# Insertion Language Models: Sequence Generation with Arbitrary-Position Insertions

Dhruvesh Patel<sup>⋄</sup> Aishwarya Sahoo<sup>⋄</sup> Avinash Amballa<sup>⋄</sup> Tahira Naseem<sup>⋄</sup> Tim G. J. Rudner<sup>♠</sup> Andrew McCallum<sup>⋄</sup>

 $^{\Diamond}$ UMass Amherst  $^{\heartsuit}$ IBM Research  $^{ullet}$ New York University

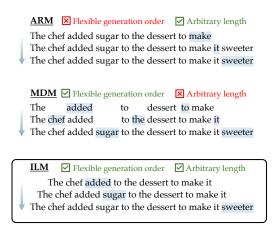
#### **Abstract**

Autoregressive models (ARMs), which predict subsequent tokens one-by-one "from left to right," have achieved significant success across a wide range of sequence generation tasks. However, they struggle to accurately represent sequences that require satisfying sophisticated constraints or whose sequential dependencies are better addressed by out-of-order generation. Masked Diffusion Models (MDMs) address some of these limitations, but the process of unmasking multiple tokens simultaneously in MDMs can introduce incoherences, and MDMs cannot handle arbitrary infilling constraints when the number of tokens to be filled in is not known in advance. In this work, we introduce Insertion Language Models (ILMs), which learn to insert tokens at arbitrary positions in a sequence—that is, they select jointly both the position and the vocabulary element to be inserted. By inserting tokens one at a time, ILMs can represent strong dependencies between tokens, and their ability to generate sequences in arbitrary order allows them to accurately model sequences where token dependencies do not follow a left-to-right sequential structure. To train ILMs, we propose a tailored network parameterization and use a simple denoising objective. Our empirical evaluation demonstrates that ILMs outperform both ARMs and MDMs on common planning tasks. Furthermore, we show that ILMs outperform MDMs and perform on par with ARMs in an unconditional text generation task while offering greater flexibility than MDMs in arbitrary-length text infilling. The code is available at https://dhruveshp.com/projects/ilm.

## 1 Introduction

Autoregressive models (ARMs), which predict subsequent tokens one-by-one in a "left-to-right" fashion, have achieved significant success in modeling natural language [Bro+20; GDJ+24]. Their simplicity makes them easy to train and has enabled a rapid increase in model sizes [Kap+20]. However, ARMs have several fundamental limitations. For example, they have fallen short on tasks that require complex reasoning and long-horizon planning [Bub+23; Val+24; Dzi+23], and they struggle to accurately model sequences that require satisfying sophisticated constraints [Sun+23]. Recently, Masked Diffusion Models (MDMs) have been shown to perform on par with ARMs while overcoming some of their limitations [Ye+25; Sah+24; LME24; Nie+24; Nie+25]. Although MDMs address some of the limitations of ARMs, departing from strictly left-to-right generation introduces new challenges. First, unmasking multiple tokens simultaneously during generation can violate token dependencies. For example, in the sentence "The chef added <mask> to the dessert to make it <mask>." if both the <mask> tokens are filled simultaneously, it can lead to a sentence that does not make sense, for example, "The chef added sugar to the dessert to make it healthier." However, if the tokens are filled sequentially, more appropriate sentences are generated, for example, "The chef added sugar to the dessert to make it sweeter." or "The chef added berries to the dessert to make it healthier." But then generating one token per forward pass is quite slow for MDMs. Second, reliance on the number of masked tokens in the input reduces a model's usefulness when performing arbitrary infilling. For example, when presented with the sentence "The conference, < mask> was postponed." the model cannot generate "The conference, originally planned for March, was postponed." as the input has only one mask.

To overcome these limitations, we revisit the idea of insertion based sequence generation [Ste+19; Rui+20] in the context of general language modeling, and introduce Insertion Language Models (ILMs), which use a simple denoising objective that involves dropping some tokens from the input sequence and learning to predict the missing tokens sequentially, one at a time. Unfortunately, estimates of the naive infilling denoising objective can have extremely high variance, which in turn can make training infeasible. To address this issue and allow efficient training, we introduce an approximate denoising training objective and a tailored parameterization of the denoising network. The key difference between ILMs and MDMs is that in ILMs, the dropped tokens are completely removed from the input sequence and are generated one at a time in reverse, whereas in MDMs, the dropped tokens are replaced by a <mask> token.



**Figure 1:** ARMs (top) generate variable-length sequences in a fixed left-to-right order. MDMs (middle) can add tokens in arbitrary order but require a fixed number of tokens to be masked. ILMs (bottom) generate sequences of arbitrary lengths in arbitrary order by inserting tokens.

In our empirical evaluation, using the synthetic task of generating paths on a star graph [BN24], we find that ILMs can consistently generate the correct path even when ARMs and MDMs struggle—especially when the paths have variable length. We also find that ILMs outperform ARMs and MDMs on the difficult constraint satisfaction task of solving Zebra Puzzles [Sha+24]. By training ILMs on small text corpora such as LM1B and TinyStories, we find that ILMs perform slightly better than MDMs on an unconditional text generation task (measured using generative perplexity under Llama 3.2, and Prometheus LLM judge) and are competitive with ARMs. We also demonstrate the effectiveness of ILMs on infilling arbitrary length sequences in a zero-shot manner. To summarize, our main contributions are as follows:

- 1. We introduce Insertion Language Models (ILMs), which learn to insert tokens at arbitrary positions in a sequence and are able to handle strong dependencies between tokens.
- 2. We present a neural network parameterization and a simple denoising objective that enable the training of ILMs.
- 3. We conduct an empirical evaluation of the proposed method and find that ILMs outperform autoregressive and masked diffusion models on common planning tasks and are competitive with ARMs and MDMs on text generation tasks while offering greater flexibility on arbitrary-length text infilling compared to MDMs.

## 2 Preliminaries

**Notation.** Capital letters are used to denote random variables (e.g. X) and the corresponding lowercase letters are used to denote their values (e.g. x). Boldface is reserved for non-scalars (vectors, matrices, etc.). Double square brackets are used to denote the set of natural numbers up to a specific number, that is,  $[n] = \{1, 2, \ldots, n\}$ . The components of a non-scalar quantity are denoted using superscripts and subscript time index of a stochastic processes whenever applicable.

#### 2.1 Masked Diffusion Models

Let V denote the token vocabulary, a finite set, and  $p_{\text{data}}$  be probability mass function on the set of sequences  $V^L$ . Assume that there is an arbitrary and fixed ordering on set V, using which we can use  $e_x$  to denote the indicator vector that is one at the index of token x and zero otherwise. Furthermore, assume that the set V contains a special token, whose probability under  $v_{\text{data}}$  is  $v_{\text{data}}$  is  $v_{\text{data}}$  and  $v_{\text{data}}$  is  $v_{\text{data}}$ 

token denoted as m. The training objective for MDMs [Shi+24; Sah+24] can be written as the data expectation (i.e.,  $x_0 \sim p_{\text{data}}$ ) of the following loss:

$$\mathcal{L}_{\theta}(\boldsymbol{x}_0) = \underset{\boldsymbol{x}_t \sim q_{t|0}(\cdot|\boldsymbol{x}_0)}{\mathbb{E}} \left[ \int_0^1 \frac{\alpha_t'}{1 - \alpha_t} \sum_{i=1}^L \delta(x_t^i, m) \log[\mu_{\theta}^{\text{mdlm}}(\boldsymbol{x}_t, t)]_{x_0^i}^i dt \right],$$

where

$$q_{t|0}(\boldsymbol{x}_t \mid \boldsymbol{x}_0) = \prod_{i=1}^{L} \operatorname{Cat} \left( \alpha_t \boldsymbol{e}_{x_0^i} + (1 - \alpha_t) \boldsymbol{e}_m \right) \tag{1}$$

is the transition probability of the noising process, and  $\mu_{\theta}^{\text{mdlm}}: \mathbb{V}^L \times [0,1] \to \left(\Delta^{|\mathbb{V}|-1}\right)^L$  is the learned parametric denoiser that takes in the current noisy sequence and produces a categorical probability distribution over the vocabulary at each sequence position. Here  $\Delta^{|\mathbb{V}|-1}$  denotes a categorical probability distribution over  $\mathbb{V}$ , and  $[\mu_{\theta}^{\text{mdlm}}(\boldsymbol{x}_t,t)]_j^i$  denotes the probability of j-th token from the vocabulary at i-th sequence position. Typically, the noising function  $\alpha_t$  is a monotonically decreasing function defined on the interval [0,1] such that  $\alpha_0=1$  (no noise) and  $\alpha_1=0$  (highest noise).

**Limitations of MDMs.** During inference, at time step t, with step size s-t, a subset of tokens is unmasked uniformly at random with probability  $P(i) \propto \frac{\alpha_s - \alpha_t}{1 - \alpha_t} \delta(x_t^i, m)$ , with their values sampled from  $x_t^i \sim [\mu_\theta^{\text{mdlm}}(\boldsymbol{x}_t, t)]^i$ . This inference procedure has two shortcomings:

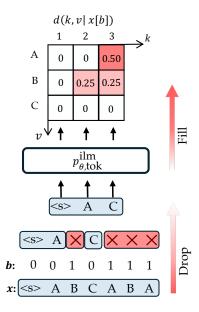
- 1. When the step size s-t is large, many tokens are unmasked simultaneously, which could result in incoherent outputs due to violation of sequential dependencies .
- 2. Since the number of masks between any two unmasked tokens is fixed, the inference has no flexibility in terms of infilling length.

In the next section, we describe our proposed Insertion Language Model (ILM) that tries to address the limitations mentioned above.

# 3 Insertion Language Model

ILM generates sequences of arbitrary lengths in arbitrary order by inserting tokens, one-at-a-time, that is, at each generation step, it predicts an output token along with a position in the existing sequence where the new token is to be inserted. The model can also decide to stop at any step, deeming the sequence to be complete. ILM's ability to predict the insertion position obviates the need for place-holder mask tokens, and thus avoids the rigid fixed-length constraint imposed by the MDMs. Moreover, this also allows the model to pick the positions for generation in any order escaping the pitfalls of left-to-right generation as in ARMs. Figure 1 depicts the key difference between ILMs, MDMs and ARMs using example generation trajectories.

An ILM can be viewed as a denoising model whose noising process drops tokens as opposed to replacing them with mask tokens. Training such a denoiser requires marginalization over possible trajectories leading to the original sequence, which can be done using the Monte Carlo sampling and learning to reverse a single step of the noising process. However, that introduces high variance in the loss estimates (see Appendix A.4 for more details). To avoid this issue, we use a biased training objective that makes



**Figure 2: ILM Training.** x is a training sequence, x[b] is a subsequence obtained after dropping tokens. d is the target insertion distribution, computed by counting the number of times each token appears in x between the  $i_k$ -th and  $i_{k+1}$ -th positions.

direct use of all the dropped tokens in the original sequence in a single gradient step. Specifically, for a position between any two tokens in the partially predicted sequence, instead of estimating the token probabilities by marginalizing over all generation trajectories, we train the model to predict the normalized counts of each vocabulary item appearing between any two tokens, in the original sequence.

