Separable Convex Cost Flow Problems

Valentinas Sungaila and Eric Olberding

Convex Objective Function

- Min Cost Flow: minimize $\sum c_{ij} x_{ij}$.
- Convex Cost Flow: minimize $\sum C_{ij}(x_{ij})$
- The cost on each arc, $C_{ij}(x_{ij})$ is a convex (bath tub shaped) function of flow on that arc. We assume $C_{ij}(0) = 0$.
- Convex function: $\theta \in [0,1]$

$$\mathrm{C_{ij}}\left(\theta x_{ij} + (1-\theta) x'_{ij}\right) \leq \theta C_{ij}\left(x_{ij}\right) + (1-\theta) C_{ij}\left(x'_{ij}\right)$$

Separable Cost

- The total cost is the sum of the cost on each arc.

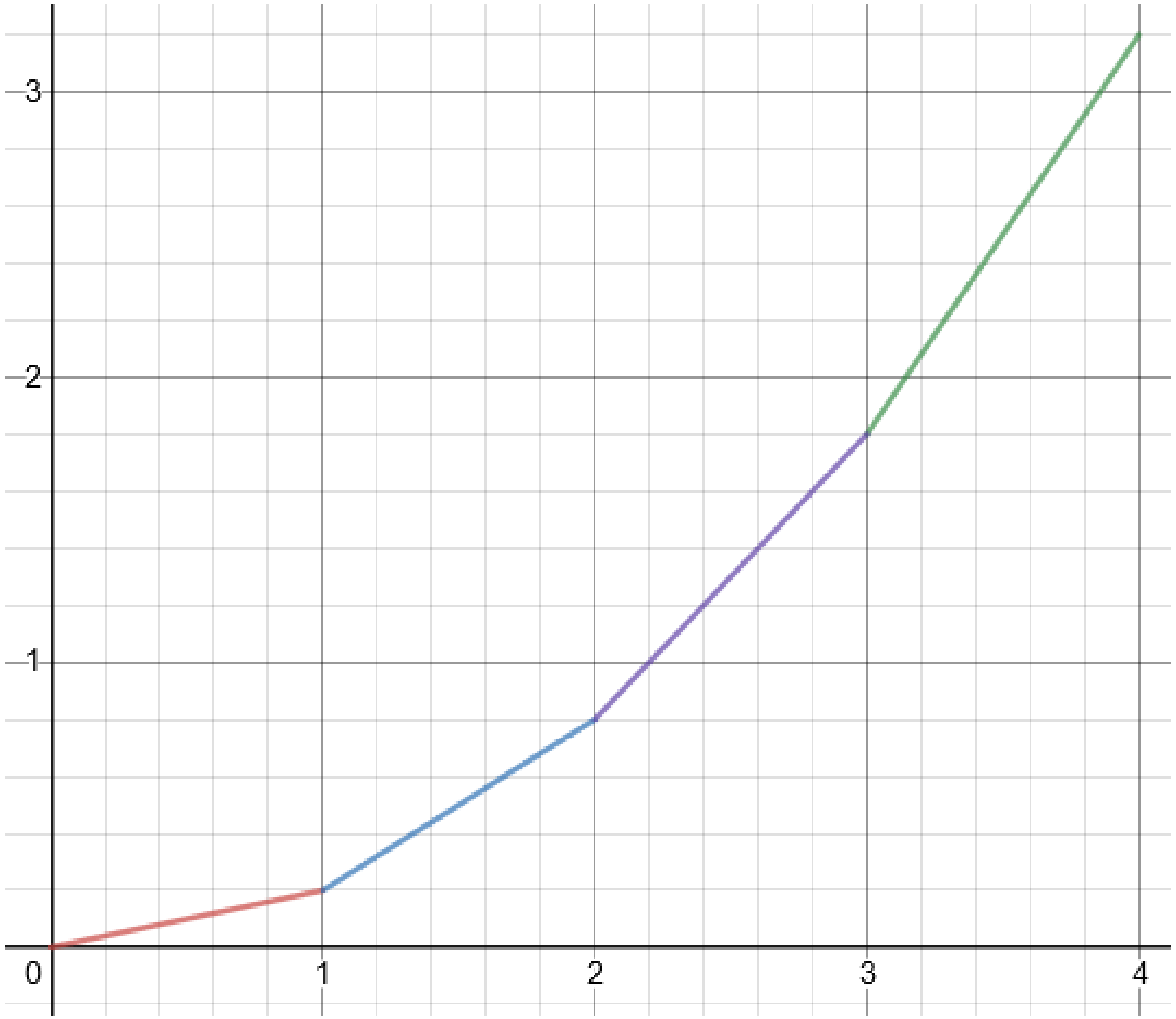
$$\mathrm{T} = \sum C_{ij}(x_{ij})$$

- Example of inseparable cost for network with two arcs with cost c_1 and c_2 per unit flow.

