5xs^2) = \frac{1}{\sqrt{x^2+3s^2}}x(-3x^2-5s^2)$$

Clearly, since $x^2 \geq 0, s^2 > 0$, we have that $x^2 + 3s^2$ and $-3x^2 - 5s^2$ never vanish. Thus, we have only the critical point $x = 0$. Since at $x = s$, f vanishes, because this is also a boundary of our domain, this must be the maximum. Thus, we have that the maximum of f is equal to:

$$f(0) = s^2\sqrt{3s^2} = \sqrt{3}s^3$$

where $2s = |a-b| = |b-c| = |c-a|$

□

Question 3. Suppose $f \in \mathcal{H}(\Pi^+)$, where $\Pi^+ = \{z = x + yi : y > 0\}$, and $|f| \leq 1$. How large can $|f'(i)|$ be? Find the extremal functions.

Solution. First, for $U = \{z : |z| < 1\}$, we consider the map $\psi : U \to \Pi^+$ via:

$$\psi(z) = i\frac{1-z}{1+z}$$

.

On U, this map is holomorphic. Further, this is injective. Suppose we have that $\psi(z) = \psi(w)$. Then, since on U, $z, w \neq -1$:

$$i\frac{1-z}{1+z} = i\frac{1-w}{1+w} \implies (1+w)(1-z) = (1+z)(1-w) \implies 1+w-z-wz = 1+z-w-wz \implies 2w = 2z \implies w = z$$

Further, we have that this map is surjective onto Π^+. Let $\zeta = a + bi \in \Pi^+$. Then, we have that, for $z = x + yi$:

$$f(z) = \zeta \iff i\frac{1-x-yi}{1+x+yi} = a + bi \iff 1 - x - yi = -ai - axi + ay + b + bx + byi$$
$$\iff \begin{cases} 1 - x = ay + b + bx \\ -y = -a - ax + by \end{cases} \iff x = \frac{1 - ay - b}{1+b}$$

where we've used the fact that $z \in U$ so $1 + x + yi \neq 0$ and $\zeta = \Pi^+$, so $b \neq -1$. Now, substituting into the second equation, this would enforce that:

$$-y = -a - a\frac{1-ay-b}{1+b} + by \iff -y\left(1 + b + \frac{a^2}{b+1}\right) = -a - \frac{a-ab}{1+b} = \frac{-2a}{1+b} \iff$$
$$y = \frac{2a}{1+b} \cdot \frac{b+1}{a^2 + (b+1)^2} = \frac{2a}{a^2 + (b+1)^2}$$

Now, substituting back in for x, we find that:

$$x = \frac{1 - a\frac{2a}{a^2+(b+1)^2} - b}{1+b} = \frac{1}{1+b} \cdot \frac{a^2 + (b+1)^2 - 2a^2 - a^2 b - b(b+1)^2}{a^2 + (b+1)^2} = \frac{1}{b+1}\frac{-a^2(b+1) + (b+1)^2(1-b)}{a^2 + (b+1)^2} =$$
$$\frac{1 - a^2 - b^2}{a^2 + (b+1)^2}$$

Now, we need only check that this lives within U. Well:

$$x^2 + y^2 = \frac{1}{(a^2 + (b+1)^2)^2}[(1 - a^2 - b^2)^2 + 4a^2]$$

It should be clear that this is always less than the denominator. If we expand everyyhing out, we see that we have the numerator as:

$$1 + a^4 + b^4 + 2a^2 - 2b^2 + 2a^2 b^2$$

and the denominator as:

$$a^4 + 2a^2(b+1)^2 + (b+1)^4 = a^4 + 2a^2b^2 + 4a^2b + 2a^2 + b^4 + 4b^3 + 8b^2 + 4b + 1$$

Subtracting the numerator from the denominator, we see:

$$(a^4 + 2a^2b^2 + 4a^2b + 2a^2 + b^4 + 4b^3 + 8b^2 + 4b + 1) - (1 + a^4 + b^4 + 2a^2 - 2b^2 + 2a^2b^2) =$$
$$4a^2b + 4b^3 + 10b^2 + 4b$$

Now, because (a, b) are chosen from the upper half plane, we have that this number must be positive, since $a^2 \geq 0$, and $b > 0$. Thus, we have that $x^2 + y^2 < 1$, and therefore $z \in U$. Thus, ψ is surjective.

