Eric Altenburg
7/17/20

Content Check 3
Diff Eq

1. $2y''' + 9y'' + 12y' + 5y = 0$

aux eq: $2m^3 + 9m^2 + 12m + 5 = 0$

$m_1 = \frac{-5}{2} \quad m_2 = -1 \quad m_3 = -1$

$y_1 = e^{(-5/2)x} = e^{-5x/2}$
$y_2 = e^{-x}$
$y_3 = xe^{-x}$

$$\boxed{\therefore\ y(x) = c_1 e^{-\frac{5x}{2}} + c_2 e^{-x} + c_3 x e^{-x}}$$

2. $3y''' + 10y'' + 15y' + 4y = 0$

aux eq: $3m^3 + 10m^2 + 15m + 4 = 0$

$m_1 = \frac{-3}{2} + \frac{\sqrt{7}}{2} \cdot i \qquad m_2 = \frac{-3}{2} - \frac{\sqrt{7}}{2} \cdot i \qquad m_3 = \frac{-1}{3}$

$y_1 = e^{-\frac{3}{2}x} \cos\left(\frac{x\sqrt{7}}{2}\right)$

$y_2 = e^{-\frac{3}{2}x} \sin\left(\frac{x\sqrt{7}}{2}\right)$

$y_3 = e^{-\frac{x}{3}}$

$$y(x) = c_1 e^{-\frac{3x}{2}} \cos\left(\frac{x\sqrt{7}}{2}\right) + c_2 e^{-\frac{3x}{2}} \sin\left(\frac{x\sqrt{7}}{2}\right) + c_3 e^{-\frac{x}{3}}$$
$$= e^{-\frac{3x}{2}} \left(c_1 \cos\left(\frac{x\sqrt{7}}{2}\right) + c_2 \sin\left(\frac{x\sqrt{7}}{2}\right)\right) + c_3 e^{-\frac{x}{3}}$$

$$\boxed{\therefore\ y(x) = e^{-\frac{3x}{2}} \left(c_1 \cos\left(\frac{x\sqrt{7}}{2}\right) + c_2 \sin\left(\frac{x\sqrt{7}}{2}\right)\right) + c_3 e^{-\frac{x}{3}}}$$

3. $2y^{(4)} + 3y''' + 2y'' + 6y' - 4y = 0$

aux eq: $2m^4 + 3m^3 + 2m^2 + 6m - 4 = 0$

$m_1 = -i\sqrt{2} \quad m_2 = i\sqrt{2} \quad m_3 = -2 \quad m_4 = \frac{1}{2}$

$y_1 = e^{0x} \cos(x\sqrt{2}) = \cos(x\sqrt{2})$
$y_2 = e^{0x} \sin(x\sqrt{2}) = \sin(x\sqrt{2})$
$y_3 = e^{-2x}$
$y_4 = e^{\frac{x}{2}}$

$$\boxed{\therefore\ y(x) = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) + c_3 e^{-2x} + c_4 e^{\frac{x}{2}}}$$

4. $y''' - 5y'' + 6y' = 8 + 2\sin x$

Find y_c: $y''' - 5y'' + 6y' = 0$

aux eq: $m^3 - 5m^2 + 6m = 0$

$\Rightarrow m(m^2 - 5m^2 + 6) = 0$

$m((m-3)(m-2)) = 0$

$m_1 = 3 \quad m_2 = 2 \quad m_3 = 0$

$y_1 = e^{3x}$

$y_2 = e^{2x}$

$y_3 = e^{0x} = 1$

$y_c = c_1 e^{3x} + c_2 e^{2x} + c_3$

Find y_p: $g(x) = 8 + 2\sin x$

$y_{p1} = Ax$, not just A because $c_3 \in y_c$

$y_{p2} = B\cos x + C\sin x$

Find coefficients:

$$\left.\begin{aligned} y_{p1} &= Ax \\ y_{p1}' &= A \\ y_{p1}'' &= 0 \\ y_{p1}''' &= 0 \end{aligned}\right\} \quad \begin{aligned} &0 + 5(0) + 6(A) = 8 \\ \Rightarrow\ &6A = 8 \\ \Rightarrow\ &A = \tfrac{8}{6} = \tfrac{4}{3} \end{aligned} \qquad \therefore y_{p1} = \tfrac{4}{3}x$$

