Eric Altenburg Midterm

8/7/20 Diff.

(1)

1. $F(s) = \frac{5s+2}{s^2(s^2-1)}$ , $\frac{5s+2}{s^2(s^2-1)} = \frac{5s+2}{s^2(s+1)(s-1)}$

$\mathcal{L}^{-1}\left\{\frac{5s+2}{s^2(s^2-1)}\right\}$

$\frac{5s+2}{s^2(s+1)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{E}{s-1}$

$A = -5$

$B = -2$

$C = \frac{3}{2}$

$E = \frac{7}{2}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{-5}{s} - \frac{2}{s^2} + \frac{3}{2}\cdot\frac{1}{s+1} + \frac{7}{2}\cdot\frac{1}{s-1}\right\}$

$\Rightarrow -5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$

$\Rightarrow -5 - 2(t) + \frac{3}{2}e^{-t} + \frac{7}{2}e^{t}$

$\boxed{F(t) = -5 - 2t + \frac{3}{2}e^{-t} + \frac{7}{2}e^{t}}$

2. $y = e^{mx}$ $y'' - 5y' + 6y = 0$

$y' = me^{mx}$ $y'' = m^2e^{mx}$

$(m^2e^{mx}) - 5(me^{mx}) + 6(e^{mx}) = 0$

$m^2e^{mx} - 5me^{mx} + 6e^{mx} = 0$

$e^{mx}(m^2 - 5m + 6) = 0$

$e^{mx} \neq 0$ , $m_1 = 2$, $m_2 = 3$

$\boxed{\text{values of } m: \; m = 2;\, m = 3}$

3. $y = x^m \qquad x^2y'' - 7xy' + 12y = 0$

$y' = mx^{m-1} \qquad y'' = m^2x^{m-2} - mx^{m-2}$

$x^2(m^2x^{m-2} - mx^{m-2}) - 7x(mx^{m-1}) + 12(x^m) = 0$

$x^2m^2x^{m-2} - mx^2x^{m-2} - 7xmx^{m-1} + 12x^m = 0$

$m^2x^m - mx^m - 7mx^m + 12x^m = 0$

$m^2x^m - 8mx^m + 12x^m = 0$

$x^m(m^2 - 8m + 12) = 0$

$x^m[(m-6)(m-2)] = 0$

$x^m \neq 0, \; m_1 = 6 \quad m_2 = 2$

$\boxed{\text{values of } m: \; m = 2, \; m = 6}$

4. $y' = \frac{xe^y}{y}$

$y' = x \cdot \frac{e^y}{y}$ $\boxed{\text{seperable}}$

5. $y'' + \frac{y'}{x} + \frac{y(x-2)}{x-3} = 0$

$P(x) = \frac{1}{x} \qquad Q(x) = \frac{x-2}{x-3}$

$x = 0 \qquad x = 3$ $\boxed{\text{singular pts: } x = 0; \; x = 3}$

6. $\mathcal{L}\{\sin t \, u(t-\pi)\}$ $\qquad \mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t-a)\}$

$= e^{-\pi s}\mathcal{L}\{\sin(t-\pi)\}$ $\qquad g(t) = \sin t \quad a = \pi$

$= e^{-\pi s} \cdot \frac{s \sin(-\pi) + \cos(-\pi)}{s^2+1}$ $\boxed{\text{second translation theorem}}$

$= e^{-\pi s}\frac{(-1)}{s^2+1} = \boxed{\frac{-e^{-\pi s}}{s^2+1}}$

7. $\frac{dP}{dt} = kP(1 - \frac{P}{K}) \qquad K = \text{carrying cap}$

$\frac{dP}{dt} = P(12 - 3P) \cdot 12 \cdot \frac{1}{12}$

$= 12P(1 - \frac{1}{4}P)$ $\boxed{\text{carrying capacity} = 4}$

(3)

