Exercise session 1

- Smooth manifolds, Maps of manifolds
- Tangent spaces, immersions, submersions, embeddings

Recollection of definitions:

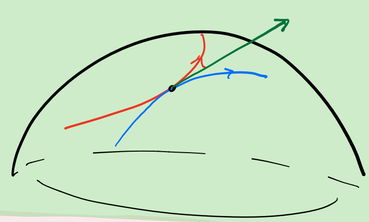
- Smooth manifold (of dim n): Second count., top. Hausdorff space with a smooth structure = equivalence class of smooth atlases.

- Smooth map: $f: X \longrightarrow Y$ cont. and for every $x \in X$ $\exists$ charts $(U,\varphi),(V,\psi)$ with $x \in U$, $f(x) \in V$, $U \subseteq f^{-1}(V)$ s.t.

$$\begin{tikzcd} U \arrow[r, "f|_U"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "\psi"] \\ \varphi(U) \arrow[r, dashed, "\psi\circ f|_U \circ \varphi^{-1}"'] & \psi(V) \end{tikzcd}$$

(φ, ψ homeomorphisms) is smooth.

- Tangent space: For $x \in X$ pick chart $x \in U$ with $\varphi(x) = 0$.
$T_xM = \{\text{smooth } c: \mathbb{R} \longrightarrow M,\ c(0) = x\} / \sim$, where $c_1 \sim c_2$ if $(\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$:



Fact: This is indep. of the chart (U, φ).

- $f: M \longrightarrow N$, $df_x: T_xM \longrightarrow T_{f(x)}N$, $[c] \longmapsto [f \circ c]$
  i) Immersion: df_x inj. $\forall x \in X$
  ii) Submersion: df_x surj. $\forall x \in X$

iii) Embedding: (inj. immersion + homeo. onto image.)
= Diffeomorphism onto its image

(*) proof: $\varphi : M \to N$ diffeo. $\Rightarrow$ $\varphi^{-1} : N \to M$ smooth

$$\Rightarrow \quad d(\varphi^{-1})_{\varphi(x)} \circ d(\varphi)_x = d(\varphi^{-1} \circ \varphi)_x = d(id_M)_x = id$$

$\Rightarrow$ $d(\varphi)_x$ injective. Hence φ is an embedding.

Conversely, if inj. immersion + homeo. onto image: Use inverse function theorem to conclude that φ is a local diffeomorphism and hence a diffeo. since homeo.

Warm up: If M and N are manifolds of dim m and n resp. with $m \neq n$ then $M \not\cong N$.

means: not diffeomorphic

Solution:

Consequence: $\mathbb{R}^m \not\cong \mathbb{R}^n$ if $m \neq n$.

Question: Is the cuspidal curve  $\subseteq \mathbb{R}^2$ given by the equation $y^2 = x^3$ a smooth manifold?

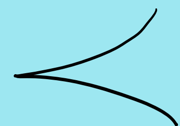
Answer: It is not a submanifold of $\mathbb{R}^2$ since the tangent space dim at $(0,0)$ is two dim.

But! As an abstract manifold it has a smooth str.

Determine in the following examples if the map f is an immersion, submersion, or an embedding.

1. <u>Cuspidal cubic:</u>

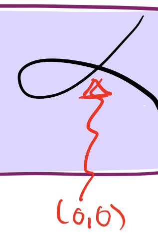
$$f : \mathbb{R}^1 \longrightarrow \mathbb{R}^2$$
$$x \longmapsto (x^2, x^3)$$



<u>Solution:</u> $df_x = \begin{pmatrix} 2x \\ 3x^2 \end{pmatrix}$ is zero at $x=0$. Hence not injective. So f is <u>not</u> an immersion.

$y^2 = x^2(x+1)$

2. <u>Nodal cubic:</u> $f : \mathbb{R} \longrightarrow \mathbb{R}^2$
$$x \longmapsto (x^2-1, \; x(x^2-1))$$



<u>Solution:</u> $df_x = \begin{pmatrix} 2x \\ x^2-1+x(2x) \end{pmatrix}$

$\Rightarrow df_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ which is injective.

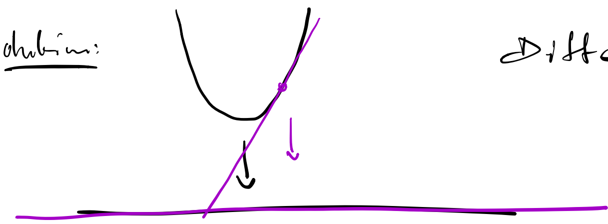
$df_{a\neq 0} = \begin{pmatrix} 2a \\ ? \end{pmatrix}$ — " — .

Hence f is an immersion.