$$T = (c_1 x_1 + c_2 x_2)^2$$

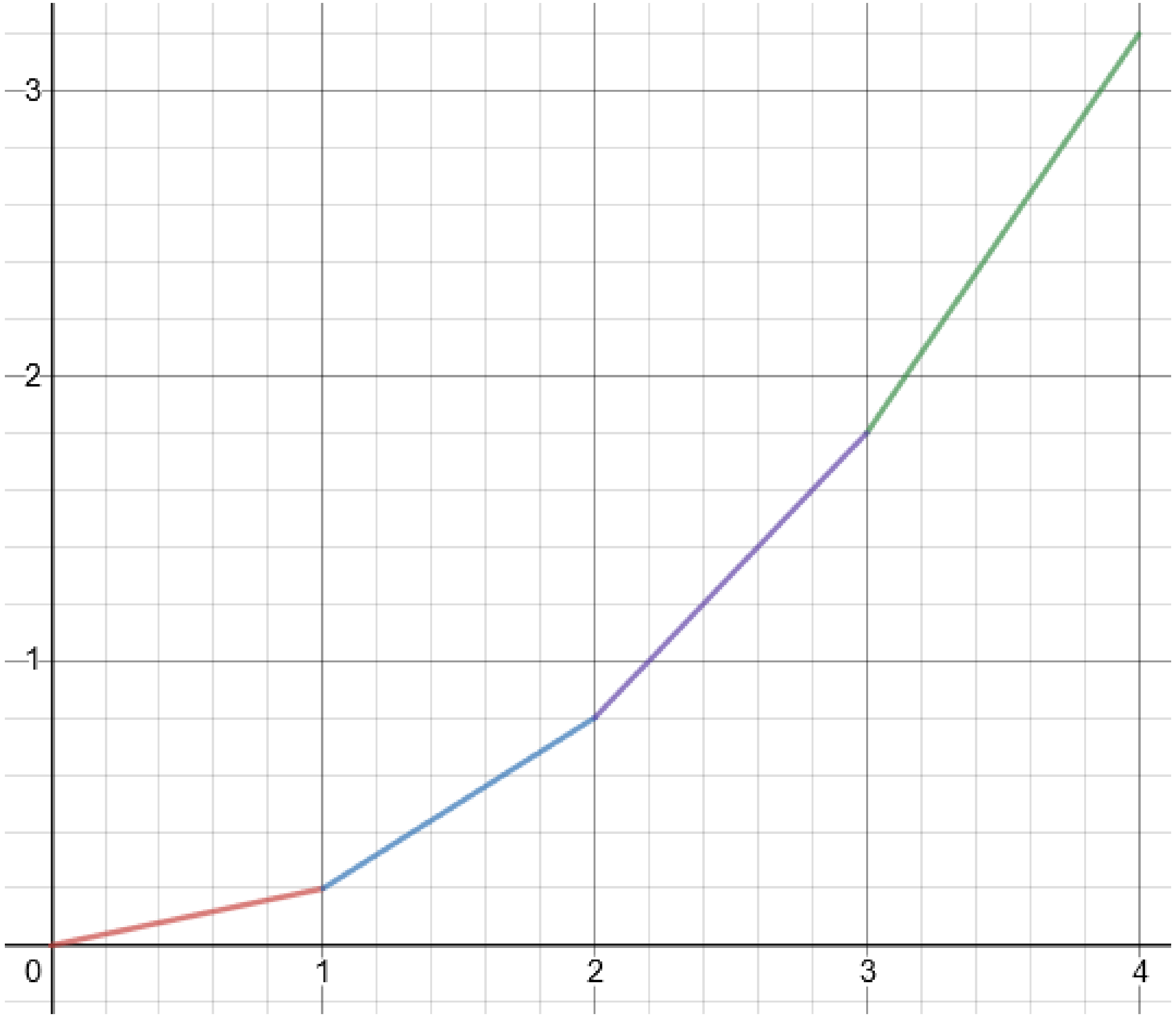
Piecewise Continuous Linear Costs

- y is cost, x is flow
- Continuous linear piece-wise functions.
    - arc cost specified with p pieces of information.
- Total information for cost will be total number of pieces for all arc cost functions
    - 4 pieces of information for this arc



