1. (i) $M_X(t) = Ee^{tX} = \int_0^\infty e^{tx}\, 3e^{-3x}\, dx = \int_0^\infty 3e^{(t-3)x}\, dx$

$$= 3 \cdot \frac{e^{(t-3)x}}{t-3}\Big|_0^\infty = \frac{-3}{t-3} \quad (t<3)$$

(ii) $E(X) = M_X'(0) \qquad E(X^2) = M_X''(0)$

$$M_X(t) = \frac{3}{(t-3)^2}$$

$$M_X''(t) = 3 \times (-2) \times (t-3)^{-3}.$$

$$E(X) = \frac{3}{(0-3)^2} = \frac{1}{3}$$

$$E(X^2) = 3 \times (-2) \times (-3)^{-3} = \frac{2}{9}.$$

(iii) $E(e^{\frac{X}{2}}) = M_X(\tfrac{1}{2}) = \dfrac{-3}{\frac{1}{2}-3} = \dfrac{6}{5}.$

$$E(e^X) = M_X(1) = \frac{3}{2}$$

$E(e^{4X})$ doesn't exist!

(i) $L(\lambda, \beta) = f(X_1, \cdots, X_n, Y_1, \cdots, Y_n ; \lambda, \beta)$

$$= \left(\prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{X_i}}{X_i !} \right) \left(\prod_{i=1}^{n} \frac{e^{-\beta\lambda} (\beta\lambda)^{Y_i}}{Y_i !} \right)$$

$$= \frac{e^{-n(1+\beta)\lambda} \; \beta^{\Sigma Y_i} \; \lambda^{\Sigma X_i + \Sigma Y_i}}{\prod_{i=1}^{n}(X_i ! \, Y_i !)}$$

(ii) If $\beta = 1$, then

✡

$f(x_1, \cdots, x_n, y_1, \cdots, y_n ; \lambda)$

$$= \frac{1}{\prod_{i=1}^{n}(x_i ! \, y_i !)} \cdot e^{-n \cdot 2\lambda} \cdot \lambda^{\Sigma x_i + \Sigma y_i} .$$

By factorization theorem, $\Sigma X_i + \Sigma Y_i$ is a sufficient statistic for λ.

Or ✡ $(X_i, Y_i)_{i=1}^{n}$ can be seemed as iid random sample from a p.d.f. having the form:

$$f(x, y ; \lambda) = \frac{e^{-\lambda} \lambda^{x}}{x!} \cdot \frac{e^{-\lambda} \lambda^{y}}{y!} = \frac{e^{-2\lambda} \lambda^{x+y}}{x! \, y!}$$

This belongs to the exponential family with

$h(\underset{\sim}{x}) = \frac{1}{x! \, y!}$ $\quad c(\underset{\sim}{\theta}) = e^{-2\lambda}$.

$\exp\left(\sum_{i=1}^{s} P_i(\theta) t_i(\underset{\sim}{x}) \right) = \lambda^{x+y} = \exp\{ (x+y) \cdot \log \lambda \}$.

$s = 1$, $P_1(\theta) = \log \lambda$. $\quad t_1(\underset{\sim}{x}) = x + y$.

So $\sum_{i=1}^{n} t(\underset{\sim}{X}) = \sum_{i=1}^{n}(X_i+Y_i)$ is a sufficient statistic for λ.

(iii) If both λ and β are unknown,

☆ $f(x_1, \cdots, x_n, y_1, \cdots, y_n; \lambda, \beta)$

$$= \frac{1}{\prod_{i=1}^{n}(x_i!\, y_i!)} e^{-n(1+\beta)\lambda} \cdot \beta^{\Sigma y_i} \lambda^{\Sigma x_i + \Sigma y_i}$$

By factorization theorem, $T = (\Sigma X_i, \Sigma Y_i)$ is sufficient for (λ, β).

☆ Or use the property of exponential family.

$$f(x, y; \lambda, \beta) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!} \frac{e^{-\beta\lambda}(\beta\lambda)^{y}}{y!}$$

with $h(\underset{\sim}{x}) = \frac{1}{x!\, y!}$, $c(\underset{\sim}{\theta}) = e^{-\lambda} \cdot e^{-\beta\lambda}$,

$$\beta^{y}\lambda^{x+y} = \exp\left(\sum_{i=1}^{s} p_i(\underset{\sim}{\theta})\, t_i(\underset{\sim}{x})\right) = \exp\left\{y \log\beta + (x+y)\log\lambda\right\}$$

$$\begin{cases} p_1(\underset{\sim}{\theta}) = \log\beta \\ t_1(\underset{\sim}{x}) = y \end{cases} \qquad \begin{cases} p_2(\underset{\sim}{\theta}) = \log\lambda \\ t_2(\underset{\sim}{x}) = x+y. \end{cases}$$

So $T(\underset{\sim}{X}) = \left(\sum_{i=1}^{n} t_1(\underset{\sim}{X}), \sum_{i=1}^{n} t_2(\underset{\sim}{X})\right) = \left(\sum_{i=1}^{n} Y_i, \sum_{i=1}^{n}(X_i+Y_i)\right)$

is sufficient for (λ, β).

