The first two problems are from Hartshorne.

1.3. Let Y be the algebraic set in $\mathbf{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

<u>Solution:</u>

$$
\begin{aligned}
Y &= V(xz-x) \cap V(x^2-yz) \\
&= \big(V(x) \cup V(z-1)\big) \cap V(x^2-yz) \\
&= \big(V(x) \cap V(x^2-yz)\big) \cup \big(V(z-1) \cap V(x^2-yz)\big) \\
&= V(x, yz) \cup V(z-1, x^2-y) \\
&= V(x,y) \cup V(x,z) \cup V(z-1, x^2-y)
\end{aligned}
$$

1.7. (a) Show that the following conditions are equivalent for a topological space X: (i) X is noetherian; (ii) every nonempty family of closed subsets has a minimal element; (iii) X satisfies the ascending chain condition for open subsets; (iv) every nonempty family of open subsets has a maximal element.

(b) A noetherian topological space is *quasi-compact*, i.e., every open cover has a finite subcover.

(c) Any subset of a noetherian topological space is noetherian in its induced topology.

(d) A noetherian space which is also Hausdorff must be a finite set with the discrete topology.

<u>Solution:</u>

a)

$$
\begin{array}{ccc}
(i) & \overset{\text{Zorn's lemma}}{\Longleftrightarrow} & (ii) \\
\text{complements}\ \Updownarrow & & \\
(iii) & \overset{\text{Zorn's lemma}}{\Longleftrightarrow} & (iv)
\end{array}
$$

b) Let $X = \bigcup_{i \in I} U_i$. Consider the family of finite unions $U_{i_1} \cup \dots \cup U_{i_n}$, $i_1, \dots, i_n \in I$. By a)(iv) it has a maximal element.

c) Let $S \subseteq X$ be a subset of a Noetherian space X. Then $U \subseteq S$ is open in the induced topology iff $U = \tilde{U} \cap S$ for $\tilde{U} \overset{\text{open}}{\subseteq} X$. An chain $U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots$ in S gives a chain $\tilde{U}_1 \subseteq \tilde{U}_1 \cup \tilde{U}_2 \subseteq \tilde{U}_1 \cup \tilde{U}_2 \cup \tilde{U}_3 \subseteq \dots$ which must finally stabilize, i.e., $\exists N \in \mathbb{N}$ s.t. $\tilde{U}_n \subseteq \bigcup_{i=1}^{N} \tilde{U}_i \;\; \forall n \geq N$. Hence

$$\begin{aligned} U_n = \tilde{U}_n \cap S &\subseteq \left(\bigcup_{i=1}^{N} \tilde{U}_i\right) \cap S \\ &= \bigcup_{i=1}^{N} \tilde{U}_i \cap S \\ &= \bigcup_{i=1}^{N} U_i \\ &= U_N \quad \forall n \geq N. \end{aligned}$$

Thus S is Noetherian in the induced topology by a).

d) Let $S \subseteq X$ be any subset. By c) S is Noetherian and by b) S is compact. Since X is Hausdorff this means ~~that~~ S is closed. Hence X has the discrete topology. Since X is compact (by b)) it must be finite ($X = \bigcup_{x \in X} \{x\}$ is an open cover).

AM 2. Let A be a Noetherian ring. Show that $f = \sum_{k=0}^{\infty} a_k x^k \in A[[x]]$ is nilpotent iff each a_k is nilpotent.

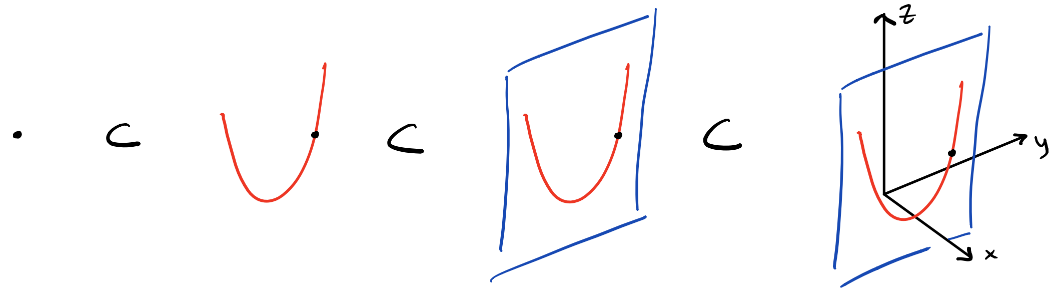
Solution: We start with the "only if" part: Suppose $\exists n \in \mathbb{N}$ such that $f^n = 0$. We have

$$f^n = a_0^n + n a_0^{n-1} a_1 x + \left(\binom{n}{2} a_0^{n-2} a_1^2 + n a_2 a_0^{n-1}\right) x^2 + \cdots$$

So a_0 is nilpotent. But a finite sum of nilpotent elements is nilpotent (prove it yourself) and hence $f - a_0$ is nilpotent. Write $f - a_0 = xp$ with $p \in A[[x]]$. Then p is nilpotent and has constant term a_1. Hence a_1 is nilpotent by the same argument as for a_0. By induction we see that a_k is nilpot. for all k.

Conversely, assume that all a_k are nilpotent. By [DF, Proposition 15.2.14] the nilradical N of A is nilpotent, i.e., there is an $m \in \mathbb{N}$ s.t. $N^m = 0$. Every coefficient in f^m is a sum of elem. of the form $a_{k_1} \cdot \ldots \cdot a_{k_m}$ which is an element of $N^m = 0$ and hence zero. Thus $f^m = 0$.

Example: The picture shows a chain of closed irred. subsets of $\mathbb{A}^3$



$$(x-1, z, y) \supset (x-1, z^2-y) \supset (x-1) \supset (0)$$

The dimension of $\mathbb{A}^3$ is 3.

 # 1.6: Show that any nonempty open subset of an irred. space is dense.

Solution:

Let $U \subseteq V$ be nonempty open and V irred. If $\bar{U} \neq V$ then $V = \bar{U} \cup (V \setminus U)$ $\lightning$.