CS225 Assignment 3 Solution Set

EECS, Oregon State University

Summer, 2012

2.2 Contradiction and Other Techniques

Textbook 1.6

8. Prove that if n is a perfect square, then n + 2 is not a perfect square.
Solution:

Let p be "n is a perfect square" and q be "n + 2 is not a perfect square". To construct a proof by contradiction, assume that both p and $\neg q$ are true. That is, assume that n is a perfect square and n + 2 is a perfect square. First, because n + 2 is a perfect square, there is an integer k such that $n + 2 = k^2 (k \geq 2)$, which implies that $n = k^2 - 2$. Since $(k-1)^2 = k^2 - 2k + 1 < n < (k+1)^2 = k^2 + 2k + 1$ when $k \geq 2$, besides,$n \neq k^2$, we can draw the conclusion that n is not a perfect square. Note that the statement "n is not a perfect square" is $\neg p$, so both p and $\neg p$ are true, we have a contradiction. This completes the proof by contradiction, proving that if n is a perfect square, then n + 2 is not a perfect square.

22. Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.
Solution:

Let p be "You have picked three socks from a drawer containing just blue and black socks" and q be "you will get either a pair of blue socks or a pair of black socks". To construct a proof by contradiction, assume that both p and $\neg q$ are true. That is, assume that "You have picked three socks from a drawer containing just blue and black socks and you will not get a pair of either color socks". Because a pair of socks indicates that there would be two socks of the same color, if you want to guarantee to avoid a pair, at most two socks can be picked out. This contradicts the hypothesis that three socks under consideration. That is, if r is the statement that three socks are chosen, then we have shown that $\neg p \rightarrow (r \wedge \neg r)$. Consequently, we know that p is true. We have proved that if three socks are picked from a drawer containing just blue and black socks, you must either get a pair of blue socks or a pair of black socks.

24. Show that at least 3 of any 25 days chosen must fall in the same month of the year.
Solution:

Let p be the proposition "At least 3 of any 25 days chosen must fall in the same month of the year". Suppose that $\neg p$ is true. This means that at most 2 of 25 days fall on the same month. Because there are twelve months in a year, this implies that at most 24 days could have been chosen because for each of the days of the month, at most two of the chosen days could fall on the same month. This contradicts the hypothesis that we have 25 days underconsideration. That is, if r is the statement that 25 days are chosen, then we have shown that $\neg p \rightarrow (r \wedge \neg r)$. Consequently, we know that p is true. We have proved that at least 3 of any 25 days chosen must fall in the same month of the year.

26. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.
Solution:

The statement above has the form "p if and only if q", where p is "n is even" and q is "$7n + 4$ is even". To prove the statement, we need to show that both "$p \rightarrow q$" and "$q \rightarrow p$" are true.

First, we will show that $p \rightarrow q$ is true. According to the definition of an even integer, n can be represented as $n = 2k$, where k is a positive integer if n is positive. Therefore, $7n + 4 = 7 * 2k + 4 = 2(7k + 2)$. By the definition of an even integer, we can conclude that $7n + 4$ is also an even integer. Consequently, we have proved that if n is even, then $7n + 4$ is also even.

Second, we will show that $q \rightarrow p$ is true. Since $7n + 4$ is an even integer, $7n + 4$ can be represented as $7n + 4 = 2k$, where k is a positive integer. Therefore, $7n = 2k - 4 = 2(k - 2)$ follows the definition of even integer; in other words, $7n$ is also even. Since $7n$ is an even integer, it must be a multiple of 2; however, 7 is a prime number and is not a multiple of 2, so we can draw the conclusion that n can be divide by 2 for sure, that is, n is even.

Because we have shown that both $p \rightarrow q$ and $q \rightarrow p$ are true, we have shown that the statement is true.

30. Show that these statements are equivalent, where a and b are real numbers: (i) a is less than b, (ii) the average of a and b is greater than a, and (iii) the average of a and b is less than b.
Solution:

We give proofs by contraposition of $(i) \rightarrow (ii)$, $(ii) \rightarrow (i)$, $(i) \rightarrow (iii)$ and $(iii) \rightarrow (i)$. For the first of these, suppose the average of a and b is smaller or equal to a, that is, $\frac{a+b}{2} \leq a$, then we can easily conclude that $b \leq a$. For the second conditional statement, suppose that a is greater than b, namely, $a \geq b$, it is easy to show that $a+b \leq 2a$; in other words, $\frac{a+b}{2} \leq a$. For the third conditional statement, suppose, the average of a and b is greater than b, which implies $\frac{a+b}{2} \geq b$, so obviously we can conclude that $a \geq b$. And for the fourth conditional statement, suppose that $a \geq b$, namely, $a + b \geq 2b$. This is equivalent to $\frac{a+b}{2} \geq b$. To summarize, we have proved that $(i) \leftrightarrow (ii)$ and $(i) \leftrightarrow (iii)$, so all the three statements are equivalent.

Textbook 1.7

6. Prove that there is a positive integer that equals the sum of the positive integers not exceeding it. Is your proof constructive or nonconstructive?
Solution:

Because we can find that $1 = \sum_{i=1}^{1} i$, which shows a positive integer that equals the sum of the positive integers not exceeding it, we are done.
Our proof is constructive.

28. Prove that there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$.
Solution:

We prove by leveraging proof by cases and exhaustive proof. First, we can reduce a proof to checking just a few simple cases because $2x^2 > 14$ when $|x| \geq 3$ and $5y^2 > 14$ when $|y| \geq 2$. This leaves cases when x takes on one of the values $-2, -1, 0, 1$, or 2 and y takes on one of the values $-1, 0$, or 1. We can finish using an exhaustive proof. For the remaining cases, we note that possible values for $2x^2$ are 0, 2, and 8 and possible values for $5y^2$ are 0, and 5, and the largest sum of possible values for $2x^2$ and $5y^2$ is 13. Consequently, it is impossible for $2x^2 + 5y^2 = 14$ to hold when x and y are integers.

