Q1) (i) $X_i \overset{iid}{\sim} U[-\theta, 0]$.

$$f(x) = \begin{cases} \frac{1}{\theta} & \text{if } x \in [-\theta, 0] \\ 0 & \text{otherwise} \end{cases}$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i) = \frac{1}{\theta^n} I(-\theta \leq x_{min})$$

$$\frac{dL}{d\theta} = -\frac{n}{\theta^{n+1}} < 0, \quad \forall \theta < 0$$

$\therefore$ As θ decreases, $L(\theta)$ increases.

$\because$ $\theta \geq -x_{min}$ for $L(\theta)$ to be non-zero

$\therefore$ $\hat{\theta}_{MLE} = -x_{min}$.

(ii) $L(\theta) = I(-\theta \leq x_{min}) \cdot \frac{1}{\theta^n} = g(T; \theta) h(x_1, \ldots, x_n)$

where $g(T; \theta) = I(-\theta \leq x_{min}) \frac{1}{\theta^n}$ with $T = x_{min}$,

and $h(x_1, \ldots, x_n) = 1$.

$\therefore$ By factorization theorem, $T = x_{min}$ is a sufficient statistic for θ.

Q2) The MLE of θ depends on the realization of X. That is, the actual observed value of X.

$$\hat{\theta}_{MLE} = \begin{cases} 1 & \text{if } x=0 \\ 1 & \text{if } x=1 \\ 2 \text{ or } 3 & \text{if } x=2 \\ 3 & \text{if } x=3 \\ 3 & \text{if } x=4 \end{cases}$$

Q3) (i) $L(\theta) = \prod_{i=1}^{n} f(x_i) = \dfrac{\theta^n}{\prod_{i=1}^{n} x_i^2} I(\theta \le x_{min})$

$$\frac{dL}{d\theta} = \frac{n\theta^{n-1}}{\prod_{i=1}^{n} x_i^2}, \quad \forall \theta > 0$$

$\therefore$ As $\theta \uparrow$, L always $\uparrow$

$\because$ $\theta \le x_{min}$ for $L(\theta)$ to be non-zero.

$\therefore$ $\hat{\theta}_{MLE} = x_{min}$

(ii) $E(X^{\frac{1}{3}}) = \int_\theta^\infty x^{\frac{1}{3}} \frac{\theta}{x^2} dx = \theta \int_\theta^\infty x^{-\frac{5}{3}} dx = \theta\left(-\frac{3}{2}\right)\left[x^{-\frac{2}{3}}\right]_\theta^\infty$

$= -\frac{3}{2}\theta\left[0 - \theta^{-\frac{2}{3}}\right] = \frac{3}{2}\theta^{\frac{1}{3}}$

(iii) $\frac{3}{2}\theta^{\frac{1}{3}} = E(X^{\frac{1}{3}}) = \frac{1}{n}\sum_{i=1}^{n} x_i^{\frac{1}{3}} \Rightarrow \theta^{\frac{1}{3}} = \frac{2}{3}\frac{1}{n}\sum_{i=1}^{n} x_i^{\frac{1}{3}} \Rightarrow \hat{\theta}_{MME} = \frac{8}{27}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\frac{1}{3}}\right)^3$

$\because$ $X_1^{\frac{1}{3}}, X_2^{\frac{1}{3}}, X_3^{\frac{1}{3}}, \ldots, X_n^{\frac{1}{3}}$ are i.i.d. with $E(X_i^{\frac{1}{3}}) < \infty$

$\therefore$ By WLLN, $\frac{1}{n}\sum_{i=1}^{n} X_i^{\frac{1}{3}} \xrightarrow{P} E(X_i^{\frac{1}{3}}) = \frac{3}{2}\theta^{\frac{1}{3}}$ ...... ①

$\because$ $g(y) = \frac{8}{27}y^3$ is a continuous function

$\therefore$ By the continuous mapping theorem and ①,

$$\hat{\theta}_{MLE} \xrightarrow{P} \frac{8}{27}\left(\frac{3}{2}\theta^{\frac{1}{3}}\right)^3 = \theta.$$

$\therefore$ $\hat{\theta}_{MLE}$ is a consistent estimator for θ.

Q4) Solution:

(i)

Let X be number of failures until and up to the first success. Then, $X \sim \text{Geo}(p)$, p is the success probability.

$$\Pr(X = x) = p(1-p)^x, \qquad \text{for } x = 0, 1, 2, \ldots, \text{ and } 0 < \theta < 1.$$

Note: This is an alternative definition of Geometric distribution and the support of X is $S_X = \{0, 1, 2, \ldots\}$.

Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. Geo(p).

Let $T = \sum_{i=1}^n X_i$,

$$\begin{aligned} \Pr(X_1 = x_1, \ldots, X_n = x_n) &= \prod_{i=1}^n \Pr(X_i = x_i) = \prod_{i=1}^n \left[p(1-p)^{x_i}\right] \\ &= p^n (1-p)^{\sum_{i=1}^n x_i} \times 1 = g(T; p) \times h(\overrightarrow{x}^\top), \end{aligned}$$

where $g(T; p) = p^n(1-p)^{\sum_{i=1}^n x_i}$ and $h(\overrightarrow{x}^\top) = 1$.

