

A Theoretical and Empirical Investigation into the Equivalence of Graph Neural Networks and the Weisfeiler-Leman Algorithm

From the faculty of Mathematics, Physics, and Computer Science approved for the purpose of obtaining
the academic degree of Bachelor of Sciences.

Eric Tillmann Bill

Supervision:
Prof. Dr. rer. nat. Christopher Morris

Informatik 6
RWTH Aachen University

July 21, 2023

Abstract

 Lorem Ipsum :D

Acknowledgements

Luis, Christopher und HPC RWTH

Contents

1 Introduction

This work aims to shed light on the representations learned by Graph Neural Networks. In this section, we will discuss the prevalence of graphs and the crucial role GNNs play in analyzing them. We will delve into the methods we use to gain insights and highlight the significance of our approach. Lastly, we will provide an overview of the structure of this work.

1.1 Motivation

Graphs are ubiquitous in various fields of life. Despite not always being explicitly identified as such, the graph data model's flexibility and simplicity make it an effective tool for modeling a diverse range of data. Examples of graph modeling applications include unexpected instances, such as modeling text or images as a graph, as well as more complex instances like chemical molecules, citation networks, or connectivity encodings of the World Wide Web ??.

Although machine learning has achieved remarkable success with image classification (e.g., ??) and text generation (e.g., ??) in the last decade, the lack of a significant breakthrough in machine learning for graphs can be attributed to the graph data model's inherent flexibility and simplicity. While, for example, an image classifier constrains its input data to be a grid-like image or a text generator expects its input to be a linear sequence of words, machine learning models working on graphs cannot leverage any constraints on the format or size of their input graphs without limiting their generality.

To put this flexibility of the graph data model into perspective and give an idea of how ubiquitous graphs are in various fields, we refer back to the examples of image classifiers and text generators and demonstrate how seemingly natural the graph data model can encode their input data. For example, images can be encoded by a graph, such that each pixel of the image corresponds to a node in the graph holding its color value, and each node is connected to its neighboring pixel nodes. Similarly, for sequential data like text files, one can encode a directed graph where each word in this file is represented as a node with the word as a feature and connected outgoingly to the next following word. See ?? for an illustrative example of these encodings.

In recent years, a significant amount of research has been conducted to investigate Graph Neural Networks (GNNs). Among the most promising approaches are those utilizing the message-passing architecture, which was introduced by ? and ?. Empirically, this framework has demonstrated strong performance across various applications ????. However, its expressiveness is limited, as has been proved by the works of ?, as well as ?. These works establish a connection to the commonly used Weisfeiler-Leman¹ algorithm (1-WL), originally proposed by ? as a simple heuristic for the graph isomorphism problem. In particular, it has been proven that a GNN based on the message-passing architecture can, at most, be as good as the 1-WL algorithm in distinguishing non-isomorphic graphs. Furthermore, the 1-WL method demonstrates numerous similarities with the fundamental workings of the GNN architecture. It is, therefore, commonly believed that both methods are, to some extent, equivalent in their capacity to capture information in a graph.

OUTLOOOK WHAT IS THE MOTIVATION NOW?

¹Based on <https://www.iti.zcu.cz/wl2018/pdf/leman.pdf>, we will use the spelling “Leman” here, as requested by A. Leman, the co-inventor of the algorithm.

Footnotes
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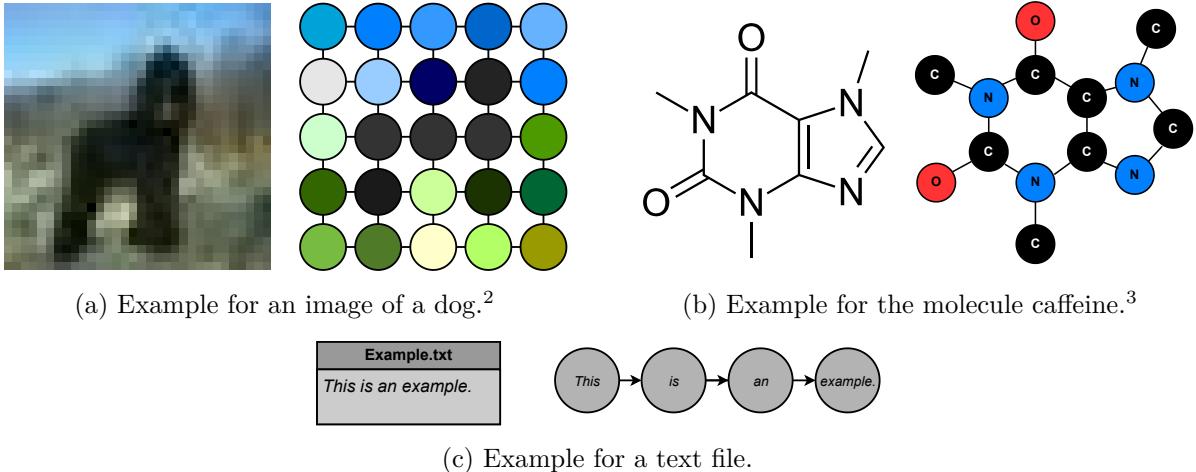


Figure 1: Here are some examples of how graphs can be used to encode information in a variety of domains. Please note that these examples are just a sample, and in actual practice, more detailed encodings are usually utilized to capture additional information.⁴

1.2 Research Questions & Contributions

Test

1.3 Methodology

In this work, we introduce a novel framework, which we coined “1-WL+NN,” which involves applying the 1-WL algorithm to an input graph and further processing the resulting information using a feedforward neural network. Thereby, we obtain a trainable framework suited for all kinds of graph-related tasks, such as graph classification, node regression, and more. We will prove later on that both frameworks, 1-WL+NN and GNN, are theoretically equivalent, such that each function computed by a 1-WL+NN model can also be computed by a GNN model and vice versa. With this framework in hand, we can investigate the representations learned by a GNN.

The interesting property of this framework compared to GNNs, which is also the original idea that inspired this work, is the fundamental difference in how both frameworks learn and optimize themselves when applied to a specific task. Take, for example, an arbitrary classification task. While the first part of a 1-WL+NN model starts by applying the 1-WL algorithm to its input graph and retrieves a highly informative representation of this graph, the second part, the learnable feedforward neural network, must find common patterns in this very detailed representation that coincides with the classes of the task, such that the model makes good predictions. In comparison, while a GNN is theoretically as expressive as the 1-WL algorithm, we imagine that such a model first leans to find common patterns as early as possible in the input graph and will drop irrelevant information as soon as possible before making a class prediction.

To put both learning behaviors into perspective: while the 1-WL+NN is given a maximally

²The image is from the CIFAR-10 collection made available by ?.

³The illustration of the skeletal formula of caffeine is taken from <https://commons.wikimedia.org>.

⁴All graphics were created using the free open source platform <https://www.draw.io>.

informative representation and needs to find the important information to make a good prediction, a GNN works the other way around and needs to learn how it constructs its own informative representation of the input graph and how to leverage this information for making a prediction. Therefore, we expect 1-WL+NN models to perform sufficiently well on their training data but expect poor generalization capabilities due to the too-informative representations computed by the 1-WL algorithm. In contrast, we expect more promising generalization capabilities of GNNs in comparison since they optimize for the best representation of their input graphs and might leverage this information more efficiently.

We will use this novel framework and various empirical experiments to compare both frameworks on multiple datasets to establish a deeper understanding of the representations learned by GNNs.

1.4 Outline

For the ease of readability, we split this work into two parts. The first part investigates and establishes the theoretical equivalence between the frameworks of 1-WL+NN and GNNs. In contrast, the second part discusses our different experiments and their empirical results and insights into GNNs.

In detail, this work starts with ??, where we discuss related work, milestones in GNNs over the past decade, essential properties of the 1-WL algorithm, and a subset of interesting connections between GNNs and the 1-WL algorithm. Afterward, we will start with ??, which begins with ???. Here, we will introduce formal definitions for both frameworks, as well as a set of notations we will use throughout the theoretical part. Afterward, in ??, we will introduce four theorems that each present a connection between both frameworks and combined prove the equivalence of both frameworks. Finally, we will prove each theorem individually after another in corresponding subsections.

The second part will deal with ...

2 Background and Related Work

In this section, we will briefly introduce the foundation of our research by explaining the origins of the two frameworks, mentioning important recent advances, and providing a brief overview of the connections between them.

2.1 Weisfeiler-Leman Algorithm

The (1-dimensional) Weisfeiler-Leman algorithm (1-WL), proposed by ?, was initially designed as a simple heuristic for the *graph isomorphism problem*, but due to its interesting properties, its simplicity, and its good performance, the 1-WL algorithm gained a lot of attention from researchers across many fields. One of the most noticeable properties is that the algorithm color codes the nodes of the input graph in such a way that in each iteration, each color encodes a learned local substructure.

It works by coloring all nodes in each iteration the same color that fulfills two properties: 1. the nodes already share the same color, and 2. each color appears equally often in the set of the nodes direct neighbors. The algorithm continues as long as the number of colors changes in each iteration. For determining whether two graphs are non-isomorphic, the heuristic is applied to both graphs simultaneously. It concludes that the graphs are non-isomorphic as soon as the

number of occurrences of a color differs between them. We present a more formal definition of the algorithm in the following part in ??.

Since the *graph isomorphism problem* is difficult to solve due to the best known complete algorithm only running in deterministic quasipolynomial time (?), the 1-WL algorithm, running in deterministic polynomial time, cannot solve the problem completely. Moreover, ? constructed counterexamples of non-isomorphic graphs that the heuristic fails to distinguish, e.g., see ?? . However, following the work of ?, this simple heuristic is still quite powerful and has a very low probability of failing to distinguish non-isomorphic graphs when both graphs are uniformly chosen at random as the number of nodes tends to infinity.

To overcome the limited expressiveness of the 1-WL algorithm, it has been generalized to the k -dimensional Weisfeiler-Leman algorithm (k -WL) by ?? , as well as ?⁵. This version works with k -tuples over the k -ary Cartesian product of the set of nodes. Interestingly, this created a hierarchy for the expressiveness of determining non-isomorphism, such that for all $k \in \mathbb{N}$ there exists a pair of non-isomorphic graphs that can be distinguished by the $(k+1)$ -WL but not by the k -WL (?).

2.2 Graph Neural Networks

The idea of leveraging machine learning techniques, previously proven effective in various domains, for graph-related tasks has been a well-established topic in the literature for the past decades. However, researchers faced challenges in effectively adapting these techniques to graphs of diverse sizes and complexities in the early stages. Notably, the works by ?? , and ? were the first prominent examples of successful applications in this regard.

However, it was not until the emergence of more advanced models that the scientific community truly recognized the significance and potential of **Graph Neural Networks** (GNNs). Noteworthy among these advancements were the work of ?, who introduced a differentiable approach for generating unique fingerprints of arbitrary graphs, as well as ?, who applied gated recurrent units to capture graphs of various sizes, while ? utilized diffusional convolutions for the same purpose. Of particular significance, however, were the contributions of ?? and ?, which extended the concept of convolution from its traditional application on images to the domain of arbitrary graphs.