Piecewise Continuous Linear Costs

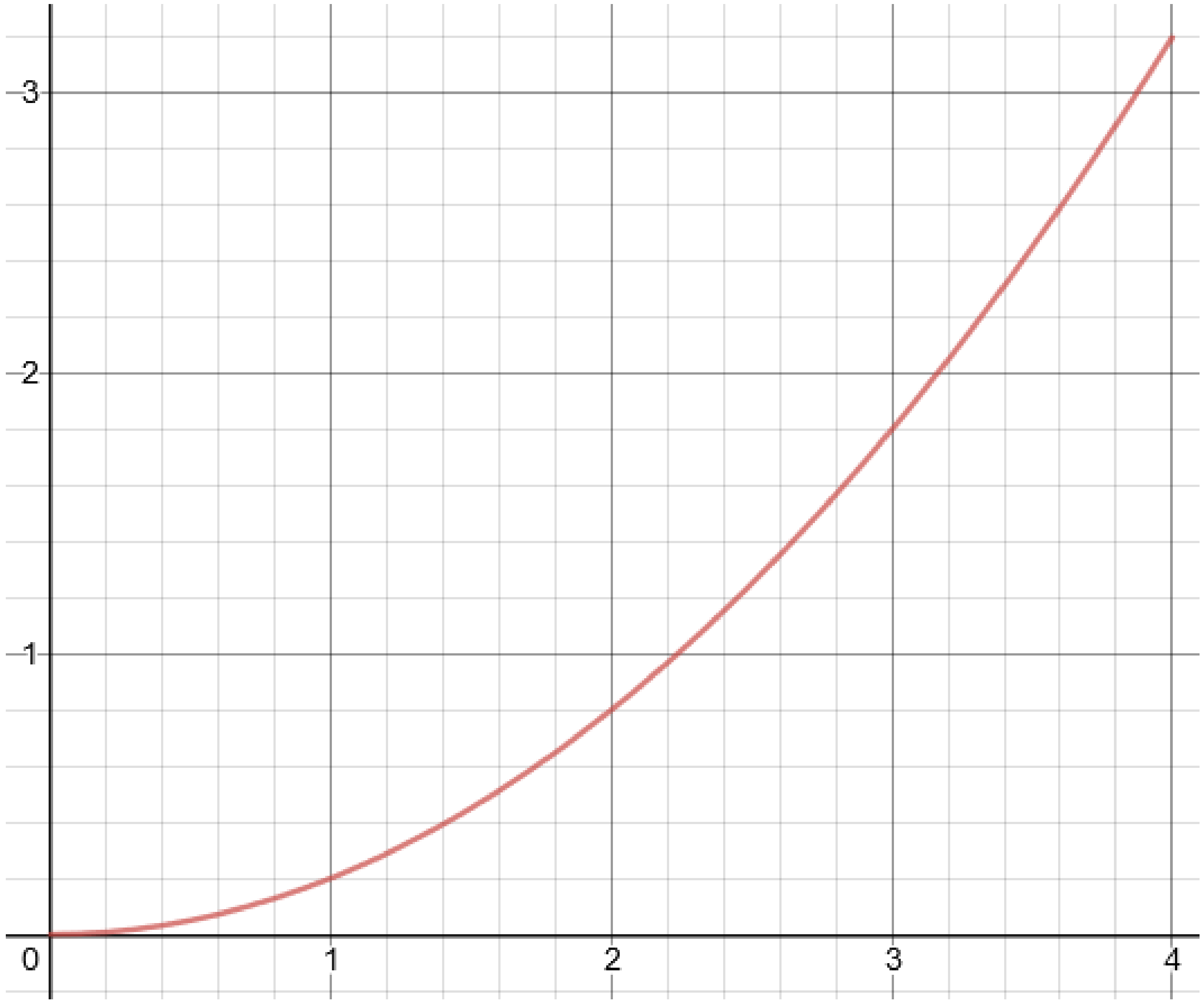
$$C(x) = \begin{cases} 0.2x, & \text{if } 0 \leq x < 1 \\ 0.6x - 0.4, & \text{if } 1 \leq x < 2 \\ x - 1.2, & \text{if } 2 \leq x < 3 \\ 1.4x - 2.4, & \text{if } 3 \leq x \leq 4 \end{cases}$$