$\therefore$ By factorization theorem,

$T = \sum_{i=1}^n x_i$ is a sufficient statistic of p.

Now, let's prove the completeness of T.

Note: To prove completeness, you can also use Property 3.4 of Lecture 3. I have used the following method here because I want you to have a deeper understanding on the definition of completeness.

Since $X_1, X_2, \ldots, X_n$ are i.i.d. Geometric random variables, their sum must follow Negative Binomial distribution. Geo(p), it is possible show that using mgf.

Note: You can prove this using mgf.

$\therefore T \sim \text{NB}(n, p)$.

Note: T is defined as the number of failures until and up to the n-th successes, as we have used the alternative defintions of Negative Binomial & Geometric distributions.

Then, the pmf of T is

$$\Pr(T = t) = \binom{n+t-1}{t} p^n (1-p)^t, \qquad \text{for } y = 0, 1, 2, 3, \ldots \text{ and } 0 < p < 1.$$

Let $h(T)$ be any function of T s.t. $\mathrm{E}(h(T)) = 0$ for $\forall p \in \Theta$.

Let's prove $h(T) = 0$ for $\forall T$, i.e. $\Pr(h(T) = 0) = 1$.

Note: $T \sim \text{NB}(n, p)$.

$$\begin{aligned} 0 &= \mathrm{E}(h(T)) = \sum_{t=0}^{\infty} h(t) \Pr(T = t) = \sum_{t=0}^{\infty} h(t) \binom{n+t-1}{t} \theta^n (1-\theta)^t \\ \overset{\theta \neq 0}{\Longrightarrow} 0 &= \sum_{t=0}^{\infty} C_t u^t, \qquad \text{where } C_t = h(t)\binom{n+t-1}{t} \text{ and } u = (1-\theta). \end{aligned}$$

$\therefore$ We have $0 = \sum_{t=0}^{\infty} C_t u^t$ for $\forall u \in (0, 1)$.

Thereom: If $0 = \sum_{t=0}^{\infty} C_t u^t$, for all $u \in (0,1)$, then

$$C_t = 0, \forall t \in \{0,1,2,3,\ldots\}.$$

By theorem above, $C_t = 0, \forall t \in \{0,1,2,\ldots\}$.
$\because$

$$\binom{n+t-1}{t} \neq 0, \forall t \in \{0,1,2,\ldots\}.$$

$\therefore h(t) = 0, \qquad \text{for } \forall t \in \{0,1,2,3,\ldots\}.$
$\therefore$ We have the following result:

$$\mathrm{E}(h(T)) = 0,\ \forall p \in \Theta \Longrightarrow h(T) = 0, \forall T.$$

$\therefore T$ is complete for p.

(ii)
Now, let's use Lehmann-Scheffé Theorem to find the UMVUE of θ.
First of all, noticed that $\frac{T}{n}$ is not an unbiased estimator of θ because we have used the alternative definitions of Geometric & Negative Binomial distributions.

$$\mathrm{E}(T) = n\mathrm{E}(X_i) = \frac{n(1-p)}{p},$$

and

$$\mathrm{E}\left(\frac{T}{n}\right) = \frac{(1-p)}{p}.$$

Now, let's find an unbiased estimator of θ.
Define

$$I_{(0)}(X_j) = \begin{cases} 1, & \text{if } X_j = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} \mathrm{E}\left(I_{(0)}(X_j)\right) &= 1\,\mathrm{Pr}\left(X_j = 0\right) + 0\,\mathrm{Pr}\left(X_j \neq 0\right) \\ &= 1 \cdot p \\ &= p. \end{aligned}$$

$\therefore I_{(0)}(X_j)$ is an unbiased estimator of p.
Let $g(T) = \mathrm{E}\left(I_{(0)}(X_j)|T\right)$.
Then,

$$\begin{aligned} \mathrm{E}\left(g(T)\right) &= \mathrm{E}_T\left[\mathrm{E}\left(I_{(0)}(X_j)|T\right)\right] \\ &= \mathrm{E}\left(I_{(0)}(X_j)\right) \\ &= p. \end{aligned}$$

$\therefore g(T)$ is an unbiased estimator of p.
$\because T$ is sufficient and complete for p.

$\therefore g(T)$ is unique UMVUE for p.

