Eric Han
eric_han@nus.edu.sg
https://eric-han.com
Computer Science

NUS | Computing
National University of Singapore

T01 – Week 2

Introduction and Asymptotic Analysis

CS3230 TG19

Your Tutor Dr Eric Han

- **[Pioneer JC 2009-2010]** Took 'A' levels and fell in love with Computing
  - H2 Computing, Interested in research and in AI.
- **[B.Com. NUS 2013-2018]** Not so long ago I was in your seat
  - A*STAR Scholarship, Turing Programme
  - [University of Southern California, 2016] Student Exchange
- **[PhD. NUS 2018-2023]** PhD in AI/ML
  - My research is in AI/Machine Learning regarding optimization, scaling and robustness.
- **[Lecturer]** Applied for jobs and got 2 - Industry and another in NUS SoC
  - If you are interested to do a FYP, feel free to chat with me!
  - You are welcome to check my profile & research: https://eric-han.com
  - Some of the courses I taught: CS2109S(3), CS3244(1), CS1010(1), CS3217(1), CS3243(2), CS3203(5), CS2030(1)
  - Teaching this semester: CS3230(1), CS2103/T(1)

About CS3230: <insert personal exp. here>; Goal: Learn algorithms, do well.

Expectations / Commitment

Expectations of you

1. Fill seats from the front.
2. Good students are always prepared:
   a. Attempt your Tutorial
   b. Review lecture content
   c. Be on time
3. Refrain from taking pictures of the slides.
   a. Learn to take good notes.
   b. Slides/notes will be distributed; Created the main deck, with customizations...

Commitment from me

1. Be available for your learning as much as possible.
2. Strive to make the lessons interesting and fun.
3. Pass on a good foundation in Algorithms (not just the `A+`).

Any comments or suggestions for the lessons welcome!

CS3230 S2 24/25 Admin

- **Grading** (20 marks, 5% of final grade):
  - **Attendance** (12 marks): *1 mark* per session (12 total).
  - **Participation** (10 marks): *5 marks* for each of two presentations. Identified by **[P1/2/3]**.
- **Tutorial Discussion**: Learning is **social** and I hope that we are able to build friendships in this class. Identified by **[G]**.
- **Policies**: Tutorials & Assignment Policy: Plagiarism (no AI tools).
- **Assignments**: Graded by me.
- **Consultations**: Wed 10-11 AM.
- **Telegram Group**: Join for updates.
- **Welcome Survey**: Get to know you better.
- **Questions?**: Use ChatGPT, Telegram Group / PM `@Eric_Vader`.

Telegram / Welcome Survey

Figure 1: Telegram Group

Figure 2: Welcome Survey

Figure 3: PPT Schedule

Asymptotic Analysis Ω, Θ, O

We say	if $\exists c, c_1, c_2, n_0 > 0$ s.t. $\forall n \geq n_0$	In other words
$f(n) \in O(g(n))$	$0 \leq f(n) \leq c \cdot g(n)$	g is an **upper** bound on f
$f(n) \in \Omega(g(n))$	$0 \leq c \cdot g(n) \leq f(n)$	g is a **lower** bound on f
$f(n) \in \Theta(g(n))$	$0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$	g is a **tight** bound on f

Asymptotic Analysis o, ω

We say	if $\forall c > 0$, $\exists n_0 > 0$ s.t. $\forall n \geq n_0$	In other words
$f(n) \in o(g(n))$	$0 \leq f(n) < c \cdot g(n)$	g is a **strict upper** bound on f
$f(n) \in \omega(g(n))$	$0 \leq c \cdot g(n) < f(n)$	g is a **strict lower** bound on f

Question 1 [G]

Assume $f(n), g(n) > 0$, show:

a. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \Rightarrow f(n) \in o(g(n))$ — this has already been shown in lec01b.

b. $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \Rightarrow f(n) \in O(g(n))$ [P1]

c. $0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \Rightarrow f(n) \in \Theta(g(n))$

d. $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \Rightarrow f(n) \in \Omega(g(n))$

e. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \Rightarrow f(n) \in \omega(g(n))$

Recap

- Definition of Limit for functions

Question 1 [G]

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d. $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \Rightarrow f(n) \in \Omega(g(n))$

e. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \Rightarrow f(n) \in \omega(g(n))$

Recap

- Definition of Limit for functions

$$\lim_{n\to\infty} f(n) = L \iff (\forall \epsilon > 0)(\exists n_0 > 0)\,(n > n_0 \implies |f(n) - L| < \epsilon).$$

- How to proof?