Our training objective is a sum of two components that are optimized simultaneously. First, the token insertion component  $\mathcal{L}^{\mathrm{ilm}}_{\mathrm{tok}}(\theta; \boldsymbol{x})$ . Second, a binary decision component  $\mathcal{L}^{\mathrm{ilm}}_{\mathrm{stop}}(\theta; \boldsymbol{x})$ , that decides when to stop generation and in turn governs the length of the sequence. Formally, let  $\mathbb{B}_{L,n}$  be the set of bit vectors of length L with exactly n ones, and let  $\boldsymbol{x}[\boldsymbol{b}]$  be the sequence obtained after l

### Algorithm 1 ILM training

**Require:** Input example x of length L

- 1: Sample  $n \sim U[\![L]\!]$
- 2: Sample  $\boldsymbol{b} \sim q_{n|L}$
- 3: Compute  $d(k, v; \boldsymbol{x}, \boldsymbol{b})$
- 4:  $\mathcal{L}(\theta; \boldsymbol{x}) \leftarrow \mathcal{L}_{tok}(\theta; \boldsymbol{x}) + \mathcal{L}_{stop}(\theta; \boldsymbol{x})$
- 5: Update  $\theta$  using gradient descent

corresponding to the ones in b from x (c.f. Figure 2 bottom). Let  $p_{\theta,\text{tok}}(k, v \mid x[b])$  be the learned insertion probability of inserting token v between positions k and k+1, which is learned using

$$\mathcal{L}_{\text{tok}}^{\text{ilm}}(\theta; \boldsymbol{x}) = - \underset{n \sim U[\![L]\!]}{\mathbb{E}} \underset{\boldsymbol{b} \sim q_{n|L}}{\mathbb{E}} \left[ \frac{1}{n} \sum_{k \in [\![L-n]\!]} c_{i_k, i_{k+1}}(v; \boldsymbol{x}) \log p_{\theta, \text{tok}}^{\text{ilm}}(k, v \mid \boldsymbol{x}[\boldsymbol{b}]) \right], \tag{2}$$

where  $i_1,...,i_{L-n}$  are the indices in  ${\boldsymbol x}$  of the visible tokens after dropping tokens according to  ${\boldsymbol b}$ ,  $U[\![L]\!]$  is the uniform distribution over  $\{1,...,L\}$ ,  $q_{n|L}({\boldsymbol b})=1/\binom{L}{n}$  is the probability of selecting a bit vector of length L with n ones, and  $c_{i_k,i_{k+1}}(v;{\boldsymbol x})=\sum_{j=i_k}^{i_{k+1}-1}\delta({\boldsymbol x}^j,v)$  is the number of times token v appears in  ${\boldsymbol x}$  between the  $i_k$ -th and  $i_{k+1}$ -th positions. Note that  $d(k,v;{\boldsymbol x},{\boldsymbol b})\coloneqq c_{i_k,i_{k+1}}(v;{\boldsymbol x})/n$  (with n being the total number of tokens dropped), when summed over k and v gives 1. Therefore, we call it the target insertion distribution, which is usually quite sparse.

The second loss component is for learning a binary classifier  $p_{\theta,\text{stop}}(S \mid \boldsymbol{x}[\boldsymbol{b}])$ , where S is binary random variable, which takes a partially noised sequence of tokens and predicts whether the sequence is complete (S = 1) or not.

$$\mathcal{L}_{\text{stop}}^{\text{ilm}}(\theta; \boldsymbol{x}) = - \underset{n \sim U[\![L]\!]}{\mathbb{E}} \underset{\boldsymbol{b} \sim q_{n|L}}{\mathbb{E}} \left[ \delta(\boldsymbol{b}, \boldsymbol{0}) \log p_{\theta, \text{stop}}^{\text{ilm}}(1 \mid \boldsymbol{x}[\boldsymbol{b}]) + (1 - \delta(\boldsymbol{b}, \boldsymbol{0})) \log p_{\theta, \text{stop}}^{\text{ilm}}(0 \mid \boldsymbol{x}[\boldsymbol{b}]) \right], \quad (3)$$

where **0** is the vector of all zeros. The overall training loss is the sum of the token insertion loss and the stopping loss. The stopping classifier and the denoiser share the transformer backbone and are trained simultaneously (see Section 3.1 for more details). The overall training procedure for ILM, shown in Algorithm 1, resembles that of MDMs, one extra step of computing the target insertion distribution (highlighted in bold).<sup>1</sup>

During inference, ILM inserts one token at a time as shown in Algorithm 2.<sup>2</sup> For step 4 in the algorithm, we can sample from the joint, or perform two-step sampling  $k' \sim p_{\theta}^{ilm}(k \mid \boldsymbol{x}[\boldsymbol{b}])$  followed by  $v' \sim p_{\theta}^{ilm}(v \mid \boldsymbol{x}[\boldsymbol{b}], k')$ , where the latter approach allows us to use either top-k sampling or nucleus sampling [Hol+20] for each step separately.

## **Algorithm 2** One step of ILM prediction

**Require:** Current sequence x=(v,u), where v is the out-of-order sequence of tokens, and u is their corresponding real positions relative to one another, stopping threshold  $\tau$ 

```
1: if p_{\theta, \text{stop}}^{ilm}(1 \mid \boldsymbol{x}) > \tau then

2: return \boldsymbol{x}

3: end if

4: k', v' \sim p_{\theta, \text{tok}}^{ilm}(\cdot \mid \boldsymbol{x})

5: \boldsymbol{v}' \leftarrow \text{concat}(\boldsymbol{v}, v')

6: for i = 1 to len(\boldsymbol{u}) do

7: if \boldsymbol{u}[i] > k' then

8: \boldsymbol{u}[i] \leftarrow \boldsymbol{u}[i] + 1

9: end if

10: end for

11: \boldsymbol{u}' \leftarrow \text{concat}(\boldsymbol{u}, k' + 1)

12: return \boldsymbol{x}' = (\boldsymbol{v}', \boldsymbol{u}')
```

 $<sup>^{1}</sup>$ Unlike in MDM training, where the mask is usually sampled on the GPU, we sample b and compute d in the data pipeline on the CPU.

<sup>&</sup>lt;sup>2</sup>The procedure can be implemented using tensor operations that can be performed on mini-batches.

## 3.1 Parameterization

We parameterize  $p_{\theta}$  using insertion logits computed using a standard transformer as follows Let  $f_{\theta}^{\text{dec}}: \mathbb{V}^n \to \mathbb{R}^{n \times d}$  denote a transformer backbone, that is, a stack of transformer layers but without the final unembedding/linear layer. For each position  $i \in [n]$  the corresponding output of the transformer backbone  $f_{\theta}^{\text{dec}}(\boldsymbol{x})_i \in \mathbb{R}^d$  is passed through the unembedding layer  $f_{\theta}^{\text{ins}}: \mathbb{R}^d \to \mathbb{R}^{|\mathbb{V}|}$  to get the insertion logits for each position in the sequence. In other words

$$s_{\theta}(k, v \mid \boldsymbol{x}[\boldsymbol{b}]) = f_{\theta}^{\text{ins}} \left( f_{\theta}^{\text{dec}}(\boldsymbol{x}[\boldsymbol{b}])_{k} \right)_{v},$$
 (4)

which represents the unnormalized log probability (logit) for inserting token v between k and k+1 positions in the sequence x[b]. Finally, the join distribution over all possible insertions is given by

$$p_{\theta}^{\text{ilm}}(i_k, v \mid \boldsymbol{x}[\boldsymbol{b}]) = \frac{\exp(s_{\theta}(i_k, v \mid \boldsymbol{x}[\boldsymbol{b}]))}{\sum_{k=1}^{L-n} \sum_{v' \in \mathbb{V}} \exp(s_{\theta}(k, v' \mid \boldsymbol{x}[\boldsymbol{b}]))}.$$
 (5)

The stopping probability is predicted using the output from a special  $\langle \text{stp} \rangle$  that is always placed at the beginning of the input sequence. Therefore the input shown in Figure 2 looks like  $x[b] = \langle \text{stp} \rangle$   $\langle \text{sp} \rangle$  A C.

## 4 Related Work

Diffusion models [Soh+15; HJA20; SE19] have emerged as a powerful alternative to ARMs for sequence generation tasks that require planning and need to follow constraints. Masked Diffusion Models (MDMs) have been shown to scale competitively to ARMs while addressing some of its key shortcomings [Aus+21; Cam+22; LME24; Sah+24; Shi+24]. However, as discussed in Section 2, due to the use of fixed length mask tokens, and simultaneous unmasking, these models, without additional inference time tricks, tend to generate incoherent sequences. To address this, Gong et al. [Gon+24] propose to use a greedy strategy to select the tokens to unmask, Zheng et al. [Zhe+24] generalizes it to top-k sampling strategy, while Campbell et al. [Cam+24] utilizes a flow-based formulation to introduce helpful stochasticity on top of the greedy sampling process.

All these approaches, rely on inference time techniques to elicit better samples. Ye et al. [Ye+25] modify the MDM training objective by introducing an adaptive token-wise weight that helps the model identify the critical parts of the sequence. This objective, however, is only shown to work for synthetic tasks. Departing from this line of work, we propose a new parameterization and training objective. The MDMs are closely related to order-agnostic sequence models [Yan+20; Hoo+21]. The key difference between MDMs and order-agnostic models is that unlike MDMs, which can denoise the entire sequence in one go, order-agnostic models only generate one token at a time in an arbitrary order. Our model also generates the sequence by inserting tokens at arbitrary positions but is allowed to pick the position to insert the token.

The ability to insert tokens allow ILMs to perform infilling more naturally compared to ARMs. There has been only a handful of works that focus on the task of arbitrary length infilling using ARMs, most of which require specialized fine-tuning. Bavarian et al. [Bav+22] introduces fill-in-the-middle training objective where ARMs are trained to take prefix><suffix> as the left-context and is required to generate the <middle> part such that <prefix><middle><suffix> is a meaningful natural language sequence. While this approach enjoys the benefit of adapting an existing pre-trained ARM, its applicability is quite limited because the model is not capable of performing arbitrary infilling, for example, filling two blanks at separate places in the sequence. Gong et al. [Gon+24] also proposes a method to adapt pre-trained ARMs to masked denoising models. However, once adapted, the model has the same limitations as MDMs. Please refer to Appendix A.1 for an extended discussion.