Now, let's work out the explicit expression of $g(T)$.

$$\begin{aligned} g(T) &= \mathrm{E}\left(I_{(0)}(X_j|T)\right) \\ &= 1 \cdot \Pr\left(X_j = 0|T\right) + 0 \Pr\left(X_j \neq 0|T\right) \\ &= \Pr(X_j = 0|T) \qquad \ldots\ldots(*) \end{aligned}$$

Let $T' = \sum_{\forall i: i \neq j} X_i$, then from part (b), we know

$$T \sim \mathrm{NB}(n, p), \text{ and } T' \sim \mathrm{NB}(n-1, p)$$

$\because$

$$\begin{aligned} \Pr(X_j = 0|T = t) &= \frac{\Pr\left(X_j = 0 \cap T = t\right)}{\Pr(T = t)} \\ &= \frac{\Pr\left(T = t|X_j = 0\right)\Pr\left(X_j = 0\right)}{\Pr(T = t)} \\ &= \frac{\Pr\left(T' = t\right)\Pr\left(X_j = 0\right)}{\Pr(T = t)} \\ &= \frac{\binom{n-1+t-1}{t} p^{n-1}(1-p)^t p}{\binom{n+t-1}{t} p^n (1-p)^t} \\ &= \frac{(n+t-2)!}{(n-2)!t!} \div \left[\frac{(n+t-1)!}{(n-1)!t!}\right] \\ &= \frac{n-1}{n+t-1} \\ &= \frac{n-1}{t+n-1}. \end{aligned}$$

$\therefore$ By (*), $g(T) = \Pr\left(X_j = 0|T\right) = \frac{n-1}{T+n-1} = \frac{n-1}{\sum_{i=1}^n X_i + n - 1}$.

$\because T$ is complete and sufficient for p and $g(T)$ is unbiased for p.

$\therefore$By Lehmann-Scheffé Theorem, $g(T) = \frac{n-1}{T+n-1} = \frac{n-1}{\sum_{i=1}^n X_i + n - 1}$ is the UMVUE of p.

Note:

$\because T = T' + X_j$.

$\therefore \Pr\left(T = t|X_j = 0\right) = \Pr\left(T' = t\right)$.

Q5) (i) $E(T_1) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i Y_i\right)$

$$= \frac{1}{n}\sum E(X_i Y_i)$$

$$\overset{\text{ind.}}{=} \frac{1}{n}\sum E(X_i)E(Y_i)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \frac{p}{q}\, q$$

$$= p$$

$\therefore$ T_1 is an unbiased estimator for p.

(ii) $\because$ $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ are independence

$\therefore$ $Var(T_1) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i Y_i\right)$

$$\overset{\text{ind.}}{=} \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i Y_i)$$

$$\overset{\text{identical}}{=} \frac{1}{n} Var(X_i Y_i)$$

$$Var(XY) = E(X^2Y^2) - E(XY)^2$$

$$\overset{\text{ind}}{=} E(X^2)E(Y^2) - E(X)^2E(Y)^2$$

$$= \left(\sigma_1^2 + \frac{p^2}{q^2}\right)(\sigma_2^2 + q^2) - \frac{p^2}{q^2}\, q^2$$

$$= \sigma_1^2\sigma_2^2 + \frac{p^2\sigma_2^2}{q^2} + \sigma_1^2 q^2 + p^2 - p^2$$

$$= \sigma_1^2\sigma_2^2 + \frac{p^2\sigma_2^2}{q^2} + \sigma_1^2 q^2$$

$\therefore$ $Var(T_1) = \frac{1}{n}\left(\sigma_1^2\sigma_2^2 + \frac{p^2\sigma_2^2}{q^2} + \sigma_1^2 q^2\right)$

Q5) (iii) $E(T_2) = E\left[\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}Y_i\right)\right]$

$$\overset{\text{ind.}}{=} E\left[\frac{1}{n}\sum X_i\right] E\left[\frac{1}{n}\sum Y_i\right]$$

$$\overset{\text{iid}}{=} E(X_i)\,E(Y_i)$$

$$= \frac{p}{q}\,q$$

$$= p$$

$\therefore$ T_2 is also an unbiased estimator for p.

(iv) $\because$ X_i's are i.i.d. with $E(X_i) = \frac{p}{q} < \infty$

$\therefore$ By WLLN, $\frac{1}{n}\sum_{i=1}^{n}X_i \xrightarrow{p} \frac{p}{q}$.

$\because$ Y_i's are i.i.d. with $E(Y_i) = q < \infty$

$\therefore$ By WLLN, $\frac{1}{n}\sum_{i=1}^{n}Y_i \xrightarrow{p} q$.

$\therefore$ By Slutsky's theorem,

$$T_2 = \left(\frac{1}{n}\sum X_i\right)\left(\frac{1}{n}\sum Y_i\right) \xrightarrow{p} \frac{p}{q}\cdot q = p.$$

$\therefore$ T_2 is a consistent estimator for p.

