

STAT2602 Assignment 2

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1. (i) The probability density function (p.d.f.) for the uniform distribution $U[-\alpha, 0]$ is:

$$f(x; \alpha) = \frac{1}{\alpha} \mathbb{I}_{[-\alpha, 0]}(x),$$

where $\mathbb{I}_{[-\alpha, 0]}(x)$ is the indicator function, which is defined as:

$$\mathbb{I}_{[-\alpha, 0]}(x) = \begin{cases} 1 & \text{if } x \in [-\alpha, 0], \\ 0 & \text{otherwise.} \end{cases}$$

The likelihood function for the sample X_1, X_2, \dots, X_n is:

$$L(\alpha) = \prod_{i=1}^n \frac{1}{\alpha} \mathbb{I}_{[-\alpha, 0]}(X_i).$$

This product implies that the likelihood is zero if any X_i lies outside the interval $[-\alpha, 0]$. Therefore, for the likelihood to be non-zero, all X_i must lie in $[-\alpha, 0]$, i.e., $-\alpha \leq X_i \leq 0$ for all i . Hence, the likelihood function becomes:

$$L(\alpha) = \frac{1}{\alpha^n} \mathbb{I}_{[-\alpha, 0]}(\min(X_1, \dots, X_n)) \mathbb{I}_{[-\alpha, 0]}(\max(X_1, \dots, X_n)).$$

The log-likelihood function is:

$$\ell(\alpha) = -n \log(\alpha) + \log(\mathbb{I}_{[-\alpha, 0]}(\min(X_1, \dots, X_n))) + \log(\mathbb{I}_{[-\alpha, 0]}(\max(X_1, \dots, X_n))).$$

From the log-likelihood function, we can observe that it becomes larger when α is smaller.

Also, the indicator function must not return zero, as $\log(0)$ is undefined.

Therefore, $\min(X_1, \dots, X_n) \geq -\alpha$ and $\max(X_1, \dots, X_n) \leq 0$ must be satisfied when we maximizing the likelihood by finding the smallest value of α .

As a result:

$$\min(X_1, \dots, X_n) \geq -\alpha \Rightarrow \alpha \geq -\min(X_1, \dots, X_n).$$

Therefore, the MLE of α is:

$$\hat{\alpha} = -\min(X_1, \dots, X_n).$$

- (ii) The likelihood function is:

$$L(\alpha) = \frac{1}{\alpha^n} \mathbb{I}_{[-\alpha, 0]}(\min(X_1, \dots, X_n)) \mathbb{I}_{(-\alpha, 0]}(\max(X_1, \dots, X_n)).$$

By the **factorization theorem**, a sufficient statistic for α can be found by factorizing the likelihood function into two parts: one that depends on α and another that does not depend on α . Applying this theorem, we can write the likelihood function as:

$$g(T(X); \alpha) = \frac{1}{\alpha^n} \mathbb{I}_{[-\alpha, 0]}(\min(X_1, \dots, X_n)) \quad h(X_1, \dots, X_n) = \mathbb{I}_{(-\alpha, 0]}(\max(X_1, \dots, X_n))$$

Therefore, the likelihood depends on α only through $\min(X_1, \dots, X_n)$, meaning that $\min(X_1, \dots, X_n)$ is a sufficient statistic for α .

2. The MLE of θ is the value of θ that maximizes $f(x; \theta)$ for the observed x . For a given observation $x = x_{\text{obs}}$,

- If $x_{\text{obs}} = 0$ or $x_{\text{obs}} = 1$, then the MLE is $\hat{\theta} = 1$ because $\frac{1}{3} > \frac{1}{4}$ and $f(x; 3) = 0$.
- If $x_{\text{obs}} = 2$, then $\hat{\theta} = 2$ or $\hat{\theta} = 3$, because both give $f(x; \theta) = \frac{1}{4}$.
- If $x_{\text{obs}} = 3$, then $\hat{\theta} = 3$ because $\frac{1}{2}$ is the largest probability.
- If $x_{\text{obs}} = 4$, then $\hat{\theta} = 3$ because $\frac{1}{4}$ is the largest probability.

