Exercise session 5

Homework 5 – Main ideas for solutions / Presentations

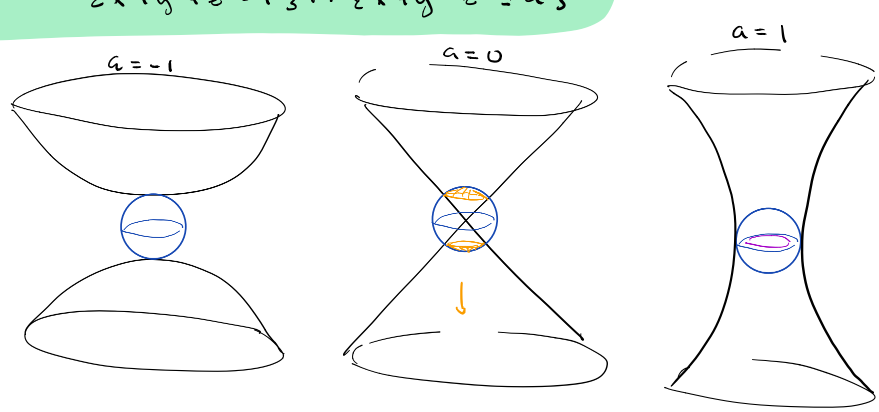
1. Diffeo. of cmpct mnflds "stable":

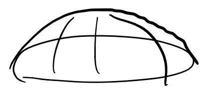
Solution: Embeddings are stable and local diffeo. are stable (see GP). Hence $\exists \varepsilon > 0$ s.t. f_t is an open embedding $\forall t < \varepsilon$. But f_t is also closed since $f_t(M)$ compact and hence closed. Hence f_t is a diffeo. to a union of conn. comp.

For any $y \in N$, $\exists x \in M$ s.t. $f_0(x) = y$. But then $H(x, -) : [0,t] \longrightarrow N$ is a path from y to $f_t(x)$. Thus f_t hits all components $\Rightarrow f_t$ surj. $\square$

2. Intersection $\{x^2+y^2+z^2 = 1\} \cap \{x^2+y^2-z^2 \leq a\}$:

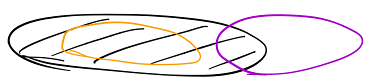
Solution:





<u>Ludvig</u>

$$x, y \longmapsto x, y, \pm\sqrt{1-x^2-y^2}$$



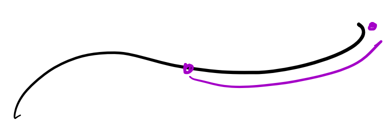
$$(\tfrac{1}{2}, 1] \times (0, 2\pi) \longrightarrow U \subseteq D$$
$$(r, \theta) \longmapsto (r\cos\theta, r\sin\theta)$$
$$(\tfrac{1}{2}, 1] \times (-\pi, \pi) \longrightarrow V \subseteq D$$
$$(r, \theta) \longmapsto (r\cos\theta, r\sin\theta)$$

3. $\mathbb{R} \cong L \subset M$ 1-dim: $\#(\bar{L} - L) \leq 2$.

<u>Solution:</u>

$$f: (0,1) \longrightarrow (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$
$$x \longmapsto \pi x - \tfrac{\pi}{2}$$
$$g: (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \longrightarrow \mathbb{R}$$
$$x \longmapsto \tan(x) \implies g \circ f \text{ diffeo.}$$

We write $\varphi: (0,1) \xrightarrow{\sim} L \subseteq M$. Take $p \in \bar{L} \setminus L$. Then p lies in the same comp. as $L \implies \exists \gamma: [0,1] \longrightarrow M$ from $\varphi(1/2)$ to p (which is an embedding). Denote its image by P.



$\operatorname{im}\gamma = P$ is closed since compact.

<u>Claim:</u> $\operatorname{im}(\gamma|_{(0,1)})$ is open.

Indeed, $\gamma|_{(a,1)}$ is an immersion so $d(\gamma|_{(0,1)})_t$ is inj. $\forall t \in (0,1)$. But $(0,1)$ and M have the same dim. $\Rightarrow d(\gamma|_{(0,1)})_t$ surj. Hence $\gamma|_{(0,1)}$ is a local diffeo $\rightarrow$ open. We have

$$\mathrm{im}(\varphi|_{(0,1/2)}) \cap P = \mathrm{im}(\varphi|_{(0,1/2)}) \cap \mathrm{im}(\gamma|_{(0,1)})$$

since $p \notin L$ $\gamma(0) = \varphi(1/2) \notin \mathrm{im}(\varphi|_{(0,1/2)})$.

$$\Rightarrow (\varphi|_{(0,1/2)})^{-1}(P) \text{ is open and closed.}$$

$$\Rightarrow (\varphi|_{(0,1/2)})^{-1}(P) = \begin{cases} (0,1/2) & \text{or} \\ \emptyset . \end{cases}$$

