STAT2602 Assignment 3

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Q1

(a)

Since $X_i \sim \text{Poisson}(\lambda)$, we have $\mathbb{E}(X_i) = \lambda$ and $\text{Var}(X_i) = \lambda$.
For large n, by the Central Limit Theorem, the sample mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is approximately normally distributed:

$$\overline{X} \sim N\left(\lambda, \frac{\lambda}{n}\right)$$

By standardizing, we obtain the pivotal quantity:

$$Z = \frac{\overline{X} - \lambda}{\sqrt{\frac{\lambda}{n}}} \sim N(0,1)$$

which its distribution does not depend on the unknown parameter λ.

(b)

From (a), we have:

$$Z = \frac{\overline{X} - \lambda}{\sqrt{\frac{\lambda}{n}}} \sim N(0,1)$$

Since $\overline{X}$ is an consistent estimator of λ (which $\overline{X} \to_p \lambda$ for large n), by the Slutsky's theorem,

$$\frac{\overline{X} - \lambda}{\sqrt{\frac{\bar{X}}{n}}} \to_d Z \sim N(0,1)$$

For a $(1-\alpha)$ confidence interval, we have:

$$\begin{aligned}
1 - \alpha &= P\left(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}\right) \\
&= P\left(-z_{\alpha/2} \leq \frac{\overline{X} - \lambda}{\sqrt{\frac{\lambda}{n}}} \leq z_{\alpha/2}\right) \\
&\approx P\left(-z_{\alpha/2} \leq \frac{\overline{X} - \lambda}{\sqrt{\frac{\bar{X}}{n}}} \leq z_{\alpha/2}\right) \\
&\approx P\left(\overline{X} - z_{\alpha/2}\sqrt{\frac{\bar{X}}{n}} \leq \lambda \leq \overline{X} + z_{\alpha/2}\sqrt{\frac{\bar{X}}{n}}\right)
\end{aligned}$$

$\therefore$ an approximate $(1-\alpha)$ confidence interval for λ is:

$$\left(\overline{X} - z_{\alpha/2}\sqrt{\frac{\overline{X}}{n}},\ \overline{X} + z_{\alpha/2}\sqrt{\frac{\overline{X}}{n}}\right)$$

(c)

Given that:

$$\overline{X} = 1.5, \quad S^2 = 4, \quad n = 36$$

For a 95% confidence interval, $z_{\alpha/2} = z_{0.025} = 1.96$. Using the approximate confidence interval from part (b):

$$\lambda \in \left(1.5 - 1.96 \times \sqrt{\frac{1.5}{36}},\ 1.5 + 1.96 \times \sqrt{\frac{1.5}{36}}\right)$$

$$\Rightarrow \lambda \in (1.099916675, 1.900083325)$$
$$\Rightarrow \lambda \in (1.10, 1.90)$$

Q2

(a)

As $X_i \sim U(0, \theta)$, $Pr(X_i \leq x) = \frac{x}{\theta}$ for $0 \leq x \leq \theta$. Since $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} U(0, \theta)$,

$$\begin{aligned}
Pr(X_{(n)} \leq x) &= Pr(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x) \\
&= (\frac{x}{\theta})^n \quad \text{for } 0 \leq x \leq \theta \\
Pr(\frac{X_{(n)}}{\theta} \leq \frac{x}{\theta}) &= (\frac{x}{\theta})^n \quad \text{for } 0 \leq x \leq \theta \\
Pr((\frac{X_{(n)}}{\theta})^n \leq (\frac{x}{\theta})^n) &= (\frac{x}{\theta})^n \quad \text{for } 0 \leq x \leq \theta \\
\text{Let } U &= (\frac{X_{(n)}}{\theta})^n, \\
Pr(U \leq u) &= u \quad \text{for } 0 \leq u \leq 1 \\
&= F_U(u) \quad \text{which is strictly increasing} \\
f_U(u) &= \frac{d}{du} F_U(u) = 1 \quad \text{for } 0 \leq u \leq 1 \\
\therefore U &\sim U(0,1)
\end{aligned}$$

As $U = (\frac{X_{(n)}}{\theta})^n$ does not depend on the unknown parameter θ.
$\therefore (\frac{X_{(n)}}{\theta})^n$ is a pivotal variable for $X_{(n)}$.