Lastly, we consider the action of ψ on $T = \{z : |z| = 1\}$, or really, $T \setminus \{-1\}$. Well, if $|z| = 1$, we may write it as $z = e^{i\varphi}$. First, we notice that:

$$\begin{cases} \sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \\ \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \end{cases} \implies \tan(x) = i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = i\frac{e^{2ix} - 1}{e^{2ix} + 1}$$

Then, we have that:

$$\psi(e^{i\varphi}) = i\frac{1 - e^{i\varphi}}{1 + e^{i\varphi}} = -i\frac{e^{i\varphi} - 1}{1 + e^{i\varphi}} = -i \cdot i \tan\left(\frac{\varphi}{2}\right) = \tan\left(\frac{\varphi}{2}\right)$$

Since on $T \setminus \{-1\}$, $\varphi \in (-\pi, \pi)$, and on $x \in (-\pi/2, \pi/2)$, $\tan(x) \in (-\infty, \infty)$, $\tan(\varphi/2)$ covers the real line.

Now, let f be as given, and consider the map $g = f \circ \psi : U \to \mathbb{C}$. Because $|f| \leq 1$ on the upper half plane, and the work we've done above, we have that $g \in \mathcal{H}^\infty(U)$, $\|g\|_\infty \leq 1$, and since g is defined on U, we have that, as stated in 12.5, we may take $\alpha = 0 < 1$. Further, if $g(0) = \beta$, then we may assume that $|\beta| < 1$, as otherwise, by the maximum modulus principle, since $|g| \leq 1$ on U, this extends to the boundary by continuity. So, if $|g(0)| = 1$, then g is constant everywhere and the derivative is 0.

Then, by the discussion in 12.5, we have that:

$$|g'(0)| \leq 1 - |\beta|^2$$

However, here, we notice that because $g = f \circ \psi$, $\psi(0) = i\frac{1-0}{1+0} = i$, so $g'(0) = f'(i), g(0) = \beta = f(i)$.

Thus, restated in terms of f, we have that:

$$|f'(i)| \leq 1 - |f(i)|^2$$

Thus, we have two conditions to realize the maximum value here across all functions f. Firstly, we require $f(i) = 0$, and secondly, by Theorem 12.2, if $f(i) = \beta = 0$, then we have that $|g'(0)| = 1$ occurs if and only if $g = \lambda z$, for some $\lambda \in \mathbb{C} : |\lambda| = 1$, that is, f composed with ψ acts as a rotation by some λ on the unit disk U.

This means that, we need only take an inverse to ψ, with some scale factor for the rotation, and a translation such that $f(i) = 0$. Well, I claim that $f(z) = \frac{iz+1}{-iz+1}$ acts as a left inverse to ψ:

$$f\left(i\frac{1-z}{1+z}\right) = \frac{-\frac{1-z}{1+z} + 1}{\frac{1-z}{1+z} + 1} = \frac{-1 + z + z + 1}{1 - z + 1 + z} = \frac{2z}{2} = z$$

Further, we see that $f(i) = \frac{i^2+1}{-i^2+1} = \frac{0}{2} = 0$. So that part is all set.

Then, the maximal functions take on exactly the form $f_\lambda(z) = \lambda\frac{iz+1}{-iz+1}$ for $\lambda \in \mathbb{C} : |\lambda| = 1$.

□

Question 4. Suppose $f \in \mathcal{H}(\Omega)$. Under what conditions can $|f|$ have a local minimum in Ω?

Solution. □

Question 5. (a) Suppose that Ω is a region, D is a disc, $\overline{D} \subset \Omega, f \in \mathcal{H}(\Omega)$, non-constant, and $|f|$ is constant on ∂D. Prove that f has at least one zero in D.

(b) Find all entire functions f such that $|f(z)| = 1$ for all $|z| = 1$.

Solution. □