Homework Problem Set 1

Ari Bruck
Eric Addison

Social Computing
Fall 2018

1

Consider a bipartite graph with a set of boys and girls. Suppose that every girl likes at least $d \geq 1$ boys and every boy likes at most d girls. Show that there exists a perfect matching between boys and girls.

Let S be a subset of girls $S \subseteq G$, with $|S| = n_g$. Let the $N(S)$ be the neighbor set of S with $|N(S)| = n_b$. From the problem statement, we know that the set S must contain at least dn_g outgoing edges (connections to boys), and the set $N(S)$ must contain at most dn_b incoming edges. Additionally, *all* of the outgoing edges from S must also be incoming edges to $N(S)$. Let n_S be the number of edges between S and $N(S)$. From this, we can conclude:

$$\begin{aligned} & dn_g \leq n_S \leq dn_b \\ \Rightarrow & n_g \leq n_b \\ \Rightarrow & |S| \leq |N(S)| \\ \Rightarrow & S \text{ is not a constricted set} \end{aligned}$$

Since this applies to any set $S \subseteq G$, there are **no constricted subsets in** G, and so a perfect matching must exist by Hall's marriage theorem.

2

Show that König-Egerváry Theorem implies Hall's Theorem.

König-Egerváry Theorem states that in any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover.

Then for bipartite graph G with bipartite sets X and Y such that $G := (X, Y, E)$ $\exists$ a minimum cover C that maximizes matching. Suppose there is no matching that matches X. Then by König-Egerváry Theorem $|C| < |X|$ vertices. Consider $C \cap X$ and $C \cap Y$. There is no edge between $X \setminus C$ and $Y \setminus C$, so $N(X \setminus C) \subset C \cap Y$. But then

$$|N(X \setminus C)| \leq |C \cap Y| = |C| - |C \cap X| < |X| - |C \cap X| = |X \setminus C|,$$

which violates Hall's condition and therefore means a perfect match does not exist.

Using König-Egerváry Theorem, if a perfect matching exists then

$$\forall X \text{ is matched to a } Y \implies \forall X, N(X) \geq X.$$

So a constricted set does not exist, thus Hall's Theorem:

$$\exists \text{ perfect matching } \iff N(X) \geq X$$

is proven.

3

An $r \times n$ Sudoku is an $r \times n$ matrix with entries $\{1, ..., n\}$ such that each number appears at most once in each row and column. Show that any $r \times n$ Sudoku can be extended to an $(r+1) \times n$ Sudoku whenever $r < n$.

Let S be an $(r+1) \times n$ matrix where the submatrix $\hat{S} \equiv \{S_{ij}\}_{i=1}^{r}$ is a valid $r \times n$ Sudoku, with $r < n$. Let $B \equiv \{1, ..., n\}$. In order for S to also be a valid Sudoku, each entry in the last row must take a value such that:

$$S_{r+1,j} \in P_j \equiv B - \hat{S}_j$$

Where P_j is the set of potential values for cell $S_{r+1,j}$, with $|P_j| = n - r$, and $\hat{S}_j$ is the set of values in the jth column of $\hat{S}$.

We can cast this as a matching problem where the matrix entries $G \equiv \{S_{r+1,j}\}_{j=1}^{n}$ form the left-hand side of a bipartite graph (the "girls"), and the set B forms the right-hand side (the "boys").

Given the size of the potential value set P_j, we know that each new cell in G connects to exactly $n - r$ elements in B.

Also, since each value $b \in B$ must be present in each row, and every column in $\hat{S}$ contains r of the values in B, every value in B is matched to exactly $n - r$ of the cells in G. That is, every value b has $n - r$ potential cells in G that it could fit as a valid entry.

Now, **by application of the result of problem 1**, since every cell $g \in G$ is connected to at least (in this case, exactly) $n - r \geq 1$ values $b \in B$, and every value b is connected to at most (exactly) $n - r$ cells in G, we know that there exists a perfect matching between cells in G and values in B. Therefore, the Sudoku $\hat{S}$ can be extended.

Note that for $r \geq n$, we fail to meet the condition that $n - r \geq 1$, and cannot guarantee a perfect matching.