3. $P = \{(x,y) \in \mathbb{R}^2 : y = x^2\} \xrightarrow{f} \mathbb{R}$

$(x,y) \longmapsto x.$

<u>Solution:</u>

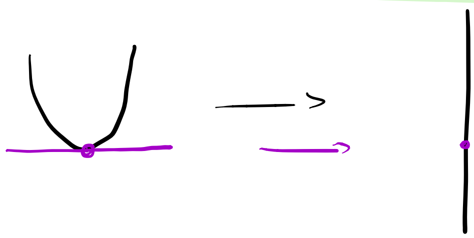


Diffeomorphism!

4. $P = \{(x,y) \in \mathbb{R}^2 : y = x^2\} \xrightarrow{f} \mathbb{R}$

$(x,y) \longmapsto y.$

<u>Solution:</u>



$$\mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$x \longmapsto (x, x^2) \longmapsto x^2$$

$\Longrightarrow \; df_x = 2x$

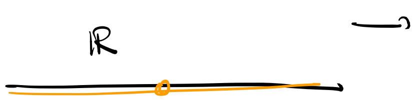
which vanishes at $x=0$.

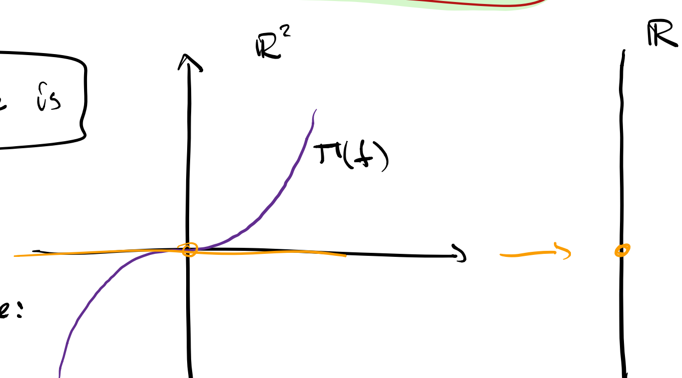
$\Longrightarrow$ Not an immersion.

5. $\mathbb{R} \xrightarrow{f} \mathbb{R}$ Diffeomorphism?

$x \longmapsto x^3.$

<u>Solution:</u> No! The inverse is not smooth





Tangent space perspective:

<u>not</u> an immersion.

$$x \longmapsto (x, x^3) \longmapsto x^3$$

Rmk: we defined (for a chart $x \in (U, \varphi)$)

$$T_xM = \{ c : \mathbb{R} \to M : c(0) = x \} / \sim$$

with $c_1 \sim c_2 \iff (\varphi \circ c_1)'(x) = (\varphi \circ c_2)'(x)$.

As you saw in the lecture, the map

$$(n = \dim M) \qquad \begin{aligned} T_xM &\longrightarrow \mathbb{R}^n \\ [c] &\longmapsto (\varphi \circ c)'(0) \end{aligned}$$

For each chart at x we have a canonical basis for T_xM!

is an isomorphism.

Furthermore, if (U, φ) and (V, ψ) are charts at x then we have a commutative diagram

$$\begin{tikzcd}
 & T_xM \arrow[dl, "{[c] \mapsto (\varphi \circ c)'(0)}"'] \arrow[dr, "{[c] \mapsto (\psi \circ c)'(0)}"] \arrow[d, phantom, "\circlearrowleft"] & \\
\mathbb{R}^n \arrow[rr, "d(\psi \circ \varphi^{-1})"'] & & \mathbb{R}^n
\end{tikzcd}$$

We saw that the differentials satisfy

$$d(g \circ f)_x = dg_{f(x)} \circ df_x \quad \text{and} \quad d(id)_x = id_{T_xM}.$$

Remark: We often don't need to know what T_xM is explicitely.

All you need to know is that for every entry in the left column we have the following data:

1. $(\overset{\dim n}{M}, x)$	T_xM n-dim vector space
2. Chart $(U,\varphi)\ni x$	iso. $b_\varphi : T_xM \xrightarrow{\cong} \mathbb{R}^n$ (a basis's)
3. $(U,\varphi)\ni x \in (V,\psi)$	$\begin{tikzcd} & T_xM \arrow[dl, "b_\varphi"'] \arrow[dr, "b_\psi"] & \\ \mathbb{R}^n \arrow[rr, "d(\psi\circ\varphi^{-1})"'] \arrow[rr, phantom, shift left=4, "\circlearrowleft"] & & \mathbb{R}^n \end{tikzcd}$
4. $\begin{tikzcd} f: \underset{\dim m}{M} \arrow[r] & \underset{\dim n}{N} \\ U \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "f\vert_U"] \arrow[d, "\varphi"'] & V \arrow[u, phantom, "\cup"] \arrow[d, "\psi"] \\ \varphi(U) \arrow[r] & \psi(V) \end{tikzcd}$	$\begin{tikzcd} df_x : T_xM \arrow[r] \arrow[d, "b_\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & T_{f(x)}N \arrow[d, "b_\psi"] \\ \mathbb{R}^m \arrow[r, "\boxed{d(\psi\circ f\vert_U\circ(\varphi\vert_U)^{-1})}"'] & \mathbb{R}^n \end{tikzcd}$
5. $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$	$\begin{tikzcd} T_xM_1 \arrow[r, "df_x"] \arrow[dr, "d(g\circ f)_x"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & T_{f(x)}M_2 \arrow[d, "dg_{f(x)}"] \\ & T_{g(f(x))}M_3 \end{tikzcd}$
6. $id: M \to M$	$d(id)_x = id_{T_xM} : T_xM \to T_xM$

The explicit construction of tangent spaces is necessary in order to show that such data exists.

