Metronome Synchronization Using Feedback Control

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***Abstract*—This paper concerns controlled in-phase synchronization of two mechanically coupled metronomes. A Proportional and Derivative feedback control as well as a feedback linearizing controller are proposed as control laws in order to minimize the synchronization time. Experiments in a laboratory prototype show the feasibility of the proposed control law.**

***Index Terms*—Pendulum Synchronization, Feedback Linearizing Control, PD Control.**

I. Introduction

Synchronization is the process that occurs when two or more coupled systems tend to display the same motion at the same time. We can find examples of this kind of phenomena in biological systems such as fireflies from South-East Asia that gather together in trees at night to flash in synchrony [1]. Other examples include: pacemaker cells in the mammalian heart as well as in the nervous system, collective oscillations of pancreatic beta cells and synchronized menstrual cycles in behaviorally coupled women [2].

The first documented scientific description of synchronization in mechanical systems was made by Christian Huygens in 1657 when he was developing a pendulum clock in order to locate the longitude of a ship. Huygens found that two pendulums attached to the same beam supported by two chairs would swing in exact opposite directions. He observed this anti-phase synchronization behavior of the pendulums after some time regardless of the pendulums initial conditions. A close version of the Huygens setup conceived to illustrate some mechanical properties in physics classroom is shown in [3]. In this case the in-phase and anti-phase pendulum phenomena are studied via the characterization of the system parameters in the time course for synchronization. The experimental setup employs two Super-Mini-Taktell metronomes resting on a light wooden board that sit on two empty soda cans. The pendulum phases are measured using microphones that allow the recording of the metronomes ticks. Each pendulum uses a dedicated microphone. Data acquisition card and signal analysis software is then employed to process the recording in order to measure the metronomes phases. The implementation does not include the measurement of the board displacement.

Reference [4] reconsiders Huygens observations and reproduces his original results building an updated version of the two clock system. In the performed experiments the coupling strength is studied by changing the ratio of the mass of the pendulums to the system mass. Authors show that the coupling strength influences the behavior of the clocks for in-phase, anti-phase and a “beating death”, this is, when one or both pendulums cease to run, allowing a better understanding of Huygens observations. Authors use Poincare maps to study the non-linear dynamics of the system. The experimental setup includes two pendulum clocks, a wood beam attached to the two clocks that are mounted on a low friction cart and then the whole system mounted on a slotted cart. The angular position of each clock pendulum is measured using a tracking laser. The voltage signal from the laser is then read by an analog to digital converter via a dedicated acquisition card.

Reference [5] is inspired in Huygens observations and in what is discussed in [4] and [3]. Authors pay attention to the different regimes: in-phase synchronization, anti-phase synchronization and intermediate regimes. The experimental setup uses two standard Wittner Maezel metronomes. The prototype is suspended by leaf springs, allowing frictionless horizontal displacement with linear damping and stiffness. The pendulum angles are measured using anisotropic magneto resistance (AMR) sensor delivering an analog voltage proportional to the pendulum angle. The velocity of the platform is measured using a laser vibrometer. Experimental results are obtained for in-phase, anti-phase synchronization depending on the system parameters such as coupling and damping ratio.

In this paper we study the in-phase pendulum synchronization using control laws allowing a faster synchronization of the pendulums. Similar to [3] and [5] we employ two metronomes; however, the metronomes are mounted on a low friction moving cart as in [4]. The cart is coupled to a direct current motor via a timing belt and pulleys allowing then the reduction of the synchronization time via feedback-based control of the motion of the cart. In order to measure the metronomes angular motion as well as the cart position we use a vision system, inscribing then our exposition in the visual servoing field domain of study.

The paper is organized as follows. Section II presents the model of the pendulum system. Section III describes the experimental setup. Section IV is dedicated to the control design. Section V presents both the simulation and the experimental results. We end with some concluding remarks.

II. Metronome System model

In this paper we consider two metronomes resting on a surface that couples them mechanically mounted on a cart allowing only horizontal displacement. The synchronization of the metronomes pendulums will occur naturally due to energy transmission via the movable platform and when the difference in natural frequencies is small. Fig. 1 depicts the system configuration. We consider the metronome as a regular pendulum with rod and a bob at the end of the arm. In the non-linear system model we suppose that the rod does not have mass.

