NUS | Computing
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CS3230

Computer Science

T08 – Week 9

Post-Midterm Exam Discussion

CS3230 – Design and Analysis of Algorithms

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Admin

- TG19 stats: mean: 22.07, median: 21.75, 25th: 16.5, 75th: 28.5
- Course stats: mean: 21.27, median: 19.75, 25th: 15, 75th: 26.5

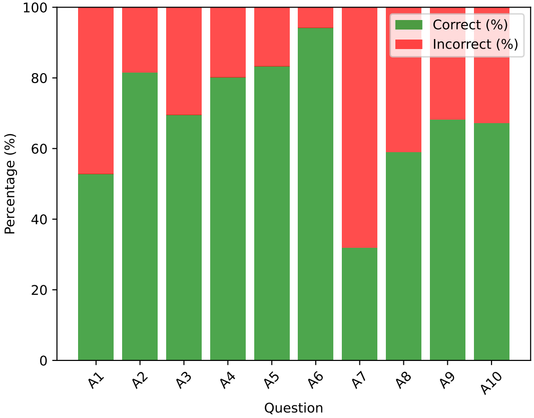

Figure 1: MCQ Correctness Statistics from course.

Question A.1 [P1]

$n^{10} - n^9$ is in

- A $\Omega(n^{11})$
- B $o(n^{10})$
- C $\Theta(n^9)$
- D $O(n^8)$
- E None of the above

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Solution

Since $\lim_{n\to\infty} \frac{n^{10}-n^9}{n^{10}} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1 \implies n^{10} - n^9 \in \Theta(n^{10})$, *the correct answer is None of the above.*

Question A.2

$(n+1)!$ is in

- A $O(n!)$
- B $\omega(n!)$
- C $\Theta(n!)$
- D $o(n!)$
- E None of the above

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Solution

$\lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty \implies (n+1)! \in \omega(n!)$.

Question A.3

$2^{\log_3 n}$ is in

A $O(\log_2 n)$

B $\Theta(n^2)$

C $\omega(n)$

D $\Omega(\sqrt{n})$

E None of the above

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- B $\Theta(n^2)$
- C $\omega(n)$
- D $\Omega(\sqrt{n})$
- E None of the above

Solution

$2^{\log_3 n} = n^{\log_3 2} = n^{0.6309\ldots}$, *so options A, B, and C are incorrect. We check for D:* $\lim_{n\to\infty} \frac{2^{\log_3 n}}{\sqrt{n}} = \lim_{n\to\infty} \frac{n^{\log_3 2}}{n^{1/2}} = \lim_{n\to\infty} n^{\log_3 2 - 1/2} = \infty \implies 2^{\log_3 n} \in \Omega(\sqrt{n})$.

Question A.4

Suppose $f(n) \in \Theta(n^2(\log n)^5)$ and $g(n) \in \Theta(n^5(\log n)^2)$. Then, $f(n) + g(n)$ is in

A $\Theta(n^5(\log n)^5)$

B $\Theta(n^2(\log n)^5)$

C $\Theta(n^5(\log n)^2)$

D $\Theta(n^7(\log n)^7)$

E None of the above

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- B $\Theta(n^2(\log n)^5)$
- C $\Theta(n^5(\log n)^2)$
- D $\Theta(n^7(\log n)^7)$
- E None of the above

Solution

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^2(\log n)^5}{n^5(\log n)^2} = \lim_{n\to\infty} \frac{(\log n)^3}{n^3} = 0 \implies f(n) \in o(g(n))$. *Hence,* $\lim_{n\to\infty} \frac{f(n)+g(n)}{g(n)} = \lim_{n\to\infty} \frac{f(n)}{g(n)} + 1 = 0 + 1 = 1 \implies f(n) + g(n) \in \Theta(n^5(\log n)^2)$.

