Question: Show $\log(n!) = \Theta(n \log n)$

(a) Upper bound: find a constant c1, such that for large enough n, $\log(n!) \leq c_1 n \log n$, or $n! \leq c_1 n^n$ Because $n! = 1 \times 2 \times 3 \ldots \times n \leq (n \times n \times n \ldots \times n) = n^n$. By letting $c_1 = 1$, we have $n! \leq c_1 n^n$ thus $\log(n!) = O(n \log n)$

(b) Lower bound: $n! = 1 \times 2 \times 3 \ldots \times n$. By ignoring the first half of the series and keeping only the larger $\lceil n/2 \rceil$ items, we have

$$n! \geq \lceil n/2 \rceil \times \lceil n/2 \rceil + 1 \times \lceil n/2 \rceil + 2 \ldots \times n \geq \lceil n/2 \rceil \times \lceil n/2 \rceil \ldots \times \lceil n/2 \rceil \geq \lceil n/2 \rceil^{\lfloor n/2 \rfloor}$$

Therefore $\log(n!) \geq \log((n/2)^{n/2}) = n/2 \log n/2$ We can ignore the floor and ceiling function in asymptotic notations, so $\log(n!) = \Omega(n \log n)$

Question: Show $\sum_{i=1}^{n} (1/i) = \Theta(\log n)$

(a) Upper bound:

$$\sum_{i=1}^{n} (1/i) \leq \sum_{i=1}^{n} (1/i_{p2})$$

where i_{p2} is the largest power of 2 that's less than or equal to i.

So:

$$\sum_{i=1}^{n} (1/i) \leq (1/1 + 1/2 + 1/2 + 1/4 + 1/4 + 1/4 + 1/4 + 1/8 + \ldots)$$

$$= 1 + \sum_{i=1}^{2} (1/2) + \sum_{i=1}^{4} (1/4) + \sum_{i=1}^{8} (1/8) + \ldots + \sum_{i=1}^{2^k} (1/2^k) = k.$$

Because there are only n items in the original series, the constraint on k would be $1+2+4+8+\ldots+2k-1 \leq (n-1)$. Using geometric series lemma, $2k \leq n$, so $k = O(\log n)$, meaning

$$\sum_{i=1}^{n} (1/i) = O(k) = O(\log n).$$

(b)Similarly for lower bound:

$$\sum_{i=1}^{n} (1/i) \leq \sum_{i=1}^{n} (1/i_{p2})$$

where i_{p2} is the smallest power of 2 that's greater than i.

So:

$$\sum_{i=1}^{n} (1/i) \geq (1/1 + 1/2 + 1/4 + 1/4 + 1/8 + 1/8 + 1/8 + 1/8 + \ldots)$$

$$= 1 + \sum_{i=1}^{1} (1/2) + \sum_{i=1}^{2} (1/4) + \sum_{i=1}^{4} (1/8) + \ldots + \sum_{i=1}^{2^k} (1/2^{k+1}) = 1 + k/2.$$

Again, we have $1 + 1 + 2 + 4 + ... + 2^k \geq n$, therefore $k \geq \log n - 1$, and $\sum_{i=1}^{n}(1/i) = \Omega(k) = \Omega(\log n)$.

Question 0.1 from DPV

(a) f(n)=Θ(g(n))

(b) f(n)= O(g(n))

(c) f(n)=Θ(g(n))

(d) f(n)=Θ(g(n))

(e) f(n)=Θ(g(n))

(f) f(n)=Θ(g(n))

(g) f(n)=Ω(g(n))

(h) f(n)=Ω(g(n))

(i) f(n)=Ω(g(n))

(j) f(n)=Ω(g(n))

(k) f(n)=Ω(g(n))