After the early success of these GNN models, ? introduced a unified architecture for GNNs. The authors observed a recurring pattern in how information is exchanged and processed among many of these works, including many mentioned in the paragraph above. Leveraging these observations, ? devised the message-passing architecture as a generalized framework for GNNs. Models using this architecture can be referred to as Message-Passing-Neural-Network (MPNN); however, throughout this thesis, we will use the term GNN and MPNN interchangeably, as the focus of this thesis is solely on the message-passing architecture. This architecture uses the input graph as its basis for computation and computes new node features for the graph in each layer. The computation of each new node feature involves aggregating all the features of the neighboring nodes and the node's own feature. After applying each layer of a GNN model, a representation of the entire graph is obtained by applying a pooling function (e.g. ?). This representation is then further processed by common machine learning techniques like a multilayer perceptron for the final output. We will present a more formal definition of this

⁵In ? on page 27, László Babai explains that he, together with Rudolf Mathon, first introduced this algorithm in 1979. He adds that the work of ? introduced this algorithm independently of him.

architecture in the following part in ??; however, important to note is that the information exchange in the graph across nodes is limited to a one-hop neighbor per layer.

With this general framework and the empirical success of some models using this message-passing architecture, the question of how expressive models based on this architecture can be gained a lot of attention in the scientific community. Many papers immediately established connections to the 1-WL algorithm, among the most prominent being ? and ?. These connections seem natural, as both methods share similar properties in terms of how they process graph data. Most strikingly, both methods never change the graph structurally since they only compute new node features in each iteration. Moreover, both methods use a one-hop neighborhood aggregation as the basis for computing the new node feature. Following this intuition, ?, as well as ?, showed that the expressiveness of GNNs is upper-bounded by the 1-WL in terms of distinguishing non-isomorphic graphs. Moreover, ? proposed a new k -GNN architecture that operates over the set of subgraphs of size k . Interestingly, ? has shown that the proposed hierarchy over $k \in \mathbb{N}$ is equivalent to the k -WL hierarchy in terms of its ability to distinguish non-isomorphic graphs, i.e., if there is a k -GNN that can distinguish two non-isomorphic graphs, it is equivalent to say that the k -WL algorithm can also distinguish these graphs.

Although there are other modifications of the message-passing architecture besides the theoretical concept of k -GNN to increase the expressiveness of GNNs in terms of distinguishing non-isomorphism, e.g., using node identifiers ?, adding random node features ??, adding directed flows ? and many more. Relatively few works have been published that attempt to understand the representation learned from a standard GNN.

Notable works include ?, where the author, in addition to the normal learning process, optimized GNNs to preserve a notion of distance in their representation and examined the effectiveness of GNNs in utilizing such representations. However, their insights can only be applied to these specially trained GNN models and not be generalized. In another publication, ? presented mathematical proof and empirical confirmation showing how much structural information is encoded by modern GNN models. Their research highlights that GNN models like DGCNN (?) and GAT (?) encode all nodes with the same feature vector, while in contrast, models like GCN (?) and GIN (?) encode nodes after k layers of message-passing with features that relate with the number of walks of length k over the input graph form each node, disregarding the local structure within the nodes are contained.

Part I

Theoretical Equivalence

This part of the thesis focuses on the equivalence between 1-WL+NN and GNN. We will begin by providing a preliminary section that formalizes all the concepts used in the proof and introduces a general notation. Afterward, we will dedicate a separate section to present and prove three theorems. These theorems combined conclude the equivalence.

3 Preliminaries

This section will introduce and formalizes all concepts used throughout the proof and the rest of the thesis. We start with general notations, introduce a general graph definition, and familiarize the reader with the Weisfeiler-Leman algorithm. We will introduce each framework independently, first the 1-WL+NN and then GNN. In the end, we will briefly introduce important properties of collections of functions computed by both methods.

3.1 General Notation

Let \mathbb{N} denote the set of natural numbers such that $\mathbb{N} := \{0, 1, 2, \dots\}$. By $[n]$, we denote the set $\{0, \dots, n\} \subset \mathbb{N}$ for each $n \in \mathbb{N}$. Further, with $\{\!\!\{ \dots \}\!\!\}$, we denote a multiset formally defined as a 2-tuple (X, m) , where X is a set of all unique elements and $m : X \rightarrow \mathbb{N}_{\geq 1}$ a mapping that maps each element in X to the number of its occurrences in the multiset.

3.2 Graphs

We will briefly introduce a formal definition for graphs and coloring on graphs. Starting with the definition of a graph.

Definition 1 (Graph). A graph G is defined as a 3-tuple denoted by $G := (V, E, l)$. This tuple consists of a set of nodes $V \subset \mathbb{N}$, a set of edges $E \subseteq V \times V$, and a labeling function $l : M \rightarrow \Sigma$. The domain M of the labeling function can be either V , $V \cup E$, or E , and the codomain Σ is an alphabet with $\Sigma \subseteq \mathbb{N}^k$, where $k \in \mathbb{N}$ is arbitrary. In the context of this thesis, the assigned values by the labeling function are referred to as either labels or features, depending on the dimension of Σ . In detail, if $k = 1$, we usually refer to the values as labels, otherwise as features. Additionally, the set of all graphs is denoted by \mathcal{G} .

Furthermore, a graph G can be either directed or undirected based on the definition of its set of edges E . If $E \subseteq \{(v, u) \mid v, u \in V\}$, it represents a directed graph, whereas if $E \subseteq \{(v, u) \mid v, u \in V, v \neq u\}$ such that for every $(v, u) \in E$ there exists $(u, v) \in E$, it defines an undirected graph. Additionally, for ease of notation, we will use $V(G)$ and $E(G)$ to denote the set of nodes and the set of edges of G , respectively, as well as l_G to denote the label function of G . Further, with $\mathcal{N}(v)$ for $v \in V(G)$ we denote the set of neighbors of v defined as $\mathcal{N}(v) := \{u \mid (u, v) \in E(G)\}$, and with $d(v)$ for $v \in V(G)$ the degree of node v , defined as $d(v) := |\mathcal{N}(v)|$.

We continue with the definition of a graph coloring.

Definition 2 (Graph Coloring). A coloring of a Graph G is a function $C : V(G) \rightarrow \mathbb{N}$ that assigns each node in the graph a color (here, a positive integer). Further, a coloring C induces a partition \mathcal{P} on the set of nodes, for which we define C^{-1} being the function that maps each color $c \in \mathbb{N}$ to its class of nodes with $C^{-1}(c) = \{v \in V(G) \mid C(v) = c\}$. In addition, we define $h_{G,C}$ as the histogram of graph G with coloring C that maps every color in the image of C under $V(G)$ to the number of occurrences. In detail, $\forall c \in \mathbb{N} : h_{G,C}(c) := |\{v \in V(G) \mid C(v) = c\}| = |C^{-1}(c)|$.

Permutation-invariance and -equivariance

We use S_n to denote the symmetric group over the elements $[n]$ for any $n \in \mathbb{N}$. S_n consists of all permutations over these elements. Let G be a graph with $V(G) = [n]$, applying a permutation $\pi \in S_n$ on G , is defined as $G_\pi := \pi \cdot G$ where $V(G_\pi) = \{\pi(0), \dots, \pi(n)\}$ and $E(G_\pi) = \{(\pi(v), \pi(u)) \mid (v, u) \in E(G)\}$. We will now introduce two key concepts for classifying functions on graphs.

Definition 3 (Permutation Invariant). Let $f : \mathcal{G} \rightarrow \mathcal{Y}$ be an arbitrary function, then f is *permutation-invariant* if and only if for all $G \in \mathcal{G}$, where $n_G := |V(G)|$ and for every $\pi \in S_{n_G}$: $f(G) = f(\pi \cdot G)$.

Definition 4 (Permutation Equivariant). Let $f : \mathcal{G} \rightarrow \mathcal{Y}$ be an arbitrary function, then f is *permutation-equivariant* if and only if for all $G \in \mathcal{G}$, where $n_G := |V(G)|$ and for every $\pi \in S_{n_G}$: $f(G) = \pi^{-1} \cdot f(\pi \cdot G)$.

3.3 Weisfeiler and Leman Algorithm

The Weisfeiler-Leman algorithm consists of two main parts: the coloring algorithm and the graph isomorphism test. We will introduce each part individually and present some implications afterward.

The Weisfeiler-Leman Graph Coloring Algorithm

The 1-WL algorithm computes a node coloring of its input graph in each iteration. In detail, a color is assigned to each node based on the colors of its neighbors and its own current color. The algorithm continues until convergence is reached, resulting in the final coloring of the graph. We will now formally define this procedure and provide an illustrative example in ??.

Definition 5 (1-WL Algorithm). Let $G = (V, E, l)$ be a labeled graph. In each iteration i , the 1-WL algorithm computes a node coloring $C_i : V(G) \rightarrow \mathbb{N}$. In the initial iteration $i = 0$, the coloring is set to $C_0 = l$ if l exists. Otherwise, for all $v \in V(G) : C_0(v) = c$ with $c \in \mathbb{N}$ being an arbitrary but fixed constant. For $i > 0$, the algorithm assigns a color to $v \in V(G)$ as follows:

$$C_i(v) = \text{RELABEL}(C_{i-1}(v), \{C_{i-1}(u) \mid u \in \mathcal{N}(v)\}),$$

where `RELABEL` injectively maps the above pair to a unique, previously not used, color. The algorithm terminates when the number of colors between two iterations does not change, meaning the algorithm terminates after iteration i if the following condition is satisfied:

$$\forall v, w \in V(G) : C_i(v) = C_i(w) \iff C_{i+1}(v) = C_{i+1}(w).$$

Upon terminating we define $C_\infty := C_i$ as the stable coloring, such that $1\text{-WL}(G) := C_\infty$.

The colorings computed in each iteration always converge to the final one, such that the algorithm always terminates. In more detail, ? showed that it always holds after at most $|V(G)|$ iterations. For an illustration of this algorithm, see ???. Moreover, based on the work of ? about efficient refinement strategies, ? proved that the stable coloring C_∞ can be computed in time $\mathcal{O}(|V(G)| + |E(G)| \cdot \log |V(G)|)$.

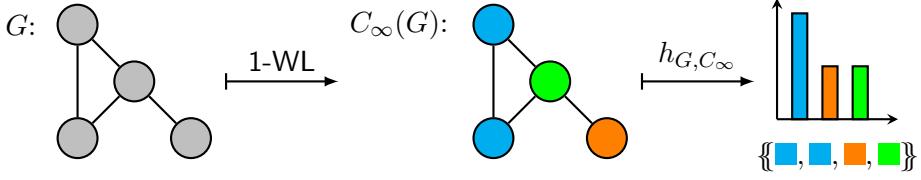


Figure 2: An example of the final coloring computed by applying the 1-WL algorithm on the graph G . The graph G consists of 4 nodes with all their labels being set to the same color.

It is important to understand that since the algorithm was originally developed as a simple heuristic for the *graph isomorphism problem*, which is an inherently discrete problem, the 1-WL algorithm in its simplest form, as we presented it here, does only work on graphs with discrete, one-dimensional node labels. Although it is quite easy to adapt the algorithm to respect discrete edge labels of a graph by using them as weights in the neighborhood aggregation (?), modifying its definition to work with continuous graph features is more complex. Numerous proposed modifications have been put forward to address this integration in the literature, such as those discussed by ?. However, note that this particular topic will not be further investigated in this thesis, although its mention holds value for the following section.

The Weisfeiler-Leman Graph Isomorphism Test

The isomorphism test uses the 1-WL coloring algorithm and is defined as follows.

