COMP2119 Introduction to Data Structures and Algorithms
Assignment 1 - Recursion, Mathematical Induction and Algorithm Analysis

Due Date: Sept 27, 2024 7:00pm

Question 1 - Asymptotic Bounds [20%]

Rank the following functions by order of growth in increasing order and partition your list into equivalence classes such that two functions $f(n)$ and $g(n)$ are in the same class if and only if $f(n) = \Theta(g(n))$. You do not need to prove your answer. [In this question, we assume $\log n$ stands for $\log_2 n$.]

(a) n^{π},

(b) π^n,

(c) n^n,

(d) $\log n$,

(e) $\pi^{\log n}$,

(f) $n^{\log \pi}$,

(g) $\frac{n}{\log \pi}$,

(h) $\frac{n}{\log n}$,

(i) $\frac{n}{\log \log n}$,

(j) $\log \frac{n}{\log n}$,

(k) $\pi^{\log(2n)}$,

(l) $n^{\log(2\pi)}$,

(m) $\sqrt{\sum_{i=1}^{n}(i+1)}$,

(n) $1910n! + 316n^n$

Solution: (20 points)

- $\Theta(\log n)$: $\log n$ (d), $\log \frac{n}{\log n}$ (j)
- $\Theta(\frac{n}{\log n})$: $\frac{n}{\log n}$ (h)
- $\Theta(\frac{n}{\log \log n})$: $\frac{n}{\log \log n}$ (i)
- $\Theta(n)$: $\frac{n}{\log \pi}$ (g), $\sqrt{\sum_{i=1}^{n}(i+1)}$ (m)
- $\Theta(n^{\log \pi})$: $\pi^{\log n}$ (e), $n^{\log \pi}$ (f), $\pi^{\log(2n)}$ (k)
- $\Theta(n^{\log \pi + 1})$: $n^{\log(2\pi)}$ (l)

- $\Theta(n^\pi)$: n^π (a)
- $\Theta(\pi^n)$: π^n (b)
- $\Theta(n^n)$: n^n (c), $1910n! + 316n^n$ (n)

Question 2 - Recurrence Relations [20%]

For each of the following recurrence relations, solve the recurrence and state the Θ bound.

(a) $T(n) = T(n-1) + 3$; $T(0) = 0$.

(b) $T(n) = 3T(\frac{n}{3}) + n$; $T(1) = 0$, you may assume that n is a power of 3.

(c) $T(n) = 4T(\frac{n}{3}) + 1$; $T(1) = 0$, you may assume that n is a power of 3.

(d) $T(n) = nT(\frac{n}{2}) + n - 1$; $T(1) = 1$, you may assume that n is a power of 2.

Solution: (5 points each)

(a)
$$\begin{aligned} T(n) &= T(n-1) + 3 \times 1 \\ &= T(n-2) + 3 \times 2 \\ &= T(n-3) + 3 \times 3 \\ &= \cdots \\ &= T(0) + 3 \times n \\ &= 3n \\ &\rightarrow \Theta(n) \end{aligned}$$

(b)
$$\begin{aligned} T(n) &= 3T(\frac{n}{3}) + n \\ &= 3(3T(\frac{n}{3^2}) + \frac{n}{3}) + n \times 1 \\ &= 3^2T(\frac{n}{3^2}) + n \times 2 \\ &= 3^2(3T(\frac{n}{3^3} + \frac{n}{3^2})) + n \times 2 \\ &= 3^3T(\frac{n}{3^3}) + n \times 3 \\ &= \cdots \\ &= 3^{\log_3 n}T(\frac{n}{3^{\log_3 n}}) + n \times \log_3 n \\ &= nT(1) + n \times \log_3 n \\ &\rightarrow \Theta(n \log n) \end{aligned}$$

(c) $$\begin{aligned}T(n) &= 4T(\frac{n}{3}) + 1\\
&= 4(4T(\frac{n}{3^2}) + 1) + 1\\
&= 4^2T(\frac{n}{3^2}) + 4 + 1\\
&= 4^2(4T(\frac{n}{3^3}) + 1) + \frac{4^2-1}{3}\\
&= 4^3T(\frac{n}{3^3}) + \frac{4^3-1}{3}\\
&= \cdots\\
&= 4^{\log_3 n}T(\frac{n}{3^{\log_3 n}}) + \frac{4^{\log_3 n}-1}{3}\\
&= \frac{n^{\log_3 4}-1}{3}\\
&\to \Theta(n^{\log_3 4})\end{aligned}$$

where the last equality is due to $4^{\log_3 n} = 3^{(\log_3 n)(\log_3 4)} = n^{\log_3 4}$.