(b)

Since $U = (\frac{X_{(n)}}{\theta})^n \sim U(0,1)$, we can calculate the confidence interval for θ:

$$\begin{aligned}
1 - \alpha &= Pr\left(\frac{\alpha}{2} \leq U \leq 1 - \frac{\alpha}{2}\right) \\
&= Pr\left(\frac{\alpha}{2} \leq (\frac{X_{(n)}}{\theta})^n \leq 1 - \frac{\alpha}{2}\right) \\
&= Pr\left((\frac{\alpha}{2})^{1/n} \leq \frac{X_{(n)}}{\theta} \leq (1 - \frac{\alpha}{2})^{1/n}\right) \\
&= Pr\left(\frac{(\frac{\alpha}{2})^{1/n}}{X_{(n)}} \leq \frac{1}{\theta} \leq \frac{(1 - \frac{\alpha}{2})^{1/n}}{X_{(n)}}\right) \\
&= Pr\left(\frac{X_{(n)}}{(1 - \frac{\alpha}{2})^{1/n}} \leq \theta \leq \frac{X_{(n)}}{(\frac{\alpha}{2})^{1/n}}\right)
\end{aligned}$$

Therefore, the $(1 - \alpha)$ confidence interval for θ is:

$$\left(\frac{X_{(n)}}{(1 - \frac{\alpha}{2})^{1/n}}, \frac{X_{(n)}}{(\frac{\alpha}{2})^{1/n}}\right)$$

Q3

(i)

Since $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$, we construct a pivotal quantity for σ^2: $T = \frac{nS^2}{\sigma^2} \sim \chi^2_{n-1}$.
We have:

$$\begin{aligned}
1-\alpha &= Pr\left(\chi^2_{1-\alpha/2, df=n-1} \leq \frac{nS^2}{\sigma^2} \leq \chi^2_{\alpha/2, df=n-1}\right) \\
&= Pr\left(\frac{nS^2}{\chi^2_{\alpha/2, df=n-1}} \leq \sigma^2 \leq \frac{nS^2}{\chi^2_{1-\alpha/2, df=n-1}}\right) \\
&= Pr\left(\frac{\sqrt{n}S}{\sqrt{\chi^2_{\alpha/2, df=n-1}}} \leq \sigma \leq \frac{\sqrt{n}S}{\sqrt{\chi^2_{1-\alpha/2, df=n-1}}}\right)
\end{aligned}$$

$$\text{Given that } 1-\alpha = Pr\left(\frac{\sqrt{n}S}{\sqrt{b}} \leq \sigma \leq \frac{\sqrt{n}S}{\sqrt{a}}\right), \text{We have } a = \chi^2_{1-\alpha/2, df=n-1}, b = \chi^2_{\alpha/2, df=n-1},$$

$$\text{So, } G(b) - G(a) = F_{\chi^2_{n-1}}(b) - F_{\chi^2_{n-1}}(a)$$

$$\text{Since } 1-\alpha/2 \text{ is a tail probability}, F_{\chi^2_{n-1}}(a) = \alpha/2, F_{\chi^2_{n-1}}(b) = 1-\alpha/2$$

$$G(b) - G(a) = (1-\alpha/2) - (\alpha/2) = 1-\alpha$$

(ii)

Given that $1-\alpha = Pr(\frac{\sqrt{n}S}{\sqrt{b}} \leq \sigma \leq \frac{\sqrt{n}S}{\sqrt{a}}) = Pr(\frac{1}{\sqrt{a}} \leq \frac{\sigma}{\sqrt{n}S} \leq \frac{1}{\sqrt{b}})$, we let $Y = \frac{\sigma}{\sqrt{n}S}$.
Since $T = \frac{nS^2}{\sigma^2} \sim \chi^2_{n-1}$, $Y = \frac{1}{\sqrt{T}} \Rightarrow T = \frac{1}{Y^2}$. Note that Y has a hump-shaped curve similar to normal distribution.
To minimize $k = \frac{\sqrt{n}S}{\sqrt{a}} - \frac{\sqrt{n}S}{\sqrt{b}}$, we will minimize $\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}$, which is minimized when $f_Y(\frac{1}{\sqrt{a}}) = f_Y(\frac{1}{\sqrt{b}})$ and $F_Y(\frac{1}{\sqrt{b}}) - F_Y(\frac{1}{\sqrt{a}}) = 1-\alpha$.
Expressing f_Y in terms of f_T: $f_Y(y) = f_T(t)\ |\frac{dt}{dy}| = f_T(t) \times \frac{1}{2}y^{-3}$.
As $\frac{1}{(\frac{1}{\sqrt{a}})^2} = a$, $\frac{1}{(\frac{1}{\sqrt{b}})^2} = b$, we have:

