Pledge: *I pledge my honor that I have abided by the Stevens Honor System.* -Eric Altenburg

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1: Take a look at the various sets of real numbers listed in the reading above. Do these sets havea least element? If so, what is it?

$(X = \mathbb{R})$

I don't think this has a lowest element simply because you can choose a number x to be the lowest element, but then you can simply divide it by 2 and you would have a smaller element. Therefore, since it does not have a defined least element.

$(X = \mathbb{Z})$

Same as $\mathbb{R}$, this would not have a least element because you can have a number x as the least element, but you can subtract 1 from it and have a more least element. This is not well ordered.

$(X = \mathbb{N})$

This does have a least element: 1. Because it has a least element it is well ordered.

$(X = (0, 1))$

This does not have a least element because it does not have a hard limit to 0 like the range $[0, 1]$ would. You can always take a least number x, then divide it by 10 and get something smaller. Therefore, it is not well ordered.

$(X = (1, 2) \cup [-1, 0))$

This has the least element -1 because that is the smallest number possible in the set which does include -1 in it. Since it has a least element, it is well ordered.

$(X = \mathbb{Q} \cap [0, 1])$

This has the least element 0 which is a rational number and is therefore a part of the set $\mathbb{Q}$. Since it the range includes 0 to be the smallest element, it is the least element. It is well ordered.

$(X = \{n^{-1} | n \in \mathbb{N}\})$

There is no least element in here because you can say x is the largest number in $\mathbb{N}$ which would give the lowest element in the set X. However, $x + 1$ is still in $\mathbb{N}$ and would give an even smaller number in the set X, therefore there is no least element and it is not well ordered.

2: Use induction to prove that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

Proof. Base case: $n = 1$

$1 = \frac{1(1+1)}{2} = 1$

Inductive Hypothesis:

Assume $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for $n = k$ where $k \geq 1 | k \in \mathbb{N}$. $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$.

Under this assumption, then if $n = k + 1$ we get:

$$\begin{aligned}1 + 2 + 3 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1)\\ &= \frac{k^2+k}{2} + \frac{2(k+1)}{2}\\ &= \frac{k^2+3k+2}{2}\\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

This proves the claim for any $n \in \mathbb{N}$. □

3: Use induction to prove that the $1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$ for all n.

Proof. Base Case: $n = 0$

$1 = 2^1 - 1 = 1$

Inductive Hypothesis:

Assume $1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$ for $n = k$ where $k \geq 0 | k \in \mathbb{N}$. $1 + 2 + 4 + \cdots + 2^k = 2^{k+1} - 1$.

Under this assumption, then if $n = k + 1$ we get:

$$\begin{aligned}1 + 2 + 4 + \cdots + 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1}\\ &= (2^k \cdot 2^1) + (2^k \cdot 2^1) - 1\\ &= 2(2^k \cdot 2^1) - 1\\ &= (2^k \cdot 2^1 \cdot 2^1) - 1\\ &= (2^k \cdot 2^2) - 1\\ &= 2^{k+2} - 1\end{aligned}$$

This proves the claim for any $n \in \mathbb{N}$. □

4: Did you know that, despite the fact that they appear to be distinct, all real numbers are actually equal to one another? Here's a proof! (*insert proof here*). Does this proof look fishy to you? If so, can you explain what, exactly, has gone wrong?

The base case seems fine to me, however, by definition, a set is only composed of unique elements. While the base case is technically true since there are no duplicates in a set of size 1, if there is a set of size n+1 where they are all equal then technically the set would only be of size 1. In short, a set of n real number that are all equal would be of size 1, and a set of $n + 1$ real numbers would also be of size 1 which is not induction.

5: Use the well-ordering principle to prove that induction works. (That is, try to prove the "Mathematical Induction" theorem stated above.)

Theorem 1. *For each $n \in \mathbb{N}$, let $P(n)$ be a proposition, and suppose the following two conditions hold:*

1. *$P(1)$ is true.*
2. *The implication $P(n) \Rightarrow P(n+1)$ is true for all $n \in \mathbb{N}$.*

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. (Contradiction) Suppose a non-empty set S is composed of $\mathbb{N}$ and it has no least element. That would mean 1 does not exist in the set because then S would have a least element. Now suppose, $1, 2, 3, 4, \cdots, n$ is not in S. Then, $n+1$ does not exist in the set either because if it did, then it would be the least element. With this, that means the set S would end up being empty by induction which would be a contradiction. S needs to have a least element in order for induction to work. $\square$