Concise Function

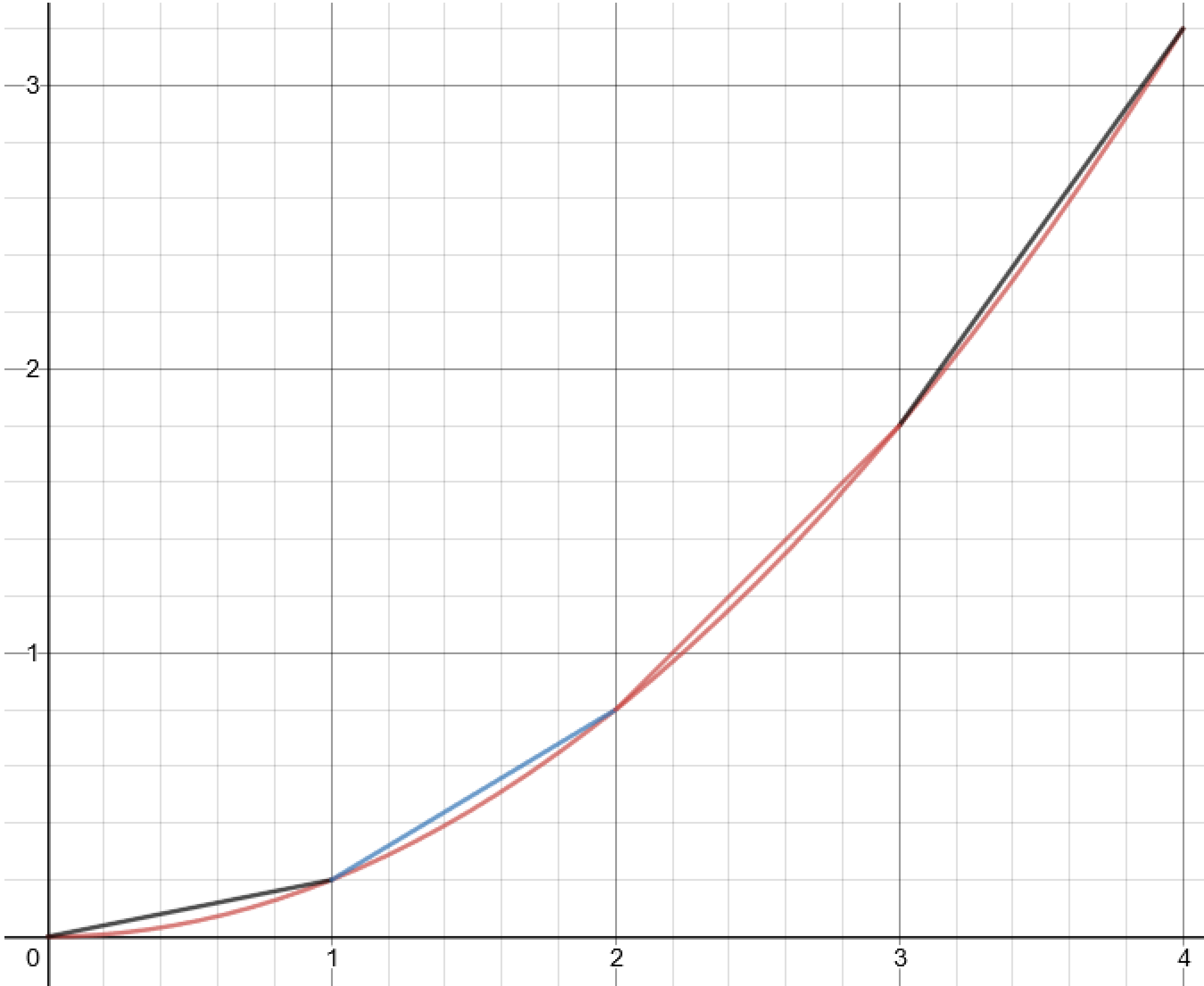
- Specified with $O(1)$ pieces of information per arc

$$C(x) = \frac{1}{5}x^2 \quad \text{if } 0 \leq x \leq 4$$



Approximation of Concise function with a Piecewise Linear function

- Assume feasible solutions for concise function are integers
- Equivalent to piecewise linear convex cost
- Optimal integer solution can be worse than continuous solution



Scaling to improve accuracy

- Replace x_{ij} in original problem with $\frac{y_{ij}}{M}$ for large M

- Integer optimal solution y_{ij}^* gives solution to original problem is $x_{ij}^* = \frac{y_{ij}^*}{M}$ within $1/M$ of the optimal solution.

- Stretching the x-axis to include more integer breakpoints

Issues with Approximation

- Algorithm running time on piecewise approximation may not be in terms of input data for concise cost graph

- Running time depends on number of segments used to approximate the concise function.

Application: Urban Traffic Flow (1)

- As the number of cars increases, the road becomes more congested
  - delay in travel time
- Can be modeled as

$$Delay = \frac{\alpha x}{(u - x)}$$

  - x is the flow of traffic
  - u theoretical road capacity
  - α a constant

- Finding flow that achieves a minimum overall delay for all roads is a convex cost network flow problem

Application: Urban Traffic Flow (2)

- People travel from their starting point to their destination with minimum delay
- The objective function is given by