$$\begin{aligned}
f_Y(\frac{1}{\sqrt{a}}) &= f_Y(\frac{1}{\sqrt{b}}) \\
f_T(a) \times \frac{1}{2}(a^{-\frac{1}{2}})^{-3} &= f_T(b) \times \frac{1}{2}(b^{-\frac{1}{2}})^{-3} \\
\frac{1}{2^{(n-1)/2}\Gamma(\frac{n-1}{2})} a^{(n-3)/2} e^{-a/2} \times \frac{1}{2}a^{3/2} &= \frac{1}{2^{(n-1)/2}\Gamma(\frac{n-1}{2})} b^{(n-3)/2} e^{-b/2} \times \frac{1}{2}b^{3/2} \\
a^{(n-3)/2} e^{-a/2} \times a^{3/2} &= b^{(n-3)/2} e^{-b/2} \times b^{3/2} \\
a^{n/2} e^{-a/2} - b^{n/2} e^{-b/2} &= 0
\end{aligned}$$

Q4

(a)

Man	X_i	Y_i	$W_i = Y_i - X_i$
1	120	128	8
2	124	131	7
3	130	131	1
4	118	127	9
5	140	132	-8
6	128	125	-3
7	140	141	1
8	135	137	2
9	126	118	-8
10	130	132	2
11	126	129	3
12	127	135	8

$$\overline{W} = \frac{\sum_{i=1}^{12} W_i}{12} = \frac{8+7+1+9-8-3+1+2-8+2+3+8}{12} = \frac{22}{12} \approx 1.833333$$

$$S_W = \sqrt{\frac{\sum_{i=1}^{12}(W_i - \overline{W})^2}{12}} \approx \sqrt{\frac{373.6666667}{12}} \approx 5.580223$$

We want to construct a 95% confidence interval for $\mu_X - \mu_Y$, which is same as $-\mu_W$.
We construct a pivotal quantity for W, which is $T = \frac{\overline{W}-\mu_W}{S_W/\sqrt{n-1}} \sim t_{n-1}$.
Given that $n = 12$, $1 - \alpha = 95\% \Rightarrow \alpha = 0.05$, the confidence interval is

$$\begin{aligned}
1 - \alpha &= Pr(-t_{\alpha/2,df=n-1} \leq T \leq t_{\alpha/2,df=n-1}) \\
&= Pr(-t_{0.025,11} \leq \frac{\overline{W} - \mu_W}{S_W/\sqrt{n-1}} \leq t_{0.025,11}) \\
&\approx Pr(-2.201 \leq \frac{1.833333 - \mu_W}{5.580223/\sqrt{11}} \leq 2.201) \\
&= Pr(-2.201 \times \frac{5.580223}{\sqrt{11}} \leq 1.833333 - \mu_W \leq 2.201 \times \frac{5.580223}{\sqrt{11}}) \\
&= Pr(-2.201 \times \frac{5.580223}{\sqrt{11}} - 1.833333 \leq -\mu_W \leq 2.201 \times \frac{5.580223}{\sqrt{11}} - 1.833333) \\
&= Pr(-5.536517 \leq -\mu_W \leq 1.869851)
\end{aligned}$$

$\therefore$ the 95% confidence interval for $\mu_X - \mu_Y$ is $(-5.537, 1.870)$ (correct to 3 d.p.).
If the stimulus is effective, we expect $\mu_W \neq 0$. Since $0 \in (-5.536517, 1.869851)$, we cannot reject the null hypothesis that the stimulus is ineffective.
Therefore, I don't think the stimulus has an effect on the blood pressure.

