Eric Altenburg Midterm

7/20/20 Diff. Eq.

①

3. $y'' - 2y' + y = e^x$

Find y_c: $y'' - 2y' + y = 0$

aux eq: $m^2 - 2m + 1 = 0$

$m_1 = 1 \quad m_2 = 1$

$y_1 = e^x \quad y_2 = xe^x$

$y_c = c_1 e^x + c_2 x e^x$

Find y_p: $g(x) = e^x$

$y_{p_1} = Ax^2e^x$

$y_{p_1}' = Ax^2e^x + 2Axe^x$

$y_{p_1}'' = Ax^2e^x + 4Axe^x + 2Ae^x$

$(Ax^2e^x + 4Axe^x + 2Ae^x) - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = e^x$

$Ax^2e^x + \cancel{4Axe^x} + 2Ae^x - 2Ax^2e^x - \cancel{4Axe^x} + Ax^2e^x = e^x$

$2Ax^2e^x + 2Ae^x - 2Ax^2e^x = e^x$

$2Ae^x = e^x$

$2A = 1$

$A = \frac{1}{2}$

$y_p = \frac{1}{2}x^2e^x = \frac{x^2e^x}{2}$

$\therefore\ y(x) = c_1 e^x + c_2 x e^x + \frac{x^2 e^x}{2}$

4. $y'' - 5y' - 6y = 0$ ②

aux eq: $m^2 - 5m - 6 = 0$

$m_1 = -1$ $m_2 = 6$

$y_1 = e^{-x}$ $y_2 = e^{6x}$

$\therefore y(x) = c_1 e^{-x} + c_2 e^{6x}$

③

5. a) Set up IVP: $\frac{dT}{dt} = K(T-35)$ $\quad \underbrace{T(0) = 65}_{\text{initial condition}}$

b) Solve the IVP: $\frac{dT}{dt} = K(T-35)$

$\Rightarrow \frac{dT}{(T-35)} = K\,dt$

$\Rightarrow \ln|T-35| = Kt$

$\Rightarrow T-35 = ce^{Kt}$

$\Rightarrow T = ce^{Kt} + 35$

IC: $T(0) = 65$

$T(0) = ce^{K(0)} + 35 = 65$

$\Rightarrow ce^{0} + 35 = 65$

$\Rightarrow c = 30, \quad T(t) = 30e^{Kt} + 35$

$T(5) = 63$

$T(5) = 30e^{K(5)} + 35 = 63$

$\Rightarrow 30e^{5K} = 28$

$\Rightarrow e^{5K} = \frac{28}{30} = \frac{14}{15}$

$\Rightarrow 5K = \ln\left(\frac{14}{15}\right)$

$\Rightarrow K = \frac{\ln(14/15)}{5}$

$\therefore T(t) = 30\exp\left(\left(\frac{\ln(14/15)}{5}\right)t\right) + 35$

c) 1 hr = 60 min. $T(60) = 30\exp\left(\left(\frac{\ln(14/15)}{5}\right)(60)\right) + 35$

$= 30\exp\left(12\ln\left(\frac{14}{15}\right)\right) + 35$

$= 48.1°F$

8. a) setup IVP: $\frac{dA}{dt} = kA$ $\qquad \underbrace{A(0) = 5}_{\text{initial condition}}$

b) Solve IVP: gen sol. is $A(t) = ce^{kt}$

$A(0) = 5$,

$A(0) = ce^{k(0)} = 5$

$\Rightarrow ce^{0} = 5$

$\Rightarrow c = 5$, $\quad A(t) = 5e^{kt}$

$A(1700) = \frac{1}{2} A_0 \quad , A_0 = 5$

$A(1700) = \cancel{5}e^{k(1700)} = \frac{1}{2}(\cancel{5})$

$\Rightarrow e^{1700k} = \frac{1}{2}$

$\Rightarrow 1700k = \ln\left(\frac{1}{2}\right)$

$\Rightarrow k = \frac{\ln(1/2)}{1700}$

$\therefore A(t) = 5\exp\left(\left(\frac{\ln(1/2)}{1700}\right)t\right)$

9. $y' + \frac{y}{x} = y^2$ (5)

$\Rightarrow \frac{dy}{dx} = y^2 - \frac{y}{x}$

$\Rightarrow dy = (y^2 - \frac{y}{x})dx$

$\Rightarrow -(y^2 - \frac{y}{x})\,dx + dy = 0$

$\Rightarrow (y^2 - \frac{y}{x})\,dx + (-1)dy = 0$

$M = y^2 - \frac{y}{x}$ $\qquad N = -1$

$\frac{\partial M}{\partial y} = 2y - \frac{1}{x}$ $\qquad \frac{\partial N}{\partial x} = 0$ $\qquad$ not exact

$y' = (-\frac{1}{x})y + y^2$ ← Bernoulli

$P(x) = -\frac{1}{x}$ $\qquad Q(x) = 1$ $\qquad n = 2$

$u = y^{-1}$, $u' = -y^{-2}y'$

$y^{-2}\left[y' = (-\frac{1}{x})y + y^2\right]$

$\Rightarrow y^{-2}y' = (-\frac{1}{x})(y^{-2})(y) + 1$

$\Rightarrow y^{-2}y' = (-\frac{1}{x})y^{-1} + 1$ $\qquad -u' = y^{-2}y'$

$\Rightarrow -u' = (-\frac{1}{x})u + 1$

$\Rightarrow -u' + (\frac{1}{x})u = 1$

$\Rightarrow u' + (-\frac{1}{x})u = -1$

$P(x) = -\frac{1}{x}$, $\mu = e^{\int -1/x\,dx} = e^{-\ln|x|} = |x^{-1}| = x^{-1}$

w/o loss of gen. assume pos.

$\Rightarrow (x^{-1}) + (-\frac{1}{x})(x^{-1})u = (x^{-1})(-1)$

$\Rightarrow \frac{d}{dx}\left[x^{-1}u\right] = -x^{-1}$

$\Rightarrow x^{-1}u = \int -x^{-1}\,dx$

$\Rightarrow \frac{1}{x}u = -\ln|x| + c$

$\Rightarrow u = x(-\ln|x| + c)$ $\qquad , u = \frac{1}{y}$

$\Rightarrow \frac{1}{y} = x(-\ln|x| + c)$

$\Rightarrow y = \frac{1}{x(-\ln|x| + c)}$

$\therefore y(x) = \frac{1}{x(-\ln|x| + c)}$