**Pledge:** *I pledge my honor that I have abided by the Stevens Honor System.* -Eric Altenburg

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**1**: Prove that if two sets $A$ and $B$ have the same cardinality, then for any set $C$, the sets $A \times C$ and $B \times C$ have the same cardinality as well.

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*Proof.* Since $|A| = |B|$, there must exist a bijective function $f$ such that $f : A \to B$, and it is defined as $f(a) \to b$ where $a \in A$ and $b \in B$. Now to show that $|A \times C| = |B \times C|$, we can create another bijective function $g$ such that $g : A \times C \to B \times C$, and it is defined as $g(a, c) \to (f(a), c)$ where $c \in C$. Because this bijective function exists, it is proven that $|A \times C| = |B \times C|$. $\square$

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**2**: Is it true that a set $A$ is countable if and only if there exists a surjection $f : \mathbb{N} \to A$? (Recall that a surjection is an onto function.) If so, prove it. If not, state and prove a closely related theorem.

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The way the theorem is currently written cannot be proven because it is possible for $A$ to be an empty set. Therefore, the theorem should be rewritten to account for this.

**Theorem 1.** *The set $A$ is countable if and only if $A$ is empty or there exists a surjection $f : \mathbb{N} \to A$.*

*Proof.* $(\Longrightarrow)$
Suppose $A$ is countable. Then this means $A$ is either finite or infinitely countable.
Case 1: $A$ is infinitely countable.
If $A$ is infinitely countable, then there exists a bijective function from $\mathbb{N}$ to $A$. And since it is bijective, then the function from $\mathbb{N}$ to $A$ is also surjective.
Case 2: $A$ is finite and is empty.
The empty set is countable.
Case 3: $A$ is finite and not empty.
Since $A$ is finite and countable, then there is a bijective function for some $n$ where $f : [n] \to A$ such that $n \geq 1$ and $0 \in [n]$. We can create a surjection between $\mathbb{N}$ and $A$ by defining $g : \mathbb{N} \to A$ as,

$$g(x) \to \begin{cases} f(x) & \text{if } x < n \\ f(0) & \text{else} \end{cases}.$$

The function $g$ simply returns each element in the set $A$, and then it will repeat $f(0)$. This is surjective.
$(\Longleftarrow)$
Case 1: $A$ is finite.
If $A$ is finite, then it is countable.
Case 2: $A$ is infinite.
If $A$ is infinite, then it is not empty, and there is a surjection $f : \mathbb{N} \to A$. To make $f$ bijective, we need to account for the issue of an element in $A$ being enumerated more than once because if it does, then it is not injective. To fix this, we can create another function $g$ that will enumerate $f$'s output and does not include the output that has been repeated. Define $g$ as $g(0) = f(0)$ and for every $i$, $g(i + 1) = f(j)$ such that $j$ is the least number in $\mathbb{N}$ and $f(j) \notin \{g(0), g(1), \ldots, g(j)\}$. The

function $g$ is a bijection because for every $i$, $g(i+1) \notin \{g(0), g(1), \ldots, g(i)\}$ which makes it injective, and since $f$ is surjective, $g$ is as well. This means $A$ countable. $\square$

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**3**: Prove that the infinite strip $S = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{R} \text{ and } 0 \leq y \leq 1\}$ and the Cartesian plane $\mathbb{R}^2$ have the same cardinality. *Hint:* You may assume the Schröder-Bernstein theorem.

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*Proof.* Assuming the Schröder-Bernstein theorem, we can create an injective function $f : S \to \mathbb{R}^2$ by defining $f$ as $f(x, y) \to (x, y)$ where any pair $(x, y) \in S$ will give itself in $\mathbb{R}^2$. This shows that $|S| \leq |\mathbb{R}^2|$. To create an injective function $g : \mathbb{R}^2 \to S$, we can define $g$ as,

$$g(x,y) \to \begin{cases} \left(x, \left(\frac{1}{2} + 2^{-y-1}\right)\right), & y > 0 \\ \left(x, \frac{1}{2}\right), & y = 0 \\ \left(x, 2^{y-1}\right), & y < 0 \end{cases}.$$

This shows that $|\mathbb{R}^2| \leq |S|$, and because of the Schröder-Bernstein theorem, we must have that $|S| = |\mathbb{R}^2|$. $\square$

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**4**: Prove the theorem whose proof we didn't complete in class: For any set $A$, we have $|A| < |2^A|$.

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*Proof.* (Contradiction) First we can construct a surjective function $f : A \to 2^A$ such that $f(a) \to a$ where $a \in A$. This implies that $|A| \leq |2^A|$, however, to prove that the cardinality of $A$ is strictly less than the cardinality of $2^A$ we must show that a bijection does not exist from $A$ to $2^A$. Assuming the opposite and that such a function exists, let the function be $g : A \to 2^A$. Using this, we can construct the set $S = \{x \in A \mid x \notin g(x)\}$, note that $S \subseteq A$. Because $S \subseteq 2^A$ and our function $g$ is also surjective, this must mean that $g(y) \in S$ for some $y \in A$. This is a contradiction because if we have some $y \in S$, then that means $y \notin g(y)$ but like we stated earlier, $g(y) \in S$. Additionally, if $y \notin S$, then this means $y \in g(y)$ but $g(y) \in S$. So regardless of $y$'s membership with $S$ there is a contradiction which must mean that no bijective function $g$ exists. Therefore, $|A| < |2^A|$. $\square$