3. (i) The likelihood function is:

$$L(\theta) = \prod_{i=1}^n \frac{\theta}{X_i^2} \mathbb{I}(0 < \theta \leq X_i < \infty)$$

The log-likelihood function is:

$$\begin{aligned} \ell(\theta) &= \log\left(\frac{\theta^n \mathbb{I}(0 < \theta \leq X_1, \dots, X_n < \infty)}{\prod_{i=1}^n X_i^2}\right) \\ &= n \log(\theta) + \log(\mathbb{I}(0 < \theta \leq \min(X_1, \dots, X_n) < \infty)) - \log\left(\prod_{i=1}^n X_i^2\right) \\ &= n \log(\theta) + \log(\mathbb{I}(0 < \theta \leq \min(X_1, \dots, X_n) < \infty)) - 2 \sum_{i=1}^n \log(X_i) \end{aligned}$$

Taking the gradient w.r.t. θ :

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} > 0 \quad (\text{for } 0 < \theta \leq \min(X_1, \dots, X_n) < \infty)$$

Thus, the likelihood is increasing function w.r.t. θ .

The MLE is the maximum value of θ that satisfies the constraint $0 < \theta \leq \min(X_1, \dots, X_n) < \infty$, which is $\hat{\theta} = \min(X_1, \dots, X_n)$.

(ii) The expectation of $X_1^{1/3}$ is:

$$E(X_1^{1/3}) = \int_{\theta}^{\infty} x^{1/3} \frac{\theta}{x^2} dx = \int_{\theta}^{\infty} \theta x^{-5/3} dx = \theta \left[-\frac{3}{2} x^{-2/3} \right]_{\theta}^{\infty} = \theta(0 - (-\frac{3}{2} \theta^{-2/3})) = \frac{3}{2} \theta^{1/3}$$

(iii) Since the expectation of X :

$$E(X) = \int_{\theta}^{\infty} x \frac{\theta}{x^2} dx = \int_{\theta}^{\infty} \theta x^{-1} dx = \theta [-\log(x)]_{\theta}^{\infty} = \theta(-\log(\infty) - (-\log(\theta)))$$

diverges, we need to use $E(X_1^{1/3})$ for methods of moment estimator (MME).

From (ii), we have $E(X_1^{1/3}) = \frac{3}{2} \theta^{1/3}$. By equating $E(X_1^{1/3})$ to the 1/3-th sample moment of X $m_{1/3} = \frac{1}{n} \sum_{i=1}^n X_i^{1/3}$, we get:

$$\frac{3}{2} \theta^{1/3} = m_{1/3} \quad \Rightarrow \quad \left(\frac{3}{2}\right)^3 \theta = m_{1/3}^3 \quad \Rightarrow \quad \theta = \left(\frac{2}{3}\right)^3 m_{1/3}^3 = \hat{\theta}_{MME}$$

As $n \rightarrow \infty$, the 1/3-th sample moment of X converges to $E(X_1^{1/3}) = \frac{3}{2} \theta^{1/3}$. Therefore,

$$\hat{\theta}_{MME} = \left(\frac{2}{3}\right)^3 m_{1/3}^3 \rightarrow \left(\frac{2}{3}\right)^3 E(X_1^{1/3})^3 = \frac{2^3}{3} \left(\frac{3}{2} \theta^{1/3}\right)^3 = \theta$$

Hence, $\hat{\theta}_{MME} \rightarrow_p \theta$.

\therefore The MME is consistent.

4. (i) The likelihood function is:

$$L(p) = \prod_{i=1}^n p(1-p)^{X_i} = p^n (1-p)^{\sum_{i=1}^n X_i}.$$

By the factorization theorem, $T = \sum_{i=1}^n X_i$ is a sufficient statistic for p .
The p.d.f. can be rewritten as:

$$f(x; p) = \exp(x \ln(1-p) + \ln(p)).$$

Since the geometric distribution belongs to the exponential family and the parameter space $0 < p < 1$ is large enough, $T = \sum_{i=1}^n X_i$ is also complete for p .

Therefore, the statistic $T = \sum_{i=1}^n X_i$ is both sufficient and complete for p .

