8. Prove that if $s_1, \dots, s_n \in S$ are integral over R, then the ring $R[s_1, \dots, s_n]$ is a finitely generated R-module.

Solution: Since s_1 is integral there is an $n \in \mathbb{N}$ and $a_0, \dots, a_{n-1} \in R$ s.t.

$$s_1^n + a_{n-1}s_1^{n-1} + \dots + a_1 s_1 + a_0 = 0$$

Hence $s_1^n = -(a_{n-1}s_1^{n-1} + \dots + a_1 s_1 + a_0)$ so $R[s_1]$ is generated by $\{s_1^0, s_1^1, \dots, s_1^{n-1}\}$. By induction $R[s_1, \dots, s_k]$ is finitely generated over $R[s_1, \dots, s_{k-1}]$ and hence $R[s_1, \dots, s_n]$ is finitely gen.

(If B gen. by $\{b_1, \dots, b_n\}$ over A and

C — " — $\{c_1, \dots, c_m\}$ over B then

C — " — $\{b_i c_j\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}$ over A.)

3. Suppose k is a field and i and j are relatively prime positive integers. Find the normalization of the integral domain $R = k[x, y]/(x^i - y^j)$ (cf. Exercise 14, Section 9.1).

Solution: Since i and j are relatively prime there are integers a, b s.t.

$$ai + bj = 1.$$

Hence $(x^b y^a)^j = x^{bj} x^{ai} = x$ and

$$(x^b y^a)^i = y^{bj} y^{ai} = y.$$

Note that either

$(a > 0) \wedge (b < 0)$ or $(a < 0) \wedge (b > 0)$ so

$x^b y^a \notin k[x,y]/(x^i - y^j)$ but $x^b y^a \in \mathrm{Frac}(k[x,y]/(x^i-y^j))$.
Hence $x^b y^a$ is in the normalization.
It remains to show that $R[x^b y^a]$ is integrally closed in Frac(R).

Consider the morphism

$$\begin{array}{ccc} k[t] \supset k[t^i, t^j] & \xrightarrow{\varphi} & R \\ t^j & \longmapsto & x \\ t^i & \longmapsto & y \end{array}$$

which is well-def. since we define it on the generators and extend it to a ring map. The preimage of $(x^i - y^j)$ is $((t^j)^i - (t^i)^j) = 0$ so it is also injective. It is obviously surjective since $x, y \in \mathrm{Im}(\varphi)$. Hence φ is an isomorphism.
Since t satisfies the equation

$$(t)^i - t^i = 0$$

we see that $k[t]$ is in the integral closure of $k[t^i, t^j]$. But $k[t]$ is a UFD and hence normal:

<u>Claim:</u> A UFD $\Longrightarrow$ A normal

<u>proof:</u> If A is a UFD and $x \in \mathrm{Frac}(A)$, write $x = \frac{a}{b}$ (with $b \neq 0$). Suppose x is integral over A. Then it satisfies some eq.

$$x^n + \alpha_{n-1} x^{n-1} + \dots + \alpha_0 = 0 \text{ with all } \alpha_i\text{'s in } A.$$

Multiply by b^n to get

$$a^n + \alpha_{n-1} b a^{n-1} + \dots + \alpha_0 b^n = 0 .$$

Hence $b \mid a$ and $\frac{a}{b} \in A$. $\square$