

16-811: Math Fundamentals for Robotics, Fall 2020

Assignment 5

DUE: Thursday, November 12, 2020

HANDIN: There is no code, so in Autolab please submit the writeup as `hw5.pdf` (not `hw5.tar`).

1. Consider a plane curve $y(x)$ over the interval $[x_0, x_1]$, with specified endpoints $y_0 = y(x_0)$ and $y_1 = y(x_1)$. Assume that $y_0 > 0$ and $y_1 > 0$ and that $y(x) \geq 0$ for $x_0 \leq x \leq x_1$. Now imagine rotating the curve about the x -axis to obtain a surface of revolution. Find the C^2 curve $y(x)$ with specified endpoints that minimizes the surface area of this surface of revolution.

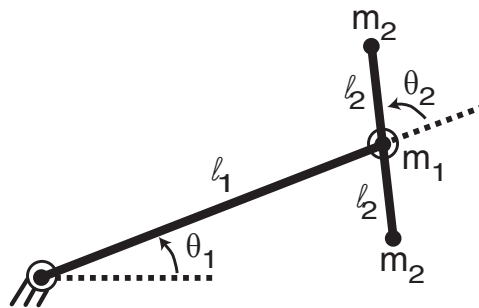
[Hint: This problem explores further some of the limitations of the Calculus of Variations. Depending on the endpoint conditions there may or may not be a C^2 solution. What does the optimal “curve” look like when it is not C^2 ? Can you say how the endpoint conditions matter? Be aware: There are many subtleties; don’t expect to cover all, but explore what you can.]

2. Using Calculus of Variations, show that the shortest curve between two points on a sphere is an arc of a great circle. [Hints: There are different ways to solve this problem. Here is one approach: Use spherical (u, v) coordinates, where $x = R \sin v \cos u$, $y = R \sin v \sin u$, $z = R \cos v$, with R the radius of the sphere. (Do not worry about singularities in the representation.) Cast 3D arclength $\sqrt{dx^2 + dy^2 + dz^2}$ into (u, v) space. Parametrize the curve in terms of the coordinate u . Observe that u does not appear directly in the integrand for arclength, so replace the Euler-Lagrange equation with another, as in lecture. Finally, you may find the following identity useful:

$$\int \frac{a dw}{\sqrt{\sin^4 w - a^2 \sin^2 w}} = -\sin^{-1} \left(\frac{\cot w}{\sqrt{\frac{1}{a^2} - 1}} \right) + k,$$

where a and k are appropriate constants.]

3. In the brachistochrone problem, suppose the right endpoint is constrained only to touch some curve given implicitly by an equation of the form $g(x, y) = 0$. Show that the optimizing curve $y(x)$ must intersect the iso-contour $g(x, y) = 0$ orthogonally. [Hint: Use an equation from lecture rather than derive everything from scratch.]
4. (a) Using Lagrangian Dynamics, derive the relationship between joint torques and the angular state (angles, velocities, and accelerations) of the following balanced manipulator:



There is no gravity (in practice, gravity is perpendicular to the sheet of the paper).

Legend: All of link #1’s mass, m_1 , is concentrated at distance l_1 from its rotational joint (which is attached to the ground). In turn, link #2 rotates around this distal point, with two masses, m_2 , located symmetrically, each at distance l_2 , from the joint. In practice, these two masses might constitute one counter-balanced end-effector or two different but equally weighted end-effectors. — This is a variation of a basic SCARA-type robot arm, used in industrial assembly.

- (b) When $\ddot{\theta}_2 = 0$, explain the terms relating $\ddot{\theta}_1$ to τ_1 .