$y_{p2} = B\cos x + C\sin x$

$y_{p2}' = -B\sin x + C\cos x$

$y_{p2}'' = -B\cos x - C\sin x$

$y_{p2}''' = B\sin x - C\cos x$

$B\sin x - C\cos x - 5(-B\cos x - C\sin x) + 6(-B\sin x + C\cos x) = 2\sin x$

$\Rightarrow B\sin x - C\cos x + 5B\cos x + 5C\sin x - 6B\sin x + 6C\cos x = 2\sin x$

$\Rightarrow (B + 5C - 6B)\sin x + (-C + 5B + 6C)\cos x = 2\sin x$

$\Rightarrow (-5B + 5C)\sin x + (5B + 5C)\cos x = 2\sin x + 0\cos x$

$$\left.\begin{aligned} -5B + 5C &= 2 \\ 5B + 5C &= 0 \end{aligned}\right\} \quad \begin{aligned} B &= -\tfrac{1}{5} \\ C &= \tfrac{1}{5} \end{aligned}$$

$\therefore y_{p2} = -\frac{1}{5}\cos x + \frac{1}{5}\sin x$

$\therefore y_p = \frac{4x}{3} - \frac{1}{5}\cos x + \frac{1}{5}\sin x$

Add y_c & y_p:

$$\boxed{\therefore y(x) = c_1 e^{3x} + c_2 e^{2x} + c_3 + \frac{4x}{3} - \frac{1}{5}\cos x + \frac{1}{5}\sin x}$$

5. $y'' - 2y' + 2y = e^x \tan x$

Find y_c: $y'' - 2y' + 2y = 0$

$m^2 - 2m + 2 = 0$

$m_1 = 1+i \qquad m_2 = 1-i$

$y_1 = e^{1x}\cos x = e^x \cos x$
$y_2 = e^{1x}\sin x = e^x \sin x$

$\therefore y_c = c_1 e^x \cos x + c_2 e^x \sin x$
$= e^x (c_1 \cos x + c_2 \sin x)$

Find y_p: $f(x) = e^x \tan x$ "variation of parameters"

$$w = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \cos x + e^x \sin x \end{vmatrix}$$
$= e^x \cos x (e^x \cos x + e^x \sin x) - e^x \sin x (e^x \cos x - e^x \sin x)$
$= e^{2x} \cos^2 x + e^{2x} \cos x \sin x - e^{2x} \cos x \sin x + e^{2x} \sin^2 x$
$= e^{2x} \cos^2 x + e^{2x} \sin^2 x$
$= e^{2x} (\cos^2 x + \sin^2 x)$
$= e^{2x} (1) = e^{2x}$

$$w_1 = \begin{vmatrix} 0 & e^x \sin x \\ e^x \tan x & e^x \cos x + e^x \sin x \end{vmatrix}$$
$= 0 - e^x \tan x (e^x \sin x)$
$= -e^x \tan x (e^x \sin x)$
$= -e^{2x} \sin x \tan x$

$$w_2 = \begin{vmatrix} e^x \cos x & 0 \\ e^x \cos x - e^x \sin x & e^x \tan x \end{vmatrix}$$
$= e^x \cos x (e^x \tan x) - 0$
$= e^x \cos x (e^x \tan x)$
$= e^{2x} \cos x \tan x = e^{2x} \cos x \cdot \frac{\sin x}{\cos x} = e^{2x} \sin x$

$u_1' = \frac{-e^{2x} \sin x \tan x}{e^{2x}} = -\sin x \tan x$

$u_1 = \int -\sin x \tan x \, dx = \sin x - \ln|\sec x + \tan x|$

$u_2' = \frac{e^{2x} \sin x}{e^{2x}} = \sin x$

$u_2 = \int \sin x \, dx = -\cos x$

$y_p = e^x \cos x (\sin x - \ln|\sec x + \tan x|) + e^x \sin x (-\cos x)$
$= e^x \cos x \sin x - e^x \cos x (\ln|\sec x + \tan x|) - e^x \cos x \sin x$
$= -e^x \cos x (\ln|\sec x + \tan x|)$

Add $y_c + y_p$: $y(x) = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x (\ln|\sec x + \tan x|)$
$= e^x \left[(c_1 \cos x + c_2 \sin x) - \cos x (\ln|\sec x + \tan x|) \right]$

$$\boxed{\therefore\ y(x) = e^{x}\left[(c_1\cos x + c_2\sin x) - \cos x\left(\ln|\sec x + \tan x|\right)\right]}$$