STAT2602/3902 Mid-term Class Test (2:30pm - 3:30pm)

[Total: 50 marks]

1. Recall that if $X \sim \chi^2_\nu$, $M_X(t) = \dfrac{1}{(1-2t)^{\frac{\nu}{2}}}$ for $t < \frac{1}{2}$, where $M_X(t)$ is the moment generating function of X. Let X_1, X_2, X_3 be an independent random sample such that $X_i \sim \chi^2_i$ for $i = 1, 2, 3$.

   (i) Calculate the moment generating function of $Y = \sum_{i=1}^{3} X_i$. **[5 marks]**

   (ii) Specify the distribution of Y. **[2 marks]**

   (iii) Calculate $\mathrm{E}(Y^s)$ for $s = 1,\ 2$. **[8 marks]**

[Total: 15 marks]

2. Let $X_1, X_2, \cdots, X_n$ be an independent random sample from $N(0, \theta)$, where $\theta > 0$.

   (i) Find $\hat{\theta}_1$, the MLE of θ on the space $\Omega_1 = \{\theta : \theta > 0\}$. **[5 marks]**

   (ii) Show that $\hat{\theta}_1$ is an unbiased estimator of θ. **[5 marks]**

   (iii) Show that $\hat{\theta}_1$ is a consistent estimator of θ. **[5 marks]**

   (iv) Recall that $EX_i^4 = 3\theta^2$. Will $\hat{\theta}_1$ be the UMVUE of θ? Explain it. **[5 marks]**

   (v) Find $\hat{\theta}_2$, the MLE of θ on the space

   $$\Omega_2 = \{\theta : \theta > \theta_*\},$$

   where θ_* is a given finite positive constant. **[10 marks]**

   (vi) Show that $\hat{\theta}_2$ is a biased estimator of θ. **[5 marks]**

[Total: 35 marks]

A LIST OF STATISTICAL FORMULAE

1. $M_X(t) = \mathrm{E}(e^{tX})$. $\quad \mathrm{E}(X^r) = \left(\dfrac{\mathrm{d}^r}{\mathrm{d}t^r} M_X(t)\right)\Big|_{t=0}$.
2. For $X \sim \mathrm{Uniform}(a,b)$,
$$f(x) = \frac{x}{b-a}$$
3. For $X \sim \mathrm{N}(\mu, \sigma^2)$,
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).$$
4. $\overline{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$. $\quad S^2 = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$.
5. $\mathrm{Bias}(\hat{\theta}) = \mathrm{E}(\hat{\theta}) - \theta$. $\quad \mathrm{E}\left[(\hat{\theta}-\theta)^2\right] = \mathrm{Var}(\hat{\theta}) + \left[\mathrm{Bias}(\hat{\theta})\right]^2$.
6. $I(\theta) = \mathrm{E}\left[\left(\dfrac{\partial \ln f(X;\theta)}{\partial\theta}\right)^2\right] = \mathrm{E}\left[-\dfrac{\partial^2 \ln f(X;\theta)}{\partial\theta^2}\right]$. $\quad \mathrm{Var}(\hat{\theta}) \geq \dfrac{1}{nI(\theta)}$.
7. Factorization: $\mathbf{f}(x_1, \cdots, x_n; \theta) = g\big(T(x_1, \cdots, x_n); \theta\big) h(x_1, \cdots, x_n)$.
8. Expontial family: $f(x;\theta) = h(x)c(\theta)\exp\left(\displaystyle\sum_{i=1}^{s} p_i(\theta) t_i(x)\right)$.
9. CLT: $\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \approx N(0,1)$ for large n.
10. Normal population $\Longrightarrow \dfrac{nS^2}{\sigma^2} \sim \chi^2_{n-1}$, $\dfrac{\overline{X}-\mu}{S/\sqrt{n-1}} \sim t_{n-1}$.

********** **END OF PAPER** **********