8. $\frac{dT}{dt} = K(T - T_m)$

$\frac{dT}{dt} = K(T-20)$ , $T(0) = 0$

$\frac{dT}{T-20} = K\,dt$

$\ln|T-20| = Kt$

$T - 20 = ce^{Kt}$

$T(t) = 20 + ce^{Kt}$

$T(0) = 0 = 20 + ce^{0}$

$0 = 20 + c$

$c = -20$

$T(t) = 20 - 20e^{Kt}$

$T(10) = 2 = 20 - 20e^{K(10)}$

$2 = 20 - 20e^{10K}$

$18 = 20e^{10K}$

$\frac{18}{20} = e^{10K}$

$\ln(\frac{9}{10}) = 10K$

$K = \frac{\ln(\frac{9}{10})}{10}$

$T(t) = 20 - 20e^{\frac{\ln(9/10)}{10}t}$

$T(t) = 15 = 20 - 20e^{\frac{\ln(9/10)t}{10}}$

$5 = 20e^{\frac{\ln(9/10)t}{10}}$

$\frac{1}{4} = e^{\frac{\ln(9/10)t}{10}}$

$\ln(\frac{1}{4}) = \frac{\ln(9/10)}{10}t$

$\frac{\ln(\frac{1}{4})10}{\ln(9/10)} = t = 131.57 \approx \boxed{132 \text{ minutes}}$

9. $W = \begin{vmatrix} e^{-2t} & 3e^{6t} \\ e^{-2t} & 5e^{6t} \end{vmatrix} = \boxed{8e^{4t}}$

10. $(x^2-4)^2 y'' - 2xy' + y = 0$

$a_2(x) = (x^2-4)^2$

$a_2(2) = (4-4)^2 = 0$ so singular

$y'' - \frac{2xy'}{(x^2-4)^2} + \frac{y}{(x^2-4)^2} = 0$

$y'' - \frac{2xy'}{(x+2)^2(x-2)^2} + \frac{y}{(x+2)^2(x-2)^2} = 0$

$P(x) = \frac{-2x}{(x+2)^2\underbrace{(x-2)^2}}$ $\quad Q(x) = \frac{1}{(x+2)^2(x-2)^2}$ $\quad x=2$

$2 \neq 1$ $\boxed{\text{at most power 1 so irregular singular point}}$

11. $\int_0^t e^{\tau} \sin(t-\tau)\, d\tau$

$\boxed{\text{convolution theorem}}$

$\mathcal{L}\{\int_0^t e^{\tau} \sin(t-\tau)\,d\tau\}$

$\mathcal{L}\{\frac{1}{2}(e^t - \sin t - \cos t)\}$

$\frac{1}{2}\mathcal{L}\{e^t\} - \frac{1}{2}\mathcal{L}\{\sin t\} - \frac{1}{2}\mathcal{L}\{\cos t\}$

$\frac{1}{2} \cdot \frac{1}{s-1} - \frac{1}{2} \cdot \frac{1}{s^2+1} - \frac{1}{2} \cdot \frac{s}{s^2+1}$

$\frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s^2+1} - \frac{s}{s^2+1}\right)$

$\frac{1}{2}\left(\frac{s^2+1}{s^2+1(s-1)} - \frac{(s-1)}{s^2+1(s-1)} - \frac{s(s-1)}{s^2+1(s-1)}\right)$

$\frac{1}{2}\left(\frac{s^2+1-s+1-s^2+s}{s^2+1(s-1)}\right)$

$\frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{s^2+1(s-1)}\right) = \boxed{\frac{1}{s^2+1(s-1)}}$

12. Tm is the temperature of the medium the object is in.

So maybe this is the same as the temperature of the environment?

13. $x' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} t \\ 1 \end{bmatrix}$ (5)

$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -3 \\ -2 & \lambda - 1 \end{vmatrix} = \lambda^2 - 3\lambda - 4 = 0$

$\lambda_1 = -1 \quad \lambda_2 = 4$

$\lambda_1 = -1: \begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad x_1 = -x_2$

$x_2 = \mathbb{R}$

$\vec{\Lambda}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \vec{\Phi}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$

$\lambda_2 = 4: \begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix} \quad x_1 = \frac{3}{2} x^2$

$x_2 = \mathbb{R}$

$\vec{\Lambda}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \vec{\Phi}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t}$

$\Phi_c = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t}$

$\Phi = \begin{bmatrix} -e^{-t} & 3e^{4t} \\ e^{-t} & 2e^{4t} \end{bmatrix} \quad \Phi^{-1} = \begin{bmatrix} -2/5\, e^{t} & 3/5\, e^{t} \\ 1/5\, e^{-4t} & 1/5\, e^{-4t} \end{bmatrix}$

$\Phi_p = \begin{bmatrix} -e^{-t} & 3e^{4t} \\ e^{-t} & 2e^{4t} \end{bmatrix} \cdot \int \begin{bmatrix} -2/5\, e^{t} & 3/5\, e^{t} \\ 1/5\, e^{-4t} & 1/5\, e^{-4t} \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} dt$

