Exercise session 2

- Regular values, Sard's theorem, Transversality, Whitney embedding thm, ...

Recall: If $f: X \to Y$ is a smooth map then $y \in Y$ is a regular value if df_x is surjective for all $x \in f^{-1}(y)$.

If df_x is not surjective then x is a critical point and $y = f(x)$ a critical value.

(From Bredon:)

1. $\phi: \mathbb{R}^2 \longrightarrow \mathbb{R}$; $(x,y) \longmapsto \sin(x^2+y^2)$.
What are the regular/critical values?

Solution: The Jacobian is $\left(2x\cos(x^2+y^2), 2y\cos(x^2+y^2) \right)$ which is:

not surjective $\iff$ it is zero

$\iff$ $x = y = 0$ or $\cos(x^2+y^2) = 0$

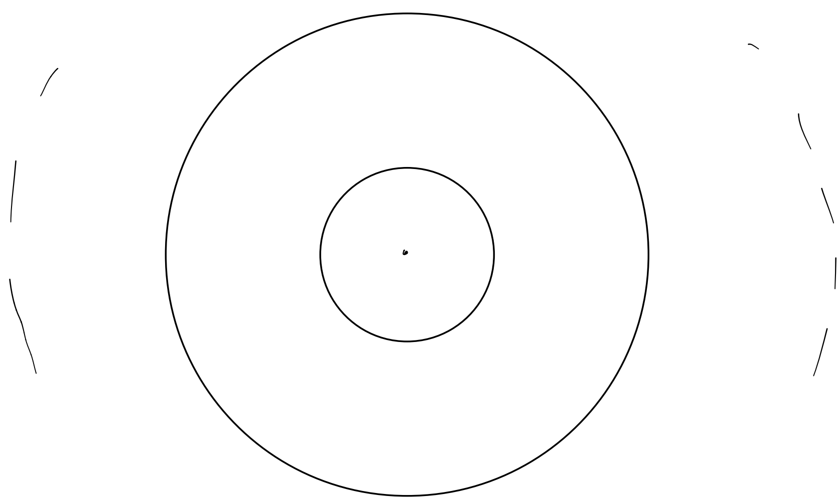
$x = y = 0$ or $x^2+y^2 = \frac{\pi}{2} + \pi n$, $n \in \mathbb{Z}$

$\Rightarrow$ critical values: $\{\sin(0), \sin(\frac{\pi}{2}), \sin(\frac{3\pi}{2})\} = \{0, 1, -1\}$

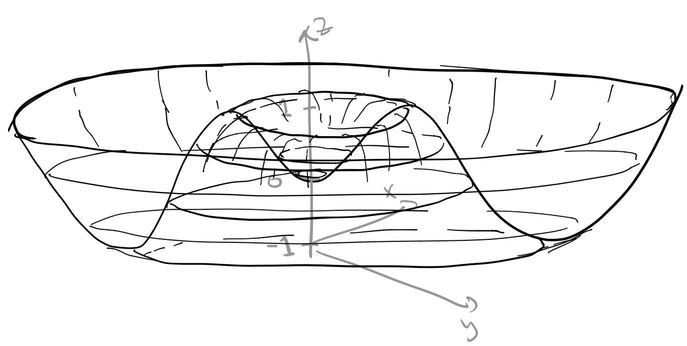
1. what are the critical points?

Solution: The set of (x,y) satisfying
$$x^2+y^2 = \frac{\pi}{2}+\pi n \quad \text{for } n\in\mathbb{Z} \text{ fixed}$$
is empty if $n<0$ and a circle if $n\geq 0$!

Critical points:



Graph: $(x,y,\phi(x,y))\in\mathbb{R}^3$

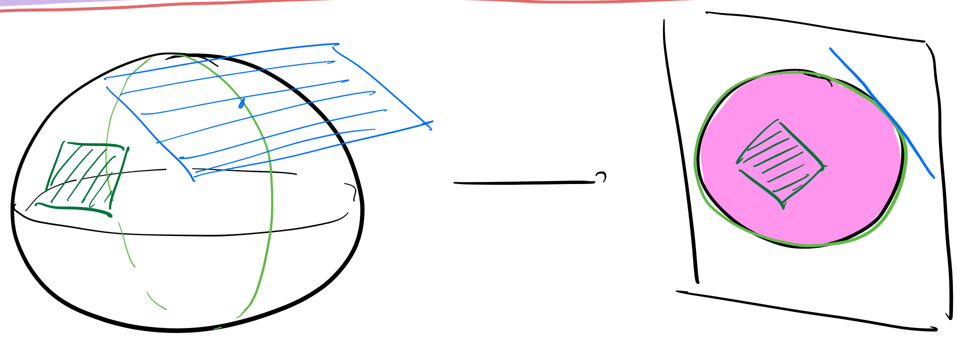


2. Consider $S^2 \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

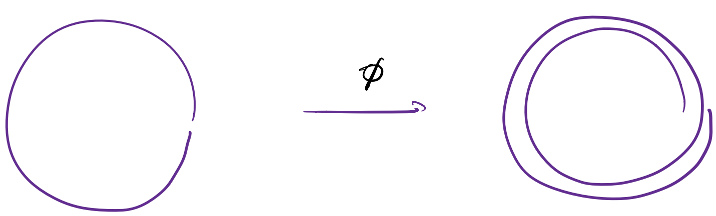
$(x,y,z) \longmapsto (x,y)$.

What is the set of critical values and the set of regular values?

