COMP2119 Introduction to Data Structures and Algorithms Assignment 1 - Recursion, Mathematical Induction and Algorithm Analysis

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1 Asymptotic Bounds

$n^{\pi}, \pi^{n}, n^{n}, \log n, \pi^{\log n}, n^{\log \pi}, \frac{n}{\log \pi}, \frac{n}{\log n}, \frac{n}{\log \log n}, \log \frac{n}{\log n}, \pi^{\log(2n)}, n^{\log 2\pi}, \sqrt{\sum_{i=1}^{n}(i+1)}, 1910n! + 316n^{n}$

2 Recurrence Relations

(a)
$$\begin{aligned} T(n) &= T(n-1) + 3 \quad \text{for } n > 0 \\ &= T(n-2) + 3 + 3 \quad \text{for } n > 1 \\ &= \dots \\ \therefore T(n) &= T(0) + 3n \\ &= 3n \\ \therefore T(n) &= \Theta(n) \end{aligned}$$

(b) Assume that n is a power of 3, i.e. $n = 3^k$ for $k \in \mathbb{N}$, and $\log_3 n = k$,

$$\begin{aligned} \because T(n) &= 3T(\frac{n}{3}) + n \quad \text{for } n \neq 1 \\ &= 3(3T(\frac{n}{9}) + \frac{n}{3}) + n \quad \text{for } n >= 9 \\ &= 9T(\frac{n}{9}) + n + n \quad \text{for } n >= 9 \\ &= ... \\ \therefore T(n) &= 3^k T(\frac{n}{3^k}) + k * n \\ &= 3^k T(1) + k * n \\ &= 0 + k * n \\ &= kn \\ &= n * \log_3 n \\ \therefore T(n) &= \Theta(n \log n) \end{aligned}$$

(c) Assume that n is a power of 3, i.e. $n = 3^k$ for $k \in \mathbb{N}$, and $\log_3 n = k$,

$$\begin{aligned}
\because T(n) &= 4T(\frac{n}{3}) + 1 \quad \text{for } n >= 3 \\
&= 4(4T(\frac{n}{9}) + 1) + 1 \quad \text{for } n >= 9 \\
&= 16T(\frac{n}{9}) + 4^{k-1} + ... + 4^0 \quad \text{for } n >= 9 \\
&= ... \\
\therefore T(n) &= 4^k T(\frac{n}{3^k}) + \frac{1 * (1 - 4^k)}{1 - 4} \\
&= 4^k T(1) + \frac{4^k - 1}{3} \\
&= 0 + \frac{4^{log_3 n} - 1}{3} \\
&= \frac{1}{3} * 4^{log_3 n} - \frac{1}{3} \\
&= \frac{1}{3} * 3^{(log_3 4)(log_3 n)} - \frac{1}{3} \\
&= \frac{1}{3} * n^{log_3 4} - \frac{1}{3} \\
\therefore T(n) &= \Theta(n^{log_3 4})
\end{aligned}$$

(d) Assume that n is a power of 2, i.e. $n = 2^k$ for $k \in \mathbb{N}$, and $\log_2 n = k$,

$$\begin{aligned}
\because T(n) &= nT(\frac{n}{2}) + n - 1 \quad \text{for } n >= 2 \\
&= n * (\frac{n}{2} * T(\frac{n}{4}) + \frac{n}{2} - 1) + n - 1 \quad \text{for } n >= 4 \\
&= \frac{n^2}{2} * T(\frac{n}{4}) + \frac{n^2}{2} - n + n - 1 \quad \text{for } n >= 4 \\
&= \frac{n^2}{2} * T(\frac{n}{4}) + \frac{n^2}{2} - 1 \quad \text{for } n >= 4 \\
&= \frac{n^2}{2} * (\frac{n}{4} * T(\frac{n}{8}) + \frac{n}{4} - 1) + \frac{n^2}{2} - 1 \quad \text{for } n >= 8 \\
&= \frac{n^3}{2 * 4} * T(\frac{n}{8}) + \frac{n^3}{2 * 4} - \frac{n^2}{2} + \frac{n^2}{2} - 1 \quad \text{for } n >= 8 \\
&= \frac{n^3}{2^1 * 2^2} * T(\frac{n}{2^1 * 2^2}) + \frac{n^3}{2^1 * 2^2} - 1 \quad \text{for } n >= 8 \\
&= \dots \\
\therefore T(n) &= \frac{n^k}{2^0 * 2^1 * 2^2 * ... * 2^{k-1}} * T(1) + \frac{n^k}{2^0 * 2^1 * 2^2 * ... * 2^{k-1}} - 1 \\
&= \frac{n^k}{2^{\frac{(k-1)*k}{2}}} * 2 - 1 \\
&= \frac{n^k}{n^{\frac{k-1}{2}}} * 2 - 1 \\
&= n^{\frac{2k}{2} - \frac{k-1}{2}} * 2 - 1 \\
&= n^{\frac{k+1}{2}} * 2 - 1 \\
&= 2n^{\frac{log_2 n + 1}{2}} - 1 \\
\therefore T(n) &= \Theta(n^{\frac{log_2 n + 1}{2}})
\end{aligned}$$