Q5) (v) $Var(T_2) = Var(\bar{X}\bar{Y})$

$$= E(\bar{X}^2\bar{Y}^2) - E(\bar{X})^2 E(\bar{Y})^2$$

$$\overset{ind}{=} E(\bar{X}^2)E(\bar{Y}^2) - \frac{p^2}{q^2}\, q^2$$

$$= \left[Var(\bar{X}) + E(\bar{X})^2\right]\left[Var(\bar{Y}) + E(\bar{Y})^2\right] - p^2$$

$$= \left(\frac{\sigma_1^2}{n} + \frac{p^2}{q^2}\right)\left(\frac{\sigma_2^2}{n} + q^2\right) - p^2$$

$$= \frac{\sigma_1^2\sigma_2^2}{n^2} + \frac{p^2\sigma_2^2}{nq^2} + \frac{\sigma_1^2 q^2}{n}$$

$$= \frac{2\sigma_1^2\sigma_2^2}{n^2}$$

Note: $p = 0$

$q^2 = \frac{\sigma_2^2}{n}$

$$Var(T_1) = \frac{1}{n}\left(\sigma_1^2\sigma_2^2 + \frac{\sigma_1^2\sigma_2^2}{n}\right) = \frac{\sigma_1^2\sigma_2^2}{n}\left(\frac{n+1}{n}\right)$$

$\therefore$ $Var(T_1) \div Var(T_2) = \frac{n+1}{2} > 1$ iff $n \geq 2$.

$\therefore$ T_2 is more efficient if $n \geq 2$.

When $n = 1$, T_1 and T_2 are identical and they are equally efficient.

Q6) (i) $X_i \overset{iid}{\sim} Poi(\lambda)$.

$$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(X_i)$$

$$\overset{identical}{=\!=} \frac{1}{n}\, n\, E(X_i)$$

$$= \lambda$$

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2$$

$$\therefore E(S^2) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2\right)$$

$$\overset{identical}{=\!=} \frac{1}{n}\, n\, E(X_i^2) - E(\bar{X}^2)$$

$$= \left[Var(X_i) + E(X_i)^2\right] - \left[Var(\bar{X}) + E(\bar{X})^2\right]$$

$$\overset{iid}{=} (\lambda + \lambda^2) - \left(\frac{\lambda}{n} + \lambda^2\right)$$

$$= \left(\frac{n-1}{n}\right)\lambda$$

$$\therefore E\left(\frac{n}{n-1} S^2\right) = \frac{n}{n-1} E(S^2) = \lambda.$$

Q6) (ii) $\Pr(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{1}{x!} e^{-\lambda} \exp\{\ln(\lambda^{x})\}$

$$= \frac{1}{x!} e^{-\lambda} \exp\{(\ln\lambda)\, x\}$$

$$= h(x)\, c(\lambda) \exp\{P(\lambda)\, t(x)\},$$

where $h(x) = \frac{1}{x!}$, $c(\lambda) = e^{-\lambda}$, $P(\lambda) = \ln\lambda$ and $t(x) = x$

$\therefore$ By property 3.2, $\sum_{i=1}^{n} t(x_i) = \sum_{i=1}^{n} x_i$ is sufficient for the estimation of λ.

$\because$ $\Theta = \{\lambda : \lambda \in \mathbb{R}^{+}\}$ contains an open see in $\mathbb{R}^{1}$

$\therefore$ $\sum_{i=1}^{n} x_i$ is also complete for λ.

(iii) $\ln \Pr(X=x) = \ln\left[\frac{e^{-\lambda}\lambda^{x}}{x!}\right]$

$$= -\lambda + x\ln\lambda + \ln\frac{1}{x!}$$

$$\frac{d}{d\lambda} \ln \Pr(X=x) = -1 + \frac{x}{\lambda}$$

$$\frac{d^2}{d\lambda^2} \ln \Pr(X=x) = -\frac{x}{\lambda^2}$$

$$I(\lambda) = -E\left[\frac{d^2}{d\lambda^2} \ln \Pr(X=x)\right] = \frac{1}{\lambda^2} E(X) = \frac{1}{\lambda}$$

$$I_n(\lambda) = n I(\lambda) = \frac{n}{\lambda}$$

Q6) (iv) $CRLB = \frac{1}{nI(\lambda)} = \frac{\lambda}{n}$.

(v) $\because$ $\overline{X}$ is an unbiased estimator for λ and is also a function of the complete and sufficient statistic $\sum_{i=1}^{n} X_i$

$\therefore$ By Theorem 3.2, $\overline{X}$ is the UMVUE for λ

$\therefore$ $\overline{X}$ is preferred.

Q7) (i) $X_i \overset{iid}{\sim} N(\theta, \theta^2)$. $f(x_i) = \frac{1}{\sqrt{2\pi\theta^2}} \exp\left\{-\frac{(x_i-\theta)^2}{2\theta^2}\right\}$

$$L(\theta) = \prod_{i=1}^{n} f(x_i) = (2\pi\theta^2)^{-\frac{n}{2}} \exp\left\{-\frac{\sum_{i=1}^{n}(x_i-\theta)^2}{2\theta^2}\right\}$$

$$= (2\pi\theta^2)^{-\frac{n}{2}} \exp\left\{-\frac{\sum x_i^2 - 2\theta\sum x_i + n\theta^2}{2\theta^2}\right\}$$

