Exercise session 7

Problem 1 (2p)**.** Let M be a compact submanifold of $\mathbf{R}^N$. Show that there is an $\epsilon > 0$ and a submersion $p\colon M^\epsilon \to M$, where $M^\epsilon = \{x \in \mathbf{R}^N \mid d(x, M) < \epsilon)\}$ and d denotes the Euclidean distance.

Solution: We use the tubular nbhd thm: There is a function $\varepsilon : M \longrightarrow (0,\infty)$ s.t.

$$\varphi : \nu_M^{<1} := \{(x,v) \in \nu_M : |v| < 1\} \longrightarrow \mathbb{R}^N$$

$$(x,v) \longmapsto x + \varepsilon(x) v$$

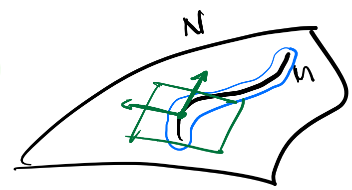
is a diffeo. onto its image M^ε. Since M is compact, the function ε obtains a minimal value $\varepsilon_0 \in (0,\infty)$. The subset $M^{\varepsilon_0} \subseteq M^\varepsilon$ is an open submanifold and hence

$$M^{\varepsilon_0} \xrightarrow{\varphi^{-1}|_{M^{\varepsilon_0}}} \nu_M^{<1} \xrightarrow{pr} M$$

is a submersion.

Submersion.

Notation: $\quad s_\varepsilon : M^\varepsilon \xrightarrow{\sim} \nu_M^{<1} \xrightarrow{pr} M.$



Problem 2 (4p)**.** Prove the following strengthening of the tubular neighborhood theorem: let $M \subset N$ be a submanifold, and define its normal bundle ν_M such that for $x \in M$, $\nu_{M,x}$ is a complement of T_xM in T_xN. Then there is a diffeomorphism between an open neighborhood of M in ν_M with an open neighborhood of M in N. (*Taken from Guillemin-Pollack, ex. II.3.16. There are some hints there as well.*)

Solution: Let $\varepsilon : N \longrightarrow (0,\infty)$ a cont. function s.t. $N^\varepsilon \subseteq \nu^{<1}_{N\subseteq \mathbb{R}^n}$. Consider

$$f : \nu_{M\subseteq N} \longrightarrow \mathbb{R}^n \supseteq N \supseteq M$$
$$(x,v) \longmapsto x+v$$

Then $f^{-1}(N^\varepsilon)$ is open and contains $M \subseteq \nu_{M\subseteq N}$. The composition $f^{-1}(N^\varepsilon) \xrightarrow{f} N^\varepsilon \xrightarrow{s_\varepsilon} N$ is the identity on $M \subseteq N$ and $d(s_\varepsilon \circ f)_x$ is an isomorphism for $x\in M$. Hence it suffices to prove:

Improved inverse function theorem: If $f : X \longrightarrow Y$ is a smooth map s.t. df_x is an isomorphism on a submanifold $Z \subseteq X$ and $f|_Z : Z \xrightarrow{\sim} f(Z)$ then f maps an open neighborhood of Z diffeomorphically onto an open neighborhood of $f(Z)$.

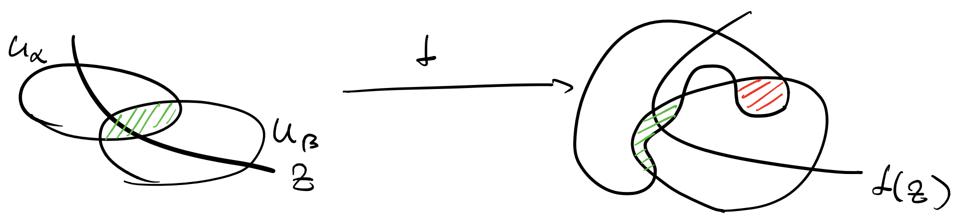
proof: By the classical inverse function theorem we have for every $x\in Z$ an open neighborhood $x \in U_x \subseteq X$ s.t. $f|_{U_x} : U_x \longrightarrow f(U_x)$

is a diffeomorphism. Let $g_x : f(U_x) \longrightarrow U_x$ denote the local inverse. By second-countability we may choose a locally finite subcollection

$$\{V_\alpha, g_\alpha\}_{\alpha \in I} \subseteq \{f(U_x), g_x\}_{x \in Z}$$

still covering $f(Z)$. The goal is to use the fact that $f|_Z$ is a diffeomorphism onto $f(Z)$ to show that the g_α's glue on some open nbhd of $f(Z)$.

