Linear Regression as Linear Optimization

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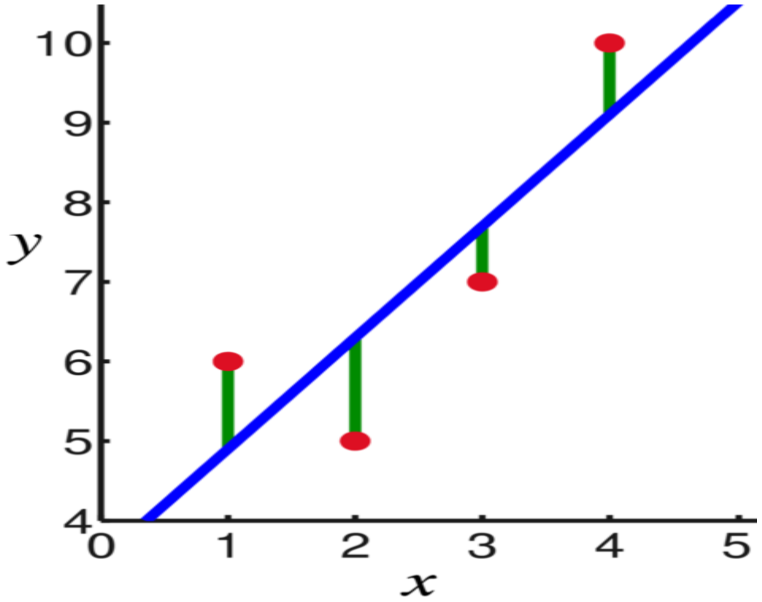
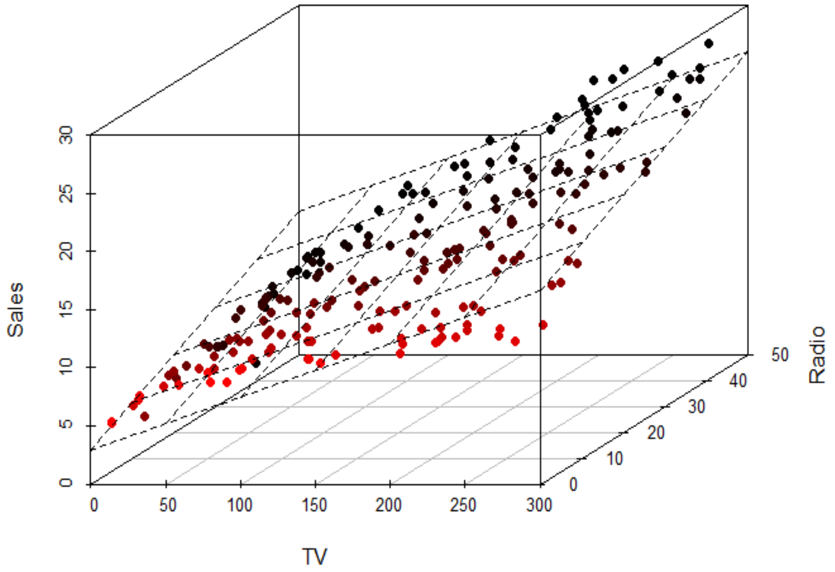
Linear Regression

- Model a dependent variable y as a linear function of some independent variables x_i. The model usually won't be perfect so there is some error term ϵ.
- Example for 1 independent variable: $y = \beta_0 + \beta x + \epsilon$



Linear Regression: General Form

- $y_i = \beta_0 + \beta_1 x_{i,1} + \cdots + \beta_p x_{i,p} + \epsilon_i; \qquad i = 1, \ldots, n$
- N datapoints $(y_i, x_{i,1}, \ldots, x_{i,p})$
- 3-dimensional example (y, x_1, x_2)



Minimizing Error

- $\epsilon_i = y_i \; - \beta_0 + \sum_{j=1}^{p} \beta_j x_{i,j}\,; \qquad i = 1, \ldots, n$
- Want to find β_j values that minimize "total error"
- Ordinary Least Squares(OLS) Regression minimizes Squared Error

$$\sum_{i=1}^{n} \left(y_i - \beta_0 + \sum_{j=1}^{p} \beta_j x_{i,j} \right)^2 = \sum_{i=1}^{n} \epsilon_i^2 \text{ (quadratic optimization)}$$

