Example 1 Write the CNOT gate matrix in 2-qubit basis

On the left hand side we have two arbitrary qubits

$$|\psi\rangle_{LHS} = |\psi\rangle_1 \otimes |\psi\rangle_2 = (a_1|0\rangle + b_1|1\rangle) \otimes (a_2|0\rangle + b_2|1\rangle)$$
$$= a_1a_2|00\rangle + a_1b_2|01\rangle + b_1a_2|10\rangle + b_1b_2|11\rangle$$

$$\text{(in vector form)} = \begin{array}{c|l} \begin{bmatrix} a_1a_2 \\ a_1b_2 \\ b_1a_2 \\ b_1b_2 \end{bmatrix} & \begin{array}{l} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \end{array}$$

Applying CNOT flips state of qubit 2 if qubit 1 is $|1\rangle$:

$$|\psi\rangle_{LHS} = \begin{bmatrix} a_1a_2 \\ a_1b_2 \\ b_1a_2 \\ b_1b_2 \end{bmatrix} \begin{array}{l} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \rightarrow$$

$$\rightarrow |\psi\rangle_{RHS} = \begin{bmatrix} a_1a_2 \\ a_1b_2 \\ b_1a_2 \\ b_1b_2 \end{bmatrix} \begin{array}{l} |00\rangle \\ |01\rangle \\ |11\rangle \\ |10\rangle \end{array} = \begin{bmatrix} a_1a_2 \\ a_1b_2 \\ b_1b_2 \\ b_1a_2 \end{bmatrix} \begin{array}{l} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array}$$

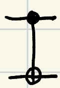
Rewrite Problem:

$$\underbrace{\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}}_{CNOT} \begin{bmatrix} a_1a_2 \\ a_1b_2 \\ b_1a_2 \\ b_1b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 \\ a_1b_2 \\ b_1b_2 \\ b_1a_2 \end{bmatrix}, \text{ find } x_{ij}$$

Top row example:

$$x_{11}a_1a_2 + x_{12}a_1b_2 + x_{13}b_1a_2 + x_{14}b_1b_2 = a_1a_2 \Rightarrow CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

Repeat for rows 2-4 and get final form of CNOT:



$$= \begin{array}{cc} \begin{array}{cccc} \langle 00| & \langle 01| & \langle 10| & \langle 11| \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \begin{array}{l} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \end{array}$$

Example 2 Write matrix form of 2 Hadamards in parallel:

–H–
–H–

Since each –H– operation acts on each qubit separately, we can write the combination as a tensor product of the two one-qubit operations:

$$|H_1\rangle \otimes |H_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & 1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ 1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -1\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{bmatrix} = \boxed{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{array}{c} -H- \\ -H- \end{array}}$$