☆ Moment Generating Function.

$\boxed{\text{property 2.1}}$ $\mu_r = EX^r$.

proof. (1) For a discrete random variable $X$,

$$M_X(t) = Ee^{tX} = \sum_x e^{tx} P(X=x) = \sum_x \sum_{r\geq 0}^{\infty} \frac{(tx)^r}{r!} P(X=x).$$

$$= \sum_{r\geq 0}^{\infty} \frac{t^r}{r!} \sum_x x^r P(X=x) = \sum_{r\geq 0}^{\infty} \frac{t^r}{r!} \mu_r$$

$$\boxed{e^{tx} = \sum_{r\geq 0}^{\infty} \frac{t^r x^r}{r!} \quad \langle \text{Taylor Expansion} \rangle}$$

(2). $$M_X'(t) = \sum_{r\geq 1}^{\infty} \frac{t^{r-1} \cdot r}{r!} \mu_r = \sum_{r\geq 1}^{\infty} \frac{t^{r-1}}{(r-1)!} \mu_r.$$

$$\Rightarrow M_X'(0) = \mu_1.$$

$$M_X''(t) = \sum_{r=2}^{\infty} \frac{r(r-1)t^{r-2}}{r!} \mu_r.$$

$$\Rightarrow M_X''(0) = \mu_2.$$

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(3). $M_X(t) = Ee^{tX}$. $\mu_r = EX^r$.

$$M_{aX+b}(t) = E e^{t(aX+b)}$$

$$= E[e^{taX} e^{tb}]$$

$$= e^{tb} E[e^{taX}]$$

$$= e^{tb} \cdot M_X(at).$$

☆ $\boxed{\text{Example 2.5.}}$ Gamma$(\alpha, \lambda)$

$$M_X(t) = Ee^{tX}$$

$$= \int_0^{\infty} e^{tx} \cdot \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx$$

$$= \int_0^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \left[\int_0^{\infty} \frac{(\lambda-t)^\alpha x^{\alpha-1} e^{-(\lambda-t)x}}{\Gamma(\alpha)} dx\right] \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha}$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^\alpha. \qquad (t<\lambda)$$

If $t \geq \lambda$, $M_X(t)$ does not exist.

Example 2.6 $X_i \sim \text{Gamma}(\alpha_i, \lambda)$

$Y = X_1 + X_2 + \cdots + X_n$.

$M_Y(t) = Ee^{tY} = \prod_{i=1}^{n} M_{X_i}(t)$

$M_{X_i}(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_i}$.

$M_Y(t) = \prod_{i=1}^{n} \left(\frac{\lambda}{\lambda-t}\right)^{\alpha_i} = \left(\frac{\lambda}{\lambda-t}\right)^{\sum_{i=1}^{n}\alpha_i}$.

So $Y \sim \text{Gamma}\left(\sum_{i=1}^{n} \alpha_i, \lambda\right)$

Example 2.7 $X_1, \ldots X_n \sim$ Bernoulli.

$M_{X_i}(t) = Ee^{tX_i} = e^{0 \cdot t} P(X_i = 0) + e^{1 \cdot t} P(X_i = 1)$

$= (1-p) + pe^t$.

So the MGF for $Y = X_1 + \cdots + X_n$ is

$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n}(1-p+pe^t) = (1-p+pe^t)^n$.

On the other hand, the MGF for $Z \sim \text{Binomial}(n, p)$ is

$M_Z(t) = Ee^{tZ} = \sum_{z=0}^{n} e^{tz} \cdot \binom{n}{z} p^z (1-p)^{n-z} = \sum_{z=0}^{n} \binom{n}{z} (pe^t)^z (1-p)^{n-z} = (pe^t + 1-p)^n$.

Example 2.8 $X_1, \ldots X_n \sim N(0, 1)$.

$M_X(t) = e^{\frac{t^2}{2}}$. $\quad Y = X_1^2 + \cdots + X_n^2$.

$M_Y(t) = Ee^{tY} = \prod_{i=1}^{n} M_{X_i^2}(t) = \left(\frac{1}{\sqrt{1-2t}}\right)^n$.

Note $M_{X_i^2}(t) = E(e^{tX_i^2}) = \int_{-\infty}^{\infty} e^{tx^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(1-2t)x^2}{2}} dx \qquad (y \equiv \sqrt{1-2t}\, x)$.

$= \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1-2t}} dy$.

$= \frac{1}{\sqrt{1-2t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \frac{1}{\sqrt{1-2t}}$.

The MGF of $\chi^2_n$ is $\left(\frac{1}{\sqrt{1-2t}}\right)^n$.

So $Y = X_1^2 + \cdots + X_n^2 \sim \chi^2_n$.

