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Individual Assignments #58

Assignment: *4.1: 6 (see definition of n! pg. 145), 8, 14, 18, 28, 38, 40*

Q6

Prove that: $\sum_{k=1}^{n} n * n! = (n+1)! - 1$ using induction.

Base Case

Check at k=1.

$P(1)$: $1 * 1! = (1+1)! - 1 = 2 - 1 = 2 \Longrightarrow$ OK

Inductive Case

Assume that $\sum_{k=1}^{m} m * m! = (m+1)! - 1$ is true and that it implies $\sum_{k=1}^{m+1} (m+1) * (m+1)! = ((m+1)+1)! - 1$.

So, $\sum_{k=1}^{m+1} = \sum_{k=1}^{m} m * m! + (m+1) * (m+1)! = (m+1)! - 1 + (m+1) * (m+1)!$

$$= (m+1)!\,[(m+1)+1] - 1$$

$$= (m+1)!\,(m+2) - 1$$

$$= (m+2)! - 1$$

Thus, as assumed P(m)→P(m+1).

Q8

Prove that $P(n) \equiv \sum_{k=0}^{n} 2 * (-7)^k = \frac{1-(-7)^{n+1}}{4}$ using induction.

Base Case

Where n=0:$P(0)$: $2 * (-7)^0 = \frac{1-(-7)^{0+1}}{4} \Longrightarrow 2 = \frac{1+7}{4} = 2 \Longrightarrow$OK

Inductive Case

Assume $P(m) \equiv \sum_{k=0}^{m} 2 * (-7)^k = \frac{1-(-7)^{m+1}}{4} \rightarrow P(m+1) \equiv \sum_{k=0}^{m+1} 2 * (-7)^k = \frac{1-(-7)^{m+1+1}}{4}$

So, $\sum_{k=0}^{m+1} = \sum_{k=0}^{m} 2 * (-7)^m + 2 * (-7)^{m+1} = \frac{1-(-7)^{m+1}}{4} + 2 * (-7)^{m+1}$

$$= \frac{1-(-7)^{m+1}}{4} + \frac{8 * (-7)^{m+1}}{4}$$

$$= \frac{1 + 7 * (-7)^{m+1}}{4} = \frac{1 - (-7) * (-7)^{m+1}}{4}$$

$$= \frac{1 - (-7)^{m+2}}{4}$$

Thus, as assumed P(m)→P(m+1).

Q14

Prove that $P(n) \equiv \sum_{k=1}^{n} k * 2^k = (n-1) * 2^{n+1} + 2$ using induction.

Base Case

Where n=1: $P(1)$: $1 * 2 = 0 + 2 \implies 2 = 2 \implies$OK

Inductive Case

Assume $P(m) \equiv \sum_{k=1}^{m} k * 2^k = (m-1) * 2^{m+1} + 2 \to P(m+1) \equiv \sum_{k=0}^{m+1} k * 2^k = (m) * 2^{m+2} + 2$

So, $\sum_{k=0}^{m+1} k * 2^k = \sum_{k=1}^{m} k * 2^k + (m+1) * 2^{m+1} = (m-1) * 2^{m+1} + 2 + (m+1) * 2^{m+1}$

$$= (m-1) * 2^{m+1} + 2 + (m+1) * 2^{m+1}$$

$$= (m - 1 + m + 1) * 2^{m+1} + 2 = 2 * m * 2^{m+1} + 2$$

$$= (m) * 2^{m+2} + 2$$

Thus, as assumed P(m)→P(m+1).

Q18

a) $P(2) = 2! < 2^2$.
b) $P(2) = 2! < 2^2$ is true because $2 < 4$.
c) $P(m) \equiv m! < m^m$
d) We must assume the inductive hypothesis is correct. For each $m \geq 2$ that P(m) implies P(m+1).
e) $m! < m^m \longrightarrow (m+1)! < (m+1)^{m+1}$
   $(m+1)! = m!\,(m+1)$
   $(m+1)! < m^m(m+1)$ by inductive hypothesis
   $(m+1)! < (m+1)^m(m+1)$
   $(m+1)! < (m+1)^{m+1}$
f) Both the basis and inductive step are completed so by principle of mathematical induction the statement is true for every integer greater than 1.

Q28

Prove that $P(n) \equiv n^2 - 7n + 12 \geq 0$ when $n \geq 3$ using induction.

Base Case

Where n=3: $P(3): 3^2 - 7 * 3 + 12 \geq 0 \Longrightarrow 9 - 21 + 12 \geq 0 \Longrightarrow 0 \geq 0 \Longrightarrow$OK

Inductive Case

Assume $P(m) \equiv m^2 - 7m + 12 \geq 0 \rightarrow P(m+1) \equiv (m+1)^2 - 7(m+1) + 12 \geq 0$

Note: $m^2 - 7m + 12 = (m-3)(m-4) \geq 0.$

Note: $(m+1)^2 - 7(m+1) + 12 = (m-2)(m-3)) \geq 0.$

Since $(m-2) > (m-4)$ the equality holds for $n \geq 3$.

Thus, as assumed P(m)→P(m+1).

Q38

Base Case

$\bigcup_{j=1}^{n} A_1 \subseteq \bigcup_{j=1}^{n} B_1$ is always true by definition.

Inductive Case

Assume

$$\bigcup_{j=1}^{k} A_j \subseteq \bigcup_{j=1}^{k} B_j \rightarrow \bigcup_{j=1}^{k+1} A_j \subseteq \bigcup_{j=1}^{k+1} B_j$$

Let x be an arbitrary element of $\bigcup_{j=1}^{k+1} A_j = (\bigcup_{j=1}^{k} A_j) \cup A_{k+1}$.

Because $x \in \bigcup_{j=1}^{k} A_j$ then by the inductive hypothesis $x \in \bigcup_{j=1}^{k} B_j$. We also know that $x \in A_{k+1}$ so by the given fact that $A_{k+1} \subseteq B_{k+1}$ thus $x \in B_{k+1}$. Therefore $x \in \bigcup_{j=1}^{k+1} B_j$.

Q40

Base Case

P(1) $A_1 \cup B = A_1 \cup B$

Inductive Case

P(k)→P(k+1)

$(A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1}) \cup B =$

$((A_1 \cap A_2 \cap \ldots \cap A_k) \cap A_{k+1}) \cup B =$

$[(A_1 \cap A_2 \cap \ldots \cap A_k) \cup B) \cap (A_{k+1} \cup B) =$

$[(A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_k \cup B)] \cap (A_{k+1} \cup B) =$

$(A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_k \cup B) \cap (A_{k+1} \cup B)$ □