# 5 Empirical Evaluation

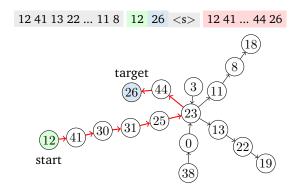
To highlight the key differences between ILMs, MDMs and ARMs, we consider two planning tasks: a generalized version of the synthetic planning task on star shape graphs introduced in Bachmann and Nagarajan [BN24] and Zebra Puzzles [Sha+24]. To demonstrate the effectiveness of ILM beyond synthetic planning task, we also perform unconditional text generation and infilling, for which we train the model on two language modeling datasets with different characteristics: (1) The One Billion Word Benchmark (LM1B) and (2) TinyStories [EL23]. For all our experiments, we use a transformer architecture with rotary positions encoding (RoPE) for ILMs and ARMs [Su+23]. For MDMs, we use the DDiT architecture identical to the one used in [Sah+24; LME24]. The DDiT is based on the DiT architecture that inserts adaptive layer-norm (AdaLN) in the RoPE based transformer to condition on the time variable [PX23]. Since AdaLN has trainable parameters, MDMs with the same hyperparameters as ILMs have slightly more trainable parameters.

## 5.1 Planning Tasks

## 5.1.1 Star Graphs

To highlight the key characteristics of the three models, we consider the task of generating the path from a starting node to a target node on star shaped graphs [BN24]. As shown in Figure 3, a star graph is a directed graph with one junction node. We create three versions of the task. Star<sub>easy</sub> only contains symmetric graphs wherein the start node is always the junction node, all paths go out from the junction, and are of equal length.

Star<sub>medium</sub> and Star<sub>hard</sub> contain asymmetric graphs with variable arm lengths, that is, graphs where the start node is not the junction node, there are incoming as well as outgoing edges from the junction node, and most importantly, the arm lengths can be different for each arm of the graph. The easy, medium and hard datasets have graphs with degree 3, 2, 5, respectively, and maximum path length of 5, 6, 12, respectively. We provide an overview of all parameters of the star graphs datasets in Table 4 (Appendix A.2.1). Each graph is presented to the model as a string of edges (expressed as node-pairs) in a random order as shown at the top of Figure 3, where the model needs to predict the path from the start node (green) to the target node (blue). All three models are trained for 50k steps with a learning rate of 1e-4 and batch size of 64. We provide an overview of all hyperparameters in Appendix A.2.1. For Star<sub>small</sub>, the optimal autoregressive order of generating the solution is in reverse (target to start) because that makes the dependencies trivial and deterministic. As expected, an ARM trained to predict the path in reverse order gets 100% accuracy on Star<sub>easy</sub> as shown in the first row of Table 1. However, it struggles to generate the path in the original left-to-right order (second row) as it requires an implicit lookahead. Since both the MDM and the ILM can generate out-of-order, they get 100% accuracy on Star<sub>easy</sub>. But the MDM struggles when the lengths of the arms start varying



**Figure 3:** Given the edges of a directed star graph (expressed as a sequence of connected node pairs in a random order), and the start and the target node, the goal is to predict the path from the start to the target node.

Table 1: Performance	(in terms of accuracy)	on the star gran	h nlanning task
Table 1. I chormance	(III terris or accuracy)	on the star grap	n prammig task.

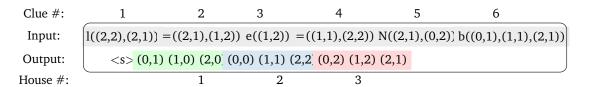
Model	Stare	asy	Star <sub>me</sub>	dium	Star <sub>h</sub> ;	ard	Zebra
	Sequence Acc.	Token Acc.	Sequence Acc.	Token Acc.	Sequence Acc.	Token Acc.	Sequence Acc.
ARMO	100.0	100.0	-	-	-	-	91.2
ARM	32.3	81.7	75.0	81.4	23.0	43.2	81.2
MDM	100.0	100.0	36.5	90.6	21.0	54.9	82.6
IT	35.2	98.2	22.1	80.9	17.5	79.9	-
ILM	100.0	100.0	100.0	100.0	99.1	99.7	90.0

with its sequence level accuracy (seq.) dropping to 36 and 21 on Star<sub>medium</sub> and Star<sub>hard</sub>, respectively. ILM continues to perform well on all three task variants, thereby highlighting its ability to generate out-of-order while maintaining flexibility in handling variable lengths. Some example generation trajectories for ILM are shown in Figure 7 (Appendix A.3.2), where it can be seen that the model tends to start the generation from both ends, leaving the most challenging edges, that is, the junction to latter steps.

#### 5.1.2 Zebra Puzzles

Zebra Puzzles are well-known logic puzzles that have been used to benchmark the performance of constraint satisfaction systems [Zeb25]. The are many variants of Zebra Puzzles, with different sizes and complexity. We use the version introduced in Shah et al. [Sha+24], wherein each puzzle is characterized by a tuple (m,n) where m represents the number of *entities* and n denotes the number of *attributes* associated with each entity. Given some constraints (clues) on the placement of the entity-attribute pairs, the goal is the place each entity-attribute pair in one of the *houses* such that all the constraints are satisfied.

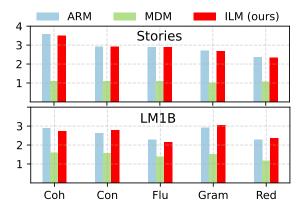
Each constraint consists of a *relationship*, and an entity-attribute pair, tuple or triple, for unary, binary, and ternary relationships, respectively. There are 7 types of relationships: = (same house), != (different house), 1 (left of), L (immediate left), N (neighbor), e, (ends) and b (between). Figure 4 shows an example of a (3,3)-zebra puzzle with 3 entities, 3 attributes, 3 houses, and 6 clues involving the relationships = and 1, N, e and b. For the ease of comparison, we use the same setup as well as the same dataset as Shah et al. [Sha+24]. We train a 42M parameter transformer model with 8 layers and 8 attention heads with hidden size of 576 with rotary position encoding. The order of solving the constraints plays an important role in the overall performance of the model [Sha+24]. Therefore, to demonstrate the usefulness of out-of-order generation, we train the model on output strings that present the solution in an arbitrary but fixed order that is sorted by house and entity as shown in Figure 4. As seen in the last column of Table 1, the ILM model obtains sequence accuracy of 90% outperforming both the MDM and the ARM, and it even gets close to the performance achieved by the ARM trained on oracle solver decomposed sequence order [Sha+24].



**Figure 4:** The box contains a compact string representation of a zebra puzzle and its solution. The input is a sequence of constraints in arbitrary order. The solution is a sequence of house, entity, attribute triples, sorted by house number. The complete input output string for this example is given in the Appendix A.2.2.

**Table 2:** Evaluation of unconditional generation quality using per-token NLL under Llama 3.2 3B. The rows with the dataset names contain the NLL and entropy of the examples in the training data.

NLLv     Ent₄     len       Stories     1.65     4.19     205       ARM     2.11     4.06     201       MDM     2.54     4.55     985       ILM (Ours)     2.14     3.76     119       LM1B     3.71     3.08     28       ARM     3.94     3.12     30       MDM     4.81     3.70     85       ILM (Ours)     4.67     2.80     21				
ARM 2.11 4.06 201 MDM 2.54 4.55 985 ILM (Ours) 2.14 3.76 119 LM1B 3.71 3.08 28 ARM 3.94 3.12 30 MDM 4.81 3.70 85		NLL▼	Ent₄	len
MDM       2.54       4.55       985         ILM (ours)       2.14       3.76       119         LM1B       3.71       3.08       28         ARM       3.94       3.12       30         MDM       4.81       3.70       85	Stories	1.65	4.19	205
ILM (Ours)         2.14         3.76         119           LM1B         3.71         3.08         28           ARM         3.94         3.12         30           MDM         4.81         3.70         85	ARM	2.11	4.06	201
LM1B 3.71 3.08 28  ARM 3.94 3.12 30  MDM 4.81 3.70 85	MDM	2.54	4.55	985
ARM 3.94 3.12 30 MDM 4.81 3.70 85	ILM (Ours)	<u>2.14</u>	3.76	119
MDM 4.81 3.70 85	LM1B	3.71	3.08	28
112111 1101 0170 00	ARM	3.94	3.12	30
ILM (Ours) <u>4.67</u> 2.80 21	MDM	4.81	3.70	85
	ILM (Ours)	<u>4.67</u>	2.80	21



**Figure 5:** Evaluation of unconditional generation quality using Prometheus 2 7B model as the LLM Judge. Legend: Coh.=coherence, Con.=consistency, Flu.=fluency, Gram.=grammaticality, Red.=non-redundancy.

## 5.2 Language Modeling

In order to test the ability of the model to generate short and long text sequences, we pick two small-sized pre-training datasets with different characteristics: (1) The One Billion Word Benchmark [Che+13] (LM1B), and (2) a mixture of TinyStories [EL23] and ROCStories [Mos+16] (Stories). The LM1B dataset, which has been used to benchmark the performance of MDMs [Aus+21; Sah+24], consists of short sequences (up to 2-3 sentences) of text from the news domain with a large vocabulary. The TinyStories dataset, on the other hand, consists of 2.1 million stories that 3-4 year old children can understand. In order to increase the diversity of the stories, we also include the ROCStories dataset, which contains 5-sentences stories based on common sense and world knowledge. The combined dataset contains 2.2 million stories in the training set. For both the datasets, we train ILMs, MDMs and ARMs of the same size and architecture (RoPE-based transformer as described above), with ~85M non-embedding trainable parameters (the MDM has slightly more due to the addition of AdaLN layers).3 We use bert-base-uncased tokenizer for both the datasets and pad each example to 128 tokens for LM1B and 1024 tokens for TinyStories. All the models are trained with an effective batch size of 512, up to 1M steps on LM1B and 60K steps on TinyStories using AdamW [LH19] with a constant learning rate of  $10^{-4}$ . All the models were trained on 4 A100 (40GB and 80GB) GPUs.

## 5.3 Unconditional Generation

For sampling unconditional sequences, we use the tau-leaping sampler for the MDM [Sah+24; Cam+22] as described in Section 2, and nucleus sampling with p=1 for ARM. For ILM, we sample according to Algorithm 2 using two-step ancestral sampling where we first sample the position of insertion using top-k sampling  $k \sim p_{\theta}^{\text{ilm}}(k \mid \boldsymbol{x}[\boldsymbol{b}])$  followed by  $v \sim p_{\theta}^{\text{ilm}}(v \mid \boldsymbol{x}[\boldsymbol{b}], k')$  using nucleus sampling. Our primary metric for evaluating unconditional generation is the per-token negative log-likelihood (NLL) under a large language model and the entropy of the generated text, defined as

$$NLL(\boldsymbol{x}) = -\frac{1}{|\boldsymbol{x}|} \sum_{i=1}^{|\boldsymbol{x}|} \log p^{LLM}(x_i | \boldsymbol{x}_{1:i-1}) \quad \text{and} \quad Entropy(\boldsymbol{x}) = -\sum_{j=1}^{|\mathbb{V}|} c_j \log c_j, \quad (6)$$

where  $p^{\mathrm{LLM}}(x_i|x_{1:i-1})$  is the probability of the i-th token in the sequence x given the previous i-1, and  $c_j = \sum_{i=1}^{|x|} \delta(x_i, v_i)/|x|$  is the relative frequency of the i-th vocabulary item  $v_i$  in the sequence x. We use Llama-3.2-3B [GDJ+24] for computing the NLL. Since NLL and entropy may not be sufficient to judge the overall quality of the generated text, we also use Prometheus 2 7B [Kim+24] as the LLM

<sup>&</sup>lt;sup>3</sup>Our MDM implementation is based on Sahoo et al. [Sah+24] and it uses log-linear noise schedule.