Question A.5

Suppose $T(n) = 36T(n/6) + 2n + n^{8/3}$. Then, $T(n)$ is in

- A $\Theta(n^{8/3})$
- B $\Theta(n^{8/3} \log n)$
- C $\Theta(n^2)$
- D $\Theta(n^2 \log n)$
- E None of the above

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- C $\Theta(n^2)$
- D $\Theta(n^2 \log n)$
- E None of the above

Solution

Since $a = 36$, $b = 6$, $d = \log_6 36 = 2$, and $f(n) = 2n + n^{8/3} \in \Omega(n^{2+\epsilon})$ with $\epsilon = \frac{8}{3} - 2 = \frac{2}{3}$, and the regularity condition holds (e.g., $36 \cdot f(n/6) \leq \frac{1}{6^{2/3}} f(n)$ for large n, with $\frac{1}{6^{2/3}} < 1$), by Master Theorem Case 3 we have $T(n) \in \Theta(n^{8/3})$.

Question A.6

Suppose $T(n) = 64T(n/4) + 3n^{1.5}$. Then, $T(n)$ is in

- A $\Theta(n^2)$
- B $\Theta(n^3)$
- C $\Theta(n^{1.5})$
- D $\Theta(n^{1.5} \log n)$
- E None of the above

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- E None of the above

Solution

Since $a = 64$, $b = 4$, $d = \log_4 64 = 3$, *and* $f(n) = 3n^{1.5} \in O(n^{3-\epsilon})$ *with* $\epsilon = 1.5$, *by Master Theorem Case 1 we have* $T(n) \in \Theta(n^3)$.

Question A.7 [P2]

Suppose $T(n) = T(n/5) + 2T(n/3) + n$. Then, $T(n)$ is in

- A $\Theta(n)$
- B $\omega(n^2)$
- C $\Omega(n \log n)$
- D $o(n)$
- E None of the above

Question A.7 [P2]

Suppose $T(n) = T(n/5) + 2T(n/3) + n$. Then, $T(n)$ is in

- **A** $\Theta(n)$
- **B** $\omega(n^2)$
- **C** $\Omega(n \log n)$
- **D** $o(n)$
- **E** None of the above

Solution

Clearly, $T(n) \geq n$. Let $c \geq \frac{15}{2}$ be such that $T(n) \leq cn$ for all $n \leq 100$. We will show by induction that $T(n) \leq cn$ for all n. Assuming that this is true for all $n < n_0$ where $n_0 > 100$, we have $T(n_0) \leq c \cdot \frac{n_0}{5} + 2c \cdot \frac{n_0}{3} + n_0 \leq cn_0$, where the last inequality follows from the assumption that $c \geq \frac{15}{2}$. Hence, $T(n) \in \Theta(n)$.

Question A.8

For any randomized algorithm, let $E(n)$ and $T(n)$ denote the expected and worst-case running time, respectively, for inputs of length n. Then, which of the following statement is always **TRUE**, irrespective of the randomized algorithm being considered?

- A For every n, $E(n) < T(n)$
- B For every n, $E(n) = T(n)$
- C For every n, $E(n) > T(n)$
- D For at least one n, $E(n) < T(n)$, and for at least one n, $E(n) > T(n)$
- E None of the above

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- D For at least one n, $E(n) < T(n)$, and for at least one n, $E(n) > T(n)$
- E None of the above

Solution

Since $T(n)$ is the maximum running time over all inputs and random choices, we always have $E(n) \leq T(n)$. However, it can happen that $E(n) < T(n)$ for some n (possibly none), and $E(n) = T(n)$ for the remaining n (possibly none).

Question A.9

Suppose we throw 3 balls independently and uniformly at random into 5 bins. Then,

A The probability that all the balls fall into the same bin is 0.

B The probability that all the balls fall into the same bin is $\frac{3}{5}$.

C The probability that all the balls fall into the same bin is $\frac{1}{25}$.

D The probability that all the balls fall into the same bin is $\frac{1}{9}$.

E None of the above.

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- D The probability that all the balls fall into the same bin is $\frac{1}{9}$.
- E None of the above.

Solution

*Let B_i be the event that all 3 balls fall in bin i. Then, **Pr**$(B_i) = (1/5)^3 = 1/125$, and since the events $B_1, \ldots, B_5$ are disjoint (never occur at the same time), **Pr**$(\bigcup_{i=1}^{5} B_i) = 5 \cdot (1/125) = 1/25$.*

Question A.10

Consider an undirected graph $G = (V, E)$ with $n = |V|$ vertices and $m = |E|$ edges. A randomized algorithm selects a vertex $v \in V$ uniformly at random and returns $\deg(v)$, where $\deg(v)$ denotes the degree of vertex v. Let X be the random variable that denotes the output of this algorithm. What is the expected value of X, i.e., $\mathbb{E}[X]$?