1. The circle $S^1 \subseteq \mathbb{R}^2$ is a manifold.

Solution:

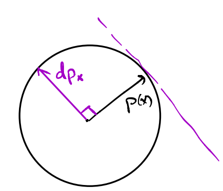
2. If X, Y are manifolds of dim m and n resp. then $X \times Y$ is a manifold of dim $m+n$.
In particular, $S^1 \times S^1 =: \mathbb{T}$ is a manifold of dim 2.

Solution:

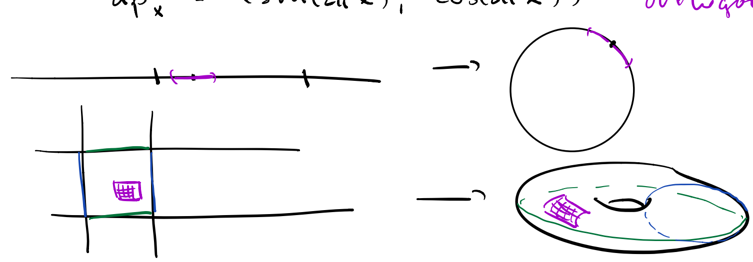
3. $\mathbb{R}^1 \longrightarrow S^1$ is a local diffeomorphism and so is the induced map $\mathbb{R}^1 \times \mathbb{R}^1 \longrightarrow S^1 \times S^1 = \mathbb{T}$.

Solution: $\mathbb{R}^1 \xrightarrow{p} S^1$

$x \longmapsto (\cos(2\pi x), \sin(2\pi x)) = p(x)$



$\Rightarrow \; dp_x = (\sin(2\pi x), -\cos(2\pi x))$ orthogonal to $p(x)$!



4. If $L \subseteq \mathbb{R}^1 \times \mathbb{R}^1$ is a line, then

$$L \hookrightarrow \mathbb{R}^1 \times \mathbb{R}^1 \longrightarrow \mathbb{T}$$

is an immersion.

<u>Solution:</u> Wlog we may assume that L passes through the origin and is spanned by some $(x_0, y_0) \in \mathbb{R}^1 \times \mathbb{R}^1$: $\mathbb{R} \xrightarrow{\cong} L$; $a \longmapsto (ax_0, ay_0)$.

The differential of
$$\mathbb{R}^1 \times \mathbb{R}^1 \xrightarrow{f} \mathbb{T} \subseteq \mathbb{R}^4$$
$$(x,y) \longmapsto (\cos(2\pi x), \sin(2\pi x), \cos(2\pi y), \sin(2\pi y))$$

given by the Jacobian
$$\begin{pmatrix} \sin(2\pi x) & 0 \\ -\cos(2\pi x) & 0 \\ 0 & \sin(2\pi y) \\ 0 & -\cos(2\pi y) \end{pmatrix}$$
and the restriction to L is given by
$$\begin{pmatrix} \sin(2\pi a x_0) \\ -\cos(2\pi a x_0) \\ \sin(2\pi a y_0) \\ -\cos(2\pi a y_0) \end{pmatrix} =: A$$
which is injective for all $a \in \mathbb{R}$ since $\sin(\theta)$ and $\cos(\theta)$ can never vanish simult.

Now chain rule: $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \Rightarrow d(f_2 \circ f_1)_x = df_{2\,f_1(x)} \circ df_{1\,x}$ so $d(f_2 \circ f_1)_x$ inj. $\Rightarrow df_{1\,x}$ inj.

5. If $L \subseteq \mathbb{R}^1 \times \mathbb{R}^1$ is a line of irrational slope, then the composition

$$L \hookrightarrow \mathbb{R}^1 \times \mathbb{R}^1 \longrightarrow \mathbb{T}$$

is injective.

<u>Solution:</u>

$$\mathbb{R}^1 \times \mathbb{R}^1 \longrightarrow \mathbb{T}$$
$$(x, y) \longmapsto (e^{2\pi i x}, e^{2\pi i y}).$$

If L irrat. slope s ~~then~~ $\mathbb{R} \xrightarrow{\cong} L$
$x \longmapsto (x, s(x))$.

Want to show that

$$\mathbb{R} \longrightarrow \mathbb{T}$$
$$x \longmapsto (e^{2\pi i x}, e^{2\pi i s x}) \text{ injective.}$$

This map is clearly a group map and if $(e^{2\pi i x}, e^{2\pi i s x}) = (1, 1)$ then $x \in \mathbb{Z}$ and $sx \in \mathbb{Z}$ $\Rightarrow$ $x = 0$. Hence injective.