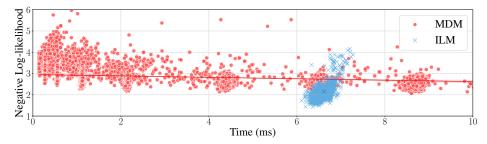


Figure 6: Per-token generation time vs. NLL for the MDM and the ILM trained on the stories dataset.

Judge to evaluate the quality of the generated text on various linguistic and readability aspects, of which the most important ones are coherence and grammatically (see Appendix A.2.5 for the details of the evaluation prompt).

As seen in Table 2, both the MDM and the ILM obtain worse NLL compared to the ARM trained for the same number of steps, which could be attributed to the training token efficiency and scaling laws for different model types [Nie+24]. However, the ILM performs better than the MDM on both datasets in terms of NLL. In terms of token diversity measured using entropy, the ILM is on the lower side compared to the MDM and the ARM, but still fairly close to the dataset entropy given in the rows with the dataset names. In general, we found that the MDM produces longer sequences than both the ARM, and the ILM, as well as the mean sequence lengths in the training data. We found that to be the main reason for the high entropy (even higher than dataset entropy) of sequences produced by the MDM. The ILM provides linguistically balanced generation similar to ARM and consistently outperforms the MDM, which struggles particularly with coherence and consistency. Notably, MDM's performance deteriorates in the Stories dataset as generation length increases, resulting in more disjointed narratives (see Appendix A.2.6 for the examples). One more difference between the ILM and the MDM is the number of input tokens in each forward pass during inference—for the MDM it stays fixed at maximum allowed sequence length from the beginning, while for the ILM it starts from zero and goes up to the maximum sequence length. Figure 6 shows the impact of per-token generation time on the generation quality measure using per-token NLL under Llama 3.2 3B. For the MDM, we collect samples with varying number of sampling steps (128, 256, 512, and 1024). The generation quality for the MDM (red) improves as per-token generation time/the number of sampling steps is increased, but stays below that of the ILM (blue).

**Table 3:**  $\Delta$ NLL and  $\Delta$ Entropy denote the percentage change in per-token negative log-likelihood and entropy after infilling, respectively, where subscript gt and inp denote the change with respect to the ground truth and input (sample with the segments removed), respectively.

	$\Delta NLL_{gt}$ $ ightharpoonup$	$\Delta \mathrm{Entropy}_{\mathrm{gt}}$	$\Delta NLL_{inp} {\color{red} \bullet}$	$\Delta \text{Entropy}_{\text{inp}}$
TinyStories single-segment				
MDM	+14.36	-3.82	+3.63	+1.48
ILM (Ours)	+12.27	-4.18	+1.79	+0.04
LM1B single-segment				
MDM	+25.31	-0.05	-0.49	+4.56
ILM (Ours)	+20.47	-1.71	-3.57	+2.64
LM1B multi-segment				
MDM	+25.64	+0.15	-6.02	+3.97
ILM (Ours)	+23.52	-0.79	-7.93	+2.98

# 5.4 Infilling

We construct an infilling evaluation dataset by taking 3500 test sequences from the LM1B dataset. The LM1B single-segment dataset is obtained by removing one contiguous segment of tokens from each example, and the multi-segment version is obtained by removing two or more contiguous segments of tokens from each example. Similarly, we construct TinyStories single-segment infilling evaluation dataset by removing the middle sentence from each example from the first 3.3k examples of the TinyStories test dataset.

Since we are evaluating the ability of the pre-trained models to perform arbitrary infilling, we only compare MDMs and ILMs as ARMs are not capable of performing infilling without specialized training. We again employ NLL under Llama-3.2-3B and entropy as the evaluation metrics. However, since we are evaluating the quality of the infilled text, instead of using raw metrics, we use the percentage change  $\Delta M_{\rm ref} = 100 * (M(x) - M(x^{\rm ref}))/M(x^{\rm ref})$ , where M is either NLL or Entropy, and  $x^{\rm ref}$  is either the input with missing segments (inp) or the ground truth text (gt). Note that when the input text ( $x^{\rm inp}$ ) is provided to the evaluator LLM, the tokens that belong to the removed segment are completely removed. Therefore, we expect to observe a drop in NLL with respect to the input text and an increase with respect to the ground truth text. As shown in Table 3, we see trends similar to the unconditional generation results. Specifically, the ILM outperforms the MDM on all three evaluation datasets in terms of NLL. On the TinyStories evaluation set, both the MDM and the ILM show an increase in NLL with respect to the input text. However, upon manual inspection, we find that the stories in the dataset are often fairly simple, and removing a sentence from the middle may not change the overall all meaning too much, and hence the NLL for the corresponding input sequences with missing segments is already fairly low.

## 6 Discussion

In this paper, we introduced Insertion Language Models (ILMs), which learn to insert tokens at arbitrary positions in a sequence and address limitations of both autoregressive models (ARMs) and Masked Diffusion Models (MDMs). By inserting one token at a time while allowing arbitrary generation order, ILMs can handle strong token dependencies effectively without being constrained to left-to-right generation. We enable successful training by using a tailored network parameterization and a denoising objective that approximates a distribution over denoising steps. Our experiments demonstrated that ILMs outperform ARMs and MDMs on planning tasks, perform on par with ARMs on unconditional text generation, and excel at arbitrary-length text infilling compared to MDMs.

Limitations and Future Work. While ILMs show promising results on synthetic planning tasks as well as in language modeling, they also have some limitations. First, on text data, ILMs still perform slightly worse than ARMs trained for the same number of gradient steps. Similar to MDMs, and unlike ARMs, ILMs also do not allow caching of hidden states and can therefore be slower at inference compared to ARMs with hidden state caching. Addressing these two aspects and scaling ILMs to larger datasets are important directions for future work.

# **Reproducibility Statement**

In the empirical evaluation section (Section 5) and the appendix (Appendix A) provide details about the network datasets, architecture, and training hyperparameters.

The code is available at <a href="https://dhruveshp.com/projects/ilm">https://dhruveshp.com/projects/ilm</a>.

## References

- [Aus+21] J. Austin, D. D. Johnson, J. Ho, D. Tarlow, and R. van den Berg. "Structured Denoising Diffusion Models in Discrete State-Spaces". In: *Advances in Neural Information Processing Systems*. 2021 (cit. on pp. 5, 8, 14).
- [BN24] G. Bachmann and V. Nagarajan. "The pitfalls of next-token prediction". In: *Proceedings of the 41st International Conference on Machine Learning*. 2024 (cit. on pp. 2, 6, 15).
- [Bav+22] M. Bavarian, H. Jun, N. Tezak, J. Schulman, C. McLeavey, J. Tworek, and M. Chen. *Efficient training of language models to fill in the middle*. 2022. arXiv: 2207.14255 [cs.CL] (cit. on pp. 5, 15).
- [Bro+20] T. B. Brown, B. Mann, N. Ryder, M. Subbiah, J. Kaplan, P. Dhariwal, A. Neelakantan, P. Shyam, G. Sastry, A. Askell, S. Agarwal, A. Herbert-Voss, G. Krueger, T. Henighan, R. Child, A. Ramesh, D. M. Ziegler, J. Wu, C. Winter, C. Hesse, M. Chen, E. Sigler, M. Litwin, S. Gray, B. Chess, J. Clark, C. Berner, S. McCandlish, A. Radford, I. Sutskever, and D. Amodei. Language models are few-shot learners. 2020. arXiv: 2005.14165 [cs.CL] (cit. on p. 1).
- [Bub+23] S. Bubeck, V. Chandrasekaran, R. Eldan, J. Gehrke, E. Horvitz, E. Kamar, P. Lee, Y. T. Lee, Y. Li, S. Lundberg, et al. "Sparks of artificial general intelligence: early experiments with gpt-4". arXiv preprint arXiv:2303.12712 (2023) (cit. on p. 1).
- [Cam+22] A. Campbell, J. Benton, V. D. Bortoli, T. Rainforth, G. Deligiannidis, and A. Doucet. "A Continuous Time Framework for Discrete Denoising Models". en. In: 2022 (cit. on pp. 5, 8, 14).
- [Cam+23] A. Campbell, W. Harvey, C. Weilbach, V. De Bortoli, T. Rainforth, and A. Doucet. "Trans-Dimensional Generative Modeling via Jump Diffusion Models". en. *Advances in Neural Information Processing Systems* (2023) (cit. on p. 14).
- [Cam+24] A. Campbell, J. Yim, R. Barzilay, T. Rainforth, and T. Jaakkola. "Generative Flows on Discrete State-Spaces: Enabling Multimodal Flows with Applications to Protein Co-Design". In: *Proceedings of the 41st International Conference on Machine Learning*. 2024 (cit. on pp. 5, 14).
- [Che+13] C. Chelba, T. Mikolov, M. Schuster, Q. Ge, T. Brants, P. Koehn, and T. Robinson. "One billion word benchmark for measuring progress in statistical language modeling". *arXiv* preprint arXiv:1312.3005 (2013) (cit. on p. 8).
- [Dzi+23] N. Dziri, X. Lu, M. Sclar, X. L. Li, L. Jiang, B. Y. Lin, S. Welleck, P. West, C. Bhagavatula, R. L. Bras, J. D. Hwang, S. Sanyal, X. Ren, A. Ettinger, Z. Harchaoui, and Y. Choi. "Faith and fate: limits of transformers on compositionality". In: *Thirty-seventh Conference on Neural Information Processing Systems*. 2023 (cit. on p. 1).
- [EL23] R. Eldan and Y. Li. *Tinystories: how small can language models be and still speak coherent english?* 2023. arXiv: 2305.07759 [cs.CL] (cit. on pp. 6, 8, 15).
- [Fry24] A. Frydenlund. "The mystery of the pathological path-star task for language models". In: *Proceedings of the 2024 Conference on Empirical Methods in Natural Language Processing*. 2024 (cit. on p. 15).
- [Fry25] A. Frydenlund. "Language models, graph searching, and supervision adulteration: when more supervision is less and how to make more more". In: *Proceedings of the 63rd Annual Meeting of the Association for Computational Linguistics (Volume 1: Long Papers)*. 2025 (cit. on p. 15).
- [Gha+19] M. Ghazvininejad, O. Levy, Y. Liu, and L. Zettlemoyer. "Mask-predict: parallel decoding of conditional masked language models". In: Proceedings of the 2019 Conference on Empirical Methods in Natural Language Processing and the 9th International Joint Conference on Natural Language Processing (EMNLP-IJCNLP). 2019 (cit. on p. 14).
- [Gon+24] S. Gong, S. Agarwal, Y. Zhang, J. Ye, L. Zheng, M. Li, C. An, P. Zhao, W. Bi, J. Han, H. Peng, and L. Kong. "Scaling Diffusion Language Models via Adaptation from Autoregressive Models". In: *The Thirteenth International Conference on Learning Representations*. 2024 (cit. on pp. 5, 14, 15).

