CS225 Assignment 5 Solution Set

EECS, Oregon State University

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4.1 Weak Induction

Textbook 4.1

6. Prove that $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

Solution:

To prove this formula using mathematical induction, let P(n) be the statement that the sum of the first n terms of the left hand side in the formula is correct.

BASIS STEP: P(1) is true, because 1 · 1! = (1 + 1)! - 1 = 1.

INDUCTIVE STEP: The inductive hypothesis is the statement that P(k) is true, where k is a positive integer. That is, P(k) is the statement that

$$1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! = (k+1)! - 1$$

To complete the inductive step, we must show that if P(k) is true, then P(k + 1) is also true. To show that this is the case, we first add (k + 1) · (k + 1)! to both sides of the equality asserted by P(k). We find that

$$1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! + (k+1) \cdot (k+1)! = (k+1)! - 1 + (k+1) \cdot (k+1)!$$

Rewriting the right-hand side of this equation shows that

$$\begin{aligned} (k+1)! - 1 + (k+1) \cdot (k+1)! &= (k+1)! \cdot 1 + (k+1)! \cdot (k+1) - 1 \\ &= (k+1)![1 + k + 1] - 1 \\ &= (k+1)!(k+2) - 1 \\ &= (k+2)! - 1 \end{aligned}$$

This shows that if the inductive hypothesis P(k) is true, then P(k + 1) must also be true, which completes the inductive argument.

We have completed the basis step and the inductive step, so by mathematical induction P(n) is true for all positive integers n. This shows that the formula for the sum of the terms in the formula is correct.

8. Prove that $2 - 2 \cdot 7 + 2 \cdot 7^2 - \ldots + 2 \cdot (-7)^n = \dfrac{(1-(-7)^{n+1})}{4}$ whenever n is a nonnegative integer.

Solution:

To prove this formula using mathematical induction, let P(n) be the statement that the sum of the first n terms of the left hand side in the formula is correct.

BASIS STEP: P(0) is true, because $2 = \dfrac{1-(-7)^1}{4} = 2$.

INDUCTIVE STEP: The inductive hypothesis is the statement that P(k) is true, where k is a nonnegative integer. That is, P(k) is the statement that

$$2 - 2 \cdot 7 + 2 \cdot 7^2 - \ldots + 2 \cdot (-7)^k = \frac{(1-(-7)^{k+1})}{4}$$

To complete the inductive step we must show that if P(k) is true, then P(k + 1) is also true. We first add $2 \cdot (-7)^{k+1}$ to both sides of the equality asserted by P(k). Here we have:

$$2 - 2 \cdot 7 + 2 \cdot 7^2 - \ldots + 2 \cdot (-7)^k + 2 \cdot (-7)^{k+1} = \frac{(1-(-7)^{k+1})}{4} + 2 \cdot (-7)^{k+1}$$

Rewriting the right-hand side of the equation shows that:

$$\begin{aligned}
\frac{(1-(-7)^{k+1})}{4} + 2 \cdot (-7)^{k+1} &= \frac{(1-(-7)^{k+1})}{4} + \frac{2 \cdot (-7)^{k+1} \cdot 4}{4} \\
&= \frac{(1-(-7)^{k+1})}{4} + \frac{8 \cdot (-7)^{k+1}}{4} \\
&= \frac{(1-(-7)^{k+1}) + 8 \cdot (-7)^{k+1}}{4} \\
&= \frac{1 + 7 \cdot (-7)^{k+1}}{4} \\
&= \frac{1 + (-1)(-7) \cdot (-7)^{k+1}}{4} \\
&= \frac{1 + (-1) \cdot (-7)^{k+2}}{4} \\
&= \frac{1 - (-7)^{k+2}}{4}
\end{aligned}$$

Combining the last two equations gives:

$$2 - 2 \cdot 7 + 2 \cdot 7^2 - \ldots + 2 \cdot (-7)^k + 2 \cdot (-7)^{k+1} = \frac{1-(-7)^{k+2}}{4}$$

This shows that if the inductive hypothesis P(k) is true, then P(k + 1) must also be true, which completes the inductive argument. So far, we have completed the basis step and the inductive step, so by mathematical induction P(n) is true for all nonnegative integers n.