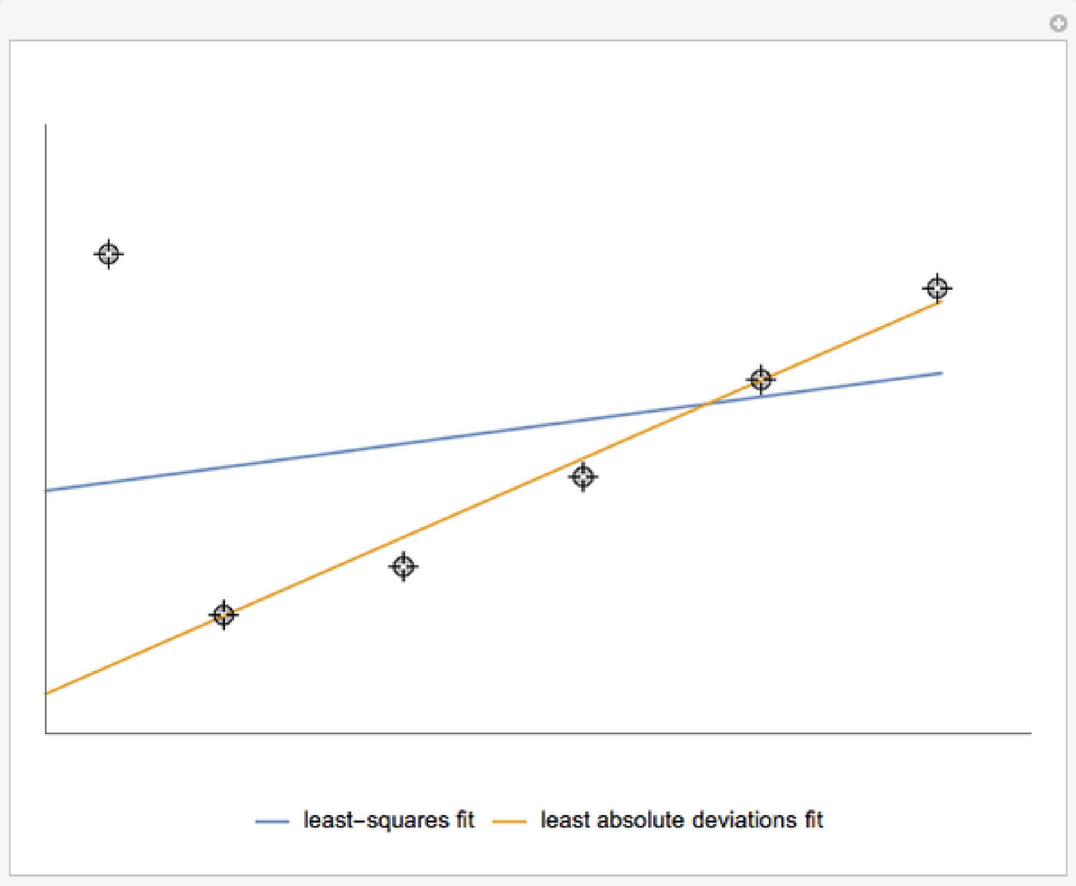
- Least Absolute Deviation(LAD) Regression minimizes Absolute Error

$$\sum_{i=1}^{n} \left| y_i - \beta_0 + \sum_{j=1}^{p} \beta_j x_{i,j} \right| = \sum_{i=1}^{n} |\epsilon_i| \quad \text{(linear optimization)}$$