Definition 6 (1-WL Isomorphism Test). To determine if two graphs $G, H \in \mathcal{G}$ are non-isomorphic ($G \not\cong H$), one applies the 1-WL coloring algorithm on both graphs “in parallel” and checks after each iteration if the occurrences of each color are equal, else the algorithm would terminate and conclude non-isomorphic. Formally, the algorithm concludes non-isomorphic in iteration i if there exists a color c such that:

$$|\{v \in V(G) \mid C_i(v) = c\}| \neq |\{v \in V(H) \mid C_i(v) = c\}|.$$

Note that this test is only sound and not complete for the *graph isomorphism problem*. Counterexamples can be easily constructed where the algorithm fails to distinguish non-isomorphic graphs. See ?? for a straightforward example of where this test fails that was discovered and proven by ?.

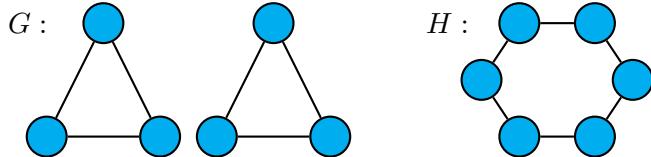


Figure 3: This is an example of two graphs G and H that are non-isomorphic but cannot be distinguished by the 1-WL isomorphism test.

Implications of the 1-WL Algorithm

One intriguing implication of the 1-WL algorithm and its isomorphism test is that, due to it not being complete for solving the *graph isomorphism problem*, it gives rise to a related but weaker relation than the isomorphism relation (\simeq). We define this relation as follows.

Definition 7 (1-WL Relation). Let $\mathcal{X} \subseteq \mathcal{G}$. For any graphs $G, H \in \mathcal{X}$ we will denote $G \simeq_{1WL} H$ if the 1-WL isomorphism test can not distinguish both graphs. Note that due to the soundness of this algorithm, if $G \not\simeq_{1WL} H$, we always can conclude that $G \not\simeq H$.

The \simeq_{1WL} relation can further be classified as an equivalence relation, as it is reflexive, symmetric and transitive. With this, we introduce a notation of its equivalence classes. Let $\mathcal{X} \subseteq \mathcal{G}$ and $G \in \mathcal{X}$, then we denote with $\mathcal{X}/\simeq_{1WL}(G) := \{G' \in \mathcal{X} \mid G \simeq_{1WL} G'\}$ its equivalence class.

Similarly, we define the notion 1-WL-Discriminating for collections of permutation invariant functions.

Definition 8 (1-WL-Discriminating). Let $\mathcal{X} \subseteq \mathcal{G}$. Further, let \mathcal{C} be a collection of permutation invariant functions from \mathcal{X} to \mathbb{R} . We say \mathcal{C} is 1-WL-Discriminating if for all graphs $G_1, G_2 \in \mathcal{X}$ for which the 1-WL isomorphism test concludes non-isomorphic ($G_1 \not\simeq_{1WL} G_2$), there exists a function $h \in \mathcal{C}$ such that $h(G_1) \neq h(G_2)$.

3.4 1-WL+NN

As the previous section shows, the 1-WL algorithm is quite powerful in identifying a graph's substructures and distinguishing non-isomorphic graph pairs. With the 1-WL+NN framework, we define functions that utilize this structural information to derive application-specific insights.

Definition 9 (1-WL+NN). A 1-WL+NN model consists of three components that are applied sequentially to its input: 1. the 1-WL algorithm, 2. an encoding function f_{enc} operating on graph colorings, and 3. an arbitrary multilayer perceptron MLP. In detail, a 1-WL+NN model computes the function \mathcal{B} , that is defined as follows:

$$\mathcal{B} : \mathcal{G} \rightarrow \mathbb{R}^k, \quad G \mapsto \text{MLP} \circ f_{enc}(\{\{1\text{-WL}(G)(v) \mid v \in V(G)\}\}),$$

where “1-WL(G)” is the coloring computed by the 1-WL algorithm when applied on G , and $k \in \mathbb{N}$ is an arbitrary constant. For a better understanding and an illustrative explanation, see ??.

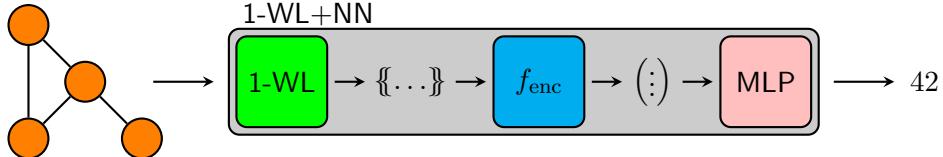


Figure 4: This simplified illustration explains the components that make up a 1-WL+NN model and how each one processes the input further. In detail, the model takes the graph on the left as input and first applies the 1-WL algorithm, thereby obtaining a multiset of the colors assigned by the algorithm. Then the encoding function f_{enc} is applied, resulting in a fixed-sized vector that is further processed by the multilayer perceptron MLP. The output of the MLP is then propagated as the 1-WL+NN models output, here the number 42.

It is worth noting that this definition can be easily adjusted to accommodate node or edge-related tasks by applying the encoding function f_{enc} and the multilayer perceptron MLP

elementwise to the colors of the multiset. However, for the purposes of this thesis, we will not delve into these variations, as our main focus will be on graph-wide tasks such as graph classification or regression, which possess greater theoretical interest and are more prevalent in most datasets. Furthermore, all the theoretical findings presented in this thesis can also be applied to 1-WL+NN models designed for node or edge tasks.

3.5 Graph Neural Networks (Message-Passing)

A Graph Neural Network (GNN) is a composition of multiple layers, where each layer computes a new feature for each node and edge. Each GNN layer thus technically obtains a new graph structurally identical to the previous one but contains new feature information. After an input graph has been passed through all layers, a final readout function is applied that pools all graph features and derives a task-related output. With this, it is possible to apply a GNN to every graph, regardless of its size, as the “computation” will only take place on the nodes and edges of the graph.

Note that in the following, we will restrict the definition only to consider node features; however, one can easily extend it to include edge features as well.

Definition 10 (Graph Neural Network). Let $G = (V, E, l)$ be an arbitrary graph. A GNN is a composition of multiple layers and a final readout function where each layer t is represented by a function $f^{(t)}$. The initial layer at $t = 0$ is a functioning of the format $f^{(0)} : V(G) \rightarrow \mathbb{R}^{1 \times d}$ that is consistent with l and translates all labels into a vector representation. In contrast, for every $t > 0$, $f^{(t)}$ is recursively defined as:

$$f^{(t)}(v) = f_{\text{merge}}^{(t)}(f^{(t-1)}(v), f_{\text{agg}}^{(t)}(\{f^{(t-1)}(w) \mid w \in \mathcal{N}(v)\})),$$

where $f_{\text{merge}}^{(t)}$ is an arbitrary function that maps the aforementioned tuple to a vector, effectively “merging” them, while $f_{\text{agg}}^{(t)}$ is an arbitrary function that maps the multiset to a vector, effectively “aggregating” it.

The readout function, referred to as **Readout**, is applied after the input graph has been passed subsequently through all layers and is defined as follows:

$$\text{Readout}(\{f^{(t)}(v) \mid v \in V(G)\}).$$

This function pools the information from every node feature, processes it, and calculates a fixed-sized output vector for the entire graph.

In summary, a GNN model will compute the function \mathcal{A} defined as follows:

$$\mathcal{A} : \mathcal{G} \rightarrow \mathbb{R}^k, G \mapsto \text{Readout}(\{f^{(T)}(v) \mid v \in V(G)\}),$$

where T is the number of layer of the GNN, and $k \in \mathbb{N}$ an arbitrary constant. To enable end-to-end training of a GNN, it is essential that all its components are differentiable. Therefore, we require all $f_{\text{merge}}^{(t)}$ and $f_{\text{agg}}^{(t)}$ functions for all $t \in [T]$, along with the final **Readout** function, to be differentiable.

Note that, due to our definition of the “aggregation” and the “readout” function to operate over multisets, both functions are permutation invariant by definition. With this, we can conclude that the total composition \mathcal{A} is permutation invariant, and with similar reasoning, it is also differentiable. This property enables us to train \mathcal{A} like any other machine learning method

in an end-to-end fashion, regardless of the underlying encoding used for graphs. Furthermore, GNNs following this definition are regarded as **Message-Passing-Neural-Network (MPNN)**. This designation stems from each node exchanging information with its direct neighbors in each layer. As a result, information during the processing of a graph is propagated by passing many messages across the graph; thus, these layers are also referred to as *message-passing* layers. As outlined in the introduction to this thesis, we will solely focus on GNNs utilizing the *message-passing* architecture. Therefore we will use the term **GNN** and **MPNN** interchangeably throughout this thesis. The definition and notation used here are inspired by ? and ?.

To bridge the gap from the theoretical definition to practical instances of the definition, we will now introduce three distinct GNN architectures. Specifically, we will explore the **Graph Attention Network (GAT)** developed by ?, **Graph Convolutional Network (GCN)** proposed by ?, and the **Graph Isomorphism Network (GIN)** introduced by ?. These architectures will serve as empirical baselines in ?. Additionally, we will also elaborate on the reasons for this choice of models in ?? in ?. We listed the definitions of the *message-passing* layers of each model in ??.

Table 1: Overview of the construction of the *message-passing* layers and their respective learnable parameters by popular GNN model. This format is inspired by ?.

Model	Message-Passing Layer Definition	Learnable Parameters
GAT	$f^{(t)}(v) = \sigma \left(\sum_{u \in \mathcal{N}(v)} \alpha_{vu} \cdot W^{(t)} \cdot f^{(t-1)}(u) \right)$ with $\alpha_{vu} = \frac{\exp(\text{LeakyReLU}(\vec{a}^T \cdot \text{concat}[W^{(t)} f^{(t-1)}(v), W^{(t)} f^{(t-1)}(u)]))}{\sum_{k \in \mathcal{N}(v)} \exp(\text{LeakyReLU}(\vec{a}^T \cdot \text{concat}[W^{(t)} f^{(t-1)}(v), W^{(t)} f^{(t-1)}(k)]))}$	$\vec{a}, W^{(t)}$
GCN	$f^{(t)}(v) = \text{ReLU} \left(\sum_{u \in \mathcal{N}(v) \cup \{v\}} \frac{W^{(t)}}{\sqrt{(1+d(v)) \cdot (1+d(u))}} f^{(t-1)}(u) \right)$	$W^{(t)}$
GIN	$f^{(t)}(v) = \text{MLP}^{(t)} \left((1 + \epsilon^{(t)}) \cdot f^{(t-1)}(v) + \sum_{u \in \mathcal{N}(v)} f^{(t-1)}(u) \right)$	$\epsilon^{(t)}, \text{MLP}^{(t)}$

Commonly employed **Readout** functions in this context often involve straightforward pooling techniques like elementwise summation, mean calculation, or maximum extraction. These pooling operations are typically followed by a multilayer perceptron, which performs additional processing on the aggregated information. Although more sophisticated pooling operations exist, such as **SET2SET** developed by ?, ? showed that given the correct configuration, the elementwise summation pooling function combined with a following multilayer perceptron suffices to create a GNN that is as expressive as the 1-WL algorithm in distinguishing non-isomorphism.

4 Theoretical Connection

This section is the main part of our theoretical investigation of the two frameworks 1-WL+NN and GNN. We will present three intriguing theorems, which will be proven separately afterward. In detail, the first two theorems will establish an equivalence between the two frameworks when the input set of graphs is finite. While the last theorem will take this a step further and demonstrate how powerful the 1-WL algorithm is by establishing a connection between 1-WL+NN and GNN for continuous functions.