Through an invertable transformation, $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right)$ is also suff.

(iv) Now $\beta = \lambda$.

(a) $$f(x_1, \text{—}, x_n, y_1, \text{—}, y_n ; \lambda) = \frac{e^{-n(1+\lambda)\lambda} \lambda^{\sum x_i + 2\sum y_i}}{\prod_{i=1}^{n} (x_i ! \, y_i !)}$$

So $\sum_{i=1}^{n} X_i + 2\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} (X_i + 2Y_i)$ is sufficient for λ.

(b) $$I_{2n}(\lambda) = E\left[-\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2}\right]$$

$$\ell(\lambda) = \log L(\lambda) = \log\left[\frac{e^{-n(1+\lambda)\lambda} \lambda^{\sum_{i=1}^{n} X_i + 2\sum_{i=1}^{n} Y_i}}{\prod_{i=1}^{n} (X_i ! \, Y_i !)}\right]$$

$$= -n(1+\lambda)\lambda + \left(\sum_{i=1}^{n} (X_i + 2Y_i)\right) \log \lambda - \log\left(\prod_{i=1}^{n} (X_i ! \, Y_i !)\right)$$

(#) $$\frac{\partial \ell(\lambda)}{\partial \lambda} = -n(1+\lambda) - n\lambda + \frac{T}{\lambda} \qquad \left(T = \sum_{i=1}^{n} (X_i + 2Y_i)\right)$$

$$\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2} = \cancel{\frac{n}{1+\lambda}} - n - n - \frac{T}{\lambda^2} = -2n - \frac{T}{\lambda^2}.$$

$$-E\left[\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2}\right] = 2n + \frac{E(T)}{\lambda^2} = 2n + \frac{1}{\lambda^2}\left(n\lambda + 2n\lambda^2\right) = n\left(4 + \frac{1}{\lambda}\right)$$

$$\left(\begin{aligned} & E(T) = E\left(\sum_{i=1}^{n} (X_i + 2Y_i)\right) = n\lambda + 2n\lambda^2 \\ & E(X) = \lambda; \quad E(Y) = \beta\lambda = \lambda^2 \end{aligned}\right)$$

☆ Or $I(\lambda) = -E\left[\frac{\partial^2 \ln f(\underset{\sim}{x};\underset{\sim}{\theta})}{\partial \theta^2} \right]$

$$f(\underset{\sim}{x};\theta) = f(x, y; \lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \frac{e^{-\lambda^2}(\lambda^2)^y}{y!}$$

$$= \frac{e^{-(\lambda+\lambda^2)} \lambda^{x+2y}}{x!\, y!}$$

$$\ln f(x, y; \lambda) = -(\lambda+\lambda^2) + (x+2y)\ln\lambda - \ln(x!\, y!)$$

$$\frac{\partial \ln f(x, y; \lambda)}{\partial \lambda} = -(1+2\lambda) + \frac{x+2y}{\lambda}$$

$$\frac{\partial^2 \ln f(x, y; \lambda)}{\partial \lambda^2} = -2 - \frac{x+2y}{\lambda^2}$$

$$I(\lambda) = -E\left[\frac{\partial^2 \ln f(\underset{\sim}{x};\lambda)}{\partial \lambda^2} \right] = -E\left[-2 - \frac{X+2Y}{\lambda^2} \right]$$

$$= 2 + \frac{1}{\lambda^2} E(X+2Y)$$

$$= 2 + \frac{1}{\lambda^2}\cdot(\lambda + 2\lambda^2)$$

$$= 4 + \frac{1}{\lambda^2}$$

So $I_{2n}(\lambda) = \left(4 + \frac{1}{\lambda^2}\right) n$.

(iv) (c) According to equation (#) in Page 4,

$$\frac{\partial l(\lambda)}{\partial\lambda} = -n(1+\lambda) - n\lambda + \frac{T}{\lambda} = 0.$$

$$\Rightarrow \quad 2n\lambda^2 + n\lambda - T = 0. \quad \text{w.t.} \quad T = \sum_{i=1}^{n}(X_i + 2Y_i)$$

$$\hat{\lambda} = \frac{-n + \sqrt{n^2 + 4\times 2n\times T}}{2\times 2n}$$

$$= \frac{-n + \sqrt{n^2 + 8nT}}{4n}$$

$$= \sqrt{\frac{1}{16} + \frac{T}{2n}} - \frac{1}{4}.$$

(d) $\hat{\lambda} \sim N\left(\lambda, \frac{1}{I_{2n}(\lambda)}\right)$

Mean : λ

Varience : $\frac{1}{I_{2n}(\lambda)} = \frac{1}{n(4+\frac{1}{\lambda})}$

(e) $\hat{\lambda} = \sqrt{\frac{1}{16} + \frac{T}{2n}} - \frac{1}{4} = \sqrt{\frac{1}{16} + \frac{40+2\times 10}{2\times 20}} - \frac{1}{4}$

$= 1$

Varience : $\frac{1}{I_{2n}(\hat{\lambda})} = \frac{1}{20\times(4+1)} = \frac{1}{100}$

Standard error : $\frac{1}{10}$.