Mathematics without Numbers

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Mathematics Without Numbers

John G. Kemeny

A hundred years ago a mathematician would have defined mathematics as "the study of number and space." Indeed, the Thorndike-Barnhart *Dictionary* published in 1956 still defines mathematics as the "science dealing with numbers and the measurement, properties, and relationships of quantities." The study of numbers led to the development of algebra, and the study of space to geometry. These two disciplines merged in the calculus, the crowning glory of classical mathematics. A significant feature of modern mathematics is that such a definition is much too narrow to include its newer branches.

Classical mathematics was ideally suited for the development of physics. Indeed, it arose from physics in many cases. For innumerable problems in physics on which measurements are readily available, the physicist may use a numerical model. On other problems the physicist is concerned with the nature of physical space, and thus classical geometry is suitable as a model. Even when Euclidean geometry proved to be no longer adequate for the needs of modern physics, Einstein was able to use a mathematical model which combined a non-Euclidean geometry with methods of the calculus. This type of model still fits the description of mathematics as the study of number and space.

The social sciences may be characterized by the fact that in most of their problems numerical measurements seem to be absent and considerations of space are irrelevant. I would like to consider in this paper ways in which mathematical models may be used in connection with typical problems in the social sciences.

Let us consider why a scientist employs mathematical models. First of all, the language of mathematics is a convenient form in which to formulate scientific theories. It forces the theoretician in various sciences to formulate his hypotheses in a precise and unambiguous form. It also forces him to strip the scientific problem of all accidental details. Once the model is formulated in its abstract form, it becomes a branch of mathematics. If the scientist is fortunate, this branch of mathematics will have been studied by mathematicians previously, and then theorems proven in this field become available as predictions for the scientist. For the axioms of the mathematical system, when interpreted, represent scientific theories, and hence the theorems, when interpreted in the same way, are logical consequences of the scientist's theories. In this way the mathematician will have accomplished the logical analysis of scientific theories for the scientist.

It has often been pointed out that the mathematical theorem adds nothing to the hypotheses from which it is deduced. Indeed, if a theorem added to the content of the hypotheses, it would not follow logically from them, and hence it would not be a theorem of the branch of mathematics. However, theorems, though not new in content, may be psychologically new to the scientist, and very often are. In effect, the mathematician says to the scientist, "Did you know that your assumptions imply such and such?" And very often this will come as a pleasant (or unpleasant) surprise to the scientist. The mathematician has bridged the gap between original assumptions and verifiable predictions. He has enabled the scientist to test his hypotheses, and often enables him to make pragmatically significant predictions about the future.

But it sometimes happens that the mathematical model formulated by the scientist does not correspond to any known branch of mathematics. In this case the scientist either must create a new branch of mathematics or must appeal to the mathematician to undertake this task for him. For example, when Newton formulated his Laws of Motion, he found that there was no branch of mathematics suitable for the treatment of his new model. He had to turn to the method of the calculus that he invented. The social scientist today often finds the mathematician unable to enlighten him on the particular model of interest to him. Many mathematicians have the impression that mathematical problems in the social sciences are entirely trivial. On the contrary, most problems in the social sciences are too difficult for present-day mathematics. It is because the problems arising in the social sciences rapidly become difficult that only some of the very simplest mathematical problems have been solved so far.

There is every reason to expect that the various social sciences will serve as incentives for the development of great new branches of mathematics and that some day the theoretical social scientist will have to know more mathematics than the physicist needs to know today.

There are essentially two different ways in which a mathematical model may be formed for a problem that does not involve numbers or space. The first method is to use a branch of mathematics which itself does not employ numbers and does not deal with space. The second method is to introduce numbers by a more or less arbitrary method, where no numbers were at first apparent. Then it may be possible to form a significant numerical model of a nonnumerical problem. Two examples of each of these approaches will be discussed in detail below.

The examples to be discussed will employ methods either from modern algebra or from modern geometry. To give a maximum variety to these examples, one algebraic and one geometric model will be discussed for each of the two possible approaches to nonnumerical problems.

Model No. 1

Our first model will employ graph theory, which is a branch of modern geometry, but it may be said to be in no way relevant to the study of space. Thus we will discuss a geometric example of a model in which the problem, to start with, is nonnumerical and nonspatial, and the model formed remains so. The problem to be considered is that of structural balance in a social group.[1]

We consider a social group with certain information concerning likes and dislikes between pairs of individuals. A graph is a convenient mathematical language in which to represent such a structure.