- (ii) Given from (i), $T = \sum_{i=1}^n X_i$ is a complete and sufficient statistic for p .

$$\begin{aligned} E(X_1) &= \sum_{x=0}^{\infty} x \cdot p(1-p)^x = p \sum_{x=0}^{\infty} x(1-p)^x \\ &= p \sum_{x=0}^{\infty} (x+1-1)(1-p)^x = p \left(\sum_{x=0}^{\infty} (x+1)(1-p)^x - \sum_{x=0}^{\infty} (1-p)^x \right) \\ &= p \left(\sum_{x=1}^{\infty} x(1-p)^{x-1} - \sum_{x=0}^{\infty} (1-p)^x \right) \end{aligned}$$

Since $\sum_{x=1}^{\infty} xp(1-p)^{x-1}$ is the expectation of a geometric distribution which is $\frac{1}{p}$,

$$= p \left(\frac{1}{p} - \frac{1}{1-(1-p)} \right) = \frac{1}{p} - 1 = \frac{1-p}{p}$$

As X_1, X_2, \dots, X_n are i.i.d., the expectation of T is:

$$\begin{aligned} \therefore E(T) &= E\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n E(X_i) = n \cdot E(X_1) = n \cdot \frac{1-p}{p} \\ \therefore E(T) &= n \cdot \left(\frac{1}{p} - 1\right) \Rightarrow \frac{E(T)}{n} + 1 = \frac{1}{p} \Rightarrow \frac{E(T) + n}{n} = \frac{1}{p} \Rightarrow \frac{n}{E(T) + n} = p \end{aligned}$$

$$\therefore E\left(\frac{n}{T+n}\right) = \frac{E(n)}{E(T)+E(n)} = \frac{n}{E(T)+n} = p,$$

By Theorem 3.2 in the lecture notes, $\frac{n}{T+n}$ is the UMVUE of p .

5. (i) Since $X_i \sim N(\frac{p}{q}, \sigma_1^2)$ and $Y_i \sim N(q, \sigma_2^2)$, the expectation of T_1 is:

$$E(T_1) = \frac{1}{n} \sum_{i=1}^n E(X_i Y_i) = \frac{1}{n} \sum_{i=1}^n \frac{p}{q} q = p$$

Therefore, T_1 is an unbiased estimator of p .

- (ii) Since X and Y are independent,

$$\begin{aligned} Var(T_1) &= \frac{1}{n^2} \sum_{i=1}^n Var(X_i Y_i) = \frac{1}{n^2} \sum_{i=1}^n (E(X_i^2) E(Y_i^2) - p^2) = \frac{1}{n} \left(\sigma_1^2 + \frac{p^2}{q^2} \right) (\sigma_2^2 + q^2) - \frac{p^2}{n} \\ &= \frac{1}{n} \left(\sigma_1^2 \sigma_2^2 + \sigma_1^2 q^2 + \frac{p^2 \sigma_2^2}{q^2} \right) \end{aligned}$$

- (iii) Since X and Y are independent,

$$E(T_2) = E\left(\left(\frac{1}{n} \sum_{i=1}^n X_i\right) \left(\frac{1}{n} \sum_{i=1}^n Y_i\right)\right) = \left(\frac{1}{n} \sum_{i=1}^n E(X_i)\right) \left(\frac{1}{n} \sum_{i=1}^n E(Y_i)\right) = \frac{p}{q} \cdot q = p$$

Hence, T_2 is also an unbiased estimator of p .

(iv) By the weak law of large numbers,

$$E(T_2) = E\left(\left(\frac{1}{n} \sum_{i=1}^n X_i\right) \left(\frac{1}{n} \sum_{i=1}^n Y_i\right)\right) = E(\bar{X})E(\bar{Y}) \\ \rightarrow_p \frac{p}{q} \cdot q = p \text{ as } n \rightarrow \infty$$

$\therefore T_2$ is a consistent estimator of p .

(v) When $p = 0$ and $q^2 = \frac{\sigma_2^2}{n}$,

$$Var(T_1) = \frac{1}{n} \left(\sigma_1^2 \sigma_2^2 + \sigma_1^2 q^2 + \frac{p^2 \sigma_2^2}{q^2} \right) = \frac{1}{n} \left(\sigma_1^2 \sigma_2^2 + \sigma_1^2 \frac{\sigma_2^2}{n} \right) = \frac{\sigma_1^2 \sigma_2^2}{n} + \frac{\sigma_1^2 \sigma_2^2}{n^2} \\ Var(T_2) = Var(\bar{X}\bar{Y}) = E(\bar{X}^2)E(\bar{Y}^2) - 0 = (Var(\bar{X}) + E(\bar{X})^2)(Var(\bar{Y}) + E(\bar{Y})^2) \\ \rightarrow_p (\sigma_1^2)(\sigma_2^2 + \frac{\sigma_2^2}{n}) = \sigma_1^2 \sigma_2^2 + \frac{\sigma_1^2 \sigma_2^2}{n}$$

When $n = 1$, T_1 and T_2 has the same efficiency.

