Eric Altenburg   Content Check 2

7/9/20   Diff. Eq.

1. $(y^2+1)\,dx = y\sec^2 x\,dy$ Hint: $\cos^2 x = \frac{1+\cos 2x}{2}$

$$\Rightarrow \frac{(y^2+1)}{y}\,dx = \sec^2 x\,dy$$

$$\Rightarrow \frac{(y^2+1)}{y}\,dx = \frac{1}{\cos^2 x}\cdot dy$$

$$\Rightarrow \cos^2 x\,dx = \frac{y}{(y^2+1)}\,dy$$

$$\Rightarrow \int \cos^2 x\,dx = \int \frac{y}{y^2+1}\,dy$$

$$\Rightarrow \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + c = \frac{1}{2}\ln|y^2+1|$$

$$\Rightarrow \frac{x}{2} + \frac{\sin 2x}{4} + c = \frac{1}{2}\ln|y^2+1|$$

$$\Rightarrow x + \frac{\sin 2x}{2} + c = \ln|y^2+1|$$

$$\Rightarrow e^{x + \sin 2x/2 + c} = |y^2+1|$$

$$\Rightarrow |y^2+1| = e^x e^{\frac{\sin 2x}{2}} e^c$$

$$\Rightarrow |y^2+1| = ce^x e^{\frac{\sin 2x}{2}}$$

$$\Rightarrow y^2+1 = \pm c e^x e^{\frac{\sin 2x}{2}} = ce^x e^{\frac{\sin 2x}{2}}$$

$$\Rightarrow y^2 = ce^x e^{\frac{\sin 2x}{2}} - 1$$

$$\Rightarrow y = \pm\sqrt{ce^x e^{\frac{\sin 2x}{2}} - 1}$$

$$\boxed{\therefore\ y(x) = \pm\sqrt{ce^x e^{\frac{\sin 2x}{2}} - 1}}$$

2. $y(\ln x - \ln y)\,dx = (x\ln x - x\ln y - y)\,dy$

$\Rightarrow \quad y(\ln x - \ln y)dx - (x\ln x - x\ln y - y)\,dy = 0$

$\Rightarrow \quad y(\ln x - \ln y)dx + (-x\ln x + x\ln y + y)\,dy = 0$

$M(x,y) = y(\ln x - \ln y)$

$M(tx,ty) = ty(\ln tx - \ln ty)$

$\qquad = ty(\ln t + \ln x - (\ln t + \ln y))$

$\qquad = ty(\ln x - \ln y) = tM$ ✓

$N(x,y) = -x\ln x + x\ln y + y$

$N(tx,ty) = -tx\ln tx + tx\ln ty + ty$

$\qquad = t(-x\ln tx + x\ln ty + y)$

$\qquad = t(-x(\ln t + \ln x) + x(\ln t + \ln y) + y)$

$\qquad = t(-\cancel{x\ln t} - x\ln x + \cancel{x\ln t} + x\ln y + y)$

$\qquad = t(-x\ln x + x\ln y + y) = tN$ ✓

Homogeneous!

Let $y = ux$

$dy = u\,dx + x\,du$

$\Rightarrow \quad (ux)(\ln x - \ln(ux))\,dx + (-x\ln x + x\ln(ux) + (ux))(u\,dx + x\,du) = 0$

$\Rightarrow \quad (-ux\ln u)\,dx + (\cancel{-x\ln x} + x\ln u + \cancel{x\ln x} + (ux))(u\,dx + x\,du) = 0$

$\Rightarrow \quad (-ux\ln u)\,dx + (x\ln u + ux)(u\,dx + x\,du) = 0$

$\Rightarrow \quad -u\cancel{x\ln u\,dx} + (u\cancel{x\ln u\,dx} + x^2\ln u\,du + u^2x\,dx + ux^2\,du) = 0$

$\Rightarrow \quad x^2\ln u\,du + u^2x\,dx + ux^2\,du = 0$

$\Rightarrow \quad x^2\ln u\,du + ux^2\,du + u^2x\,dx = 0$

$\Rightarrow \quad du\,(x^2\ln u + ux^2) + u^2x\,dx = 0$

$\Rightarrow \quad u^2x\,dx + du\,(x^2(\ln u + u)) = 0$

$\Rightarrow \quad \dfrac{x}{x^2(\ln u + u)}\,dx + \dfrac{1}{u^2}\,du = 0$

$\Rightarrow \quad \dfrac{x}{x^2}\,dx + \dfrac{\ln u + u}{u^2}\,du = 0$

2. (cont.)