In the first two theorems, we focus on a finite collection of graphs, which we denote by \mathcal{X} with $\mathcal{X} \subset \mathcal{G}$.

Theorem 11 (Finite Case: “GNN \subseteq 1-WL+NN”). Let \mathcal{C} be a collection of functions from \mathcal{X} to \mathbb{R} computable by GNNs, then \mathcal{C} is also computable by 1-WL+NN.

Theorem 12 (Finite Case: “1-WL+NN \subseteq GNN”). Let \mathcal{C} be a collection of functions from \mathcal{X} to \mathbb{R} computable by 1-WL+NN, then \mathcal{C} is also computable by GNNs.

With these two theorems, the equivalence between both frameworks follows. Specifically, every function computed by 1-WL+NN working over any arbitrary, but finite $\mathcal{X} \subset \mathcal{G}$ is also computable by a GNN, and vice versa. As we move towards the empirical evaluation in ??, it is evident that if we test a 1-WL+NN model on any of the benchmark datasets, it can achieve theoretically the same level of performance as a GNN model. Notice that we did not leverage any constraints on the encoding of graphs throughout the first two theorems and their corresponding proves but instead kept it general.

Having established a connection between the two frameworks on a finite subset of graphs, we wanted to further demonstrate the expressive power of the 1-WL algorithm and in particular of collections of functions that are 1-WL-Discriminating by investigating a connection between the two frameworks for continuous feature spaces and continuous functions. However, since the *graph isomorphism problem* is an inherently discrete problem, the 1-WL algorithm, as outlined in ??, is only defined as a discrete and discontinuous function operating on discrete colors, such that extending the definition of the 1-WL algorithm to a continuous function working over continuous values is not very trivial and, to our knowledge, has not yet been widely investigated. Therefore, we assume in the proof and the following theorem that such a continuous version of the 1-WL algorithm exists.

We define the set of graphs with continuous labels using the following definition.

Definition 13. Let X be a compact subset of \mathbb{R} including 0. We decode graphs with n nodes as a matrix $G \in X^{n \times n}$, where $G_{i,i}$ decodes the label of node i for $i \in [n]$, and $G_{i,j}$ with $i \neq j \in [n]$ decodes an edge from node i to j and a corresponding edge features. Furthermore, we say that there is an edge between node i and j if and only if $G_{i,j} \neq 0$. Additionally, if G encodes an undirected graph, G is a symmetric matrix. For simplicity, we denote $\mathcal{X} := X^{n \times n}$ throughout the next theorem.

Then the following theorem can be derived.

Theorem 14 (Continuous Case: “GNN \subseteq 1-WL+NN”). Let \mathcal{C} be a collection of continuous functions from \mathcal{X} to \mathbb{R} computable by 1-WL+NN. If \mathcal{C} is 1-WL-Discriminating, then there exists a collection of functions \mathcal{C}' computable by 1-WL+NN that is GNN-Approximating.

The notion of a collection of functions capable of approximating any GNN function is defined as follows.

Definition 15 (GNN-Approximating). Let \mathcal{C} be a collection of permutation invariant functions from \mathcal{X} to \mathbb{R} . We say \mathcal{C} is GNN-Approximating if for all permutation-invariant functions \mathcal{A} computed by a GNN, and for all $\epsilon \in \mathbb{R}$ with $\epsilon > 0$, there exists $h_{\mathcal{A},\epsilon} \in \mathcal{C}$ such that $\|\mathcal{A} - h_{\mathcal{A},\epsilon}\|_\infty := \sup_{G \in \mathcal{X}} |\mathcal{A}(G) - h_{\mathcal{A},\epsilon}(G)| < \epsilon$

Since we only wanted to show the expressive power of 1-WL-Discriminating and made the major assumption of the existence of a continuous 1-WL algorithm, we have included the proof of ?? in the Appendix in ??.

By putting all theorems into perspective, we can conclude that the ability 1-WL-Discriminating is very powerful, so we can assume that 1-WL+NN is sufficiently expressive for the upcoming empirical part.

4.1 Proof of ??: “GNN \subseteq 1-WL+NN”

We will prove ?? by introducing a couple of small lemmas, which combined prove the theorem. In detail, in ??, we show the existence of a collection computed by 1-WL+NN that is 1-WL-Discriminating. In ?????? we derive properties of 1-WL+NN functions we will use throughout ?????? with which we prove the theorem. We took great inspiration for ?????? from the proof presented in section 3.1 in the work of ?.

Lemma 16. There exists a collection \mathcal{C} of functions from \mathcal{X} to \mathbb{R} computable by 1-WL+NN that is 1-WL-Discriminating.

Proof. We define f_c for $c \in \mathbb{N}$ as the encoding function that returns the number of nodes colored as c . With this, we can construct the collection of functions C as follows:

$$C := \{\mathcal{B}_c : \mathcal{X} \rightarrow \mathbb{R}, G \mapsto \text{MLP}_{\text{id}} \circ f_c(\{\{1\text{-WL}(G)(v) \mid v \in V(G)\}\}) \mid c \in \mathbb{N}\},$$

where MLP_{id} is a dummy multilayer perceptron that returns its input. Since every function $\mathcal{B}_c \in C$ is composed of the 1-WL algorithm, an encoding function, and a multilayer perceptron, each function is computable by 1-WL+NN, and consequently also the whole collection.

Proof of correctness: Let $G_1, G_2 \in \mathcal{X}$ with $G_1 \not\simeq_{1\text{-WL}} G_2$. Further, let C_1, C_2 be the final colorings computed by the 1-WL algorithm when applied on G_1, G_2 respectively. Due to $G_1 \not\simeq_{1\text{-WL}} G_2$, there exists a color $c \in \mathbb{N}$ such that $h_{G_1, C_1}(c) \neq h_{G_2, C_2}(c)$. Such that $\mathcal{B}_c \in C$ exists with $\mathcal{B}_c(G_1) \neq \mathcal{B}_c(G_2)$. \square

Lemma 17 (1-WL+NN Equivalence). Let \mathcal{B} be a function over \mathcal{X} computable by 1-WL+NN, then for every pair of graphs $G_1, G_2 \in \mathcal{X}$: if $G_1 \simeq_{1\text{-WL}} G_2$ than $\mathcal{B}(G_1) = \mathcal{B}(G_2)$.

Proof. Let \mathcal{B} be an arbitrary function over \mathcal{X} computable by 1-WL+NN, then \mathcal{B} is composed as follows: $\mathcal{B}(\cdot) = \text{MLP} \circ f_{\text{enc}}\{\{1\text{-WL}(\cdot)(v) \mid v \in V(\cdot)\}\}$. Further, let $G_1, G_2 \in \mathcal{X}$ be arbitrary graphs with $G_1 \simeq_{1\text{-WL}} G_2$, then by definition of the relation $\simeq_{1\text{-WL}}$ we know that $1\text{-WL}(G_1) = 1\text{-WL}(G_2)$. With this, the equivalence follows immediately. \square

Lemma 18 (1-WL+NN Permutation Invariance). Let \mathcal{B} be a function over \mathcal{X} computable by 1-WL+NN, then \mathcal{B} is permutation invariant.

Proof. Let $G_1, G_2 \in \mathcal{X}$ be arbitrary graphs with $G_1 \simeq G_2$ and \mathcal{B} an arbitrary function computable by 1-WL+NN. Since the 1-WL algorithm is sound, we know that $G_1 \simeq G_2$ implies $G_1 \simeq_{1\text{-WL}} G_2$. Using ??, we can therefore conclude that: $\mathcal{B}(G_1) = \mathcal{B}(G_2)$. \square

Lemma 19 (1-WL+NN Composition). Let \mathcal{C} be a collection of functions computable by 1-WL+NN. Further, let $h_1, \dots, h_n \in \mathcal{C}$ and MLP^* an multilayer perceptron, than the function \mathcal{B} composed of $\mathcal{B}(\cdot) := \text{MLP}^*(h_1(\cdot), \dots, h_n(\cdot))$ is also computable by 1-WL+NN.

Proof Sketch. Assume the above and let f_1, \dots, f_n be the encoding functions, as well as $\text{MLP}_1, \dots, \text{MLP}_n$ be the multilayer perceptrons used by h_1, \dots, h_n respectively. The idea of this proof is that we construct an encoding function f^* that “duplicates” its input and applies each encoding function f_i individually. We also construct a multilayer perceptron MLP^* that takes in the output of f^* and simulates all $\text{MLP}_1, \dots, \text{MLP}_n$ simultaneously. Afterward, the given MLP^* will be applied on the concatenation of the output of all MLP_i ’s. See ?? for a sketch of the proof idea. For the complete proof, please refer to the Appendix in ??.

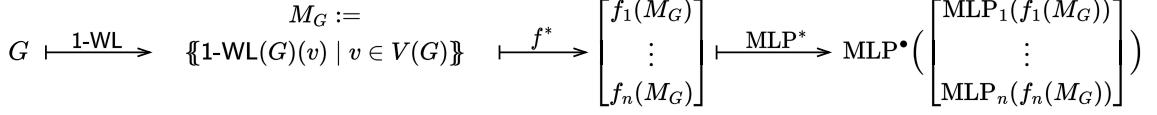


Figure 5: The proof idea for ??, how the constructed functions f^* and MLP^* will work on input $G \in \mathcal{X}$. Here we denote with M_G the multiset of colors of the nodes of G after applying the 1-WL algorithm.

Lemma 20. Let \mathcal{C} be a collection of functions from \mathcal{X} to \mathbb{R} computable by 1-WL+NN that is 1-WL-Discriminating. Then for all $G^* \in \mathcal{X}$, there exists a function h_{G^*} from \mathcal{X} to \mathbb{R} computable by 1-WL+NN, such that for all $G \in \mathcal{X}$: $h_{G^*}(G) = 0$, if and only if, $G \simeq_{1\text{WL}} G^*$.

Proof. Assume the above. For any $G_1, G_2 \in \mathcal{X}$ with $G_1 \not\simeq_{1\text{WL}} G_2$, let $h_{G_1, G_2} \in \mathcal{C}$ be the function distinguishing them, with $h_{G_1, G_2}(G_1) \neq h_{G_1, G_2}(G_2)$. We define the function \bar{h}_{G_1, G_2} working over \mathcal{X} as follows:

$$\begin{aligned} \bar{h}_{G_1, G_2}(\cdot) &= |h_{G_1, G_2}(\cdot) - h_{G_1, G_2}(G_1)| \\ &= \max(h_{G_1, G_2}(\cdot) - h_{G_1, G_2}(G_1), 0) + \max(h_{G_1, G_2}(G_1) - h_{G_1, G_2}(\cdot), 0) \\ &= \text{ReLU}(h_{G_1, G_2}(\cdot) - h_{G_1, G_2}(G_1)) + \text{ReLU}(h_{G_1, G_2}(G_1) - h_{G_1, G_2}(\cdot)) \end{aligned} \quad (0.1)$$

Note, that in the equations above “ $h_{G_1, G_2}(G_1)$ ” is a fixed constant and the resulting function \bar{h}_{G_1, G_2} is non-negative. Let $G_1 \in \mathcal{X}$ now be fixed, we will construct the function h_{G_1} with the desired properties as follows:

$$h_{G_1}(\cdot) = \sum_{G_2 \in \mathcal{X}, G_1 \not\simeq_{1\text{WL}} G_2} \bar{h}_{G_1, G_2}(\cdot). \quad (0.2)$$

Since \mathcal{X} is finite, the sum is finite and therefore well-defined. Next, we will prove that for a fixed graph $G_1 \in \mathcal{X}$, the function h_{G_1} is correct on input $G \in \mathcal{X}$:

1. If $G_1 \simeq_{1\text{WL}} G$, then for every function \bar{h}_{G_1, G_2} of the sum with $G_1 \not\simeq_{1\text{WL}} G_2$, we know, using ??, that $\bar{h}_{G_1, G_2}(G)$ is equal to $\bar{h}_{G_1, G_2}(G_1)$ which is by definition 0, such that $h_{G_1}(G) = 0$.
2. If $G_1 \not\simeq_{1\text{WL}} G$, then $\bar{h}_{G_1, G}(G)$ is a summand of the overall sum, and since $\bar{h}_{G_1, G}(G) > 0$, we can conclude $h_{G_1}(G) > 0$ due to the non-negativity of each function \bar{h}_{G_1, G_2} .