14. Prove that for every positive integer n, $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$.

Solution:

To prove this formula using mathematical induction, let P(n) be the statement that the sum of the first n terms of the left hand side in the formula is correct.

BASIS STEP: P(1) is true, because $1 \cdot 2^1 = (1-1)2^{1+1} + 2 = 0 + 2 = 2$.

INDUCTIVE STEP: The inductive hypothesis is the statement that P(k) is true, where k is a positive integer. That is, P(k) is the statement that

$$\sum_{i=1}^{k} i2^i = (k-1)2^{k+1} + 2$$

To complete the inductive step we must show that if P(k) is true, then P(k + 1) is also true. We first add $(k+1) \cdot 2^{k+1}$ to both sides of the equality asserted by P(k). Here we have:

$$\sum_{i=1}^{k} i2^i + (k+1) \cdot 2^{k+1} = (k-1)2^{k+1} + 2 + (k+1) \cdot 2^{k+1}$$

Rewriting the right-hand side of the equation shows that:

$$\begin{aligned} (k-1)2^{k+1} + 2 + (k+1) \cdot 2^{k+1} &= 2^{k+1} \cdot (k-1) + 2^{k+1} \cdot (k+1) + 2 \\ &= 2^{k+1} \cdot [k-1+k+1] + 2 \\ &= 2^{k+1} \cdot 2k + 2 \\ &= k \cdot 2^{k+2} + 2 \end{aligned}$$

Combining the last two equations gives:

$$\sum_{i=1}^{k+1} k2^k = k \cdot 2^{k+2} + 2$$

This shows that if the inductive hypothesis P(k) is true, then P(k + 1) must also be true. This completes the inductive argument. So far, we have completed the basis step and the inductive step, so by mathematical induction P(n) is true for all positive integers n.

18. Let P(n) be the statement that $n! < n^n$, where n is an integer greater than 1.

a) What is the statement P(2)?

b) Show that P(2) is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step.

f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

Solution:

a) P(2) is $2! < 2^2$.

b) This is true because 2 is smaller than 4.

c) The inductive hypothesis P(k) is $k! < k^k$.

d) For each $k > 2$ that P(k) implies P(k + 1); in other words, we want to show that assuming the inductive hypothesis(part c), we can show $(k+1)! < (k+1)^{k+1}$.

e) Multiply $(k+1)$ to both sides of the inequality asserted by P(k). Here we have:

$$k! \cdot (k+1) < k^k \cdot (k+1) < (k+1)^k \cdot (k+1) = (k+1)^{k+1}$$

In other words, we have already proved that $(k+1)! < (k+1)^{k+1}$.

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every integer n greater than 1.

28. Prove that $n^2 - 7n + 12$ is nonnegative whenever n is an integer with $n \geq 3$.

Solution:
To prove this formula using mathematical induction, let P(n) be the statement that $n^2 - 7n + 12$ is nonnegative whenever n is an integer with $n \geq 3$.

BASIS STEP: P(3) is true, because $3^2 - 7 \cdot 3 + 12 = 9 - 21 + 12 = 0 \geq 0$.

INDUCTIVE STEP: The inductive hypothesis is the statement that P(k) is true, where k is a positive integer greater than 3. That is, P(k) is the statement that

$$n^2 - 7n + 12 \geq 0 (n \geq 3)$$

To complete the inductive step we must show that if P(k) is true, then P(k + 1) is also true. Now we want to express P(k+1) in terms of P(k). That is, if we express $(n+1)^2 - 7(n+1) + 12$ in terms of $n^2 - 7n + 12$ and show that the difference between them is nonnegative, then we prove that $(n+1)^2 - 7(n+1) + 12$ is nonnegative, since $n^2 - 7n + 12$ is nonnegative by the inductive hypothesis. We do this in the following way

$$\begin{aligned} (k+1)^2 - 7(k+1) + 12 &= k^2 + 2k + 1 - 7k - 7 + 12 \\ &= (k^2 - 7k + 12) + (2k - 6) \\ &= (k^2 - 7k + 12) + 2(k - 3) \end{aligned}$$

Because the difference $2(k-3)$ is nonnegative whenever $k \geq 3$, and $k^2-7k+12$ is nonnegative by the inductive hypothesis that P(k) is true, $(k+1)^2-7(k+1)+12$ must also be nonnegative, which means P(k + 1) must also be true. This completes the inductive argument.