Solution:



3. $S^1 \xrightarrow{\phi} S^1$ that wraps the circle twice around itself. Regular / critical values critical points?



Solution:

- Smooth str. $\gamma : \mathbb{R} \longrightarrow S^1 \subseteq \mathbb{R}^2$ $\cong \mathbb{R}/\mathbb{Z}$

$x \longmapsto (\cos(2\pi x), \sin(2\pi x))$

- As in HW1 ex 3:

γ is a local diffeo.

- The diagram

$$\begin{tikzcd} x \arrow[r, mapsto] & 2x \\[-2em] \mathbb{R} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R} \arrow[d] \\ S^1 \arrow[r, "\phi"'] & S^1 \end{tikzcd}$$

commutes:

$\Longrightarrow$ The map $\phi : S^1 \to S^1$ is an immersion since $\mathbb{R} \xrightarrow{2\cdot(-)} \mathbb{R}$ is so.

- An immersion betw. manif. of the same dim. has surj. differential at all points.

$\longrightarrow$ ϕ has <u>no</u> critical values.

An application of Sard's theorem:

Thm (Fundamental theorem of algebra): Let $p(z) \neq$ const. be a complex polynomial. Then $p(z)$ has a root in $\mathbb{C}$.

proof (Milnor): Write $z = x+iy$: $p(x+iy) = u(x,y) + i\,v(x,y)$.

Hence p defines a smooth function

$$\tilde{p} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x,y) \longmapsto (u(x,y), v(x,y)).$$

(*) We will show that $\tilde{p}$ is surjective.

The differential is

Cauchy-Riemann

$$D(\tilde{p}) = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} u_x & v_x \\ -v_x & u_x \end{pmatrix} \rightsquigarrow \det D(\tilde{p}) = u_x^2 + v_x^2$$

which is not invertible at $z_0 \Longleftrightarrow p'(z_0) = 0$

But $p'(z)$ has only a finitely number of zeros.

$\Longrightarrow$ $\tilde{p}$ has a finite number of critical values.

Writing $p(z) = a_n z^n + \dots + a_0$

$$= z^n (a_n + \dots + a_0 z^{-n})$$

we see that $|p(z)| \leq |z^n| \, |a_n + \dots + a_0 z^{-n}| \longrightarrow \infty$

as $|z| \to \infty$. Hence $\tilde{p}$ may be extended as a continuous map to

(one pt compactification)

$$\begin{tikzcd} S^2 \arrow[r, dashed, "\varphi"] & S^2 \\ \mathbb{R}^2 \cup \{\infty\} \arrow[u, phantom, "\shortparallel"] & \mathbb{R}^2 \cup \{\infty\} \arrow[u, phantom, "\shortparallel"] \\ \mathbb{R}^2 \arrow[u, phantom, "\cup"] \arrow[r, "\tilde{p}"] & \mathbb{R}^2 \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\implies$ φ proper $\to$ φ closed $\Rightarrow$ $\tilde{p}$ closed.

At every non-critical point z, $d\tilde{p}_z$ is inj. and hence an iso. $\Rightarrow$ Inv. func. thm:

$\tilde{p}$ local diffeo. at z.

Consider the function

$$k : \mathbb{R}^2 \longrightarrow \mathbb{Z}$$
$$c \longmapsto |\tilde{p}^{-1}(c)|.$$

It is enough to show that $\mathrm{im}(k) \subseteq \mathbb{Z}_{\geq 1}$.

Let $c \in \mathbb{R}^2$ be a regular value. Then if $c \in \tilde{p}(\mathbb{R}^2)$ we can find $c \in V \overset{\text{open}}{\subseteq} \mathbb{R}^2 \setminus \{\text{critical values}\}$ and $U_i \ni z_i$ for each $z_i \in \tilde{p}^{-1}(c)$ s.t. $\tilde{p}|_{U_i} : U_i \to V$ is a diffeo. for all i.

$\implies$ k is locally const. on $\mathbb{R}^2 \setminus \{\text{crit. val}\}$

But $\mathbb{R}^2 \setminus \{\text{critical values}\}$ is connected since $\{\text{crit. values}\}$ is finite.

$\longrightarrow$ k is constant on $\mathbb{R}^2 \setminus \{\text{crit. val}\}$.
If k is zero on $\mathbb{R}^2 \setminus \{\text{crit. val}\}$ then $\tilde{p}$ lands in $\{\text{crit. val}\}$.

But $\tilde{p}(\mathbb{R}^2)$ is connected $\Longrightarrow$ $\tilde{p}(\mathbb{R}^2) = \{*\}$ $\lightning$

Hence k is constant and not zero on $\mathbb{R}^2 \setminus \{\text{crit. val.}\}$.

$\longrightarrow$ $\tilde{p}$ is surjective.
which means, 0 lies in the image. $\square$

<u>Transversality:</u>

<u>Recall:</u> For $Z \subseteq N$ submanifold and $f: M \to N$ smooth

$f \pitchfork Z \iff \forall\, x \in M$ with $f(x) \in Z$ we have
$$\operatorname{im}(df_x) + T_{f(x)} Z = T_{f(x)} M.$$

<u>G-P 9.</u> For V a v.sp. and $A: V \longrightarrow V$ a linear map, show that $\Delta \pitchfork \operatorname{Graph}(A) \iff 1$ is not an eigenvalue of A.

Presentations:

(Lina, Daniel, Jonathan, Ludvig)

1.

2. (Lina, Daniel, Jonathan, Ludvig)

3. (Lina, Jonathan, Ludvig).