Homework 3.

1) Are the following maps automorphisms of their respective groups?

1. $G = \mathbb{Z}$ (i.e. the group of integers under addition), and $f(x) = -x$.
2. $G =$ the group of positive reals under multiplication, and $f(x) = x^2$.
3. $G =$ the cyclic group of order 12, and $f(x) = x^3$.
4. G is a non Abelian group and $f(x) = x^{-1}$.

2) Let G be a group, f an automorphism of G, and N a normal subgroup of G. Prove that $f(N)$ is a normal subgroup of G.

3) A subgroup C of G is said to be a *characteristic* subgroup of G if $f(C) \subset C$ for all automorphisms f of G. Prove that

1. a characteristic subgroup of G must be a normal subgroup of G.
2. Prove that the converse of the above is false.

4) If G is a group, N a normal subgroup of G, M a characteristic subgroup of N, prove that M is a normal subgroup of G.

5) Let G be a finite group, f an automorphism of G with the property that $f(x) = x$ for $x \in G$ if and only if $x = 1_G$. Prove that every $g \in G$ can be represented as $g = x^{-1}f(x)$ for some $x \in G$.

6) Let G be a finite group, f an automorphism of G with the property that $f(x) = x$ if and only if $x = e$. Suppose further that $f^2 = Id$. Prove that G must be Abelian.

7) Let G be a group and $\mathcal{Z}(G)$ the center of G. If f is any automorphism of G, prove that $f(\mathcal{Z}(G)) \subset \mathcal{Z}(G)$.

8) Let G be a group. Consider the map

$$\lambda_g : G \longrightarrow G, \qquad \lambda_g(x) = xg$$

for all $x \in G$. Prove that λ_g is one-to-one and onto, and that

$$\lambda_{gh} = \lambda_h \lambda_g.$$

9) Prove that

$$\mathrm{GL}(n, R) / \mathrm{SL}(n, \mathbb{R}) \cong \mathbb{R}^{\star}.$$

10) Prove that $\mathbb{Z}_{15}/\mathbb{Z}_3 \cong \mathbb{Z}_5$.