☆ Ex 1.1. Uniform distribution. $X \sim U(a,b)$

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

$$F(x) = \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases}$$

☆ properey 1.3.

$X \sim U(0,1)$.

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}.$$

c.d.f.

$$\begin{aligned} P(Y \le y) &= P(F^{-1}(X) \le y) \\ &= P(F(F^{-1}(X)) \le F(y)) \\ &= P(X \le F(y)) = F_X(F(y)) \\ &= F(y). \end{aligned}$$

~~$\Rightarrow Y \sim U(0,1)$~~

☆ Ex 1.3.

$$F_{1000}(x) = \frac{1}{1000} \sum_{k=1}^{1000} I(X_k \le x)$$

$$= \begin{cases} \frac{1}{1000} \times 0 = 0 & x \in (-\infty, 0) \\ \frac{1}{1000} \times 65 = 0.065 & x \in [0,1) \\ \frac{1}{1000} \times (65+246) = 0.311 & x \in [1,2) \\ \frac{1}{1000} \times (65+246+358) = 0.669 & x \in [2,3) \\ \frac{1}{1000} \times (65+246+358+272) = 0.941 & x \in [3,4) \\ \frac{1}{1000} \times (65+246+358+272+59) = 1 & x \in [4,\infty) \\ \frac{1}{1000} \times (65+246+358+272+59 & \end{cases}$$

Binomial$(4, \frac{1}{2})$. $n=4$, $p=\frac{1}{2}$

P.m.f. $f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$

$$f(x) = \begin{cases} 1/16 = 0.0625 & x=0 \\ 4/16 = 0.25 & x=1 \\ 6/16 = 0.375 & x=2 \\ 4/16 = 0.25 & x=3 \\ 1/16 = 0.0625 & x=4 \end{cases}$$

$$F(x) = \begin{cases} 0 & (-\infty, 0) \\ 0.0625 & [0,1) \\ 0.3125 & [1,2) \\ 0.6875 & [2,3) \\ 0.9375 & [3,4) \\ 1 & [4,\infty) \end{cases}$$

$$f_{1000}(x) = \frac{1}{1000} \sum_{k=1}^{1000} I(X_k = x)$$

☆ Properey 1.7.

(i) $E(ax+b) = \sum_{x\in S}(ax+b)f(x) = a\sum_{x\in S} x f(x) + b$

$= aE(X)+b.$

(ii) If $P(X=b)=1$, then

$E(X) = \sum_{x\in S} x\cdot f(x) = b\cdot 1 = b$

(iii) $P(a<X\le b)=1.$

$a \le E(X) = \sum_{x\in S} x\cdot f(x) \cancel{\le \sum_{x\in S} b\, P(X\le b)}$

$\le \sum_{x\in S} b\cdot f(x) = b$

(iv) $E[g(x)+h(x)] = \sum_{x\in S}(g(x)+h(x))f(x) = E[g(x)] + E[h(x)]$

☆ Properey 1.8. If $X\ge 0$ takes integer values,

then $E(X) = \sum_{n\ge 1}^{\infty} P(X\ge n) = \sum_{n\ge 0}^{\infty} P(X>n)$.

proof. ① If $P(X=+\infty)>0$, then $E(X)=+\infty$

$P(X>n) \ge P(X=+\infty)>0.$

So $\sum_{n\ge 0}^{\infty} P(X>n) \to +\infty$. Equality Holds!

② If $P(X=+\infty)=0$, then

$\sum_{n=0}^{\infty} P(X>n) = \sum_{n\ge 0}^{\infty}\sum_{j=n+1}^{\infty} P(X=j)$

$= \sum_{j=1}^{\infty}\sum_{n=0}^{j-1} P(X=j) = \sum_{j=1}^{\infty} j\,P(X=j) = E(X)$

☆ properey 1.9

(iii) For $a>0$, $P(h(X)\ge a) \le E\left(\frac{h(X)}{a}\right)$

proof. $E\left(\frac{h(X)}{a}\right) = \frac{1}{a}E(h(x))$

$= \frac{1}{a}\left\{\int_{x\in\mathbb{R}} h(x)f(x)\,dx\right\}$

$\ge \frac{1}{a}\left\{\int_{h(x)\ge a} a\cdot f(x)\,dx\right\}$

$= \int_{h(x)\ge a} f(x)\,dx$

$= P(h(x)\ge a)$

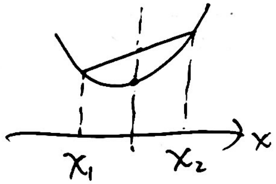
☆ Properey 1.9.

(iv) If g is convex, then $g(E(X)) \leq E(g(X))$

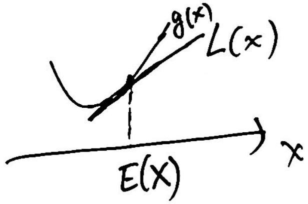
proof: △ convex function

$$g(tx_1 + (1-t)x_2) \leq t\,g(x_1) + (1-t)\,g(x_2)$$

for $(0 \leq t \leq 1)$.



