Eric Altenburg Content Check 4 ①

7/26/20 Diff.

1. $2x^3y''' + 19x^2y'' + 39xy' + 9y = 0$

Hint: You will need to first determine the auxilliary equation for a 3rd-order eq.

aux eq: $2m^3 + (19 - 3(2))m^2 + (2(2) - 19 + 39)m + 9 = 0$

$\Rightarrow 2m^3 + 13m^2 + 24m + 9 = 0$

$\Rightarrow (m+3)^2 \cdot (2m+1) = 0$

$m_1 = -3 \quad m_2 = -3 \quad m_3 = -\frac{1}{2}$

$$\boxed{\therefore y(x) = c_1 x^{-3} + c_2 x^{-3}\ln(x) + c_3 x^{-1/2} = \frac{c_1}{x^3} + \frac{c_2 \ln(x)}{x^3} + \frac{c_3}{\sqrt{x}}}$$

2. $x^2y'' - 4xy' + 6y = 2x^4 + x^2$

y_c: $x^2y'' - 4xy' + 6y = 0$

aux eq: $m^2 + (-4-1)m + 6 = 0$

$\Rightarrow m^2 - 5m + 6 = 0$

$m_1 = 2 \quad m_2 = 3$

$y_1 = x^2 \quad y_2 = x^3$

$y_c = c_1x^2 + c_2x^3$

y_p: $\frac{x^2y'' - 4xy' + 6y}{x^2} = \frac{2x^4 + x^2}{x^2}$

$\Rightarrow y'' - \frac{4}{x}y' + \frac{6}{x^2}y = \underbrace{2x^2 + 1}_{g(x)}$

$g(x) = 2x^2 + 1$

$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4$

$W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2+1 & 3x^2 \end{vmatrix} = -x^3(2x^2+1) = -2x^5 - x^3$

$W_2 = \begin{vmatrix} x^2 & 0 \\ 2x & 2x^2+1 \end{vmatrix} = 2x^4 + x^2$

$u_1' = \frac{-2x^5 - x^3}{x^4} = \frac{x^3(-2x^2-1)}{x^4} = \frac{-2x^2-1}{x}$, $u_1 = \int \frac{-2x^2-1}{x}\,dx = -\ln|x| - x^2$

$u_2' = \frac{2x^4 + x^2}{x^4} = \frac{x^2(2x^2+1)}{x^4} = \frac{2x^2+1}{x^2}$, $u_2 = \int \frac{2x^2+1}{x^2}\,dx = 2x - \frac{1}{x}$

$$
\begin{aligned}
y_p &= y_1u_1 + y_2u_2 \\
&= x^2(-\ln|x| - x^2) + x^3(2x - \tfrac{1}{x}) \\
&= -x^2\ln|x| - x^4 + 2x^4 - x^2 \\
&= -x^4 + 2x^4 - x^2 - x^2\ln|x| \\
&= x^4 - x^2 - x^2\ln|x| \\
&= x^4 - x^2(1 + \ln|x|)
\end{aligned}
$$

$$\boxed{\therefore\ y(x) = c_1x^2 + c_2x^3 + x^4 - x^2(\ln|x| + 1)}$$

3. $2xy'' + y' + y = 0$

$x_0 = 0$, $2x = 0$ so singular $\quad P(x) = \frac{1}{2x}$, at most 1^{st} power
$Q(x) = \frac{1}{2x}$, at most 2^{nd} power $\therefore$ regular

$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad y' = \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}$

$$2x\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2} + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} + \sum_{n=0}^{\infty} c_nx^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2c_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} + \sum_{n=0}^{\infty} c_nx^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[2(n+r)(n+r-1) + (n+r)\right]c_nx^{n+r-1} + \sum_{n=0}^{\infty} c_nx^{n+r} = 0$$

$$\Rightarrow x^r\left[\sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_nx^{n-1} + \sum_{n=0}^{\infty} c_nx^n\right] = 0$$

$$\Rightarrow x^r\left[r(2r-1)c_0x^{-1} + \sum_{n=1}^{\infty}(n+r)(2n+2r-1)c_nx^{n-1} + \sum_{n=0}^{\infty} c_nx^n\right] = 0$$

$$\Rightarrow x^r\left[r(2r-1)c_0x^{-1} + \sum_{n=0}^{\infty}(n+r+1)(2n+2r+1)c_{n+1}x^n + \sum_{n=0}^{\infty} c_nx^n\right] = 0$$

$$\Rightarrow x^r\left[\underline{r(2r-1)}\,c_0x^{-1} + \sum_{n=0}^{\infty}\left[\underline{(n+r+1)(2n+2r+1)c_{n+1} + c_n}\right]x^n\right] = 0$$