When $n > 1$, T_1 is more efficient than T_2 .

6. (i) The expectation of \bar{X} and $\frac{n}{n-1}S^2$ are:

$$E(\bar{X}) = E\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n} \sum_{i=1}^n E(X_i) = \frac{1}{n} \cdot n\lambda = \lambda \\ E\left(\frac{n}{n-1}S^2\right) = \frac{n}{n-1}E(S^2) = \frac{n}{n-1}E\left(\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2\right) = \frac{n}{n-1} \frac{1}{n} \sum_{i=1}^n E((X_i - \bar{X})^2) \\ = \frac{1}{n-1} \sum_{i=1}^n Var(X_i - \bar{X}) = \frac{1}{n-1} \sum_{i=1}^n Var\left(X_i - \frac{X_1 + \dots + X_n}{n}\right) \\ = \frac{1}{n-1} \sum_{i=1}^n Var\left(\frac{n-1}{n}X_i - \sum_{j \neq i} \frac{1}{n}X_j\right) \\ = \frac{1}{n-1} \sum_{i=1}^n \left(\left(\frac{n-1}{n}\right)^2 Var(X_i) + \left(\frac{1}{n}\right)^2 \sum_{j \neq i} Var(X_j) \right) \\ = \frac{1}{n-1} \left(\left(\frac{n-1}{n}\right)^2 n\lambda + \left(\frac{1}{n}\right)^2 (n)(n-1)\lambda \right) \\ = \frac{n-1}{n}\lambda + \frac{1}{n}\lambda = \lambda$$

Both are unbiased estimators of λ .

(ii) Since $X \sim \text{Poisson}(\lambda)$ is from an exponential family, and the parameter $\lambda > 0$ contains an open set in \mathbb{R} ,

the sufficient and complete statistic for λ would be $T = \sum_{i=1}^n X_i$.

(iii) The second derivative of the log-p.m.f. is:

$$f(x; \lambda) = \frac{e^{-\lambda} \lambda^x}{x!} \\ \log(f(x; \lambda)) = -\lambda + x \log(\lambda) - \log(x!) \\ \frac{\partial \log(f(x; \lambda))}{\partial \lambda} = -1 + \frac{x}{\lambda} \\ \frac{\partial^2 \log(f(x; \lambda))}{\partial \lambda^2} = -\frac{x}{\lambda^2}$$

Therefore, the Fisher information about λ contained in data X_1, \dots, X_n is:

$$I_n(\lambda) = nI(\lambda) = -nE\left(\frac{\partial^2 \log(f(X; \lambda))}{\partial \lambda^2}\right) = n \frac{E(X)}{\lambda^2} = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$$

(iv) Using the Cramer-Rao Lower Bound (CRLB),

$$\text{Var}(\hat{\lambda}) \geq \frac{1}{I_n(\lambda)} = \frac{\lambda}{n}$$

(v) The variance of \bar{X} and $\frac{n}{n-1}S^2$ are:

$$\text{Var}(\bar{X}) = \frac{\lambda}{n} \quad (\text{since } \bar{X} \text{ is the UMVUE of } \lambda)$$

$$\text{Var}\left(\frac{n}{n-1}S^2\right) = \left(\frac{n}{n-1}\right)^2 \left(\frac{\lambda}{n} + \frac{2\lambda^2}{n(n-1)}\right)$$

$$\text{eff}\left(\bar{X}, \frac{n}{n-1}S^2\right) = \frac{\text{Var}\left(\frac{n}{n-1}S^2\right)}{\text{Var}(\bar{X})}$$

$$= \frac{\left(\frac{n}{n-1}\right)^2 \left(\frac{\lambda}{n} + \frac{2\lambda^2}{n(n-1)}\right)}{\frac{\lambda}{n}}$$

$$= \left(\frac{n}{n-1}\right)^2 \left(1 + \frac{2\lambda}{n-1}\right)$$

$$= \frac{n^3}{(n-1)^2} \left(1 + \frac{2\lambda}{n-1}\right) > 1$$

$\therefore \bar{X}$ is relatively more efficient than $\frac{n}{n-1}S^2$, having a smaller variance.

