Recall that if V is a smooth hypersurface in k^n cut out by an equation

$$f(x_1, \dots, x_n) = 0$$

then the gradient $\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_i) e_i$ evaluated at a point $v \in V$ is a vector which is normal to V at v. If we define

$$D_v(f)(x_1, \dots, x_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(v) x_i$$

then if $\bar{a} = (a_1, \dots, a_n)$ is a point in k^n, we have

$$\langle \nabla f(v) \mid \bar{a} \rangle = D_v(f)(a_1, \dots, a_n).$$

This implies that

$$Z(D_v(f)(x_1, \dots, x_n))$$

consists exactly of those points (vectors) that are orthogonal to ∇f. Hence it makes sense to define the <u>tangent space</u> at $v \in V$ by

$$T_{V,v} = Z(D_v(f)(x_1, \dots, x_n)),$$

or more generally, if $V = Z(I)$ then

$$T_{V,v} = Z(\{D_v(f)(x_1, \dots, x_n) \mid f \in I\}).$$

<u>Remark:</u> At a singularity, the gradient will vanish.

26. (*Differentials of Morphisms*) Let $\varphi : V \to W$ be a morphism of affine varieties over the algebraically closed field k and suppose $\varphi(v) = w$.

(a) Show that φ induces a linear map from the k-vector space M_w/M_w^2 to the k-vector space M_v/M_v^2, and use this to show that φ induces a linear map $d\varphi$ (called the *differential* of φ) from the k-vector space $\mathbb{T}_{v,V}$ to the k-vector space $\mathbb{T}_{w,W}$.

(b) Prove that if $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ and $\varphi = (F_1(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n))$ then $d\varphi : \mathbb{T}_{v,V} \to \mathbb{T}_{w,W}$ is given explicitly by

$$(d\varphi)(a_1, \dots, a_n) = (D_v(F_1)(a_1, \dots, a_n), \dots, D_v(F_m)(a_1, \dots, a_n)).$$

[If $g = g(y_1, \dots, y_m)$ show that the chain rule implies

$$\frac{\partial(g \circ \varphi)}{\partial x_i}(v) = \sum_{j=1}^{m} \frac{\partial g}{\partial y_j}(w)\,\frac{\partial F_j}{\partial x_i}(v),$$

so that $D_v(g \circ \varphi)(a_1, \dots, a_n) = D_w(g)(b_1, \dots, b_m)$ where $b_j = D_v(F_j)(a_1, \dots, a_n)$. Then use the fact that $g \circ \varphi \in \mathcal{I}(V)$ if $g \in \mathcal{I}(W)$.]

(c) If $\psi : U \to V$ is another morphism with $\psi(u) = v$, prove that the associated $d(\varphi \circ \psi) : \mathbb{T}_{u,U} \to \mathbb{T}_{w,W}$ is the same as $d\varphi \circ d\psi$.

(d) Prove that if φ is an isomorphism then $d\varphi$ is a vector space isomorphism from $\mathbb{T}_{v,V}$ to $\mathbb{T}_{w,W}$ for every $\varphi(v) = w$.

<u>Solution:</u>

a) The morphism $\varphi: V \to W$ is determined by a morphism $\tilde{\varphi}: k[W] \to k[V]$ and $\varphi(v) = w$ means that $\tilde{\varphi}^{-1}(I(v)) = I(w)$ where $I(v) = M_v/I(V)$ and $I(w) = M_w/I(W)$. Hence we get an induced map

$$\underset{\text{local}}{\varphi_L}: k[W]_{I(w)} = \mathcal{O}_{w,W} \longrightarrow \mathcal{O}_{v,V} = k[V]_{I(v)}$$
$$\frac{f}{g} \longmapsto \frac{\tilde{\varphi}(f)}{\tilde{\varphi}(g)}.$$

We have $m_v = I(v)k[V]_{I(v)}$ and $m_w = I(w)k[W]_{I(w)}$, and hence $m_w \subseteq \varphi_L^{-1}(m_v)$. This gives an induced map $m_w \longrightarrow m_v$ and since m_w^2 must map into m_v^2 we have a map

$$\Psi: m_w/m_w^2 \longrightarrow m_v/m_v^2.$$

The map of tangent spaces is defined by

$$d\varphi: T_{v,V} \cong \mathrm{Hom}_k(m_v/m_v^2, k) \longrightarrow \mathrm{Hom}_k(m_w/m_w^2, k) \cong T_{w,W}$$
$$\alpha \longmapsto \alpha \circ \Psi.$$

we have for $u \in m_w/m_w^2$ that

$$d\varphi(a\alpha+b\beta)(u) = (a\alpha+b\beta)(\psi(u))$$
$$= a\alpha(\psi(u)) + b\beta(\psi(u))$$
$$= a(\alpha\circ\psi)(u) + b(\beta\circ\psi)(u)$$
$$= a\,d\varphi(\alpha)(u) + b\,d\varphi(\beta)(u)$$

and hence $d\varphi$ is k-linear.

b) The map

$$\tilde{\varphi}: k[W] \longrightarrow k[V] \quad \text{is given by}$$
$$p \longmapsto p\circ F$$

where $p\circ F(x_1,\dots,x_n) = p(F_1(x_1,\dots x_n),\dots,F_m(x_1,\dots,x_n))$. We have $m_w/m_w^2 \cong M_w/M_w^2+I(W) \xrightarrow{\tilde{\varphi}} m_v/m_v^2 \cong M_v/M_v^2+I(V)$.

The iso.

$$D: M_v/M_v^2+I(V) \xrightarrow{\sim} (T_{V,v})^*$$

takes $x_i - v_i$ to the element x_i which is the projection onto the i:th coordinate. Hence the induces map $T_{V,v} \cong ((T_{V,v})^*)^* \longrightarrow (M_v/M_v^2+I(V))^*$ takes

$$(f:(T_{V,v})^* \to k) \mapsto f\circ D$$

and $f\circ D(x_i - v_i) = f(x_i)$. But the iso.

$$((T_{V,v})^*)^* \xrightarrow{\sim} T_{V,v} \quad \text{takes a function}$$

$f:(T_{V,v})^* \to k$ to the point $(f(x_1),\dots,f(x_n))$. Hence

$$T_{V,v} \xrightarrow{\sim} (M_v/M_v^2+I(V))^* \quad \text{is given by}$$
$$a \longmapsto e_v(a+v),$$

the evaluation map at $a+v$. Thus we want to show that

$$d\varphi(e_v(a+v)) = e_v(D_v(F)(a)+w)$$

where $D_v(F)(a) := (D_v(F_1)(a), \ldots, D_v(F_m)(a))$. Note that

$$e_v(D_v(F)(a)+w)(y_i - w_i) = D_v(F_i)(a).$$

We have

$$d\varphi(e_v(a+v))(y_i - w_i) = e_v(a+v)(F_i(x) - w_i)$$

and $F_i(\bar{x})$ has Taylor expansion

$$F_i(x) = F_i(v) + \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j}(v)(x_j - v_j) + \text{"Higher order terms"}$$

where all higher order terms lie in M_v^2.

Hence $F_i(x) = F_i(v) + D_v(F_i)(x-v) \pmod{M_v^2 + I(V)}$ and

$$e_v(a+v)(F_i(v) + D_v(F_i)(x-v) - w_i) = e_v(a+v)(D_v(F_i)(x-v))$$
$$= D_v(F_i)(a). \qquad \square$$