16-811 hw 4

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November 12, 2020

1

the surface area of a curve y rotated about the x-axis is given by the integral:

$$\int_{x_0}^{x_1} 2\pi y\sqrt{1+y'^2}\,dx$$

let's take the constant 2π out of the integral and then we have:

$$F = y\sqrt{1+y'^2}$$

we also have:

$$F_{y'} = \frac{yy'}{\sqrt{1+y'^2}}$$

from pg. 26 of the Calculus of Variations notes:

$$y'F_{y'} - F = c_0$$

plugging things in:

$$\frac{yy'^2}{\sqrt{1+y'^2}} - y\sqrt{1+y'^2} = c_0$$

simplify:

$$c_0 + y(1+y'^2) - yy'^2 = 0$$

simplifying some more:

$$y = -c_0\sqrt{1+y'^2}$$

$$\sqrt{\frac{y^2-c_0^2}{c_0^2}} = y'$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2-c_0^2}}{c_0}$$

$$dy = \frac{\sqrt{y^2-c_0^2}}{c_0}\,dx$$

$$\int \frac{c_0}{\sqrt{y^2 - c_0^2}}\, dy = \int dx$$

$$c_0 \cosh^{-1}\left(\frac{y}{c_0}\right) + c_1 = x$$

$$y = c_0 \cosh\left(\frac{x - c_1}{c_0}\right)$$

plug in initial conditions to solve for c_0 and c_1:

$$y_0 = c_0 \cosh\left(\frac{x_0 - c_1}{c_0}\right)$$

$$y_1 = c_0 \cosh\left(\frac{x_1 - c_1}{c_0}\right)$$

Not really sure about how to solve these implicit equations for c_0 and c_1. probably can't. so basically, there's not always a C^2 solution. the solution forms a catenary between the points (x_0, y_0) and (x_1, y_1). because y is constrained to be > 0, in the case where (x_0, y_0) and (x_1, y_1) causes the catenary that connects them to drop below the x-axis, then it's hard to say what the minimizing solution will be.

2

objective is 3D arc length, given by the integral:

$$\int_{x_0}^{x_1} \sqrt{dx^2 + dy^2 + dz^2}$$

change of variables to make life easier (change of vars allows us to weave the constraint that the points must be on a sphere into the change of variables; we get treat the variable R as a constant to constrain the points to lie on the sphere of radius R):

$$x = R \sin v \cos u$$

$$dx = R \cos v \cos u dv - R \sin v \sin u du$$

$$y = R \sin v \sin u$$

$$dy = R \cos v \sin u dv + R \sin v \cos u du$$

$$z = R \cos v$$

$$dz = -R \sin v dv$$

plug things in:

$$\int_{\cos^{-1}(\frac{x_0}{R \sin v})}^{\cos^{-1}(\frac{x_1}{R \sin v})} \sqrt{(R \cos v \cos u dv - R \sin v \sin u du)^2 + (R \cos v \sin u dv + R \sin v \cos u du)^2 + (-R \sin v dv)^2}$$

simplify:

$$R\int_{\cos^{-1}(\frac{x_0}{R\sin v})}^{\cos^{-1}(\frac{x_1}{R\sin v})} \sqrt{dv^2 + \sin^2 v\ du^2}$$

$$R\int_{\cos^{-1}(\frac{x_0}{R\sin v})}^{\cos^{-1}(\frac{x_1}{R\sin v})} \sqrt{v'^2 + \sin^2 v}\ du$$

So we have:

$$F = \sqrt{v'^2 + \sin^2 v}$$

as there is no u in our F we will be plugging things into this ELE:

$$v' F_{v'} - F = c$$

$$F_{v'} = \frac{v'}{\sqrt{v'^2 + \sin^2 v}}$$

plug things in:

$$\frac{v'^2}{\sqrt{v'^2 + \sin^2 v}} - \sqrt{v'^2 + \sin^2 v} = c$$

simplify:

$$v'^2 - (v'^2 + \sin^2 v) = c\sqrt{v'^2 \sin^2 v}$$

$$\sin^4 v = c^2(v'^{+} \sin^2 v)$$

$$v' = \sqrt{\frac{\sin^4 v - c^2 \sin^2 v}{c^2}}$$

$$\frac{c}{\sqrt{\sin^4 v - c^2 \sin^2 v}}\ dv = du$$

integrate both sides using the helpfully provided identity:

$$u = \int \frac{c}{\sqrt{\sin^4 v - c^2 \sin^2 v}}\ dv = -\sin^{-1}\left(\frac{\cot v}{\sqrt{\frac{1}{c^2} - 1}}\right) + c_1$$

substitute $c_0 = \frac{1}{\sqrt{\frac{1}{c^2}-1}}$:

$$c_0 \cot v = \sin(c_1 - u)$$

using the trig identity $\sin(a-b) = \sin a \cos b - \cos a \sin b$:

$$c_0 \cot v = \sin c_1 \cos u - \cos c_1 \sin u$$

$$c_0 \cos v = \sin c_1 \sin v \cos u - \cos c_1 \sin v \sin u$$

changing variables back to x, y, and z:

$$\frac{c_0 z}{R} = \frac{x}{R}\sin c_1 - \frac{y}{R}\cos c_1$$

$$c_0 z = x \sin c_1 - y \cos c_1$$

$$x \sin c_1 - y \cos c_1 - c_0 z = 0$$

this is a plane that passes through the origin. by the way we parametrized the sphere, its center is at the origin. thus, this plane intersects the sphere along a great circle, which is where our minimizing curve must lie on.

