Pledge: *I pledge my honor that I have abided by the Stevens Honor System.* -Eric Altenburg

---

1: Prove that, given any two real numbers x and y such that $x < y$, there exists an irrational number z such that $x < z < y$.

Proof. Let z be a real number equally distant from x and y, $z = \frac{x+y}{2}$. To show that $x < z$, substitute z for $\frac{x+y}{2}$, and

$$\begin{aligned} x &< z \\ x &< \frac{x+y}{2} \\ 2x &< x + y \\ x &< y. \end{aligned}$$

Given that $x < y$, this proves the inequality to be true for $x < z$. To show that $z < y$, substitute z for $\frac{x+y}{2}$ again, and

$$\begin{aligned} z &< y \\ \frac{x+y}{2} &< y \\ x + y &< 2y \\ x &< y. \end{aligned}$$

Once again, given $x < y$ this proves the inequality to be true for $z < y$. $\square$

2: Let $S \subset \{1, 2, \ldots, 1000\}$ be a set of 100 natural numbers. Prove that there exists distinct nonempty subsets $X, Y \subset S$ such that the sum of the elements of X equals the sum of the elements of Y.

Proof. The possible number of subsets of 100 natural numbers are 2^{100}, and the largest possible sum of a subset of numbers of S is $901 + 902 + \ldots + 999 + 1000 = 95050$. This means there are 95050 possible sums of numbers, and due to the Pigeonhole Principle, since there are more subsets (2^{100}) than there are possible sums (95050), there exists at least one sum which can be made from two subsets.

Let A and B be the two subsets that form the same sum, and let $C = A \cap B$ which are all the elements that are found in both A and B. If we remove the set of C from A and B, then $A' = A - C$ and $B' = B - C$. Since we removed the same elements from both sets, the sum of A' and B' will remain the same while being distinct and non-empty. $\square$

3: Make a conjecture about which numbers $n \in \mathbb{N}$ can be expressed as a sum of two or more consecutive natural numbers. (Note that the numbers int he sum don't have to start at 1. For example, 12 is such a number since $12 = 3 + 4 + 5$.) Then prove your conjecture.

Conjecture 1. *Every number $n \in \mathbb{N}$ where $n \neq 2^k$ and $k \in \mathbb{N}$ can be expressed as the sum of two or more consecutive natural numbers.*

Proof. (Contradiciton) Assume a number n that is a power of 2 can be written as a sum of consecutive natural numbers. The amount of numbers that can be added up to make n can be an odd and even amount.

Case 1: The summation has an odd amount of consecutive numbers.
A sum of consecutive numbers having an odd amount of numbers would have one exact middle number being added together (i.e. $m + (m+1) + \ldots + (m+n)$ will have an element that is equally distant from m and $(m+n)$, this is known as the average of the two numbers). Then the sum can be expressed as sum $=$ average $\cdot$ amount of consecutive number added together, the latter of which is odd. This would mean the sum has an odd number as a factor, however, a power of 2 will always be even which is a contradiction.

Case 2: The summation has an even amount of consecutive numbers.
Because the sum will have an even amount of consecutive numbers, there will not be a number that is the average like with Case 1, instead the middle two numbers must be summed and then divided by 2. This means the sum can be expressed as
sum $=$ middle two numbers summed $\cdot \frac{1}{2} \cdot$ amount of consecutive numbers. We know the amount of consecutive numbers divided by 2 will still be an even number, however, two consecutive numbers added together will always form an odd number. This means the sum has an odd number as a factor, and since a power of 2 will always be even, this is a contradiction. $\square$