A *graph* is defined as a set of points with lines connecting some, though not necessarily all, pairs of points. We may allow some of these lines to have arrows on them indicating directions, in which case we speak of a *directed graph.* We may also allow plus and minus signs on some of these lines, in which case we speak of a *signed graph.* If persons A and B are represented by two points, then an arrow from A to B with a plus sign might indicate that A likes B, and one with a minus sign might indicate that A dislikes B. If there is no arrow from A to B, then A is indifferent to B (see Figure 1).

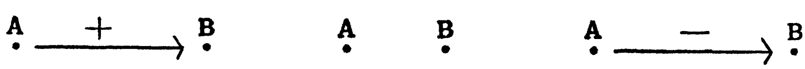

Figure 1

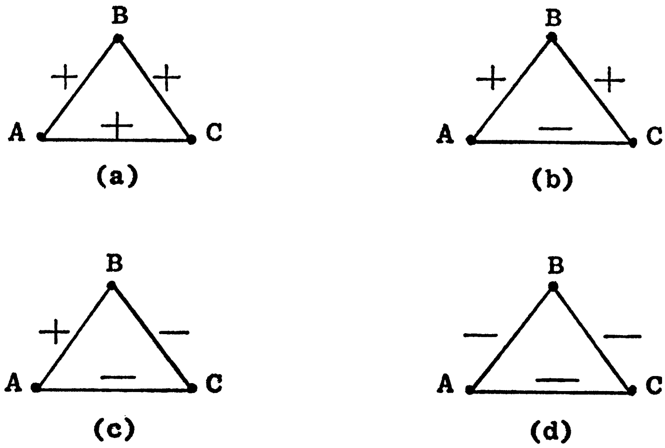

Figure 2

In the problem to be considered we will be interested in conditions under which a social group is in "balance." If A likes B but B does not like A, then there is a lack of balance. The first necessary condition for balance will be that B should always hold the same relation to A that A holds to B. Therefore we need not employ directed graphs; ordinary signed graphs will serve our purpose. These graphs, which have no arrows on the line segments, are suitable for symmetric relations.

Figure 2 represents the possible signed graphs for three people if no person is indifferent to any other person. In (a), where everyone likes everyone else, the social group is balanced. In (b), where person B likes both the others, but these two dislike each other, there is an unbalanced situation. In (c), A and B like each other and each of them dislikes the third person. This is again a balanced situation. Graph (d) represents a situation in which everyone dislikes everyone else. This may be considered to be unbalanced, as there will be strong forces for pairs of individuals to form a coalition against the third one. It may be observed that the graphs with an even number of minus signs are balanced and the graphs with an odd number of minus signs are unbalanced.

Cartwright and Harary searched the literature for examples in which social scientists had labeled social groups "balanced" or "unbalanced." They noted that all these situations satisfied the following definition: if we take a *cycle* within a graph to be a path starting at A and ending at A, we then define a signed graph to be *balanced* if every cycle in it has an even number of minus signs.

Since this definition subsumed every example they found in the literature, and since it provided a complete definition of balanced social structures, they proposed it as a general definition of structural balance in a social group. Of course it remains for the social scientists to decide whether this is a general satisfactory definition. Let us assume for the moment that it is.

We now have a mathematical model for structural balance in a social group. Since we have the tools of graph theory available to us, let us search for a theorem of graph theory that would lead to an interesting conclusion concerning social groups. Such an example is the *structure theorem* for signed graphs. This theorem may be stated as follows: a signed graph is balanced if, and only if, it is possible to subdivide the points into two sets, such that all positive connections occur between points in the same set and all negative connections occur between points in different sets.

This theorem has a most interesting interpretation in terms of political science. Let us suppose that we have a political body with likes, dislikes, and indifferences between pairs of members. Or, if we prefer, we may replace likes by "ability to get along politically." Let us say that "it is possible to impose a two-party structure" on the political body if there is a method of dividing the members of the political body into two parties, so that any one member likes members only of his own party and dislikes members only of the other party. This holds under the alternate interpretation if any one member can get along only with members of his own party and fails to get along politically only with members of the opposing party. Then the structure theorem says simply that a political body is balanced if, and only if, it is possible to impose a two-party structure on it.

This result, which may be surprising to the social scientist, is a good example of the pure mathematician contributing a useful theorem.

Model No. 2

The second model employs group theory, a branch of modern algebra in which numbers need play no role at all. Specifically, we will be concerned with a group of transformations.