Rmk: We need to resolve situations like:



Define $B_{\alpha\beta} = \{y \in V_\alpha \cap V_\beta : g_\alpha(y) \neq g_\beta(y)\} \subseteq V_\alpha \cap V_\beta$ and put $\tilde{V}_\alpha = V_\alpha \setminus \bigcup_\beta \overline{B_{\alpha\beta}}$ and $W = \bigcup_\alpha \tilde{V}_\alpha$.

We claim that $f(Z) \subseteq W$. Indeed, the V_α's cover $f(Z)$ and if $y \in V_\alpha \cap V_\beta \cap f(Z)$ then $g_\alpha(y) = g_\beta(y)$ since $f|_Z$ is injective. Hence $y \notin B_{\alpha\beta}$. Now choose an open nbhd $y \in N_y$ intersecting only finitely many V_α's. If $N_y \cap V_\beta = \emptyset$ then $N_y \cap \overline{B_{\alpha\beta}} = \emptyset$ and hence we have a finite number of sets say $\overline{B_{\alpha\beta_1}}, \dots, \overline{B_{\alpha\beta_n}}$ which y could potentially live in. But if we take N_y

to be the image of some $N_x \subseteq U_\alpha \cap U_{\beta_i}$ with

$f(x) = y$ then $g_\alpha(y) = g_{\beta_i}(y) = f(x)$ and $N_y \cap B_{\alpha\beta_i} = \emptyset$.

This is where we use the local finiteness.

Choosing $N_x \subseteq U_\alpha \cap \left(\bigcap_{i=1}^{n} U_{\beta_i}\right)$ open! we get $N_y \cap B_{\alpha\beta_i} = \emptyset \ \forall i$.

Hence $y \notin \bigcup_\beta \overline{B_{\alpha\beta}}$ and $f(Z) \subseteq W$. In fact, choosing N_y small enough we may assume $N_y \cap \overline{B_{\alpha\beta_i}} = \emptyset$ and hence we have

$$f(Z) \subseteq \bigcup_y N_y =: \tilde{W} \overset{\text{open}}{\subseteq} W.$$

Define

$$g : \tilde{W} \longrightarrow X$$
$$y \longmapsto \sum_\alpha \lambda_\alpha(y)\, g_\alpha(y). \quad (*)$$

<u>Q</u>: <u>So why is this well-defined</u>?

We have that $\lambda_\alpha(y) = 0$ for all but finitely many α, say $\alpha_1, \ldots, \alpha_e$, and

$$g_{\alpha_1}(y) = \cdots = g_{\alpha_e}(y)$$

so
$$\sum_\alpha \lambda_\alpha(y) g_\alpha(y) = \left(\sum_\alpha \lambda_\alpha(y)\right) g_{\alpha_1}(y) = g_{\alpha_1}(y).$$

Problem 3 (4p). Let $\Delta\colon M \to M \times M$ be the diagonal embedding of M into $M \times M$, i.e. $\Delta(x) = (x, x)$. Show that the map $TM \to \nu_M$ sending (x, v) to $((x, x), (v, -v))$ is a diffeomorphism. (*Taken from Guillemin-Pollack, ex. II.3.17-18. There are some hints there as well.*)

Solution: The diagonal $M \xrightarrow{\Delta} M \times M$ induces the diagonal

$$\Delta : TM \longrightarrow T(M\times M)$$
$$(x,v) \longmapsto (x,x,v,v)$$

and the vector $(v,v) \in T_xM \times T_xM$ is orthogonal to $(v,-v)$.

Hence $\nu_M = \{(x,x,v,-v) \in T(M\times M)\}$ and

$$TM \longrightarrow \nu_M$$
$$(x,v) \longmapsto (x,x,v,-v)$$ is a bijection.