So far, we have completed the basis step and the inductive step, so by mathematical induction P(n) is true for all positive integers $n \geq 3$.

38. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets that $A_j \subseteq B_j$ for $j = 1, 2, \ldots, n$ then $\bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{n} B_j$.

Solution:

Let P(n) be $\bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{n} B_j$ if $A_i \subseteq B_i$ for $i = 1, 2, \ldots, n$.

BASIS STEP: P(1) asserts that $A_1 \subseteq B_1$, which directly implies that $\bigcup_{j=1}^{1} A_j \subseteq \bigcup_{j=1}^{1} B_j$.

INDUCTIVE STEP: The inductive hypothesis is the statement that P(k) is true, where k is a positive integer. That is, P(k) is the statement that if $A_j \subseteq B_j$ for $j = 1, 2, \ldots, k$, then $\bigcup_{j=1}^{k+1} A_j \subseteq \bigcup_{j=1}^{k+1} B_j$.

Let x be an arbitrary element of $\bigcup_{j=1}^{k+1} A_j = (\bigcup_{j=1}^{k} A_j) \bigcup A_{k+1}$. There will be two possible situations, $x \in \bigcup_{j=1}^{k} A_j$ or $x \in A_{k+1}$. If $x \in \bigcup_{j=1}^{k} A_j$, from the inductive hypothesis, it's safe to draw the conclusion that $x \in \bigcup_{j=1}^{k} B_j$; on the other hand, if $x \in A_{k+1}$, we know from the given fact that $A_{k+1} \subseteq B_{k+1}$ that $x \in B_{k+1}$. To summarize, $x \in (\bigcup_{j=1}^{k} B_j) \bigcup B_{k+1} = \bigcup_{j=1}^{k+1} B_j$.

So far, we have shown that if the inductive hypothesis P(k) is true, then P(k + 1) must also be true. This completes the inductive argument. So based on the basis step and the inductive step, we have uesed mathematical induction to prove P(n) is true for all positive integers.

40. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cap A_2 \cap \ldots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_n \bigcup B).$$

Solution:

Let P(n) be $(A_1 \cap A_2 \cap \ldots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_n \cup B)$.

BASIS STEP: P(1) asserts that $A_1 \cup B = A_1 \cup B$, which is obviously true.

INDUCTIVE STEP: The inductive hypothesis is the statement that P(k) is true, where k is a positive integer. That is, P(k) is the statement that $(A_1 \cap A_2 \cap \ldots \cap A_k) \cup B =$

$(A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_k \cup B)$. Assume that P(k) is true. Then for P(k + 1), we will have the following:

$$\begin{aligned}
&(A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1}) \cup B \\
&\quad = [(A_1 \cap A_2 \cap \ldots \cap A_k) \cap A_{k+1}] \cup B && \text{By 2nd Associative Law} \\
&\quad = [(A_1 \cap A_2 \cap \ldots \cap A_k) \cup B] \cap (A_{k+1} \cup B) && \text{By 2nd Distributive Law} \\
&\quad = (A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_k \cup B) \cap (A_{k+1} \cup B) && \text{By Inductive Hypothesis}
\end{aligned}$$

That is, P(k+1) is true.

So far, we have shown that if the inductive hypothesis P(k) is true, then P(k + 1) must also be true. This completes the inductive argument. So based on the basis step and the inductive step, we have used mathematical induction to prove P(n) is true for all positive integers.

4.2 Strong Induction

Textbook 4.2

2. Use strong induction to show that all dominoes fall in an infinite arrangement of dominoes if you know that the first three dominoes fall, and that when a domino falls, the domino three farther down in the arrangement also falls.

Solution:

Let P(n) be the statement that the dominoes at the n, $n+1$ and $n+2$ position in the infinite arrangement fall.

BASIS STEP: Since we are told that the first three dominoes falls, so P(1) is true.

