Assume $n \geq 3$ and identitiy as ground truth. Set of elements: $0, 1, \ldots, n-1$. Set of weights: $w_0, \ldots, w_{n-2}$.

Let

$$\delta_j = \sum_{\ell > j} \mathbb{1}(\ell <_\pi j).$$

For $j \in \{0, 1, \ldots, n-1\}$, suppose that, having ranked items $j+1, \ldots, n-1$, we would like to rank j. The correct (unknown) position for j, among the existing items, is at the top. The probability that item j is instead put after δ_j items is given by

$$\Pr(\delta_j) \propto \exp\left(-\phi \sum_{\ell=j}^{j+\delta_j-1} w_\ell\right)$$

and

$$\Pr(\delta_j) = \frac{\exp\left(-\phi \sum_{\ell=j}^{j+\delta_j-1} w_\ell\right)}{Z_j},$$

where $Z_j = \sum_{i=0}^{n-j} \exp\left(-\phi \sum_{\ell=j}^{j+i-1} w_\ell\right)$.

For arithmetic weights where $w_\ell = 1 - \ell b$,

$$\sum_{\ell=j}^{j+\delta_j-1} w_\ell = \sum_{\ell=j}^{j+\delta_j-1} (1 - \ell b) = \delta_j - b\delta_j(j - 1/2) - \frac{b\delta_j^2}{2},$$

$$\begin{aligned} Z_j &= \sum_{i=0}^{n-1-j} \exp\left[-\phi\left(i - bi(j-1/2) - \frac{bi^2}{2}\right)\right] \\ &= \sum_{i=0}^{n-1-j} \exp\left[-\phi\left(i - b\left[\binom{i}{2} + ij\right]\right)\right] \end{aligned}$$

$$\begin{aligned} Z_j' = \frac{dZ_j}{db} &= \sum_{i=0}^{n-1-j} \exp\left[-\phi\left(i - b\left[\binom{i}{2} + ij\right]\right)\right] \phi \left[\binom{i}{2} + ij\right] \\ Z_j'' &= \sum_{i=0}^{n-1-j} \exp\left[-\phi\left(i - b\left[\binom{i}{2} + ij\right]\right)\right] \phi^2 \left[\binom{i}{2} + ij\right]^2 \end{aligned}$$

For geometric weights, i.e., $w_\ell = b^\ell$,

$$\sum_{\ell=j}^{j+\delta_j-1} w_\ell = \frac{b^j\left(1 - b^{\delta_j}\right)}{1-b}$$

So

$$p(\pi) = \prod_{j=0}^{n-1} \frac{\exp\left(-\phi \sum_{\ell=j}^{j+\delta_j-1} w_\ell\right)}{Z_j}$$

$$\log p(\pi) = \sum_{j=0}^{n-1} \left(-\phi \sum_{\ell=j}^{j+\delta_j-1} w_\ell - \log Z_j\right)$$

With N samples, the log-likelihood is

$$\log p\left(\pi_1^N\right) = \sum_{i=1}^{N}\sum_{j=0}^{n-1}\left(-\phi\sum_{\ell=j}^{j+\delta_{ji}-1} w_\ell - \log Z_j\right)$$
$$= -\phi\sum_{i=1}^{N}\sum_{j=0}^{n-1}\sum_{\ell=j}^{j+\delta_{ji}-1} w_\ell - N\sum_{j=1}^{n}\log Z_j.$$

Let $s_j = \frac{1}{N}\sum_{i=1}^{N}\delta_{ji}$, and $v_j = \frac{1}{N}\sum_{i=1}^{N}\delta_{ji}^2$. For arithmetic weights

$$\log p\left(\pi_1^N\right) = -N\phi\sum_{j=0}^{n-1}\left(s_j - b\left(js_j + (v_j - s_j)/2\right)\right) - N\sum_{j=0}^{n-1}\log Z_j,$$

where

$$Z_j = \sum_{i=0}^{n-1-j}\exp\left[-\phi\left(i - b\left[\binom{i}{2} + ij\right]\right)\right].$$

