# Exercise session 6

## Homework 6:

**Problem 1** (4p). Show that Morseness is stable. That is, if $F: M \times [0,1] \to \mathbf{R}$ is smooth with $F_0$ Morse and $M$ compact, then there is an $\epsilon > 0$ such that $F_t$ is Morse for all $t < \epsilon$. (Hints given in Guillemin-Pollack, exercises I.7.17 and I.7.18).

**Solution:** We use the hints given in G-P:

$$f := F_0.$$

**I.7.16:** Let $\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$. By def.

$f$ Morse $\iff$ $\forall x$ s.t. $(\nabla f)(x) = 0$ we have

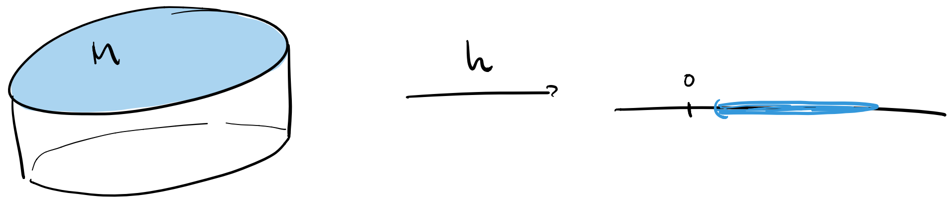
$$\det(H_f(x)) \neq 0.$$

$$f \text{ Morse} \iff \varphi_f(x) := \underbrace{\det(H_f(x))^2 + \|(\nabla f)(x)\|^2}_{\text{smooth since pol. of part. derivatives.}} > 0 \quad \forall x \in M.$$

**I.7.17,18:** ($M$ compact $\Rightarrow$ $\exists$ open $U$ : $M \subseteq U \subseteq \mathbb{R}^n$ s.t. $\varphi_f(x) > 0$ on $U$). Want to show that

$$h: M \times I \longrightarrow \mathbb{R}$$
$$(x,t) \longmapsto \varphi_{f_t}(x), \quad f_t = F(-,t)$$

is continuous. Note that $\frac{\partial f_{t_0}}{\partial x_i} = \frac{\partial F}{\partial x_i}\Big|_{t=t_0}$ $\Rightarrow$ $h$ is constructed as a polynomial of partial derivatives of $F$ $\Rightarrow$ Smooth.




We have that $h^{-1}(0)$ is closed and disjoint from $M \times \{0\}$ which is compact. Hence there exists an $\varepsilon > 0$ s.t.

$$(M \times [0,\varepsilon)) \cap h^{-1}(0) = \emptyset.$$

$$\Rightarrow \quad \varphi_{f_t} = h(-,t) > 0 \qquad \forall t < \varepsilon.$$

$$\Rightarrow \quad f_t \text{ is Morse}. \qquad \square$$

**Problem 2** (3p). An *orientation* on a smooth manifold $M$ (with boundary) is an equivalence class of atlases such that all change-of-coordinate functions $\alpha = \phi \circ \psi^{-1}$ satisfy that $\det(d\alpha_x) > 0$ for all $x$ where $\alpha$ is defined.

Classify all compact oriented 1-manifolds with boundary.

Solution: WARNING: with the def. of orientation as given here, $[0,1]$ is not orientable. However, it is with the definition from G-P: This is because in the def. of manifold with boundary as in lecture notes, we don't allow $H^n_{\leq 0} \subseteq \mathbb{R}^n$ "as charts".

Both answers are accepted.

Prop 3.2. (G-P): An orientable connected mnfld with bndry admits exactly two orientations.

Hence every oriented compact 1-mnfld with bndy is diffeomorphic (as mnfld without orient.) to a finite (since compact) union $\bigcup_\alpha M_\alpha$ with each $M_\alpha$ being $[0,1]$ or $S^1$ with one of the two orientations.

Alternatively: Consider "up to diffeo. of oriented mnflds". Then we don't see the orientations at all ...

**Problem 3** (3p). Using Brouwer's fixed point theorem, show that any square matrix with nonnegative entries must have a nonnegative eigenvalue.

Solution: If 0 is an eigenv. then we are done so we may assume the matrix $A$ is invertible.

Think of $A$ as a lin. map

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Let $S^{n-1}_{\geq 0} = \{(x_1, \dots, x_n) \in S^{n-1} : x_i \geq 0 \quad \forall i\}$.

Since all entries of $A$ are non-neg. we get that $A(\mathbb{R}^n_{\geq 0}) \subseteq \mathbb{R}^n_{\geq 0} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0 \quad \forall i\}$.

Define

$$f : S^{n-1}_{\geq 0} \longrightarrow S^{n-1}_{\geq 0}$$
$$x \longmapsto \frac{Ax}{\|Ax\|}.$$

By Brouwer's fix. pt thm, $f$ has a fix pt. say $x$. Then $x$ is an eigenvector.

**G-P. 2.3.5:** $X \subseteq Y$ compact submanifold intersecting a submanifold $Z \subseteq Y$ s.t. $\dim X + \dim Z < \dim Y$. Show that for every $\varepsilon > 0$ there exists a deformation $i_t : X \hookrightarrow Y$ ($i_0 = \text{incl}: X \to Y$) s.t. $i_t(X) \cap Z = \emptyset$ and $|x - i_t(x)| < \varepsilon$.

proof: By def. $X$ and $Z$ are transv. iff

$$T_yX + T_yZ = T_yY \qquad \forall y \in Y.$$

Hence $X \pitchfork Z \iff X \cap Z = \emptyset$.

Hence it is enough to show that there is an $\varepsilon$-small def. $X_t$ s.t. $X_t \pitchfork Z$.

Case 1: $Y = \mathbb{R}^n$.

Def. 
$$F : X \times B_\varepsilon(0) \longrightarrow Y$$
$$(x, s) \longmapsto x + s.$$

Then since $\dim B_\varepsilon(0) = n$ we have that $F \pitchfork Z$ and $F|_{\partial X \times B_\varepsilon(0)} \pitchfork Z$. Thus by the transv. thm, the set of pts $s$ s.t. at least one of $F(-, s)$ and $F|_{\partial X \times B_\varepsilon(0)}(-, s)$ is <u>not</u> transv. to $Z$ has meas. 0. In particular, $\exists s_0 \in B_\varepsilon(0)$ s.t. $F(-, s_0)$ and $F|_{\partial X \times B_\varepsilon(0)}(-, s)$ are tr. to $Z$. Let $\gamma : I \longrightarrow B_\varepsilon(0)$ be a path from $0$ to $s_0$ and define

$$H : X \times I \longrightarrow Y$$
$$(x, t) \longmapsto F(x, \gamma(t)).$$

Then $H(X,1)$ is an $\varepsilon$-small deform. of $X$ and transv. to $Z$, i.e., $H(X,1) \cap Z = \emptyset$.

Generalize to arbitr. $Y$ . . .