△ Define $L(x) = a + bx$. is the linear function tangential to $g(x)$ at the point $E(X)$.



So $g(x) \geq L(x)$ for all x.

So $E[g(X)] \geq E[L(X)] = E(a+bX)$

$= a + bE(X) = L(E(X))$

$= g(E(X))$

☆ Properey. 1.10. X non-negative

$$E(X) = \int_0^\infty x \cdot f(x)\,dx \overset{?}{=} \int_0^\infty (1-F(x)) \cdot dx$$

proof. $1 - F(x) = 1 - P(X \leq x) = P(X > x)$

$$= \int_x^\infty f(t)\,dt$$

$$\int_0^\infty (1-F(x))\,dx = \int_0^\infty \int_x^\infty f(t)\,dt\,dx$$

$t \geq x$

$$= \int_0^\infty \left(\int_0^t 1\,dx \right) f(t)\,dt$$

$$= \int_0^\infty t\,f(t)\,dt = E(X).$$

☆ Property 1.11.

(i) ① If X is discrete,

$E(X^2)=0 \overset{?}{\Rightarrow} P(X=0)=1$.

If $P(X=0)<1$, then $P(X\neq 0)>0$.

$E(X^2)=\sum_{x\neq 0} x^2 P(X=x^2)>0$. contradicts!

② If X is continuous.

$0=E(X^2)=\int_{-\infty}^{\infty} x^2 f(x)\,dx$

$\geq \int_{\varepsilon}^{\infty} x^2 f(x)\,dx+\int_{-\infty}^{-\varepsilon} x^2 f(x)\,dx \quad (\varepsilon>0)$

$\geq \varepsilon^2\left[\int_{\varepsilon}^{\infty} f(x)\,dx+\int_{-\infty}^{-\varepsilon} f(x)\,dx\right]$

$=\varepsilon^2\left[1-F(\varepsilon)+F(-\varepsilon)\right]$

$=\varepsilon^2 P(X\geq\varepsilon \text{ or } X\leq-\varepsilon)\geq 0$

So for any $\varepsilon>0$, $P(|X|\geq\varepsilon)=0$.

$\Rightarrow P(X=0)=1$

(ii) $Cov(aX+b, cY+d) \overset{?}{=} ac\, Cov(X,Y)$

(iii) $Var(X+Y)=Var(X)+Var(Y)+2Cov(X,Y)$.

(iv) $-1\leq\rho(X,Y)\leq 1$.

proof. $\rho^2(X,Y)=\dfrac{Cov^2(X,Y)}{Var(X)\ Var(Y)}$

$Cov^2(X,Y)=\{E[(X-E(X))(Y-E(Y))]\}^2$

$\leq E[X-E(X)]^2\ E[Y-E(Y)]^2$ (property 1.12)

$=Var(X)\cdot Var(Y)$

(vi) $|\rho(X,Y)|=1 \Leftrightarrow P(X=aY+b)=1$.

proof. $|\rho(X,Y)|=1 \Leftrightarrow$ property 1.12.

let $X'\equiv X-E(X)$ $\quad Y'\equiv Y-E(Y)$

$P\left(X'=Y'\dfrac{E(X'Y')}{E(Y'^2)}\right)=1$

$\Leftrightarrow P(X=aY+b)=1$.

★ Properey 1.12. $E(XY) \le \sqrt{E(X^2)\,E(Y^2)}$.

proof. $0 \le E[X \cdot E(Y^2) - Y E(XY)]^2$

$$= E\{X^2 [E(Y^2)]^2 + Y^2 E^2(XY) - 2XY E(Y^2) E(XY)\}$$

$$= E(Y^2)\{E(X^2) \cdot E(Y^2) + [E(XY)]^2 - 2[E(XY)]^2\}$$

$$= E(Y^2) \cdot \{E(X^2)E(Y^2) - [E(XY)]^2\}$$

$$\Rightarrow E(X^2)E(Y^2) \geq [E(XY)]^2.$$

"=" Holds $\Leftrightarrow P(X E(Y^2) - Y E(XY) = 0) = 1$

$$\Leftrightarrow P\left(X - Y \cdot \frac{E(XY)}{E(Y^2)} = 0\right) = 1.$$

★ Properey 1.11

(vii) $\rho(aX+b, cY+d) = \operatorname{sgn}(ac)\,\rho(X, Y)$

proof. $\rho(aX+b, cY+d)$

$$= \frac{Cov(aX+b, cY+d)}{\sqrt{Var(aX+b)\ Var(cY+d)}}$$

$$= \frac{ac\ Cov(X, Y)}{\sqrt{Var(X)\ Var(Y)} \cdot |ac|}$$

$$= \operatorname{sgn}(ac)\,\rho(X, Y)$$

(viii) If X & Y are independent, then $Cov(X, Y) = 0$, then $\rho(X, Y) = 0$.