Pledge: *I pledge my honor that I have abided by the Stevens Honor System.* -Eric Altenburg

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1: Use the epsilon-delta definition of continuity to prove that the function $f(x) = 1 - 5x$ is continuous at $x = 2$.

Proof. Let $\epsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{5}$. Then for $|x - 2| < \delta$, this gives

$$\begin{aligned} |f(x) - f(2)| &= |(1 - 5x) - (1 - 5(2))| \\ &= |1 - 5x + 9| \\ &= |-5x + 10| \\ &= |-5||x - 2|, \end{aligned}$$

and now we have

$$\begin{aligned} |-5||x - 2| &< |-5| \cdot \delta \\ |-5||x - 2| &< 5 \cdot \frac{\varepsilon}{5} \\ |-5||x - 2| &< \varepsilon \end{aligned}$$

thus showing that $f(x)$ is continuous at $x = 2$. $\square$

2: Prove that every nonempty subset $A \subseteq \mathbb{R}$ that has a lower bound has a greatest lower bound, also known as the *infimum* and denoted $\inf(A)$.

Proof. Let a be a lower bound in A, then $a \leq x$ for all $x \in A$. For every $x \in A$, multiplying both this and a by -1 would give $-x \leq -a$ which would be the same as saying $y \leq -a$ for all $y \in -A$. This then implies that the new set $-A$, where every $x \in A$ has been multiplied by -1, is nonempty and now bounded above. Therefore, by the Least Upper Bound Principle, there is a least upper bound $b \in -A$. Now we show that $-b$ is the greatest lower bound of A.

To show that $-b$ is a lower bound for A, assume the opposite; that $-b$ is not a lower bound for A. Then there is a $z \in A$ such that $z < -b$, but multiplying both sides by -1 gives $-z > b$. Since $-z \in -A$, this contradicts the fact that b is a least upper bound of $-A$.

To show there is no number greater than $-b$ that is a lower bound for A, assume the opposite; that there exists a number greater than $-b$ that is a lower bound for A. Let j be a lower bound of A such that $-b < j$. Multiplying both sides by -1 gives $b > -j$. This then implies that $y \leq -j < b$ for all $y \in -A$, however, this is a contradiction as well because we said that b was the least upper bound in $-A$.

Since $-b$ is a lower bound for A, and there is no greater number that is also a lower bound of A, then $-b$ must be the greatest lower bound of A. $\square$

3: Let $\mathbb{Q} = \{q_1, q_2, \ldots, q_n, \ldots\}$ be an enumeration of the rational numbers (note that the existence of such an enumeration follows from the fact that $\mathbb{Q}$ is countable). Then define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = q_n \\ 0 & \text{otherwise} \end{cases} .$$

Prove that $f(x)$ is continuous if and only if x is irrational.

Proof. ($\Longrightarrow$) Let $x_0 \in \mathbb{R}$ be a rational number, then based on f, we know $f(x_0) > 0$. Since we know the irrationals are dense in $\mathbb{R}$ including leading up to x_0, we say there is a sequence $a_1, a_2, a_3, \ldots \in \mathbb{R}$ of irrational numbers such that a_n approaches x_0 as n approaches infinity. This then gives us $\lim\limits_{n\to\infty} a_n = x_0$, however, we know that $\lim\limits_{n\to\infty} f(a_n) = 0 \neq f(x_0)$. Therefore, this shows that f is not continuous when x_0 is rational.

($\Longleftarrow$) Let $x_0 \in \mathbb{R}$ be irrational, then $f(x_0) = 0$. To show that f is continuous at x_0 we then need to show that for all $\varepsilon > 0$, there exists a $\delta > 0$ where $|x - x_0| < \delta$ implies that $f(x) < \varepsilon$ for some x.

Now, choose y where $y > \frac{1}{\varepsilon}$ and $y \in \mathbb{Z}$. Among all the rational numbers, for any given n where $n = \frac{p}{q}$ such that $q < y$, choose an n_0 whose distance is closest to x_0. With this, let δ be the distance from x_0 to the chosen rational number n_0. Then we can construct an interval $(x_0 - \delta, x_0 + \delta)$ where if $x \in (x_0 - \delta, x_0 + \delta)$, x is either an irrational number or its denominator is greater than y.

From this interval, if x is irrational, then $f(x) = 0$, but if x is rational, then this forms the inequality $0 < f(x) \leq \frac{1}{y} < \varepsilon$. Regardless of whether x is rational or irrational, we note that $|f(x)| < \varepsilon$. Therefore, with $x \in (x_0 - \delta, x_0 + \delta)$, $|x - x_0| < \delta$, and $|f(x)| < \varepsilon$, this implies $|f(x) - f(x_0)| < \varepsilon$ proving that f is continuous at x_0. $\square$