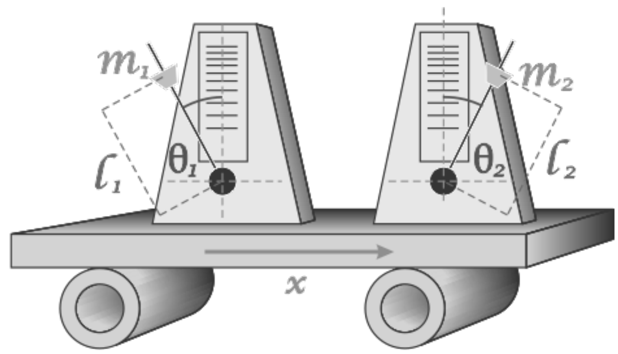

Fig. 1. Two metronomes placed over a movable surface.

The mathematical model is inspired from [3]. The mathematical model for a single metronome on the cart is given by:

$$\ddot{\theta}+\frac{mlg}{I}\sin\theta+\epsilon\left(\left(\frac{\theta}{\theta_0}\right)^2-1\right)\dot{\theta}+\left(\frac{ml\cos\theta}{I}\right)\ddot{x}=0, \quad (1)$$

where θ is the angle between the metronome arm and the vertical line, I is the inertia moment of the metronome bob, m is the mass of the metronome bob, l is the length of the arm of the metronome. g is the acceleration due to gravity, and x is the horizontal position of the cart. The third member of the eq. 1 is the escapement, *i.e.*, the force transfered from the metronome spring to the pendulum of the metronome. This term is of the van der Pol type and it increases the angular velocity when $\theta > \theta_0$. For small values of ϵ the term will produce stable oscilations. If we consider the bob of the metronome as a point mass, its inertia moment is:

$$I = ml^2.$$

So the mathematical model for one metronome results in

$$m\left(l\ddot{\theta}+g\sin\theta+\cos\theta\ddot{x}\right)+ml\epsilon\left(\left(\frac{\theta}{\theta_0}\right)^2-1\right)\dot{\theta}=0, \quad (2)$$

For the equations of two metronomes we have

$$\begin{aligned} m\left(l_1\ddot{\theta}_1+gs_{\theta_1}+c_{\theta_1}\ddot{x}\right)+ml_1\epsilon\left(\left(\tfrac{\theta_1}{\theta_{1,0}}\right)^2-1\right)\dot{\theta}_1 &= 0,\\ m\left(l_2\ddot{\theta}_2+gs_{\theta_2}+c_{\theta_2}\ddot{x}\right)+ml_2\epsilon\left(\left(\tfrac{\theta_2}{\theta_{2,0}}\right)^2-1\right)\dot{\theta}_2 &= 0, \end{aligned}$$

where $\sin\theta_i = s_{\theta_i}$ and $\cos\theta_i = c_{\theta_i}$. The center of mass for the two pendulums is:

$$x_{cm}=\frac{Mx+m\left(x+l_1s_{\theta_1}\right)+m\left(x+l_2s_{\theta_2}\right)}{M+2m}, \quad (3)$$

where M is de mass of the cart.

We assume that the total external force is cero, then the equation of motion of the cart is:

$$\frac{d^2x_{cm}}{dt^2}=0. \quad (4)$$

By substituting eq. 3 into eq. 4 and considering the torque input force given by de DC motor τ we have that:

$$\ddot{x}\left(M+2m\right)+\sum_{i=1}^{2}ml_i\left(c_{\theta_i}\ddot{\theta}_i-s_{\theta_i}\dot{\theta}_i^2\right)=\tau. \quad (5)$$

The coupled system equation are then given by:

$$\begin{aligned} m\left(l_1\ddot{\theta}_1+gs_{\theta_1}+c_{\theta_1}\ddot{x}\right)+ml_1\epsilon\left(\left(\tfrac{\theta_1}{\theta_{1,0}}\right)^2-1\right)\dot{\theta}_1 &= 0\\ m\left(l_2\ddot{\theta}_2+gs_{\theta_2}+c_{\theta_2}\ddot{x}\right)+ml_2\epsilon\left(\left(\tfrac{\theta_2}{\theta_{2,0}}\right)^2-1\right)\dot{\theta}_2 &= 0 \end{aligned}$$

$$\ddot{x}\left(M+2m\right)+\sum_{i=1}^{2}ml_i\left(c_{\theta_i}\ddot{\theta}_i-s_{\theta_i}\dot{\theta}_i^2\right)=\tau, \quad (6)$$

