CS225 Assignment 9 Solution Set

EECS, Oregon State University

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6.1 Basic Graph Definitions and Properties

Textbook 9.1

24.a) Explain how graphs can be used to model electronic mail messages in a network. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

Solution: The e-mail message network can be modeled as a directed graph where each email address is represented by a vertex and where an edge starts at the e-mail address a and ends at the e-mail address b if there is a message sending from a to b. There could be multiple directed edges connecting $\{a, b\}$ since multiple messages can be sent from a to b. Loops are allowed because an email address can send messages to itself.

28. Describe a graph model that represents a subway system in a large city. Should edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

Solution: The subway system can be modeled as a directed graph where each station stop is represented by a vertex and where an edge starts at station stop a and ends at b if there is a subway line that goes from station a to b. Multiple edges are allowed because there could be multiple subway lines goes from one station to another. However, loops should not be allowed, since we do not need to go from a station to itself.

Textbook 9.2

6. Show that the sum, over the set of people at a party, of the number of people a person has shaken hands with, is even. Assume that no one shakes his or her own hand.

Solution: We can first model the set of people as an undirected graph $G(V, E)$ where each person is represented by a vertex and where two people a and b is connected by an edge if they shake hands. Then the number of a person v has shaken hands with is the degree of v. The sum of the number of people a person has shaken hands with is equal to $\sum_{v \in V} \deg(v)$. According to *THE HANDSHAKING THEOREM*, $\sum_{v \in V} \deg(v) = 2e$, where e is the number of edges in the graph (e must be an integer). Therefore, the sum is even by the definition of an even number.

16. What do the in-degree and the out-degree of a vertex in the Web graph, as described in Example 8 of Section 9.1, represent?

Solution: The in-degree of a vertex $\deg^-(a)$ is the number of web pages that have a link pointing to a. The out-degree of a vertex $\deg^+(a)$ is the number of web pages that a has a link pointing to.

18. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree.

Solution: Let G be a simple graph with V vertices. Since G is simple, the highest degree of a vertex v is $V-1$ (edges exist between v and all the other $V-1$ vertices). The lowest degree of a vertex v is 0 (no edges exist between v and other vertices). We prove by contradiction that the lowest degree can not possible be 0. If all vertices have different degrees, it is possible that a vertex v has degree zero and a vertex u has degree $V-1$. However, in order for u to have degree $V-1$, there must be an edge between u and v, contradicting the fact that v has degree zero. Therefore, the possible degrees of vertices in the graph are from 1 to $V-1$. Since there are V vertices in the graph, by the pigeonhole principle, there must be two vertices that have the same degree.

26. For which values of n are these graphs bipartite?

Solution: By Theorem 4, we can use coloring to determine the value of n. We have the following graphs that are bipartite.

a). K_n
$n = 1$ and $n = 2$.
For $K_n, n \geq 3$ we can not assign one of two different colors to each vertex because each two distinct vertices are connected by an edge.
(Note that K_1 can be viewed as bipartite since the vertices can be partitioned into a set that contains only v and an empty set, and no edge in K_1 connects either two vertices in these two sets. One can also consider whether K_1 can be colored such that no two adjacent vertices share a color (trivially possible).)

b). C_n
n is even.
Suppose we try to assign one of two colors, say red and green, to each vertex in C_n. We can start with coloring one of the vertices with red. Then we color the adjacent vertices with green. We go on coloring in in this way. As long as n is even, we will stop at the end that the last two vertices that have been colored has the common adjacent vertex. We can finish the process by coloring the last vertex properly.

c). W_n

None of W_n is bipartite.

Suppose we try to assign one of two colors, say red and green, to each vertex in W_n so that no edge in W_n connects a red vertex and a green vertex. There are two kinds of edges cases that we can start coloring: vertices that are originally in C_n, and the additional vertex, say c.

Suppose we start with coloring c with red. Then we need to color all the other vertices with green since they are all adjacent to c. Because these vertices in C_n are adjacent to each other, we will have edges that connects two green vertices. So we can not succeed in this case.

Suppose we start with coloring one of the vertices that are originally in C_n, say vertex u, with red. Then we need to color the two vertices in C_n that is adjacent to u, say v and w, and the c with green. Since c is adjacent with v and w, now we have two edges that connects two green vertices. So we can not succeed in this case, either.

6.2 Connectivity

Textbook 9.4

12. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

Solution:

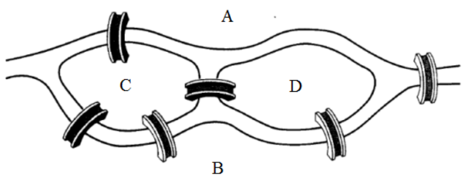
a). It is not strongly connected. For example, there is not a path from b to e. However, it is weakly connected. (Students should verify this).

b). It is strongly connected (as should be verified).

