Exercise session 4

First!

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HW 4 Exercise 1 Show that every Lie group is parallelizable.

Solution: Let $(U,\varphi) \ni e$ be a chart. This yields an iso. $d\varphi_e : T_eG \xrightarrow{\sim} \mathbb{R}^n$. Define

$$f : TG \longrightarrow G \times \mathbb{R}^n$$

$$T_eG \xleftarrow{dg^{-1}_g} T_gG \ni (g,v) \longmapsto (g, d\varphi_e \circ dg^{-1}_g(v)).$$

This restricts to

(U,φ) chart on G

$$\begin{tikzcd} p^{-1}(U) \subseteq TG \arrow[r, "p"] & G \end{tikzcd}$$

$$(x,v) \longmapsto (\varphi(U), d\varphi_x(v)), \qquad p^{-1}(U) \longrightarrow \varphi(U)\times\mathbb{R}^n$$

$$\begin{tikzcd}
 & TG \arrow[r] & G\times\mathbb{R}^n & \\
 & \cup & \cup & \\
(x,v) \arrow[d, mapsto] & p^{-1}(gU) \arrow[r] \arrow[d] & gU\times\mathbb{R}^n \arrow[d] & (x, d\varphi_e\circ dx^{-1}_x(v)) \arrow[d, mapsto] \\
(\varphi(g^{-1}x), d(\varphi\circ g^{-1})_x(v)) & \tilde{U}\times\mathbb{R}^n \arrow[r, dashed] & \tilde{U}\times\mathbb{R}^n & (\varphi(g^{-1}x), d\varphi_e\circ dx^{-1}_x(v)) \\
\end{tikzcd}$$

$$d(\varphi\circ g^{-1})_x(v) = d\varphi_{g^{-1}x}\circ dg^{-1}_x(v), \qquad \tilde{U}\times\mathbb{R}^n \subset \mathbb{R}^{2n}$$

$$L(d(\varphi\circ g^{-1})_x(v)) = d\varphi_e \circ dx^{-1}_x(v)$$

$$\Rightarrow \quad L = d\varphi_e\circ dx^{-1}_x \circ dg_{g^{-1}x}\circ d\varphi^{-1}_{\varphi(g^{-1}x)}$$
$$= d\varphi_e\circ d(x^{-1}g)_{g^{-1}x}\circ d\varphi^{-1}_{\varphi(g^{-1}x)} \implies f \text{ smooth}$$

Simil. show that f^{-1} exists and is smooth.

HW4 Exercise 2 Show that

$$f: \mathbb{RP}^2 \longrightarrow \mathbb{R}^4$$

$$[x,y,z] \longmapsto (x^2-y^2, xy, xz, yz)$$

is an embedding.

Solution: Show

1. proper
2. injective
3. immersion (in particular smooth).

1.

$$\begin{tikzcd}
 & \mathbb{R}^3\setminus\{0\} \arrow[ddr, bend left, "\text{cont.}"] & \\
 & & \\
S^2 \arrow[uur, phantom, "\subseteq"] \arrow[rr] \arrow[d] & \circlearrowleft & \mathbb{R}^4 \arrow[d, phantom, "\supseteq"] \\
\mathbb{RP}^2 \arrow[urr, "f"'] \arrow[d, phantom, "\supseteq"] & & K \\
f^{-1}(K) & &
\end{tikzcd}$$

K compact $\overset{\text{Hausd.}}{\Longrightarrow}$ closed $\overset{f \text{ cont.}}{\Longrightarrow}$ $f^{-1}(K)$ closed $\underset{\mathbb{RP}^2 \text{ compact.}}{\Longrightarrow}$ $f^{-1}(K)$ compact

$\Longrightarrow$ f is proper.

2. We define an inverse locally on $\mathbb{R}^4$.

Q: When can $[x,y,z]$ be determined from $(x^2-y^2, xy, xz, yz) =: (a_1, a_2, a_3, a_4)$?

Consider cases depending on if a_2, a_3, a_4 are zero or not. This gives 8 cases: $(\{1,2,3\} \longrightarrow \{0,1\})$

- $(1,1,1)$: $a_2, a_3 \neq 0$ $(\Rightarrow a_4 \neq 0)$ then $x \neq 0$ and

$$[x,y,z] = [1, \tfrac{y}{x}, \tfrac{z}{x}] = [1, \tfrac{yz}{xz}, \tfrac{yz}{xy}] = [1, \tfrac{a_4}{a_3}, \tfrac{a_4}{a_2}].$$

- $(1,1,0)$: Not possible.
- $(1,0,1)$: Not possible.

$$a_1 = x^2 - y^2 = 2x^2 - 1$$

- $(1,0,0)$: $a_2 \neq 0$, $a_3 = a_4 = 0$ $\Rightarrow$ $x, y \neq 0$, $z = 0$: $x^2 + y^2 = 1$ $\Rightarrow$ $y^2 = 1 - x^2$

  and $x^2 - y^2 = 2x^2 - 1$ $\Rightarrow$ $[x, y, z] = [x^2, xy, 0] = [(a_1+1)/2, a_2, 0]$.

- $(0,1,1)$: Not possible

  fill in yourself!

- $(0,1,0)$: $a_3 \neq 0$, $a_2 = a_4 = 0$ $\Rightarrow$ $x, z \neq 0$, $y = 0$:

  $[x, y, z] = [\qquad] = [\qquad] = [\qquad]$.

- $(0,0,1)$: $a_2 = a_3 = 0$, $a_4 \neq 0$: $x = 0$, $y, z \neq 0$:

  $[x, y, z] = [\qquad] = [\qquad] = [\qquad]$.

- $(0,0,0)$:

$\Rightarrow$ f is injective.