$= \begin{bmatrix} -e^{-t} & 3e^{4t} \\ e^{-t} & 2e^{4t} \end{bmatrix} \cdot \int \begin{bmatrix} \frac{3e^{t}}{5} - \frac{2te^{t}}{5} \\ \frac{te^{-4t}}{5} + \frac{e^{-4t}}{6} \end{bmatrix} dt$

$= \begin{bmatrix} \frac{t}{4} - \frac{19}{16} \\ \frac{7}{8} - \frac{t}{2} \end{bmatrix} = \begin{bmatrix} \frac{t}{4} - \frac{19}{16} \\ -\frac{t}{2} + \frac{7}{8} \end{bmatrix}$

$$\boxed{\therefore \vec{\Phi} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} + \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \end{bmatrix} t + \begin{bmatrix} -\frac{19}{16} \\ \frac{7}{8} \end{bmatrix}}$$

14. $\mathcal{L}\{y' + 6y + 9\int_0^t y(\tau)\,d\tau\} = \mathcal{L}\{1\}$ $\quad y(0)=1$ 6

$\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} + 9\mathcal{L}\{y(t) * 1\} = \frac{1}{s}$

$sY(s) - y(0) + 6Y(s) + 9Y(s)\cdot\frac{1}{s} = \frac{1}{s}$

$sY(s) - 1 + 6Y(s) + 9Y(s)\frac{1}{s} = 1/s$

$(s + 6 + \frac{9}{s})Y(s) - 1 = \frac{1}{s}$

$(s + 6 + \frac{9}{s})Y(s) = \frac{1}{s} + 1 = \frac{1}{s} + \frac{s}{s} = \frac{s+1}{s}$

$(s^2 + 6s + 9)Y(s) = s+1$

$$Y(s) = \frac{s+1}{s^2+6s+9} = \frac{s+1}{(s+3)^2}$$

$$\frac{s+1}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2} \qquad A=1, \quad B=-2$$

$$Y(s) = \frac{1}{s+3} - \frac{2}{(s+3)^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$

$$= e^{-3t} - 2te^{-3t}$$

$$\boxed{\therefore y(t) = e^{-3t} - 2te^{-3t}}$$

15. $x^2y'' - 2xy' + 2y = 0$

aux: $m^2 + (-2-1)m + 2 = 0$

$m^2 - 3m + 2 = 0$

$m_1 = 1 \quad m_2 = 2$

$$\boxed{\therefore y(x) = c_1x + c_2x^2}$$

16. $y'' + 2y' + y = e^{3t}$ $\quad y(0) = 1, \quad y'(0) = 2$

$\mathcal{L}\{y'' + 2y' + y\} = \mathcal{L}\{e^{3t}\}$

$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \frac{1}{s-3}$

$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = \frac{1}{s-3}$

$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) = \frac{1}{s-3}$

$s^2Y(s) - s - 2 + 2sY(s) - 2 + Y(s) = \frac{1}{s-3}$

$(s^2 + 2s + 1)Y(s) - s - 4 = \frac{1}{s-3}$

$(s^2 + 2s + 1)Y(s) = \frac{1}{s-3} + s + 4 = \frac{1 + s(s-3) + 4(s-3)}{s-3} = \frac{1 + s^2 - 3s + 4s - 12}{s-3}$

$(s^2 + 2s + 1)Y(s) = \frac{s^2 + s - 11}{s-3}$

$Y(s) = \frac{s^2 + s - 11}{s-3} \cdot \frac{1}{s^2 + 2s + 1}$

$Y(s) = s^2 + s - 11 / (s-3)(s+1)^2$

$\frac{s^2 + s - 11}{(s-3)(s+1)^2} = \frac{A}{s-3} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$

$A = \frac{1}{16} \quad B = \frac{15}{16} \quad C = \frac{11}{4}$

$Y(s) = \frac{1}{16} \cdot \frac{1}{s-3} + \frac{15}{16} \cdot \frac{1}{s+1} + \frac{11}{4} \cdot \frac{1}{(s+1)^2}$

$y(t) = \frac{1}{16}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{15}{16}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{11}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$

$= \frac{1}{16} \cdot e^{3t} + \frac{15}{16}e^{-t} + \frac{11}{4}te^{-t}$

$$\boxed{\therefore\ y(t) = \frac{1}{16} e^{3t} + \frac{15}{16} \cdot e^{-t} + \frac{11}{4} \cdot te^{-t}}$$