A *group of transformations* may be characterized as follows: we are given a set of objects S and a certain collection G of "changes" on S. That is, each element of G may be used to change an object of S into some other (or possibly the same) object of S. For these transformations G to form a *group*, two conditions must be satisfied. First, the changes must come in pairs: for every transformation g_1 there must be a transformation g_2, so that g_2 always undoes what g_1 did, and vice versa. That is, if g_1 changes an object s into an object t, then g_2 must change the object t into the object s. The second condition is that the result of performing two transformations, one after the other, should again be a transformation within G. Thus if g_1 takes s into t, and g_2 takes t into u, then g_3 will change s directly into u. This may be thought of simply as a mental attitude on our part, in that we always decide to include the "combined transformation" g_3 in our collection G.

The reader will note the extremely general nature of the concept of a group of transformations. Yet there is a vast literature on groups of transformations, and hence a tremendous number of theorems that may be employed any time such a group is available to us.

Marriage rules in primitive societies have been studied from a mathematical point of view by André Weil and Robert R. Bush.[2] The marriage rules in certain primitive societies are designed to prevent marriage between close relatives, even when these relatives are not aware of the fact that they are related. This is desirable in a society where no exact records are kept and where family ties may soon be forgotten. The basic rule is that each person in the society is assigned a certain "marriage type" and that a man may marry a woman only if she is of his own type. Given the type of the parents, each son is assigned one definite type and each daughter is assigned another definite type.

We immediately see that brother-sister marriages are automatically forbidden in this society, since a son from a given marriage is always assigned a different type from that of a daughter.

Our basic set of objects is the set of marriage types. Our transformations will be rules according to which we find the type of a relative of a person, knowing the type of the person. Since a relative of a relative is again a relative, the result of applying two transformations will again be a transformation. Furthermore, if there is a transformation changing the type of an uncle to that of a nephew, there should also be a transformation that changes the type of a nephew to that of an uncle, and hence both conditions are satisfied for having a group of transformations.

Among the conditions of reasonableness for marriage types, the two most important conditions not yet mentioned are: "For any two individuals it is permissible for some of their descendants to intermarry"; and "The rule as to whether a man is allowed to marry a female relative of a given kind depends only on the kind of relationship." The former assures that the society does not split into castes, and the latter assures that there is no discrimination against a given marriage type.

We now have a mathetical model for marriage rules in primitive societies, and we may search the mathematical literature for appropriate theorems applicable to this problem. The basic result is that the marriage group must be a regular permutation group which is generated by the parent-to-son transformation and by the parent-to-daughter transformation. Since regular permutation groups are relatively rare, this theorem enables one to find easily all possible marriage rules for a given number of marriage types. For example, it is shown that there are but six possible sets of rules for a society having four marriage types. It is then interesting to note that two of these are actually in use in the Tarau and the Kariera societies respectively.

For example, in the Kariera society, the parent-to-son transformation interchanges types 1, 2, and 3, 4, while the parent-to-daughter transformation reverses the order of the types (see the table below). If we have parents of type 2, a son will have type 4, and his daughters will have type 1. A daughter of the original parents will have type 3, and her sons will also have type 1. Hence a son of the daughter of given parents will be allowed to marry the daughter of a son. The same is true no matter what type the grandparents are.

TYPE NUMBERS IN KARIERA SOCIETY

Parents	Son	Daughter
1	3	4
2	4	3
3	1	2
4	2	1

The model also suggests certain additional questions one may not have thought of in an informal formulation of the problem. For example, both the above-mentioned societies allow certain first-cousin marriages, though other first-cousin marriages are forbidden. It would be reasonable to impose an additional restriction that first-cousin marriages should always be forbidden. In this case one can prove that the necessary and sufficient condition for this is that parent-to-son and parent-to-daughter transformations should not commute and that their squares should not be equal. These additional conditions eliminate all regular groups with less than six types. Therefore we find that the Kariera and Tarau societies could not possibly have eliminated all first-cousin marriages if they wanted to use only four types.

This example is historically very interesting and illuminating. It is most impressive that a society that is unable to keep precise records should have been able to solve, through trial and error, a problem that requires fairly intricate mathematical operations for formal analysis. It also shows, however, that their procedures could have been considerably improved if they had been in a position to use modern algebra to design the rules. For example, they could have eliminated all first-cousin marriages.