- A m
- B n
- C m/n
- D $2m/n$
- E None of the above

Question A.10

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- **A** m
- **B** n
- **C** m/n
- **D** $2m/n$
- **E** None of the above

Solution

From the handshaking lemma, we have $\sum_{u \in V} \deg(u) = 2m$. *Since the vertex is chosen uniformly at random from* V, *the expected value is* $\mathbb{E}[X] = \frac{1}{n} \sum_{v \in V} \deg(v) = \frac{2m}{n}$.

Question B.1 [P3]

There are three rods and $n \geq 1$ disks of different diameters, all stacked on the first rod, smallest on top and largest at the bottom.

Alice must move all disks to the third rod, following these rules:

1. Move one disk at a time from the top of any rod.
2. No disk may be placed on a smaller disk.
3. (Variant) Disks can only move between adjacent rods.

Let $f(n)$ be the number of moves Alice needs to complete this task (minimize moves, but no proof required).

a. Write a recurrence for $f(n)$, including the base case(s), and explain how you derived it.
b. Solve the recurrence from part (a) (i.e., give a closed-form formula for $f(n)$, with justification).

Solution a

Alice can make the following moves[1]:

Step	Move	From	To	Moves
1	Top $n-1$ disks	First rod	Third rod	$f(n-1)$
2	Bottom disk	First rod	Second rod	1
3	Top $n-1$ disks	Third rod	First rod	$f(n-1)$
4	Bottom disk	Second rod	Third rod	1
5	Top $n-1$ disks	First rod	Third rod	$f(n-1)$

Hence,

- Base case: $f(1) = 2$ moves (first rod to the second, second rod to the third)
- Recursive step: $3f(n-1) + 2$ moves

$$f(n) = \begin{cases} 2, & \text{if } n = 1, \\ 3f(n-1) + 2, & \text{if } n > 1. \end{cases}$$

[1]Think about another alternative way!

Solution b

We claim that $f(n) = 3^n - 1$ and prove this by induction.

Base case

$$f(1) = 2 = 3^1 - 1.$$

Inductive step

Assume the claim holds for some $n \geq 1$, i.e., $f(n) = 3^n - 1$. Then:

$$\begin{aligned} f(n+1) &= 3f(n) + 2 \\ &= 3(3^n - 1) + 2 \\ &= 3^{n+1} - 3 + 2 \\ &= 3^{n+1} - 1. \end{aligned}$$

Thus, the claim holds for all $n \geq 1$ by induction.

Solution b (via expansion)

We rearrange $f(n)+1 = 3f(n-1)+3 = 3(f(n-1)+1)$ and $f(1)+1=3$. By expansion, we get

$$f(n)+1 = 3\left(3\left(\cdots 3\left(f\left(1\right)+1\right)\cdots\right)\right) = 3^n,$$

hence $f(n) = 3^n - 1$.

Question B.2

Teacher Bob has 10 students and 20 candies.
Each student assigns a distinct value to each candy, summing to 3230.

- Bob sees all students' values and selects an ordering.
- Students pick candies in **two rounds** based on Bob's order.
- On each turn, a student picks their highest-value available candy.
- Each student gets **2 candies**, with a **final value** equal to their sum.

Prove that Bob can always choose an ordering such that the sum of all 10 students' final values is at least 3230.

Solution

If the ordering is chosen **uniformly at random**, the expected total final value is at least 3230, implying that such an ordering must exist.

Expected Value Calculation

Fix a student with values for the candies $a_1 > a_2 > \cdots > a_{20}$, where:

$$a_1 + a_2 + \cdots + a_{20} = 3230.$$

- In position j, the student picks at least a_j first.
- In position $10 + j$, she picks at least a_{10+j}.
- Her final value is at least $a_j + a_{10+j}$.

Since each position is equally likely $(1/10)$, her expected final value is *at least*:

$$\frac{1}{10}\sum_{j=1}^{10}(a_j + a_{10+j}) = \frac{1}{10} \cdot 3230 = \frac{3230}{10}.$$

By **linearity of expectation**, the total expected value is *at least*: $10 \cdot \frac{3230}{10} = 3230.$

Question B.3

Charlie has **100 coins**, knowing that **4 are fake** but not which ones.