$$= (2\pi\theta^2)^{-\frac{n}{2}} \exp\left\{-\frac{\sum x_i^2}{2\theta^2} + \frac{\sum x_i}{\theta}\right\} \exp\left\{-\frac{n}{2}\right\}$$

$$= g(T_1, T_2; \theta) \cdot h(x_1, \ldots, x_n),$$

where $g(T_1, T_2; \theta) = (2\pi\theta^2)^{-\frac{n}{2}} \exp\left\{-\frac{\sum x_i^2}{2\theta^2} + \frac{\sum x_i}{\theta}\right\}$

with $T_1 = \sum_{i=1}^{n} x_i^2$ and $T_2 = \sum_{i=1}^{n} x_i$,

and $h(x_1, \ldots, x_n) = \exp\left\{-\frac{n}{2}\right\}$.

$\therefore$ By factorization theorem, T_1 and T_2 are jointly sufficiently for the estimation of θ.

Let's prove the sufficient statistics are <u>not</u> complete.

Define $g(T_1, T_2) = \frac{2\left[T_1 - n\left(\frac{T_2}{n}\right)^2\right]}{n-1} - \frac{1}{n} T_1$.

Then, $g(T_1, T_2) = \frac{2\left[\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right]}{n-1} - \frac{1}{n}\sum_{i=1}^{n} x_i^2$

Q7) (i) cont.

$$\therefore\ E[g(T_1, T_2)] = \frac{2[nE(X_i^2) - nE(\bar{X}^2)]}{n-1} - E(X_i^2)$$

$$= \frac{2[n(\theta^2+\theta^2) - n(\frac{\theta^2}{n} + \theta^2)]}{n-1} - (\theta^2+\theta^2)$$

$$= \frac{2[2n\theta^2 - \theta^2 - n\theta^2]}{n-1} - 2\theta^2$$

$$= \frac{2(n-1)\theta^2}{n-1} - 2\theta^2$$

$$= 0, \quad \forall \theta \in \mathbb{R} \setminus \{0\}$$

However, $Pr(g(T_1, T_2) = 0) \neq 1$.

$\therefore$ $\{T_1, T_2\}$ is not complete for θ.

Note 1: $g(T_1, T_2) = \frac{2\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1} - \frac{1}{n}\sum_{i=1}^{n} X_i^2$.

Obviously, $g(T_1, T_2)$ is <u>not</u> always equal to zero.

Note 2: $E(X^2) = Var(X) + E(X)^2 = \theta^2 + \theta^2 = 2\theta^2$

$E(\bar{X}^2) = Var(\bar{X}) + E(\bar{X})^2 = \frac{\theta^2}{n} + \theta^2$

Q7) (ii) $L(\theta) = \prod_{i=1}^{n} f(x_i) = (2\pi\theta^2)^{-\frac{n}{2}} \exp\left\{ - \frac{\sum_{i=1}^{n} (x_i - \theta)^2}{2\theta^2} \right\}$

$$= (2\pi\theta^2)^{-\frac{n}{2}} \exp\left\{ - \frac{\sum x_i^2}{2\theta^2} + \frac{\sum x_i}{\theta} - \frac{n}{2} \right\} \cdots\cdots \text{①}$$

$$l(\theta) = \ln L(\theta) = -\frac{n}{2} \ln \theta^2 - \frac{\sum x_i^2}{2\theta^2} + \frac{\sum x_i}{\theta} - \frac{n}{2} + \ln\left[(2\pi)^{-\frac{n}{2}}\right]$$

$$\frac{dl}{d\theta} = -\frac{n}{2}\frac{2\theta}{\theta^2} + \frac{\sum x_i^2}{\theta^3} - \frac{\sum x_i}{\theta^2} = 0$$

$$\Rightarrow n\theta^2 + \sum x_i \theta - \sum x_i^2 = 0$$

$$\Rightarrow \theta = \frac{-\sum x_i \pm \sqrt{(\sum x_i)^2 + 4n\sum x_i^2}}{2n} = \frac{-\bar{x} \pm \sqrt{\bar{x}^2 + 4\overline{x^2}}}{2}$$

$\therefore \theta_1 = \dfrac{-\bar{x} + \sqrt{\bar{x}^2 + 4\overline{x^2}}}{2}$ and $\theta_2 = \dfrac{-\bar{x} - \sqrt{\bar{x}^2 + 4\overline{x^2}}}{2}$, where $\overline{x^2} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$.

It is possible to show that the 2nd derivative of $l(\theta)$ is negative when $\theta = \theta_1$ or θ_2.

$$\therefore \hat{\theta}_{MLE} = \begin{cases} \theta_1 & \text{if } l(\theta_1) > l(\theta_2) \\ \theta_1 \text{ or } \theta_2 & \text{if } l(\theta_1) = l(\theta_2) \\ \theta_2 & \text{if } l(\theta_1) < l(\theta_2) \end{cases}$$

Note: By equation ①, we can easily see that as $|\theta| \to \infty$, $L(\theta) \to 0$ and that $\lim_{\theta \to 0^+} L(\theta) = \lim_{\theta \to 0^-} L(\theta) = 0$ (as the exponential term dominates). Since $L(\theta)$ is always non-negative and always differentiable for any $\theta \neq 0$ ($L(\theta)$ is undefined when $\theta = 0$), the only two stationary points $L(\theta_1)$ and $L(\theta_2)$ must be the local maximums.

