Pledge: *I pledge my honor that I have abided by the Stevens Honor System.* -Eric Altenburg

1: Prove that each of the following relations is an equivalence relation. Then describe the corresponding equivalence classes, e.g. by giving a geometric description.

(The relation R defined on $\mathbb{R}^2$ by $((a,b),(c,d)) \in R$ if $|a| + |b| = |c| + |d|$.)

Proof. We must show that R is reflexive, symmetric, and transitive.

1. Reflexive: Let (a,b) be a pair of real numbers. To show that $((a,b),(a,b)) \in R$, we can set $|a| + |b|$ is equal to itself. This gives, $|a| + |b| = |a| + |b|$, so $((a,b),(a,b)) \in R$. This shows that R is reflexive.

2. Symmetric: Let (a,b) and (c,d) be pairs of real numbers where $((a,b),(c,d)) \in R$. Then $|a| + |b| = |c| + |d|$. This is the same as $|c| + |d| = |a| + |b|$ which means $((c,d),(a,b) \in R$. This shows that R is symmetric.

3. Transitive: Let $(a,b),(c,d),(e,f)$ be pairs of real numbers where $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$. Then $|a| + |b| = |c| + |d|$ and $|c| + |d| = |e| + |f|$. Because $|c| + |d|$ is common in both equations, then we can say $|a| + |b| = |e| + |f|$ which means $((a,b),(e,f)) \in R$. This shows that R is transitive.

□

Description: Geometrically, the equivalence classes formed for the relation $((a,b),(c,d)) \in R$ take the shape of a diamond centered at the point $(0,0)$. All of its sides are the same length n, where $n = |a| + |b|$.

(The relation S defined on the set of positive rational numbers $\mathbb{Q}_{>0}$ by $(a,b) \in S$ if $\frac{a}{b} = 2^n$ for some $n \in \mathbb{Z}$.)

Proof. We must show that S is reflexive, symmetric, and transitive.

1. Reflexive: Let (a,a) be a pair of positive rational numbers. We can then say, $\frac{a}{a} = 1 = 2^0$. Because the ratio of $\frac{a}{a}$ is a power of 2, this means $(a,a) \in S$. This shows that S is reflexive.

2. Symmetric: Let (a,b) be a pair of positive rational numbers where $(a,b) \in S$. Then,

$$\begin{aligned} \frac{a}{b} &= 2^n \\ a \cdot 2^{-n} &= b \\ 2^{-n} &= \frac{b}{a}. \end{aligned}$$

   Since $\frac{b}{a}$ is still equal to a power of 2, we can say $(b,a) \in S$. This shows that S is symmetric.

3. Transitive: Let (a, b) and (b, c) be pairs of positive rational numbers where $(a, b) \in S$ and $(b, c) \in S$. Then,

$$\frac{a}{c} = \frac{a}{b} \cdot \frac{b}{c}$$
$$\frac{a}{c} = 2^n \cdot 2^m, \quad \text{where } n, m \in \mathbb{Z}$$
$$\frac{a}{c} = 2^{n+m}.$$

Since $\frac{a}{c}$ is equal to a power of 2, this means $(a, c) \in S$. This shows that S is transitive.

$\square$

Description: There are a series of lines that begin at $(2^n, 1)$ where $n \in \mathbb{Z}$. The slope of each line is its corresponding 2^n. For example:

1: Line 1 begins at $(2^0, 1)$ and has a slope of 2^0

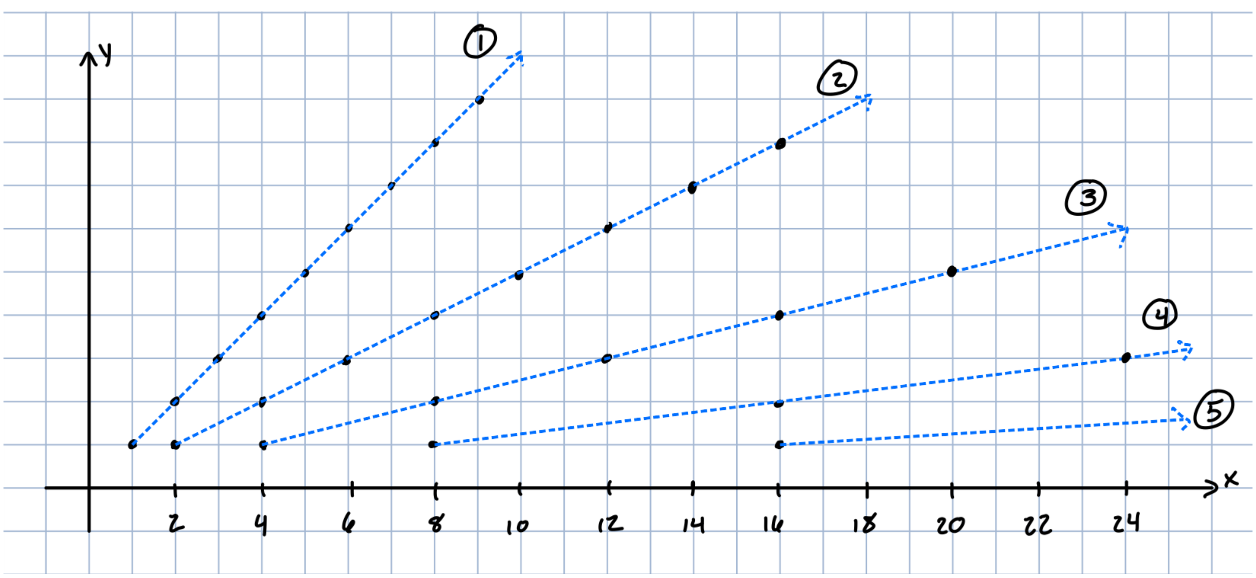
2: Line 2 begins at $(2^1, 1)$ and has a slope of 2^1