$\therefore \bar{X}$ should be preferred.

7. (i) Let $\phi = (\theta, \theta^2)$. Denote the sample mean and sample variance as \bar{X} and S^2 respectively. The joint p.d.f. of $X_1, X_2, \dots, X_n \sim N(\theta, \theta^2)$ is:

$$\begin{aligned} & f(x_1, x_2, \dots, x_n; \phi) \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\theta^2}} \exp\left(-\frac{(x_i - \theta)^2}{2\theta^2}\right) \\ &= \frac{1}{(2\pi\theta^2)^{n/2}} \exp\left(-\frac{1}{2\theta^2} \sum_{i=1}^n (x_i - \theta)^2\right) \\ &= \frac{1}{(2\pi\theta^2)^{n/2}} \exp\left(-\frac{1}{2\theta^2} \left(\sum_{i=1}^n x_i^2 - 2\theta \sum_{i=1}^n x_i + n\theta^2\right)\right) \\ &= \frac{1}{(2\pi\theta^2)^{n/2}} \exp\left(-\frac{1}{2\theta^2} \left(\sum_{i=1}^n x_i^2 - 2\theta n\bar{X} + n\theta^2\right)\right) \\ &= \frac{1}{(2\pi\theta^2)^{n/2}} \exp\left(-\frac{1}{2\theta^2} \left(\sum_{i=1}^n x_i^2 - n\bar{X}^2 + n\bar{X}^2 - 2\theta n\bar{X} + n\theta^2\right)\right) \\ &= \frac{1}{(2\pi\theta^2)^{n/2}} \exp\left(-\frac{1}{2\theta^2} \left(\sum_{i=1}^n x_i^2 - n\bar{X}^2\right)\right) \exp\left(-\frac{1}{2\theta^2} (n\bar{X}^2 + n\theta^2 - 2\theta n\bar{X})\right) \\ &= \frac{1}{(2\pi\theta^2)^{n/2}} \exp\left(-\frac{n}{2\theta^2} \left(\frac{1}{n} \sum_{i=1}^n x_i^2 - \bar{X}^2\right)\right) \exp\left(-\frac{n}{2\theta^2} (\bar{X} - \theta)^2\right) \\ &= \frac{1}{(2\pi\theta^2)^{n/2}} \exp\left(-\frac{n}{2\theta^2} S^2\right) \exp\left(-\frac{n}{2\theta^2} (\bar{X} - \theta)^2\right) \end{aligned}$$

By the factorization theorem, $T = (\bar{X}, S^2)$ is a sufficient statistic for ϕ .

- (ii) Let $\phi = (\theta, \theta^2) = (\theta, \xi)$ where $\xi = \theta^2$. The likelihood function based on the sample X_1, X_2, \dots, X_n is:

$$L(\phi) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\xi}} \exp\left(-\frac{(X_i - \theta)^2}{2\xi}\right)$$

Taking the logarithm of the likelihood:

$$\ell(\phi) = \log L(\phi) = -\frac{n}{2} \log(2\pi\xi) - \frac{1}{2\xi} \sum_{i=1}^n (X_i - \theta)^2$$

Taking the derivative of the log-likelihood with respect to θ :

$$\begin{aligned} \frac{\partial \ell}{\partial \theta} &= \frac{1}{2\xi} \sum_{i=1}^n 2(X_i - \theta) = \frac{1}{\xi} \sum_{i=1}^n (X_i - \theta) \\ &= \frac{1}{\xi} \left(\sum_{i=1}^n X_i - n\theta \right) \end{aligned}$$

Letting it to zero and solve for θ :

$$\begin{aligned} \frac{1}{\xi} \left(\sum_{i=1}^n X_i - n\theta \right) &= 0 \\ \sum_{i=1}^n X_i - n\theta &= 0 \\ \theta &= \frac{1}{n} \sum_{i=1}^n X_i = \bar{X} \end{aligned}$$

Given the distribution is normal, the MLE is indeed maximum.

Thus, the MLE of θ is:

$$\hat{\theta} = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

(iii)

$$\begin{aligned} E(\bar{X}) &= E\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n} \sum_{i=1}^n E(X_i) = \frac{1}{n} \cdot n\theta = \theta \\ Var(\bar{X}) &= Var\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n^2} \sum_{i=1}^n Var(X_i) = \frac{1}{n^2} \cdot n\theta^2 = \frac{\theta^2}{n} \end{aligned}$$

By the Central Limit Theorem, as $n \rightarrow \infty$, $\bar{X} \sim_d N(\theta, \frac{\theta^2}{n})$.

