# Assignment 2

**Course: STAT2602** **Due date: 18 October 2024 (11:59pm)**

**Q1**. Let $\mathbf{X} = \{X_1, X_2, \cdots, X_n\}$ be an independent random sample from a uniformly distribution over the interval $[-\alpha, 0]$.

(i) Find the MLE of $\alpha$.

(ii) Find a sufficient statistic for $\alpha$.

**Q2**. One observation is taken on a discrete random variable $X$ with p.d.f. (p.m.f.) $f(x;\theta)$ in the following table, where $\theta \in \{1, 2, 3\}$. Find the MLE of $\theta$.

| $x$ | $f(x;1)$ | $f(x;2)$ | $f(x;3)$ |
|---|---|---|---|
| $0$ | $1/3$ | $1/4$ | $0$ |
| $1$ | $1/3$ | $1/4$ | $0$ |
| $2$ | $0$ | $1/4$ | $1/4$ |
| $3$ | $1/6$ | $1/4$ | $1/2$ |
| $4$ | $1/6$ | $0$ | $1/4$ |

**Q3**. Let $X_1, \cdots, X_n$ be an independent random sample from the p.d.f. given by

$$f(x;\theta) = \frac{\theta}{x^2}, \quad 0 < \theta \leq x < \infty.$$

(i) Find the MLE of $\theta$;

(ii) Calculate $E(X_1^{1/3})$;

(iii) Find one MME of $\theta$ and show that it is consistent.

**Q4**. Let $X_1, X_2, \cdots X_n$ be an independent random sample from the distribution with p.d.f.

$$f(x;p) = p(1-p)^x, \quad x = 0, 1, 2, \cdots, \text{where } 0 < p < 1.$$

(i) Find a complete and sufficient statistic of $p$;

(ii) Find the UMVUE of $p$.

**Q5**. Let $\mathbf{X} = \{X_1, X_2, \cdots, X_n\}$ be an independent random sample from the population $N\left(\frac{p}{q}, \sigma_1^2\right)$, and $\mathbf{Y} = \{Y_1, Y_2, \cdots, Y_n\}$ be an independent random sample from the population $N(q, \sigma_2^2)$. Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are independent, $q \neq 0$, $\sigma_1^2 > 0$, and $\sigma_2^2 > 0$.

(i) Show that $T_1 = \frac{1}{n}\sum_{i=1}^{n} X_i Y_i$ is an unbiased estimator of $p$;

(ii) Calculate $\mathrm{Var}(T_1)$ ;

(iii) Show that $T_2 = \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)$ is also an unbiased estimator of $p$;

(iv) Show that $T_2$ is a consistent estimator of $p$;

(v) When $p = 0$ and $q^2 = \frac{\sigma_2^2}{n}$, compare the efficiency of $T_1$ and $T_2$.

**Q6**. Let $X_1, \cdots, X_n$ be an independent random sample from $\mathrm{Poisson}(\lambda)$.

(i) Show that both $\overline{X}$ and $\frac{n}{n-1}S^2$ are unbiased estimators of $\lambda$;

(ii) Find a sufficient and complete statistic for $\lambda$.

(iii) Find the Fisher information about $\lambda$ contained in data $X_1, \cdots, X_n$.

(iv) Calculate the Cramer-Rao Lower Bound for estimation of $\lambda$;

(v) Which estimator ($\overline{X}$ and $\frac{n}{n-1}S^2$) should be preferred and why?

**Q7**. Let $X_1, X_2, \cdots X_n$ be an independent random sample from $\mathrm{N}(\theta, \theta^2)$ with $\theta \neq 0$.

(i) Find a sufficient statistic of $\theta$;

(Optional: Show that the sufficient statistic is not complete statistic of $\theta$.)

(ii) Derive the maximum likelihood estimator of $\theta$;

(iii) Derive the asymptotic distribution of the estimator found in part (ii).

**Q8**. Let $X_1, \cdots, X_n$ be an independent random sample, where $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, \cdots, n$. Suppose that $n = 2m$ $(m \geq 2)$ is an even number, the value of $\sigma_i$ (for $i = 1, 2, \cdots, n$) is known, and

$$\mu_1 = \mu_2 = \cdots = \mu_m = s_1, \quad \mu_{m+1} = \mu_{m+2} = \cdots = \mu_n = s_2.$$

(i) Show that $\widetilde{s}_1 = \dfrac{X_1 + 2X_2}{3}$ is unbiased estimator of $s_1$.

(ii) Find MLEs of $s_1$ and $s_2$.

Furthermore, we set $\sigma_i^2 = \dfrac{m}{i}$.

(iii) Will the MLE of $s_1$ be relative more efficient than $\widetilde{s}_1$ and why?

(iv) State the asymptotic distribution of both MLEs of $s_1$ and $s_2$.

(v) Show that both MLEs of $s_1$ and $s_2$ are consistent.

**Q9**. Let $X_1, X_2, \cdots, X_n$ be an independent random sample from a distribution, which has the density given by

$$f(x; \theta) = \frac{\theta}{x^{\theta+1}} \quad \text{for} \quad x \geq 1,$$

where $\theta > 0$ is an unknown parameter.

(i) Write down the likelihood function of $\theta$ based on $X_1, X_2, \cdots, X_n$.

(ii) Find a scalar sufficient statistic for $\theta$.

(iii) Find the Fisher information about $\theta$ contained in the data $X_1, X_2, \cdots, X_n$, i.e., $I_n(\theta)$.

(iv) Find the Cramer-Rao Lower Bound for estimation of $\theta$.

(v) What is the MLE of $\theta$?

(vi) State the asymptotic distribution of the MLE you found in (v).