CS225: Quiz 5

Question 1 (10 pts)

a) Give a recursive definition of the sequence $a_n = 5n - 10, n = 0, 1, 2, \ldots$

a_0 = 5(0) – 10 = -10

a_(n + 1) = 5(n+1) – 10 = 5(n) + 5 – 10 = 5(n) – 10 + 5 = a_n + 5

b) Give a recursive definition of the sequence $a_n = n^2 - n\,, n = 0, 1, 2, \ldots$

a_0 = 0^2 – 0 = 0

a_1 = 1^2 – 1 = 0

a_(n + 1) = (n + 1)(n + 1) – (n + 1) = n^2 + 2n + 1 - n - 1 = n^2 – n +2n = a_n + 2n, for n >= 1

Question 2 (10 pts)

a) Give a recursive definition of the set of positive odd integers that are greater than 6.

Basis step: 7 is an element of the odd positive integers

Recursive step: If x is an element of the odd positive integers, then x + 2 is an element of the odd positive integers

b) Give a recursive definition of the set of binary strings that have even length. Recall that the empty string

(Note: Has some similarities to solution listed for Assignment 4.3, section 4.3, problem #32a, but relies on evenness of the length of two strings of bits)

λ has length 0 and hence has even length.

Let *Alphabet* = {0, 1}, define + as being the concatenation of two strings formed from *Alphabet*, and define *Pair* as all possible length 2 combinations of the elements of *Alphabet* (that is: 00, 01, 10, 11).

Basis step: λ has length 0, which is an even length

Recursive step: If x is a binary string with even length then x + *Pair* has even length because x has an even length, *Pair* is defined as always being even length, and the parity of the length of the concatenation of two even strings is even (similar to the proof that adding two even numbers results in an even number).

Question 3 (10 pts)

Consider the following recursive definition of a set *S* of ordered pairs of integers.

Base Case: $(0,1) \in S, (1,0) \in S$

Recursive Case: $If\ (a,b) \in S, then\ (a+1, b+1) \in S$

Use structural induction to prove that for any $(a,b) \in S$, it is the case that *a*+*b* is odd.

Lets use induction to prove this.

Show base cases are true: (0 + 1, 1 + 1) = (1, 2), 1 + 2 = 3, which is odd

(1 + 1, 0 + 1) = (2, 1), 2 + 1 = 3, which is odd

Show a way to get additional elements recursively and show that they also follow a + b is odd:

by the recursive case, $If\ (a,b) \in S, then\ (a+1, b+1) \in S$, so for any (a,b), for which a + b is odd, if we add 1 to each part in the pair (switching parity of each part in the pair), then we are adding 2 to the total (changing both parities results in no change in the overall parity) and we still have the same property of oddness in the total number (a+1 + b + 1 = a + b + 2, where a + b is odd, 2 is even, and an even + an odd is odd).

Question 4 (10 pts)

How many strings of length 10 constructed from characters in the set {a, b, c, d, e, f} contain at least one occurrence of the letter 'a'. (As usual the strings do not need to contain all 6 characters and can contain characters multiple times.)

A straight forward counting problem:

total strings, from this alphabet, of length 10: 6^10

total strings, from this alphabet, of length 10 **not** containing an 'a' in any position: 5^10

number of strings, from this alphabet, of length 10 that contain at least one 'a' in any position:

6^10 – 5^10

Question 5 (10 pts)

Consider a bag that contains 10 blue balls, 10 red balls, 20 green balls, and 20 yellow balls. Now suppose that we take *n* balls from the bag at random.

a) How large does n need to be in order to guarantee that we get at least 3 balls of the same color?

4 possible outcomes per take, worst case before satisfying is 2 balls of each (2*4 is maximum before drawing a third of a previous color taken), so 9 balls (generalized pigeonhole principle will solve this).

b) How large does n need to be in order to guarantee that we get at least 3 blue balls?

4 possible outcomes per take, worst case before satisfying is that we take all balls not blue and 2 blue balls (10 red + 20 green + 20 yellow, + 2 blue is maximum before meeting the condition (52)), so 53 balls is the maximum before guaranteeing that at least 3 blue balls have been drawn (again generalized pigeonhole principle should solve this).