Assignment 3

Course: STAT2602 **Due date: 17 November 2024 (11:59pm)**

Q1. Let $\{X_1, X_2, \cdots, X_n\}$ be an independent random sample from a Poisson(λ) distribution where λ is an unknown parameter. Suppose that the sample size n is large.

(a) Find a pivotal variable in terms of $\overline{X}$ and the parameter λ using normal approximation based on the Central Limit Theorem.

(b) Derive an approximate $(1-\alpha)$ confidence interval for λ using the pivotal variable found in part (a) and the Slutsky Theorem. You may further approximate the unknown parameter λ in the variance term by its estimator.

(c) Construct a 95% confidence interval for λ if $\overline{X} = 1.5$ and $S^2 = 4$ from a random sample of size $n = 36$.

Q2. Let$X_1, X_2, \cdots, X_n$ be an independent random sample from the uniform distribution $U(0, \theta)$.

(a) Construct a pivotal variable based on the sample maximum $X_{(n)}$.

(b) Use the result in part (a), or otherwise, to construct a $1-\alpha$ confidence interval for θ.

Q3. Let $X_1, X_2, \cdots, X_n$ be an independent random sample from $N(\mu, \sigma^2)$, where μ and σ^2 are unknown. Suppose that

$$1 - \alpha = \mathrm{P}\left(\frac{\sqrt{n}S}{\sqrt{b}} \le \sigma \le \frac{\sqrt{n}S}{\sqrt{a}}\right).$$

Hence, $\left[\frac{\sqrt{n}S}{\sqrt{b}}, \frac{\sqrt{n}S}{\sqrt{a}}\right]$ is the $1-\alpha$ confidence interval of σ, and it has the length

$$k = \frac{\sqrt{n}S}{\sqrt{a}} - \frac{\sqrt{n}S}{\sqrt{b}}.$$

Show that the values of a and b for σ of minimum length satisfy the following equations:

$$(i)\ G(b) - G(a) = 1 - \alpha;$$

$$(ii)\ a^{n/2}e^{-a/2} - b^{n/2}e^{-b/2} = 0,$$

where $G(\cdot)$ is the c.d.f. of χ^2_{n-1}.

Q4. A certain stimulus is to be tested for its effect on blood pressure. Twelve men have their blood pressure measured before and after the stimulus. The results are

Man	1	2	3	4	5	6	7	8	9	10	11	12
Before (X)	120	124	130	118	140	128	140	135	126	130	126	127
After (Y)	128	131	131	127	132	125	141	137	118	132	129	135

In this particular scenario, we define $W = Y - X$ as the improvement or increase in blood pressure due to the stimulus and assume that W_i's ($i = 1, 2, \cdots, 12$) are independent and distributed according to a $N(\mu_W, \sigma^2_W)$ distribution. Let $E(X) = \mu_X$ and $E(Y) = \mu_Y$.

(a) Construct a 95% confidence interval for $\mu_X - \mu_Y$, the average effect of the stimulus on blood pressure. Do you think the stimulus does have an effect on the blood pressure based on this 95% confidence interval? Briefly explain your answer.

(b) Construct a 95% confidence interval for σ_W.

Q5. Let X and Y be the times in days required for maturation of Guardiola seeds from narrow-leaved and broad-leaved parents, respectively. Assume that both X and Y are distributed as normal and they are independent to each other. A sample of size 13 yield $s^2_X = 9.88$ and the other sample of size 9 yield $s^2_Y = 4.08$.

(i) Find a 95% confidence interval estimate for σ^2_x.

(ii) Find a 95% confidence interval estimate for σ^2_x/σ^2_y.