CS225 Assignment 4 Solution Set

EECS, Oregon State University

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3.2 Set Operations

Textbook 2.2

2. Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.
Solution:

a). the set of sophomores taking discrete mathematics in your school
$A \cap B$

b). the set of sophomores at your school who are not taking discrete mathematics
$A - B$ (or equivalently, $A \cap \overline{B}$)

c). the set of students at your school who either are sophomores or are taking discrete mathematics
$A \cup B$

d). the set of students at your school who either are not sophomores or are not taking discrete mathematics.
$\overline{A} \cup \overline{B}$ (or equivalently, $\overline{A \cap B}$)

4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find
Solution:

a). $A \cup B = \{a, b, c, d, e, f, g, h\}$

b). $A \cap B = \{a, b, c, d, e\}$

c). $A - B = \phi$

d). $B - A = \{f, g, h\}$

12. Prove the first absorption law from Table 1 by showing that if A and B are sets, then $A \cup (A \cap B) = A$.
Proof: We will prove this identity by showing that each side is a subset of the other side.

Suppose that $x \in A \cup (A \cap B)$. By the definition of union, $x \in A$ or $x \in A \cap B$. By the definition of intersection, we know that either it is the case that $x \in A$ or that $x \in A$ and $x \in B$. In both case, it follows $x \in A$. Consequently, we know that if $x \in A \cup (A \cap B)$, then $x \in A$, which conclude that $A \cup (A \cap B) \subseteq A$.

Suppose that $x \in A$. Then, it follows that either $x \in A$ or $x \in A$, and $x \in B$. By the definition of union, $x \in A \cup (A \cap B)$, which conclude that $A \subseteq A \cup (A \cap B)$. This completes the proof of the identity.

(Note that one can also prove the identity using membership tables.)

16. Let A and B be sets, show that

Solution:

a). $(A \cap B) \subseteq A$.
Proof: Suppose $x \in (A \cap B)$. By the definition of intersection, $x \in A$ and $x \in B$. Since $x \in A$, we conclude that $(A \cap B) \subseteq A$.

b). $A \subseteq (A \cup B)$
Proof: Suppose $x \in A$. Then it follows that either $x \in A$ or $x \in B$. By the definition of union, $x \in A \cup B$, which concludes our proof.

c). $A - B \subseteq A$
Proof: Suppose $x \in A - B$. By the definition of difference, $x \in A$ and $x \notin B$. Hence, we know that $x \in A$. This completes our proof that $A - B \subseteq A$.

d). $A \cap (B - A) = \phi$
Proof: We use a proof by contradiction. Suppose for the sake of contradition $A \cap (B - A) \neq \phi$, then there exists an $x \in A \cap (B - A)$. By the definition of union, we know that $x \in A$ and $x \in (B - A)$. By the definition of difference, it follows $x \in A, x \in B$ and $x \notin A$. This leads to the contradiction that x must be $x \in A$ and $x \notin A$. Hence, our assumption is false and we proved that $A \cap (B - A) = \phi$.

e). $A \cup (B - A) = A \cup B$
Proof: We use membership tables to prove this identity. One can also prove by showing each side is a subset of the other side.

A	B	$B - A$	$A \cup (B - A)$	$A \cup B$
1	1	0	1	1
1	0	0	1	1
0	1	1	1	1
0	0	0	0	0

18. Let A, B and C be sets, show that

Solution:

a). $(A \cup B) \subseteq (A \cup B \cup C)$.
Proof: Suppose $x \in (A \cup B)$. By the definition of union, $x \in A$ or $x \in B$. It follows that $x \in A$, $x \in B$, or $x \in C$ holds. Using the definition of union, we conclude that $x \in (A \cup B \cup C)$. Hence, we conclude that $(A \cup B) \subseteq (A \cup B \cup C)$.

b). $(A \cap B \cap C) \subseteq (A \cap B)$

Proof: Suppose $x \in (A \cap B \cap C)$. By the definition of intersection, $x \in A$, $x \in B$, and $x \in C$. It is obvious that $x \in A$ and $x \in B$ holds. By the definition of intersection, $x \in A \cap B$. So we conclude that $(A \cap B \cap C) \subseteq (A \cap B)$.