34. Prove that between every rational number and every irrational number there is an irrational number.
Solution:

We prove by construction. Let r be an rational number and x be an irrational number. We construct $\frac{x+r}{2}$ and prove that $\frac{x+r}{2}$ is an irrational number by contradiction.

Suppose $\frac{x+r}{2}$ is a rational number. By the definition of rational number, we have $\frac{x+r}{2} = s/t$ and $r = p/q$, where s, t, p and q are integers and $t \neq 0, q \neq 0$. Then

$$x = 2 * \frac{x+r}{2} - r = \frac{2s}{t} - \frac{p}{q} = \frac{2sq - pt}{qt}$$

, and $2sq - pt, qt$ are integers, $qt \neq 0$. By the definition of rational number, x is rational, which contradicts with the proposition that x is irrational. So we proved that $\frac{x+r}{2}$ is an irrational number. This completes the proof that between every rational number and every irrational number there is an irrational number.

3.1 Set Notation (Textbook 2.1)

6. For each of the sets in Exercise 5, determine whether $\{2\}$ is an element of that set.
Solution:

a). $\{x \in \mathbf{R} | x \text{ is an integer greater than } 1\}$
No.

b). $\{x \in \mathbf{R} | x \text{ is the square of an integer}\}$
No.

c). $\{2, \{2\}\}$
Yes.

d). $\{\{2\}, \{\{2\}\}\}$
Yes.

e). $\{\{2\}, \{2, \{2\}\}\}$
Yes.

f). $\{\{\{\{2\}\}\}\}$
No.

8. Determine whether each of these statements are true or false.
Solution:

a). $\phi \in \{\phi\}$
True.

b). $\phi \in \{\phi, \{\phi\}\}$
True.

c). $\{\phi\} \in \{\phi\}$
False.

d). $\{\phi\} \in \{\{\phi\}\}$
True.

e). $\{\phi\} \subset \{\phi, \{\phi\}\}$
True.

f). $\{\{\phi\}\} \subset \{\phi, \{\phi\}\}$
True. ($\{\{\phi\}\}$ is a proper set of $\{\phi, \{\phi\}\}$ because it is a subset of $\{\phi, \{\phi\}\}$ and the two sets are not equal.)

g). $\{\{\phi\}\} \subset \{\{\phi\}, \{\phi\}\}$
False. (Note that $\{\{\phi\}, \{\phi\}\} = \{\{\phi\}\}$ because the two elements listed in the braces are the same. Hence $\{\{\phi\}\}$ is equal to $\{\{\phi\}, \{\phi\}\}$. According to the definition of proper set, A is not a proper subset of B if $A = B$. Consequently, the statement is False.)

12. Use a Venn diagram to illustrate the relation $A \subseteq B$ and $B \subseteq C$.
Solution: See Figure 1.
16. Find two sets A and B such that $A \in B$ and $A \subseteq B$.
Solution: $A \in B$ means that set A is an element of set B. Hence, one of element of B must be A. $A \subset B$ means that every element of A is also an element of B. This requires that B should also contain all elements of A beyond the set A. Any two sets that have the above property would satisfy the requirement. One example is:

$$A = \{2\}, B = \{2, \{2\}\}.$$

18. What is the cardinality of each of these sets?
Solution:

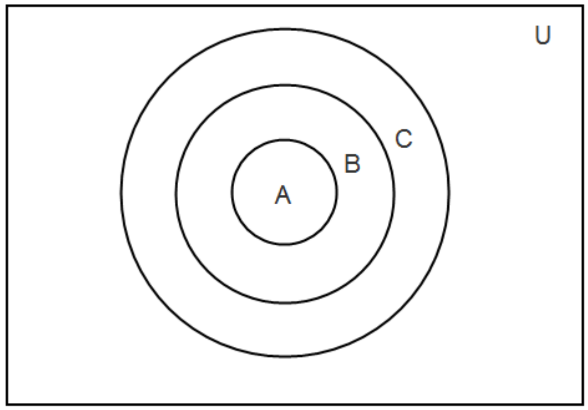

Figure 1: Venn diagram of relation $A \subseteq B$ and $B \subseteq C$

a). $|\phi| = 0$

b). $|\{\phi\}| = 1$

c). $|\{\phi, \{\phi\}\}| = 2$

d). $|\{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}| = 3$

28. Let $A = \{a, b, c\}, B = \{x, y\}$ and $C = \{0, 1\}$. Find
Solution:

a). $A \times B \times C$
$\{(a, x, 0), (a, x, 1), (a, y, 0), (a, y, 1), (b, x, 0), (b, x, 1), (b, y, 0), (b, y, 1),$
$(c, x, 0), (c, x, 1), (c, y, 0), (c, y, 1)\}$

b). $C \times B \times A$
$\{(0, x, a), (0, x, b), (0, x, c), (1, x, a), (1, x, b), (1, x, c), (0, y, a), (0, y, b),$
$(0, y, c), (1, y, a), (1, y, b), (1, y, c)\}$

c). $C \times A \times B$
$\{(0, a, x), (0, a, y), (0, b, x), (0, b, y), (0, c, x), (0, c, y), (1, a, x), (1, a, y),$
$(1, b, x), (1, b, y), (1, c, x), (1, c, y)\}$

d). $B \times B \times B$
$\{(x, x, x), (x, x, y), (x, y, x), (x, y, y), (y, x, x), (y, x, y), (y, y, x), (y, y, y)\}$