$$\Rightarrow \frac{1}{x}\,dx + \frac{\ln u + u}{u^2}\,du = 0$$

$$\Rightarrow \ln|x| + \left(-\frac{\ln u + u}{u} - \frac{1}{u} + \ln|u| \right) = c \qquad u = \frac{y}{x}$$

$$\Rightarrow \ln|x| + \left(\frac{-\ln(y/x) + (y/x)}{y/x} - \frac{x}{y} + \ln\left|\frac{y}{x}\right| \right) = c$$

$$\Rightarrow \cancel{\ln|x|} + \left(\frac{-x\ln(y/x) - y}{y} - \frac{x}{y} + \ln|y| - \cancel{\ln|x|} \right) = c$$

$$\Rightarrow -\frac{x\ln(y/x) - y - x}{y} + \ln|y| = c$$

$$\Rightarrow \frac{-x\ln(y/x)}{y} - \frac{y}{y} - \frac{x}{y} + \ln|y| = c$$

$$\Rightarrow \frac{-x\ln(y/x)}{y} - 1 - \frac{x}{y} + \ln|y| = c$$

$$\Rightarrow \frac{-x\ln(y/x)}{y} - \frac{x}{y} + \ln|y| = c + 1 = c$$

$$\Rightarrow \frac{-x\ln(y/x) - x}{y} + \ln|y| = c$$

$$\Rightarrow \frac{-x(\ln(y/x) + 1)}{y} + \ln|y| = c$$

$$\boxed{\therefore \frac{-x(\ln(y/x)+1)}{y} + \ln|y| = c}$$

← solution since isolating y is rather difficult.

3. $(6x+1)y^2 \dfrac{dy}{dx} + 3x^2 + 2y^3 = 0$

$\Rightarrow (6x+1)y^2\,dy + (3x^2+2y^3)\,dx = 0$

$\Rightarrow (3x^2+2y^3)\,dx + (6x+1)y^2\,dy = 0$

$M = 3x^2+2y^3 \qquad N = 6xy^2 + y^2$

$\dfrac{\partial M}{\partial y} = 6y^2 \qquad \dfrac{\partial N}{\partial x} = 6y^2 \qquad \checkmark$ Exact

$\dfrac{\partial F}{\partial x} = 3x^2+2y^3$

$\Rightarrow F = \int 3x^2+2y^3\,dx = x^3 + 2y^3x + g(y)$

$\Rightarrow \dfrac{\partial F}{\partial y} = 6y^2x + g'(y) \quad , \quad \dfrac{\partial F}{\partial y} = N = 6xy^2 + y^2$

$\Rightarrow g'(y) = y^2$

$\Rightarrow g(y) = \int y^2 = \dfrac{y^3}{3}$

$F = x^3 + 2y^3x + \dfrac{y^3}{3} = c$

$\boxed{\therefore \quad x^3 + 2y^3x + \dfrac{y^3}{3} = c}$ $\leftarrow$ solution

4. $\frac{dx}{dy} = -\frac{4y^2 + 6xy}{3y^2 + 2x}$

$\Rightarrow \quad dx\,(3y^2 + 2x) = -(4y^2 + 6xy)\,dy$

$\Rightarrow \quad (3y^2 + 2x)\,dx + (4y^2 + 6xy)\,dy = 0$

$M = 3y^2 + 2x \qquad\qquad N = 4y^2 + 6xy$

$\frac{\partial M}{\partial y} = 6y \qquad\qquad \frac{\partial N}{\partial x} = 6y \qquad \checkmark$ Exact

$\frac{\partial F}{\partial x} = 3y^2 + 2x$

$\Rightarrow \quad F = \int 3y^2 + 2x\,dx = 3y^2x + x^2 + g(y)$

$\Rightarrow \quad \frac{\partial F}{\partial y} = 6yx + g'(y) \quad , \quad \frac{\partial F}{\partial y} = N = 4y^2 + 6xy$

$g'(y) = 4y^2$

$\Rightarrow g(y) = \int 4y^2\,dy = \frac{4y^3}{3}$

$\Rightarrow \quad F = 3y^2x + x^2 + \frac{4y^3}{3} = c$

$\boxed{\therefore \quad 3xy^2 + x^2 + \frac{4y^3}{3} = c}$ ← solution

5. $t \dfrac{dQ}{dt} + Q = t^4 \ln t$

$\Rightarrow t \dfrac{dQ}{dt} = t^4 \ln t - Q$

$\Rightarrow \dfrac{dQ}{dt} = t^3 \ln t - \dfrac{Q}{t}$

$\Rightarrow \dfrac{dQ}{dt} + \dfrac{Q}{t} = t^3 \ln t$

$\Rightarrow \dfrac{dQ}{dt} + \underbrace{\dfrac{1}{t}}_{P} \cdot Q = t^3 \ln t \quad , \quad \mu = e^{\int \frac{1}{t} dt} = e^{\ln|t|} = |t|$

$\Rightarrow |t| \dfrac{dQ}{dt} + |t| \cdot \dfrac{1}{t} \cdot Q = |t| t^3 \ln t \quad \leftarrow$ w/o loss of generality, we assume t is positive

$\Rightarrow t \dfrac{dQ}{dt} + Q = t^4 \ln t$

$\Rightarrow \dfrac{d}{dt}\left[tQ \right] = t^4 \ln t$

$\Rightarrow tQ = \int t^4 \cdot \ln t \, dt$

$\Rightarrow tQ = \dfrac{t^5 (5 \ln t - 1)}{25} + c$

$\Rightarrow Q = \dfrac{1}{t} \dfrac{t^5 (5 \ln t - 1)}{25} + \dfrac{c}{t}$

$\Rightarrow Q = \dfrac{t^4 (5 \ln t - 1)}{25} + \dfrac{c}{t} \qquad t \neq 0$

$$\boxed{\therefore \quad Q(t) = \frac{t^4 (5 \ln t - 1)}{25} + \frac{c}{t} \quad , \quad t \neq 0}$$