Homework 3 Solutions

MS&E 220 (Probabilistic Analysis)

October 17, 2025

Due: October 31 at 10:00 PM

Show your work and justify each step.
Answers submitted without explanation will not receive full credit.

1. **[8 points]** An oil company conducts a geological study indicating that an exploratory oil well has a 20% chance of striking oil. Assume independence across wells.

   (a) **[2 points]** What is the probability that the first strike occurs on the third well drilled?

   (b) **[2 points]** What is the probability that the third strike occurs on the seventh well drilled?

   (c) **[4 points]** If the company wants to establish three producing wells, compute the mean and variance of the total number of wells that must be drilled.

Solution.

Let the strike (success) probability be $p = 0.2$ and the miss (failure) probability be $q = 1 - p = 0.8$.

(a) This is a geometric waiting-time event: two failures followed by one success.

$$\Pr(\text{first strike on 3rd}) \;=\; q^2\, p \;=\; (0.8)^2(0.2) \;=\; 0.128.$$

(b) This is a negative binomial point probability: we must have exactly 2 strikes (in any order) by the 6th well, and the 7th must be a strike.

$$\Pr(\text{3rd strike on 7th}) = \binom{7-1}{3-1} p^3\, q^{7-3} = \binom{6}{2}(0.2)^3(0.8)^4 = 0.049152.$$

(c) Let T be the total number of wells drilled to obtain $r = 3$ strikes. Then $T \sim \text{NegBin}(r = 3, p = 0.2)$ (counting total trials needed to achieve r successes). The standard formulas are

$$E[T] \;=\; \frac{r}{p} \quad \text{and} \quad \text{Var}(T) \;=\; \frac{r(1-p)}{p^2}.$$

Plugging in $r = 3$ and $p = 0.2$ gives

$$E[T] \;=\; \frac{3}{0.2} \;=\; 15, \qquad \text{Var}(T) \;=\; \frac{3(0.8)}{(0.2)^2} \;=\; \frac{2.4}{0.04} \;=\; 60.$$

2. **[6 points]** Suppose X is a discrete random variable that follows a Poisson distribution, i.e., with the following probability mass function:

$$\Pr(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \text{for } k = 0, 1, 2, \ldots$$

(a) **[3 points]** Prove that $E[X] = \lambda$;

(b) **[3 points]** What is $\mathrm{Var}[X]$?

Solution.

Let $X \sim \text{Poisson}(\lambda)$ so that $\Pr(X = k) = e^{-\lambda}\dfrac{\lambda^k}{k!}$ for $k = 0, 1, 2, \ldots$.

We will use the power-series identity $e^z = \sum_{m=0}^{\infty} \dfrac{z^m}{m!}$.

(a)

$$\begin{aligned}
E[X] &= \sum_{k=0}^{\infty} k \Pr(X = k) = \sum_{k=0}^{\infty} k\, e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=1}^{\infty}\frac{k\,\lambda^k}{k!} \\
&= e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!} = e^{-\lambda}\lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} \quad \text{(factor out } \lambda\text{)} \\
&= e^{-\lambda}\lambda\sum_{m=0}^{\infty}\frac{\lambda^m}{m!} \quad (m = k-1) \\
&= e^{-\lambda}\lambda\, e^{\lambda} = \lambda.
\end{aligned}$$

(b)

$$\begin{aligned}
E[X(X-1)] &= \sum_{k=0}^{\infty} k(k-1)\, e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=2}^{\infty}\frac{\lambda^k}{(k-2)!} \\
&= e^{-\lambda}\lambda^2\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!} = e^{-\lambda}\lambda^2\sum_{m=0}^{\infty}\frac{\lambda^m}{m!} = \lambda^2.
\end{aligned}$$

Then,

$$E[X^2] = E[X(X-1)] + E[X] = \lambda^2 + \lambda,$$

so,

$$\mathrm{Var}(X) = E[X^2] - \big(E[X]\big)^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda.$$

3. [**6 points**] A random variable X has the following probability mass function:

$$P(X = k) = \frac{1}{H_n} \cdot \frac{1}{k}, \quad \text{for } k = 1, 2, \ldots, n,$$

where $H_n = \sum_{k=1}^{n} \dfrac{1}{k}$ is the n-th harmonic number.
Compute:

(a) [**3 points**] $E[X]$ in terms of n;

(b) [**3 points**] $\mathrm{Var}[X]$ in terms of n.