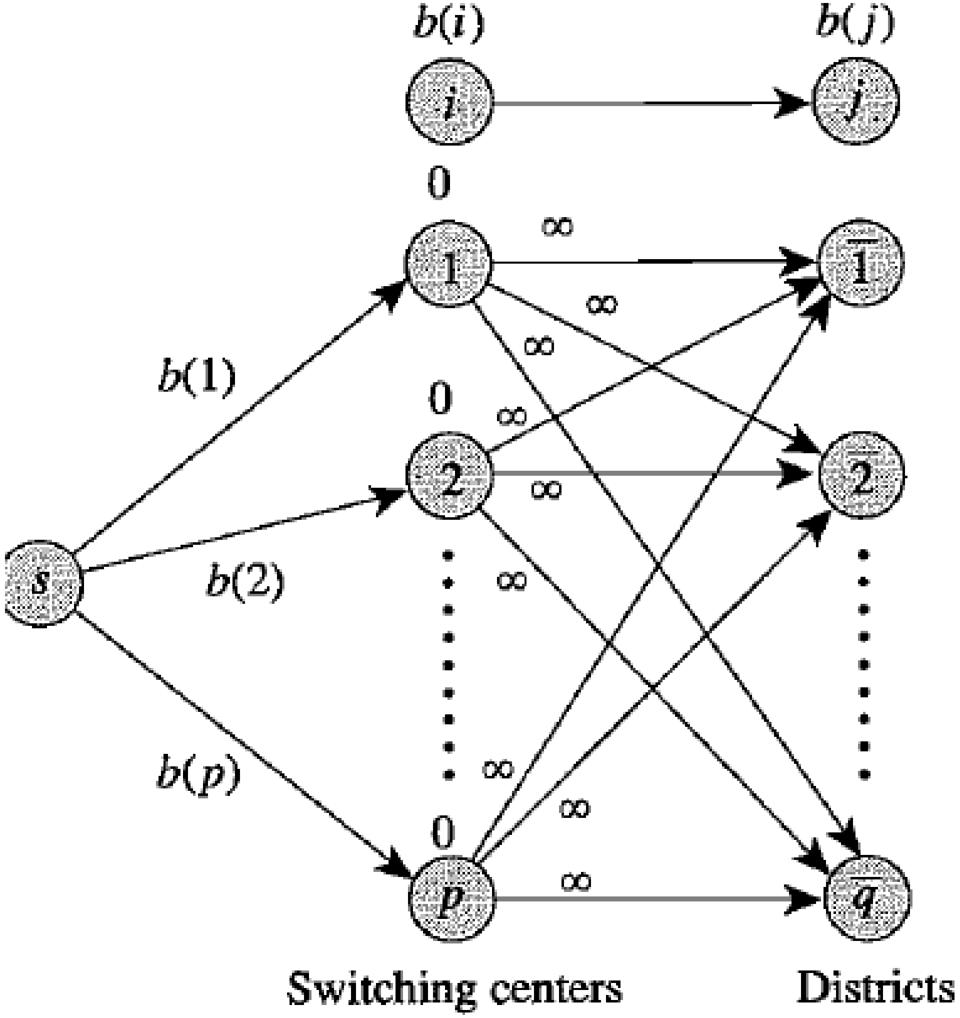
$$\sum_{(i,j)\epsilon\ A} \int_{0}^{x_{ij}} C_{ij}(y)\, dy$$

  - x_{ij} is the flow on arc *(i,j)*
  - $C_{ij}(y)$ is the delay cost on the arc
  - The delay a person incurs depends on others
  - If delay function is non decreasing then the objective function is convex

Application: Area Transfers in Communication Networks

- In the past equipping every phone with a routing ability was expensive. Thus requiring switch centers to be used when routing phone calls.

- *d(j)* current demand of district *j*
- *b(i)* represents the capacity of switch center *i*



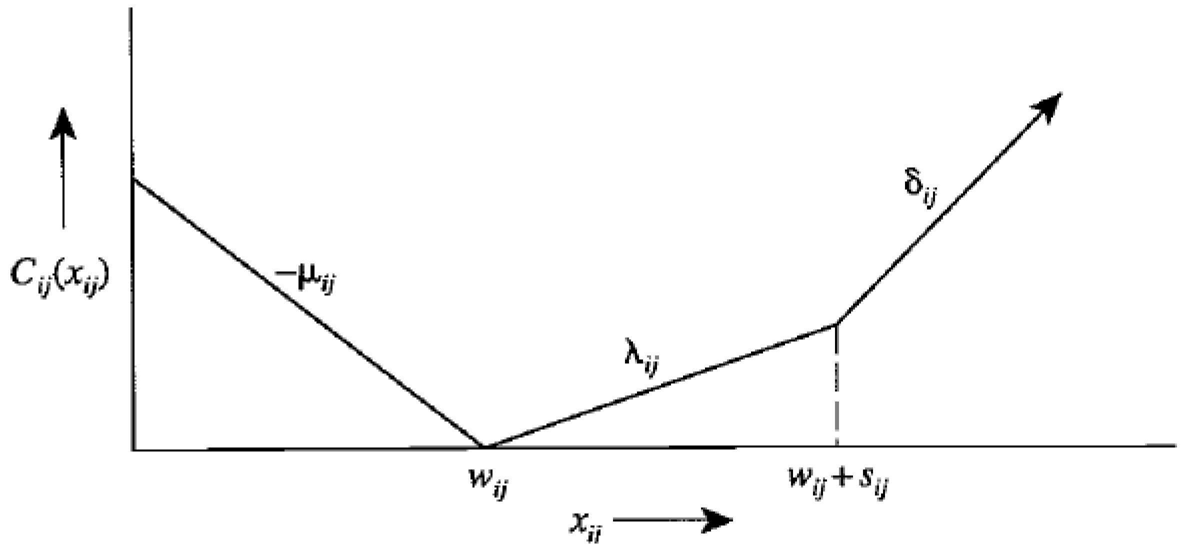
Application: Area Transfers in Communication Networks

