Exercise session 3

– Tangent bundle, Lie groups

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The hint in HW 3:

Claim: Let $A \subseteq M$ be a connected subset of a manifold and let $f: M \to M$ be a smooth retraction: $f(M) = A$, $f|_A = id_A$. Then the rank of df is constant in an open nbhd of A.

proof: Reduce to the case when M is an open subset of $\mathbb{R}^n$.

operator norm $\sup_{x \neq 0} \frac{\|Lx\|}{\|x\|}$

Since $f: M \to A \subseteq M$ is smooth, the derivative $f': M \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n)$ is continuous, and can be computed as the Jacobian $J = \left(\frac{\partial f_i}{\partial x_j}\right)$.

Each $\frac{\partial f_i}{\partial x_j}$ is a continuous map

$$M \longrightarrow \mathbb{R} \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, \mathbb{R}).$$

usual norm $\longleftarrow$ oper. norm

The continuity of f' implies that there is an open nbhd $V \ni x$ s.t.

(*) $\operatorname{rk} J_y \geq \operatorname{rk} J_x \quad \forall y \in V \cap A.$

Since $f(M) = A$ and $f|_A = id_A$ we get that $f \circ f = f$ and $df_x \circ df_x = df_x$.

Note: If $L: V \to V$ is a linear operator on a vector space V s.t. $L \circ L = L$, then $V = \operatorname{im}(L) \oplus \operatorname{im}(I - L)$.

Hence we have $T_x M = \operatorname{im}(df_x) \oplus \operatorname{im}(d(id_M)_x - df_x)$ for all $x \in A$. Thus $\dim M = \operatorname{rk} df_x + \operatorname{rk}(I - df_x)$.

Now (*) says that

$\operatorname{rk} df_x \leq \operatorname{rk} df_y$ and $\operatorname{rk} I - df_x \leq \operatorname{rk} I - df_y$.

But the sum of the two is constant $\dim M$ and hence $\operatorname{rk} df_x = \operatorname{rk} df_y \quad \forall y \in V$.

Hence $\operatorname{rk} df$ is locally const. at A and since A is connected we get that df has const. rank on A.

Now extend to nbhd of A using: $df_p = df_{f(p)} \circ df_p$.

The tangent bundle

The smooth structure on TM is the one generated by charts

$$d\varphi : p^{-1}(U) \xrightarrow{(\varphi, d\varphi_{(-)})} \mathbb{R}^n \times \mathbb{R}^n$$

where (U, φ) is a chart on M.

The topology on TM is the coarsest topology s.t. all $d\varphi$'s are continuous.

A (smooth) vector field on M is a (smooth) section $s : M \to TM$ of the projection $TM \to M$.

A manifold is parallelizable if there is a diffeomorfi $TM \xrightarrow{\sim} M \times \mathbb{R}^n$ restricting to a isomorphism of vector sp. on the fibers.

Exercise: Every Lie group is parallelizable.

Example: S^2 is not parallelizable. This follows from the hairy ball theorem:

There is no nowhere vanishing smooth vector field on S^2.

If $\begin{array}{c} TS^2 \\ \downarrow \\ S^2 \end{array}$ was $\begin{array}{c} S^2 \times \mathbb{R}^2 \\ \downarrow \\ S^2 \end{array} \Rightarrow$ many smooth nowhere vanish. vect. field.

Exercise: Is $\mathbb{R}P^1$ parallelizable?

Is $\mathbb{R}P^2$ parallelizable?

Lie groups:

A **Lie group** is

a topological group (G, m, i) s.t.

(1) G is a smooth manifold,

(2) $m: G \times G \longrightarrow G$, $(g,h) \longmapsto gh$ and

$i: G \longrightarrow G$, $g \longmapsto g^{-1}$ are smooth.

Exercise: Let G be a Lie group and (U, φ) a chart containing the identity $e \in G$. Show that $(gU, \varphi \circ g^{-1})$ is a chart for all $g \in G$.

Solution: The map

$g: G \longrightarrow G$;

is a . Hence we only need to prove the following:

Fact: If $M \xrightarrow{f} N$ is a diffeomorphism and (U, φ) a chart on M, then

$(f(U), \varphi \circ f^{-1})$ is a chart for the smooth str. on N.

proof:

$$\begin{tikzcd} \alpha : M \arrow[r, "\sim"] & N \\ U \arrow[u, phantom, "\cup"] \arrow[r, "\alpha|_U"] \arrow[d, "\varphi"] & V \arrow[u, phantom, "\cup"] \arrow[d, "\psi"] \\ U' \arrow[r, dashed, "\psi \circ \alpha|_U \circ \varphi^{-1}"] & V' \end{tikzcd}$$

Consider the chart $(\alpha(U), \varphi \circ (\alpha|_U)^{-1})$. We know that $(\alpha(U), \psi|_{\alpha(U)})$ is a chart and since $\varphi \circ (\alpha|_U)^{-1} = (\psi \circ \alpha|_U \circ \varphi^{-1})^{-1} \circ \psi$ it suffices to observe that if (V, ψ) is a chart and $\theta : \psi(V) \xrightarrow{\sim} W \subset \mathbb{R}^n$ is a diffeomorphism, then $(V, \theta \circ \psi)$ is a chart.

$$\begin{tikzcd} & V \cap W \arrow[dl, "\beta"'] \arrow[dr, "\psi"] \arrow[d, phantom, "\circlearrowleft"] & \\ W' \cap \beta(V) \arrow[rr, dashed] \arrow[drr, dashed] & & V' \cap \psi(W) \arrow[d, "\wr"] \arrow[dl, phantom, "\circlearrowleft"] \\ & & \theta(V') \cap \theta(\psi(W)) \end{tikzcd}$$

$\square$

Conclusion: The smooth structure of a Lie group, is determined by a single chart.

Exercise: Show that there is a canonical diffeomorphism $T(M \times N) \xrightarrow{\sim} TM \times TN$.