3

The objective to minimize is the same as that used during lecture, so we will copy it here:

$$\frac{1}{\sqrt{-2g}} \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y_0 - y}}\ dx$$

$$F = \frac{\sqrt{1+y'^2}}{\sqrt{y_0 - y}}$$

$$F_{y'} = \frac{y'}{\sqrt{(y_0 - y)(1+y'^2)}}$$

We have the equation that was given during lecture:

$$F_{y'}\Bigg|_{x_1} = \frac{g_y F}{g_x + y' g_y}$$

plug things in (for $x = x_1$:

$$\frac{y'}{\sqrt{(y_0 - y)(1+y'^2)}} = \frac{g_y}{g_x + y' g_y} \frac{\sqrt{1+y'^2}}{\sqrt{y_0 - y}}$$

simplify:

$$y' = \frac{g_y}{g_x + y' g_y}(1 + y'^2)$$

$$g_y(1+y'^2) = y'(g_x + y' g_y)$$

$$g_y = g_x y'$$

$$y' = \frac{g_y}{g_x}$$

this means that the slope of the curve y is orthogonal to the curve $g(x, y)$ at $x = x_1$. (this is bc the gradient of $g(x, y) = \langle g_x, g_y \rangle$ is orthogonal to the curve; the slope of the tangent line written in terms of x is $\frac{g_y}{g_x}$, and this is orthogonal to the curve.)

4

4a

our Lagrangian equation is:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i$$

we will substitute in the given qs and L for our given manipulator into the above equilibrium equation to obtain the relationship between torque and the state of the system.

$$L = T - V$$

we use v_{21} to refer to the upper m_2's velocity and v_{22} for the lower m_2's velocity, as they have the same mass but different velocities:

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_{21}^2 + \frac{1}{2}m_2 v_{22}^2$$

There is no gravity, so:

$$V = 0$$

putting it together:

$$L = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_{21}^2 + \frac{1}{2}m_2 v_{22}^2$$

here are our equations for v in terms of the time derivatives of ℓ and θ:

$$v_1 = \ell_1 \dot{\theta}_1$$

$$v_1^2 = \ell_1^2 \dot{\theta}_1^2$$

here, we use x_{21} and y_{21} to refer to the x- and y-position of the upper mass, as measured from the manipulator's point of contact with the ground, and analogously for x_{22} and y_{22}. Similar to as we did class, we will use c_i as stand-in for $\cos\theta_i$, c_{12} for $\cos(\theta_1 + \theta_2)$, s_i for $\sin\theta_i$, and s_{12} for $\sin(\theta_1 + \theta_2)$.

$$x_{21} = \ell_1 c_1 + \ell_2 c_{12}$$

$$y_{21} = \ell_1 s_1 + \ell_2 s_{12}$$

$$\dot{x}_{21} = -\ell_1 s_1 \dot{\theta}_1 - \ell_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$

$$\dot{y}_{21} = \ell_1 c_1 \dot{\theta}_1 + \ell_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$

$$v_{21} = \dot{x}_{21}^2 + \dot{y}_{21}^2$$

$$v_{21}^2 = \ell_1^2 \dot{\theta}_1^2 + \ell_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2\ell_1\ell_2 c_2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1$$

$$x_{22} = \ell_1 c_1 - \ell_2 c_{12}$$

$$y_{22} = \ell_1 s_1 - \ell_2 s_{12}$$
$$\dot{x}_{22} = -\ell_1 s_1 \dot{\theta}_1 + \ell_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$
$$\dot{y}_{22} = \ell_1 c_1 \dot{\theta}_1 - \ell_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$
$$v_{22} = \dot{x}_{22}^2 + \dot{y}_{22}^2$$
$$v_{22}^2 = \ell_1^2 \dot{\theta}_1^2 + \ell_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 - 2\ell_1 \ell_2 c_2 (\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1$$

so plugging in v_1^2, v_{22}^2, and v_{22}^2 into our equation for L, we have:

$$L = \frac{1}{2} m_1 \ell_1^2 \dot{\theta}_1^2 + \ell_1^2 \dot{\theta}_1^2 + \ell_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

taking the necessary derivatives:

$$\frac{\partial L}{\partial \theta_1} = 0$$
$$\frac{\partial L}{\partial \theta_2} = 0$$
$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 \ell_1^2 \dot{\theta}_1 + 2m_2 \ell_1^2 \dot{\theta}_1 + 2m_2 \ell_2^2 (\dot{\theta}_1 + \dot{\theta}_2)$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = m_1 \ell_1^2 \ddot{\theta}_1 + 2m_2 \ell_1^2 \ddot{\theta}_1 + 2m_2 \ell_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2)$$
$$\frac{\partial L}{\partial \dot{\theta}_2} = 2m_2 \ell_2^2 (\dot{\theta}_1 + \dot{\theta}_2)$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = 2m_2 \ell_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2)$$

finally, plugging things into the Lagrangian equation ($\tau_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}$):

$$\tau_1 = m_1 \ell_1^2 \ddot{\theta}_1 + 2m_2 \ell_1^2 \ddot{\theta}_1 + 2m_2 \ell_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2)$$
$$\tau_2 = 2m_2 \ell_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2)$$

In matrix form:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} m_1 \ell_1^2 + 2m_2(\ell_1^2 + \ell_2^2) & 2m_2 \ell_2^2 \\ 2m_2 \ell_2^2 & 2m_2 \ell_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

4b

if $\ddot{\theta}_2 = 0$, then:

$$\tau_1 = (m_1 \ell_1^2 + 2m_2 \ell_1^2 + 2m_2 \ell_2^2)\,\ddot{\theta}_1$$

the first term is the moment of inertia of m_1. the second term is the moment of inertia for the sum of the two m_2s about their center of mass (the point where m_1 is, thus ℓ_1). the third term is the moment of inertia required to give the two m_2s their existing centripetal acceleration as they rotate about the point where m_1 is located.