Answer 1b

Proof $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \Rightarrow f(n) \in O(g(n))$.

$$
\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} = z \implies & \quad (\forall \epsilon > 0)(\exists n_0 > 0) \left| \frac{f(n)}{g(n)} - z \right| < \epsilon, && \text{(By definition)} \\
\implies & \quad -\epsilon < \frac{f(n)}{g(n)} - z < \epsilon \\
\implies & \quad z - \epsilon < \frac{f(n)}{g(n)} < z + \epsilon \\
\implies & \quad \frac{f(n)}{g(n)} \le c && \text{(Let } c = z + \epsilon + \cdots) \\
\implies & \quad (\forall \epsilon > 0)(\exists n_0 > 0)\, f(n) \le c \cdot g(n) && (g(n) > 0) \\
\implies & \quad (\exists n_0 > 0)\, f(n) \le c \cdot g(n) && (\epsilon > 0 \text{ arbitrary}) \\
\implies & \quad f(n) \in O(g(n)). && \text{(By definition of } O \text{ notation)}
\end{aligned}
$$

Answer 1c

Proof $0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \implies f(n) \in \Theta(g(n))$.

$$
\begin{aligned}
0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \implies &\quad f(n) \in O(g(n)) && \text{(Upper bound)}\\
\text{and} &\quad f(n) \in \Omega(g(n)). && \text{(Lower bound)}\\
\implies &\quad f(n) \in \Theta(g(n)). && \text{(By definition of } \Theta \text{ notation)}
\end{aligned}
$$

Answer 1d

Proof $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \implies f(n) \in \Omega(g(n))$.

$$
\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} = z \implies &\quad \lim_{n\to\infty} \frac{g(n)}{f(n)} = \frac{1}{z}, && \text{(Flip } f(n) \text{ and } g(n))\\
\implies &\quad g(n) \in O(f(n)), && \text{(By the explanation above)}\\
\implies &\quad f(n) \in \Omega(g(n)). && \text{(By complementarity property)}
\end{aligned}
$$

Answer 1e

Proof $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \implies f(n) \in \omega(g(n))$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty &\implies \lim_{n\to\infty} \frac{g(n)}{f(n)} = 0, && \text{(Flip } f(n) \text{ and } g(n)\text{)} \\
&\implies g(n) \in o(f(n)), && \text{(By the definition of } o \text{ notation)} \\
&\implies f(n) \in \omega(g(n)). && \text{(By complementarity property)}
\end{aligned}$$

Moral of the story

- Be lazy, proof a single case and use properties.

Question 2 [G]

Assume $f(n), g(n) > 0$, show:

a. Reflexivity
   - $f(n) \in O(f(n))$
   - $f(n) \in \Omega(f(n))$
   - $f(n) \in \Theta(f(n))$

b. Transitivity [P2]
   - $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ implies $f(n) \in O(h(n))$
   - Do the same for Ω, Θ, o, ω

c. Symmetry
   - $f(n) \in \Theta(g(n))$ iff $g(n) \in \Theta(f(n))$

d. Complementarity
   - $f(n) \in O(g(n))$ iff $g(n) \in \Omega(f(n))$
   - $f(n) \in o(g(n))$ iff $g(n) \in \omega(f(n))$