We have just considered models in which numbers are not used and in which no geometry occurs. Now we shall consider others in which numbers or geometric concepts are artificially introduced: Model No. 3 will be numerical and Model No. 4 geometrical.

Model No. 3

Let us consider a communication network. By this we mean a set of people with certain means of sending messages from one to the other. For each pair of people i and j it may be possible to send a message from i to j, from j to i, in both directions, or in neither direction. It would at first appear that this is a situation in which no numbers could ever be usefully introduced. However, a simple numerical model for communication networks has proved fruitful.

We introduce a square array of numbers, known as a matrix, which has as many rows and columns as there are people in our network. Let us call this matrix C and let us call the entry in the ith row and the jth column $c_{i,j}$; $c_{i,j}$ will be chosen to be 1 if it is possible to send a message directly from i to j; otherwise $c_{i,j} = 0$. In particular, we will always choose $c_{i,i} = 0$, which is merely a convention. (That is, by definition, a person cannot send himself a message.)

It is immediately clear that all the information available to us about the communication network is furnished by the matrix. However, any number of other methods could be thought of that would represent this information just as well. Has anything been gained by introducing numbers? Numbers are truly useful only if arithmetical operations are introduced. For example, matrices can be multiplied; in particular, we can multiply the matrix C by itself. According to the customary rules of matrix multiplication, we will then find that the entry in the ith row in the jth column of the new matrix will give us the number of different ways in which i can send a message to j in two steps.

In Figure 3 we show communication matrix C for a network of four people in which 1 can communicate directly with 2, 2 can communicate directly with all three of the others, 3 can communicate directly with 4, and 4 can communicate directly with 1 and 3. We also show in the same figure C^2, which indicates the number of ways a given man can communicate with other men in two steps. For example, 2 can communicate with each man in two steps in just one way.

$$C = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{Bmatrix} \qquad C^2 = \begin{Bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{Bmatrix}$$

Figure 3

The fruitfulness of such a model might of course be judged in terms of theorems that can be proved about it. An interesting theorem[3] concerns a complete communication network. By this we mean that, for each pair of people i and j, it is possible to send a message either from i to j, or from j to i, or in both directions. For such a complete communication network there is a simple interpretation for having the largest number of ones in a given row. For example, in Figure 3 (which shows a complete network) man 2 has the largest row sum, namely 3. The proof shows that the person whose row in the matrix has the largest row sum can communicate with everyone in the network in one or two steps. Of course, in Figure 3 man 2 can actually do this in a single step.

This system has an interesting mathematical feature known as duality—namely, by interchanging rows and columns it is possible to change a matrix of "can send a message to" to a matrix of "can receive a message from." The above-quoted theorem is still applicable to the dual matrix, and hence we know that if a given person's column sum is a maximum, then this person can receive a message from everybody in one or two steps. In our example, columns 1, 3, and 4 all have maximal column sum 2, and hence all three of these men can be reached by any man in one or two steps.

These results do not appear very surprising when there are only four men in our network; but when we consider a complex network of a hundred people, they may be very useful indeed. For example, an efficiency expert studying a large firm may find a communication matrix a convenient means for representing either the communication system or the table of organization of the company. Should he find that the company forms a complete communication network, he could immediately search for the hub of command from which instructions can be given in one or two steps to any employee. And even if the network is incomplete, studying powers of the communication matrix would provide valuable information.

This example illustrates in very simple terms how numerical tools may be introduced in a problem where no numbers are apparent. Our final model will show how geometrical tools may sometimes be useful in a problem that at first appears completely nongeometrical.

Model No. 4

The problem confronting us deals with the ranking of a set of objects. Suppose that ten experts are each asked to rank a set of 50 objects, in order of preference. To allow a maximum amount of freedom, we will allow ties in the rankings. We are then supposed to arrive at a consensus ranking. How are we to do this? This problem can be reduced to a problem analogous to classical statistics problems[4] if we are able to introduce a measure of distance between rankings. So our problem is that of taking the set of all possible rankings of 50 objects and of turning them into a geometrical space, one in which a definite distance is defined between any two rankings. I will here summarize the results of some as yet unpublished research.[5]

Let us agree on some notation. We will have in mind a fixed number of objects to be ranked. We will denote by capital letters, A, B, C, etc., possible rankings. For example, if we have three objects, a, b, and c, in mind, then A may be the ranking where b is first, a is second, and c is third; and B may be the ranking where c is first and a and b are tied for second place. We want to introduce a measure of distance between pairs A and B, which will be denoted by $d(A,B)$. Let us try to agree on certain conditions that such a definition must satisfy.