Q7) (iii) $f(x) = \frac{1}{\sqrt{2\pi\theta^2}} \exp\left\{-\frac{(x-\theta)^2}{2\theta^2}\right\} = (\theta^2)^{-\frac{1}{2}} \exp\left\{-\frac{x^2 - 2\theta x + \theta^2}{2\theta^2}\right\} \frac{1}{\sqrt{2\pi}}$

$$\ln f(x) = -\frac{1}{2}\ln\theta^2 - \frac{x^2}{2\theta^2} + \frac{x}{\theta} - \frac{1}{2} + \ln\frac{1}{\sqrt{2\pi}}$$

$$\frac{d}{d\theta}\ln f(x) = -\frac{1}{2}\frac{2\theta}{\theta^2} + \frac{x^2}{\theta^3} - \frac{x}{\theta^2}$$

$$\frac{d^2}{d\theta^2}\ln f(x) = \frac{1}{\theta^2} - \frac{3x^2}{\theta^4} + \frac{2x}{\theta^3}$$

$$I(\theta) = -E\left(\frac{d^2}{d\theta^2}\ln f(X)\right) = -\frac{1}{\theta^2} + \frac{3E(X^2)}{\theta^4} - \frac{2E(X)}{\theta^3}$$

$$= -\frac{1}{\theta^2} + \frac{3(\theta^2+\theta^2)}{\theta^4} - \frac{2\theta}{\theta^3}$$

$$= -\frac{1}{\theta^2} + \frac{6}{\theta^2} - \frac{2}{\theta^2}$$

$$= \frac{3}{\theta^2}$$

$\therefore$ By the asymptotic properties of MLE,

$$\sqrt{n}(\hat{\theta}_{MLE} - \theta) \xrightarrow{d} N\left(0, \frac{\theta^2}{3}\right)$$

Note: $CRLB = \frac{1}{I_n(\theta)}$

Q8) (i) $E(\hat{S}_1) = E\left(\frac{X_1 + 2X_2}{3}\right)$

$$= \frac{E(X_1) + 2E(X_2)}{3}$$

$$= \frac{S_1 + 2S_1}{3}$$

$$= S_1$$

$\therefore$ $\hat{S}_1$ is an unbiased estimator for S_1.

(ii) $L(S_1, S_2) = f(x_1, \ldots, x_n) \overset{ind}{=} f(x_1, \ldots, x_m) f(x_{m+1}, \ldots, x_n)$

$$\overset{ind}{=} \prod_{i=1}^{m} \left[\frac{1}{\sqrt{2\pi \sigma_i^2}} e^{-\frac{(x_i - S_1)^2}{2\sigma_i^2}} \right] \prod_{i=m+1}^{n} \left[\frac{1}{\sqrt{2\pi \sigma_i^2}} e^{-\frac{(x_i - S_2)^2}{2\sigma_i^2}} \right]$$

$$= \left(\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi \sigma_i^2}} \right) \exp\left\{ -\sum_{i=1}^{m} \frac{(x_i - S_1)^2}{2\sigma_i^2} \right\} \left(\prod_{i=m+1}^{n} \frac{1}{\sqrt{2\pi \sigma_i^2}} \right) \exp\left\{ -\sum_{i=m+1}^{n} \frac{(x_i - S_2)^2}{2\sigma_i^2} \right\}$$

$\therefore$ $l(S_1, S_2) = \ln L(S_1, S_2) = -\sum_{i=1}^{m} \frac{(x_i - S_1)^2}{2\sigma_i^2} - \sum_{i=m+1}^{n} \frac{(x_i - S_2)^2}{2\sigma_i^2} + c,$

where $c = \ln\left(\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi \sigma_i^2}} \right) + \ln\left(\prod_{i=m+1}^{n} \frac{1}{\sqrt{2\pi \sigma_i^2}} \right)$

$$\frac{\partial l}{\partial S_1} = \sum_{i=1}^{m} \frac{(x_i - S_1)}{\sigma_i^2} = 0 \Rightarrow \sum_{i=1}^{m} \frac{x_i}{\sigma_i^2} - \sum_{i=1}^{m} \frac{S_1}{\sigma_i^2} = 0 \Rightarrow S_1 \sum_{i=1}^{m} \frac{1}{\sigma_i^2} = \sum_{i=1}^{m} \frac{x_i}{\sigma_i^2}$$

$$\Rightarrow S_1 = \left(\sum_{i=1}^{m} \frac{1}{\sigma_i^2} \right)^{-1} \sum_{i=1}^{m} \frac{x_i}{\sigma_i^2}.$$

$$\frac{\partial l}{\partial S_2} = \sum_{i=m+1}^{n} \frac{(x_i - S_2)}{\sigma_i^2} = 0 \Rightarrow S_2 \sum_{i=m+1}^{n} \frac{1}{\sigma_i^2} = \sum_{i=m+1}^{n} \frac{x_i}{\sigma_i^2} \Rightarrow S_2 = \left(\sum_{i=m+1}^{n} \frac{1}{\sigma_i^2} \right)^{-1} \sum_{i=m+1}^{n} \frac{x_i}{\sigma_i^2}$$