We will show that

$$\Psi: TM \longrightarrow T(M\times M)$$
$$(x,v) \longmapsto (x,x,v,-v)$$

is an open (smooth) immersion.

Let (U,φ) be a chart for M giving a chart $\pi^{-1}(U) \xrightarrow[(\varphi, d\varphi_{(\cdot)})]{} \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$ for TM and a chart

$$\pi^{-1}(U\times U) \xrightarrow[(\varphi,\varphi, d\varphi_{(\cdot)}, d\varphi_{(\cdot)})]{} \varphi(U)\times\varphi(U)\times\mathbb{R}^{2n} \quad \text{for } T(M\times M).$$

In these charts the differential takes the form

$$\begin{bmatrix} I & \\ & I \\ I & \\ & -I \end{bmatrix} : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{4n}$$ which is clearly injective.

Hence Ψ is an immersion.

Now let $U \subseteq M$ be an open set. $TM \supseteq \pi^{-1}(M) \cong U \times \mathbb{R}^n$ and let $V \subseteq \mathbb{R}^n$ be open. Then we may view $U \times V$ as an open set of TM and the open sets of this form generates the topology. Hence it is enough to show that $\Psi(U \times V)$ is open. But $\Psi(U \times V) = (U \times U \times V \times (-V)) \cap \nu_M$ which is open. $\square$

Problem 4 (bonus problem, 3p). The kth *homotopy group* $\pi_k(X, x_0)$ of a pointed space (X, x_0) is defined as the set of homotopy classes of maps $\mathbf{S}^k \to X$ sending a chosen base point $*$ of $\mathbf{S}^k$ to x_0. The homotopies are also to be taken *rel* $*$.

Using smooth approximation of continuous maps, show that $\pi_k(\mathbf{S}^n) = 0$ (i.e. that every map is homotopic to the constant map) when $k < n$. (*Hint:* Every smooth map must miss a point of $\mathbf{S}^n$.)

Solution: If $k < n$ and $f : S^k \longrightarrow S^n$ is smooth then df_x is non-surjective $\forall x \in S^k$. By Sard's theorem $f(S^k)$ has measure zero. In particular, f is not surjective.

The smooth approx. theorem implies that every class in $\pi_k(S^n)$ is repr. by a smooth map $f : S^k \longrightarrow S^n$.

Let $f : S^k \longrightarrow S^n$ be smooth. Then it is not surjective. Suppose that $x \in S^n \setminus f(S^k)$. Then f

factorizes as

$$f : S^k \xrightarrow{f'} \underbrace{S^n \setminus \{x\}}_{\cong\, \mathbb{R}^{n-1} \simeq *} \xhookrightarrow{f''} S^n .$$

Hence $[f] \in \pi_k(S^n)$ lies in the image of

$$0 = \pi_k(\mathbb{R}^{n-1}) \longrightarrow \pi_k(S^n)$$

by functoriality. Hence $[f] = 0$. $\square$

$$[f] = \pi_k(f)([\mathrm{id} : S^k \to S^k]) = \pi_k(f'')([f']).$$

<u>Functoriality:</u> If $g : X \longrightarrow Y$ then we want to define

$$\begin{aligned} \pi_n(X) &\longrightarrow \pi_n(Y) \\ [f] &\longmapsto [g \circ f] \qquad (f : S^n \to X). \end{aligned}$$

This is well-defined since if $H : S^n \times I \longrightarrow X$ is a nullhomotopy for f, then $g \circ H : S^n \times I \longrightarrow Y$ is — " — $g \circ f$.

(1) <u>Composition:</u> Given $X \xrightarrow{g} Y \xrightarrow{h} Z$ we get

$$\begin{aligned} \pi_n(h \circ g)([f]) &= [h \circ g \circ f] \\ &= \pi_n(h)([g \circ f]) \\ &= \pi_n(h) \circ \pi_n(g)([f]). \end{aligned}$$

(2) <u>Identity:</u> Obvious.

Hence $\pi_k : (\mathrm{Top}_*) \longrightarrow (\mathrm{Grp})$ is a functor.