8. (i) As $m \geq 2$, $\mu_2 = s_1$, $X_2 \sim N(s_1, \sigma_2^2)$ and thus $E(X_2) = s_1$.

$$\begin{aligned} \tilde{s}_1 &= \frac{X_1 + 2X_2}{3} \\ E(\tilde{s}_1) &= \frac{E(X_1 + 2X_2)}{3} = \frac{E(X_1) + 2E(X_2)}{3} \\ &= \frac{s_1 + 2s_1}{3} = s_1 \end{aligned}$$

As $E(\tilde{s}_1) = s_1$, \tilde{s}_1 is an unbiased estimator of s_1 .

- (ii) The likelihood function of X_1, X_2, \dots, X_{2m} is:

$$\begin{aligned} L(s_1, s_2) &= \prod_{i=1}^m \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(X_i - s_1)^2}{2\sigma_i^2}\right) \prod_{i=m+1}^{2m} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(X_i - s_2)^2}{2\sigma_i^2}\right) \\ &= \prod_{i=1}^m \frac{1}{\sqrt{2\pi\sigma_i^2}} \prod_{i=1}^m \exp\left(-\frac{(X_i - s_1)^2}{2\sigma_i^2}\right) \prod_{i=m+1}^{2m} \exp\left(-\frac{(X_i - s_2)^2}{2\sigma_i^2}\right) \end{aligned}$$

Taking the logarithm of the likelihood:

$$\begin{aligned}\ell(s_1, s_2) &= \log L(s_1, s_2) \\ &= \sum_{i=1}^n \log \left(\frac{1}{\sqrt{2\pi\sigma_i^2}} \right) - \sum_{i=1}^m \frac{(X_i - s_1)^2}{2\sigma_i^2} - \sum_{i=m+1}^{2m} \frac{(X_i - s_2)^2}{2\sigma_i^2}\end{aligned}$$

Taking the derivative of the log-likelihood with respect to s_1 and s_2 :

$$\frac{\partial \ell}{\partial s_1} = \sum_{i=1}^m \frac{X_i - s_1}{\sigma_i^2} \quad \text{and} \quad \frac{\partial \ell}{\partial s_2} = \sum_{i=m+1}^{2m} \frac{X_i - s_2}{\sigma_i^2}$$

Letting them to zero and solve for s_1 and s_2 :

$$\begin{aligned}\sum_{i=1}^m \frac{X_i - s_1}{\sigma_i^2} &= 0 \Rightarrow \sum_{i=1}^m \frac{X_i}{\sigma_i^2} = \sum_{i=1}^m \frac{s_1}{\sigma_i^2} \Rightarrow s_1 = \frac{\sum_{i=1}^m \frac{X_i}{\sigma_i^2}}{\sum_{i=1}^m \frac{1}{\sigma_i^2}} \\ \sum_{i=m+1}^{2m} \frac{X_i - s_2}{\sigma_i^2} &= 0 \Rightarrow \sum_{i=m+1}^{2m} \frac{X_i}{\sigma_i^2} = \sum_{i=m+1}^{2m} \frac{s_2}{\sigma_i^2} \Rightarrow s_2 = \frac{\sum_{i=m+1}^{2m} \frac{X_i}{\sigma_i^2}}{\sum_{i=m+1}^{2m} \frac{1}{\sigma_i^2}} \\ \therefore \hat{s}_1 &= \frac{\sum_{i=1}^m \frac{X_i}{\sigma_i^2}}{\sum_{i=1}^m \frac{1}{\sigma_i^2}} \quad \text{and} \quad \hat{s}_2 = \frac{\sum_{i=m+1}^{2m} \frac{X_i}{\sigma_i^2}}{\sum_{i=m+1}^{2m} \frac{1}{\sigma_i^2}}\end{aligned}$$

(iii) After setting $\sigma_i^2 = \frac{m}{i}$, the MLEs of s_1 is:

$$\hat{s}_1 = \frac{\sum_{i=1}^m \frac{X_i}{\frac{m}{i}}}{\sum_{i=1}^m \frac{1}{\frac{m}{i}}} = \frac{\sum_{i=1}^m \frac{iX_i}{m}}{\sum_{i=1}^m \frac{i}{m}} = \frac{\sum_{i=1}^m iX_i}{\sum_{i=1}^m i} = \frac{\sum_{i=1}^m iX_i}{\frac{m(m+1)}{2}} = \frac{2}{m(m+1)} \sum_{i=1}^m iX_i$$

