Pledge: *I pledge my honor that I have abided by the Stevens Honor System.* -Eric Altenburg

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1: Write a proof for the following theorem:

Theorem 1. *Let $n \in \mathbb{Z}$. Prove that $2n^2 + n$ is odd if and only if $cos(\frac{n\pi}{2})$ is even.*

Proof. Because of the nature of the *if and only if* it must be proven bidirectionally. First, consider: $2n^2 + n$ is odd $\Rightarrow cos(\frac{n\pi}{2})$ is even.

Case 1: n is odd.

Let $n = 2m + 1$ where $m \in \mathbb{Z}$.

$$\begin{aligned}
2n^2 + n &= 2(2m+1)^2 + (2m+1) \\
&= 2(4m^2 + 4m + 1) + (2m+1) \\
&= 8m^2 + 8m + 2 + 2m + 1 \\
&= 8m^2 + 10m + 2 + 1 \\
&= 2(4m^2 + 5m + 1) + 1
\end{aligned}$$

This an odd number by definition.

Case 2: n is even.

Let $n = 2m$ where $m \in \mathbb{Z}$.

$$\begin{aligned}
2n^2 + n &= 2(2m)^2 + (2m) \\
&= 2(4m^2) + 2m \\
&= 8m^2 + 2m \\
&= 2(4m^2 + m)
\end{aligned}$$

This is an even number by definition. Since we want $2n^2 + n$ to be odd for this implication, we can see that n must be odd as well. Now let $n = 2m + 1$ where $m \in \mathbb{Z}$. Therefore,

$$\begin{aligned}
cos(\frac{n\pi}{2}) &= cos(\frac{(2m+1)\pi}{2}) \\
&= cos(\frac{2m\pi + \pi}{2}) \\
&= cos(m\pi + \frac{\pi}{2})
\end{aligned}$$

This evaluates to $0 \ \forall \ m \in \mathbb{Z}$.

The other direction of the implication that needs to be proven is: $cos(\frac{n\pi}{2})$ is even $\Rightarrow 2n^2 + n$ is odd. Once again if $cos(\frac{n\pi}{2})$ is even, then it must evaluate to 0 which means n must be odd. And because of case 1 previously shown, it is known that $2n^2 + n$ will always be odd as well. □

2: Write a proof for the following theorem:

Theorem 2. *Let $x, y \in \mathbb{Z}$. Prove that if xy is odd, then $x^2 + y^2$ is even.*

Proof. Because xy is odd, then x and y must be odd numbers. To show this, we can break it into cases.

Case: x, y are both odd.

Let $x = 2m + 1$ and $y = 2n + 1$ where $m, n \in \mathbb{Z}$.

$$\begin{aligned} xy &= (2m+1)(2n+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

This is the definition of an odd number.

Case: x is odd and y is even.

Let $x = 2m + 1$ and $y = 2n$ where $m, n \in \mathbb{Z}$.

$$\begin{aligned} xy &= (2m+1)(2n) \\ &= 2(2mn + n) \end{aligned}$$

This is the definition of an even number.

Case: x is even and y is odd.

Mutatis mutandis for the previous case.

Case: x, y are both even.

Let $x = 2m$ and $y = 2n$ where $m, n \in \mathbb{Z}$.

$$\begin{aligned} xy &= (2m)(2n) \\ &= 4mn \\ &= 2(2mn) \end{aligned}$$

This is the definition of an even number.

Since it is shown that both x and y must both be odd for xy to be odd, we can define $x = 2m + 1$ and $2n + 1$ where $m, n \in \mathbb{Z}$.

$$\begin{aligned} x^2 + y^2 &= (2m+1)^2 + (2n+1)^2 \\ &= (4m^2 + 4m + 1) + (4n^2 + 4n + 1) \\ &= 4m^2 + 4n^2 + 4m + 4n + 2 \\ &= 2(2m^2 + 2n^2 + 2m + 2n + 1) \end{aligned}$$

This is the definition of an even number.

$\square$

3: Write a proof for the following theorem:

Theorem 3. *Prove that there is a student S in MA 240 for whom the following is true: If S eats a chocolate on Valentine's Day, then everyone in MA 240 will eat a chocolate on Valentine's Day.*

Proof. (Contrapositive) Let S be the student who eats a chocolate on Valentine's Day and E be everyone eating a chocolate on Valentine's Day. Instead of proving $S \Rightarrow E$, this can be proven by rewriting it as $\neg E \Rightarrow \neg S$.

Case: No one in the class will eat a chocolate on Valentine's Day.

Since no one in the class will be eating any chocolate on Valentine's Day, then there exists the student S in the class who will not be eating chocolate on Valentine's Day either. This means $\neg E$ and $\neg S$ are both true making the implication $\neg E \Rightarrow \neg S$ true.

Case: Everyone in the class will eat a chocolate on Valentine's Day.

Since everyone in the class will eat a chocolate on Valentine's Day, then there exists the student S in the class who will be eating a chocolate on Valentine's Day. This means $\neg E$ and $\neg S$ are both false making the implication $\neg E \Rightarrow \neg S$ true. $\square$