- w_{ij} represents the number of lines currently in use between center *i* and district *j*.

- μ_{ij} represents the cost to disconnect a line from switch center i to district j and connect it to another switch center.

- s_{ij} represents the spare lines connecting to the switch center *i* to district j at a cost of λ_{ij} per line.

- δ_{ij}represents a cost for an additional line beyond the spare ones and the assumption is $\delta_{ij} > \lambda_{ij}$.



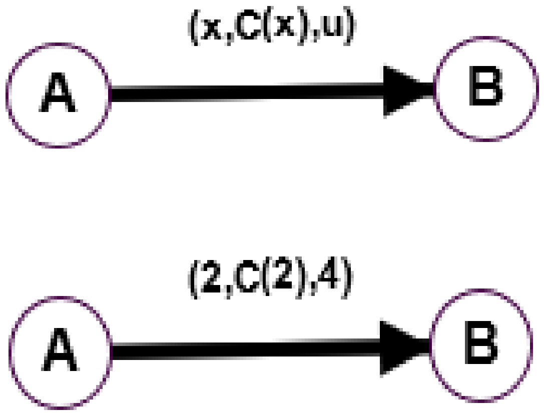
Network Transformation for Piecewise Convex Costs

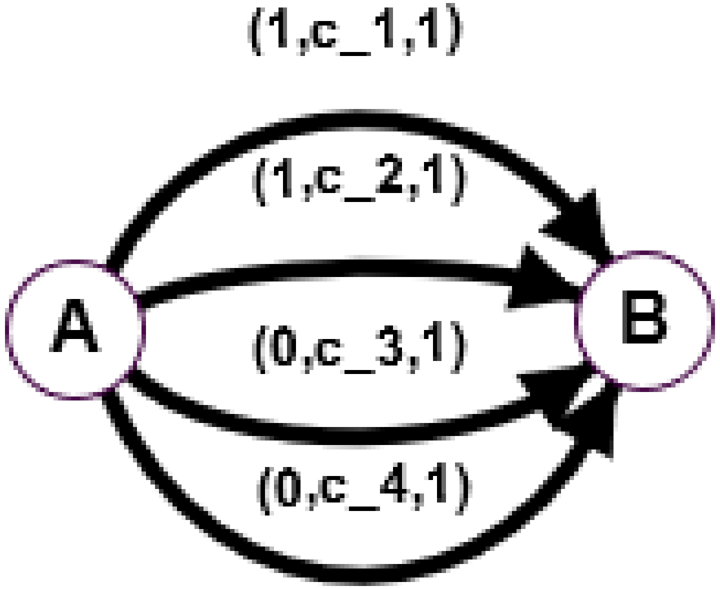
- Transform convex cost flow problem into format where standard min cost flow algorithms work

- Idea: arc (i, j) with p pieces for its cost function becomes multiple arcs $(i, j)_k$, then we use the residual network to remove the multi-arc issue

Network Transformation for Piecewise Convex Costs

- Arc with cost per unit flow of c_1 when $x_{ij} \in [0,1)$, c_2 when $x_{ij} \in [1,2)$, and c_3, c_4 are similar. Capacity is 4. We send 2 units of flow along this arc.





Issues with Network Transformation

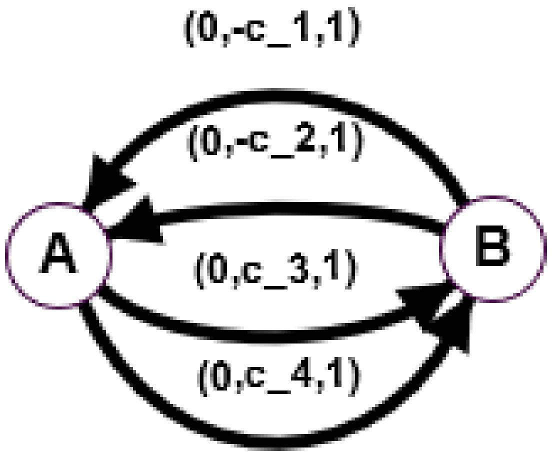
- Now we have a graph with multi-arcs!
- Same amount of information if cost was originally piecewise.
- Added information if cost was concise.