(d) Note that $\frac{T(n)+1}{T(\frac{n}{2})+1} = n$, we have

$$\begin{aligned}T(n)+1 &= \frac{T(n)+1}{T(\frac{n}{2})+1}\cdot(T(\frac{n}{2})+1)\\
&= \frac{T(n)+1}{T(\frac{n}{2})+1}\cdot\frac{T(\frac{n}{2})+1}{T(\frac{n}{2^2})+1}\cdot(T(\frac{n}{2^2})+1)\\
&= \frac{T(n)+1}{T(\frac{n}{2})+1}\cdot\frac{T(\frac{n}{2})+1}{T(\frac{n}{2^2})+1}\cdot\cdots\cdot\frac{T(2)+1}{T(1)+1}\cdot(T(1)+1)\\
&= n\cdot\frac{n}{2}\cdot\frac{n}{2^2}\cdot\cdots\cdot 2\cdot(T(1)+1)\\
&= \frac{n^{\log_2 n}}{2^{0+1+2+\cdots+(\log_2 n-1)}}\cdot 2\\
&= \frac{n^{\log_2 n}}{2^{\frac{\log_2 n(\log_2 n-1)}{2}}}\cdot 2\\
&= \frac{n^{\log_2 n}}{n^{\frac{\log_2 n-1}{2}}}\cdot 2\\
&= n^{\frac{\log_2 n+1}{2}}\cdot 2\\
&\to \Theta(n^{\frac{\log_2 n+1}{2}})\end{aligned}$$

Question 3 - Mathematical Induction [30%]

Prove the following equations with mathematical induction.

(a) $1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2$ for all positive integers n.

(b) $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1} - 1$ for all positive integers n.

Solution: (15 points each)

(a) Let $P(n)$ be the proposition that $1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2$ for all positive integers n.

Base case:
When $n = 1$, $P(1)$ is true since $1 \cdot 2^1 = (1-1) \cdot 2^2 + 2 = 2$

Inductive step:
Assume that $P(k)$ is true, where k is some positive integers,
i.e. $1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + k \cdot 2^k = (k-1) \cdot 2^{k+1} + 2$

Now consider the case of $P(k+1)$,

$$\begin{aligned}
\text{L.H.S.} &= (1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + k \cdot 2^k) + (k+1) \cdot 2^{k+1} \\
&= (k-1) \cdot 2^{k+1} + 2 + (k+1) \cdot 2^{k+1} \\
&= (k-1+k+1) \cdot 2^{k+1} + 2 \\
&= (2k) \cdot 2^{k+1} + 2 \\
&= k \cdot 2^{k+2} + 2 \\
&= ((k+1)-1) \cdot 2^{(k+1)+1} + 2 \\
&= \text{R.H.S.}
\end{aligned}$$

Therefore, $P(k+1)$ is true.

According to M.I., $P(n)$ is true for all positive integers n.

(b) Let $P(n)$ be the proposition that $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1} - 1$ for all positive integers n.

Base case:
When $n = 1$, $P(1)$ is true since $\frac{1}{\sqrt{1}+\sqrt{2}} = \frac{\sqrt{2}-\sqrt{1}}{(\sqrt{1}+\sqrt{2})(\sqrt{2}-\sqrt{1})} = \frac{\sqrt{2}-1}{(\sqrt{2})^2-(\sqrt{1})^2} = \frac{\sqrt{2}-1}{1} = \sqrt{2} - 1$

Inductive step:
Assume that $P(k)$ is true, where k is some positive integers,
i.e. $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{k}+\sqrt{k+1}} = \sqrt{k+1} - 1$

