Eric Altenburg Final

8/9/20 Linear

1. $u = (5, -1, 2)$ $v = (2, -1, -3)$ (1)

a) $|v| = \sqrt{2^2 + (-1)^2 + (-3)^2} = \sqrt{4+1+9} = \sqrt{14}$

b) $|u| = \sqrt{5^2 + (-1)^2 + (2)^2} = \sqrt{25+1+4} = \sqrt{30}$

c) $u \cdot v = (5)(2) + (-1)(-1) + (2)(-3) = 10 + 1 - 6 = 5$

d) $\mathrm{proj}_u v = \frac{\langle u, v \rangle}{\langle u, u \rangle} u = \frac{10+1-6}{25+1+4} (5, -1, 2)$

$= \frac{5}{30} (5, -1, 2) = \left(\frac{25}{30}, \frac{-5}{30}, \frac{10}{30} \right)$

e) $\cos\theta = \frac{\langle u, v \rangle}{\|u\| \|v\|} \Rightarrow \cos\theta = \frac{5}{\sqrt{30} \cdot \sqrt{14}}$

$\theta = \cos^{-1} \frac{5}{\sqrt{30} \cdot \sqrt{14}} = \theta = 75.88°$

2. $S = \{(1,2,2), (-1,0,2), (0,0,1)\}$

$w_1 = v_1 = (1,2,2)$

$w_2 = v_2 - \frac{\langle v_2 w_1 \rangle}{\langle w_1 w_1 \rangle} w_1 = (-1,0,2) - \frac{\langle (-1,0,2)(1,2,2) \rangle}{\langle (1,2,2)(1,2,2) \rangle} (1,2,2)$

$= (-1,0,2) - \frac{3}{9}(1,2,2) = (-1,0,2) - \left(\frac{3}{9}, \frac{6}{9}, \frac{6}{9}\right)$

$= \left(\frac{-12}{9}, \frac{-6}{9}, \frac{12}{9}\right)$

$w_3 = v_3 - \frac{\langle v_3 w_1 \rangle}{\langle w_1 w_1 \rangle} w_1 - \frac{\langle v_3 w_2 \rangle}{\langle w_2 w_2 \rangle} w_2$

$= (0,0,1) - \frac{\langle (0,0,1)(1,2,2) \rangle}{\langle (1,2,2)(1,2,2) \rangle}(1,2,2) - \frac{\langle (0,0,1)\left(\frac{-12}{9}, \frac{-6}{9}, \frac{12}{9}\right) \rangle}{\langle \left(\frac{-12}{9}, \frac{-6}{9}, \frac{12}{9}\right)\left(\frac{-12}{9}, \frac{-6}{9}, \frac{12}{9}\right) \rangle} \left(\frac{-12}{9}, \frac{-6}{9}, \frac{12}{9}\right)$

$= (0,0,1) - \frac{2}{9}(1,2,2) - \frac{1}{3}\left(\frac{-12}{9}, \frac{-6}{9}, \frac{12}{9}\right)$

$= (0,0,1) - \left(\frac{2}{9}, \frac{4}{9}, \frac{4}{9}\right) - \left(-\frac{4}{9}, \frac{-2}{9}, \frac{4}{9}\right)$

$= \left(\frac{2}{9}, \frac{-2}{9}, \frac{1}{9}\right)$

$B' = \left\{ (1,2,2), \left(\frac{-12}{9}, \frac{-6}{9}, \frac{12}{9}\right), \left(\frac{2}{9}, \frac{-2}{9}, \frac{1}{9}\right) \right\}$

$B'' = \left\{ \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(\frac{-12}{18}, \frac{-6}{18}, \frac{12}{18}\right), \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right) \right\}$

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3. $T(v) = Av \quad A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$

a) $Ker(T) = span\{(0,0)\}$ $\quad A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad x_1 = 0$, $x_2 = 0$

b) $nullity(T) = 1$

c) $range(T) = \{(1,-1),(2,0)\}$

d) $rank(T) = 2$

4. $T(1,-1) = (2,-3)$ Find $T(2,4)$

$T(0,2) = (0,8)$

$2(1,-1) + 4(0,2) = (2,4)$

$2T(1,-1) + 4T(0,2) = T(2,4)$

$2(2,-3) + 4(0,8) = T(2,4)$

$(4,-6) + (0,16) = T(2,4)$

$(4,10) = T(2,4)$

5. $A = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{bmatrix}$

$det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 0 & 0 \\ 8 & \lambda-4 & 6 \\ -8 & -1 & \lambda-9 \end{vmatrix} = (\lambda-1)(\lambda^2-13\lambda+42) = 0$