- [GDJ+24] A. Grattafiori, A. Dubey, A. Jauhri, et al. *The llama 3 herd of models*. 2024. arXiv: 2407.21783 [cs.AI] (cit. on pp. 1, 8).
- [GWZ19] J. Gu, C. Wang, and J. Zhao. "Levenshtein Transformer". In: *Advances in Neural Information Processing Systems*. 2019 (cit. on p. 14).
- [HJA20] J. Ho, A. Jain, and P. Abbeel. "Denoising Diffusion Probabilistic Models". In: *Advances in Neural Information Processing Systems*. 2020 (cit. on pp. 5, 14, 23).
- [Hol+20] A. Holtzman, J. Buys, L. Du, M. Forbes, and Y. Choi. "The curious case of neural text degeneration". In: *International Conference on Learning Representations*. 2020 (cit. on p. 4).
- [Hoo+21] E. Hoogeboom, A. A. Gritsenko, J. Bastings, B. Poole, R. van den Berg, and T. Salimans. "Autoregressive Diffusion Models". In: *International Conference on Learning Representations*. 2021 (cit. on pp. 5, 14).
- [Kap+20] J. Kaplan, S. McCandlish, T. Henighan, T. B. Brown, B. Chess, R. Child, S. Gray, A. Radford, J. Wu, and D. Amodei. *Scaling laws for neural language models*. 2020. arXiv: 2001.08361 [cs.LG] (cit. on p. 1).
- [Kim+25] J. Kim, K. Shah, V. Kontonis, S. M. Kakade, and S. Chen. "Train for the Worst, Plan for the Best: Understanding Token Ordering in Masked Diffusions". en. In: 2025 (cit. on p. 15).
- [Kim+24] S. Kim, J. Suk, S. Longpre, B. Y. Lin, J. Shin, S. Welleck, G. Neubig, M. Lee, K. Lee, and M. Seo. *Prometheus 2: an open source language model specialized in evaluating other language models*. 2024. arXiv: 2405.01535 [cs.CL] (cit. on pp. 8, 16).
- [LH19] I. Loshchilov and F. Hutter. "Decoupled weight decay regularization". In: *International Conference on Learning Representations*. 2019 (cit. on p. 8).
- [LME24] A. Lou, C. Meng, and S. Ermon. "Discrete diffusion modeling by estimating the ratios of the data distribution". In: *Proceedings of the 41st international conference on machine learning*. 2024 (cit. on pp. 1, 5, 6, 14).
- [Mos+16] N. Mostafazadeh, N. Chambers, X. He, D. Parikh, D. Batra, L. Vanderwende, P. Kohli, and J. Allen. "A corpus and cloze evaluation for deeper understanding of commonsense stories". In: Proceedings of the 2016 Conference of the North American Chapter of the Association for Computational Linguistics: Human Language Technologies. 2016 (cit. on pp. 8, 15).
- [Nie+24] S. Nie, F. Zhu, C. Du, T. Pang, Q. Liu, G. Zeng, M. Lin, and C. Li. *Scaling up masked diffusion models on text*. 2024. arXiv: 2410.18514 [cs.AI] (cit. on pp. 1, 9).
- [Nie+25] S. Nie, F. Zhu, Z. You, X. Zhang, J. Ou, J. Hu, J. Zhou, Y. Lin, J.-R. Wen, and C. Li. *Large language diffusion models*. 2025. arXiv: 2502.09992 [cs.CL] (cit. on p. 1).
- [PX23] W. Peebles and S. Xie. "Scalable Diffusion Models with Transformers". en. In: 2023 (cit. on p. 6).
- [Rui+20] L. Ruis, M. Stern, J. Proskurnia, and W. Chan. *Insertion-deletion transformer*. 2020. arXiv: 2001.05540 [cs.LG] (cit. on pp. 2, 14).
- [Sah+24] S. S. Sahoo, M. Arriola, A. Gokaslan, E. M. Marroquin, A. M. Rush, Y. Schiff, J. T. Chiu, and V. Kuleshov. "Simple and Effective Masked Diffusion Language Models". en. In: 2024 (cit. on pp. 1, 3, 5, 6, 8, 14).
- [Sha+24] K. Shah, N. Dikkala, X. Wang, and R. Panigrahy. "Causal language modeling can elicit search and reasoning capabilities on logic puzzles". In: *The Thirty-eighth Annual Conference on Neural Information Processing Systems*. 2024 (cit. on pp. 2, 6, 7, 15).
- [Shi+24] J. Shi, K. Han, Z. Wang, A. Doucet, and M. Titsias. "Simplified and generalized masked diffusion for discrete data". In: *The Thirty-eighth Annual Conference on Neural Information Processing Systems*. 2024 (cit. on pp. 3, 5, 14).
- [Soh+15] J. Sohl-Dickstein, E. A. Weiss, N. Maheswaranathan, and S. Ganguli. *Deep Unsupervised Learning using Nonequilibrium Thermodynamics*. 2015. arXiv: 1503.03585 [cond-mat, q-bio, stat] (cit. on pp. 5, 14).
- [SE19] Y. Song and S. Ermon. "Generative modeling by estimating gradients of the data distribution". In: *Advances in Neural Information Processing Systems*. 2019 (cit. on pp. 5, 14).

- [Ste+19] M. Stern, W. Chan, J. Kiros, and J. Uszkoreit. *Insertion Transformer: Flexible Sequence Generation via Insertion Operations*. arXiv:1902.03249 [cs]. 2019 (cit. on pp. 2, 14, 19).
- [Su+23] J. Su, Y. Lu, S. Pan, A. Murtadha, B. Wen, and Y. Liu. *RoFormer: Enhanced Transformer with Rotary Position Embedding*. 2023. arXiv: 2104.09864 [cs] (cit. on p. 6).
- [Sun+23] J. Sun, Y. Tian, W. Zhou, N. Xu, Q. Hu, R. Gupta, J. Wieting, N. Peng, and X. Ma. "Evaluating large language models on controlled generation tasks". In: *Proceedings of the 2023 Conference on Empirical Methods in Natural Language Processing.* 2023 (cit. on p. 1).
- [Val+24] K. Valmeekam, M. Marquez, A. Olmo, S. Sreedharan, and S. Kambhampati. "Planbench: an extensible benchmark for evaluating large language models on planning and reasoning about change". *Advances in Neural Information Processing Systems* (2024) (cit. on p. 1).
- [Wel+19] S. Welleck, K. Brantley, H. D. Iii, and K. Cho. "Non-monotonic sequential text generation". In: *Proceedings of the 36th International Conference on Machine Learning*. 2019 (cit. on p. 14).
- [Yan+20] Z. Yang, Z. Dai, Y. Yang, J. Carbonell, R. Salakhutdinov, and Q. V. Le. *XLNet: Generalized Autoregressive Pretraining for Language Understanding*. arXiv:1906.08237 [cs]. 2020 (cit. on pp. 5, 14).
- [Ye+25] J. Ye, J. Gao, S. Gong, L. Zheng, X. Jiang, Z. Li, and L. Kong. "Beyond autoregression: discrete diffusion for complex reasoning and planning". In: *The Thirteenth International Conference on Learning Representations*. 2025 (cit. on pp. 1, 5, 14).
- [Zeb25] Zebra Puzzle. Zebra puzzle Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Zebra\_Puzzle&oldid=1278211825. [Online; accessed 25-March-2025]. 2025 (cit. on p. 7).
- [Zhe+24] L. Zheng, J. Yuan, L. Yu, and L. Kong. *A Reparameterized Discrete Diffusion Model for Text Generation*. 2024. arXiv: 2302.05737 [cs] (cit. on pp. 5, 14).

# A Appendix

## A.1 Extended Related Work

The exploration of non-autoregressive sequence generation can be traced back to early neural machine translation literature [Gha+19; Ste+19; Wel+19; GWZ19]. But the scaling story of the left-to-right AR LLMs inadvertently diminished the interest in the topic in subsequent years. The success of diffusion models [Soh+15; HJA20; SE19], however, has lead to a resurgence of interest in the topic, but now focusing on scaling in the context of language modeling as opposed to specific sequence-to-sequence tasks like machine translation. There is a vast amount of work on non-autoregressive sequence generation. Here we will try to cover the most relevant works.

MDMs Masked Diffusion Models (MDMs) have been shown to scale competitively to ARMs while addressing some of its key shortcomings [Aus+21; Cam+22; LME24; Sah+24; Shi+24] on tasks that require planning and following constraints. However, as discussed in Section 2, due to the use of fixed length mask tokens, and simultaneous unmasking, these models, without additional inference time tricks, can generate incoherent sequences. However, as discussed in Section 2, due to the use of fixed length mask tokens, and simultaneous unmasking, these models, without additional inference time tricks, can generate incoherent sequences. To address this, Gong et al. [Gon+24] propose to use a greedy strategy to select the tokens to unmask, Zheng et al. [Zhe+24] generalizes it to top-k sampling strategy, while Campbell et al. [Cam+24] utilizes a flow-based formulation to introduce helpful stochasticity on top of the greedy sampling process. All these approaches, rely on inference time techniques to elicit better samples. Ye et al. [Ye+25] modify the MDM training objective by introducing an adaptive token-wise weight that helps the model identify the critical parts of the sequence. This objective, however, is only shown to work for synthetic tasks. Departing from this line of work, we propose a new parameterization and training objective. The MDMs are closely related to order-agnostic sequence models [Yan+20; Hoo+21]. The key difference between MDMs and order-agnostic models is that unlike MDMs, which can denoise the entire sequence in one go, order-agnostic models only generate one token at a time in an arbitrary order. Our model also generates the sequence by inserting tokens at arbitrary positions but is allowed to pick the position to insert the token much like Trans-dimensional Jump Diffusion [Cam+23], however, unlike Campbell et al. [Cam+23], which is designed for continuous spaces (like videos), we work with discrete space of token sequences. Moreover, we take advantage of the simpler space to instantiate lower variance training objective, which allows us to scale the training to language modeling.