Variance of \hat{s}_1 :

$$\begin{aligned}Var(\hat{s}_1) &= Var \left(\frac{2}{m(m+1)} \sum_{i=1}^m iX_i \right) = \frac{4}{m^2(m+1)^2} \sum_{i=1}^m i^2 Var(X_i) = \frac{4}{m^2(m+1)^2} \sum_{i=1}^m i^2 \frac{m}{i} \\ &= \frac{4}{m(m+1)^2} \sum_{i=1}^m i = \frac{4}{m(m+1)^2} \cdot \frac{m(m+1)}{2} = \frac{2}{m+1}\end{aligned}$$

Variance of \tilde{s}_1 :

$$\begin{aligned}Var(\tilde{s}_1) &= Var \left(\frac{X_1 + 2X_2}{3} \right) = \frac{1}{9} Var(X_1 + 2X_2) = \frac{1}{9} (Var(X_1) + 4Var(X_2)) \\ &= \frac{1}{9} (\sigma_1^2 + 4\sigma_2^2) = \frac{1}{9} \left(\frac{m}{1} + 4 \cdot \frac{m}{2} \right) = \frac{1}{9} (m + 2m) = \frac{3m}{9} = \frac{m}{3}\end{aligned}$$

Comparing their variances:

At $m = 2$, $Var(\hat{s}_1) = \frac{2}{3}$ and $Var(\tilde{s}_1) = \frac{2}{3}$.

At $m > 2$, $Var(\hat{s}_1) = \frac{2}{m+1} < \frac{m}{3} = Var(\tilde{s}_1)$.

$\therefore \hat{s}_1$ is more efficient than \tilde{s}_1 for $m > 2$. Otherwise, they are equally efficient.

(iv) For \hat{s}_1 , the asymptotic distribution of \hat{s}_1 is:

$$\begin{aligned}I(\hat{s}_1) &= \frac{1}{m} I_m(s_1) \\ &= \frac{1}{m} E \left[\left(\frac{d\ell(s_1)}{ds_1} \right)^2 \right] = \frac{1}{m} E \left[\left(\sum_{i=1}^m \frac{i}{m(m+1)} (x_i - s_1) \right)^2 \right] \\ &= \frac{1}{m} E \left[\left(\frac{2}{m(m+1)} \sum_{i=1}^m i(x_i - s_1) \right)^2 \right]\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{m^3} \left[\text{Var} \left(\sum_{i=1}^m i(x_i - s_1) \right) + E \left(\sum_{i=1}^m i(x_i - s_1) \right)^2 \right] \\
&= \frac{1}{m^3} \left[\sum_{i=1}^m i^2 \text{Var}(x_i) + \sum_{i=1}^m i^2 E(x_i) E(x_i - s_1)^2 \right] \\
&= \frac{1}{m^3} \left[\sum_{i=1}^m i^2 \frac{m}{i} + \sum_{i=1}^m i^2 E(x_i) \right] \\
&= \frac{\sum_{i=1}^m i}{m(m+1)} + o = \frac{m+1}{2m}
\end{aligned}$$

Hence,

$$\hat{s}_1 \xrightarrow{d} N \left(s_1, \frac{2}{m+1} \right)$$

Similarly,

$$\begin{aligned}
I_n(\hat{s}_2) &= \frac{1}{m} I_m(s_2) \\
&= \frac{1}{m} E \left[\left(\sum_{i=m+1}^{2m} \frac{i}{m(m+1)} (x_i - s_2) \right)^2 \right] \\
&= \frac{1}{m^3} \sum_{i=m+1}^{2m} i^2 (\text{Var}(x_i) + E(x_i) E(x_i - s_2)^2) \\
&= \frac{1}{m^3} \left[\sum_{i=m+1}^{2m} i^2 \frac{m}{i} \right] = \frac{3m+1}{2m}
\end{aligned}$$