INDUCTIVE STEP: Assume the inductive hypothesis that P(j) is true for any positive integer $j < k$, we must show that P(k+1) is true. If $k = 2$, because the first dominoes falls, we know that the the domino three farther down in the arrangement also falls. Namely, the dominoes in position 4 falls. Since we are told that the dominoes at 2 and 3 fall, we know that P(2) is true. If $k > 2$, then the inductive hypothesis told us that P(k-2) is true, which means that the dominoes at $n-2$, $n-1$ and n position fall. Because when a domino falls, the domino three farther down in the arrangement also falls, we know that the dominoes at the $n+1$, $n+2$, $n+3$, which shows that P(n) is true.

So we have proved that P(n) is true for all positive integers. Because far all possible integer n, the dominoes at the n, n+1 and n+2 position in an infinite arrangement fall, we show that all dominoes in the infinite arrangement fall.

4. Let P(n) be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of the exercise outline a strong induction proof that P(n) is true for $n \geq 18$.

a) Show statements P(18), P(19), P(20), P(21) are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 21$.

e) Explain why these steps show that this statement is true whenever $n \geq 18$.

Solution:

a) P(18) is true, because we can form a 18 cent postage with a 4-cent stamp and two 7-cent stamps; P(19) is true, because we can form a 19 cent postage with a 7-cent stamp and three 4-cent stamps; P(20) is true, because we can form a 20 cent postage with five 4-cent stamps; P(21) is true, because we can form a 21 cent postage with three 7-cent stamps.

b) The inductive hypothesis P(k) is that just use 4-cent and 7-cent stamps, we can form i cents postage for all i with $18 \leq i \leq k$, where we assume that $k \geq 18$.

c) Assuming the inductive hypothesis, we can form k + 1 cents postage using just 4-cent and 7-cent stamps.

d) Because $k \geq 21$, we know that P(k - 3) is true, that is, we can form k - 3 cents postage. But one more 4-cent stamp on the envelop, and we have formed k + 1 cents of postage.

e) We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 18.

12. Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1, 2^1 = 2, 2^2 = 4$, and so on. [*Hint:* For the inductive step, separately consider the case where k + 1 is even and where it is odd. When it is even, note that $\frac{k+1}{2}$ is an integer.]

Solution: Let P(n) be the statement that a positive integer n can be written as a sum of distinct powers of two.
BASIS STEP: P(1) implies that $1 = 2^0$; P(2) implies that $2 = 2^1$.
INDUCTIVE STEP: Assume P(j) is true for all $j \leq k$, where k is an arbitrary positive integer. We can discuss whether k + 1 is even or not separately.

1. On one hand, when k + 1 is even, clearly, $\frac{k+1}{2}$ is also an integer; besides, $\frac{k+1}{2} \leq k$ for any positive integer k. Since P(j) is true for all $j \leq k$ and $\frac{k+1}{2} \leq k$, we can conclude that $\frac{k+1}{2}$ can be represented as a sum of distinct powers of two. Clearly,

$(k+1) = 2 \cdot \dfrac{k+1}{2}$, which means to add power one to each of the component that sums up to $\dfrac{k+1}{2}$. Therefore, if k + 1 is an even number, P(k + 1) is true.

2. On the other hand, when k + 1 is odd, then it is apparent that k is even. Among all the powers of 2, only $2^0 = 1$ is a odd number. Since in this case, k is an even number, it cannot have 2^0 as a component in the summation. Therefore, $k+1 = k+2^0$, which is a sum of distinct powers of two.

So far, we have proved that P(k + 1) can be represented as a sum of distinct powers of two, which shows that if the inductive hypothesis P(j) ($j \leq k$) is true, then P(k + 1) must also be true. This completes the inductive argument. So far, we have completed the basis step and the inductive step, so by strong induction P(n) is true for all positive integers $n \geq 3$.

30. Find out the flaw with the following "proof" that "$a^n = 1$" for all nonnegative integers n, whenever a is nonzero real number.
BASIS STEP: $a^0 = 1$ is true by the definition of a^0.
INDUCTIVE STEP: Assume that $a^j = 1$ for all nonnegative integers j with $j \leq k$. Then note that $a^{k+1} = \dfrac{a^k \cdot a^k}{a^{k-1}} = \dfrac{1 \cdot 1}{1} = 1$.

Solution:

The error comes from the basis step $n = 0$ to the next case; we cannot assume that $a^1 = 1$. In the inductive proof steps, it involves $a^k$and a^{k-1}, which is wrong.