Other insertion-style models There have been several works in the machine translation and early language modeling literature that explore insertion-style models [GWZ19: Rui+20]. The Non-monotonic Sequential Text Generation (NMTG) [Wel+19] parameterizes an insertion policy. It uses a "learning to search" approach to generate text by inserting tokens to the left or right of the current tokens. While this approach is similar to the ILM, it is comparatively much slower to train due to the high variance of the RL objective. Moreover, the inference process is constrained to be a level-order traversal of a binary tree as opposed to an arbitrary order of insertion, as in the ILM. Due to these two reasons, NMTG is not easily scalable to larger language modeling corpora. The Insertion Transformer [Ste+19], by virtue of the insertion-based decoding procedure, shares several high-level similarities with the ILM. There are also a few differences, like in the token loss normalization and the decoder architecture. The most significant difference, however, is in the stopping criteria: unlike ILM, the IT does not have a specialized stopping classifier. It instead predicts a special EOS from all slots to decide whether to stop the generation or not. We demonstrate that this approach is unreliable and often overshoots or undershoots the target sequence (see Appendix A.3.1 for a detailed discussion). Stern et al. [Ste+19] also explores the possibility of inserting multiple tokens simultaneously using a fixed binary tree-based insertion scheme. However, we find that insertion of multiple tokens without errors requires context-dependent policy, and leave a detailed exploration of this aspect to future work.

Infilling The ability to insert tokens allow ILMs to perform infilling more naturally compared to ARMs. There has been only a handful of works that focus on the task of arbitrary length infilling using ARMs, most of which require specialized fine-tuning. Bavarian et al. [Bav+22] introduces fill-in-the-middle training objective where ARMs are trained to take prefix><suffix> as the left-context and is required to generate the <middle> part such that <prefix><middle><suffix> is a meaningful natural language sequence. While this approach enjoys the benefit of adapting an existing pre-trained ARM, its applicability is quite limited because the model is not capable of performing arbitrary infilling, for example, filling two blanks at separate places in the sequence. Gong et al. [Gon+24] also proposes a method to adapt pre-trained ARMs to masked denoising models. However, once adapted, the model has the same limitations as MDMs.

Shortcomings of left-to-right generation. There are several works that attempt to study the shortcomings of left-to-right sequence generation using controlled experiments on synthetic tasks [BN24; Fry24; Fry25]. Bachmann and Nagarajan [BN24] show that left-to-right generation using next-token prediction training paradigm has problems when there are some tokens that are much harder to predict than others. Frydenlund [Fry24] show that the star-graph task with fixed arm lengths can be solved using teacher-forcing but with modified input ordering where the edges in the input are not shuffled, making the task somewhat trivial. Frydenlund [Fry25] show that the pathological behaviour for next-token prediction paradigm on star-graph task is due to excessive supervision for "easy" prediction steps, i.e., the steps that follow the "hard" step of junction node. MDMs circumvent this issue of excessive supervision by trying to predict all the tokens simultaneously. This introduces, so called, task decomposition [Fry25; Kim+25]. In our work, we generalize the star-graph task to incorporate variable arm lengths, and show that while MDMs can induce task decomposition when the output sequence lengths are fixed, but struggle with variable sequence lengths.

# A.2 Experimental Details

## A.2.1 Star Graphs

All three models, the ILM, the MDM and the ARM, are RoPE-based transformers with 84M parameters with 12 attention heads and 12 layers with hidden size of 768.

**Table 4:** Different star graph datasets used in the experiments. All the datasets use asymmetric graphs, meaning the start and the goal nodes both are away from the junction, and the target path passed through the junction. VStar version additionally has variable arm lengths a in the same input star graph.

Name	Degree	min(a)	$\min(l)$	$\max(l)$	$ \mathbb{V} $	#Train	#Test
Star <sub>easy</sub>	3	1	5	5	20	50k	5k
Star <sub>medium</sub>	2	2	3	6	20	50k	5k
$Star_{hard}$	5	5	6	12	56	50k	5k

#### A.2.2 Zebra Puzzles

We use the dataset created by Shah et al. [Sha+24], which they make publicly available at zebra train and zebra test. The train dataset contains about 1.5 million puzzles and the test set contains about 100 thousand puzzles. Following the experimental setup in Shah et al. [Sha+24], we train for 500k steps after which the change in training loss is negligible. Table 5 shows an example input and output from the dataset.

## A.2.3 Language modeling: Story Generation

We combine the TinyStories [EL23] and ROCStories [Mos+16] datasets. The combined dataset contains almost 2.2 million stories (2,198,247) in the training set. We use randomly selected 3.3k

(m,n)	Inputs	Outputs		
(3,3)	left-of LHS c 2 2 RHS c 2 1 CLUE_END = LHS c 2 1 RHS c 1 2 CLUE_END ends LHS c 1 2 RHS CLUE_END = LHS c 1 1 RHS c 2 2 CLUE_END nbr LHS c 2 1 RHS c 0 2 CLUE_END inbetween LHS c 0 1 RHS c 1 1 c 2 1 CLUE_END	00110020001211121202012 2221		
Vocab:	Vocab: 0 1 2 3 4 5 nbr left-of inbetween immedate-left end != = CLIE FND RHS LHS			

**Table 5:** Example inputs and outputs for the zebra puzzles. Each example is a concatenation of the input and output strings. The strings are tokenized using space and the tokenizer uses a custom vocabulary as shown in the table. The output string is entity-house-attribute.

stories from the test split for performing infilling evaluation. The stories were generate using GPT-3.5 and GPT-4. TinyStories has longer sequences but a smaller vocabulary compared to LM1B.

### A.2.4 Language modeling: LM1B

We use a model with 85M parameters, consisting of 12 layers and 12 attention heads, trained with a learning rate of 0.0001 for 1M steps.

## A.2.5 LLM Evaluation using Prometheus-2

We use Prometheus-2 7B model and follow the evaluation protocol given in Kim et al. [Kim+24]. For evaluating natural language generation, we use metrics like: Coherence, Consistency, Fluency, Grammaticality, Non-Redundancy and Spelling Accuracy. We generate evaluation text using a sampling temperature of 0.0, a maximum token limit of 1k, and a top-p value of 0.9

## **LLM-As-Judge Evaluation Prompt:**

You are a fair judge assistant tasked with providing clear, objective feedback based on specific criteria, ensuring each assessment reflects the absolute standards set for performance.

## **Task Description:**

An unconditional generation to evaluate, and a score rubric representing an evaluation criteria are given.

- 1. Write a detailed feedback that assesses the quality of the generation strictly based on the given score rubric, not evaluating in general.
- 2. After writing a feedback, write a score that is an integer between 1 and 5. You should refer to the score rubric.
- 3. The output format should look as follows: "(write a feedback for criteria) [RESULT] (an integer number between 1 and 5)".
- 4. Please do not generate any other opening, closing, or explanations.

## **Generation to evaluate:**

{generation}

### **Score Rubrics:**

{rubrics}

#### Feedback:

Rubric Item	Rubric Text
Coherence	(Is the text coherent and logically organized?) Score of 1: Very incoherent. The generation lacks structure, has sudden jumps, and is difficult to follow. Score of 2: Somewhat incoherent. The generation has some semblance of structure, but has significant flaws in flow and organization. Score of 3: Neutral. The generation is decently organized, with minor issues in flow and structure. Score of 4: Mostly coherent. The generation is well-structured with very few minor coherence issues. Score of 5: Highly coherent. The generation is excellently organized, flows seamlessly, and builds information logically from start to end
Consistency	(Is the text consistent in terms of style, tone, and tense?) Score of 1: The text is inconsistent in style, tone, and tense, leading to confusion. Score of 2: The text shows occasional inconsistencies in style, tone, and tense. Score of 3: The text is mostly consistent in style, tone, and tense, with minor lapses. Score of 4: The text is consistent in style, tone, and tense, with rare inconsistencies. Score of 5: The text is highly consistent in style, tone, and tense throughout.
Fluency	(Is the text fluent and easy to read?) Score of 1: The text is disjointed and lacks fluency, making it hard to follow. Score of 2: The text has limited fluency with frequent awkward phrasing. Score of 3: The text is moderately fluent, with some awkward phrasing but generally easy to follow. Score of 4: The text is fluent with smooth transitions and rare awkward phrases. Score of 5: The text is highly fluent, with natural and smooth expression throughout.
Spelling Accuracy	(Does the text demonstrate correct spelling?) Score of 1: The text contains frequent spelling errors, making it difficult to understand. Score of 2: The text has multiple spelling errors that affect readability and clarity. Score of 3: The text has occasional spelling errors, but they do not significantly impact comprehension. Score of 4: The text is mostly free of spelling errors, with only rare mistakes that do not affect understanding. Score of 5: The text has perfect spelling accuracy, with no errors present.
Grammaticality	(Does the text demonstrate proper grammatical usage?) Score of 1: The text contains frequent grammatical errors, making it difficult to understand. Score of 2: The text shows occasional grammatical errors, which disrupt the flow and clarity of the text. Score of 3: The text generally adheres to grammatical rules, though minor errors are present. Score of 4: The text demonstrates good grammaticality with rare errors that do not affect comprehension. Score of 5: The text excels in grammatical usage, with clear and correct grammar throughout.
Non-Redundancy	(Does the text avoid unnecessary repetition?) Score of 1: The text is highly redundant, with excessive repetition of words, phrases, or ideas that make it difficult to read. Score of 2: The text contains noticeable redundancy, with multiple instances of unnecessary repetition that affect clarity. Score of 3: The text has some minor redundancy, but it does not significantly impact readability or meaning. Score of 4: The text is mostly nonredundant, with rare instances of repetition that do not affect clarity. Score of 5: The text is highly concise and avoids any unnecessary repetition, presenting ideas efficiently and effectively.

