16-811: Math Fundamentals for Robotics, Fall 2020

Assignment 3

DUE: Thursday, October 15, 2020

Regarding MATLAB/Python functions: It is possible that MATLAB/Python has some advanced functions that solve some of these problems for you directly from the data we have supplied. You may not use such functions. Instead, you should come up with your own solutions, based on the basic core functions available in MATLAB/Python. (When in doubt, ask.)

1. Consider the function $f(x) = \sin x - 0.5$ over the interval $[-\pi/2, \pi/2]$.

   (a) What is the Taylor series expansion for $f(x)$ around $x = 0$?

   (b) Graph $f(x)$ over the interval $[-\pi/2, \pi/2]$.

   (c) Determine the best uniform approximation by a quadratic to the function $f(x)$ on the interval $[-\pi/2, \pi/2]$. What are the L_∞ and L_2 errors for this approximation?

   (d) Determine the best least-squares approximation by a quadratic to the function $f(x)$ over the interval $[-\pi/2, \pi/2]$. What are the L_∞ and L_2 errors for this approximation?

   (Please show all your hand derivations and explain how you got results if you used numerical methods. No code submission required.)

   Terminology: Suppose an approximation has error function $e(x)$ over an interval $[a, b]$. The L_∞ error is $||e(x)||_\infty = \max_{a \le x \le b} |e(x)|$ and the L_2 error is $||e(x)||_2 = \sqrt{\int_a^b |e(x)|^2 dx}$.

2. Suppose very accurate values of some function $f(x)$ are given at the points $0 = x_0, x_1, \ldots, x_{100} = 1$, with the $\{x_i\}$ uniformly distributed over the interval $[0, 1]$. (So $x_i = i/100$, $i = 0, \ldots, 100$.) The values $\{f(x_i)\}$ are given in the file '`problem2.txt`' in sequential order (so, for example, $f(0.27) = f(x_{27}) = -0.964603914513021$).

   What is the function $f(x)$? Provide a succinct description. Submit any code you use.

   [Hint: Try "basis" functions, $1, x, x^2, \ldots, \cos(\pi x), \sin(\pi x), \cos(2\pi x), \sin(2\pi x), \ldots$, giving an over- and under-constrained system of equations. Try to find a relatively simple description of the function $f(x)$ by determining which function coefficients may be set to zero. There may be multiple candidate answers; find one with the fewest nonzero coefficients.]

3. The *Chebyshev polynomials of the first kind*, $T_n(x)$, are defined indirectly on $[-1, 1]$ by:

   $$T_n(\cos\theta) = \cos(n\theta), \text{ for } n \ge 0.$$

   Expanding cosine, one finds the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, for $n > 0$.

   (a) Derive T_4 and T_5. (As always, please show your handwritten work.)

   (b) Without actually computing or working out any integrals, show that T_4 and T_5 are orthogonal polynomials relative to the inner product

   $$< g, h > = \int_{-1}^{1} (1 - x^2)^{-1/2} g(x) h(x) dx.$$

(c) Relative to this inner product, all the T_n, with $n > 0$, have the same length. Establish that fact by handcomputing the length of T_n (leave n symbolic, assume $n > 0$).

(d) Finally, show that $< T_i, T_j > = 0$ for all i and j such that $i \geq 0$, $j \geq 0$, and $i \neq j$. (There are different ways to prove this, e.g., by working out an integral explicitly or by combining known facts from above and lecture.)

4. After weeks of work you have finally completed construction of a gecko robot. It is a quadruped robot with suctioning feet that allow it to walk on walls. It is also equipped with a Kinect-like sensor, providing a 3D point cloud observation of the world. You want to use these point clouds to reason about the environment and aid in navigation.

   (a) You boot up the robot and place it on a table, taking an initial observation. The observation is saved in the provided `clear_table.txt`, and lists (x, y, z) locations in the following format:

$$\begin{array}{ccc} x_1 & y_1 & z_1 \\ & \vdots & \\ x_n & y_n & z_n \end{array}$$

   Points are in units of meters and the positive x-direction is right, positive y-direction is down, and positive z-direction is forward. Find the least-squares approximation plane that fits the data. Visualize your fitted plane along with the data. What is the average distance of a point in the data set to the fitted plane?

   (b) Interested in your gecko robot, your cat jumps up on the table. You take a second observation, saved as the provided `cluttered_table.txt`. Using the same method as above, find the least-squares fit to the new data. How does it look? Why?

   (c) Can you suggest a way to still find a fit to the plane of the table regardless of clutter? Verify your idea by writing a program that can successfully find the dominant plane in a list of points regardless of outliers. [Hint: You may assume that the number of points in the plane is much larger than the number of points not in the plane.] Visualize `cluttered_table.txt` with your new plane.

   (d) Encouraged by your results when testing on a table, you move your geckobot into the hallway and take an observation saved as the provided `clean_hallway.txt`. Describe an extension to your solution to part (c) that finds the four dominant planes shown in the scene, then implement it and visualize the data and the four planes.

   You may assume that there are roughly the same amount of points in each plane.

   (e) You decide it is time to test your gecko robot's suction feet and move it to a different hallway. The feet are strong enough to ignore the force of gravity, allowing the robot to walk on the floor, walls, or ceiling. However, the locomotion of the legs works best on smooth surfaces with few obstacles. Using your solution from part (d), describe how you can mathematically characterize the smoothness of each surface. Load the provided scan `cluttered_hallway.txt`, find and plot the four wall planes, describe which surface is safest for your robot to traverse, and provide the smoothness scores from your mathematical characterization.

   Note that you may *no longer* assume that there are roughly the same amount of points in each plane.

(Please submit code for all parts of this problem.)