Using ??, we can conclude that for any $G \in \mathcal{X}$, h_G is computable by 1-WL+NN, as we can encode ?? via a multilayer perceptron where the factor “ $h_{G_1, G_2}(G_1)$ ” of ?? is just a constant. \square

Lemma 21. Let \mathcal{C} be a collection of functions from \mathcal{X} to \mathbb{R} computable by 1-WL+NN so that for all $G^* \in \mathcal{X}$, there exists $h_{G^*} \in \mathcal{C}$ satisfying $h_{G^*}(G) = 0$ if and only if $G \simeq_{1\text{WL}} G^*$, for all $G \in \mathcal{X}$. Then for every $G^* \in \mathcal{X}$, there exists a function φ_{G^*} computable by 1-WL+NN such that for all $G \in \mathcal{X}$: $\varphi_{G^*}(G) = \mathbf{1}_{G \simeq_{1\text{WL}} G^*}$.

Proof. Assuming the above. Due to \mathcal{X} being finite, we can define for every graph G^* the constant:

$$\delta_{G^*} := \frac{1}{2} \min_{G \in \mathcal{X}, G \not\simeq_{1\text{WL}} G^*} |h_{G^*}(G)| > 0.$$

With this constant, we can use a so-called ‘‘bump’’ function working from \mathbb{R} to \mathbb{R} that is similar to the indicator function. We define this function for parameter $a \in \mathbb{R}$ with $a > 0$ as:

$$\begin{aligned}\psi_a(x) &:= \max\left(\frac{x}{a} - 1, 0\right) + \max\left(\frac{x}{a} + 1, 0\right) - 2 \cdot \max\left(\frac{x}{a}, 0\right) \\ &= \text{ReLU}\left(\frac{x}{a} - 1\right) + \text{ReLU}\left(\frac{x}{a} + 1\right) - 2 \cdot \text{ReLU}\left(\frac{x}{a}\right)\end{aligned}\quad (0.3)$$

The interesting property of ψ_a is that it maps every value x to 0, except when x is being drawn from the interval $(-a, a)$. In particular, it maps x to 1 if and only if x is equal to 0. See ?? for a plot of the relevant part of this function with exemplary values for a .

We use these properties to define for every graph $G^* \in \mathcal{X}$ the function $\varphi_{G^*}(\cdot) := \psi_{\delta_{G^*}}(h_{G^*}(\cdot))$. We will quickly demonstrate that this function is equal to the indicator function, for this let G^* be fixed and G , an arbitrary graph from \mathcal{X} , the input:

1. If $G \simeq_{1\text{-WL}} G^*$, then $h_{G^*}(G) = 0$ resulting in $\varphi_{G^*}(G) = \psi_{\delta_{G^*}}(0) = 1$.
2. If $G \not\simeq_{1\text{-WL}} G^*$ then $h_{G^*}(G) \neq 0$, such that $|h_{G^*}(G)| > \delta_{G^*}$ so that $h_{G^*}(G) \notin (-\delta_{G^*}, \delta_{G^*})$ resulting in $\varphi_{G^*}(G) = 0$.

Note that we can encode φ_{G^*} using ?? via a multilayer perceptron, where δ_{G^*} is a constant. With ?? we can therefore conclude that φ_{G^*} is computable by 1-WL+NN for every graph $G^* \in \mathcal{X}$. \square

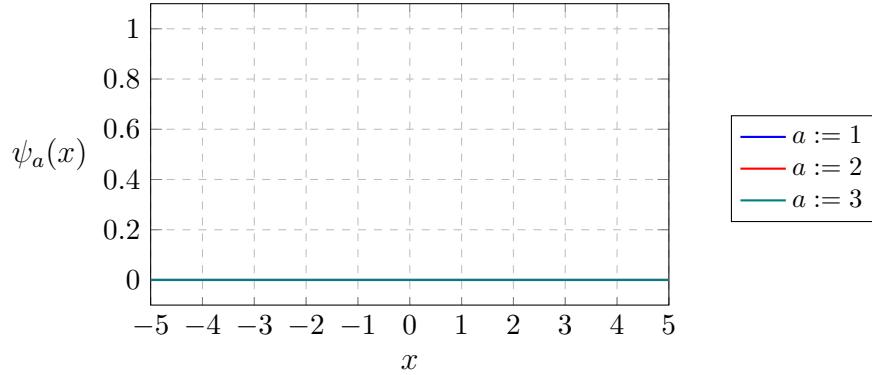


Figure 6: Illustration of the so-called ‘‘bump’’ function $\psi_a(x)$ used in the proof of ?? with different exemplary values for a .

Lemma 22. Let \mathcal{C} be a collection of functions from \mathcal{X} to \mathbb{R} computable by 1-WL+NN such that for all $G^* \in \mathcal{X}$, there exists $\varphi_{G^*} \in \mathcal{C}$ satisfying $\forall G \in \mathcal{X} : \varphi_{G^*}(G) = \mathbf{1}_{G \simeq_{1\text{-WL}} G^*}$. Then every function computable by a GNN is also computable by 1-WL+NN.

Proof. Assume the above. For any function \mathcal{A} computed by a GNN that works over \mathcal{X} to \mathbb{R} , we

show that it can be decomposed as follows for any $G \in \mathcal{X}$ as input:

$$\begin{aligned}
\mathcal{A}(G) &= \left(\frac{1}{|\mathcal{X}/\simeq_{1WL}(G)|} \sum_{G^* \in \mathcal{X}} \mathbb{1}_{G \simeq_{1WL} G^*} \right) \cdot \mathcal{A}(G) \\
&= \frac{1}{|\mathcal{X}/\simeq_{1WL}(G)|} \sum_{G^* \in \mathcal{X}} \mathcal{A}(G^*) \cdot \mathbb{1}_{G \simeq_{1WL} G^*} \\
&= \sum_{G^* \in \mathcal{X}} \frac{\mathcal{A}(G^*)}{|\mathcal{X}/\simeq_{1WL}(G^*)|} \cdot \varphi_{G^*}(G)
\end{aligned} \tag{0.4}$$

where we denote with $\mathcal{X}/\simeq_{1WL}(G^*)$ the set of all graphs G over \mathcal{X} that are equivalent to G^* according to the \simeq_{1WL} relation.

Since \mathcal{A} is permutation-invariant and GNNs are, at most, as good as the 1-WL algorithm in distinguishing non-isomorphic graphs, we can use the fact that for every pair of graphs $G, H \in \mathcal{X}$ with $G \simeq_{1WL} H$: $\mathcal{A}(G) = \mathcal{A}(H)$. Therefore, we can decompose \mathcal{A} as indicated in ?? and encode it using a multilayer perceptron where $\frac{\mathcal{A}(G^*)}{|\mathcal{X}/\simeq_{1WL}(G^*)|}$ is a constant, and $\varphi_{G^*} \in \mathcal{C}$ encodes the indicator function. Combined with the ??, we can conclude that \mathcal{A} is computable by 1-WL+NN. Important to note, we can only do this since \mathcal{X} is finite, making the overall sum finite and the cardinality of $\mathcal{X}/\simeq_{1WL}(G^*)$ well-defined for all graphs. \square

4.2 Proof of ??: “1-WL+NN \subseteq GNN”

In this section we will prove the converse direction. We start with ??, where we introduce an upper bound that we will use in ?? to show that there exists a collection of GNN-computable functions that is 1-WL-Discriminating. After that, we will prove a composition lemma with ?? which is similar to the one we introduced in the previous section. From this point on, the proof continues as in the previous section and concludes the property to be proved in ??.

Lemma 23. Let G be an arbitrary graph with $n := |V(G)|$ the number of nodes and $C : V(G) \rightarrow \mathbb{N}$ an arbitrary coloring of the graph G . Then the total number of possible tuples of the form:

$$(C(v), \{\{C(u) \mid u \in \mathcal{N}(v)\}\}),$$

for all $v \in V(G)$ can be upper bounded by:

$$n \cdot \sum_{i=0}^{n-1} \binom{n+i-1}{i}.$$

Proof. Assume the above. For the first entry of the tuple, at most n different colors exist since there are n nodes. For the second entry, each node $v \in V(G)$ can have between 0 and $n-1$ neighbors, such that the total number of possibilities is the sum over each cardinality of a multiset with n different colors. In the end, we soundly combine both results by multiplying both together. \square

Lemma 24 (GNN 1-WL-Discriminating). There exists a collection \mathcal{C} of functions from \mathcal{X} to \mathbb{R} computable by GNNs that is 1-WL-Discriminating. Meaning for every $G_1, G_2 \in \mathcal{X}$ with $G_1 \not\simeq_{1WL} G_2$ there exists $\mathcal{A} \in \mathcal{C}$ such that $\mathcal{A}(G_1) \neq \mathcal{A}(G_2)$.

Proof. Since \mathcal{X} is finite, we define $n := \max\{|V(G)| \mid G \in \mathcal{X}\}$ to be the maximum number of nodes of a graph in \mathcal{X} , and $k := \max\{l_G(v) \mid v \in V(G), G \in \mathcal{X}\}$ to be the largest label of any node of a graph in \mathcal{X} . Using ??, we can compute an upper bound m using n for the number of distinct tuples. Note that, this bound holds true for all graphs in \mathcal{X} . We will now construct a GNN with n layers working on input G as follows:

$$\begin{aligned} f^{(0)}(v) &:= l_G(v), \text{ and} \\ f^{(t)}(v) &:= \text{RELABEL}_{m,t}(f^{(t-1)}(v), \{\{f^{(t-1)}(u) \mid u \in \mathcal{N}(v)\}\}), \quad 0 < t < n. \end{aligned}$$

Here $\text{RELABEL}_{m,t}$ is a function, parametrized by m and t , that maps the tuples injectively to an integer of the set:

$$\{i \in \mathbb{N} \mid k + (t-1) \cdot m + 1 \leq i \leq k + t \cdot m\}.$$

This function exists as by the soundness of the upper bound of ??, the cardinality of its co-domain is greater or equal than the one of its domain. Thereby, and with the injectiveness of $\text{RELABEL}_{m,t}$, we ensure that each GNN layer maps a tuple to a new, previously unused color. Therefore, every layer of this GNN computes a single iteration of the 1-WL algorithm. Further, since the 1-WL algorithm converges after at most $|V(G)| \leq n$ iterations, we set the number of layers to n , such that we ensure that the coloring computed by this GNN after n layers when applied on any graph $G \in \mathcal{X}$ is similarly expressive as the coloring computed by the 1-WL algorithm when applied on G .