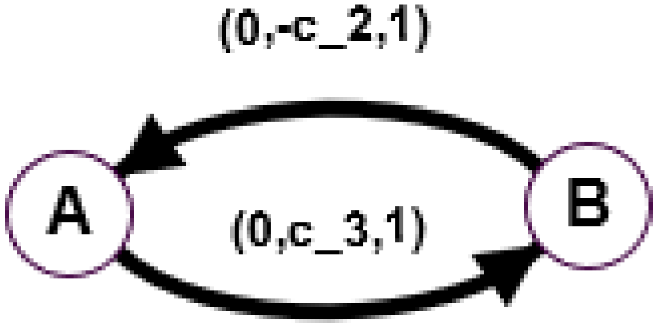
Residual Network Modification

- Since convex, $c_1 < c_2 < c_3 < c_4$.

- If we sent a flow of 2, only need backward arc that saves that most and forward arc that costs the least per unit flow.

- Now usual min cost algorithms work

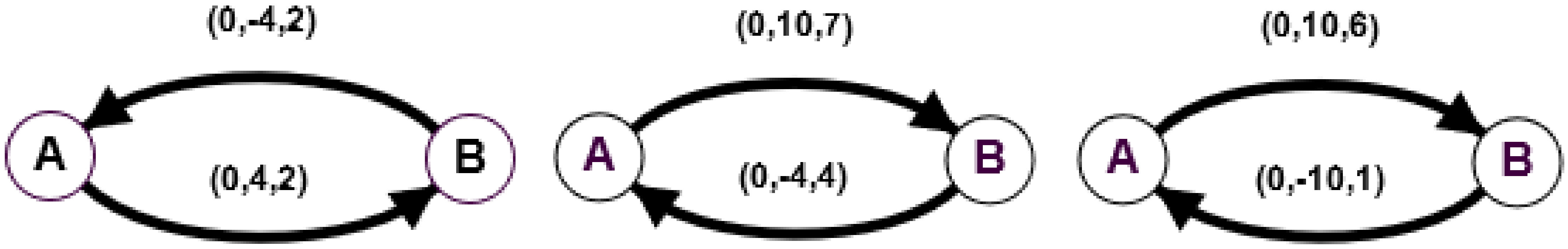




General Residual Network Transformation

- Cost per unit flow may not change for each unit of flow. Let $c_1 = 1$ for $x_{ij} \in [0,2)$, $c_2 = 4$ when $\mathrm{x_{ij}} \in [2,6)$, and $c_3 = 10$ when $x_{ij} \in [6,13]$.

- Send 4 units of flow, cost per unit is c_2. 2 capacity left until cost per unit changes either way. Now send another 3 units of flow so $x_{ij} = 7$.



Pseudopolynomial Algorithms

- Cyclecanceling Algorithm: Similar to before but with modified residual network. When augmenting, may not be able to augment full amount along a negative cycle in the residual network. (-c_2 only present in residual network)

- To fix this, send flow along the same negative cycle until its cost becomes nonnegative. This avoids unnecessarily searching for negative cost cycles.

Pseudopolynomial Algorithms

- Successive shortest path algorithm: Again, use modified residual network.
- However, we may augment along the same path multiple times as the cost per unit flow of an arc changes.
- Might send only 1 unit of flow per augmentation with concise function approximation

Polynomial Time Algorithm

- Capacity Scaling Algorithm: Modification of successive shortest paths. Instead of using integer breaks of 1 unit from the start, we start with larger intervals and then scale downward.

- Let $C_{ij}(x_{ij}) = x_{ij}^4$ with $u_{ij} = 12$.

- Model function with breakpoints 8 apart, then 4 apart. Do this until we have breakpoints that are 1 apart.

- First phase can augment 8 units of flow, then 4, etc. Faster decrease of excess.

Algorithm

- Initially: 0 pseudoflow and 0 node potentials

- Let Δ be the interval between breakpoints. We have a phase for each Δ. During each phase, we create a Δ residual network, constructed as we did before. (Different approximations require different modified residual networks)

- If arc (i, j) or (j, i) violates the reduced cost optimality conditions, increase of decrease flow x_{ij} by Δ to correct this.

Algorithm Cont.