- **All real coins** have the same weight.
- **All fake coins** have the same weight, but are **lighter** than real coins.
- Charlie does not know these weights.

Charlie's Balance

He can compare two **disjoint** sets of coins A and B, determining:

1. $A > B$: A is heavier than B
2. $A < B$: A is lighter than B.
3. $A = B$: A and B weigh equally.

Determine, with proof, a small number k such that by using at most k weighings, Charlie can always point to one coin and say with certainty that this coin is real.

Solution

Charlie can determine the fake coins in at most $k = 2$ **weighings**.

Step 1: Initial Weighing

Divide 100 coins into $A = 33$, $B = 33$, $C = 34$.

- Weigh A vs. B (If **unequal** - at most **1 fake coin** is in heavier set)
  - Remove a coin from the heavier set (1 coins),
  - Weigh the rest in two equal sets (16 coins).
- Weigh A vs. B (If **equal** - C has **0, 2, or 4** fake coins)
  - Weigh $B \cup \{x\}$ vs. C for some $x \in A$.

Step 2.Neq: Second Weighing (If first weighing unequal, say[2] $A > B$)

Set A is split 3 ways A_1, A_2, A_3 with respective coin sizes $16, 16, 1$.

- Weigh A_1 vs. A_2 (If **unequal**)
  - Removed coin in A_3 is real.
- Weigh A_1 vs. A_2 (If **equal**)
  - Coins in both A_1 and A_2 are real.

[2]Note that $B > A$ is symmetric.

Step 2.Eq: Second Weighing (If first weighing equal)

Create set $B' = B \cup \{x\}$, by adding $x \in A$ to B.

- Weigh B' vs. C (If $B' > C$)
  - Added coin x is real.
- Weigh B' vs. C (If $B' = C$)
  - Coins in $A \setminus \{x\}$ are real.
- Weigh B' vs. C (If $B' < C$)
  - Coins in C are real.

It may be clearer to see an illustration of the decision tree of the 2 weighings with the possible configurations.

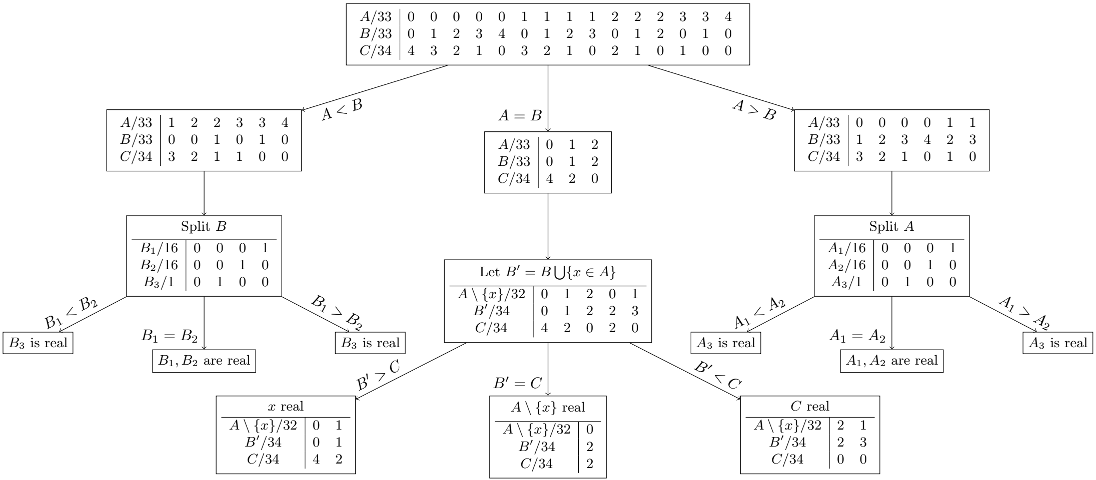

Figure 2: Configurations of fake coins across sets A, B, and C, where each column represents a unique combination. The table specifies the number of fake coins in each set (e.g., $A/33$ indicates that set A has 33 coins, with the corresponding cell showing the number of fake coins in A).