Assignment 1

Course: STAT2602 **Due date: Oct-2-2024 (11:59pm)**

$\mathbf{Q1}$. Let the p.d.f. of X be defined by $f(x) = 2\left(\frac{1}{3}\right)^x$, $x = 1, 2, 3, \cdots$.

(i) Calculate the cumulative density function of X.

(ii) Find the moment generating function (m.g.f.) of X;

(iii) Calculate the mean and variance of X.

$\mathbf{Q2}$. Let $X_1, X_2, \cdots, X_n$ be an independent random sample from the Gamma$(3, \theta)$ distribution, where 3 is the shape and θ is the unknown rate. Let $Y = \sum_{i=1}^n X_i$.

(i) Find the distribution of Y;

(ii) Determine c so that $\mathrm{E}(cY) = \theta^{-1}$;

(iii) Find the m.g.f. of $3\theta Y + 1$.

$\mathbf{Q3}$. Suppose that X has moment generating function

$$M_X(t) = \frac{1}{4}e^{-3t} + \frac{1}{2} + \frac{1}{4}e^t.$$

(i) Find the mean and variance of X;

(ii) Find the probability density function of X. Use your expression for the p.d.f. to check your answers from part (i).

$\mathbf{Q4}$. A random sample of $n = 10$ people yields the following counts of the number of times they swam in the past month: $(0, 1, 2, 2, 3, 3, 4, 6, 6, 7)$.

(i) Write down the empirical distribution;

(ii) Use the empirical distribution to estimate $P(X \leq 4)$ and $P(4 < X < 7)$.

Q5. Suppose that $X = \xi_1 + \xi_2$, where $\xi_1 \sim N(\theta, 1)$, $\xi_2 \sim N(\lambda\theta, \lambda^2)$, and ξ_1 and ξ_2 are independent. Here, $\lambda \geq 1$ is a given constant, and θ is a unknown parameter.

(i) Calculate the moment generating function of X.

(ii) Calculate $\mathrm{E}(X^3)$.

(iii) Specify the distribution of X.

Q6. Let $\overline{X}$ denote the sample mean of an independent random sample of size 25 from the distribution whose p.d.f. is $f(x) = x^3/4$, $0 < x < 2$. Find the approximated value of $\mathrm{P}(1.2 \leq \overline{X} \leq 1.6)$.

Q7. Let $X_1, \cdots, X_{12}$ be a random sample from $U(0,1)$. Let

$$Y = X_{(12)} \quad \text{and} \quad Z = \left(\sum_{i=1}^{12} X_i\right) - 6.$$

(i) Determine the probability density function of Y.

(ii) Determine the moment generating function of Z.

(iii) Show that Z is approximately distributed as standard normal.

Q8. Let U and V be two independent Cauchy random variables with the common density function

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty,$$

then for any value of p from 0 to 1, $T = pU + (1-p)V$ is also distributed as Cauchy.

(i) Suppose that $X_1, X_2, \cdots$ be a sequence of independently identically distributed random variables from the Cauchy distribution. Show that $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ has the same distribution as X_1 and therefore does not converge in probability to anything.

(ii) Explain why the law of large number does not apply in this case.