CS225 Assignment 7 Solution Set

EECS, Oregon State University

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5.1 Basic Counting Rules

Textbook 5.1

8. How many different three-letter initials with none of the letters repeated can people have?

Solution:

There will be 26, 25 and 24 choices for each of the three letters initials with none of the letters repeated. Hence, by the product rule there are a total of $26 \cdot 25 \cdot 24 = 15600$ possible combinations.

12. How many bit strings are there of length six or less?

Solution:

In order to calculate the bit strings length less or equal to six, we need to sum up the combinations of bit string length from 0 to 6(length 0 is the empty string). Product rule shows that string length from 0 to 6, different string number will be $2^0 = 1, 2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32$ and $2^6 = 64$, because each bit can be either 0 or 1. Therefore, the total number of bit strings length smaller or equal to 6 is

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = \frac{2^7 - 2^0}{2 - 1} = 127.$$

16. How many strings are there of four lowercase letters that have the letter x in the term?

Solution:

To calculate a four-letter string with x in the term, it actually requires that there is at least one x in the term. In other words, there could be 1, 2, 3 or 4 x in the term.

1. If there is only one x in the string, x can be placed to $C(4,1) = 4$ different positions, and the rest three letters each has 25 possible choices. So there will be $4 \cdot 25 \cdot 25 \cdot 25 = 62500$ different strings.

2. If there are two x in the string, there will be $C(4,2) = 6$ different combinations for the two x, while the rest two positions have 25 possible choices each. So there will be $6 \cdot 25 \cdot 25 = 3750$ different strings.

3. If there are three x in the string, there will be $C(4,3) = 4$ different combinations for the three x, while the rest one position has 25 possible choices. So there will be $4 \cdot 25 = 100$ different strings.

4. If all the four-letter string is formed by x, there will be 1 unique string.

Therefore, the number of four-lowercase-letter string with x in the term is $62500 + 3750 + 100 + 1 = 66351$.

24. How many strings of four decimal digits

a) do not contain the same digit twice?

b) end with an even digit?

c) have exactly three digits that are 9s?

Solution:

a) The FIRST interpretation of the question is that "do not contain the same digit twice" is the same as "contains distinct digits" since if a digit is contained 3 or more times then it could be said to be contained twice. Under this situation, we just need to consider the number of combinations where all the four digit are different. Since the number of four-digit string with four different numbers is $P(10,4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$; therefore, the number of strings with four decimal digits which do not contain the same digit twice is 5040.

The SECOND interpretation of the question will lead to a much more complicated solution. If "the same digit twice" means that the a digit can appear exactly twice in the four-digit string, the solution is described as follows.

The total number of four decimal digits is $10 \cdot 10 \cdot 10 \cdot 10 = 10000$, since there are 10 choices for each digit. The number of string with the same digit showing up twice contains two possible situations, one is there is only one pair of digits with the same number, the other is there are two pairs of digits with same numbers. It can be calculated as $C(10,3) \cdot 3 \cdot C(4,2) \cdot 2 + C(10,2) \cdot C(4,2) = 4590$. For the first term, it represents the situation where there is only one pair same-digit number. You have $C(10,3)$ choices for the three numbers, 3 choices for the same-digit pair, which have $C(4,2)$ combinations for taking up two positions in the four-digit string. Since the rest two digits are different number, 2 is added because of the two digit are exchangable. The second term can be interpreted as two pairs of same-digit number. You have $C(10,2)$ choices for two numbers, and $C(4,2)$ choices for the one pair to take up 2 out of 4 positions. To summarize, the number of strings with four decimal digits which do not contain the same digit twice is $10000 - 4590 = 5410$.

You can also calculate this from the other way. The number of four-digit string with four different numbers is $P(10,4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$, the number of four-digit string with 3 same digits is $C(10,2) \cdot 2 \cdot C(4,3) = 360$ and the number of four-digit string with

4 same digit is 10. Therefore, the number of strings with four decimal digits which do not contain the same digit twice is $5040 + 360 + 10 = 5410$.

b) If the four digit string ends with an even number, there will be 5 choices for the last digit and 10 choices for all other digits. Therefore, the number of four decimal string that ends with an even number is $5 \cdot 10^3 = 5000$.

c) If the string has 3 digit 9s, there will be only one position with 9 choices$(0, 1, 2, \ldots, 8)$. Therefore, the number of four decimal string with three digits 9s is $C(4, 3) \cdot 9 = 36$.

26. How many license plates can be made using either three digits followed by three letters or three letters followed by three digits?

Solution:

There are 10 choices for each of the three digits and 26 choices for each of the three letters. Hence, by the product rule there are a total of $10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 \cdot 2 = 35,152,000$ different license plates.

42. How many bit strings of length seven either begin with two $0s$ or end with three $1s$?

Solution:

Using the inclusion-exclusion principle, we can construct the following bit strings.

1. We can construct a bit string of length 7 that starts with two $0s$. Given the product rule, the first two bits can only be 0 and the rest of 5 bits can be chosen in two ways, so we will have $2^5 = 32$ different strings.