$r(2r-1) = 0$
$r_1 = 0 \quad r_2 = \frac{1}{2}$

$(n+r+1)(2n+2r+1)c_{n+1} + c_n = 0$

$$c_{n+1} = \frac{-1}{(n+r+1)(2n+2r+1)} \cdot c_n$$

$r_1 = 0$: $\quad c_{n+1} = \dfrac{-1}{(n+1)(2n+1)}c_n$

$n=0$: $\quad c_1 = \dfrac{-1}{1}c_0 = -c_0$

$n=1$: $\quad c_2 = \dfrac{-1}{(2)(3)}c_1 = \dfrac{1}{3\cdot 2}c_0$

$n=2$: $\quad c_3 = \dfrac{-1}{3\cdot 5}c_2 = \dfrac{-1}{5\cdot 3^2\cdot 2}c_0$

$n=3$: $\quad c_4 = \dfrac{-1}{4\cdot 7}c_3 = \dfrac{1}{7\cdot 5\cdot 3\cdot 4\cdot 3\cdot 2}c_0$

$n=4$: $\quad c_5 = \dfrac{-1}{5\cdot 9}c_4 = \dfrac{-1}{9\cdot 7\cdot 5\cdot 3\cdot 5\cdot 4\cdot 3\cdot 2}c_0$

$$
\begin{aligned}
y_1(x) &= x^0\left[c_0 - c_0x + \frac{1}{3\cdot 2}c_0x^2 - \frac{1}{5\cdot 3\cdot 3\cdot 2}c_0x^3 + \dots\right] \\
&= c_0\left[1 - x + \frac{1}{3\cdot 2}x^2 - \frac{1}{5\cdot 3\cdot 3\cdot 2}x^3 + \dots\right] = c_0\left[1 + \sum_{n=1}^{\infty}(-1)^n\frac{x^n}{(2n-1)!!\,n!}\right]
\end{aligned}
$$

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n-1)!!\, n!}$$

$$\left| \frac{(-1)^{n+1} x^{n+1}}{(2n+3)!!(n+1)!(-1)^n} \cdot \frac{(2n-1)!!(n!)}{x^n} \right| = \left| \frac{-x}{(2n+3)(2n+1)(n+1)} \right| = \frac{|x|}{(2n+3)(2n+1)(n+1)} \xrightarrow{n\to\infty} 0 < 1 \ \forall x$$

$|x| < \infty \mapsto x > 0$

$r_2 = \frac{1}{2}$: $c_{n+1} = \frac{-1}{(n+\frac{3}{2})(2n+2)} c_n$

$n=0$: $c_1 = \frac{-1}{\frac{3}{2}(2)} c_0 = \frac{-1}{3} c_0$

$n=1$: $c_2 = \frac{-1}{\frac{5}{2}(4)} c_1 = \frac{1}{5\cdot 2\cdot 3} c_0 = \frac{1}{5\cdot 3\cdot 2} c_0$

$n=2$: $c_3 = \frac{-1}{\frac{7}{2}(6)} c_2 = \frac{-1}{7\cdot 3\cdot 5\cdot 3\cdot 2} c_0 = \frac{-1}{7\cdot 5\cdot 3\cdot 3\cdot 2} c_0$

$n=3$: $c_4 = \frac{-1}{\frac{9}{2}(8)} c_3 = \frac{1}{9\cdot 7\cdot 5\cdot 3\cdot 4\cdot 3\cdot 2} c_0$

$$y_2(x) = x^{1/2}\left[c_0 - \tfrac{1}{3} c_0 x + \tfrac{1}{5\cdot 3\cdot 2} c_0 x^2 - \tfrac{1}{7!!\,3!} c_0 x^3 + \dots \right]$$
$$= x^{1/2}\cdot c_0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n+1)!!\, n!} \right]$$
$$= x^{1/2} + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n+1)!!\, n!}$$

$$\left| \frac{-1^{n+1} x^{n+1}}{(2n+3)!!\,(n+1)!} \cdot \frac{(2n+1)!!\, n!}{(-1)^n x^n} \right| = \left| \frac{-x}{(2n+3)(2n+2)(n+1)} \right| = \frac{|x|}{(2n+3)(2n+2)(n+1)} \xrightarrow{n\to\infty} 0 < 1 \ \forall x$$

$|x| < \infty \mapsto x > 0$

$$\boxed{\therefore y(x) = c_1\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n-1)!!\, n!}\right) + c_2\left(x^{1/2} + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n+1)!!\, n!}\right), \quad x > 0}$$

4. $(x-1)y'' + 3y = 0$

Let $x_0 = 0$. $R = 1$, $|x| < 1$

$y = \sum_{n=0}^{\infty} c_n x^n \qquad y' = \sum_{n=1}^{\infty} c_n \cdot n \cdot x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} c_n \cdot n \cdot (n-1)\, x^{n-2}$

$$(x-1)\sum_{n=2}^{\infty} c_n (n)(n-1) x^{n-2} + 3\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow x\sum_{n=2}^{\infty} c_n (n)(n-1) x^{n-2} - \sum_{n=2}^{\infty} c_n (n)(n-1) x^{n-2} + \sum_{n=0}^{\infty} 3 c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} c_n (n)(n-1) x^{n-1} - \sum_{n=2}^{\infty} c_n (n)(n-1) x^{n-2} + \sum_{n=0}^{\infty} 3 c_n x^n = 0$$

want to get start n the same, w/ x^n.

