Pledge: *I pledge my honor that I have abided by the Stevens Honor System.* -Eric Altenburg

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1: Recall that the sequence of Fibonacci numbers is defined as follows: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ when $n > 2$.

(Prove that $F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$ for all $n \in \mathbb{N}$.)

Proof. (Induction)
Base Case: $n = 1$, then

$$\begin{aligned} F_1^2 &= F_1 \cdot F_2 \\ 1 &= 1 \cdot 1 \\ 1 &= 1. \end{aligned}$$

Inductive Hypothesis: Assume $F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$ holds true for some n. We wish to prove the identity holds for $n + 1$ as well, we find that

$$\begin{aligned} F_1^2 + F_2^2 + \ldots + F_n^2 + F_{n+1}^2 &= F_n F_{n+1} + F_{n+1}^2 \\ &= F_{n+1} \left(F_n + F_{n+1} \right) \\ &= F_{n+1} F_{n+2}. \end{aligned}$$

This establishes the claim and completes the proof. □

(Let S_n, where $n \in \mathbb{N}$, be the set of all n-digit binary strings that have no consecutive 1s. For example, $S_3 = \{000, 001, 010, 100, 101\}$. Prove that S_n contains exactly F_{n+2} elements.)

Proof. (Induction)
Base Case:
$n = 1$, then $S_1 = \{0, 1\}$. $F_3 = 2$ and $|S_1| = 2$.
$n = 2$, then $S_2 = \{00, 01, 10\}$. $F_4 = 3$ and $|S_2| = 3$.
Inductive Hypothesis: Assume S_n contains exactly F_{n+2} elements where S_n is some set of all n-digit binary strings that have no consecutive 1s and $n \in \mathbb{N}$. We wish to prove the identity holds for S_{n+1} as well, we observe that

$$\begin{aligned} S_{n+1} &= S_n + S_{n-1} \\ &= F_{n+2} + F_{(n-1)+2} \\ &= F_{n+2} + F_{n+1} \\ &= F_{n+3} \\ &= F_{(n+1)+2}. \end{aligned}$$

This establishes the claim and completes the proof. □

2: Suppose you draw several straight lines on a piece of paper, thereby dividing the paper into different regions (each line should extend to the edges of the page). Prove that, no matter how your lines are drawn, it is possible to color each region red or blue in such a way that no two adjacent regions have the same color.

Proof. (Induction) Suppose n are the amount of straight lines going from edge-to-edge drawn on a piece of paper.
Base Case: $n = 1$
There are two regions on the piece of paper, one is colored red and the other is colored blue. The two regions are adjacent and do not have the same colors.
Inductive Hypothesis: Assume that for a piece of paper, no two adjacent regions have the same color for some n amount of lines on it.
We wish to prove that the identity holds for $n + 1$ lines as well, and after drawing the additional line on the paper we know that on either side of the new line, there are no two adjacent regions with the same color; the only regions where there are adjacent colors are the newly created ones on either side of the new line. To make the identity hold true, all the regions' colors on one side of the newly drawn line can be inverted; i.e. red becomes blue and blue becomes red. Doing this will still ensure that all the old regions will still follow the identity since they did before, because of this, these color changes will not break it.
Having inverted all the regions to one side of the new line, the regions on opposite sides of the new line are different colors and the identity holds true still for $n + 1$ lines. $\square$