Condition 1. d must satisfy the conditions for a distance laid down by a geometer. That is:

(1) $d(A, B) \geq 0$, and equality holds if, and only if, A and B are the same ranking.

(2) $d(A, B) = d(B, A)$.

(3) $d(A, B) + d(B, C) \geq d(A, C)$, and the equality holds if, and only if, the ranking B is between A and C.

For the last part of condition 1, we need a definition of "betweenness." We will define betweenness in terms of pairwise judgments—that is, we will say that ranking B is *between* A and C if for each pair of objects, i and j, the judgment of B is between that of A and C. In other words, for the given pair the judgment of B either agrees with A or agrees with C, or A prefers i, C prefers j, and B declares them to be tied.

Next we must assure that our measure of distance does not in any way depend on the particular objects we have chosen for our rankings.

Condition 2. The definition of the distance d should not be affected by a relabeling of the set of objects to be ranked.

This means, for example, that if A rates three objects in the order a, b, c, and B rates them in the order c, b, a, the distance between these two rankings should be the same as the distance between the ranking b, c, a and a, c, b, since the latter may be obtained from the former by changing a to b, b to c, and c to a.

Condition 3. If the two rankings are in complete agreement at the beginning of the list and at the end of the list, and differ only as to the ranking of k objects in the middle, then this distance is the same as if these k objects were the only objects under consideration.

This condition is self-explanatory. Our final condition is in the nature of a convention. It may be thought of as choosing a unit of measurement.

Condition 4. The minimum positive distance is 1.

Let us suppose that we have agreed that these are four reasonable conditions for the definition of a distance between rankings. We have then translated our scientific problem into a purely mathematical problem. We can ask a mathematician three questions: (1) Is there any distance that will satisfy all of these conditions? Or, in other words, are our conditions consistent? (2) How can we characterize all definitions that will satisfy these four conditions? (3) What additional assumptions can we make that would narrow the possible choice from many distances to one?

In this particular case we are confronted with a pleasant surprise, in that we find that our conditions are indeed consistent, and that there is one and only one possible definition of distance which will satisfy all of these conditions. Therefore, if we have agreed on the four conditions above, we must agree that this is *the* correct definition of a distance. The details of this proof are here omitted.

The resulting distance may be described as follows. Compare the rankings A and B for each pair of individuals i and j. If the two rankings agree, we write down 0. If one prefers i to j and the other j to i, we write down 2. And if one expresses a preference while the other indicates a tie, we write down 1. Once we have these numbers written down for all pairs i and j, $d(A, B)$ equals the sum of these numbers.

Had we written down this definition to start with, we might have thought it a fairly reasonable way to measure the distance between two rankings. However, had other equally reasonable-sounding definitions been suggested, we would have had no rational way of choosing among them. With our present procedure the argument is reduced to the four conditions stated above. Anyone who accepts those four conditions *must* accept the resulting definition of d. Hence, anyone who rejects our definition of a distance must specify which of our conditions he rejects, and should be forced to give conditions of his own which are reasonable and which lead to a unique choice of the distance function. In this way an argument about a sociological problem can be put on a useful plane.

As an illustration, we show in Figure 4 the possible rankings of three objects and the distances between these rankings. The notation used is such that $\left\{\begin{matrix} a \\ b\text{—}c \end{matrix}\right\}$ indicates that a is in first place, b and c are tied for second place. Distances between neighboring rankings are indicated in the figure. Thus, e.g., the (shortest) distance between $\left\{\begin{matrix} a \\ b \\ c \end{matrix}\right\}$ and $(a\text{-}b\text{-}c)$ is $1 + 2 = 3$.

If we now ask a number of experts to rank three objects, we may use Figure 4 to find the consensus ranking. This ranking may be