$x^{n-1} \to x^n$ $\qquad$ $x^{n-2} \to x^n$

$n-1 = n$ $\qquad$ $n-2 = n$

$n = n+1$, shift back 1 $\qquad$ $n = n+2$, shift back 2

$$\Rightarrow \sum_{n=1}^{\infty} c_{n+1}(n+1)(n)x^n - \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} 3c_n x^n = 0$$ get to same n now (4)

$$\Rightarrow \sum_{n=1}^{\infty} c_{n+1}(n)(n+1)x^n - \left[2c_2 + \sum_{n=1}^{\infty} c_{n+2}(n+2)(n+1)x^n\right] + 3c_0 + \sum_{n=1}^{\infty} 3c_n x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} c_{n+1}(n)(n+1)x^n - 2c_2 - \sum_{n=1}^{\infty} c_{n+2}(n+2)(n+1)x^n + 3c_0 + \sum_{n=1}^{\infty} 3c_n x^n = 0$$

$$\Rightarrow -2c_2 + 3c_0 + \sum_{n=1}^{\infty} c_{n+1}(n)(n+1)x^n - \sum_{n=1}^{\infty} c_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} 3c_n x^n = 0$$

$$\Rightarrow \underbrace{-2c_2 + 3c_0} + \underbrace{\sum_{n=1}^{\infty}\left[c_{n+1}(n)(n+1) - c_{n+2}(n+2)(n+1) + 3c_n\right]x^n} = 0$$

$-2c_2 + 3c_0 = 0$

$c_2 = \frac{3}{2}c_0$

$c_{n+1}(n)(n+1) - c_{n+2}(n+2)(n+1) + 3c_n = 0$

$$c_{n+2} = \frac{c_{n+1}(n)(n+1)}{(n+2)(n+1)} + \frac{3c_n}{(n+2)(n+1)} = \frac{n}{n+2}c_{n+1} + \frac{3}{(n+2)(n+1)}c_n$$

$n=0$: $c_2 = 0 \cdot c_1 + \frac{3}{2}c_0$ ✓

$n=1$: $c_3 = \frac{1}{3}c_2 + \frac{3}{3\cdot 2}c_1 = \frac{3}{3\cdot 2}c_0 + \frac{3}{3\cdot 2}c_1 = \frac{1}{2}c_0 + \frac{1}{2}c_1$

$n=2$: $c_4 = \frac{2}{4}c_3 + \frac{3}{4\cdot 3}c_2 = \frac{1}{2}\left(\frac{1}{2}c_0 + \frac{1}{2}c_1\right) + \frac{1}{4}\left(\frac{3}{2}c_0\right) = \frac{1}{4}c_0 + \frac{1}{4}c_1 + \frac{3}{8}c_0 = \frac{5}{8}c_0 + \frac{2}{8}c_1$

$n=3$: $c_5 = \frac{3}{5}c_4 + \frac{3}{5\cdot 4}c_3 = \frac{3}{5}\left(\frac{5}{8}c_0 + \frac{1}{4}c_1\right) + \frac{3}{5\cdot 4}\left(\frac{1}{2}c_0 + \frac{1}{2}c_1\right)$

$= \frac{3}{8}c_0 + \frac{3}{5\cdot 4}c_1 + \frac{3}{5\cdot 4\cdot 2}c_0 + \frac{3}{5\cdot 4\cdot 2}c_1 = \frac{18c_0}{40} + \frac{9c_1}{40}$

$n=4$: $c_6 = \frac{4}{6}c_5 + \frac{3}{6\cdot 5}c_4 = \frac{4}{6}\left(\frac{18}{40}c_0 + \frac{9}{40}c_1\right) + \frac{3}{6\cdot 5}\left(\frac{5}{8}c_0 + \frac{2}{8}c_1\right)$

$= \frac{18}{6\cdot 5\cdot 2}c_0 + \frac{9}{6\cdot 5\cdot 2}c_1 + \frac{15}{6\cdot 5\cdot 4\cdot 2}c_0 + \frac{6}{6\cdot 5\cdot 4\cdot 2}c_1 = \frac{87c_0}{240} + \frac{42c_1}{240}$

$y(x) = c_0 + c_1 x + \frac{3}{2}c_0 x^2 + \left(\frac{1}{2}c_0 + \frac{1}{2}c_1\right)x^3 + \left(\frac{5}{8}c_0 + \frac{2}{8}c_1\right)x^4 + \dots$ stop w/ n = 2 since messy

$y_1 = \left(1 + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{8}x^4 + \dots\right)$

$y_2 = \left(x + \frac{1}{2}x^3 + \frac{2}{8}x^4 + \dots\right)$

$$\boxed{\therefore y(x) = c_0\left(1 + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{8}x^4 + \dots\right) + c_1\left(x + \frac{1}{2}x^3 + \frac{2}{8}x^4 + \dots\right),\ |x| < 1}$$