Similarly

$$(\varphi|_{(1/2,1)})^{-1}(P) = \begin{cases} \emptyset & \text{or} \\ (1/2,1). \end{cases}$$

If $(\varphi|_{(0,1/2)})^{-1}(P) = (0,1/2)$ then the function

$$\tilde{\varphi} : [0,1/2] \longrightarrow M$$
$$x \longmapsto \begin{cases} \varphi(x) & \text{if } x \in (0,1/2) \\ p & \text{if } x = 0 . \end{cases}$$

is the unique cont. extension of $\varphi|_{(0,1/2]}$ to $[0,1/2]$. Indeed, if $\tilde{\varphi}'$ is another ext. with $\tilde{\varphi}'(0) = p' \neq p$, take an open $p \in U \not\ni p'$. Then $\tilde{\varphi}^{-1}(U)$ is open in $[0,1/2]$ and hence $\exists \delta > 0$ s.t. $\tilde{\varphi}([0,\delta)) \subseteq U \Rightarrow \tilde{\varphi}'$ is <u>not</u> continuous since there is an open $p' \in V \not\ni p$ s.t. $(\tilde{\varphi}')^{-1}(V) = \{0\} \subseteq [0, 1/2]$ which is not open.

We conclude that $\#(\bar{L} - L) \leq 2$ since each point must be the image of 0 or 1 in an extension as above.

Linas

$g : (a,b) \cong L \subseteq M$

$p \in \bar{L} \setminus L$. Sekv. $x_i \overset{\in L}{\longrightarrow} p$, g kont., g^{-1} kont.

$\longrightarrow$ $g^{-1}(x_i)$, $\exists$ delsek. $g^{-1}(x_{i_\alpha}) \longrightarrow a$ eller b.

wlog. $g^{-1}(x_i) \longrightarrow a$.

$(a,b) \ni y_i \longrightarrow a$ g bikont. $\Rightarrow$

$g(y_i) \longrightarrow p'$. M Hausd. $\Rightarrow$ $p = p'$

Orientations:

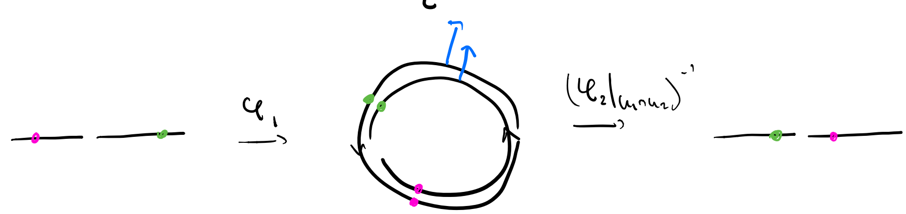
Def: An orientation on a smooth manifold M is an equiv. class of atlases for which all transition functions φ satisfies

$$\det(d\varphi_x) > 0 \quad \forall x \in \operatorname{domain}(\varphi).$$

Ex: $\varphi_1, \varphi_2 : (-\pi, \pi) \longrightarrow S^1 \subseteq \mathbb{R}^2$

1. $x \longmapsto (\cos(x), \sin(x))$
2. $x \longmapsto (\cos(x+\pi), \sin(x+\pi))$

Gives $\varphi_2|_{\dots}^{-1} \circ \varphi_1 = \begin{cases} x+\pi & \text{if } x<0 \\ x-\pi & \text{if } x>0 \end{cases}$



Hence $d(\varphi_2|_{\dots}^{-1} \circ \varphi_1)_x = 1 \quad \forall x \in U_1 \cap U_2$.

$\Rightarrow$ The atlas defines an orientation.

Non-example:

$\varphi_1, \varphi_2 : (-\pi, \pi) \longrightarrow S^1 \subseteq \mathbb{R}^2$

1. $x \longmapsto (\cos(x), \sin(x))$
2. $x \longmapsto (\cos(\pi - x), \sin(\pi - x))$

$\Rightarrow d(\varphi_2^{-1} \circ \varphi_1)_x = -1 \Rightarrow$ Not an orientation.

Morse functions:

Def: A function $U \xrightarrow{f} \mathbb{R}$ (open $U \subseteq \mathbb{R}^n$) is a Morse function if for all critical points $x \in U$, the Hessian matrix $Hf_x = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_x$ is invertible.

Ex: (Brouwer's fixed pt thm):

No retr. $r: B^2 \longrightarrow \partial B^2 = S^1$:

If such r existed, then

$$\partial B^2 \hookrightarrow B^2 \xrightarrow{r} \partial B^2 \quad \text{is the identity.}$$

By functoriality of $\pi_1(-, x)$

$$\begin{array}{ccccc} \pi_1(\partial B^2, x) & \longrightarrow & \pi_1(B^2, x) & \longrightarrow & \pi_1(\partial B^2, x) \\ \cong & & & & \\ \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \end{array} \quad \text{is the id.} \ \lightning$$