We define the collection C of functions computable by GNNs that is 1-WL-Discriminating as:

$$C := \{\mathcal{A} : \mathcal{X} \rightarrow \mathbb{R}, G \mapsto \text{Readout}_c(\{f^{(n)}(v) \mid v \in V(G)\}) \mid c \in \mathbb{N}\},$$

where Readout_c is the Readout function that returns the number of nodes colored as c in the coloring of $f^{(n)}$. \square

Similar to the proof in the previous section, we will use ?? to introduce the ability to construct GNNs that take in as input multiple GNNs and then apply a multilayer perceptron to the combined output.

Lemma 25 (GNN Composition). Let C be a collection of functions computable by GNNs. Further, let $\mathcal{A}_1, \dots, \mathcal{A}_n \in C$ and MLP^\bullet a suitable multilayer perceptron, then the function $\hat{\mathcal{A}}(\cdot) := \text{MLP}(\mathcal{A}_1(\cdot), \dots, \mathcal{A}_n(\cdot))$ is also computable by a GNN.

Proof. Before we begin the proof, we briefly introduce two notations. For any $x \in \mathbb{R}^d$, we will use the notation $x[i]$ to indicate the i .th element of the vector x . Additionally, we indicate the merge and aggregation function used in layer t by \mathcal{A}_i as $f_{\text{merge},i}^{(t)}$ and $f_{\text{agg},i}^{(t)}$. Similarly, does Readout_i indicate the Readout function and $f_i^{(0)}$ the input function of \mathcal{A}_i .

We will prove the lemma by giving a construction of a GNN model computing $\hat{\mathcal{A}}$. For the ease of readability and to reduce the complexity of the subsequent construction, we assume that for all \mathcal{A}_i its functions $f_{\text{merge},i}^{(t)}, f_{\text{agg},i}^{(t)}$ and Readout_i map into the one-dimensional space \mathbb{R} for all layers t . With this assumption, we avoid the need for a formal notation of the number of dimensions each of these functions map to.

Let T be the maximum number of layers of all $\mathcal{A}_1, \dots, \mathcal{A}_n$. We construct the GNN $\hat{\mathcal{A}}$ with T layers, with the input layer working as follows on an input graph G :

$$\forall v \in V(G) : \hat{f}^{(0)}(v) := \begin{bmatrix} f_1^{(0)}(v) \\ \vdots \\ f_n^{(0)}(v) \end{bmatrix},$$

and each other layer $0 < t \leq T$ utilizing the merge $\hat{f}_{\text{merge}}^{(t)}$ and aggregation $\hat{f}_{\text{agg}}^{(t)}$ functions as constructed in the following:

$$\begin{aligned} \hat{f}_{\text{merge}}^{(t)}(\hat{f}^{(t-1)}(v), \text{Agg}) &:= \begin{bmatrix} f_{\text{merge},1}^{(t)}(\hat{f}^{(t-1)}(v)[1], \text{Agg}[1]) \\ \vdots \\ f_{\text{merge},n}^{(t)}(\hat{f}^{(t-1)}(v)[n], \text{Agg}[n]) \end{bmatrix}, \quad \text{and} \\ \hat{f}_{\text{agg}}^{(t)}(\{\!\{ \hat{f}^{(t-1)}(w) \mid w \in \mathcal{N}(v) \}\!}) &:= \begin{bmatrix} f_{\text{agg},1}^{(t)}(\{\!\{ \hat{f}^{(t-1)}(w)[1] \mid w \in \mathcal{N}(v) \}\!}) \\ \vdots \\ f_{\text{agg},n}^{(t)}(\{\!\{ \hat{f}^{(t-1)}(w)[n] \mid w \in \mathcal{N}(v) \}\!}) \end{bmatrix}. \end{aligned}$$

Note that, not all \mathcal{A}_i will be comprised of T layers. For these cases we define the missing functions as follows:

$$\begin{aligned} f_{\text{merge},i}^{(t)}(\hat{f}^{(t-1)}(v), \text{Agg}) &:= \hat{f}^{(t-1)}(v), \quad \text{and} \\ f_{\text{agg},i}^{(t)}(\{\!\{ \hat{f}^{(t-1)}(w) \mid w \in \mathcal{N}(v) \}\!}) &:= 0. \end{aligned}$$

These functions do not change anything and only forward the result of the actual computation of \mathcal{A}_i to the last layer. Finally, we construct the Readout function of $\hat{\mathcal{A}}$ as follows:

$$\text{Readout}(\{\!\{ \hat{f}^{(T)}(v) \mid v \in V(G) \}\!}) := \text{MLP}^\bullet \circ \begin{bmatrix} \text{Readout}_1(\{\!\{ \hat{f}^{(T)}(v)[1] \mid v \in V(G) \}\!}) \\ \vdots \\ \text{Readout}_n(\{\!\{ \hat{f}^{(T)}(v)[n] \mid v \in V(G) \}\!}) \end{bmatrix}.$$

With this, the proof concludes. Note that this proof can easily be extended to work without the assumption of each function mapping into a one-dimensional space. \square

As a consequence of the previous two lemmas, we find ourselves in a similar position as at the beginning of the proof in ???. Specifically, we have established, through ???, the existence of a collection C of functions that can be computed by GNNs and can effectively distinguish any pair of graphs that are also distinguishable by the 1-WL algorithm. Furthermore, with ??, we have demonstrated that the composition of multiple GNNs and a multilayer perceptron remains computable by a single GNN. Consequently, we can also apply the findings of ???? to GNNs. Thus, we can conclude that for any fixed $G^* \in \mathcal{X}$, the indicator function φ_{G^*} working over \mathcal{X} with:

$$\forall G \in \mathcal{X} : \varphi_{G^*}(G) := \begin{cases} 1, & \text{if } G \simeq_{1\text{-WL}} G^* \\ 0, & \text{else} \end{cases},$$

is computable by a GNN.

Lemma 26. Let \mathcal{C} be a collection of functions from \mathcal{X} to \mathbb{R} computable by GNNs so that for all $G^* \in \mathcal{X}$, there exists $\varphi_{G^*} \in \mathcal{C}$ satisfying $\forall G \in \mathcal{X} : \varphi_{G^*}(G) = \mathbb{1}_{G \simeq_{1WL} G^*}$. Then every function computable by 1-WL+NN is also computable by a GNN.

Proof. Assume the above. For any function \mathcal{B} computed by 1-WL+NN that works over \mathcal{X} to \mathbb{R} , we show that it can be decomposed as follows for any $G \in \mathcal{X}$ as input:

$$\begin{aligned}\mathcal{B}(G) &= \left(\frac{1}{|\mathcal{X}/\simeq_{1WL}(G)|} \sum_{G^* \in \mathcal{X}} \mathbb{1}_{G \simeq_{1WL} G^*} \right) \cdot \mathcal{B}(G) \\ &= \frac{1}{|\mathcal{X}/\simeq_{1WL}(G)|} \sum_{G^* \in \mathcal{X}} \mathcal{B}(G^*) \cdot \mathbb{1}_{G \simeq_{1WL} G^*} \\ &= \sum_{G^* \in \mathcal{X}} \frac{\mathcal{B}(G^*)}{|\mathcal{X}/\simeq_{1WL}(G^*)|} \cdot \varphi_{G^*}(G)\end{aligned}\tag{0.5}$$

where we denote with $\mathcal{X}/\simeq_{1WL}(G^*)$ the set of all graphs G over \mathcal{X} that are equivalent to G^* according to the \simeq_{1WL} relation. Further, with ?? we know that for any $G_1, G_2 \in \mathcal{X}$ with $G_1 \simeq_{1WL} G_2 : \mathcal{B}(G_1) = \mathcal{B}(G_2)$.

We can encode \mathcal{B} as stated in ?? via a multilayer perceptron with $\frac{\mathcal{B}(G^*)}{|\mathcal{X}/\simeq_{1WL}(G^*)|}$ being constants and $\varphi_{G^*} \in \mathcal{C}$ encoding the indicator function. Combined with the ??, we can conclude that \mathcal{B} is computable by a GNN. Important to note, we can only do this since \mathcal{X} is finite, making the overall sum finite and the cardinality of $\mathcal{X}/\simeq_{1WL}(G^*)$ well-defined for all graphs. \square

Part II

Empirical Testing

The empirical part of this thesis is structured into three sections. Firstly, we delve into the methodology used for conducting our experiments, covering the selection of models and datasets, the testing procedure, and the choice of hyperparameters. Subsequently, ?? will comprehensively explore the results obtained from the testing and examine various properties of both model types in detail. Finally, ?? concludes this thesis with a discussion that consolidates all insights, highlights any identified issues, and provides recommendations for future research directions.

5 Testing Configuration

This section delves into the configuration and setup of our empirical testing. We begin by presenting our carefully selected datasets, which serve as the foundation for our evaluation. We highlight specific insights and characteristics of these datasets that make them compelling choices for our study. Afterward, we focus on the models we employ for testing and subsequent result comparison. We provide a comprehensive overview of the selected models, outlining their key features and motivations behind their inclusion in our evaluation. Subsequently, we provide a detailed description of the training pipeline and explain specific hyperparameters for which we will optimize these models.

5.1 Datasets

We will first explain our choice of datasets and introduce each dataset individually. Afterward, we will explore essential observations that are to be considered when assessing the actual empirical results.

Choice of Datasets

In selecting the datasets for our thesis, we adhered to two fundamental principles to ensure the robustness and diversity of our evaluation for GNN and 1-WL+NN models. The first principle focuses on using widely recognized benchmark datasets that have been extensively employed in previous studies. This principle enables us and readers to make meaningful comparisons with existing results. The second principle focuses on choosing datasets that are distinct from one another in terms of both their application domains and the way they encode information in graphs.

To fulfill the first principle, we opted for datasets from the TUDATASET LIBRARY. This library, curated by ?, serves as a widely recognized standard for evaluating graph-related methods.

Regarding the second principle, we incorporated the insights from ?, who developed a comprehensive taxonomy for common graph benchmark datasets. Their work examined the degree to which information is encoded in graph structures compared to node features with respect to solving the task of the datasets. Based on their taxonomy, they categorized datasets into three distinct classes:

1. Datasets in which the most crucial information for solving the task is contained in the node features.
2. Datasets similar to the first category, but with the significant exception that the node degree strongly correlates with the node features. In these datasets, utilizing simple

structural information, such as computing the node degree, is as beneficial for solving the task as using the original node features.

3. Datasets where the most crucial information is encoded within the graph structure itself.

These categories help us understand how information is encoded in various datasets, such that we aim to choose datasets from all three categories.

As a result of considering these two principles, we selected the following datasets for our thesis: ENZYMES, IMDB-BINARY, MUTAG, NCI1, PROTEINS, and REDDIT-BINARY for classification tasks, and ALCHEMY and ZINC for regression tasks. For an overview of the elemental properties of each dataset, see ???. We will now shortly introduce each dataset individually:

ENZYMES, provided by ?, is a dataset consisting of proteins in their tertiary structure, categorized into six distinct enzyme classes. Each node represents a secondary structure, and has an edge to its three spatially closest nodes. Furthermore, each node feature encodes the type of secondary structure (*helix*, *sheet* or *turn*), as well as physical and chemical information.