Using generalized coordinates $q_1=\theta_1$, $q_2=\theta_2$ and $q_3=x$ for the above equations we can define the new state variables as:

$$q=\begin{bmatrix}q_1\\q_2\\q_3\end{bmatrix}; \quad \dot{q}=\begin{bmatrix}\dot{q}_1\\\dot{q}_2\\\dot{q}_3\end{bmatrix}; \quad \ddot{q}=\begin{bmatrix}\ddot{q}_1\\\ddot{q}_2\\\ddot{q}_3\end{bmatrix}.$$

The second order system is then given by:

$$M\left(q\right)\ddot{q}+C\left(q,\dot{q}\right)\dot{q}+G\left(q\right)=\tilde{\tau}, \quad (7)$$

where

$$\begin{aligned} M\left(q\right) &= \begin{bmatrix} ml_1^2 & 0 & ml_1c_{q_1}\\ 0 & ml_1^2 & ml_2c_{q_2}\\ ml_1c_{q_1} & ml_2c_{q_2} & M+2m \end{bmatrix}\\ C\left(q,\dot{q}\right) &= \begin{bmatrix} ml_1^2\epsilon\left(\left(\frac{q_1}{q_{1,0}}\right)^2-1\right) & 0 & 0\\ 0 & ml_2^2\epsilon\left(\left(\frac{q_2}{q_{2,0}}\right)^2-1\right) & 0\\ ml_1s_{q_1}\dot{q}_1 & ml_2s_{q_2}\dot{q}_2 & 0 \end{bmatrix}\\ G\left(q\right) &= \begin{bmatrix} mgl_1s_{q_1}\\ mgl_2s_{q_2}\\ 0 \end{bmatrix}; \quad \tilde{\tau}=\begin{bmatrix}0 & 0 & \tau\end{bmatrix}^T \end{aligned}$$

III. Experimental Setup

The mechanical laboratory prototype includes two Bestmay metronomes model BCM 330, linear bearing guide THK 2560R, custom built aluminum base, DC brushed JDTH-2250-BQ-IC motor, driven by a Copley Controls analog power servoamplifier, model 413, configured in current mode. An optical encoder gives angular position of the DC motor having 10,000 pulses per revolution. The experimental setup is based on the architecture in [6]. A computer called the Vision Computer with a 3.0 GHz Intel Pentium IV processor performs image acquisition and processing using a Dalsa Camera model CA-1D-128A, which is connected to the vision computer through a National Instruments 1422 digital interface card.

The Visual C++ language, the image processing library ICE and the DIAS environment [7] allow programming the image processing algorithms. The visual sampling was performd at 110 Hz. A second computer, called the control computer that performs data logging and executes the control algorithm has a 2.0 Ghz Intel Pentium IV processor and uses a MultiQ-3 card from Quanser Consulting that performs data acquisition. The Matlab/Simulink software operating with the Wincon 5.2 real-time environment software from Quanser Consulting serves as programming platform. The cart control loop was closed at 1 Khz.

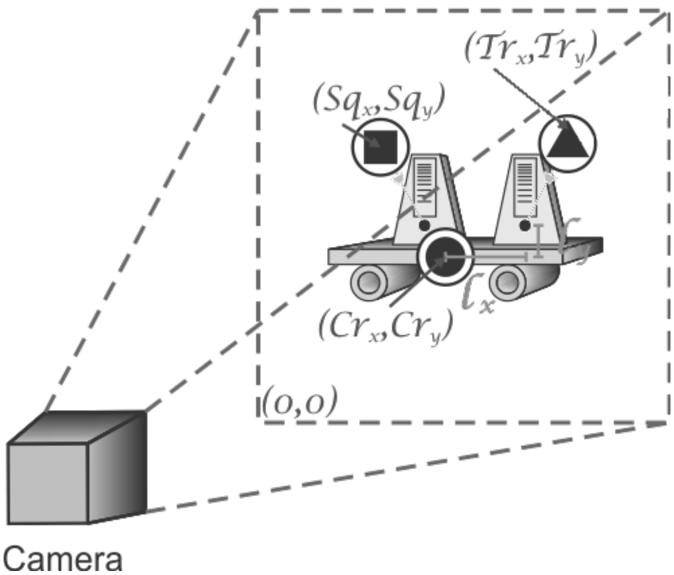

Fig. 2. Visual computing setup.

