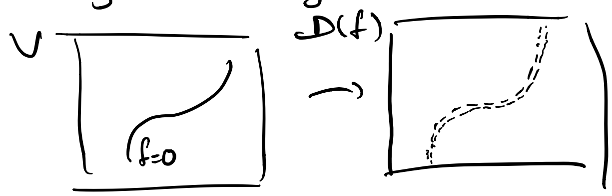
Localization (15.4)

Two main examples: (Mult. set D)

(1) For $f \in k[V]$ let $D = \{ f^{-n} : n \in \mathbb{N} \}$. Then
$k[V]_f := D^{-1}k[V]$ is the ring of regular functions
on $D(f) = V \setminus Z(f)$.



(2) For $p \subset k[V]$ prime ideal let $D = k[V] \setminus (p)$. Then
$k[V]_{(p)} := D^{-1}k[V]$ is the localization at (p).
This ring has a unique maximal ideal
$(p)k[V]_{(p)}$ and $k[V]_{(p)}$ contains info. about the
local structure around the subset corresp. to (p).
For ex. $k(x) = \mathrm{Frac}(k[x]) = k[x]_{(0)}$ and $k[x]_{(x)} = k[[x]]$.

Remark: If $p \subseteq A$ is a prime ideal
we always have a map

$$\begin{aligned} A &\longrightarrow A_p \\ a &\longmapsto \frac{a}{1} \end{aligned}$$

but this map need <u>not</u> be injective:
For ex. take $A = k[x,y]/(xy)$. Then

$$A \longrightarrow A_{(x)}$$

sends x to $\frac{x}{1} = \frac{xy}{y} = 0$.

21. $\varphi : R \longrightarrow S$ ring homo. and D' multiplicative set. Define $D = \varphi^{-1}(D')$ and show that

$$D^{-1}R \longrightarrow (D')^{-1}S$$
$$\frac{r}{d} \longmapsto \frac{\varphi(r)}{\varphi(d)}$$

is a well-def. ring homo.

<u>Solution:</u> We have $\varphi(d) \in D'$ $\forall d \in D$. ✓

We have $\frac{r}{d} + \frac{r'}{d'} = \frac{rd' + r'd}{dd'} \longmapsto \frac{\varphi(r)\varphi(d') + \varphi(r')\varphi(d)}{\varphi(d)\varphi(d')}$

$$= \frac{\varphi(r)}{\varphi(d)} + \frac{\varphi(r')}{\varphi(d')}. \quad ✓$$

<u>22.</u> Let $p \subseteq q$ be prime ideals of R. Prove that R_p is isomorphic to the localization of R_q at pR_q.

<u>Solution:</u> We have a ring map

$$\varphi : R_q \longrightarrow R_p$$
$$\frac{a}{b} \longmapsto \frac{a}{b}$$

and $\varphi^{-1}(pR_p) = pR_q$ and pR_p is the unique maximal ideal in R_p. From the previous exercise we have a ring map

$$\Upsilon : (R_q)_{pR_q} \longrightarrow (R_p)_{pR_p}.$$

But $\tilde{r} \in R_p$ is a unit iff $\tilde{r} \notin pR_p$ so $i : (R_p)_{pR_p} \cong R_p$.

if

$$i \circ \psi\left(\frac{r/d}{r'/d'}\right) = i\left(\frac{r/d}{r'/d'}\right) = \left(\frac{r'}{d'}\right)^{-1}\frac{r}{d} = 0$$

then $\frac{r}{d} = 0$ in R_p so $\exists e \in R \setminus p$: $er = 0$.
But then $\frac{e}{1} \in R_q \setminus pR_q$ and $\frac{er}{d} = 0$ so $\frac{r/d}{r'/d'} = 0$
and $i \circ \psi$ is injective.

To show that $i \circ \psi$ is surjective, take any $\frac{r}{d} \in R_p$, then $\frac{r/1}{d/1}$ is an elem. of $(R_q)_{pR_q}$ with image $\frac{r}{d}$. Hence $i \circ \psi$ is an isomorphism.

If M is an R-mod. then

$$\mathrm{Supp}(M) = \{p \in \mathrm{Spec}\,R : M_p \neq 0\}$$ (the support of M).

<u>Ex:</u> What is the support of the ring $k[x,y]/I$ viewed as a module over $k[x,y]$?

<u>Solution:</u> Let $p \in k[x,y]$ be a prime ideal. Then

$$\left(k[x,y]/I\right)_p \neq 0$$

iff $I \subseteq p$. Indeed, if $I \nsubseteq p$ then there is an $s \in I \setminus p$ and hence

$$\frac{1}{1} = \frac{s}{s} = \frac{0}{s} = 0 .$$

If $I \subseteq p$ then $\left(k[x,y]/I\right)_p = k[x,y]_p / I_p$ and I_p has no units.