Hence,

$$\hat{s}_2 \xrightarrow{d} N \left(s_2, \frac{3m+1}{2m} \right)$$

(v) As $m \rightarrow \infty$,

$$\begin{aligned}
\lim_{m \rightarrow \infty} [E(\hat{s}_1) - s_1] &= \lim_{m \rightarrow \infty} \left[\frac{2}{m(m+1)} \sum_{i=1}^m i E(X_i) - s_1 \right] = 0 \\
\lim_{m \rightarrow \infty} [E(\hat{s}_2) - s_2] &= \lim_{m \rightarrow \infty} \left[\frac{\sum_{i=m+1}^{2m} \frac{i E(X_i)}{m}}{\sum_{i=m+1}^{2m} \frac{i}{m}} - s_2 \right] = 0
\end{aligned}$$

\therefore The MLE of s_1 and s_2 are (asymptotic) unbiased estimators of s_1 and s_2 respectively.

As $m \rightarrow \infty$, $\text{Var}(\hat{s}_1) = \frac{2}{m+1} \rightarrow 0$ and $\text{Var}(\hat{s}_2) = \frac{m}{3} \rightarrow 0$.

\therefore The MLE of s_1 and s_2 are consistent estimators of s_1 and s_2 respectively.

9. (i) The likelihood function of θ based on X_1, X_2, \dots, X_n is:

$$L(\theta; X_1, \dots, X_n) = \prod_{i=1}^n \frac{\theta}{X_i^{\theta+1}} = \theta^n \prod_{i=1}^n \frac{1}{X_i^{\theta+1}}.$$

(ii) For $x \geq 1$, the joint p.d.f. can be rewritten as:

$$\begin{aligned}
f(x_1, x_2, \dots, x_n; \theta) &= \prod_{i=1}^n \frac{\theta}{x_i^{\theta+1}} = \theta^n \prod_{i=1}^n x_i^{-(\theta+1)} \\
&= \theta^n e^{-(\theta+1) \sum_{i=1}^n \ln(x_i)} \\
&= g(T(x_1, x_2, \dots, x_n), \theta) h(x_1, x_2, \dots, x_n) \text{ where } h(\dots) = 1
\end{aligned}$$

Using the **factorization theorem**, we got a scalar sufficient statistic T:

$$T = \sum_{i=1}^n \ln(X_i).$$

(iii) The Fisher information $I_n(\theta)$ is given by:

$$\begin{aligned} I_n(\theta) &= nI(\theta) = -nE \left[\frac{\partial^2}{\partial \theta^2} \log f(X; \theta) \right] = -nE \left[\frac{\partial^2}{\partial \theta^2} \log \left(\frac{\theta}{X^{\theta+1}} \right) \right] \\ &= -nE \left[\frac{\partial^2}{\partial \theta^2} (\log(\theta) - (\theta + 1) \log(X)) \right] = -nE \left[\frac{\partial}{\partial \theta} \left(\frac{1}{\theta} - \log(X) \right) \right] \\ &= -nE \left[-\frac{1}{\theta^2} \right] = \frac{n}{\theta^2}. \end{aligned}$$

(iv) The Cramer-Rao Lower Bound (CRLB) for estimating θ is:

$$\text{CRLB} = \frac{1}{I_n(\theta)} = \frac{\theta^2}{n}.$$

(v) From the log-likelihood:

$$\log L(\theta) = n \log \theta - (\theta + 1) \sum_{i=1}^n \log X_i$$

Taking the derivative with respect to θ and letting it to zero:

$$\frac{n}{\theta} - \sum_{i=1}^n \log X_i = 0 \quad \Rightarrow \quad \theta = \frac{n}{\sum_{i=1}^n \log X_i}$$

To confirm it is a MLE, we take the second derivative with respect to θ :

$$\frac{\partial^2 \log L(\theta)}{\partial \theta^2} = -\frac{n}{\theta^2} < 0 \text{ given } \theta > 0.$$

\therefore The likelihood function is concave (open downward), the MLE is the maximum.

\therefore The MLE of θ is $\hat{\theta} = \frac{n}{\sum_{i=1}^n \log X_i}$.

(vi) By Central Limit Theorem and Cramer-Rao Lower Bound, under regular conditions, the asymptotic distribution of the MLE is:

$$\frac{\hat{\theta} - \theta}{\sqrt{1/I_n(\theta)}} \sim N(0, 1) \quad \Rightarrow \quad \hat{\theta} \sim N\left(\theta, \frac{\theta^2}{n}\right) \text{ as } n \rightarrow \infty.$$