3. The group $\mathbb{Z}/2\mathbb{Z}$ acts by diffeomorphisms on S^2 and the action is free and prop. discont. (as in HW 1). The quotient is $\mathbb{RP}^2$. Hence $S^2 \to \mathbb{RP}^2$ is a local diffeo. Thus it is enough to show that

$$\begin{tikzcd} S^2 \arrow[r] \arrow[d] & \mathbb{R}^4 \\ \mathbb{RP}^2 & \end{tikzcd}$$

$$S^2 \longrightarrow \mathbb{R}^4$$

$$(x,y,z) \longmapsto (x^2 - y^2, xy, xz, yz)$$

is an immersion. The map factors through

$$\mathbb{R}^3 \setminus \{0\} \longrightarrow \mathbb{R}^4$$

and the Jacobian is $J = \begin{pmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix}$.

The plane $xe_1 + ye_2 + ze_3 = 0$ (tangent space at $(x,y,z) \in S^2$) has basis $\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix}$ if $x \neq 0$ and

$$\begin{pmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} -4xy \\ x^2 - y^2 \\ -yz \\ xz \end{pmatrix}, \quad \begin{pmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix} \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix} = \begin{pmatrix} -2xz \\ -yz \\ x^2 - z^2 \\ xy \end{pmatrix}$$

are linearly independent for all $(x,y,z) \in S^2$:

Assume the converse:

$x \neq 0 \implies -4xy = -2xz \implies z = 2y$

but $xz = xy \implies z = y \implies z = y = 0$.

but $x^2 - y^2 = -yz = 0 \implies x = 0$ $\lightning$

For $x = 0$, the plane $xe_1 + ye_2 + ze_3$ has basis
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}$ and $J\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ z \\ 0 \end{pmatrix}$, $J\begin{pmatrix} 0 \\ -z \\ y \end{pmatrix} = \begin{pmatrix} 2yz \\ 0 \\ 0 \\ y^2 - z^2 \end{pmatrix}$

which are clearly indep. since y and z are not both zero.

$\implies f$ is an immersion. $\square$

HW4. Exercise 3. Show that M is metrizable using a partition of unity (not using embedding).

Solution: Let $\mathcal{U} = (U_\alpha, \varphi_\alpha)_\alpha$ be a locally finite atlas and $(\lambda_\alpha)_\alpha : M \to [0,1]$ a part. of unity subord. to $\mathcal{U}$. paracompactness

Rmk 1: $\mathcal{U}\times\mathcal{U} = (U_\alpha \times U_\beta, \varphi_\alpha \times \varphi_\beta)_{\alpha,\beta}$ is an atlas for $M\times M$ and $(\lambda_\alpha \times \lambda_\beta) : M\times M \to [0,1]$ a part. of unity subord. to $\mathcal{U}\times\mathcal{U}$.

Rmk 2: Let $d : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ be the Euclidean metric. Then for any $\varepsilon > 0$

$$d_{\leq\varepsilon}(x,y) = \min\{\varepsilon, d(x,y)\}$$

is a metric on $\mathbb{R}^n\times\mathbb{R}^n$ which is topologically equiv. to d.

Define
$$d_\alpha(x,y) = \begin{cases} d(\varphi_\alpha(x), \varphi_\alpha(y)) & \text{if } x,y \in U_\alpha, \\ 1 & \text{otherwise}. \end{cases}$$

Define
$$d_M(x,y) = \frac{1}{2}\sum_{(\alpha,\beta)} \lambda_\alpha(x)\lambda_\beta(y)\big(d_\alpha(x,y) + d_\beta(x,y)\big).$$

Then $d_M(x,y)$ is a metric. (Check yourself).

Remains to prove: The topology induced by d_M is the one we started with.

This amounts to two things:

1. If U is open and $x \in U$ then there is an $\varepsilon > 0$ s.t. $B_\varepsilon(x) = \{ y \in M : d_M(x,y) < \varepsilon \} \subseteq U$.
2. $B_\varepsilon(x)$ is open in M.

Let $x \in M$ and $U \ni x$ open and intersecting only finitely many U_α, say $U_1, \dots, U_n$.

Claim: Let $\varepsilon > 0$ be s.t. $\varphi_i^{-1}(B_\varepsilon(\varphi_i(x))) \subseteq U$ for $i = 1, \dots, n$. Then

$$\bigcap_{i=1}^n \varphi_i^{-1}(B_\varepsilon(\varphi_i(x))) \subseteq B_\varepsilon(x) \subseteq \bigcup_{i=1}^n \varphi_i^{-1}(B_\varepsilon(\varphi_i(x)))$$

proof: If $y \in \bigcap_{i=1}^n \varphi_i^{-1}(B_\varepsilon(\varphi_i(x)))$ then $d_i(x,y) < \varepsilon \ \forall i$

$$d_M(x,y) = \frac{1}{2} \sum \lambda_i(x)\lambda_j(y)(d_i(x,y) + d_j(x,y))$$
$$< \frac{1}{2} \sum \lambda_i(x)\lambda_j(y)(\varepsilon + \varepsilon) < \varepsilon$$

This proves the first incl.

If $y \notin \bigcup_{i=1}^n \varphi_i^{-1}(B_\varepsilon(\varphi_i(x)))$ then $d_i(x,y) \geq \varepsilon \ \forall i$

and $$d_M(x,y) = \frac{1}{2} \sum \lambda_i(x)\lambda_j(y)(d_i(x,y) + d_j(x,y))$$
$$\geq \frac{1}{2} \sum \lambda_i(x)\lambda_j(y)(\varepsilon + \varepsilon) \geq \varepsilon.$$

This proves the second incl. $\square$