IMDB-BINARY, provided by ?, is a dataset comprising ego networks. Each node in the network represents an actor/actress, and a unidirectional edge exists between two nodes if and only if the corresponding actors played together in a movie. The task involves determining whether each ego network's genre is *action* or *romance*.

MUTAG, provided by ?, is a dataset comprising Nitroaromatic compounds. Each compound is represented by a graph in which nodes represent atoms, with their types encoded as node features, and edges represent atomic bonds. The task involves determining whether a given compound has a mutagenic effect on *Salmonella typhimurium* bacteria.

NCI1, provided by ?, comprises graph representations of chemical compounds. In these graphs, nodes represent atoms, and edges represent atomic bonds. Moreover, the atom types are encoded in the node features. The overall task involves determining whether a given compound is active or inactive in inhibiting non-small cell lung cancer.

PROTEINS, provided by ?, contains proteins encoded similarly to ENZYMES. The task here is to determine whether each graph represents an enzyme.

REDDIT-BINARY, provided by ?, involves graphs that are derived from popular Reddit communities. The nodes in these graphs represent users who are active in the community, while the edges represent interactions between the users. The task is to identify whether a graph belongs to a community that is focused on questions and answers, or one that is focused on discussions.

ALCHEMY, provided by ?, consists of organic molecules, with each node representing an atom and each edge representing an atomic bond. Additionally, each node feature encodes various properties for each atom, while each edge encodes the bond type and distance between atoms. The overall task is to compute 12 different continuous quantum mechanical properties for each graph.

ZINC, provided by ? and ?, consists of molecular graphs where each node represents a heavy

atom, and the corresponding node feature specifies its type. The edges in the graph encode the bonds between atoms, and their features further describe the type of bond. The task is to calculate a molecular property known as constrained solubility ($\log P - \text{SA} - \text{cycle}$).

Table 2: Dataset statistics and properties for graph-level prediction tasks. This table has been adapted from ?.

Dataset	Properties						
	Number of graphs	Number of classes/targets	\varnothing Number of nodes	\varnothing Number of edges	Node labels	Edge labels	Taxonomy Category
Classification	ENZYMEs	600	6	32.6	62.1	✓	✗
	IMDB-BINARY	1 000	2	19.8	96.5	✗	✗
	MUTAG	188	2	17.9	19.8	✓	✗
	NCI1	4 110	2	29.9	32.3	✓	✗
	PROTEINS	1 113	2	39.1	72.8	✓	✗
	REDDIT-BINARY	2 000	2	429.6	497.8	✗	✗
Reg.	ALCHEMY	202 579	12	10.1	10.4	✓	✓
	ZINC	249 456	1	23.1	24.9	✓	✓

Analysis of the Datasets

To ensure the reliability and fairness of our evaluation, one of our initial steps was to assess the balance of our selected datasets for classification. We employed the **Normalized Shannon-Index** to evaluate the balance of a dataset, where a value close to 0 indicates maximum imbalance, while a value close to 1 signifies perfect balance. See ?? in the Appendix for a formal definition of this metric.

Table 3: An overview of the **Normalized Shannon-Index** calculated for each dataset.

	Dataset					
	ENZYMEs	IMDB-MULTI	MUTAG	NCI1	PROTEINS	REDDIT-BINARY
Normalized Shannon-Index	1	1	0.920	1	0.973	1

?? provides an overview of the **Normalized Shannon-Index** computed for each selected classification dataset. Upon analyzing the data, we observed that all the datasets exhibit high balance, as all their values are close to 1. This finding assures us that the datasets do not suffer significant class imbalances, which will remain important in the following section.

In addition to the balance of all datasets, another important aspect is understanding the upper bound of performance achievable by both GNN and 1-WL+NN models on these datasets. Unlike in other areas of machine learning where multilayer perceptron models can achieve near-perfect performance due to their universal approximation capabilities (?), GNN and 1-WL+NN models have inherent limitations on their expressiveness. This restriction stems from the fact that the expressiveness of the 1-WL algorithm limits the performance of both frameworks. In detail, if the 1-WL algorithm can not distinguish a pair of graphs in a dataset, then neither a GNN nor a 1-WL+NN model can.

While we have pointed out that the 1-WL algorithm is, in general, quite powerful in distinguishing pairs of graphs in ??; we also explained that it is not complete. Therefore, assuming that GNN or 1-WL+NN models exist that can achieve almost perfect accuracy on all classification datasets is not reasonable. Thus, we calculated the theoretical maximum accuracy achievable by a perfect model for each dataset. In detail, we even investigated how many iterations of the 1-WL algorithm it takes to achieve this accuracy. For a comprehensive overview of the accuracies achievable on all datasets, please refer to ?. In this table, we have included the accuracy achievable when running the 1-WL algorithm for 0 iterations, which essentially means taking the initial node features as the coloring for iteration 0 and assessing the expressiveness of this initial coloring.

Table 4: An overview of the maximum theoretical classification accuracy achievable for each dataset based on the number of 1-WL iterations in percent. A hyphen “-” indicates that the maximum accuracy has converged with fewer iterations, implying that further iterations do not improve the accuracy.

1-WL	Dataset					
	ENZYMES	IMDB-BINARY	MUTAG	NCI1	PROTEINS	REDDIT-BINARY
Iterations: 0	81.4	60.6	93.1	91.3	91.9	83.9
Iterations: 1	100.0	88.6	95.7	99.5	99.7	100.0
Iterations: 2	-	-	99.5	99.8	-	-
Iterations: 3	-	-	100.0	99.8	-	-
Iterations: 4	-	-	-	-	-	-
Max Accuracy	100.0	88.6	100.0	99.8	99.7	100.0

Upon examining the results, we observe that all datasets exhibit perfect or near-perfect theoretical classification accuracies. This result makes interpreting results obtained from actual 1-WL+NN or GNN models later on more straightforward. Additionally, the accuracy achievable after each number of iterations provides a theoretical lower limit on the number of message-passing layers a GNN must be composed of to be even capable of achieving this accuracy.

We have conducted the same analyses on additional datasets, as their results might be valuable to the reader. For these, see ?? in the Appendix.

5.2 Choice of Models

In selecting our models, we aimed to use techniques that are relatively generic and not highly specialized for any of the datasets, such that insights we gain upon analyzing the model’s performance can be generalized better. Consequently, we opted for a basic set of models to maintain simplicity and versatility.

1-WL+NN Models

Our primary consideration for the 1-WL+NN models revolves around choosing an encoding function, as all other components are fixed. To keep things straightforward, we decided to employ encoding functions consisting of two main components. Firstly, an optional preprocessing step that operates on the colors computed by the 1-WL algorithm. Secondly, a basic pooling function for transforming the color histogram into a fixed-size vector.

In more detail, the optional preprocessing step involves a simple look-up table. If utilized, this step maps each color injectively to a vector in the range of $[-1, 1]^n$, where the value of n is a hyperparameter. By encoding the color information into higher dimensions, this approach aims to enhance efficiency during subsequent processing steps. For the second step, the pooling component, we selected elementary functions such as elementwise Max, Mean, and Summation (Sum). In summary, each of the 1-WL+NN models can be uniquely identified by their encoding functions; therefore, we will refer to each model by their encoding function as follows:

$$\text{Embedding} - \{\text{Max, Mean, Sum}\} \quad \text{or} \quad \{\text{Max, Mean, Sum}\},$$

where “Embedding” indicates the use of the optional preprocessing step.

GNN Models

As mentioned in the introduction to this section, our focus is on keeping the models basic. Hence, we opted for GIN by ?, GCN by ?, and GAT by ? as the base architecture for the message-passing layers. Each of these architectures was chosen for specific reasons. Further, we formally defined each of these architectures in ?? in ??.

Firstly, we included GIN as an obvious candidate because it has been proven to be as expressive as the 1-WL algorithm. This characteristic makes it interesting when comparing it to 1-WL+NN models. Next, we also included GCN based on its good empirical success in recent years. Furthermore, the insights provided by ? demonstrated that, in a certain sense, the node features computed by GCN and GIN are similar, making GCN a reasonable alternative to GIN in cases where the advantage of GIN’s expressiveness is not needed. Lastly, we added GAT as an alternative approach to the other two architectures. In detail, ? also showed that the node features computed by GAT are entirely different from those computed by GIN or GCN.

Regarding the choice of Readout functions, we decided to utilize functions that are composed of two components. The first component is one of the elementwise pooling functions, such as Max, Mean, and Sum, to aggregate the information from a graph representation into a fixed-sized vector. Secondly, we incorporate a multilayer perceptron to process the aggregated information further, similarly to the 1-WL+NN models.

In summary, each of the GNN models can be uniquely identified by their GNN architecture and the pooling function utilized; therefore, we will refer to each model by these two properties as follows:

$$\{\text{GAT, GCN, GIN}\} - \{\text{Max, Mean, Sum}\}.$$

Note that the selection of the Readout function creates similarities between the GNN and 1-WL+NN models. These similarities play a crucial role in ensuring that any empirical differences observed between the two frameworks are not attributed to the utilization of more powerful techniques.

5.3 Experimental Setup

For the empirical testing, we took measures to ensure the reliability of our results in terms of hyperparameter configuration. Additionally, we aimed to ensure the reproducibility of the training pipeline for each model.

Testing Procedure

We started by conducting tests on the classification datasets, which serve as the basis for our evaluation as these datasets allow for better scalability due to their smaller size than the regression datasets. Therefore, the thesis will mainly investigate these datasets.

In our testing code, we employ a 10-fold cross-validation approach. Within this framework, we further partitioned the training data randomly, allocating 10 % of it for validation purposes. Consequently, each training iteration consisted of 81 % of the original dataset as the training set, 9 % as the validation set, and 10 % as the test set. We repeat this testing procedure five times to mitigate the influence of randomness on the model’s performance. Ultimately, we recorded the mean accuracy on the test set along with its standard deviation as the primary performance metric of the model. Note that the use of standard cross-validation for generating the splits and the choice of mean accuracy as our primary metric is reasonable and justified as the datasets are highly balanced, as demonstrated in Section 2.

In the case of the regression datasets, we adopted a slightly different testing procedure. Due to their larger scale compared to the classification datasets, we opted for a more time-efficient approach. To achieve this, we utilized pre-existing training pipelines developed by ? and ?. These pipelines use a fixed training, validation, and test split, accelerating the testing process. Similar to the classification datasets, we performed five runs for each model configuration. We record the mean absolute error and its standard deviation, as well as the mean logarithmic absolute error and its standard deviation across all runs. Furthermore, we test two fixed splits for each regression dataset: one utilizing only 10 000 samples for training, while the other split uses the entire training set.

Hyperparameter Optimization

To ensure the trustworthiness of our empirical results and gain valuable insights into the optimal configuration of 1-WL+NN models, we conducted a thorough hyperparameter optimization for each type of model, both GNN and 1-WL+NN, and performed this optimization individually for each dataset.

