1. **Exercise 2.10.** Let X be a topological space, $\mathscr{F}$ a sheaf on X and let $s, t \in \mathscr{F}(U)$ be two sections of $\mathscr{F}$ over an open subset $U \subseteq X$. Show that the set of $x \in U$ such that $s_x = t_x$ is open in U.

Here s_x, t_x are the images of s, t in the stalk $\mathcal{F}_x$.

Solution: we can write

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$$

$$= \coprod_{U \ni x} \mathcal{F}(U) \Big/ \sim$$

where $f \in \mathcal{F}(U)$, $g \in \mathcal{F}(V)$ satisfies $f \sim g$ if there is an open $W \subseteq U \cap V$ s.t. $f|_W = g|_W$. This means that $s_x = t_x$ iff there is an open set U_x containing x s.t. $s|_{U_x} = t|_{U_x}$. Hence the set of $x \in X$ s.t. $s_x = t_x$ is the union of open sets U_x.

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2. Show that if $\mathcal{O}_X$ is the structure sheaf on an affine scheme $X = \operatorname{Spec} A$ then $\mathcal{O}_{X,\mathfrak{p}} \cong A_{\mathfrak{p}}$ for any point $\mathfrak{p} \in \operatorname{Spec} A$ (This is what you did in the Homework).

Solution: we have

$$\begin{aligned}
\mathcal{O}_{X,\mathfrak{p}} &= \varinjlim_{U \ni \mathfrak{p}} \mathcal{O}_X(U) \\
&= \varinjlim_{D(f) \ni \mathfrak{p}} \mathcal{O}_X(D(f)) \\
&= \varinjlim_{f \notin \mathfrak{p}} A_f \\
&\cong \coprod_{f \notin \mathfrak{p}} A_f \Big/ \sim
\end{aligned}$$

$\mathfrak{p} \in D(f) = X \setminus V(f) \iff f \notin \mathfrak{p}$.

$h \mid f \iff (f) \subseteq (h)$

$\iff V(f) \supseteq V(h)$

$\iff D(f) \subseteq D(h)$

where $\frac{a}{f^n} \sim \frac{b}{g^m}$ iff $\frac{ag^m}{f^n g^m} = \frac{bf^n}{g^m f^n}$ in $A_{f^n g^m}$.

Note that we have a map

$$\begin{aligned} A_f &\longrightarrow A_p \\ \frac{a}{f^n} &\longmapsto \frac{a}{f^n} \qquad \forall f \notin p. \end{aligned}$$

This gives a well-def. ring map

$$\Psi : \coprod A_f/_\sim \longrightarrow A_p$$

since $\frac{a}{f^n} \sim \frac{b}{g^n}$ if they have the same image in A_p. If $\frac{t}{s}$ repr. an elem. in A_p then $s \notin p$ and $\frac{t}{s}$ is in the image of $A_s \longrightarrow A_p$. Hence Ψ is surjective.

It is also injective since if $\Psi(\frac{a}{f^n}) = 0$ then $\exists s \notin p$ s.t. $sa = 0$. Hence

$$\frac{sa}{sf^n} = 0 \quad \text{in} \quad A_{sf^n}$$

which means that $\frac{a}{f^n} = 0$ in $\coprod A_f/\sim$ since

$$\frac{a}{f^n} \sim \frac{sa}{sf^n}.$$

That is, $\Psi : \mathcal{O}_{X,p} \xrightarrow{\cong} A_p$ is an isomorphism.

Exercise 2.17. Let $(X, \mathscr{O}_X)$ be locally ringed space.

(a) Let $U \subseteq X$ be an open and closed subset. Show that there exists a unique section $e_U \in \Gamma(X, \mathscr{O}_X)$ such that $e_{U|V} = 1$ for all open subsets V of U and $e_{U|V} = 0$ for all open subsets V of $X \setminus U$. Show that $U \mapsto e_U$ yields a bijection

$$\mathrm{OC}(X) \leftrightarrow \mathrm{Idem}(\Gamma(X, \mathscr{O}_X))$$

from the set of open and closed subsets of X to the set of idempotent elements of the ring $\Gamma(X, \mathscr{O}_X)$.

Solution: We have

$$\Gamma(X, \mathcal{O}_X) \cong \Gamma(U, \mathcal{O}_X) \times \Gamma(X \setminus U, \mathcal{O}_X)$$

since $U \cap (X \setminus U) = \emptyset$. Hence we let e_U be the element

$$(1, 0) \in \Gamma(U, \mathcal{O}_X) \times \Gamma(X \setminus U, \mathcal{O}_X).$$

Since the restr. morphisms are ring morphisms, they send 1 to 1 and 0 to 0. This gives a map

$$OC(X) \longrightarrow \mathrm{Idem}(\Gamma(X,\mathcal{O}_X)).$$

Conversely, if $e \in \Gamma(X,\mathcal{O}_X)$ is idempotent then so is $1-e$ since $(1-e)^2 = 1-2e+e^2 = 1-e$. Note that we have $e(1-e) = e-e^2 = 0$ and hence every elem. $x \in \Gamma(X,\mathcal{O}_X)$ may be written as

$$\begin{aligned} x &= 1\cdot x \\ &= (1-e+e)x \\ &= (1-e)x + ex \end{aligned}$$

and this representation is unique since $e(1-e)=0$. Thus

$$\Gamma(X,\mathcal{O}_X) \cong (1-e)\Gamma(X,\mathcal{O}_X) \oplus e\Gamma(X,\mathcal{O}_X).$$

Note that $e(ex) = ex$ and

$$(1-e)((1-e)x) = (1-e)x \qquad \forall x \in \Gamma(X,\mathcal{O}_X)$$

which means that e and $(1-e)$ are mult. identities and $(1-e)\Gamma(X,\mathcal{O}_X)$ and $e\Gamma(X,\mathcal{O}_X)$ are in fact rings. Define U_e to be the open subset

$$\begin{aligned} U_e &= \{x \in X : (1-e)_x = 0\} \\ &= X \setminus \{x \in X : e_x = 0\}. \end{aligned}$$

Then U_e is open and closed.