SOLUTIONS

question: water-gun induction Consider the following game for an odd number of school children. Each child stands at a point such that *no two distances between children are equal*. Each child has a water gun. Someone yells “go” and every child (simultaneously) sprays the person who is nearest to them. Prove that at least one person stays dry. Do this by proving that the statement *there is a dry survivor in every odd water-gun fight with* $2n-1$ *children* is true by induction on n. In the inductive step, how can you build a solution for $n = k+1$ from $n = k$?

solution The base case is when $n = 1$ and there is just one child playing. Of course this child stays dry!

Assume the statement is true for $n = k$, or $2k-1$ children. Using this inductive hypothesis, we show that the statement is true for $n = k+1$ or $2k+1$ children.

Let C be the set of $2k+1$ children. Let Alice and Bob be the two children in C that are closest to each other (that is, no other pair of children are closer). Certainly Alice and Bob both get wet because they fire on eachother. Consider the rest of the children C. $|C| = 2k-1$. By the inductive hypothesis, we know that if the children in C play amongst themselves, there is a survivor, Sue. In the game where Alice and Bob are now playing, Sue is still a survivor: each child has remained in place and Alice and Bob are closer to each other than either of them are to Sue (because distances are unique). No child, in the game with the set C of children, will fire on Sue.

question: internal nodes and leaves Prove the following statement by induction:

> In any complete binary tree (a tree in which every vertex either has two children or is a leaf), the number of leaves is exactly one more than the number of internal nodes.

A node is a leaf if it has no children; otherwise, it is an internal node. *Clearly indicate the three steps in your inductive proof.*

solution Let n be the number of nodes in the tree. For a tree with n nodes, let (n) be the number of leaves and let $i(n)$ be the number of internal nodes. We want to show that $(n) = 1 + i(n)$.

The three steps of the inductive proof are:

- A tree of one node $n = 1$, clearly has one leaf node and no internal nodes, so $(n) = 1 + i(n)$. (base case)
- Assume the statement is true for all trees with $n \le k$ nodes. (inductive hypothesis)
- Let T be a tree with $k+1$ nodes. Let n and n_r be the number of nodes in the left and right subtrees, respectively. Since $k+1 = n + n_r + 1$, we know that $n \le k$ and $n_r \le k$ and so we can apply the inductive hypothesis to each of these subtrees: $(n) = 1 + i(n)$ and $(n_r) = 1 + i(n_r)$.

The number of leaves of T is the sum of the number of leaves in each subtree: $(k+1) = (n) + (n_r)$. Substituting the previous two equations in, we get: $(k+1) = 2 + i(n_r) + i(n)$.

The number of internal nodes of T is the sum of the number of internal nodes of each subtree plus one (for the root of T): $i(k+1) = i(n) + i(n_r) + 1$. Substituting into the equation of the last paragraph, we get $(k+1) = i(k+1) + 1$, proving the statement.