Answer 2a

Reflexivity

- $f(n) \in O(f(n))$
  - Taking $c = 1$ (any constant ≥ 1), $n_0 = 1$, we have $\forall n \geq n_0$,
  - $f(n) \leq (1 \cdot f(n) = c \cdot f(n))$.
- $f(n) \in \Omega(f(n))$
  - Taking $c = 1$ (any positive constant ≤ 1), $n_0 = 1$, we have $\forall n \geq n_0$,
  - $(c \cdot f(n) = 1 \cdot f(n)) \leq f(n)$.
- $f(n) \in \Theta(f(n))$
  - Taking $c_1 = 1$ (any positive constant < 1), $c_2 = 1, n_0 = 1$, we have $\forall n \geq n_0$,
  - $(c_1 \cdot f(n) = 1 \cdot f(n)) \leq f(n) \leq (1 \cdot f(n) = c_2 \cdot f(n))$.

Answer 2b

Proof $f(n) \in O(g(n)), g(n) \in O(h(n)) \implies f(n) \in O(h(n))$

$$
\begin{aligned}
f(n) \in O(g(n)) \implies \quad & (\exists c_{fg} > 0, n_{0fg} > 0)(\forall n \geq n_{0fg}), && (1)\\
& f(n) \leq c_{fg} \cdot g(n). \\
g(n) \in O(h(n)) \implies \quad & (\exists c_{gh} > 0, n_{0gh} > 0)(\forall n \geq n_{0gh}), && (2)\\
& g(n) \leq c_{gh} \cdot h(n). \\
\text{From } (1),(2) \implies \quad & (\forall n \geq \max(n_{0fg}, n_{0gh})), \\
& f(n) \leq c_{fg} \cdot g(n) \leq c_{fg} \cdot c_{gh} \cdot h(n) \\
\implies \quad & f(n) \leq c_{fg} \cdot g(n) \leq c \cdot h(n) && (\text{Let } c = c_{fg} \cdot c_{gh})\\
\implies \quad & (\forall n \geq n_0) && (\text{Let } n_0 = \max(n_{0fg}, n_{0gh}))\\
& f(n) \leq c_{fg} \cdot g(n) \leq c \cdot h(n) \\
\implies \quad & f(n) \leq c \cdot h(n) \\
\implies \quad & f(n) \in O(h(n)).
\end{aligned}
$$

Do the same for Ω, Θ, o, ω

- Same, just change $\leq$ to $\geq, =, <, >$, respectively. Here, $=$ for Θ denotes that we need to do both $\geq$ and $\leq$ bounds.

Answer 2c

Prove $f(n) \in \Theta(g(n)) \iff g(n) \in \Theta(f(n))$

$$
\begin{aligned}
f(n) \in \Theta(g(n)) &\implies (\exists c_1, c_2 > 0, n_0 > 0)(\forall n \geq n_0), \\
&\quad\;\; c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n). \\
\text{From } f(n) \leq c_2 \cdot g(n) &\implies \frac{1}{c_2} \cdot f(n) \leq g(n) && \text{(Divide by } c_2\text{)} \\
&\implies c_1' \cdot f(n) \leq g(n) && \text{(Let } c_1' = \frac{1}{c_2}\text{)}, (1) \\
\text{From } c_1 \cdot g(n) \leq f(n) &\implies g(n) \leq \frac{1}{c_1} \cdot f(n). && \text{(Divide by } c_1\text{)} \\
&\implies g(n) \leq c_2' \cdot f(n) && \text{(Let } c_2' = \frac{1}{c_1}\text{)}, (2) \\
\text{From } (1), (2) &\implies c_1' \cdot f(n) \leq g(n) \leq c_2' \cdot f(n) \\
&\implies g(n) \in \Theta(f(n)).
\end{aligned}
$$

Answer 2d

Proof $f(n) \in O(g(n)) \iff g(n) \in \Omega(f(n))$

Forward Direction: Suppose $f(n) \in O(g(n))$

$$
\begin{aligned}
f(n) \in O(g(n)) \implies & \quad (\exists c > 0, n_0 > 0)(\forall n \geq n_0), \\
& \quad f(n) \leq c \cdot g(n) \\
\implies & \quad \frac{1}{c} \cdot f(n) \leq g(n) && \text{(Divide by } c\text{)} \\
\implies & \quad c' \cdot f(n) \leq g(n) \implies g(n) \in \Omega(f(n)). && \text{(Let } c' = \frac{1}{c}\text{)}
\end{aligned}
$$