$\lambda_1 = 1, \lambda_2 = 6, \lambda_3 = 7$

$\lambda_1 = 1: \begin{bmatrix} 0 & 0 & 0 \\ 8 & -3 & 6 \\ -8 & -1 & -8 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 15/16 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = \frac{-15}{16} x_3$, $x_2 = -\frac{1}{2} x_3$ $\quad x_3 = \mathbb{R}$

$\lambda_2 = 6: \begin{bmatrix} 5 & 0 & 0 \\ 8 & 2 & 6 \\ -8 & -1 & -3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = 0$, $x_2 = -3x_3 \quad x_3 = \mathbb{R}$

$\lambda_3 = 7 \begin{bmatrix} 6 & 0 & 0 \\ 8 & 3 & 6 \\ -8 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = 0$, $x_2 = -2x_3 \quad x_3 = \mathbb{R}$

$\vec{\Lambda}_1 = \begin{bmatrix} -15/16 \\ -1/2 \\ 1 \end{bmatrix} \quad \vec{\Lambda}_2 = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \quad \vec{\Lambda}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -1 & -4 \\ -1 & 4 & 1 \\ -4 & 1 & 1 \end{bmatrix}$ $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 1 & 4 \\ 1 & \lambda-4 & -1 \\ 4 & -1 & \lambda-1 \end{vmatrix} = \lambda^3 - 6\lambda^2 - 9\lambda + 54 = 0$ (4)

$\lambda_1 = -3, \lambda_2 = 3\ \lambda_3 = 6$

$\lambda_1 = -3: \begin{bmatrix} -4 & 1 & 4 \\ 1 & -7 & -1 \\ 4 & -1 & -4 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = x_3$, $x_2 = 0$, $x_3 = \mathbb{R}$ $\vec{\Lambda}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow (1,0,1)$

$\lambda_2 = 3: \begin{bmatrix} 2 & 1 & 4 \\ 1 & -1 & -1 \\ 4 & -1 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = -x_3$, $x_2 = -2x_3$, $x_3 = \mathbb{R}$ $\vec{\Lambda}_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \Rightarrow (-1,-2,1)$

$\lambda_3 = 6: \begin{bmatrix} 5 & 1 & 4 \\ 1 & 2 & -1 \\ 4 & -1 & 5 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = -x_3$, $x_2 = x_3$, $x_3 = \mathbb{R}$ $\vec{\Lambda}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow (-1,1,1)$

$w_1 = (1,0,1)$

$w_2 = v_2 - \frac{\langle v_2 w_1 \rangle}{\langle w_1 w_1 \rangle} w_1 = (-1,-2,1) - \frac{\langle (-1,-2,1)(1,0,1) \rangle}{\langle (1,0,1)(1,0,1) \rangle} (1,0,1)$

$= (-1,-2,1) - 0 = (-1,-2,1)$

$w_3 = v_3 - \frac{\langle v_3 w_1 \rangle}{\langle w_1 w_1 \rangle} w_1 - \frac{\langle v_3 w_2 \rangle}{\langle w_2 w_2 \rangle} w_2$

$= (-1,1,1) - \frac{\langle (-1,1,1),(1,0,1) \rangle}{\langle (1,0,1),(1,0,1) \rangle}(1,0,1) - \frac{\langle (-1,1,1)(-1,-2,1) \rangle}{\langle (-1,-2,1)(-1,-2,1) \rangle}(-1,-2,1)$

$= (-1,1,1) - 0 - 0 = (-1,1,1)$

orthogonalize $= (1,0,1), (-1,-2,1), (-1,1,1)$

orthonormal $= \frac{1}{\sqrt{2}}(1,0,1), \frac{1}{\sqrt{6}}(-1,-2,1), \frac{1}{\sqrt{3}}(-1,1,1)$

$= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$1^2 + 1^2 = \frac{1}{\sqrt{2}}$ $\sqrt{1^2+2^2+1^2} = \sqrt{1+4+1} = \frac{1}{\sqrt{6}}$ $\frac{1}{\sqrt{3}}$

diagonalization: $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$