3 DF. 15.1.8: Show that an R-module M over a Noetherian ring R is Noetherian iff it is finitely generated.

Solution: First assume that M is Noetherian and assume to derive a contradiction that M is not finitely generated. Choose $m_1 \in M$ s.t. $m_1 \neq 0$. Then $\langle m_1 \rangle \neq M$ since M not f.g. so choose $m_2 \in M \setminus \langle m_1 \rangle$ and so on. This gives an infinite chain of submodules. ↯

Conversely, if M is finitely generated then $\exists n \in \mathbb{N}$ and a surjection of R-modules

$$\phi : R^n \longrightarrow M.$$

Since R is Noetherian, it is by definition Noetherian as an R-module and so is R^n (Prove this!). Hence M is Noetherian since the sequence

$$0 \to \ker \phi \to R^n \to M \to 0 \quad \text{is exact}$$

and R^n is Noetherian.

5

<u>DF. 15.1.24:</u> Let $V = Z(xy-z) \subset \mathbb{A}^3$. Find an isomorphism $\varphi : Z(xy-z) \longrightarrow \mathbb{A}^2$ and the corresponding k-morphism $\bar{\varphi} : k[\mathbb{A}^2] \longrightarrow k[V]$. Also give the inverses.

Is $Z(xy-x^2)$ isomorphic to $\mathbb{A}^2$?

<u>Solution:</u> we start with $\bar{\varphi}$ and get to φ from there: Clearly,

$$\begin{aligned} \bar{\varphi} : k[x,y] &\longrightarrow k[x,y,z]/(xy-z) \\ x &\longmapsto x \\ y &\longmapsto y \end{aligned}$$

is a surjection since x, y, z is in the image. It is also injective since $xy-z$ cannot divide any elem. in the image but 0. Hence $\bar{\varphi}$ is an iso. The morphism φ is defined as

$$\begin{aligned} V = \mathrm{Hom}_k(k[x,y,z]/(xy-z), k) &\longrightarrow \mathrm{Hom}_k(k[x,y], k) = \mathbb{A}^2 \\ p &\longmapsto p \circ \bar{\varphi}. \end{aligned}$$

This takes a point $(a,b,c) \in V$ to (a,b). That is φ is just the projection onto the xy-plane. We have

$$\bar{\varphi}^{-1}(z) = xy \quad \text{and}$$

$$\varphi^{-1}(a,b) = (a,b,ab).$$

1

DF 15.1.7: Prove that submodules, quotient modules and finite direct sums of Noetherian modules are again Noetherian.

Solution: Using 15.1.6:

$N \hookrightarrow M \implies 0 \to N \to M \to M/N \to 0$ exact

$M \xrightarrow{\phi} Q \implies 0 \to \ker\phi \to M \to Q \to 0$ — " —

$M_1 \oplus M_2 \implies 0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$ — " —

4

DF 15.1.9: Let k be a field. Show ~~that~~ any subring of $k[x]$ containing k is Noetherian. Give an example showing that such rings need not be UFD's. [HINT]

Solution: we follow the hint in DF:

Take R s.t. $k \subset R \subseteq k[x]$ ($R=k$ is ~~trivial~~) and pick $y \in R \setminus k$. Then y is a polynomial in x of degree say n. A polynomial g can be written as

$$g = a_{k,s}x^k y^s + a_{k-1,s}x^{k-1}y^s + \dots + a_{0,s}y^s + a_{k,s-1}x^k y^{s-1} + \dots + a_{00}$$

where $k < n$ and $k + sn = m$. Hence $k[x]$ is gen. as a $k[y]$-module by $1, x, \dots, x^k$, i.e. $k[x]$ is a f.g. $k[y]$-mod.

By [DF, 15.1.8] $k[x]$ is a Noetherian $k[y]$-mod. By [DF, 15.1.7] R is Noetherian as a $k[y]$-module since it is a submodule of ~~the~~ the Noetherian $k[y]$-mod. $k[x]$. But every chain of ideals in R is a chain of submodules of R and hence R is Noetherian as a ring.

For the second part, the subring $k[x^2, x^3]$ is not a UFD since x^6 may be factored either as $(x^2)^3$ or as $(x^3)^2$.

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6 <u>DF 15.1.10:</u> Show that the subring
$$k[x, x^2y, x^3y^2, \ldots] \subset k[x,y]$$
is <u>not</u> Noetherian.

<u>Solution:</u> Consider the chain of ideals
$$(x) \subseteq (x, x^2y) \subseteq (x, x^2y, x^3y^2) \subseteq \cdots .$$
This chain doesn't stabilize since $x^n y^{n-1} \notin (x, x^2y, \ldots, x^{n-1}y^{n-2})$. Indeed, the exponent of x in
$$x^{a_1}(x^2y)^{a_2} \cdots (x^{n-1}y^{n-2})^{a_{n-1}}$$
is $e_x := a_1 + 2a_2 + \cdots + (n-1)a_{n-1}$ and the exponent of y is $e_y := a^2 + 2a_3 + \cdots + (n-2)a_{n-1}$ so if $e_x = n$ then $e_y = n - (a_1 + a_2 + \cdots + a_{n-1})$ which

is $n-1$ iff $a_1 + \cdots + a_{n-1} = 1$ but this is not possible since $a_i \geq 0$ and at least two a_i must must be non-zero since $e_x = n$.