2. Construct a bit string of length 7 that ends with three $1s$. Given the product rule, the last three bits can only be 1 and the rest of 4 bits can be chosen in two ways, so we will have $2^4 = 16$ different strings.

3. Finally, we can construct a string of length 7 that starts with two $0s$ and ends with three $1s$. Under this circumstance, only two digits in the middle can be chosen in two ways, so we will have $2^2 = 4$ different strings.

To summarize, the number of bit strings of length 7 that begins with two $0s$ or ends with three $1s$ equals $32 + 16 - 4 = 44$.

46. Every student in a discrete mathematics class is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are in the class if there are 38 computer science majors(including joint majors), 23 mathematics majors(including joint majors), and 7 joint majors?

Solution:

Let A_1 be the set of computer science students in the class and A_2 be the set of mathematical students in the class, so the joint major students can be represented as $A_1 \bigcap A_2$. Since the total students in the class is $A_1 \bigcup A_2$, given the formula($|X|$ to denote the number of elements in set X), there are $|A_1 \bigcup A_2| = |A_1| + |A_2| - |A_1 \bigcap A_2| = 38 + 23 - 7 = 54$ students in the class.

60. Use mathematical induction to prove the product rule for m tasks from the product rule for two tasks.

Solution:

Let $P(m)$ be the product rule for m tasks.

BASIS STEP: Start with $m = 2$. $P(2)$ is true, because if there are n_1 ways to do the first task and n_2 ways to do the second task, then there will be $n_1 n_2$ ways to do the procedure, which is just the product rule for two tasks.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, where k is a positive integer greater than 2. Consider $k+1$ tasks, $T_1, T_2, \ldots, T_{k+1}$, which can be done in $n_1, n_2, \ldots, n_{k+1}$ ways, respectively, such that it can be done separately. To finish all these tasks, let's see the first k tasks as one integral task, in which way we have n_{k+1} ways to finish the $k+1$ task. Therefore, we will have $(n_1 n_2 \ldots n_k) \cdot n_{k+1}$ to finish all the tasks.

We have completed the basis step and the inductive step, so by mathematical induction $P(m)$ is true for all positive integers $n(n \geq 2)$.

5.2 The Pigeonhole Principle

Textbook 5.2

4. A bowl contains 10 red balls an 10 blue balls. A woman selects balls at random without looking at them.

a) How many balls must she select to be sure of having at least three balls of the same color?

b) How many balls must she select to be sure of having at least three blue balls?

Solution:

a) We can view the person as drawing balls at random and putting the balls in the box of the appropriate color, so there are 2 boxes.The Pigeonhole Principle states that if N objects are placed into k boxes, there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects. Here $k = 2$ and $\lceil \frac{N}{2} \rceil = 3$, so the smallest number of N is 5.

b) To get at least three blue balls, in the worst case, she may end up with getting 10 red balls first and then 3 three blue balls. So she must pick at 13 balls to make sure of having at least three blue balls.

6. Let d be a positive integer. Show that among any group of $d+1$(not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by d.

Solution:

For a positive integer d, if it is a divisor, there will be d possible remainders, that is, $0, 1, 2, \ldots, d-1$. Therefore, according to the Pigeonhole Principle, if N objects are placed into k boxes, there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects. Given $d+1$ integers, $N = d+1$, $k = d$, so at least $\lceil \frac{d+1}{d} \rceil = 2$ will have exactly the same remainder.

14. a) Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11.

    b) Is the conclusion in part(a) true if six integers are selected rather than seven?

Solution:

a) Group the first 10 positive integers into five subsets of two integers each so that the integers of each subset add up to 11: $\{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}$. Let's start with drawing six integers. If six integers are selected from the first 10 integers, by the Pigoenhole Principle at least $\lceil \frac{6}{5} \rceil = 2$ numbers will come from the same subset; in other words, at least one pair sum up to 11. Since the first 6 integers has exactly one pair, in which case each bucket has at least one integer in it, meaning that if we add a seventh integer it will form another pair. To conclude, at least two pairs of these integers will sum up to 11.

   (Note: This can be seen as a generalized result of Pigeonhole Principle.)

b). No. For example, if you take $\{1, 2, 3, 4, 5, 6\}$ out of the first ten, you will have only one subset $\{5, 6\}$ that sums up to 11.

30. Show that if there are $100,000,000$ wage earners in the United States who earn less than $1,000,000$ dollars, then there are two who earned exacly the same amount of money, to the penny, last year.

Solution:

Since we are calculating the wage earners who earn less than $1,000,000$ dollars per year, there can be only $99,999,999$ possible wages. $\$0.01, \$0.02 \ldots$ up to $\$999,999.99$, to penny. According to the Pigeonhole Principle, so at least $\lceil \frac{100,000,000}{99,999,999} \rceil = 2$ persons earn the same amount of money last year.

32. A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

Solution:

The Pigeonhole Principle states that if N objects are placed into k boxes, there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects. Since each computer is connected to at least one of the other computers, the number of connections between a computer and others is an integers between 1 and 5 inclusively. Here we have 6 computers, $N = 6$, $k = 5$, so at least $\lceil \frac{6}{5} \rceil = 2$ computers are connected to the same number of other computers.