It's possible to show that the second total derivative of $l(S_1, S_2)$ is negative when $S_1 = \left(\sum_{i=1}^{m} \frac{1}{\sigma_i^2} \right)^{-1} \sum_{i=1}^{m} \frac{x_i}{\sigma_i^2}$ and $S_2 = \left(\sum_{i=m+1}^{n} \frac{1}{\sigma_i^2} \right)^{-1} \sum_{i=m+1}^{n} \frac{x_i}{\sigma_i^2}$.

$\therefore$ $\hat{S}_{1,MLE} = \left(\sum_{i=1}^{m} \frac{1}{\sigma_i^2} \right)^{-1} \sum_{i=1}^{m} \frac{x_i}{\sigma_i^2}$ and $\hat{S}_{2,MLE} = \left(\sum_{i=m+1}^{n} \frac{1}{\sigma_i^2} \right)^{-1} \sum_{i=m+1}^{n} \frac{x_i}{\sigma_i^2}$

Q8) (iii) $\because \sigma_i^2 = \frac{m}{i}, \ \forall i$

$$\therefore \hat{S}_{1,MLE} = \left(\sum_{i=1}^{m} \frac{1}{\frac{m}{i}}\right)^{-1} \sum_{i=1}^{m} \frac{x_i}{\frac{m}{i}}$$

$$= \left(\sum_{i=1}^{m} i\right)^{-1} m^{-1} \sum_{i=1}^{m} \frac{i x_i}{m}$$

$$= \left(\sum_{i=1}^{m} i\right)^{-1} \sum_{i=1}^{m} i x_i$$

Similarly, $\hat{S}_{2,MLE} = \left(\sum_{i=m+1}^{n} i\right)^{-1} \sum_{i=m+1}^{n} i x_i$.

$$Var(\hat{S}_{1,MLE}) = Var\left[\left(\sum_{i=1}^{m} i\right)^{-1} \sum_{i=1}^{m} i X_i\right]$$

$$\stackrel{ind}{=} \left(\sum_{i=1}^{m} i\right)^{-2} \sum_{i=1}^{m} i^2 Var(X_i)$$

$$= \left(\sum_{i=1}^{m} i\right)^{-2} \sum_{i=1}^{m} i^2 \frac{m}{i}$$

$$= \left(\sum_{i=1}^{m} i\right)^{-2} \sum_{i=1}^{m} i m$$

$$= \left(\sum_{i=1}^{m} i\right)^{-1} m$$

$$Var(\tilde{S}_1) = Var\left(\frac{X_1 + 2X_2}{3}\right) \stackrel{ind}{=} \frac{1}{9}\left[Var(X_1) + 4Var(X_2)\right]$$

$$= \frac{1}{9}\left[\frac{m}{1} + 4\frac{m}{2}\right]$$

$$= \frac{m}{3} > \left(\sum_{i=1}^{m} i\right)^{-1} m = Var(\hat{S}_{1,MLE}) \text{ iff } m > 2.$$

$\therefore \hat{S}_{1,MLE}$ is more efficient when $m > 2$, and they are identical and equally efficient when $m = 2$.

Q8) (iv) $\because$ $\sigma_i^2 = \frac{m}{i}, \forall i$

$$\therefore l(s_1, s_2) = -\sum_{i=1}^{m} \frac{(x_i - s_1)^2}{2m/i} - \sum_{i=m+1}^{n} \frac{(x_i - s_2)^2}{2m/i} + C$$

$$= -\sum_{i=1}^{m} \frac{i(x_i - s_1)^2}{2m} - \sum_{i=m+1}^{n} \frac{i(x_i - s_2)^2}{2m} + C$$

$$\frac{\partial l}{\partial s_1} = \sum_{i=1}^{m} \frac{i(x_i - s_1)}{m}$$

$$\frac{\partial^2 l}{\partial s_1^2} = -\frac{1}{m} \sum_{i=1}^{m} i$$

$$\therefore I_m(s_1) = -E\left(-\frac{\partial^2 l}{\partial s_1^2}\right) = \frac{1}{m} \sum_{i=1}^{m} i = \frac{1}{m} \frac{(1+m)m}{2} = \frac{1+m}{2}$$

Similarly, we can find $\frac{\partial^2 l}{\partial s_2^2} = -\frac{1}{m} \sum_{i=m+1}^{n} i$.

