28. If k is a field, the quotient $k[x]/(x^2)$ is called the *ring of dual numbers* over k. If V is an affine algebraic set over k, show that a k-algebra homomorphism from $k[V]$ to $k[x]/(x^2)$ is equivalent to specifying a point $v \in V$ with $\mathcal{O}_{v,V}/\mathfrak{m}_{v,V} = k$ (called a *k-rational point* of V) together with an element in the tangent space $\mathbb{T}_{v,V}$ of V at v.

<u>Solution:</u> A point $v \in V$ with $\mathcal{O}_{v,V}/\mathfrak{m}_v = k$ is given by a k-morphism $k[V] \longrightarrow k$ and since there is a unique k-morphism $k[x]/(x^2) \xrightarrow{p} k$ (sending x to 0), a k-morphism $k[V] \xrightarrow{\varphi} k[x]/(x^2)$ gives a point

$$v : k[V] \xrightarrow{\varphi} k[x]/(x^2) \xrightarrow{p} k \quad \text{by composition.}$$

The ring $k[x]/(x^2)$ is a local ring with maximal ideal (x) ($x-a$ is invertible when a a unit since $(x-a)(x+a) = -a^2$) and $\varphi^{-1}(x) = (p\circ\varphi)^{-1}(0) = \mathfrak{m}_v$. Hence $\mathfrak{m}_v^2$ is sent to $(x^2) = (0)$ in $k[x]/(x^2)$ and we have a map

$$\mathfrak{m}_v/\mathfrak{m}_v^2 \longrightarrow (x)k[x]/(x^2) \underset{\substack{\uparrow \\ \text{As a } k\text{-vector space.}}}{\cong} k$$

This is an elem. of $(\mathfrak{m}_v/\mathfrak{m}_v^2)^* \cong \mathbb{T}_{v,V}$.

<u>Ex: The prime ideals of $\mathbb{Z}[x]$.</u>

Let B be a PID and $K = \operatorname{Frac} B$. Then the prime ideals of $B[x]$ are of the form

(i) (0),

(ii) (f) for $f \in B[x]$ irreducible, or

(iii) $\mathfrak{m}$ maximal, and in this case $\mathfrak{m}$ is of the form (p, q) where $p \in B$ irreducible and $q \pmod p \in B[x]/(p)$ is irreducible.

<u>proof</u>: Let P be a prime ideal in $B[x]$ which is not principal. Then P contains two elem. $a, b \in B[x]$ that are relatively prime. By Gauss lemma they also have no common factor in $K[x]$ (clearing denominators). Hence there are elem. $s, t \in K[x]$:

$$sa + tb = 1$$

and if $d \in B$ is a common denominator for s and t then $d(sa+tb) = d \in B$ so

$$(a, b) \cap B \subseteq P \cap B \neq 0.$$

Hence $P \cap B$ is a non-zero prime ideal of B but B is a PID so $P \cap B$ is maximal. Hence $P \cap B = (p)$ for some irred. $p \in B$.

We know that $(B/(p))[x] = B[x]/(p)$ is a PID since $B/(p)$ is a field. Hence $P/(p)$ is principal and gen. by some irred. $\tilde{q}$. Let q an elem. mapping to $\tilde{q}$ under $P \longrightarrow P/(p)$. Then $P = (p, q)$. $\square$