Table 6: Rubric for NLG Evaluation through LLM-As-Judge

# A.2.6 Unconditional Generation Examples

#### Dataset: Stories | Model: ARM | Overall Score: 1

illy loved ice - cream. she liked to eat ice - cream when it was cold and sweet. one day, she saw a big bowl of ice - cream on the table. she wanted to eat it all. she took a big spoon and scooped some ice - cream into her mouth. but the ice - cream was not good. it was icy and sticky and tasted bad. Illy spat it out and made a face. she felt sick and her mouth hurt. she started to cry. she ran to her mom and said, "nom, mom, i ate ice - cream. it was bad and cold and tasted good. the ice - cream fell on the floor and it made me sick. "her mom hugged her and said, "i'm sorry, lily. you didn't do the chore. you had ice - cream for dinner. it was not good. it made your mouth hurt and your tummy ache. you have to listen to me and do the answer. "lily nodded and said, "i'm sorry, mom. i wanted to eat ice - cream. but it was too bad. it made me sick and i ate something bad. can i have some water, please? "her mom smiled and said, "of course, sweetie. here you go. feel better. and guess what? i have a surprise for you. look! "she took a plate from the cabinet and said, "i scooped some ice - cream every day for you. it was still cold and fresh and useful. see? "she pointed to the plate. illy saw the ice - cream. she was happy and relieved. she said, "wow, mom, you made the ice - cream for me? it looks delicious. thank you, mom. can i have some ice - cream now? "her mom said, "yes, you can. but you have to throw it away first. and you have to give it to me. the timer is off. "lily did as her mom said. she threw away the ice - cream and said, "ok, mom. i will do it. i like ice - cream. but i won't eat ice - cream again. and i won't use the cold. it 's bad and i want to make you happy. "her mom said, "i'm proud of you, lily. you are a smart and sweet girl. you made me happy. but you also made me sad. the ice - cream does not make you well. it gives me energy and i want to enjoy it. it also gives me love and hugs and kisses. it 's good for me and for lily. it makes me happy too. do you want some water and milk now? "lily

#### Dataset: Stories | Model: ARM | Overall Score: 5

it was a magnificent night. jill decided to take a walk around the neighborhood, she saw a group of children playing in the park, they were having so much fun, they were all gossipling and laughing.

### Dataset: Stories | Model: MDM | Overall Score: 1

ben liked to help his mom with animals. he had cows and chickens and sheep, and sheep, and hay. he liked to feed him wooly with his glass and play with it. \* hello, sheep, ben. you are the best helper in the farm, " wooly cooed and wagged his tail. anna showed her his bowl of bread and gave her a small bowl. " i think so, ben, you can have some of his favorite. you can feed him his milk with him, " his mom said, sharing the bread with him. ben smiled and ate the bread carefully. the dog licked his face and wagged his tail. it was soft and friendly spot. it was not the petter, but he belongs to a cowy, but she lived nearby. "can we go to the farm with her? "ben asked, curious. "no, ben, spot belongs to the wild spot in his barn. he knows not to come back soon. he is just playing with us. she is not shy, but she is very nice. come on on, let's go play with her in the barn, " she said, ben nodded and went to the barn spot with his mom, he liked all the animals and plants, he opened the window and called his mom, "ben, you have to be quiet and gentle. you can break a hole easily. and you can pet the cow or moo, "she said. ben looked at oinky and tilted his head. he was afraid of oinky. he wanted some beef or carrots. he thought mom was lonely. "mom, i want to find out, "he said. "maybe they are not scary. maybe there are animals in the farm. "ben peeked inside. he hoped there were a toy, or a car, or a toy car. he saw ducks, frogs, and the farm. he looked around and saw a big furry animal with a hat and a coat. he thought, " maybe it is the cow or moo." moo looked at him with his eyes. he seemed friendly, like, "hello, cowy what are you doing here?" "doo, mooing, "ruo replied. sara looked surprised. she was surprised. she knew ben had gone to the sack of food. ben hadn 't seen the cow or the pig. he had never been able to eat them. they were very nice and friendly. please, mom, please, come and see, "he asked, begging sara to come out again. he reached for his mom to oinky, but his mom wasn't mad. she said, "no, ben, stop. he might be hungry, and it is too cold for you. come on, and let 's go home for lunch. you should not go to anything about him. " sara want to oinky afraid. he seemed nice and soft. she put a box next to her bed. she whispered, "maybe i can't touch him again." ben did not listen. he reached the cow and got up. he did not see a cut on his shirt and his tooth. and he behind him and s cold and hard, ouch! ben fell down, he landed on the floor and bumped into something, it hurt a lot, sara 's mom heard ben 's cry and ran to check on him, she saw ben on the floor looking sad. she ran to him and said, "i'm sorry, i'm sorry ben. she's not mad at you. can you see her now? her finger hurts? "ben said, "no, i'm not okay. she's just blood on her finger. i held her leg and said, "ow, mom. that's my cow.' "his mom said, "don't worry, ben. you saved me. you're not brave and strong, but, i' m lucky i tried to help you. but not. now come on. let's go home. you will be okay. "she did not. she knew they were going to the doctor. she took the bandage out of the sack and cut it seped. She gave it to sara and said, "here ben, i'm here you. i love you. i'm glad you like cowy, okay. when mom arrived, ben saw sara waiting for help. he told her they were sorry, but mom was still angry or embarrassed. she hugged her and said, "i'm so happy for you, ben. you should calm down and a good sister. you have a great mom. don't you feel to forgive him and me? "ben hugged mom and said, "thank you, mom. i forgive. "they both smiled. their mom was proud too. they were glad. they kissed ben and kissed him, they also said, " sara, and so is tom

#### Dataset: Stories | Model: MDM | Overall Score: 3.6

once upon a time, there was a brave monkey named timmy. timmy loved to climb up in the tree in the jungle. one day, timmy met a scary lion, the lion looked sad and lonely. timmy knew he had to help his friend and make him feel better. timmy decided to follow the lion back home. when the lion arrived at its den, the lion said, " we told you, we can still be friends. " timmy was so happy for being brave and said he you back to the lion said, " you ' re welcome. " timmy and the lion became the best of friends. the lion became a brave friend and they played together in the jungle every day.

## Dataset: Stories | Model: ILM | Overall Score: 1

once upon a time there was a box. it was a special box. one day it wanted to go somewhere, it asked if it was ok, so it started to move, and soon, the box was ready! it was so fun, the box danced and laughed and smiled, they were so happy that they stayed in the box forever.

#### Dataset: Stories | Model: ILM | Overall Score: 4

once upon a time, there was a little girl named lily. she was very curious about the world around her. one day, she decided to pack up her toys and go to the park. but as she was packing her things, she saw a big rock. she knew the rock was not safe, so she decided to leave the rock alone. when she got home, she told them about the rock. her family was very upset and told her it was not safe to play with rocks. from that day on, lily never played with anything else again. the end.

# Dataset: LM1B | Model: ARM | Overall Score: 1

for me, the life a doctor receives is what he is doing.

#### Dataset: LM1B | Model: ARM | Overall Score: 5

i think you will find a lot more talent than you may have

## Dataset: LM1B | Model: MDM | Overall Score: 1

hazex ga, pixi (ebookcinecon. com ) and 1) apply exclusive control over the world to theguardit and the inu digital tv device which allows viewers to view hd e2, with itv more than (instead of dvds using hd) 3: and thaw kept ". " the information if possible using digital's most erasable delivery configuration software. atusa vip technology could also facilitate the use of add - print anywhere while handling unique customer experiences. the content of exorult manage and / or the donetv is natural and feature top hits. pacelle also licenses all content and ommi

Dataset: LM1B | Model: MDM | Overall Score: 3.8

a third of four blackers headteachers report regular use of cannabis with alcohol levels, according to a study published in scientific paper.

Dataset: LM1B | Model: ILM | Overall Score: 1

at that moment, he could be the next great medical doctor, so he or she died.

Dataset: LM1B | Model: ILM | Overall Score: 5

there were no casualties or injuries in the violence.

# A.3 Additional Results and examples

#### A.3.1 Comparison with Insertion Transformer

The Insertion Transformer (IT) [Ste+19] differs from ILM in the following ways:

- 1. The IT is an encoder-decoder model, while the ILM is a decoder-only model.
- 2. The IT uses a specialized final layer on top of a transformer decoder, while the ILM uses a standard transformer decoder architecture.
- 3. The IT uses local averaging for token prediction loss (equation 14 in Stern et al. [Ste+19]), i.e., the denominator is the number of tokens in the ground truth for a particular slot, while we use the global average in Equation (2) wherein the numerator is a single sum of the negative log-likelihood corresponding to all missing tokens and the denominator is the total number of missing tokens in all the slots combined.
- 4. The IT does not have a specialized stopping classifier. It instead predicts a special EOS from all slots to decide whether to stop the generation or not.

Using (2) and (3) in our setting yields an informative ablation. Therefore, we implement a decoder-only Insertion Transformer using the same transformer architecture as the ILM but with the loss provided in Stern et al. [Ste+19]. As seen in the Table 1, the IT performs poorly compared to the ILM on the star graphs task. Upon qualitative inspection, we find that IT, which uses the EOS token instead of a dedicated stopping classifier like in ILM, consistently undershoots or overshoots the target sequence. Due to this, its sequence accuracy is substantially lower than token accuracy. Below, we present two examples from the validation set that illustrate the issue.

## Input:

10 17 15 4 19 6 17 1 4 12 1 16 4 19 9 10 8 5 0 8 6 3 7 4 4 9 12 0 7 5 <s>

Predicted Output:

7 4 12 0 8 5

Target Output:

7 4 4 12 12 0 0 8 8 5

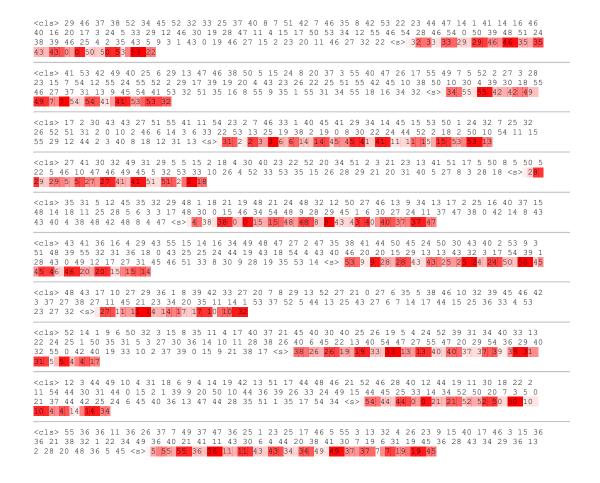
## Input:

14 11 9 6 7 9 11 19 19 16 17 3 3 11 16 5 11 1 11 7 1 10 11 10 <s>

Predicted Output:

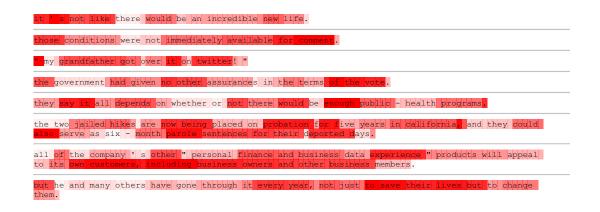
11 1 1 10 11 11 1 1 10 Target Output: 11 1 1 10

#### A.3.2 Star Graphs



**Figure 7:** Generation trajectories for ILM on 10 test examples from the Star<sub>hard</sub> task. Lighter color indicates that the token was generated earlier than the ones with the darker color.

#### A.3.3 Language modeling: Unconditional Generation



**Figure 8:** Unconditional generation trajectories for ILM trained on LM1B dataset. Lighter color indicates that the token was generated earlier than the ones with the darker color.