The cart position and the metronomes pendulum angles are measured using the above architecture. The camera is positioned in order to visualize the whole workspace. The image processing software grabs the scene image, given a fixed image threshold performs image binarization and extracts the contours in the scene. For each contour perimeter, area, form factor (area/perimeter) and contour centroid position are calculated. In order to filter correctly each metronome and the cart positions, we select different targets for the metronomes and the moving base. Each target has a different identifying geometrical figure: one pendulum has a triangle, the other has a square and the cart has a circle. The targets where mounted on cardboard and are attached to each part. Each target will have a specific perimeter and area. Thus, vision software can correctly identify convex figures and can filter each position independently. The vision computer transmits via a RS-232 link the three positions, for each pendulum and the cart position to the control computer. The angles are calculated given the position of the metronomes and the cart using trigonometric relationships. Fig. 2 depicts the position of the pendulums and the cart using image processing with the camera. Fig. 3 shows a photograph of the experimental setup.

The vision computer provides the position of the three targets: square (Sq_x, Sq_y), triangle (Tr_x, Tr_y) and circle (Cr_x, Cr_y). The pendulum angles are obtained *via*

$$\begin{aligned}\theta_1 &= \tan^{-1}\left(\frac{Sq_y-(Cr_y+l_y)}{Sq_y-(Cr_x-l_x)}\right)\\ \theta_2 &= \tan^{-1}\left(\frac{Tr_y-(Cr_y+l_y)}{Sq_y-(Cr_x+l_x)}\right)\end{aligned}$$

Velocities and accelerations where estimated using the high pass filters of the form

$$H(s)=\frac{as}{s+a}$$

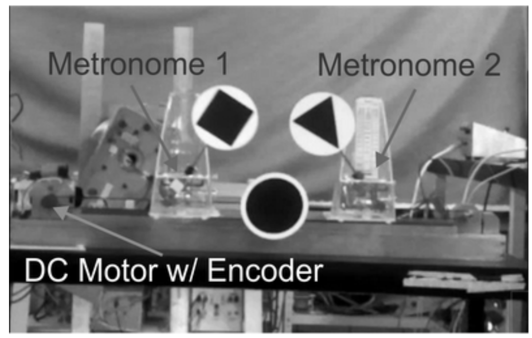

Fig. 3. Experimental setup.

IV. Control Design

Here we present two different control schemes that will be compared in the sequel: a Proportional Derivate (PD) control and a Feedback Linearizing Control.

A. PD Control

A PD control loop for the cart position is closed around x using a sinusoidal reference signal. PD control law is given by

$$\begin{aligned}\tau &= K_p\tilde{q}-K_d\dot{q}_3\\ \tilde{q} &= q^d-q_3\end{aligned}$$

Where K_p and K_d are derivative and proportional gains. The reference signal q^d is a sine function with a frequency near to the natural frequency of the metronomes. This control could be seen as an open loop control regarding θ_1 and θ_2 due to the fact that the feedback does no come from of the metronomes positions. The idea of this control is to make the cart to track a sine reference; thanks to the mechanical coupling the metronomes will in some time follow the cart motion and then the whole system will be synchronized.

B. Feedback Linearization Control

The objective of this control is to reduce the synchronization time. Now we ensure the cart to track a sinusoidal signal which is provided by one of the metronomes. As cart will follow one of the metronomes and due to the mechanical coupling to the other metronome will follow the first allwoing the system to synchronize. This should reduce the metronome synchronization time. For the system presented in equation (7), we propose the following control law

$$u = M(q)\,v + C(q,\dot{q})\,\dot{q} + G(q) \tag{8}$$

We define a desired trajectory, in this case the trajectory will be defined by one of the metronomes, then the cart will follow this metronome. Then the desired position are given by

$$\begin{aligned} q^d &= \begin{bmatrix} q_1 & q_1 & -q_1 \end{bmatrix}^T \\ \dot{q}^d &= \begin{bmatrix} \dot{q}_1 & \dot{q}_1 & -\dot{q}_1 \end{bmatrix}^T. \end{aligned}$$

We select proportional and derivative gain by:

$$K_p = \begin{bmatrix} K_{p_1} & 0 & 0 \\ 0 & K_{p_3} & 0 \\ 0 & 0 & K_{p_3} \end{bmatrix}; \quad K_p = \begin{bmatrix} K_{d_1} & 0 & 0 \\ 0 & K_{d_3} & 0 \\ 0 & 0 & K_{d_3} \end{bmatrix}$$