Reverse Direction: Suppose $g(n) \in \Omega(f(n))$

$$
\begin{aligned}
g(n) \in \Omega(f(n)) \implies & \quad (\exists c' > 0, n_0 > 0)(\forall n \geq n_0), \\
& \quad c' \cdot f(n) \leq g(n) \\
\implies & \quad f(n) \leq c \cdot g(n) \implies f(n) \in O(g(n)). && \text{(Let } c = \frac{1}{c'}\text{)}
\end{aligned}
$$

Proof $f(n) \in o(g(n)) \iff g(n) \in \omega(f(n))$

- Same as above, just change $\leq$ to $<$.

Question 3 [P3]

Which of the following statement(s) is/are True?

a. $3^{n+1} \in O(3^n)$
b. $4^n \in O(2^n)$
c. $2^{\lfloor \log n \rfloor} \in \Theta(n)$ (we assume $\log$ is in base 2)
d. For constants $i, a > 0$, we have $(n + a)^i \in O(n^i)$

Recap

How to go about showing True/False?

Answer 3a

$3^{n+1} \in O(3^n)$: True.

- Taking $c = 3, n_0 = 1$, we have $\forall n \geq n_0$,
- $3^{n+1} \leq 3 \cdot 3^n = c \cdot 3^n$.
- $3^{n+1} \in O(3^n)$.

Answer 3b

$4^n \in O(2^n)$: False.

- For all $c \geq 1, n_0 = c$, we have $\forall n \geq n_0$,
- $(4^n = (2^2)^n = (2^n)^2 = 2^n \cdot 2^n) \geq c \cdot 2^n$, i.e., we cannot upper bound 4^n with constant times 2^n.

Answer 3c

$2^{\lfloor \log n \rfloor} \in \Theta(n)$ (we assume $\log$ is in base 2): True.

- Taking $c_1 = \frac{1}{2}, c_2 = 1, n_0 = 1$ ($\log 0$ is undefined), we have $\forall n \geq n_0$,
- $(c_1 \cdot n = \frac{1}{2} \cdot n) \leq 2^{\lfloor \log n \rfloor} \leq (1 \cdot n = c_2 \cdot n)$.
- $2^{\lfloor \log n \rfloor} \in \Theta(n)$.

Answer 3d

For constants $i, a > 0$, we have $(n+a)^i \in O(n^i)$: True.

- Taking $c = 2^i, n_0 = a$, we have $\forall n \geq n_0$,
- $(n+a)^i \leq (n+n)^i = (2n)^i = 2^i \cdot n^i = c \cdot n^i$.
- $(n+a)^i \in O(n^i)$.

Question 4

Which of the following statement(s) is/are True?

$2^{\log_2 n} \in$

a. $O(n)$
b. $\Omega(n)$
c. $\Theta(\sqrt{n})$
d. $\omega(n)$

Recap

- Inverse Property of Logarithms

Question 4

Which of the following statement(s) is/are True?

$2^{\log_2 n} \in$

- a. $O(n)$
- b. $\Omega(n)$
- c. $\Theta(\sqrt{n})$
- d. $\omega(n)$

Recap

- Inverse Property of Logarithms

$$a^{\log_a b} = b$$

Question 4

Which of the following statement(s) is/are True?

$2^{\log_2 n} \in$

a. $O(n)$
b. $\Omega(n)$
c. $\Theta(\sqrt{n})$
d. $\omega(n)$

Recap

- Inverse Property of Logarithms

$$a^{\log_a b} = b$$

Answer

$2^{\log_2 n} = n \in O(n)$, and also $n \in \Omega(n)$, but $n \notin \omega(n)$ (why? - if you can answer, you understand ω, Ω).

Question 4 Variant [G]

How about $2^{\log_4 n} \in$?

Question 4 Variant [G]

How about $2^{\log_4 n} \in$?