3: Line 3 begins at $(2^2, 1)$ and has a slope of 2^2

4: Line 4 begins at $(2^3, 1)$ and has a slope of 2^3

⋮

5: Line n begins at $(2^n, 1)$ and has a slope of 2^n



2: Let R and S be equivalence relations on a set A. Prove or disprove the following statements.

(The relation $R \cup S$ is an equivalence relation on A.)

Proof. (Counter Example) Suppose $A = \{a, b, c\}$, $R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$, and $S = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$, where R and S are equivalence relations on A. Then $R \cup S = \{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)\}$, however, it does not contain (a, c). Therefore, $R \cup S$ is not transitive and not an equivalence relation on A. □

(The relation $R \cap S$ is an equivalence relation on A.)

Proof. We must show that $R \cap S$ is reflexive, symmetric, and transitive.

1. Reflexive: Because R and S are each reflexive, then for all $a \in A$, $(a, a) \in R$ and $(a, a) \in S$. This means $(a, a) \in R \cap S$. This shows that $R \cap S$ is reflexive.
2. Symmetric: Let $(a, b) \in R \cap S$, where $a, b \in A$. This means $(a, b) \in R$ and $(a, b) \in S$, and because R and S are symmetric, $(b, a) \in R$ and $(b, a) \in S$. Therefore, since both relations contain (b, a), then $(b, a) \in R \cap S$. This shows that $R \cap S$ is symmetric.
3. Transitive: Let $(a, b), (b, c) \in R \cap S$, where $a, b, c \in A$. This means $(a, b), (b, c) \in R$ and $(a, b), (b, c) \in S$. Since R and S are transitive, then $(a, c) \in R$ and $(a, c) \in S$. Therefore, because both relations contain (a, c), then $(a, c) \in R \cap S$. This shows that $R \cap S$ is transitive.

□

3: The set of integers *modulo* n, where $n > 1$ is a natural number, is denoted $\mathbb{Z}/n\mathbb{Z}$ and is defined as the set of equivalence classes under the equivalence relation on $\mathbb{Z}$ of being congruent modulo n. Prove that is it possible to define addition and multiplication operations on $\mathbb{Z}/n\mathbb{Z}$ via the formulas $[a] + [b] = [a + b]$ and $[a] \cdot [b] = [a \cdot b]$, respectively.

(Addition: $[a] + [b] = [a + b]$)

Proof. Suppose $[a] = [a']$ and $[b] = [b']$, then from this we can say $a = a' + xn$ and $b = b' + yn$, where $x, y \in \mathbb{R}$. Then, $a + b = a' + b' + n(x + y) \Rightarrow [a + b] = [a' + b']$. To show the existence of identity, $[a] + [0] = [a + 0] = [a]$. Finally, to show the existence of inverse, let's say $[a + (n - a)] = [0]$ which gives us the inverse of $[a]$ being $[n - a]$. □

(Multiplication: $[a] \cdot [b] = [a \cdot b]$)

Proof. Suppose $[a] = [a']$ and $[b] = [b']$, then from this we can say $a = a' + xn$ and $b = b' + yn$, where $x, y \in \mathbb{R}$. Then,

$$\begin{aligned}
a \cdot b &= (a' + xn) \cdot (b' + yn) \\
a \cdot b &= a'b' + a'yn + b'xn + n^2xy \\
a \cdot b &= a'b' + n(a'y + b'x + nxy).
\end{aligned}$$

From this, we can then say $[a] \cdot [b] = [a \cdot b]$. To show the existence of identity,
$[a] \cdot [1] = [a \cdot 1] = [a]$. $\square$