

COMP2119 Introduction to Data Structures and Algorithms

Assignment 1 - Recursion, Mathematical Induction and Algorithm Analysis

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September 16, 2024

1 Asymptotic Bounds

$$\begin{aligned}
 \Theta(\log n) : & (d) \log n, (j) \log \frac{n}{\log n}, \\
 \Theta\left(\frac{n}{\log n}\right) : & (h) \frac{n}{\log n}, \\
 \Theta\left(\frac{n}{\log \log n}\right) : & (i) \frac{n}{\log \log n}, \\
 \Theta\left(\frac{n}{\log \pi}\right) : & (g) \frac{n}{\log \pi}, (m) \sqrt{\sum_{i=1}^n (i+1)}, \\
 \Theta(n^{\log \pi}) : & (e) \pi^{\log n}, (f) n^{\log \pi}, (k) \pi^{\log(2n)}, \\
 \Theta(n^{\log 2\pi}) : & (l) n^{\log 2\pi}, \\
 \Theta(n^\pi) : & (a) n^\pi, \\
 \Theta(\pi^n) : & (b) \pi^n, \\
 \Theta(n^n) : & (n) 1910n! + 316n^n, (c) n^n
 \end{aligned}$$

2 Recurrence Relations

$$\begin{aligned}
 (a) \quad T(n) &= T(n-1) + 3 \quad \text{for } n > 0 \\
 &= T(n-2) + 3 + 3 \quad \text{for } n > 1 \\
 &= \dots \\
 \therefore T(n) &= T(0) + 3n \\
 &= 3n \\
 \therefore T(n) &= \Theta(n)
 \end{aligned}$$

(b) Assume that n is a power of 3, i.e. $n = 3^k$ for $k \in \mathbb{N}$, and $\log_3 n = k$,

$$\begin{aligned}
 \therefore T(n) &= 3T\left(\frac{n}{3}\right) + n \quad \text{for } n \neq 1 \\
 &= 3\left(3T\left(\frac{n}{9}\right) + \frac{n}{3}\right) + n \quad \text{for } n \geq 9 \\
 &= 9T\left(\frac{n}{9}\right) + n + n \quad \text{for } n \geq 9 \\
 &= \dots \\
 \therefore T(n) &= 3^k T\left(\frac{n}{3^k}\right) + k * n \\
 &= 3^k T(1) + k * n \\
 &= 0 + k * n \\
 &= kn \\
 &= n * \log_3 n \\
 \therefore T(n) &= \Theta(n \log n)
 \end{aligned}$$

(c) Assume that n is a power of 3, i.e. $n = 3^k$ for $k \in \mathbb{N}$, and $\log_3 n = k$,

$$\begin{aligned}
\therefore T(n) &= 4T\left(\frac{n}{3}\right) + 1 \quad \text{for } n \geq 3 \\
&= 4\left(4T\left(\frac{n}{9}\right) + 1\right) + 1 \quad \text{for } n \geq 9 \\
&= 16T\left(\frac{n}{9}\right) + 4^{k-1} + \dots + 4^0 \quad \text{for } n \geq 9 \\
&= \dots \\
\therefore T(n) &= 4^k T\left(\frac{n}{3^k}\right) + \frac{1 * (1 - 4^k)}{1 - 4} \\
&= 4^k T(1) + \frac{4^k - 1}{3} \\
&= 0 + \frac{4^{\log_3 n} - 1}{3} \\
&= \frac{1}{3} * 4^{\log_3 n} - \frac{1}{3} \\
&= \frac{1}{3} * 3^{(\log_3 4)(\log_3 n)} - \frac{1}{3} \\
&= \frac{1}{3} * n^{\log_3 4} - \frac{1}{3} \\
\therefore T(n) &= \Theta(n^{\log_3 4})
\end{aligned}$$

(d) Assume that n is a power of 2, i.e. $n = 2^k$ for $k \in \mathbb{N}$, and $\log_2 n = k$,

$$\begin{aligned}
\therefore T(n) &= nT\left(\frac{n}{2}\right) + n - 1 \quad \text{for } n \geq 2 \\
&= n * \left(\frac{n}{2} * T\left(\frac{n}{4}\right) + \frac{n}{2} - 1\right) + n - 1 \quad \text{for } n \geq 4 \\
&= \frac{n^2}{2} * T\left(\frac{n}{4}\right) + \frac{n^2}{2} - n + n - 1 \quad \text{for } n \geq 4 \\
&= \frac{n^2}{2} * T\left(\frac{n}{4}\right) + \frac{n^2}{2} - 1 \quad \text{for } n \geq 4 \\
&= \frac{n^2}{2} * \left(\frac{n}{4} * T\left(\frac{n}{8}\right) + \frac{n}{4} - 1\right) + \frac{n^2}{2} - 1 \quad \text{for } n \geq 8 \\
&= \frac{n^3}{2 * 4} * T\left(\frac{n}{8}\right) + \frac{n^3}{2 * 4} - \frac{n^2}{2} + \frac{n^2}{2} - 1 \quad \text{for } n \geq 8 \\
&= \frac{n^3}{2^1 * 2^2} * T\left(\frac{n}{2^1 * 2^2}\right) + \frac{n^3}{2^1 * 2^2} - 1 \quad \text{for } n \geq 8 \\
&= \dots \\
\therefore T(n) &= \frac{n^k}{2^0 * 2^1 * 2^2 * \dots * 2^{k-1}} * T(1) + \frac{n^k}{2^0 * 2^1 * 2^2 * \dots * 2^{k-1}} - 1 \\
&= \frac{n^k}{2^{\frac{(k-1)*k}{2}}} * 2 - 1 \\
&= \frac{n^k}{n^{\frac{k-1}{2}}} * 2 - 1 \\
&= n^{\frac{2k}{2} - \frac{k-1}{2}} * 2 - 1 \\
&= n^{\frac{k+1}{2}} * 2 - 1 \\
&= 2n^{\frac{\log_2 n + 1}{2}} - 1 \\
\therefore T(n) &= \Theta(n^{\frac{\log_2 n + 1}{2}})
\end{aligned}$$