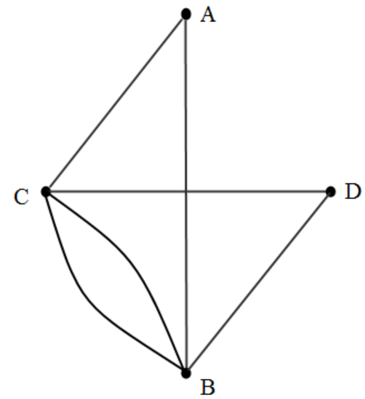
c). It is not strongly connected. For example, there is not a path from a to b. It is not weakly connected, either (a and b again).

16. Show that all vertices visited in a directed path connecting two vertices in the same strongly connected component of a directed graph are also in this strongly connected component.

Solution: Without loss of generality, suppose that there is a directed path from a to b, where a and b are nodes in the strongly connected component, visiting vertices $a, v_1, \ldots, v_n, b$. We need to show that $v_i, i = 1, \ldots, n$ is also in the strongly connected component. That is for any vertices u in the strongly connected component, there is a path from v_i to u and from u to v_i. Consider any vertex v_i on the path and any node u in the strongly connected component. We show that there is a path from v_i to u and and a path from u to v_i, showing that the v_i is also in the strongly connected component. First, from v_i we can reach b, from which we know that we can reach u, since b and u are in the strongly connected component. Second, from u we can reach a, since they are in the same component and then from a we



(a) Map



(b) Graph Model

Figure 1: The map and the corresponding graph

can also reach v_i by the directed path $a, v_1, \ldots, v_i$. Since we show that there is path from v_i to u and from u to v_1, where u is any node in the strongly connected component, we show that v_i is also in the same strongly connected component.

6.3 Euler Paths and Circuits

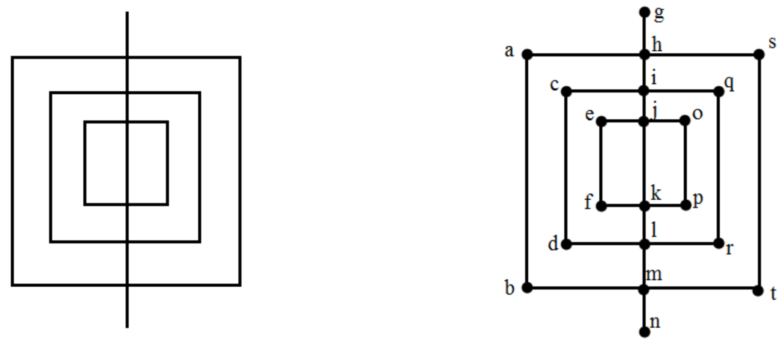
Textbook 9.5

10. Can someone cross all the bridges shown in this map exactly once and return to the starting point?

Solution: The answer is yes. We can first construct a graph model G for the map where the four zones separated by the rivers are represented as vertices and where the bridges are represented as edges (as shown in figure 1). The question is then whether there is an Euler circuit in the graph. It can be seen that each of the vertex has even degree. Hence, by Theorem 1, this graph has an Euler circuit, for example, A, C, B, D, C, B, A. Consequently, one can cross all the bridges exactly once and return to the start point by following the Euler circuit of the graph.

14. Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture?

Solution: The answer is yes. We can first construct a graph model G for the picture where each corner and each intersection is represented as a vertex and where lines between the corners and intersections are represented as edges (as shown in figure 2). We can see that the graph has exactly two vertices of odd degrees, namely, g and n. By Theorem 2, there exists an Euler path in the graph, for example, $n, m, t, s, h, a, b, m, l, r, q, i, c, d, l, k, p, o, j, e, f, k, j, i, h, g$. This gives us a way to draw the picture without lifting the pencil or retracing part of the



(a) Picture (b) Graph

Figure 2: The picture and the corresponding graph

picture.

26. For which values of n do these graphs have an Euler circuit?

Solution: By Theorem 1, we need to check that for which values of n each vertex in these graphs would have even degree.

a). K_n
For any odd integer $n > 0$.
The graph K_n has n vertices and each of them has a degree $n-1$. In order for the degrees of all the vertices to be even, namely, $n-1$ to be even, n must be odd.

b). C_n
For any positive integer $n \geq 3$.
The degree of each vertex in graph $C_n, n \geq 3$ is 2. By theorem 1, these graphs have an Euler circuit.

c). W_n
None of W_n has an Euler circuit.
Because each of the vertices that is originally in C_n has a degree of 3 (odd), it does not satisfy the condition in Theorem 2 that the graph has exactly two vertices of odd degree. Hence, none of W_n has an Euler circuit.