Pledge: *I pledge my honor that I have abided by the Stevens Honor System.* -Eric Altenburg

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1: Show that the contrapositive of an implication is logically equivalent to the implication itself.

The implication $P \Rightarrow Q$ is logically equivalent to $\neg P \vee Q$. By using this, we can take the contrapositive implication and apply the same steps:

$\neg Q \Rightarrow \neg P \equiv Q \vee \neg P$

Which is essentially the same as $\neg P \vee Q$ with the order simply switched.

2: Prove that for an integer n, if n^2 is even, then n is even.

Proof. The way the current theorem is formatted cannot be proven with a direct proof, instead, it can be proven with a contrapositive. It can then be rewritten as:

If n is not even, then n^2 is not even.

We can write an odd number as $n = 2k + 1$ where k is some integer and substitute that n into n^2 to get:

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

Since we know that the contents of the parenthesis $(2k^2 + 2k)$ will be another integer, then we can let $x = 2k^2 + 2k$. Now the final line of the equation can be rewritten as $2x + 1$ which is the definition of an odd number. $\square$

3: Let $x \in \mathbb{R}$. Prove that if $x^5 - x^4 + 7x^3 - x^2 + 5x - 8 \geq 0$, then $x \geq 0$.

Proof. Once again, proving this theorem as it stands is difficult, so we can rewrite it as a contrapositive:

Let $x \in \mathbb{R}$. If $x < 0$, then $x^5 - x^4 + 7x^3 - x^2 + 5x - 8 < 0$.

We can rearrange the even and odd powers of the second proposition so it reads $x^5 + 7x^3 + 5x < x^4 + x^2 + 8$. Because we know that $x < 0$, this inequality will always hold, therefore the original proposition $(x^5 - x^4 + 7x^3 - x^2 + 5x - 8 \geq 0)$ will not hold if $x \geq 0$. $\square$

4: Prove the following statements. Decide whether a direct proof or providing the contrapositive is more appropriate.

Theorem 1. *If n is odd, then 8 divides $n^2 - 1$.*

Proof. (Direct because it is easier)

Given that n is odd, we can let $n = 2k + 1$ where k is some integer. Substituting this into $n^2 - 1$ gives:

$$\begin{aligned} n^2 - 1 &= (2k+1)^2 - 1 \\ &= 4k^2 + 4k + 1 - 1 \\ &= 4k^2 + 4k \\ &= 4k(k+1) \end{aligned}$$

We have $4 \cdot k \cdot (k+1)$, we know $k \cdot (k+1)$ is an even number because one of the numbers is even. Because of this we can rewrite $k \cdot (k+1) = 2x$ where x is some integer. This gives the final equation $4k(k+1) = 4 \cdot 2x = 8x$ which is divisible by 8. □

Theorem 2. *If $n \in \mathbb{Z}$, then 4 does not divide $n^2 - 3$.*

Proof. (Contrapositive) The implication is rewritten as:

If 4 divides $n^2 - 3$, then $n \in \mathbb{Z}$.

For $n^2 - 3$ to be divisible there needs to be a factor of 4, so it can be rewritten as $n^2 - 3 = 4(\frac{n^2-3}{4})$. In this form there needs to be some sort of decimal to have this number be divisible by 4 and because of this requirement, n cannot be an integer. □