3 Mathematical Induction

(a) Let $f(n)$ be the predicate "$1 * 2^1 + 2 * 2^2 + 3 * 2^3 + ... + n * 2^n = (n-1)2^{n+1} + 2$" for $\forall n \in \mathbb{Z}^+$.

$$\begin{aligned}
\text{For } n = 1, \text{L.H.S.} &= 1 * 2^1 \\
&= 2 \\
\text{R.H.S.} &= (1-1)2^{1+1} + 2 \\
&= 2
\end{aligned}$$

$\because$ L.H.S. = R.H.S.

$\therefore f(1)$ is true.

Inductive Step:
Assume that $f(n)$ is true when $n = k$, for some $k \in \mathbb{Z}^+$, i.e. $f(k) = 1*2^1+2*2^2+3*2^3+...+k*2^k = (k-1)2^{k+1}+2$.

$$\begin{aligned}
\text{Consider the case } n = k+1,\ \text{L.H.S.} &= 1*2^1+2*2^2+3*2^3+...+k*2^k+(k+1)*2^{k+1} \\
&= (k-1)2^{k+1}+2+(k+1)*2^{k+1} \quad \text{(by induction hypothesis)} \\
&= 2^{k+1}*(2k)+2 \\
&= (k)2^{k+2}+2 \\
&= (k+1-1)2^{k+1+1}+2 \\
&= \text{R.H.S.}
\end{aligned}$$

$\because f(n)$ is true when $n = k+1$.

$\therefore$ By the principle of mathematical induction, $f(n)$ is true for all $n \in \mathbb{Z}^+$.

(b) Let $f(n)$ be the predicate " $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+...+\frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1}-1$ for $\forall n \in \mathbb{Z}^+$ ".

$$\begin{aligned}
\text{For } n = 1, \text{L.H.S.} &= \frac{1}{\sqrt{1}+\sqrt{2}} \\
&= \frac{1}{\sqrt{1}+\sqrt{2}} * \frac{\sqrt{1}-\sqrt{2}}{\sqrt{1}-\sqrt{2}} \\
&= \frac{\sqrt{1}-\sqrt{2}}{1-2} \\
&= \sqrt{2}-1 \\
\text{R.H.S.} &= \sqrt{1+1}-1 \\
&= \sqrt{2}-1
\end{aligned}$$

$\because$ L.H.S. = R.H.S.

$\therefore f(1)$ is true.

Inductive Step:
Assume that $f(n)$ is true when $n = k$, for some $k \in \mathbb{Z}^+$, i.e. $f(k) = \frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+...+\frac{1}{\sqrt{k}+\sqrt{k+1}} = \sqrt{k+1}-1$.

Consider the case $n = k+1$,

$$\begin{aligned}
\text{L.H.S.} &= \frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+...+\frac{1}{\sqrt{k}+\sqrt{k+1}}+\frac{1}{\sqrt{k+1}+\sqrt{k+2}} \\
&= \sqrt{k+1}-1+\frac{1}{\sqrt{k+1}+\sqrt{k+2}} \quad \text{(by induction hypothesis)} \\
&= \sqrt{k+1}-1+\frac{1}{\sqrt{k+1}+\sqrt{k+2}} * \frac{\sqrt{k+1}-\sqrt{k+2}}{\sqrt{k+1}-\sqrt{k+2}} \\
&= \sqrt{k+1}-1+\frac{\sqrt{k+1}-\sqrt{k+2}}{1-2} \\
&= \sqrt{k+1}-1+\sqrt{k+2}-\sqrt{k+1} \\
&= \sqrt{k+2}-1 \\
&= \sqrt{k+1+1}-1 \\
&= \text{R.H.S.}
\end{aligned}$$

$\because f(n)$ is true when $n = k + 1$.

$\therefore$ By the principle of mathematical induction, $f(n)$ is true for all $n \in \mathbb{Z}^+$.

4 Algorithm Design