Pledge: *I pledge my honor that I have abided by the Stevens Honor System.* -Eric Altenburg

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1: ...Eventually, each lamp will settle into a final state. When all is said and done, which lamps will be glowing?

I think that when all is said and done, the lamps left glowing will be those that are on odd numbered lamps. Initially what I think when trying to prove this would be to use some form of induction where you can consider the n lamps as a set of 0s and 1s where 0 is off and 1 is on. We can have the base cases of $n = 1$ and $n = 2$ to show that the odd number will be turned on and when the even number lamplighter starts, it will invert all the other lamps except for n so you will have a set $S = \{1, 0, \dots\}$. Then the inductive hypothesis will be typical in that you assume the property is true. Then for $n + 1$ you need to break it down into cases of even and odd and that you will form the sets $S = \{1, 0, \dots, n + 1\}$.

2: ...Can every square on the grid serve as the starting point of such a path? If not, which squares can? What can you say about the total number of such paths?

Only the 4 corners and 4 center edge squares are valid starting points and with each corner there are 2 possible opening moves and with each center there are 3 possible moves. This means there are *at least* $(3 \cdot 4 + 2 \cdot 4) = 20$ moves. The upper limit I'm not entirely sure how to calculate (using a breadth first search would definitely find the upper limit but putting it in proof notation would be hard). Also, as for a proof of starting at the 4 corners and the 4 center edges, I'm stuck trying to prove this too. For me conceptually trying to put a maze into words is difficult because they can be represented in the form of a matrix, however, trying to logically show that it's possible to win starting at these points gets confusing because of all the the possible routes that can be taken.

3: Does there exist a continuous function $f : (0, \infty) \to \mathbb{R}$ such that $f(x)$ is rational if and only if $f(2x)$ is irrational?

Proof. (Contradiction) Assume f exists and is a continuous function such that $f : (0, \infty) \to \mathbb{R}$, and $x \in (0, \infty)$.
Case 1: x is an irrational number
If x is an irrational number, then we know that given an irrational input, the result of f will be rational. Therefore, if we multiply it by 2, which is a non-zero rational number, then the input number will still be irrational and map to a rational result.
Case 2: x is a rational number
If x is rational, then we know that given a rational input, the result of f will be rational. Therefore, if we multiply it by 2, it will still be a rational number and will continue to give us a rational result.
Neither case leads to f behaving as the prompt states, and the consequent of the implication is always false, therefore, the implication itself is not true and no such function exists. □