The outer loop control v is given by:

$$v = K_p\tilde{q} + K_d\dot{\tilde{q}},$$

where:

$$\tilde{q} = q^d - q; \quad \dot{\tilde{q}} = \dot{q}^d - \dot{q}$$

Control law τ is then calculated from equation (6):

$$\tau = \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3 - ml_1 \sin q_1 \dot{q}_1^2 - ml_2 \sin q_2 \dot{q}_2^2. \tag{9}$$

where

$$\begin{aligned} \tilde{K}_1 &= ml_1 \cos q_1 \left(K_{p_1}\tilde{q}_1 + K_{d_1}\dot{\tilde{q}}_1\right); \\ \tilde{K}_2 &= ml_1 \cos q_2 \left(K_{p_2}\tilde{q}_2 + K_{d_2}\dot{\tilde{q}}_2\right); \\ \tilde{K}_3 &= (M + 2m)\left(K_{p_2}\tilde{q}_2 + K_{d_3}\dot{\tilde{q}}_3\right). \end{aligned}$$

Since the system is underactuated, the control signal in (8) is applied to the DC motor that is the only actuator in the system having state variales $q_1 = \theta_1$, $q_2 = \theta_2$ and $q_3 = x$.

V. Results

In this paper we propose two different controllers to improve the natural time of sincronization of the two metronomes, the gains of this controllers were obtained by a trial and error method.

Fig.4 shows the behavior in simulation of the model without any input control, with initial conditions $q_1 = 0.9,\ q_2 = 0.25$. This simulation of the mathematical model shows that the system is synchronized after approximately 27 seconds.

The behavior using the PD controller is shown in fig. 5, it can be seen in the figure that the metronomes synchronize in-phase after 10 seconds. Comparing with fig. 4 the sinchronization is achieved in less time.

We can observe that the simulated behavior for the model in (6) shown in fig. 6 by applying the Feedback Linearizing Control achieves in-phase synchronization in about 7 seconds. The system's parameters used for the simulation were taken from the experimental setup and from the metronomes specifications (weight, distance from metronome pendulum to pivot). The value of the mass of the cart was obtained by weighting it.

Comparing the responses obtained Fig 7 and 6 we can observe that the response to Feedback Linearizing control, the

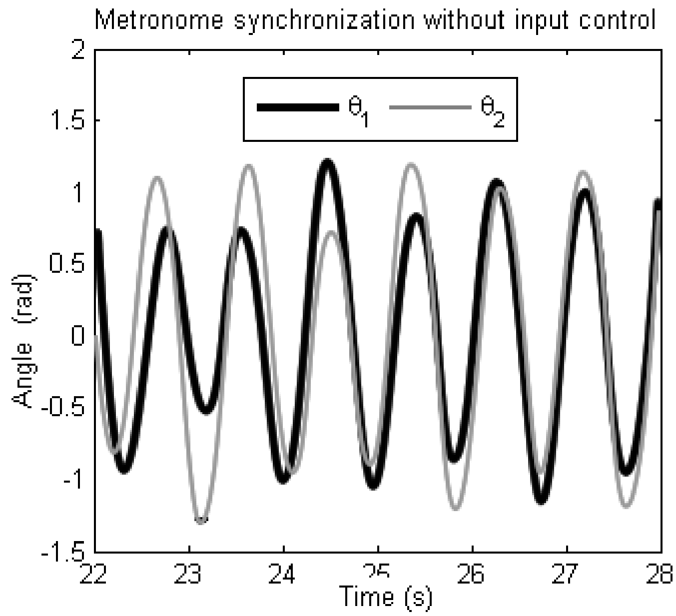

Fig. 4. Simulation result for metronome synchronization without control input.

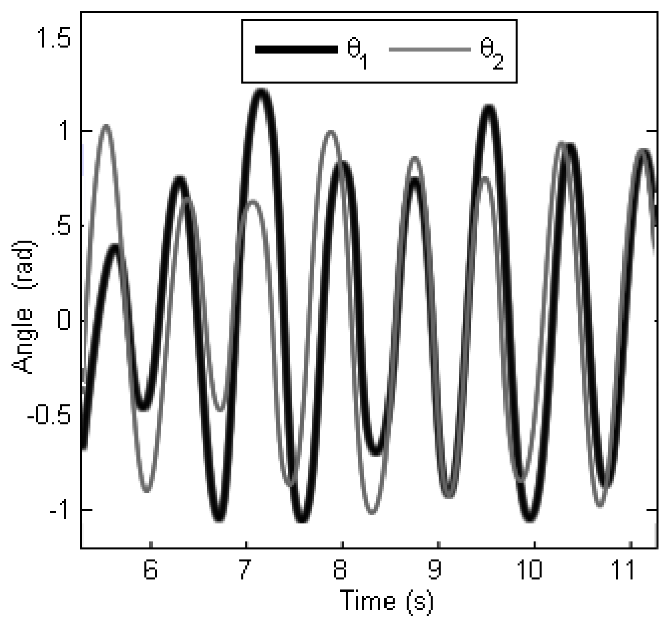

Fig. 5. Simulation results using PD control.

simulation and experimental behaviour are quite similar. The main difference is the signal amplitued and the metronome synchronization time that is between 6 and 8 seconds.