Solution.
Note that $\sum_{k=1}^{n} p(k) = \frac{1}{H_n} \sum_{k=1}^{n} \frac{1}{k} = 1$, so the probability mass function is properly normalized.

(a)

$$E[X] = \sum_{k=1}^{n} k\, p(k) = \sum_{k=1}^{n} k \left(\frac{1}{H_n} \cdot \frac{1}{k} \right) = \frac{1}{H_n} \sum_{k=1}^{n} 1 = \frac{n}{H_n}.$$

(b)

$$E[X^2] = \sum_{k=1}^{n} k^2\, p(k) = \sum_{k=1}^{n} k^2 \left(\frac{1}{H_n} \cdot \frac{1}{k} \right) = \frac{1}{H_n} \sum_{k=1}^{n} k = \frac{1}{H_n} \cdot \frac{n(n+1)}{2}.$$

Therefore,

$$\mathrm{Var}(X) = E[X^2] - \big(E[X]\big)^2 = \frac{n(n+1)}{2H_n} - \frac{n^2}{H_n^2}.$$

A slightly factored form is

$$\mathrm{Var}(X) = \frac{n}{H_n^2} \left(\frac{(n+1)H_n}{2} - n \right).$$

4. **[10 points]** The normal random variable X is with parameters $\mu = 10$ and $\sigma^2 = 36$. Compute:

(a) **[2 points]** $\mathbb{P}(X > 5)$;

(b) **[2 points]** $\mathbb{P}(4 < X < 16)$;

(c) **[2 points]** $\mathbb{P}(X < 8)$;

(d) **[2 points]** $\mathbb{P}(X < 20)$;

(e) **[2 points]** $\mathbb{P}(X > 16)$.

Solution.

Let $X \sim \mathcal{N}(\mu = 10, \sigma^2 = 36)$. Then $\sigma = 6$ and

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 10}{6} \sim \mathcal{N}(0, 1).$$

(a)

$$\Pr(X > 5) = \Pr\left(Z > \frac{5 - 10}{6}\right) = \Pr(Z > -5/6) = 1 - \Phi(-5/6) = \Phi(5/6) \approx 0.7977.$$

(b)

$$\Pr(4 < X < 16) = \Pr(-1 < Z < 1) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 \approx 0.6827.$$

(c)

$$\Pr(X < 8) = \Pr\left(Z < \frac{8 - 10}{6}\right) = \Phi(-1/3) \approx 0.3694.$$

(d)

$$\Pr(X < 20) = \Pr\left(Z < \frac{20 - 10}{6}\right) = \Phi\left(\tfrac{10}{6}\right) = \Phi\left(\tfrac{5}{3}\right) \approx 0.9522.$$

(e)

$$\Pr(X > 16) = \Pr\left(Z > \frac{16 - 10}{6}\right) = 1 - \Phi(1) \approx 0.1587.$$

5. **[6 points]** Suppose $X \sim \mathcal{N}(5, \sigma^2)$ and it is known that $\mathbb{P}(X > 9) = 0.2$. Determine an *approximate* value of $\mathrm{Var}(X) = \sigma^2$. Show your standardization step and identify the z-value you use from the table.

Solution.

Given $X \sim \mathcal{N}(5, \sigma^2)$ and $\Pr(X > 9) = 0.2$. Standardize with $Z = \dfrac{X-5}{\sigma} \sim \mathcal{N}(0,1)$:

$$\Pr(X > 9) = \Pr\left(Z > \frac{9-5}{\sigma}\right) = \Pr\left(Z > \frac{4}{\sigma}\right) = 0.2.$$

Hence, $\dfrac{4}{\sigma}$ must equal the 80th percentile of the standard normal (since $\Pr(Z > z_{0.8}) = 0.2$), i.e.

$$\frac{4}{\sigma} = z_{0.8} \approx 0.8416 \quad \Rightarrow \quad \sigma \approx \frac{4}{0.8416} \approx 4.753.$$

Therefore,

$$\sigma^2 \approx (4.753)^2 \approx 22.6.$$

(Using a table value $z_{0.8} \approx 0.8416$ would give $\sigma \approx 4/0.8416 \approx 4.753$ and $\sigma^2 \approx 22.6$.)