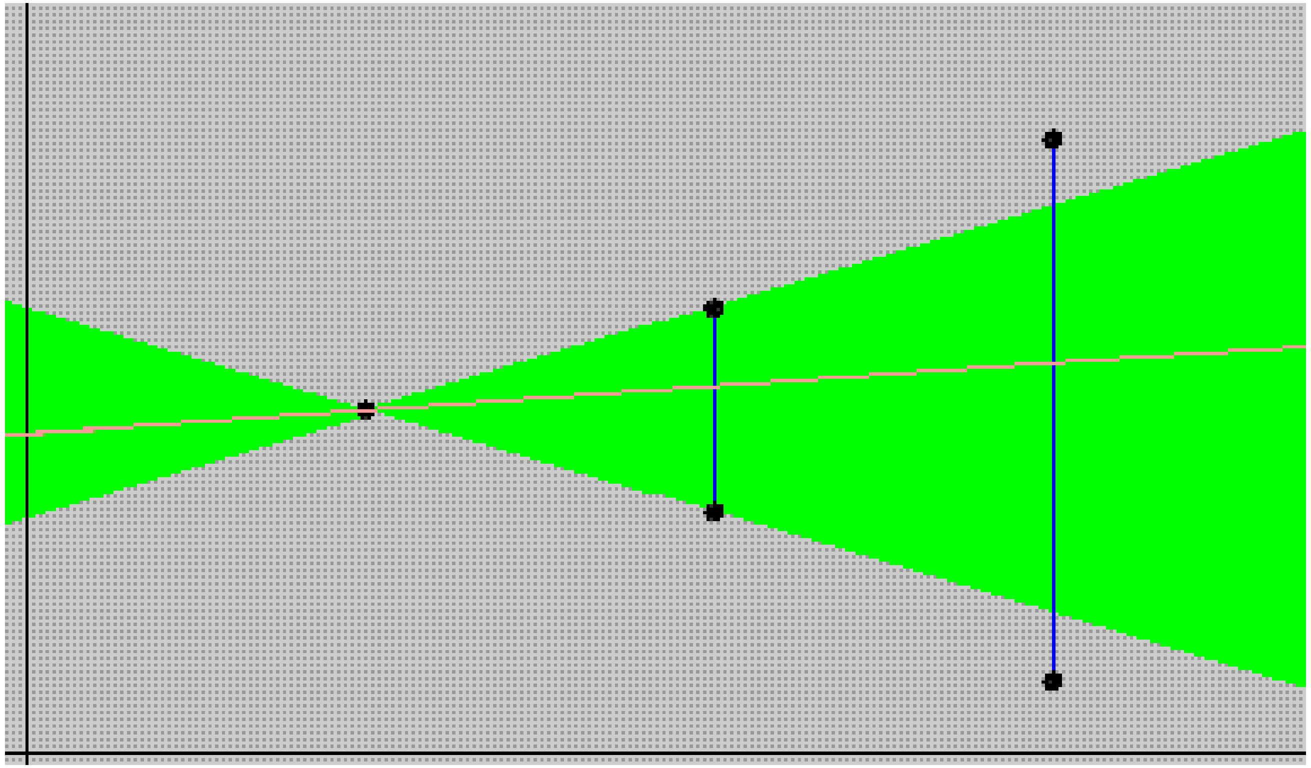
LAD vs OLS

LAD	OLS
Robust	Not very robust
Possibly many solution	Unique solution
Unstable Solution	Stable Solution

Robustness of LAD



Multiple Solutions using LAD



Least Absolute Deviations as a Linear Program

- Need fact that $|a| \leq b \Leftrightarrow -b \leq a \leq b$
- $t_i \geq 0 \quad i = 1, \ldots, n$ such that $|\epsilon_i| \leq t_i \Leftrightarrow -t_i \leq \epsilon_i \leq t_i$
- $\sum_{i=1}^{n} |\epsilon_i| \leq \sum_{i=1}^{n} t_i$

- LP formulation:

$$\min \sum_{i=1}^{n} t_i$$

$$-t_i \leq y_i - \beta_0 + \sum_{j=1}^{p} \beta_j x_{i,j} \leq t_i; \qquad i = 1, \ldots, n$$

Examples: mtcars data

- Variables in handout

- $mpg = 7.62 + 0.65cyl + 0.02disp - 0.03hp + 0.78drat - 4.56wt + 0.58qsec + 1.59vs + 1.33am + 2.37gear - 0.33carb + \epsilon$

Examples: iris data

- Variables in handout
- $petallength = -2.636 + 1.72sepallength - 1.22sepalwidth + \epsilon$

References

- https://en.wikipedia.org/wiki/File:Linear_regression.svg
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