Now consider the case of $P(k+1)$,

$$
\begin{aligned}
\text{L.H.S.} &= (\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{k}+\sqrt{k+1}}) + \frac{1}{\sqrt{k+1}+\sqrt{k+2}} \\
&= \sqrt{k+1} - 1 + \frac{1}{\sqrt{k+1}+\sqrt{k+2}} \\
&= \sqrt{k+1} - 1 + \frac{\sqrt{k+2}-\sqrt{k+1}}{(\sqrt{k+1}+\sqrt{k+2})(\sqrt{k+2}-\sqrt{k+1})} \\
&= \sqrt{k+1} - 1 + \frac{\sqrt{k+2}-\sqrt{k+1}}{(\sqrt{k+2})^2-(\sqrt{k+1})^2} \\
&= \sqrt{k+1} - 1 + \frac{\sqrt{k+2}-\sqrt{k+1}}{1} \\
&= \sqrt{k+1} - 1 + \sqrt{k+2} - \sqrt{k+1} \\
&= \sqrt{k+2} - 1 \\
&= \sqrt{(k+1)+1} - 1 \\
&= \text{R.H.S.}
\end{aligned}
$$

Therefore, $P(k+1)$ is true.

According to M.I., $P(n)$ is true for all positive integers n.

Question 4 - Algorithm Design [30%]

Given a positive integer n, please design an algorithm that calculates 2^n faster than $O(n)$ time.

You should clearly explain the algorithm, demonstrate the correctness of the algorithm with a complete proof and show the running time of the algorithm.

For example, when $n = 13$, you may calculate 2^{13} in the following steps:

$$\begin{aligned} 2^3 &= 2 \cdot 2 \cdot 2 = 8; \\ 2^6 &= 2^3 \cdot 2^3 = 8 \cdot 8 = 64; \\ 2^{13} &= 2^6 \cdot 2^6 \cdot 2 = 64 \cdot 64 \cdot 2 = 8192. \end{aligned}$$

Solution:

Algorithm. (10 points)

Algorithm $\textsc{PowerOfTwo}(n)$

1: **Input:** $n \in N^*$.
2: **if** $n == 1$ **then**
3:     **Return** 2
4: **end if**
5: $x := \textsc{PowerOfTwo}(\lfloor \frac{n}{2} \rfloor)$
6: **if** n is even **then**
7:     **Return** $x * x$
8: **else**
9:     **Return** $x * x * 2$
10: **end if**

Proof of Correctness using Mathematical Induction. (10 points)

Let $P(n)$ be the proposition that $\textsc{PowerOfTwo}(n)$ returns 2^n for all positive integers n.

1. **Base Case.** If $n = 1$, $\textsc{PowerOfTwo}(1)$ returns 2 which is 2^1.
2. **Inductive step:** Assume that $P(k)$ is true for $k < n$, where n is some positive integers, i.e. $\textsc{PowerOfTwo}(k)$ returns 2^k for $k = 1, 2, \cdots, n-1$

   Now consider the case of $P(n)$,

   - When n is even: note that $x = \textsc{PowerOfTwo}(\frac{n}{2})$ as shown in Line 5, which is equals to $2^{\frac{n}{2}}$ as assumed. Hence, we have that $\textsc{PowerOfTwo}(n)$ returns $x \cdot x = 2^{\frac{n}{2}} \cdot 2^{\frac{n}{2}} = 2^n$.
   - When n is not even: note that $x = \textsc{PowerOfTwo}(\frac{n-1}{2})$ as stated in Line 5, which is equals to $2^{\frac{n-1}{2}}$ as assumed. Hence, we have that $\textsc{PowerOfTwo}(n)$ returns $x \cdot x \cdot 2 = 2^{\frac{n-1}{2}} \cdot 2^{\frac{n-1}{2}} \cdot 2 = 2^n$.

   Therefore, $P(n)$ is true.

   According to M.I., $P(n)$ is true for all positive integers n.

Running Time. (10 poitns)

Running time $T(n)$ for input n could be calculated by $T(n) = T(\lfloor \frac{n}{2} \rfloor) + O(1)$, which gives $T(n) = \Theta(\log n)$.

Submission

Please submit your assignment (one **PDF** file) to moodle by the deadline. Make sure the content is readable. Feel free to contact the TAs if you encounter any difficulty in this assignment. We are happy to help you!