☆ $\boxed{\text{Theorem 2.1}}$ WLLN. $\bar{X} \to_p \mu$.

proof. Need to show $P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| > \varepsilon\right) \to 0$. $(n\to\infty)$

By property 1.9(iii), $P(h(X) \geq a) \leq E\left[\frac{h(X)}{a}\right]$.

we have. $P\left(\left|\frac{S_n}{n} - \mu\right|^2 > \varepsilon^2\right) \leq E\left[\left(\frac{S_n}{n} - \mu\right)^2\right]\frac{1}{\varepsilon^2}$.

$$= \frac{1}{n^2\varepsilon^2} E\left[(S_n - n\mu)^2\right]$$

$$= \frac{1}{n^2\varepsilon^2} E\left\{\left[\sum_{i=1}^{n}(X_i - \mu)\right]^2\right\}$$

$$= \frac{1}{n^2\varepsilon^2} E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}(X_i-\mu)(X_j-\mu)\right\}$$

$$= \frac{1}{n^2\varepsilon^2} E\left\{\sum_{i=1}^{n}(X_i-\mu)^2 + \sum_{i=1}^{n}\sum_{j\neq i}(X_i-\mu)(X_j-\mu)\right\}$$

$$= \frac{1}{n^2\varepsilon^2} E\left\{\sum_{i=1}^{n}(X_i-\mu)^2\right\} + E\left[\sum_{i=1}^{n}\sum_{j\neq i}(X_i-\mu)(X_j-\mu)\right]$$

<property 1.11(iv)>

$$= \frac{1}{n^2\varepsilon^2} E\left\{\sum_{i=1}^{n}(X_i-\mu)^2\right\} + 0.$$

$$= \frac{1}{n^2\varepsilon^2}\cdot n\sigma^2 = \frac{\sigma^2}{n\varepsilon^2} \to 0 \quad (\text{as } n\to\infty)$$

☆ $\boxed{\text{Example 2.9}}$

$X_i$ independent r.v.s. $\mu = E(X_1)$. $\sigma^2 = Var(X_1)$

$\mu_4 = E(X_i^4)$.

Show that $S^2 \to_p Var(X_1)$.

proof. $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n}\sum_{i=1}^{n}X_i^2 - (\bar{X})^2$

$$\frac{1}{n}\sum_{i=1}^{n}X_i^2 \to_p E(X_i^2) = \mu^2 + \sigma^2.$$

$$\bar{X} \to_p \mu.$$

$$S^2 \to_p \mu^2 + \sigma^2 - \mu^2 = \sigma^2. = Var(X_1)$$

☆ proof of CLT.

$\{X_i\}$ same $\begin{cases} \text{finite mean } E(X_i)=\mu. \\ \text{finite variance } Var(X_i)=\sigma^2. \end{cases}$

$E(\overline{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \mu.$

$Var(\overline{X}) = E[(\overline{X})^2] - [E(\overline{X})]^2 = \frac{\sigma^2}{n} = \frac{1}{n} Var(X_1)$

Need to show $Z_n = \frac{\sqrt{n}(\overline{X}-\mu)}{\sigma} \to_d N(0,1)$

Let $Y_i = \frac{X_i - \mu}{\sigma}$. Then $E(Y_i)=0$. $Var(Y_i=1)$

Taylor $M_{Y_i}(t) = M_{Y_i}(0) + t\, M_{Y_i}^{(1)}(0) + \frac{t^2}{2} M_{Y_i}^{(2)}(\varepsilon)$. $(0 \le \varepsilon \le t)$

Since $Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i$, then the MGF of $Z_n$ is

$$M_{Z_n}(t) = E e^{tZ_n} = E\left(e^{t \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i}\right) = E\left(e^{\frac{t}{\sqrt{n}} \cdot \sum_{i=1}^{n} Y_i}\right)$$

$$= \prod_{i=1}^{n} M_{Y_i}\left(\frac{t}{\sqrt{n}}\right) = \left[M_{Y_i}\left(\frac{t}{\sqrt{n}}\right)\right]^n$$

$$= \left[M_{Y_i}(0) + \frac{t}{\sqrt{n}} M_{Y_i}^{(1)}(0) + \frac{t^2}{2n} M_{Y_i}^{(2)}(\varepsilon)\right]^n$$

$$= \left[1 + \frac{t}{\sqrt{n}} \cdot E(Y_i) + \frac{t^2}{2n} M_{Y_i}^{(2)}(\varepsilon)\right]^n$$

$$= \left[1 + \frac{t^2}{2n} M_{Y_i}^{(2)}(\varepsilon)\right]^n. \quad 0 \le \varepsilon \le \frac{t}{\sqrt{n}}.$$

As $n \to \infty$, $\varepsilon \to 0$. and $M_{Y_i}^{(2)}(\varepsilon) \to M_{Y_i}^{(2)}(0)$
$= E(Y_i^2) = 1.$

So $\lim_{n\to\infty} M_{Z_n}(t) = \lim_{n\to\infty} \left[1 + \frac{t^2}{2n} \cdot M_{Y_i}(\varepsilon)\right]^n$

$$= \lim_{n\to\infty} \left[1 + \frac{t^2}{2n} \cdot 1\right]^n$$

$$= \exp\left(\frac{t^2}{2}\right)$$

↳ MGF of $N(0,1)$ random variable.

By Lemma 2.1. CLT follows.

$\begin{cases} M_X(t) = \sum_{r=0}^{\infty} \mu_r \cdot \left(\frac{t^r}{r!}\right) \\ \mu_r = M_X^{(r)}(0) = E X^r. \end{cases}$ property 2.1

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