$$\therefore I_m(s_2) = -E\left(-\frac{\partial^2 l}{\partial s_2^2}\right) = \frac{1}{m} \sum_{i=m+1}^{n} i = \frac{1}{m} \frac{(m+1+n)m}{2} = \frac{3m+1}{2}$$

$$\therefore \lim_{m\to\infty} Var(\sqrt{m}\, \hat{s}_{1,MLE}) = \lim_{m\to\infty} m\, I_m(s_1)^{-1} = \lim_{m\to\infty} \frac{2m}{1+m} = 2$$

$$\lim_{m\to\infty} Var(\sqrt{m}\, \hat{s}_{2,MLE}) = \lim_{m\to\infty} m\, I_m(s_2)^{-1} = \lim_{m\to\infty} \frac{2m}{3m+1} = \frac{2}{3}$$

$\therefore$ By the asymptotic properities of MLE, we have

$\sqrt{m}\,(\hat{s}_{1,MLE} - s_1) \xrightarrow{d} N(0, 2)$ and

$\sqrt{m}\,(\hat{s}_{2,MLE} - s_2) \xrightarrow{d} N(0, \frac{2}{3})$.

Note: When you study STAT3602, you will also learn how to work out the joint asymptotic distribution of $\hat{s}_{1,MLE}$ and $\hat{s}_{2,MLE}$.

Q8)(v) $E(\hat{S}_{1,MLE}) = E\left[\left(\sum_{i=1}^{m} i\right)^{-1} \sum_{i=1}^{m} iX_i\right] = \left(\sum_{i=1}^{m} i\right)^{-1} \sum_{i=1}^{m} iE(X_i) = s_1$,

as $E(X_i) = s_1$, $\forall i \in \{1, \dots, m\}$.

Similarly, $E(\hat{S}_{2,MLE}) = E\left[\left(\sum_{i=m+1}^{n} i\right)^{-1} \sum_{i=m+1}^{m} iX_i\right] = \left(\sum_{i=m+1}^{n} i\right)^{-1} \sum_{i=m+1}^{n} iE(X_i) = s_2$,

as $E(X_i) = s_2$, $\forall i \in \{m+1, \dots, n\}$.

Hence, $\hat{S}_{1,MLE}$ and $\hat{S}_{2,MLE}$ are unbiased estimators for s_1 and s_2, respectively.

Therefore, it suffices to show their asymptotic variances are zeros.

In part (iii), we have shown that

$$Var(\hat{S}_{1,MLE}) = \left(\sum_{i=1}^{m} i\right)^{-1} m = \left[\frac{m(m+1)}{2}\right]^{-1} m = \frac{2}{m+1} \to 0 \text{ as } m \to \infty.$$

Similarly, $Var(\hat{S}_{2,MLE}) = \left(\sum_{i=m+1}^{n} i\right)^{-1} m = \left[\frac{m(m+1+n)}{2}\right]^{-1} m = \frac{2}{3m+1}$

$\to 0$ as $m \to \infty$.

$\therefore$ $\hat{S}_{1,MLE}$ and $\hat{S}_{2,MLE}$ are consistent estimators for s_1 and s_2, respectively.

Q9. (i) **Label (LE)**

The likelihood function is

$$L(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta) = \frac{\theta^n}{\prod_{i=1}^{n} x_i^{\theta+1}}.$$

(ii) **Label (L)**

According to the factorization theorem, $T = \prod_{i=1}^{n} X_i$ is sufficient for θ.

(iii) **Label (L)**

$$\ln f(X;\theta) = \ln\theta - (\theta+1)\ln(x)$$
$$\frac{\partial \ln f(X;\theta)}{\partial\theta} = \frac{1}{\theta} - \ln X$$
$$\frac{\partial^2 \ln f(X;\theta)}{\partial\theta^2} = -\frac{1}{\theta^2}$$
$$I(\theta) = E\left[-\frac{\partial^2 \ln f(X;\theta)}{\partial\theta^2}\right] = E\left[\frac{1}{\theta^2}\right] = \frac{1}{\theta^2}$$

The Fisher information is

$$I_n(\theta) = nI(\theta) = \frac{n}{\theta^2}.$$

(iv) **Label (L)**

The Cramer-Rao Lower Bound for estimation of θ is $\frac{1}{nI(\theta)} = \frac{\theta^2}{n}$.

(v) **Label (L)**

(a) The log-likelihood is

$$l(\theta) = n\ln\theta - (\theta+1)\sum_{i=1}^{n}\ln(X_i).$$

The first derivative of log-likelihood is

$$l'(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n}\ln(X_i).$$

The MLE is

$$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n}\ln(X_i)}.$$

(b) Asymptotically the MLE $\hat{\theta}$ is distributed normal:

$$\sqrt{\frac{n}{\theta^2}}(\hat{\theta} - \theta) \to N(0,1), \text{ as } n \to \infty.$$

or $\hat{\theta} \sim N(\theta, \theta^2/n)$ asymptotically.