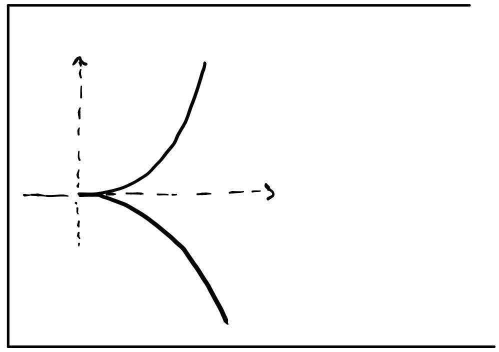
Ex (Cuspidal cubic): Let k be an infinite field and let $W = Z(x^3 - y^2)$: we have a morphism



$$f : \mathbb{A}^1 \longrightarrow W$$
$$a \longmapsto (a^2, a^3)$$

which is injective since if $b^2 = a^2$ and $b^3 = a^3$ then

$$b^2(a-b) = 0 \quad \text{and}$$

since k is a field we have $a = b$.
It is also surjective since if $\alpha^3 = \beta^2$, then $\beta = a^3$ where a is a root of α and hence $(\alpha, \beta) = (a^2, a^3)$.

The k-algebra morphism inducing f is

$$\varphi : k[x,y]/(x^3 - y^2) \longrightarrow k[t]$$
$$x \longmapsto t^2$$
$$y \longmapsto t^3$$

which is clearly <u>not</u> surjective since t is not in the image. So f is <u>not</u> an isomorphism but a bijection.

The set-theoretic inverse is given by $W \longrightarrow \mathbb{A}^1$; $(p,q) \longmapsto \frac{q}{p}$ (not polynomial)

DF 15.2.13: Prove that an affine algebraic set V is connected in the Zariski topology if and only if $k[V]$ is not a direct sum of two non-zero ideals.

Solution: If $k[V] = \bar{J}_1 \oplus \bar{J}_2$ ($\bar{J}_1 \neq \emptyset$, $\bar{J}_2 \neq \emptyset$) then let $J_1, J_2 \subset k[x_1, \ldots, x_n]$ be the corresponding ideal containing the ideal I defining V. Choose $a_1 \in J_1$, $a_2 \in J_2$ s.t. $1 = \bar{a}_1 + \bar{a}_2 \iff a_1 + a_2 \in 1 + I$, where $\bar{a}_1, \bar{a}_2$ are the images in $\bar{J}_1, \bar{J}_2$ resp. Since $k[V] = \bar{J}_1 \oplus \bar{J}_2$, we have $\bar{a}_1\bar{a}_2 = 0 \iff a_1a_2 \in I$. Hence

$$\begin{aligned}
(V \cap Z(a_1)) \cup (V \cap Z(a_2)) &= V \cap (Z(a_1) \cup Z(a_2)) \\
&= Z(I) \cap Z(a_1a_2) \\
&= Z((a_1a_2) + I) \\
&= Z(I) \\
&= V \quad \text{since } a_1a_2 \in I, \text{ and}
\end{aligned}$$

$$\begin{aligned}
(V \cap Z(a_1)) \cap (V \cap Z(a_2)) &= V \cap Z(a_1) \cap Z(a_2) \\
&= Z(I + (a_1) + (a_2)) \\
&= Z((1)) \\
&= \emptyset \quad \text{since } a_1 + a_2 \in 1 + I.
\end{aligned}$$

So V is a disjoint union of $V \cap Z(a_1)$ and $V \cap Z(a_2)$ which are closed and hence also open $\Rightarrow$ V disconnected.

Conversely, if V is disconnected, write $V = Z(I) \cup Z(J) = Z(IJ)$ with

$$Z(I) \cap Z(J) = Z(I+J) = \emptyset$$

This means that $1 \in I + J$. Hence we may write $k[V] = \frac{I}{IJ} + \frac{J}{IJ}$. But obviously, if $f \in I$ and $f \in J$ then $f \in IJ$ so we have $k[V] = \frac{I}{IJ} \oplus \frac{J}{IJ}$. $\square$