c). $(A - B) - C \subseteq A - C$

Proof: Suppose $x \in (A - B) - C$. By the definition of difference, $x \in A$, $x \notin B$, and $x \notin C$. It is obvious that $x \in A$ and $x \notin C$. By the definition of difference, $x \in A - C$. This completes our proof.

d). $(A - C) \cap (C - B) = \phi$

Proof: We prove by contradiction. Suppose, for the sake of contradiction, that $(A - C) \cap (C - B) \neq \phi$. Then there exits an x such that $x \in (A - C) \cap (C - B)$. By the definition of intersection, $x \in (A - C)$ and $x \in (C - B)$. By the definition of difference, it follows that $x \in A$ and $x \notin C$ and that $x \in C$ and $x \notin B$. This leads to the contradiction that $x \in C$ and $x \notin C$. Hence, our assumption is false and we conclude that $(A - C) \cap (C - B) = \phi$.

e). $(B - A) \cup (C - A) = (B \cup C) - A$

Proof: We use membership tables to prove this identity. One can also prove by showing each side is a subset of the other side.

A	B	C	$C - A$	$B - A$	$B \cup C$	$(C - A) \cup (B - A)$	$(B \cup C) - A$
1	1	1	0	0	1	0	0
1	1	0	0	0	1	0	0
1	0	1	0	0	1	0	0
1	0	0	0	0	0	0	0
0	1	1	1	1	1	1	1
0	1	0	1	0	1	1	1
0	0	1	0	1	1	1	1
0	0	0	0	0	0	0	0

20. Show that if A and B are sets, then $(A \cap B) \cup (A \cap \overline{B}) = A$.

Proof: We use membership tables to prove this identity. One can also prove by showing each side is a subset of the other side.

A	B	$(A \cap B)$	$\overline{B}$	$(A \cap \overline{B})$	$(A \cap B) \cup (A \cap \overline{B})$
1	1	1	0	0	1
1	0	0	1	1	1
0	1	0	0	0	0
0	0	0	1	0	0

3.3 Sequences and Sums

Textbook 2.4

4. What are the terms a_0, a_1, a_2 and a_3 of the sequence $\{a_n\}$, where a_n equals:

a) $(-2)^n$: $a_0 = 1$, $a_1 =$ -2, $a_2 = 4$, $a_3 =$ -8

b) 3: $a_0 = 3$, $a_1 = 3$, $a_2 = 3$, $a_3 = 3$

c) $7 + 4^n$: $a_0 = 8$, $a_1 = 11$, $a_2 = 23$, $a_3 = 71$

d) $2^n + (-2)^n$: $a_0 = 2$, $a_1 = 0$, $a_2 = 8$, $a_3 = 0$

13. What are the values of these sums?

a) $\sum_{k=1}^{5}(k+1) = \sum_{k=1}^{5} k + \sum_{k=1}^{5} 1 = 1+2+3+4+5+5\cdot 1 = 20$

b) $\sum_{j=0}^{4}(-2)^j = 1 + (-2) + 4 + (-8) + 16 = 11$

c) $\sum_{i=1}^{10} 3 = 10 \cdot 3 = 30$

d) $\sum_{j=0}^{8}(2^{j+1} - 2^j) = \sum_{j=0}^{8} 2^j = \frac{2^9 - 2^0}{2-1} = 511$

15. What is the value of each these sums of terms of a geometric progression?

a) $$\sum_{j=0}^{8} 3 \cdot 2^j = 3 \cdot \frac{2^9 - 2^0}{2-1} = 3 \cdot 511 = 1533$$

b) $$\sum_{j=1}^{8} 2^j = \sum_{j=0}^{8} 2^j - \sum_{j=0}^{0} 2^j = \frac{2^9 - 2^0}{2-1} - \frac{2^1 - 2^0}{2-1} = 511 - 1 = 510$$

c) $$\sum_{j=2}^{8}(-3)^j = \sum_{j=0}^{8}(-3)^j - \sum_{j=0}^{1}(-3)^j = \frac{(-3)^9 - (-3)^0}{-3-1} - \frac{(-3)^2 - (-3)^0}{-3-1} = 4923$$

d) $$\sum_{j=0}^{8} 2 \cdot (-3)^j = 2\sum_{j=0}^{8} \cdot (-3)^j = 2 \cdot \frac{(-3)^9 - (-3)^0}{-3-1} = 2 \cdot 4921 = 9842$$

