Pledge: *I pledge my honor that I have abided by the Stevens Honor System.* -Eric Altenburg

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1: Prove that, given any two real numbers x and y such that $x < y$, there exists an irrational number z such that $x < z < y$.

Proof. (Contradiction) Let (x, y) be a set where $x, y \in \mathbb{R}$ and $x < y$. Note that this set is infinitely uncountable due to Cantor's Diagonal Argument.
Assume there does not exist an irrational number $z \in (x, y)$ such that $x < z < y$. Since no irrational numbers exist in (x, y), then there must only be rational numbers in (x, y) making (x, y) a subset of $\mathbb{Q}$. Because we know that $\mathbb{Q}$ is countable, this means any subsets are also countable, including (x, y). This is a contradiction because it was previously stated that (x, y) is uncountable. Therefore, there must exist an irrational number $z \in (x, y)$ such that $x < z < y$. $\square$

2: Let $S \subset \{1, 2, \ldots, 1000\}$ be a set of 100 natural numbers. Prove that there exists distinct nonempty subsets $X, Y \subset S$ such that the sum of the elements of X equals the sum of the elements of Y.

Proof. The number of subsets of 100 natural numbers is 2^{100}, and the largest possible sum of a subset of numbers of S is $901 + 902 + \ldots + 999 + 1000 = 95050$. This means there are 95050 possible sums of numbers, and due to the Pigeonhole Principle, since there are more subsets (2^{100}) than there are possible sums (95050), there exists at least one sum which can be made from two subsets. $\square$

3: Make a conjecture about which numbers $n \in \mathbb{N}$ can be expressed as a sum of two or more consecutive natural numbers. (Note that the numbers int he sum don't have to start at 1. For example, 12 is such a number since $12 = 3 + 4 + 5$.) Then prove your conjecture.

Conjecture 1. *Every number $n \in \mathbb{N}$ can be expressed as the sum of two or more consecutive natural numbers if and only if $n \neq 2^k$ where $k \in \mathbb{N}$.*

Proof. ($\Longrightarrow$)
(Contradiction) Assume a number $n = 2^k$ where $k \in \mathbb{N}$ and can be written as a sum of two or more consecutive natural numbers. The amount of numbers that can be added up to make n can be an odd or even amount.

Case 1: The summation has an odd amount of consecutive numbers.
A sum of consecutive numbers having an odd amount of numbers would have one exact middle number being added together (i.e. $m + (m + 1) + \ldots + (m + n)$ will have an element that is equally distant from m and $(m + n)$, this is known as the average of the two numbers). Then the sum can be expressed as,

$$\text{sum} = (\text{average}) \cdot (\text{amount of consecutive number added together}),$$

the latter of which is odd. This would mean the sum has an odd number as a factor, however, 2^k where $k \in \mathbb{N}$ will always be even which is a contradiction.

Case 2: The summation has an even amount of consecutive numbers.
Because the sum will have an even amount of consecutive numbers, there will not be a number that is the average like with Case 1, instead the middle two numbers must be summed and then divided by 2. This means the sum can be expressed as,

$$\text{sum} = (\text{middle two numbers summed}) \cdot \frac{1}{2} \cdot (\text{amount of consecutive numbers}).$$

We know (amount of consecutive numbers) $\cdot \frac{1}{2}$ will still be an even number, however, two consecutive numbers added together will always form an odd number. This means the sum has an odd number as a factor, and since 2^k where $k \in \mathbb{N}$ will always be even, this is a contradiction.
($\Longleftarrow$)
(Induction) Suppose $n = 2^k$ where $k \in \mathbb{N}$ cannot be written as the sum of two or more consecutive natural numbers.
Base Case: $k = 1$
$n = 2^1 = 2$. This cannot be written as a sum of two or more consecutive natural numbers.
Inductive Hypothesis:
Assume that for $n = 2^k$, it cannot be written as the sum of two or more consecutive natural numbers. We wish to prove that the property holds for $k + 1$ as well, we observe that

$$\begin{aligned} n &= 2^{k+1} \\ &= 2^k \cdot 2^1. \end{aligned}$$

With the base case showing that 2^1 cannot be written as a sum of two or more consecutive natural numbers, and the inductive hypothesis shows that neither can 2^k, then it follows that the product $(2^1 \cdot 2^k)$ also cannot be written as a sum of two or more consecutive natural numbers. This establishes the claim and completes the proof. $\square$