<u>DF. 15.1.24</u>: Let $V = Z(xy-z) \subset \mathbb{A}^3$. Find an isomorphism $\varphi : Z(xy-z) \longrightarrow \mathbb{A}^2$ and the corresponding k-morphism $\tilde{\varphi} : k[\mathbb{A}^2] \longrightarrow k[V]$. Also give the inverses.

<u>Solution</u>: We start with $\tilde{\varphi}$ and get to φ from there: Clearly,

$$\tilde{\varphi} : k[x,y] \longrightarrow k[x,y,z]/(xy-z)$$
$$x \longmapsto x$$
$$y \longmapsto y$$

is a surjection since x, y, z is in the image. It is also injective since $xy - z$ cannot divide any elem. in the image but 0. Hence $\tilde{\varphi}$ is an iso. The morphism φ is defined as

$$V = \mathrm{Hom}_k(k[x,y,z]/(xy-z), k) \longrightarrow \mathrm{Hom}_k(k[x,y], k) = \mathbb{A}^2$$
$$p \longmapsto p \circ \tilde{\varphi} .$$

This takes a point $(a,b,c) \in V$ to (a,b). That is φ is just the projection onto the xy-plane. We have

$$\tilde{\varphi}^{-1}(z) = xy \quad \text{and}$$
$$\varphi^{-1}(a,b) = (a, b, ab) .$$

1. Find a $\overbrace{\text{k-algebra homomorphism}}^{=: \text{ k-morphism}}$ which corresponds to a morphism

$$\mathbb{A}^1_k \longrightarrow Z(y^2 - x^2(x+1)) \subset \mathbb{A}^2_k$$

sending 0 to $(0,-1)$ and $1, -1$ to $(0,0)$.

<u>Solution</u>: To give a point in $Z(I)$ is to give a k-morphism

$$k[x,y]/I \longrightarrow k \quad \left(\in \mathrm{Hom}_k(k[x,y]/I, k)\right)$$

which is to give a k-morphism

$$p : k[x,y] \longrightarrow k \quad \text{s.t.} \quad I \subseteq \ker p.$$

The morphism $\mathbb{A}^1 \longrightarrow Z(I)$ is induced by a k-morphism $\varphi : k[x,y]/I \longrightarrow k[T]$:

$$\mathbb{A}^1 = \mathrm{Hom}_k(k[T], k) \longrightarrow \mathrm{Hom}_k(k[x,y]/I, k) = Z(I)$$
$$f \longmapsto f \circ \varphi.$$

The point $a \in \mathbb{A}^1$ is the k-morphism which takes T to a. A k-morphism

$$k[x,y] \longrightarrow k$$

is completely determined by the image of x and y. So how to fill in the question marks?:

$$\begin{array}{ccccc} k[x,y] & \longrightarrow & k[T] & \longrightarrow & k \\ x & \longmapsto & ? & & \\ y & \longmapsto & ? & & \end{array}$$

$T = \pm 1$ should map to $(0,0)$ so $(T-1)(T+1)$ should be a factor in both $\varphi(x)$ and $\varphi(y)$. Also, $T = 0$ maps to $(-1,0)$ so T should be a factor in both $\varphi(x+1)$ and $\varphi(y)$. Hence we put

$$\varphi(x) = (T-1)(T+1)$$

$$\varphi(y) = T(T-1)(T+1).$$

This is well-def. since

$$\varphi(y^2 - x^2(x+1)) = T^2(T^2-1)^2 - (T^2-1)^2((T^2-1)+1) = 0.$$

DF. 15.1.8: Show that an R-module M over a Noetherian ring R is Noetherian iff it is finitely generated.

Solution: First assume that M is Noetherian and assume to derive a contradiction that M is **not** finitely generated. Choose $m_1 \in M$ s.t. $m_1 \neq 0$. Then $\langle m_1 \rangle \neq M$ since M not f.g. so choose $m_2 \in M \setminus \langle m_1 \rangle$ and so on. This gives an infinite chain of submodules. ↯

Conversely, if M is finitely generated then $\exists n \in \mathbb{N}$ and a surjection of R-modules

$$\phi : R^n \longrightarrow M.$$

Since R is Noetherian, it is by definition Noetherian as an R-module and so is R^n (Prove this!). Hence M is Noetherian since the sequence

$$0 \to \ker\phi \to R^n \to M \to 0 \quad \text{is exact}$$

and R^n is Noetherian.

DF 15.1.7: Prove that submodules, quotient modules and finite direct sums of Noetherian modules are again Noetherian.

Solution: Using 15.1.6:

$N \hookrightarrow M \implies 0 \to N \to M \to M/N \to 0$ exact

$M \xrightarrow{\phi} Q \implies 0 \to \ker\phi \to M \to Q \to 0$ —"—

$M_1 \oplus M_2 \implies 0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$ —"—

DF 15.1.9: Let k be a field. Show ~~that~~ any subring of $k[x]$ containing k is Noetherian. Give an example showing that such rings need not be UFD's. [HINT]

Solution: we follow the hint in DF: Take R st. $k \subset R \subseteq k[x]$ ($R = k$ is ~~trivial~~) and pick $y \in R \setminus k$. Then y is a polynomial in x of degree say n. A polynomial g can be written as

$$g = a_{k,s}x^k y^s + a_{k-1,s}x^{k-1}y^s + \dots + a_{0,s}y^s + a_{k,s-1}x^k y^{s-1} + \dots + a_{00}$$

where $k < n$ and $k + sn = m$. Hence $k[x]$ is gen. as a $k[y]$-module by $1, x, \dots, x^k$, i.e. $k[x]$ is a f.g. $k[y]$-mod.

By [DF, 15.1.8] $k[x]$ is a Noetherian $k[y]$-mod. By [DF, 15.1.7] R is Noetherian as a $k[y]$-module since it is a submodule of ~~the~~ Noetherian $k[y]$-mod. $k[x]$. But every chain of ideals in R is a chain of submodules of R and hence R is Noetherian as a ring.

For the second part, the subring $k[x^2, x^3]$ is not a UFD since x^6 may be factored either as $(x^2)^3$ or as $(x^3)^2$.

<u>DF 15.1.10:</u> Show that the subring

$$k[x, x^2y, x^3y^2, \dots] \subset k[x,y]$$

is <u>not</u> Noetherian.

<u>Solution:</u> Consider the chain of ideals

$$(x) \subseteq (x, x^2y) \subseteq (x, x^2y, x^3y^2) \subseteq \dots$$

This chain doesn't stabilize since $x^n y^{n-1} \notin (x, x^2y, \dots, x^{n-1}y^{n-2})$. Indeed, the exponent of x in

$$x^{a_1}(x^2y)^{a_2} \cdots (x^{n-1}y^{n-2})^{a_{n-1}}$$

is $e_x := a_1 + 2a_2 + \dots + (n-1)a_{n-1}$ and the exponent of y is $e_y := a^2 + 2a_3 + \dots + (n-2)a_{n-1}$ so if $e_x = n$ then $e_y = n - (a_1 + a_2 + \dots + a_{n-1})$ which is $n-1$ iff $a_1 + \cdots + a_{n-1} = 1$ but this is not possible since $a_i \geq 0$ and at least two a_i must must be non-zero since $e_x = n$.