Answer

First, rewrite the logarithm from one base to another base:

- $\log_4 n = \frac{\log_2 n}{\log_2 4} = \frac{\log_2 n}{2}$,

Thus, $2^{\log_4 n} = 2^{\frac{\log_2 n}{2}} = (2^{\log_2 n})^{\frac{1}{2}} = n^{\frac{1}{2}} = \sqrt{n}$.

a. $O(n)$ True.
   - $2^{\log_4 n} = \sqrt{n} \in O(n)$, taking $c = 1, n_0 = 1$.

b. $\Omega(n)$ False.

c. $\Theta(\sqrt{n})$ True.
   - $2^{\log_4 n} = \sqrt{n} \in \Theta(\sqrt{n})$, taking $c_1 = 1$ (or smaller), $c_2 = 1$ (or larger), $n_0 = 1$.

d. $\omega(n)$ False.

Question 5 [G]

Rank the following functions by their order of growth. (But if any two (or more) functions have the same order of growth, group them together).

1. $f_1(n) = \log n$
2. $f_2(n) = n!$
3. $f_3(n) = 2^n + n$
4. $f_4(n) = n^{2.3} + 16n + f_1(n)$
5. $f_5(n) = \log(n^2)$
6. $f_6(n) = \ln(n^{2n})$

Answer

1. $f_1(n) = \log n$
2. $f_2(n) = n!$
3. $f_3(n) = 2^n + n$
   - $f_3(n) \in \Theta(2^n)$
4. $f_4(n) = n^{2.3} + 16n + f_1(n)$
   - $f_4(n) \in \Theta(n^{2.3})$.
5. $f_5(n) = \log(n^2)$
   - $f_5(n) = \log(n^2) = 2\log n$, hence the same order of growth as $f_1(n)$.
6. $f_6(n) = \ln(n^{2n})$
   - $f_6(n) = 2n\ln(n) \in \Theta(n \ln n)$.

Answer

1. $f_1(n) = \log n$
2. $f_2(n) = n!$
3. $f_3(n) = 2^n + n$
   - $f_3(n) \in \Theta(2^n)$
4. $f_4(n) = n^{2.3} + 16n + f_1(n)$
   - $f_4(n) \in \Theta(n^{2.3})$.
5. $f_5(n) = \log(n^2)$
   - $f_5(n) = \log(n^2) = 2\log n$, hence the same order of growth as $f_1(n)$.
6. $f_6(n) = \ln(n^{2n})$
   - $f_6(n) = 2n\ln(n) \in \Theta(n \ln n)$.

Ordering f_2, f_3:

- $f_2(n) = n! = n \cdot (n-1) \cdot (n-2) \cdot ... \cdot 1$.
- simplified $f_3(n) = 2^n = 2 \cdot 2 \cdot 2 \cdot ... \cdot 2$.
- can show by induction that for $n \geq 4, n! \geq \frac{n}{4} \cdot 2^n$.

Therefore, with respect to order of growth, we have:

$$(f_1(n) = f_5(n)) \leq f_6(n) \leq f_4(n) \leq f_3(n) \leq f_2(n)$$

Practical [Optional]

Practical repo: To help you further your understanding, not compulsory; Work for Snack!

1. Visualize the growth of various functions in Q5 and
2. Plot the threshold ($n =?$) for which the asymptotic relations can be observed.

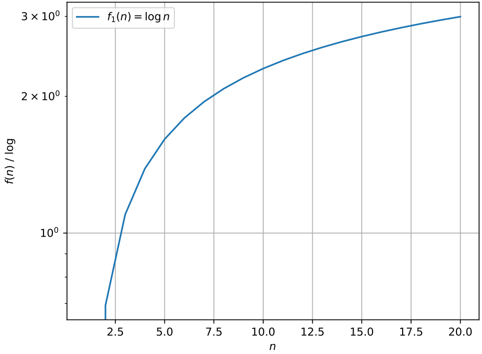

Figure 4: Functions compared - visualization to aid understanding/not proof.