Pledge: *I pledge my honor that I have abided by the Stevens Honor System.* -Eric Altenburg

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1: Recall that the sequence of Fibonacci numbers is defined as follows: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ when $n > 2$.

(Prove that $F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$ for all $n \in \mathbb{N}$.)

Proof. (Induction)
Base Case: $n = 1$, then

$$\begin{aligned} F_1^2 &= F_1 \cdot F_2 \\ 1 &= 1 \cdot 1 \\ 1 &= 1. \end{aligned}$$

Inductive Hypothesis: Assume $F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$ holds true for some n. We wish to prove the identity holds for $n + 1$ as well, and we find that

$$\begin{aligned} F_1^2 + F_2^2 + \ldots + F_n^2 + F_{n+1}^2 &= F_n F_{n+1} + F_{n+1}^2 \\ &= F_{n+1} \left(F_n + F_{n+1} \right) \\ &= F_{n+1} F_{n+2}. \end{aligned}$$

This establishes the claim and completes the proof. □

(Let S_n, where $n \in \mathbb{N}$, be the set of all n-digit binary strings that have no consecutive 1s. For example, $S_3 = \{000, 001, 010, 100, 101\}$. Prove that S_n contains exactly F_{n+2} elements.)

Proof. (Induction)
Base Case:
If $n = 1$, then $S_1 = \{0, 1\}$. $F_3 = 2$ and $|S_1| = 2$.
If $n = 2$, then $S_2 = \{00, 01, 10\}$. $F_4 = 3$ and $|S_2| = 3$.
Inductive Hypothesis: Assume S_n contains exactly F_{n+2} elements, where S_n is the set of all n-digit binary strings that have no consecutive 1s and $n \in \mathbb{N}$. We wish to prove the identity holds for S_{n+1} as well, and we observe that

$$\begin{aligned} \mid S_{n+1} \mid &= \mid S_n \mid + \mid S_{n-1} \mid \\ &= F_{n+2} + F_{(n-1)+2} \\ &= F_{n+2} + F_{n+1} \\ &= F_{n+3} \\ &= F_{(n+1)+2}. \end{aligned}$$

This establishes the claim and completes the proof. □

2: Suppose you draw several straight lines on a piece of paper, thereby dividing the paper into different regions (each line should extend to the edges of the page). Prove that, no matter how your lines are drawn, it is possible to color each region red or blue in such a way that no two adjacent regions have the same color.

Proof. (Induction) Suppose n is the number of straight lines going from edge-to-edge drawn on a piece of paper.
Base Case: $n = 1$
There are two regions on the piece of paper, one can be colored red and the other is colored blue. The two regions are adjacent and do not have the same colors.
Inductive Hypothesis: Assume that for a piece of paper with n lines on it, there are no two adjacent regions that have the same color.
We wish to prove that the property holds for $n + 1$ lines as well. After drawing the additional line on the paper, suppose one side of the new line containing all that side's regions is called A, and the other side of the new line containing all that side's regions is called B. We know that all the adjacent regions in side A and all the adjacent regions in side B satisfy the property, however, the adjacent regions where sides A and B meet will have the same color. To make the identity hold true, all the regions' colors on either side A or B can be inverted; for example, all of A's red regions become blue and blue regions become red. Doing this will still ensure that all the old regions will still follow the identity since they did prior to the inversion, because of this, these color changes will not break it.
Having inverted all the regions to one side of the new line, the regions adjacent to each other where sides A and B meet are different colors and the identity holds true still for $n + 1$ lines. $\square$