We show that this likelihood is concave in b. The first term is linear in b, and so it is concave in b. For the second term, we have

$$-\frac{d^2}{db^2}\log Z_j = \frac{\left(Z_j'\right)^2 - Z_j Z_j''}{Z_j^2},$$

which is negative if $\left(Z_j'\right)^2 < Z_j Z_j''$. Let $g_i = \binom{i}{2} + ij$. We require

$$\sum_i\sum_k e^{-i+bg_i}e^{-k+bg_k}g_k g_i < \sum_i\sum_k e^{-i+bg_i}e^{-k+bg_k}g_k^2,$$

or equivalently,

$$\sum_{i<k} 2e^{-i+bg_i}e^{-k+bg_k}g_k g_i < \sum_{i<k} e^{-i+bg_i}e^{-k+bg_k}\left(g_k^2 + g_i^2\right)$$

which holds since $g_i^2 + g_k^2 \geq 2g_i g_k$ for all i, k and $g_i^2 + g_k^2 > 2g_i g_k$ for at least one choice of i, k.

Assuming identitiy as ground truth.

Let

$$\delta_j = \sum_{\ell > j}\mathbb{1}(\ell <_\pi j).$$

Suppose that, having ranked items $j+1, \ldots, n$, we would like to rank j. The correct (unknown) position for j, among the existing items, is at the top. The probability that item j is instead put after δ_j items is given by

$$\Pr\left(\delta_j\right) \propto \exp\left(-\sum_{\ell=j}^{j+\delta_j-1} w_\ell\right)$$

and

$$\Pr\left(\delta_j\right) = \frac{\exp\left(-\sum_{\ell=j}^{j+\delta_j-1} w_\ell\right)}{Z_j},$$

where $Z_j = \sum_{i=0}^{n-j}\exp\left(-\sum_{\ell=j}^{j+i-1} w_\ell\right)$.

For arithmetic weights where $w_\ell = a - \ell b$,

$$\sum_{\ell=j}^{j+\delta_j-1} w_\ell = \sum_{\ell=j}^{j+\delta_j-1} (a - \ell b) = a\delta_j - b\delta_j(j-1/2) - \frac{b\delta_j^2}{2},$$

$$Z_j = \sum_{i=0}^{n-j} \exp\left(-ai + bi(j-1/2) + \frac{bi^2}{2}\right)$$

and if $w_\ell = ab^\ell$,

$$\sum_{\ell=j}^{j+\delta_j-1} w_\ell = \frac{ab^j\left(1-b^{\delta_j}\right)}{1-b}$$

So

$$p\left(\pi\right) = \prod_{j=1}^{n} \frac{\exp\left(-\sum_{\ell=j}^{j+\delta_j-1} w_\ell\right)}{Z_j}$$

$$\log p\left(\pi\right) = \sum_{j=1}^{n} \left(-\sum_{\ell=j}^{j+\delta_j-1} w_\ell - \log Z_j\right)$$

With n samples,

$$\begin{aligned}\log p\left(\pi_1^N\right) &= \sum_{i=1}^{N}\sum_{j=1}^{n} \left(-\sum_{\ell=j}^{j+\delta_{ji}-1} w_\ell - \log Z_j\right)\\ &= -\sum_{i=1}^{N}\sum_{j=1}^{n}\sum_{\ell=j}^{j+\delta_{ji}-1} w_\ell - N\sum_{j=1}^{n}\log Z_j.\end{aligned}$$

Let $\hat{j} = j - 1/2$, $s_j = \frac{1}{N}\sum_{i=1}^{N}\delta_{ji}$, and $v_j = \frac{1}{N}\sum_{i=1}^{N}\delta_{ji}^2$. For arithmetic weights

$$\log p\left(\pi_1^N\right) = -N\sum_{j=1}^{n}\left(\left(a - b\hat{j}\right)s_j - bv_j/2\right) - N\sum_{j=1}^{n}\log Z_j.$$