VI. Conclusions and Further Work

In this paper we presented a mathematical model of coupled mechanical system for two metronomes and a moving cart. The model allowed studying in simulation the synchronization of the metronomes without a control loop and then using a control loop for the cart. The control loop allowed the reduction the synchronization time. Two control schemes

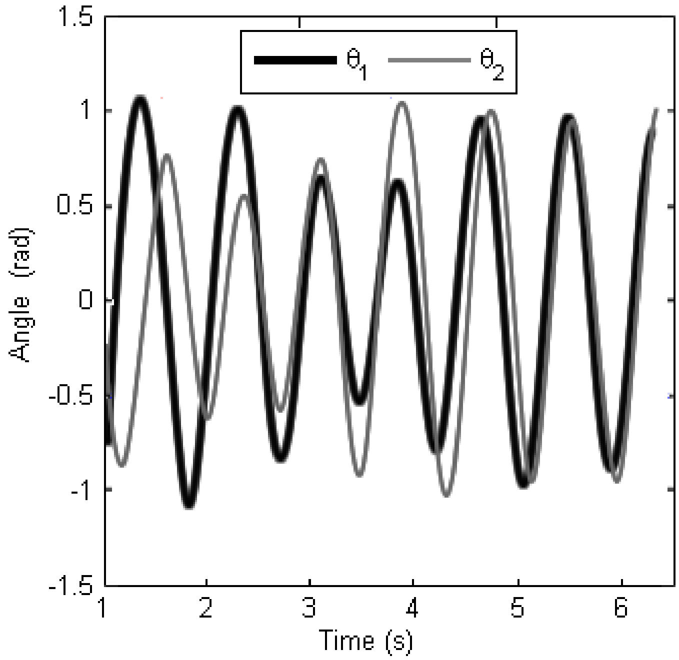

Fig. 6. Simulation results using Feedback Linearizing Control.

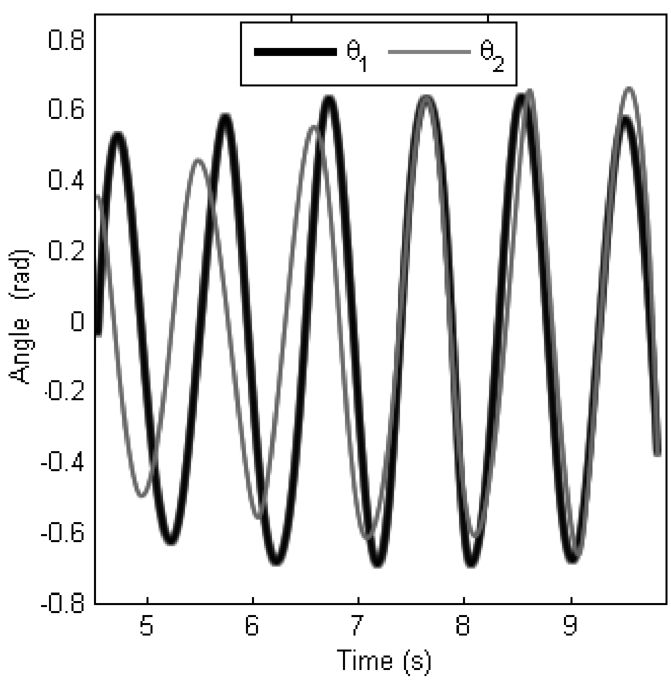

Fig. 7. Experimental resuts using Feedback Linearizing Control.

were tested, a PD control loop for the cart position and a Feedback Linearization Control for the metronome position reducing synchronization time. Experiments in a laboratory prototype show the feasibility of the proposed control laws. In future work we shall study the asymptotic stability of the synchronization error, also other control laws will be studied such as Sliding Mode Control in order to achieve better synchronization times.

VII. Acknowledgements

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