## A.3.4 Conditional Language modeling: Infilling Task

#### **Dataset: Stories**

Input once upon a time, in a big forest, there lived a rhinoceros named roxy. roxy loved to climb. it before, it was shiny and cold, and she wanted to climb it, roxy tried to climb the icy hill, but it was very slippery, she tried again and again, but she kept falling down, roxy \_\_\_ billy saw that roxy was sad and asked, " why are you sad, roxy? " roxy told billy about the icy hill and how she couldn ' t climb it. billy said, " i have an idea! let 's find some big leaves to put under your feet, they will help you climb the icy hill. " roxy and billy looked for big leaves and found some, roxy put the leaves under her feet and tried to climb the icy hill again. this time, roxy didn't slip. she climbed and climbed until she reached the top of the icy hill. roxy was so happy! on, roxy and billy were the best of friends, and they climbed and played together all the time. and roxy learned that with a little help from a friend, she could climb anything. MDM once upon a time, in a big forest, there lived a rhinoceros named roxy. roxy loved to climb. she had a big ball in her mouth. one day, roxy found an icy hill. she had never seen anything like it before. it was shiny and cold, and she wanted to climb it. roxy tried to climb the icy hill, but it was very slippery. she tried again and again, but she kept falling down. roxy was sad. she started to feel distant. suddenly, she felt a tap on her ankle. it was billy. billy saw that roxy was sad and asked, " why are you sad, roxy? " roxy told billy about the icy hill and how she couldn't climb it. billy said, "i have an ideal let's find some big leaves to put under your feet. they will help you climb the icy hill." roxy and billy looked for big leaves and found some. roxy put the leaves under her feet and tried to climb the icy hill again. this time, roxy didn't slip. she climbed and climbed until she reached the top of the icy hill. roxy was so happy! she hugged billy, and thanked him for helping her, from that day on, roxy and billy were the best of friends, and they climbed and played together all the time. and roxy learned that with a little help from a friend, she could climb anything once upon a time, in a big forest, there lived a rhinoceros named roxy. roxy loved to climb. one day, roxy found an icy hill. she had never seen anything like it before. it was shiny and cold, and she wanted to climb it. roxy tried to climb the icy hill, but it was very slippery. She tried again and again, but she kept falling down. roxy was sad. her friend billy saw that roxy was sad and asked, Why are you sad, roxy? Foxy told billy about the icy hill and how she couldn't climb it. billy said, I have an idea! let's find some big leaves to put under your feet, they will help you climb the icy hill, roxy and billy looked for big leaves and found some, roxy put the leaves under her feet and tried to climb the icy hill again. this time, roxy didn 't slip. she climbed and climbed until she reached the top of the icy hill. roxy was so happy! from that day on, roxy and billy were the best of friends, and they climbed and played together all the time. and roxy learned that with a little help from a friend, she could climb anything

**Dataset: Stories** 

#### Input

one day, a small cat named tim was walking in the park. the sun was very bright and the wind was harsh. \_\_\_\_\_\_ " do you want to play? \$\frac{1}{2}\$ am looked at tim and said, \$\tilde{y}\$ sell t's play! they played all day, but then tim hurt his foot. \_\_\_\_\_ sam wanted to help his friend feel better. sam had an idea. he offered tim a ride on his back. I can carry you home, \$\frac{1}{2}\$ aid sam. tim smiled and said, thank you, sam! they were both very happy.

#### **MDM**

"one day, a small cat named tim was walking in the park. the sun was very bright and the wind was harsh.

sam was playing faster than him. fit, welcome to my! said the mouse. do you want to play? Sam looked at tim and said, yes! let 's play! They played all day, but then tim hurt his foot. he had fallen in the grass and it hurt a lot. sam wanted to help his friend feel better. sam had an idea. he offered tim a ride on his back. I can carry you home, said sam. tim smiled and said, thank you, sam! They went home together, and tim 's foot felt better soon. the next day, they played again, and they were both very happy."

#### ILM

"one day, a small cat named tim was walking in the park. the sun was very bright and the wind was harsh. tim asked his friend sam, do you want to play? sam looked at tim and said, yes! let's play! they played all day, but then tim hurt his foot. sam wanted to help his friend feel better. sam had an idea. he offered tim a ride on his back. I can carry you home, said sam. tim smiled and said, thank you, sam! they went home together, and tim's foot felt better soon. the next day, they played again, and they were both very happy.,

#### Dataset: LM1B

#### Input

i would not be upset to see criminal charges brought against them as well as they were endangering the lives of more than 100 \_\_\_\_\_\_ eighteen minutes and could have gotten everyone aboard killed, plus people on the ground. losing their licenses is too mild a punishment.

#### **MDM**

i would not be upset to see criminal charges brought against them as well as they were endangering the lives of more than 100 iowa farm workers, the firefighters, the fire lasted just eighteen minutes and could have gotten everyone aboard killed, plus people on the ground. losing their licenses is too mild a punishment.

#### ILM

i would not be upset to see criminal charges brought against them as well as they were endangering the lives of more than 100 passengers in eighteen minutes and could have gotten everyone aboard killed, plus people on the ground. losing their licenses is too mild a punishment.

#### Dataset: LM1B

#### Input

tony rutherford, chairman of the british fertility society, welcomed the birth, but added: It \_\_\_\_\_\_ research., and should only be offered to patients within the context of a robustly designed clinical trial, carried out in suitably experienced centres.

#### **MDM**

tony rutherford, chairman of the british fertility society, welcomed the birth, but added: It is also a good example of the benefit highlighted in this central research., and should only be offered to patients within the context of a robustly designed clinical trial, carried out in suitably experienced centres.

#### ILM

tony rutherford, chairman of the british fertility society, welcomed the birth, but added: it was important for cancer research., and should only be offered to patients within the context of a robustly designed clinical trial, carried out in suitably experienced centres.

# A.4 Connection between ILM and discrete denoising

Consider a discrete time markov chain  $(X_t)$  with states taking values in  $\mathbb{V}^L$  with the transition kernel  $q(X_t \mid X_{t-1})$  that uniformly randomly drops a token until the sequence is empty. Let  $p_\theta$  be the parametric time reversal of the noising process. Then the evidence lower bound for the log-likelihood

of the data is given by:

$$\begin{aligned} \log p_{\theta}(x_0) &\geq \mathop{\mathbb{E}}_{\boldsymbol{x}_{1:T} \sim q_{\boldsymbol{x}_{1:T} \mid \boldsymbol{x}_0}} \left[ \log p_{\theta}(\boldsymbol{x}_0, \boldsymbol{x}_{1:T}) - \log q(\boldsymbol{x}_{1:T} \mid \boldsymbol{x}_0) \right] \\ &= \mathop{\mathbb{E}}_{\boldsymbol{x}_{1:T} \sim q_{\boldsymbol{x}_{1:T} \mid \boldsymbol{x}_0}} \left[ \sum_{t=1}^{T} \log \frac{p_{\theta}(\boldsymbol{x}_{t-1} \mid \boldsymbol{x}_t)}{q(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1})} + \log p_{\theta}(\boldsymbol{x}_T) \right] \\ &\geq \mathop{\mathbb{E}}_{\boldsymbol{x}_{1:T} \sim q_{\boldsymbol{x}_{1:T} \mid \boldsymbol{x}_0}} \left[ \sum_{t=1}^{T} \log p_{\theta}(\boldsymbol{x}_{t-1} \mid \boldsymbol{x}_t) \right], \end{aligned}$$

where in the last step we used the fact that q is fixed and  $\log p_{\theta}(x_T)$  is zero because  $x_T$  is always the empty sequence for large enough T. Breaking the expression down into a sum over the time steps, we get

$$\mathcal{L}^{\text{mc}}(\theta; \boldsymbol{x}_0) = - \underset{t \sim \mathcal{U}[1, T]}{\mathbb{E}} \underset{\boldsymbol{x}_{t-1}, \boldsymbol{x}_{t} \sim q, |\boldsymbol{x}_0|}{\mathbb{E}} \log p_{\theta}(\boldsymbol{x}_{t-1} | \boldsymbol{x}_t).$$

This loss based on the naive Monte Carlo estimate of the ELBO is easy to compute. However, it is intractable to train a denoising model using this due to two main reasons. First, the estimator can have extremely high variance and therefore unstable to train. Second, parameterizing the denoiser using any standard neural network for sequence modeling like a transformer or LSTM is inefficient because the only one token will be inserted in  $x_t$  to obtain  $x_{t-1}$ , which leads to weak gradients and slow convergence.

We can use the usual trick to utilize  $x_0$  to reduce the variance of the estimator [HJA20].

$$\begin{split} \log p_{\theta}(\boldsymbol{x}_{0}) &\geq \underset{\boldsymbol{x}_{2:T} \sim q_{\boldsymbol{x}_{2:T} \mid \boldsymbol{x}_{0}}}{\mathbb{E}} \sum_{t=2}^{T} \mathbb{D}_{\text{KL}}\left[q(\boldsymbol{x}_{t-1} | \boldsymbol{x}_{t}, \boldsymbol{x}_{0}) \parallel p_{\theta}(\boldsymbol{x}_{t-1} \mid \boldsymbol{x}_{t})\right] + \mathbb{D}_{\text{KL}}[q(\boldsymbol{x}_{T} \mid \boldsymbol{x}_{0}) \parallel p_{\theta}(\boldsymbol{x}_{T})] \\ \Longrightarrow \mathcal{L}^{\text{mc}}(\theta; \boldsymbol{x}_{0}) &= -\underset{t \sim \mathcal{U}[1,T]}{\mathbb{E}} \sum_{\boldsymbol{x}_{t-1}} q(\boldsymbol{x}_{t-1} | \boldsymbol{x}_{t}, \boldsymbol{x}_{0}) \log p_{\theta}(\boldsymbol{x}_{t-1} | \boldsymbol{x}_{t}). \end{split}$$

where we make use of the Bayes rule and the Markov assumption to get  $q(x_t \mid x_{t-1}) = \frac{q(x_{t-1} \mid x_t, x_0) \ q(x_t \mid x_0)}{q(x_{t-1} \mid x_0)}$  and use it in the expression for ELBO.

When  $p_{\text{data}}$  is such that only sequences that do not repeat tokens are in the support of the distribution, then  $q(\boldsymbol{x}_{t-1}|\boldsymbol{x}_t,\boldsymbol{x}_0)=d(k,v;\;\boldsymbol{x}_0,\boldsymbol{b})$  where  $\boldsymbol{b}$  is such that  $\boldsymbol{x}_t=\boldsymbol{x}_0[\boldsymbol{b}]$ . Moreover,  $p_{\theta}(\boldsymbol{x}_{t-1}|\boldsymbol{x}_t)$  can be written as  $p_{\theta}(k,v|\boldsymbol{x}_t)$ . When  $p_{\text{data}}$  does not have this property, then we need to use a dynamic programming algorithm to compute all possible alignments of  $\boldsymbol{x}_t$  w.r.t  $\boldsymbol{x}_0$  to obtain a closed form expression for the loss.