Exercise session 8

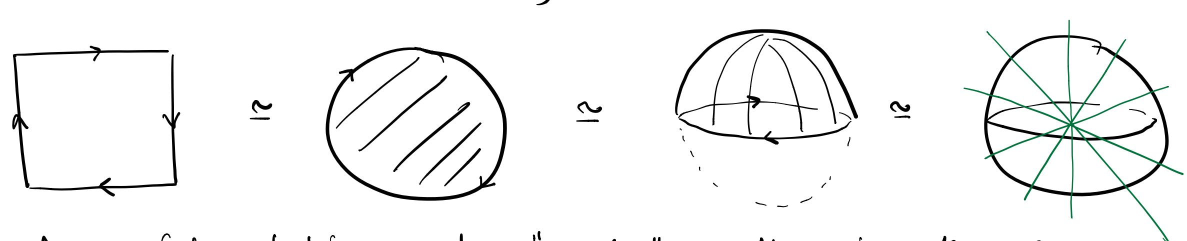
Solutions to problems

1. Consider the inclusion of the "equator"

$$X = \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^2.$$

Show that $I_2(X, X) = 1$.

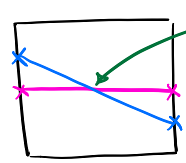
Solution: Recall the following schematic descriptions of $\mathbb{RP}^2$:



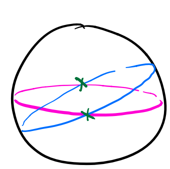
Any interpretation of "equator" will give the same answer. We illustrate the solutions with two pictures:

Unique intersection point.

1:



2:



The two intersection points in S^2 are identified in $\mathbb{RP}^2$ yielding a unique intersection point. $\square$

2. Let $M \xrightarrow{f} N \xrightarrow{g} Q$ be smooth maps between compact connected manifolds of the same dimension. Show that

$$\deg_2(g \circ f) = \deg_2(g)\deg_2(f).$$

Solution: Let $x \in Q$. Then $(g \circ f)^{-1}(x) = f^{-1}(g^{-1}(x))$.

Note that a map $h: X \to Y$ of manifolds of the same dimension is transversal to $y \in Y$ exactly when y is a regular value.

By Sard's theorem we may assume that g and $g \circ f$ are transversal to x. The chain rule

$$d(g \circ f)_p = dg_{f(p)} \circ df_p$$

shows that any point $s \in g^{-1}(x)$ is a regular value of f and hence transversal to f.

Hence
$$\begin{aligned}
\deg_2(g \circ f) &= \# f^{-1}(g^{-1}(x)) \mod 2 \\
&= \sum_{y \in g^{-1}(x)} \# f^{-1}(y) \mod 2 \\
&= \sum_{y \in g^{-1}(x)} \deg_2(f) \mod 2 \\
&= \deg_2(f) \sum_{y \in g^{-1}(x)} 1 \mod 2 \\
&= \deg_2(f) \# g^{-1}(x) \mod 2 \\
&= \deg_2(f)\deg_2(g).
\end{aligned}$$

3. Prove the following corollaries of the Borsuk-Ulam theorem:

1. Any odd smooth map $S^n \xrightarrow{f} \mathbb{R}^n$ has a zero.
2. For any map $S^n \xrightarrow{f} \mathbb{R}^n$ there is a point $x \in S^n$ such that $f(-x) = f(x)$.
3. In any covering of S^n by n closed sets $A_1, \ldots, A_n$, at least one A_i satisfies $A_i \cap (-A_i) \neq \phi$.

Solution: Let 2' be as point 2 but with the extra requirement that f is smooth.

First prove that $2' \Longleftrightarrow 1$:

$2' \Rightarrow 1$: If $f: S^n \longrightarrow \mathbb{R}^n$ is odd then $f(-x) = -f(x)$ $\forall x$ but 2 says that $\exists x_0$ s.t. $f(-x_0) = f(x_0)$.

Thus $f(x_0) = -f(x_0)$ and hence $f(x_0) = 0$.

$1 \Rightarrow 2'$: If $f: S^n \longrightarrow \mathbb{R}^n$, then $g(x) = f(x) - f(-x)$ is odd and hence has a zero:

$$0 = g(x_0) = f(x_0) - f(-x_0) \Rightarrow f(x_0) = f(-x_0).$$

$2' \Rightarrow 2$: Let $(f_i)_{i \in \mathbb{N}}$ be a sequence of smooth approximations of f s.t. $\|f_i(x) - f(x)\| < \varepsilon_i$ $\forall x \in S^n$ for some sequence $(\varepsilon_i)_{i \in \mathbb{N}}$, $\varepsilon_i > 0$, $\varepsilon_i \xrightarrow{i \to \infty} 0$.

This can be done since S^n is compact.

By 2', every f_i has an $x_i \in S^n$ s.t.

$$f_i(x_i) = f_i(-x_i)$$

and by compactness there is a subseq. of (x_i) converging to some $x \in S^n$.

Choose $\varepsilon > 0$. Then $\exists\, i \in \mathbb{N}$ s.t.

(1) $\|f(x)-f(x_i)\|, \|f(-x_i)-f(-x)\| < \varepsilon$ by cont. of f,

(2) $\|f(x_i)-f_i(x_i)\|, \|f_i(-x_i)-f(-x_i)\| < \varepsilon_i$ by constr. of f_i,

Hence $\|f(x)-f(-x)\| \leq$

$$\leq \|f(x)-f(x_i)\| + \|f(x_i)-\underbrace{f_i(x_i)}_{=f_i(-x_i)}\| + \|f_i(-x_i)-f(-x_i)\| + \|f(-x_i)-f(-x)\|$$

$$< 2\varepsilon + 2\varepsilon_i.$$

But ε and ε_i can be chosen arbitrarily small and hence $f(x) = f(-x)$.