(b)

From (a), we have $S_W = 5.580223$, $S_W^2 = 31.138889$, $n = 12$, $1 - \alpha = 95\% \Rightarrow \alpha = 0.05$.
since $\frac{nS_W^2}{\sigma_W^2} \sim \chi^2_{n-1}$, the 95% confidence interval for σ_W^2 is

$$\begin{aligned}
1 - \alpha &= Pr(\chi^2_{1-\alpha/2,df=n-1} \leq \frac{nS_W^2}{\sigma_W^2} \leq \chi^2_{\alpha/2,df=n-1}) \\
&= Pr(\chi^2_{0.975,11} \leq \frac{12 \times 31.138889}{\sigma_W^2} \leq \chi^2_{0.025,11}) \\
&=\approx Pr(3.816 \leq \frac{373.666668}{\sigma_W^2} \leq 21.920) \\
&= Pr(17.046837 \leq \sigma_W^2 \leq 97.921035) \\
&= Pr(4.128782 \leq \sigma_W \leq 9.895506)
\end{aligned}$$

$\therefore$ The 95% confidence interval for σ_W is $(4.129, 9.896)$ (correct to 3 d.p.).

Q5

(i)

Given that $n_X = 13$, $s_X^2 = 9.88$,
since $\frac{n_X s_X^2}{\sigma_x^2} \sim \chi^2_{n_X - 1}$ is a pivotal quantity around σ_X^2, we have:

$$
\begin{aligned}
1 - \alpha &= Pr(\chi^2_{1-\alpha/2, df=n_X-1} \le \frac{n_X s_X^2}{\sigma_x^2} \le \chi^2_{\alpha/2, df=n_X-1}) \\
&= Pr(\chi^2_{0.975,12} \le \frac{13 \times 9.88}{\sigma_X^2} \le \chi^2_{0.025,12}) \\
&= Pr(4.404 \le \frac{128.44}{\sigma_X^2} \le 23.337) \\
&= Pr(5.503707 \le \sigma_X^2 \le 29.164396)
\end{aligned}
$$

$\therefore$ the 95% confidence interval for σ_X^2 is $(5.504, 29.164)$ (correct to 3 d.p.).

(ii)

Given that $n_X = 13$, $s_X^2 = 9.88$, $n_Y = 9$, $s_Y^2 = 4.08$, $1 - \alpha = 95\% \Rightarrow \alpha = 0.05$,
the pivotal quantity for σ_X^2 is $\frac{n_X s_X^2}{\sigma_x^2} \sim \chi^2_{n_X-1} \Rightarrow \frac{13 s_X^2}{\sigma_X^2} \sim \chi^2_{12}$,
and the pivotal quantity for σ_Y^2 is $\frac{n_Y s_Y^2}{\sigma_Y^2} \sim \chi^2_{n_Y-1} \Rightarrow \frac{9 s_Y^2}{\sigma_Y^2} \sim \chi^2_8$.
the pivotal quantity for $\frac{\sigma_X^2}{\sigma_Y^2}$ is $\frac{\frac{n_Y s_Y^2}{\sigma_Y^2}/(n_Y-1)}{\frac{n_X s_X^2}{\sigma_X^2}/(n_X-1)} \sim F_{n_Y-1, n_X-1} \Rightarrow \frac{\frac{9 s_Y^2}{\sigma_Y^2}/8}{\frac{13 s_X^2}{\sigma_X^2}/12} \sim F_{8,12}$.
We have:

$$
1 - \alpha = Pr(F_{1-\alpha/2, df_1=8, df_2=12} \le \frac{\frac{9 s_Y^2}{\sigma_Y^2}/8}{\frac{13 s_X^2}{\sigma_X^2}/12} \le F_{\alpha/2, df_1=8, df_2=12})
$$

Since $F_{0.975, df1=8, df2=12} = 0.23811409, F_{0.025, df1=8, df2=12} = 3.51177674$,

$$
\begin{aligned}
&\approx Pr(0.23811409 \le \frac{\frac{9 \times 4.08}{\sigma_Y^2}/8}{\frac{13 \times 9.88}{\sigma_X^2}/12} \le 3.51177674) \\
&= Pr(0.23811409 \le \frac{4.59}{\sigma_Y^2} \times \frac{300 \sigma_X^2}{3211} \le 3.51177674) \\
&= Pr(0.555253698 \le \frac{\sigma_X^2}{\sigma_Y^2} \le 8.189045107)
\end{aligned}
$$

Therefore, the 95% confidence interval for $\frac{\sigma_X^2}{\sigma_Y^2}$ is $(0.555, 8.189)$ (correct to 3 d.p.).