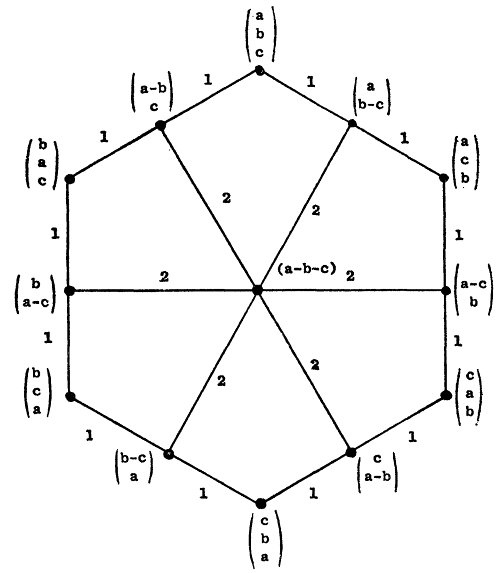

Figure 4

defined as the one such that the sum of its distances from the various experts' rankings should be a minimum. Alternatively, it is sometimes convenient to take the sum of the squares of the distances to be a minimum. This would mean, for example, that if we have three experts, two of whom hand in the ranking, *a, b, c,* and one the ranking *b, a, c,* then the method of minimizing the sums of the distances will give *a, b, c* as the consensus ranking, while the method of minimizing the sum of squares of distances will yield a tie between *a* and *b,* with *c* a definite third. On the other hand, if there is considerable disagreement among the three experts—if, for example, the first one rates *a, b, c* in that order; the second one *b, c, a;* and the third one *c, a, b*—then the method of minimizing the sums of the squares of the distances will tell us that the consensus ranking is a three-way tie. The method of minimizing the sums of the distances, on the other hand, will give us three possible consensus rankings, namely, the three rankings given by the individual judges.

It is an essential feature of either of these methods that it always yields us at least one possible consensus ranking; but, as illustrated above, it may yield us more than one consensus. These basic tools enable one to carry out a certain amount of statistical work on rankings by experts.

An interesting question to consider is the relation between this definition of distance and the selection of consensus rankings on the one hand, and the work of K. J. Arrow on the other hand.

Arrow considered conditions that any reasonable system of social choice must satisfy.[6] His major result may be stated, in our terminology, as follows: *there is no method of selecting a consensus ranking from arbitrary sets of individual rankings that satisfies all his conditions.*

We have proposed two different ways of arriving at a consensus ranking, and hence we must ask how our methods escaped from the Arrow theorem. We find, first of all, that we have violated one of his basic conditions. This is the "condition of irrelevant alternatives," which has frequently been criticized. One consequence of this condition is the following: if we have three individuals ranking our objects, and if we know that two have ranked a ahead of b while the third has ranked b ahead of a, then we should be able to tell how a and b are ranked (relative to each other) by the consensus ranking. However, if we have two rankings $\left\{\begin{matrix} a \\ c \\ b \end{matrix}\right\}$ and one $\left\{\begin{matrix} b \\ a \\ c \end{matrix}\right\}$, our consensus ranking by either method will be $\left\{\begin{matrix} a \\ b\text{—}c \end{matrix}\right\}$. While if two individuals arrive at the ranking $\left\{\begin{matrix} a \\ b \\ c \end{matrix}\right\}$ and the third at $\left\{\begin{matrix} b \\ c \\ a \end{matrix}\right\}$, then the consensus will be $\left\{\begin{matrix} a\text{—}b \\ c \end{matrix}\right\}$. In the former case a is preferred to b by the consensus, while in the latter they are tied. When we examine these two situations on Figure 4, they appear exactly alike, and it is hard to see why anyone would disagree with the consensus rankings. This lends further evidence to the thesis that the condition of irrelevant alternatives should be rejected.

But there is a second and more basic way in which our methods differ from those investigated by Arrow, namely, that we occasionally arrive at multiple consensus rankings. We conclude that the requirement of a unique social ordering is too restrictive. It suffices that we should arrive at a unique ordering "in most cases."

Conclusion

The four models discussed above illustrate various ways in which mathematics may be useful in nonnumerical and nonspatial problems in the social sciences. They illustrate how both modern algebra and modern geometry provide new techniques for these fields, and they show that these techniques may apply in situations where numbers and space are never introduced or where numbers and space are more or less artificially introduced into nonmathematical problems.

Of particular importance is the method illustrated in the fourth model. Often social scientists may be in agreement on requirements for the solution of a certain problem, even when no agreement can be reached on an actual solution. In such cases a mathematician should be consulted. He may show them that it is impossible to meet all the requirements they have laid down, and in that case they would have to agree on ways of asking for less. Or, very likely, he may tell them that there are infinitely many ways of solving their problem, and give them some indication of the kind of additional requirements they could make on a solution. Finally, in an ideal situation such as the one illustrated in Model No. 4, he may be able to prove that there is a unique solution to the problem they have laid down. In this case he would solve, once and for all, a hitherto unresolved problem, and would provide the social scientist with a most useful tool for his work.

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