6. [**6 points**] A model for the movement of a stock supposes that if the present price of the stock is s, then, after one period, it will be either us with probability p or ds with probability $1-p$. Assuming that successive movements are independent, approximate the probability that the stock's price will be up at least 30 percent after the next 1000 periods if $u = 1.012$, $d = 0.990$, and $p = 0.52$.

Solution.

Let $N = 1000$ be the number of periods and let K be the number of up moves among those N periods. Then,

$$K \sim \text{Binomial}(N = 1000,\ p = 0.52).$$

The terminal price is

$$S_N = s\, u^K d^{\,N-K}.$$

We want $\Pr(S_N \geq 1.3\, s)$. Since $s > 0$, this is equivalent to

$$u^K d^{\,N-K} \ \geq\ 1.3 \quad \iff \quad K \ln u + (N-K)\ln d \ \geq\ \ln(1.3).$$

Solving for K,

$$K(\ln u - \ln d) \ \geq\ \ln(1.3) - N \ln d \quad \Longrightarrow \quad K \ \geq\ k_* \ := \ \frac{\ln(1.3) - N\ln d}{\ln u - \ln d}.$$

With $u = 1.012$ and $d = 0.990$,

$$\ln u \approx 0.0119286, \qquad \ln d \approx -0.0100503,$$

so

$$k_* \approx \frac{\ln(1.3) - 1000(-0.0100503)}{0.0119286 - (-0.0100503)} \approx 469.21.$$

Because K is integer and checking the boundary,

$$u^{469} d^{531} \approx 1.294 < 1.3, \qquad u^{470} d^{530} \approx 1.323 \geq 1.3,$$

we need $K \geq 470$. Hence

$$\Pr(S_N \geq 1.3\, s) \ = \ \Pr(K \geq 470), \quad K \sim \text{Bin}(1000, 0.52).$$

Normal approximation (with continuity correction).

For $K \sim \text{Bin}(1000, 0.52)$, the mean and standard deviation are

$$\mu = Np = 520, \qquad \sigma = \sqrt{Np(1-p)} = \sqrt{1000 \cdot 0.52 \cdot 0.48} \approx 15.799.$$

Using continuity correction,

$$\Pr(K \geq 470) \ \approx \ \Pr\left(Z \geq \frac{469.5 - \mu}{\sigma}\right) = \Pr\left(Z \geq \frac{469.5 - 520}{15.799}\right) = \Pr(Z \geq -3.196) = \Phi(3.196),$$

From standard normal tables,

$$\Phi(3.196) \approx 0.9993.$$

7. [**8 points**] Let the repair time T (hours) be $\text{Exp}(\lambda = 1/2)$, i.e., with density $f_T(t) = \lambda e^{-\lambda t}$ for $t \geq 0$. Compute:

   (a) [**4 points**] the probability that a repair time exceeds 2 hours;

   (b) [**4 points**] the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours.

   Solution.

   Let $T \sim \text{Exp}(\lambda)$ with $\lambda = \frac{1}{2}$ (per hour). Its density and survival function are

$$f_T(t) = \lambda e^{-\lambda t} = \tfrac{1}{2}e^{-t/2}, \quad t \geq 0, \qquad S_T(t) = \Pr(T > t) = e^{-\lambda t} = e^{-t/2}.$$

   (a)

$$\Pr(T > 2) = S_T(2) = e^{-(1/2)\cdot 2} = e^{-1} \approx 0.3679.$$

   (b) Using the definition and then the exponential memoryless property:

$$\Pr(T \geq 10 \mid T > 9) = \frac{\Pr(T > 10)}{\Pr(T > 9)} = \frac{e^{-(1/2)\cdot 10}}{e^{-(1/2)\cdot 9}} = e^{-(1/2)} = \Pr(T > 1) \approx 0.6065.$$