16. Find the value of each of these sums.

a) $$\sum_{j=0}^{8}(1 + (-1)^j) = \sum_{j=0}^{8} 1 + \sum_{j=0}^{8}(-1)^j = 9 \cdot 1 + (1 + (-1) + \ldots + (-1) + 1) = 9 + 1 = 10$$

b) $$\sum_{j=0}^{8}(3^j - 2^j) = \sum_{j=0}^{8} 3^j - \sum_{j=0}^{8} 2^j = \frac{3^9 - 3^0}{3-1} - \frac{2^9 - 2^0}{2-1} = 9841 - 511 = 9330$$

c) $$\sum_{j=0}^{8}(2 \cdot 3^j + 3 \cdot 2^j) = \sum_{j=0}^{8} 2 \cdot 3^j + \sum_{j=0}^{8} 3 \cdot 2^j$$

$$= 2\sum_{j=0}^{8} 3^j + 3\sum_{j=0}^{8} 2^j$$

$$= 2 \cdot \frac{3^9 - 3^0}{3-1} + 3 \cdot \frac{2^9 - 2^0}{2-1}$$

$$= 19682 + 1533 = 21215$$

d) $$\sum_{j=0}^{8}(2^{j+1} - 2^j) = \sum_{j=0}^{8}(2^j \cdot 2 - 2^j \cdot 1) = \sum_{j=0}^{8} 2^j(2-1) = \sum_{j=0}^{8} 2^j = \frac{2^9 - 2^0}{2-1} = 511$$

18. Compute each of these double sums.

a) $$\sum_{i=1}^{3}\sum_{j=1}^{2}(i-j) = \sum_{i=1}^{3}(\sum_{j=1}^{2} i - \sum_{j=1}^{2} j)$$

$$= \sum_{i=1}^{3}(2i-3)$$

$$= 2\sum_{i=1}^{3} i - \sum_{i=1}^{3} 3$$

$$= 2(1+2+3) - 3 \cdot 3 = 3$$

b) $$\sum_{i=0}^{3}\sum_{j=0}^{2}(3i+2j) = \sum_{i=0}^{3}(\sum_{j=0}^{2} 3i + \sum_{j=0}^{2} 2j)$$

$$= \sum_{i=0}^{3}(3\sum_{j=0}^{2} i + 2\sum_{j=0}^{2} j)$$

$$= \sum_{i=0}^{3}(3 \cdot (i+i+i) + 2 \cdot (0+1+2))$$

$$= \sum_{i=0}^{3}(9i+6)$$

$$= 9 \cdot (0+1+2+3) + 6 \cdot (1+1+1+1)$$

$$= 9 \cdot 6 + 6 \cdot 4 = 78$$

c) $\sum_{i=1}^{3}\sum_{j=0}^{2} j = \sum_{i=1}^{3}(0+1+2) = 3 \cdot 3 = 9$

d) $\sum_{i=0}^{2}\sum_{j=0}^{3} i^2 j^3 = \sum_{i=0}^{2} i^2 (\sum_{j=0}^{3} j^3) = \sum_{i=0}^{2} i^2(1+8+27) = \sum_{i=0}^{2} 36i^2 = 36(1+4) = 180$

20. Use the identity $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ and exercise 19 to compute $\sum_{k=1}^{n} \frac{1}{k(k+1)}$.

Solution:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) + \ldots + (\frac{1}{n}-\frac{1}{n+1})$$

Denote $a_k = \frac{1}{k}$, for the items in the parenthesis, they can be represented as: $(a_1 - a_2), (a_2 - a_3), (a_3 - a_4), \ldots, (a_n - a_{n+1})$. Using the conclusion from question 19 of telescoping, $\sum_{j=1}^{n}(a_j - a_{j+1}) = a_1 - a_{n+1}$. In other words, $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

24. Find $\sum_{k=99}^{200} k^3$.

Solution:

$$\sum_{k=99}^{200} k^3 = \sum_{k=1}^{200} k^3 - \sum_{k=1}^{98} k^3 = \frac{200^2 \cdot 201^2}{4} - \frac{98^2 \cdot 99^2}{4} = 380477799$$