Eric Altenburg | Content Check 6 | ①
8/6/20 | Diff.

1. $y'' - 2y' + y = e^t \quad , \quad y(0) = 0 \, , \quad y'(0) = 5$

$$\mathcal{L}\{y'' - 2y' + y\} = \mathcal{L}\{e^t\}$$
$$\Rightarrow \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{e^t\}$$
$$\Rightarrow [s^2 Y(s) - s y(0) - y'(0)] - 2[s Y(s) - y(0)] + Y(s) = \frac{1}{s-1}$$
$$\Rightarrow s^2 Y(s) - s(0) - (5) - 2s Y(s) + 2(0) + Y(s) = \frac{1}{s-1}$$
$$\Rightarrow (s^2 - 2s + 1) Y(s) - 5 = \frac{1}{s-1}$$
$$\Rightarrow (s^2 - 2s + 1) Y(s) = \frac{1}{s-1} + 5 = \frac{1 + 5(s-1)}{s-1} = \frac{1 + 5s - 5}{s-1} = \frac{5s-4}{s-1}$$
$$\Rightarrow Y(s) = \frac{5s-4}{(s-1)(s^2-2s+1)} = \frac{5s-4}{(s-1)(s-1)^2} = \frac{5s-4}{(s-1)^3}$$

$$\frac{5s-4}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}$$
$$5s - 4 = A(s-1)^2 + B(s-1) + C$$
$A = 0$
$B = 5$
$C = 1$

$$\Rightarrow Y(s) = \frac{5}{(s-1)^2} + \frac{1}{(s-1)^3}$$
$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{5}{(s-1)^2} + \frac{1}{(s-1)^3}\right\} = 5\,\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\} = 5te^t + \frac{1}{2} \cdot t^2 \cdot e^t$$

$$\boxed{\therefore y(t) = 5te^t + \frac{1}{2} t^2 e^t}$$

2. $y'' - 8y' + 20y = te^t \;,\; y(0) = 0 \;,\; y'(0) = 0$

$$\mathcal{L}\{y'' - 8y' + 20y\} = \mathcal{L}\{te^t\}$$

$$\Rightarrow \mathcal{L}\{y''\} - 8\,\mathcal{L}\{y'\} + 20\,\mathcal{L}\{y\} = \frac{1}{(s-1)^2}$$

$$\Rightarrow s^2Y(s) - \cancel{sy(0)} - \cancel{y'(0)} - 8\left[sY(s) - \cancel{y(0)}\right] + 20\,Y(s) = \frac{1}{(s-1)^2}$$

$$\Rightarrow s^2Y(s) - 8sY(s) + 20\,Y(s) = \frac{1}{(s-1)^2}$$

$$\Rightarrow (s^2 - 8s + 20)\,Y(s) = \frac{1}{(s-1)^2}$$

$$\Rightarrow Y(s) = \frac{1}{(s-1)^2} \cdot \frac{1}{s^2 - 8s + 20} = \frac{1}{(s-1)^2} \cdot \frac{1}{(s^2 - 8s + 16) + 4} = \frac{1}{(s-1)^2} \cdot \frac{1}{(s-4)^2 + 4}$$

$$= \frac{1}{(s-1)^2((s-4)^2 + 4)}$$

$$\frac{1}{(s-1)^2((s-4)^2+4)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+E}{(s-4)^2+4}$$

$A = 6/169$

$B = 1/13$

$C = -6/169$

$E = 29/169$

$$\Rightarrow Y(s) = \frac{6}{169} \cdot \frac{1}{s-1} + \frac{1}{13} \cdot \frac{1}{(s-1)^2} + \left(\frac{-6}{169}\right) \cdot \frac{s}{(s-4)^2+4} + \frac{29}{169} \cdot \frac{1}{(s-4)^2+4}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{6}{169} \cdot \frac{1}{s-1} + \frac{1}{13} \cdot \frac{1}{(s-1)^2} + \left(\frac{-6}{169}\right) \cdot \frac{s}{(s-4)^2+4} + \frac{29}{169} \cdot \frac{1}{(s-4)^2+4}\right\}$$

$$\Rightarrow y(t) = \frac{6}{169} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{13} \cdot \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \frac{6}{169} \cdot \mathcal{L}^{-1}\left\{\frac{s}{(s-4)^2+2^2}\right\} + \frac{29}{169} \cdot \mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2+2^2}\right\}$$

$$\Rightarrow y(t) = \frac{6}{169} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{13} \cdot \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \frac{6}{169} \cdot \mathcal{L}^{-1}\left\{\frac{s}{(s-4)^2+2^2}\right\} + \frac{5}{338} \cdot \mathcal{L}^{-1}\left\{\frac{2}{(s-4)^2+2^2}\right\}$$

$$= \frac{6}{169} \cdot e^t + \frac{1}{13}\,te^t - \frac{6}{169}\,e^{4t}\cos 2t + \frac{5}{338}\,e^{4t}\sin 2t$$

$$\boxed{\therefore\; y(t) = \frac{6}{169}\,e^t + \frac{1}{13}\,te^t - \frac{6}{169}\,e^{4t}\cos(2t) + \frac{5}{338}\,e^{4t}\sin(2t)}$$

