Localization continued

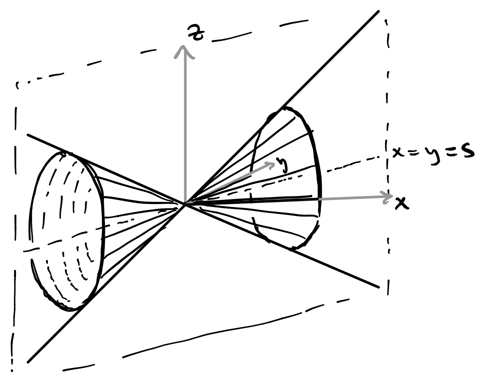
15.4.1: M f.g. R-module. Show that $D^{-1}M = 0$ iff $\exists d \in D$ s.t. $dM = 0$.

Solution: If $\exists d \in D$ s.t. $dM = 0$ then

$$\frac{m}{d'} = \frac{dm}{dd'} = 0 \qquad \forall\ m \in M,\ d' \in D.$$

Conversely, if $D^{-1}M = 0$ then if $m_1, \dots, m_n$ is a set of generators, $\exists d_1, \dots, d_n \in D$ s.t. $d_i m_i = 0$ for $1 \le i \le n$. Hence let $d = \prod_{i=1}^{n} d_i$. Then $dM = 0$.

15.4.12: Let $R = \mathbb{R}[x,y,z]/(xy - z^2)$ and $P = (\bar{x}, \bar{z})$. Show that $P^2 R_P \cap R = (\bar{x})$ and is strictly larger than P^2.



Solution: We have $\bar{z}^2 \cdot \frac{1}{\bar{y}} = \bar{x}$ in $P^2 R_P$ and hence $P^2 R_P \cap R = (\bar{x}^2, \bar{x}) = (\bar{x})$.

We have $P^2 = (\bar{x}^2, \bar{x}\bar{z}, \bar{z}^2) = (\bar{x}^2, \bar{x}\bar{z}, \bar{x}\bar{y}) \subsetneq (\bar{x})$

You can try and work out a geometric interpretation of this: R_p is the localization at the y-axis. The ring $R/p^2R_p \cap R$ corresponds to a "fat y-axis" inside the yz-plane an R/p^2 corresp. to a "fat y-axis" with some extra fatness at the origin.

<u>Claim:</u> If $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ is an exact seq. of R-modules then the $D^{-1}R$-mod. seq. $D^{-1}M' \xrightarrow{D^{-1}\alpha} D^{-1}M \xrightarrow{D^{-1}\beta} D^{-1}M''$ is exact.

<u>proof:</u> we have $D^{-1}\beta \circ D^{-1}\alpha = D^{-1}(\beta \circ \alpha) = D^{-1}(0) = 0$ and hence $\operatorname{im}(D^{-1}\alpha) \subseteq \ker(D^{-1}\beta)$. Conv., let $\frac{m}{d} \in \ker(D^{-1}\beta)$. Then $\frac{\beta(m)}{d} = 0$ so there is a $d' \in D$ s.t. $d'\beta(m) = 0$.

But $d'\beta(m) = \beta(d'm)$ so $d'm \in \ker\beta = \operatorname{Im}\alpha$ and hence $\frac{d'm}{d'd} = \frac{m}{d} \in \operatorname{im} D^{-1}\alpha$.

<u>15.4.14:</u> Let $\varphi: M \longrightarrow N$ be an R-module homomorphism. Show that the following are equivalent:

(1) $\varphi: M \longrightarrow N$ is injective (surjective)

(2) $\varphi_p: M_p \longrightarrow N_p$ —— " —— $\forall$ prime ideal $p \subset R$

(3) $\varphi_m: M_m \longrightarrow N_m$ —— " —— $\forall$ max. ideal $m \subset R$.

<u>Solution:</u> (1) $\Rightarrow$ (2) follows from the claim since $0 \to M \to N$ ($M \to N \to 0$) is exact.

(2) $\Rightarrow$ (3) follows since every max. ideal is prime.

(3) $\Rightarrow$ (1): We have that

$$0 \to \ker\varphi \to M \to N \quad (M \to N \to N/\varphi(M) \to 0) \text{ is exact}$$

and hence $0 \to (\ker\varphi)_m \to M_m \to N_m$ is exact $\forall$ max. ideals m. But $M_m \to N_m$ injective so $(\ker\varphi)_m \cong \ker\varphi_m = 0$ $\forall$ max. ideals m.

Suppose to derive a contradiction that $\ker\varphi \neq 0$ and take $x \in \ker\varphi$ non-zero. Define

$$\mathrm{Ann}(x) = \{r \in R : rx = 0\}.$$

Then $\mathrm{Ann}(x)$ is an ideal contained in some max. ideal $m \subset R$. We have $\frac{x}{1} \in (\ker\varphi)_m$ and since $(\ker\varphi)_m = 0$ $\exists\, d \in R \setminus m$ s.t. $dx = 0$. Thus $d \in \mathrm{Ann}(x) \subset m$ which contradicts. Thus $\ker\varphi = 0$.