$\underline{2 \Rightarrow 3}$: Given $A_1, \ldots, A_n$ as in 3, define functions $f_i : S^n \to \mathbb{R}$ s.t. $f_i^{-1}(0) = A_i$. Then $f = (f_1, \ldots, f_n) : S^n \to \mathbb{R}^n$ by 2 there is an $x \in S^n$ s.t. $f(x) = f(-x)$.

If $x \in A_i$ then $f_i(x) = f_i(-x) = 0$ so $x \in -A_i$.

We prove 1: If $f : S^n \to \mathbb{R}^n$ is odd with no zero then so is

$$g : S^n \xrightarrow{f} \mathbb{R}^n = \mathbb{R}^n \oplus \{0\} \hookrightarrow \mathbb{R}^{n+1}.$$

If f does not have a zero then g has no zero and $u : S^n \longrightarrow S^n$; $x \longmapsto \frac{g(x)}{\|g(x)\|}$ has degree 1 by Borsuk-Ulam. In particular u is surjective: Indeed, u is transversal to any point $y \in S^n \setminus \mathrm{im}(u)$ and if such a y exists then $\deg_2(u) = 0$. But $(0,1) \notin \mathrm{im}(u)$ so we have a contradiction. Hence $0 \in \mathrm{im}(f)$. $\square$

Orientations

New definition: An orientation on a manifold M is an equivalence class of non-vanishing sections $\sigma : M \longrightarrow \det(TM)$ where $\sigma \sim \sigma'$ iff $\frac{\sigma(x)}{\sigma'(x)} > 0$ for all (equiv: some) $x \in M$.

Comparison with old definition:

Given an orienting atlas $\{(U_i, \varphi_i)\}$ for M we have

Think about what this means for abstract manifolds.

$$(\varphi_i, (d\varphi_i)_{(-)}) : TM|_{U_i} \cong \varphi_i(U_i) \times \mathbb{R}^m \implies$$

$$(\varphi_i, \det((d\varphi_i)_{(-)})) : \det(TM)|_{U_i} \cong \varphi_i(U_i) \times \mathbb{R}.$$

Assume for simplicity that all transition functions of $\{(U_i, \varphi_i)\}$ have differential with determinant 1.

Define

$$s_i : U_i \longrightarrow \varphi_i(U_i) \times \mathbb{R} \cong \det(TM)|_{U_i}$$
$$x \longmapsto (\varphi_i(x), 1).$$

We need to show that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \forall i, j.$$

We have a commutative diagram

$$\begin{tikzcd}
 & \varphi_i(U_i \cap U_j) \times \mathbb{R} \arrow[d, "\alpha_{ij} \times \det(d(\alpha_{ij})_{(-)})"] & \\
\det(TM)|_{U_i \cap U_j} \arrow[ur, "{(\varphi_i, \det(d\varphi_i)_{(-)})}"] \arrow[r, "{(\varphi_j, \det(d\varphi_{j})_{(-)})}"'] & \varphi_j(U_i \cap U_j) \times \mathbb{R} & U_i \cap U_j \arrow[ul, "s_i|_{U_i \cap U_j}"'] \arrow[l, "s_j|_{U_i \cap U_j}"]
\end{tikzcd}$$

transition function

$= 1$ at all $x \in \mathbb{R}$.

Conversely, given a nowhere vanishing section $\sigma : M \longrightarrow \det(TM)$, we have for any transition function

$$\begin{tikzcd} \alpha_{ij} : \varphi_i(U_i \cap U_j) \arrow[rr, "\sim"] & & \varphi_j(U_i \cap U_j) \\ & U_i \cap U_j \arrow[ul, "\varphi_i"] \arrow[ur, "\varphi_j"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

a commutative diagram

$$\begin{tikzcd} & & \varphi_i(U_i \times U_j) \times \mathbb{R} \arrow[ddl] \arrow[drr, "\varphi_i \times \mathrm{id}"] \arrow[dddd, "(\alpha_{ij} \times \mathrm{id})"] & & \\ & & & & \varphi_i(U_i \cap U_j) \times \mathbb{R} \arrow[dd, "{(\alpha_{ij},\, \det(d(\alpha_{ij})_{(-)}))}"] \\ U_i \cap U_j \arrow[r, "\sigma"] & \det(TM)|_{U_i \cap U_j} \arrow[urrr] \arrow[drrr] & & & \\ & & & & \varphi_j(U_i \cap U_j) \times \mathbb{R} \\ & & \varphi_j(U_i \times U_j) \times \mathbb{R} \arrow[uul] \arrow[urr, "\varphi_j \times \mathrm{id}"'] & & \end{tikzcd}$$

$\Longrightarrow$ $\det(d(\alpha_{ij})_{(-)}) > 0$.

Warning! Lots of handwaving so verify yourself!

Rmk: A nowhere vanishing section $\sigma : M \longrightarrow \det(M)$ gives an isomorphism of vector bundles

$$\begin{aligned} M \times \mathbb{R} &\longrightarrow \det(M) \\ (x, a) &\longmapsto (x, a\sigma(x)) \end{aligned}$$

and vice versa.