3. $y' - 5y = \begin{cases} t^2 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$, $y(0) = 1$

$$\mathcal{L}\{y' - 5y\} = \mathcal{L}\{t^2 - t^2 u(t-1)\}$$

$$\Rightarrow \mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{t^2\} - \mathcal{L}\{t^2 u(t-1)\}$$

$$\Rightarrow sY(s) - y(0) - 5Y(s) = \mathcal{L}\{t^2\} - \mathcal{L}\{(t-1)^2 u(t-1)\} + \mathcal{L}\{(2t-2+1)u(t-1)\}$$

$$\Rightarrow sY(s) - y(0) - 5Y(s) = \mathcal{L}\{t^2\} - \mathcal{L}\{(t-1)^2 u(t-1)\} + 2\mathcal{L}\{(t-1)u(t-1)\} - \mathcal{L}\{u(t-1)\}$$

$$\Rightarrow (s-5)Y(s) - 1 = \frac{2}{s^3} - \frac{2}{s^3}e^{-s} - \frac{2}{s^2}e^{-s} - \frac{e^{-s}}{s}$$

$$\Rightarrow \cancel{(s-5)}Y(s) = \frac{2 - 2e^{-s} - 2se^{-s} - s^2e^{-s} + s^3}{s^3} \cdot \frac{1}{(s-5)}$$

$$\Rightarrow Y(s) = \underbrace{\frac{2+s^3}{s^3(s-5)}} - e^{-s}\underbrace{\left[\frac{s^2+2s+2}{s^3(s-5)}\right]}$$

$$\frac{2+s^3}{s^3(s-5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{E}{s-5}$$

$A = -\frac{2}{125}$

$B = -\frac{2}{25}$

$C = -\frac{2}{5}$

$E = \frac{127}{125}$

$$\frac{s^2+2s+2}{s^3(s-5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{E}{(s-5)}$$

$A = -\frac{37}{125}$

$B = -\frac{12}{25}$

$C = -\frac{2}{5}$

$E = \frac{37}{125}$

$$\Rightarrow Y(s) = \frac{-2}{125}\cdot\frac{1}{s} - \frac{2}{25}\cdot\frac{1}{s^2} - \frac{2}{5}\cdot\frac{1}{s^3} + \frac{127}{125}\cdot\frac{1}{(s-5)} - e^{-s}\left[\frac{-37}{125}\cdot\frac{1}{s} - \frac{12}{25}\cdot\frac{1}{s^2} - \frac{2}{5}\cdot\frac{1}{s^3} + \frac{37}{125}\cdot\frac{1}{(s-5)}\right]$$

$$\Rightarrow y(t) = -\frac{2}{125}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{2}{25}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} + \frac{127}{125}\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} + \frac{37}{125}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} + \frac{12}{25}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\} + \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^3}\right\} - \frac{37}{125}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s-5}\right\}$$

$$\Rightarrow y(t) = -\frac{2}{125} - \frac{2}{25}t - \frac{1}{5}t^2 + \frac{127}{125}e^{5t} + \frac{37}{125}u(t-1) + \frac{12}{25}(t-1)u(t-1) + \frac{1}{5}(t-1)^2u(t-1) - \frac{37}{125}e^{5(t-1)}u(t-1)$$

$$\Rightarrow y(t) = -\frac{2}{125} - \frac{2}{25}t - \frac{1}{5}t^2 + \frac{127}{125}e^{5t} + \left[\frac{37}{125} + \frac{12}{25}(t-1) + \frac{1}{5}(t-1)^2 - \frac{37}{125}e^{5(t-1)}\right]u(t-1)$$

$$\boxed{\therefore\ y(t) = \begin{cases} -\frac{2}{125} - \frac{2}{25}t - \frac{1}{5}t^2 + \frac{127}{125}e^{5t} & 0 \le t < 1 \\ -\frac{2}{125} - \frac{2}{25}t - \frac{1}{5}t^2 + \frac{127}{125}e^{5t} + \left[\frac{37}{125} + \frac{12}{25}(t-1) + \frac{1}{5}(t-1)^2 - \frac{37}{125}e^{5(t-1)}\right] & t \ge 1 \end{cases}}$$

4. $y'(t) = \cos t + \int_0^t y(\tau)\cos(t-\tau)\,d\tau \quad , \quad y(0) = 1$

$\mathcal{L}\{y'(t)\} = \mathcal{L}\{\cos t + \int_0^t y(\tau)\cos(t-\tau)\,d\tau\}$

$\Rightarrow sY(s) - y(0) = \mathcal{L}\{\cos t\} + \mathcal{L}\{\int_0^t y(\tau)\cos(t-\tau)\,d\tau\}$

$\Rightarrow sY(s) - 1 = \dfrac{s}{s^2+1} + \mathcal{L}\{\int_0^t y(\tau)\cos(t-\tau)\,d\tau\}$

$\Rightarrow sY(s) - 1 = \dfrac{s}{s^2+1} + \mathcal{L}\{y(t) * \cos t\}$

$\Rightarrow sY(s) - 1 = \dfrac{s}{s^2+1} + \mathcal{L}\{y(t)\} \cdot \mathcal{L}\{\cos t\}$

$\Rightarrow sY(s) - 1 = \dfrac{s}{s^2+1} + Y(s)\dfrac{s}{s^2+1}$

$\Rightarrow (s^2+1)(sY(s)-1) = s + sY(s)$

$\Rightarrow s^3Y(s) - s^2 + sY(s) - 1 = s + sY(s)$

$\Rightarrow s^3Y(s) - s^2 - 1 = s$

$\Rightarrow s^3Y(s) = s^2 + s + 1$

$\Rightarrow Y(s) = \dfrac{1}{s} + \dfrac{1}{s^2} + \dfrac{1}{s^3}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\{\frac{1}{s}\} + \mathcal{L}^{-1}\{\frac{1}{s^2}\} + \mathcal{L}^{-1}\{\frac{1}{s^3}\}$

$\Rightarrow y(t) = 1 + t + \frac{1}{2}t^2$

$$\boxed{\therefore\ y(t) = \frac{1}{2}t^2 + t + 1}$$