3 Mathematical Induction

(a) Let $f(n)$ be the predicate " $1 * 2^1 + 2 * 2^2 + 3 * 2^3 + \dots + n * 2^n = (n - 1)2^{n+1} + 2$ " for $\forall n \in \mathbb{Z}^+$.

For $n = 1$, L.H.S. $= 1 * 2^1$

$$= 2$$

$$\text{R.H.S.} = (1 - 1)2^{1+1} + 2$$

$$= 2$$

\therefore L.H.S. $=$ R.H.S.

$\therefore f(1)$ is true.

Inductive Step:

Assume that $f(n)$ is true when $n = k$, for some $k \in \mathbb{Z}^+$, i.e. $f(k) = 1 * 2^1 + 2 * 2^2 + 3 * 2^3 + \dots + k * 2^k = (k - 1)2^{k+1} + 2$.

Consider the case $n = k + 1$, L.H.S. $= 1 * 2^1 + 2 * 2^2 + 3 * 2^3 + \dots + k * 2^k + (k + 1) * 2^{k+1}$

$$= (k - 1)2^{k+1} + 2 + (k + 1) * 2^{k+1} \quad (\text{by induction hypothesis})$$

$$= 2^{k+1} * (2k) + 2$$

$$= (k)2^{k+2} + 2$$

$$= (k + 1 - 1)2^{k+1+1} + 2$$

$$= \text{R.H.S.}$$

$\therefore f(n)$ is true when $n = k + 1$.

\therefore By the principle of mathematical induction, $f(n)$ is true for all $n \in \mathbb{Z}^+$.

- (b) Let $f(n)$ be the predicate " $\frac{1}{\sqrt{1+\sqrt{2}}} + \frac{1}{\sqrt{2+\sqrt{3}}} + \frac{1}{\sqrt{3+\sqrt{4}}} + \dots + \frac{1}{\sqrt{n+\sqrt{n+1}}} = \sqrt{n+1} - 1$ for $\forall n \in \mathbb{Z}^+$ ".

$$\text{For } n = 1, \text{L.H.S.} = \frac{1}{\sqrt{1+\sqrt{2}}}$$

$$= \frac{1}{\sqrt{1+\sqrt{2}}} * \frac{\sqrt{1}-\sqrt{2}}{\sqrt{1}-\sqrt{2}}$$

$$= \frac{\sqrt{1}-\sqrt{2}}{1-2}$$

$$= \sqrt{2} - 1$$

$$\text{R.H.S.} = \sqrt{1+1} - 1$$

$$= \sqrt{2} - 1$$

\therefore L.H.S. $=$ R.H.S.

$\therefore f(1)$ is true.

Inductive Step:

Assume that $f(n)$ is true when $n = k$, for some $k \in \mathbb{Z}^+$, i.e. $f(k) = \frac{1}{\sqrt{1+\sqrt{2}}} + \frac{1}{\sqrt{2+\sqrt{3}}} + \frac{1}{\sqrt{3+\sqrt{4}}} + \dots + \frac{1}{\sqrt{k+\sqrt{k+1}}} = \sqrt{k+1} - 1$.

Consider the case $n = k + 1$,

$$\text{L.H.S.} = \frac{1}{\sqrt{1+\sqrt{2}}} + \frac{1}{\sqrt{2+\sqrt{3}}} + \frac{1}{\sqrt{3+\sqrt{4}}} + \dots + \frac{1}{\sqrt{k+\sqrt{k+1}}} + \frac{1}{\sqrt{k+1+\sqrt{k+2}}}$$

$$= \sqrt{k+1} - 1 + \frac{1}{\sqrt{k+1+\sqrt{k+2}}} \quad (\text{by induction hypothesis})$$

$$= \sqrt{k+1} - 1 + \frac{1}{\sqrt{k+1+\sqrt{k+2}}} * \frac{\sqrt{k+1}-\sqrt{k+2}}{\sqrt{k+1}-\sqrt{k+2}}$$

$$= \sqrt{k+1} - 1 + \frac{\sqrt{k+1}-\sqrt{k+2}}{1-2}$$

$$= \sqrt{k+1} - 1 + \sqrt{k+2} - \sqrt{k+1}$$

$$= \sqrt{k+2} - 1$$

$$= \sqrt{k+1+1} - 1$$

$$= \text{R.H.S.}$$

$\therefore f(n)$ is true when $n = k + 1$.

\therefore By the principle of mathematical induction, $f(n)$ is true for all $n \in \mathbb{Z}^+$.

4 Algorithm Design