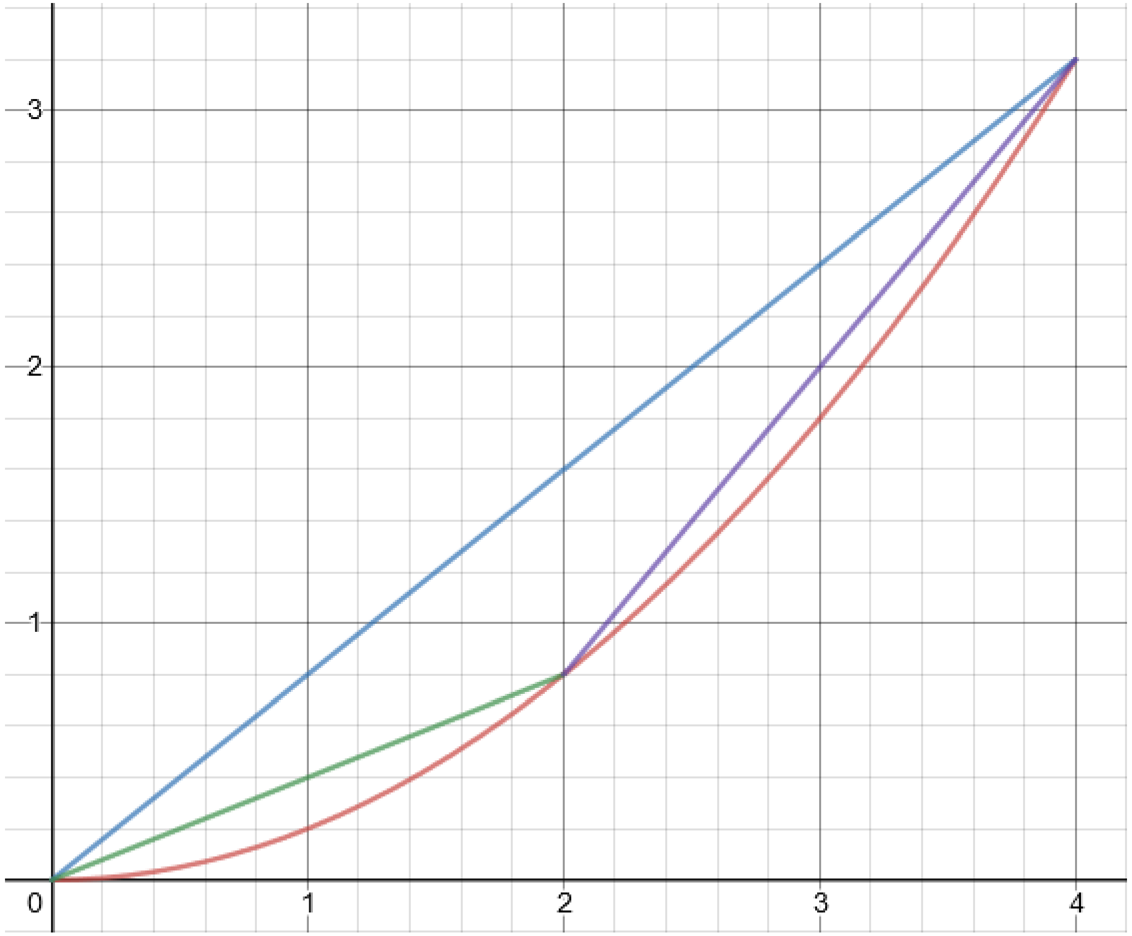
- Define sets $E(\Delta)$ and $D(\Delta)$ for nodes with excesses and deficits of at least Δ.

- We reduce excess by Δ, then $\frac{\Delta}{2}$, then $\frac{\Delta}{4}$ until we get to breakpoints that are 1 apart.

- Essentially do successive shortest path algorithm until either $E(\Delta)$ or $D(\Delta)$ is empty, then reduce Δ by half and do it again.

Capacity Scaling Algorithm - Optimality

- Flow x_{ij} satisfies reduced cost optimality within each phase
- Only needs to be adjusted by at most Δ at start of next phase to achieve optimality
- $c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j)$
- c_{ij} is slope of segment to the right of x_{ij}



Runtime

- $S(n, m, C)$ time to solve shortest path problem with nonnegative arc lengths
- At end of 2Δ phase, $E(2\Delta) = \emptyset$ or $D(2\Delta) = \emptyset$
- Less than $n * 2\Delta$ positive imbalance.

Runtime

- Start of phase, potentially adjust each arc flow by Δ.
- $(m + n) * 2\Delta$ is a bound on positive imbalance. Augment Δ each iteration within a scaling phase.
- $O(m)$ augmentations within a phase and a $\log(U)$ bound on phases. Also need to find shortest path for each augmentation.
- $O(m * \log(U)\, S(n, m, C))$

References

- Ravindra K. Ahuja, Thomas L. Magnanti, and James B. Orlin. 1993. Network flows: theory, algorithms, and applications. Prentice-Hall, Inc., USA.