In order to ensure the comparability of our results with other works and limit the number of hyperparameters requiring optimization, we followed standard practices when configuring our models. Detailed information on all hyperparameters, including the ones we optimized for, can be found in ?? for the 1-WL+NN models used in classification tasks, ?? for the GNN models employed in classification tasks, and ?? for the 1-WL+NN models utilized in the regression task in the Appendix. We will shortly discuss the most critical parameters we optimized for:

Early in our investigations, we made a significant observation regarding the learning performance of 1-WL+NN models. We noticed considerable performance discrepancies when comparing 1-WL+NN models using the standard version of the 1-WL algorithm with the results achieved by GNN’s. Consequently, we investigated the possibility of parametrizing the 1-WL algorithm to enhance control over the expressiveness of the colorings it computes. Specifically, we focused on two crucial parameters: 1) the number of iterations and 2) the usage of its convergence behavior as an early termination criterion.

The advantage of this more granular parametrization is that it enables us to apply the same number of 1-WL iterations to each graph processed by a 1-WL+NN model by deactivating the convergence behavior and fixing the number of iterations. By doing so, the number of colors used by the 1-WL algorithm remained contained, both in terms of the number of total unique

colors as well as the distance to another. Therefore, one of the most crucial hyperparameters we optimized for all 1-WL+NN models was the number of 1-WL iterations. Additionally, if employed, the size of the dimension of the look-up table was another significant parameter to optimize.

In contrast, the hyperparameter selection for the GNN models was relatively less intricate. The key parameters of interest here are the number of message-passing layers and the size of the hidden dimensions used in each layer. The reason for this is that the number of layers directly corresponds to the expressiveness of the GNN, as demonstrated in the proof, while the size of the hidden dimensions enables easier learning by allowing the model to store more information during graph processing.

Implementation Details and Result Accessibility

The implementation of our models and training procedures was carried out using PYTHON 3.10 along with the open-source library PYTORCH⁶ and its extension PYTORCH GEOMETRIC⁷. Moreover, we leveraged WEIGHTS&BIASES as our tool for coordinating and recording the results of the hyperparameter optimization. The code for our experiments is publicly available on GITHUB at <https://github.com/ericbill21/BachelorThesis>, and the corresponding results can be accessed via WEIGHTS&BIASES at <https://wandb.ai/eric-bill/BachelorThesisExperiments>. We conducted our tests on the RWTH High Performance Computing cluster by the RWTH Aachen University as well as on private hardware.

6 Empirical Testing

In this section, we present the empirical findings of our study. A total of 6 633 runs were conducted, with each run testing a specific model configuration on a designated dataset. These runs required a substantial amount of computation time, totaling 2 012 hours (over 83 days). To examine the classification accuracy achieved by the best-performing configuration of each model on each classification dataset, please refer to ???. For regression datasets, ?? provides an overview of the mean absolute error (MAE) for the best-performing configurations.

Due to the significantly larger size of the regression datasets compared to the classification datasets, running regression experiments requires considerable time for each configuration. As a result, we had to limit the number of different configurations tested. For this reason, we employed existing training pipelines, as mentioned in the previous section, which allowed us to include the results of the GINE- ε model proposed by ? and ? in ???. The GINE- ε model utilizes the GIN architecture for message-passing layers, Mean as its pooling function, and a two-layer MLP for the final processing of its input graph. This design closely resembles our GIN:Mean model, with the crucial distinction being that GINE- ε incorporates edge features in its computations.

The decision to exclude edge features from all our models, including all 1-WL+NN models, was driven by two key factors. Firstly, our focus primarily revolved around the classification datasets, all of which do not encode any information in their edge features (refer to ??). Secondly, the regression datasets we utilized in our evaluation include continuous edge features, making the

⁶A free and open-source machine learning framework that was originally developed by Meta AI and is now part of the Linux Foundation umbrella. <https://pytorch.org>

⁷An open-source library that extends PYTORCH and allows for easy development and training of graph neural networks. <https://pytorch-geometric.readthedocs.io>

Table 5: Overview of the mean classification accuracies achieved by the best configuration of each model for each dataset in percent and standard deviation.

Method	Dataset					
	ENZYMEs	IMDB-BINARY	MUTAG	NCI1	PROTEINS	REDDIT-BINARY
1-WL+NN	Max	16.7 \pm 4.2	52.0 \pm 5.3	73.8 \pm 12.4	58.6 \pm 3.3	62.9 \pm 4.9
	Mean	18.2 \pm 4.8	59.4 \pm 5.8	77.1 \pm 11.5	64.0 \pm 3.3	60.9 \pm 4.5
	Sum	18.0 \pm 6.2	57.5 \pm 5.1	66.8 \pm 13.9	56.9 \pm 3.8	65.6 \pm 4.8
Graph Neural Networks	Embedding-Max	40.5 \pm 7.4	69.4 \pm 4.9	81.1 \pm 11.2	82.7 \pm 2.0	75.2 \pm 3.9
	Embedding-Mean	42.6 \pm 9.0	72.4 \pm 4.1	84.1 \pm 9.1	83.1 \pm 1.9	72.3 \pm 4.2
	Embedding-Sum	48.3 \pm 8.1	72.0 \pm 3.8	85.1 \pm 8.6	83.6 \pm 2.2	75.2 \pm 4.5
GAT	GAT:Max	31.2 \pm 6.0	70.7 \pm 4.8	71.1 \pm 12.2	58.0 \pm 4.2	72.5 \pm 5.1
	GAT:Mean	28.9 \pm 5.9	70.9 \pm 3.7	74.8 \pm 9.1	66.1 \pm 2.8	64.9 \pm 6.4
	GAT:Sum	34.4 \pm 7.0	72.2 \pm 4.5	82.1 \pm 11.2	69.8 \pm 2.6	73.4 \pm 3.9
GCN	GCN:Max	33.1 \pm 7.5	73.5 \pm 4.1	74.5 \pm 11.3	61.1 \pm 3.6	69.8 \pm 5.9
	GCN:Mean	29.9 \pm 5.7	74.7 \pm 3.8	75.0 \pm 10.4	68.9 \pm 2.4	70.9 \pm 5.2
	GCN:Sum	31.7 \pm 7.2	73.0 \pm 4.4	81.5 \pm 10.3	70.4 \pm 2.1	3.5 \pm 3.9
GIN	GIN:Max	29.2 \pm 6.2	70.8 \pm 4.7	77.3 \pm 10.7	79.9 \pm 2.2	74.3 \pm 5.1
	GIN:Mean	31.7 \pm 6.7	71.1 \pm 5.4	82.4 \pm 9.8	70.8 \pm 2.2	72.0 \pm 4.0
	GIN:Sum	28.9 \pm 8.7	69.5 \pm 4.8	84.6 \pm 8.7	70.8 \pm 2.3	74.0 \pm 4.3

Table 6: Overview of the mean absolute error and the standard deviation (logMAE) on large-scale (multi-target) molecular regression tasks. The results marked by ¹ are adopted from ? and by ² from ?.

Method	Dataset				
	ALCHEMY	ALCHEMY (10K)	ZINC	ZINC (10K)	
GNN	GINE- ε ^{1,2}	0.103 \pm 0.001 -2.956 \pm 0.029 ¹	0.180 \pm 0.006 -1.958 \pm 0.047 ²	-	0.084 \pm 0.004 ¹
	GIN:Max	0.604 \pm 0.004 -0.578 \pm 0.009	0.353 \pm 0.003 -1.228 \pm 0.006	0.124 \pm 0.004	0.427 \pm 0.009
	GIN:Mean	0.643 \pm 0.014 -0.505 \pm 0.021	0.314 \pm 0.004 -1.469 \pm 0.044	0.110 \pm 0.005	0.341 \pm 0.037
1-WL+NN	GIN:Sum	0.523 \pm 0.016 -0.705 \pm 0.035	0.282 \pm 0.002 -1.890 \pm 0.031	0.104 \pm 0.005	0.298 \pm 0.034
	Embedding-Max	0.648 \pm 0.003 -0.511 \pm 0.009	0.409 \pm 0.003 -1.023 \pm 0.009	0.382 \pm 0.005	0.659 \pm 0.007
	Embedding-Mean	0.617 \pm 0.003 -0.564 \pm 0.005	0.355 \pm 0.004 -1.269 \pm 0.020	0.348 \pm 0.010	0.484 \pm 0.009
	Embedding-Sum	0.600 \pm 0.004 -0.625 \pm 0.032	0.305 \pm 0.001 -1.740 \pm 0.042	0.384 \pm 0.004	0.465 \pm 0.009

application of the 1-WL algorithm more complex. Although some work exists on modifying the 1-WL algorithm to incorporate continuous features, we opted for a more straightforward and more scalable approach by not considering edge features in the computations of all our models. Nonetheless, this limitation does not compromise the integrity of our results, as our main objective is to compare similar GNN and 1-WL+NN models. The inclusion of GINE- ε showcases the potential and highlights the significance of edge feature information for solving the dataset tasks of ALCHEMY and ZINC.

A notable finding when analyzing the presented results is that the performance of 1-WL+NN models is comparable to that of GNN models. In fact, the best 1-WL+NN models even outperform the GNN models in all classification datasets, with the exception of IMDB-BINARY

and REDDIT-BINARY. These two datasets stand out among the classification datasets as they lack node features. We will investigate this insight later on in more detail. Moreover, our results align with the empirical findings of other GNN models in various studies. The accuracy reported here falls within a similar range as observed in works by ??? and ?.

In the subsequent subsection, we will further analyze these results. Specifically, we will investigate the tradeoff between generality and expressiveness in 1-WL+NN models, the differences in their learning behaviors, the ability of GNN models to approximate 1-WL+NN models, the learned pooled graph representations of each model type, the impact of GNN models performance by preprocessing the data, and finally the effect of large datasets on the performance of 1-WL+NN models.

6.1 Fixed 1-WL Iterations: Tradeoff between Expressiveness and Generality

As discussed in Section ??, we explored constraining the expressiveness of the 1-WL algorithm by fixing the number of iterations it is applied to each graph in a 1-WL+NN model. The intention behind this was to ensure that the resulting colorings of graphs computed using a fixed 1-WL algorithm would fall within a similar color range, thereby limiting the total number of colors and the overall distance between each used color.

From a theoretical perspective, fixing the number of 1-WL iterations in a 1-WL+NN model only limits its expressiveness. For instance, let $k \in \mathbb{N}$ now be fixed. If the standard 1-WL algorithm converges on an input graph G after $k' < k$ iterations, the coloring computed after k iterations is as expressive as the one obtained from the standard 1-WL algorithm since the coloring convergence after k' iterations. On the contrary, if the standard 1-WL algorithm converges after $k' > k$ iterations on an input graph G , the coloring obtained after k iterations is less expressive and contains less information than the coloring derived from the standard 1-WL algorithm.

Comparing the performance of 1-WL+NN models utilizing the standard 1-WL algorithm to those using the parametrized version reveals a significant difference across all classification datasets, as seen in ???. This data confirms our belief that the standard 1-WL algorithm can be overly expressive for specific tasks, as evidenced by the considerable performance gap between the two algorithm types. Additionally, we included the mean number of iterations the 1-WL algorithm runs when being applied on each graph of a dataset. For comparison, we also included the number of 1-WL iterations used by the best-performing model utilizing the parametrized version of the 1-WL algorithm.

Table 7: Comparison between the best performing 1-WL+NN models using the standard 1-WL algorithm and the one employing the parametrized version of it in percent and standard deviation. Additionally, we put the average number of iterations of the 1-WL algorithm in brackets for each model.

Algorithm Type	Dataset					
	ENZYMEs	IMDB-BINARY	MUTAG	NCI1	PROTEINS	REDDIT-BINARY
Standard	Test Accuracy:	34.2 ± 6.7	67.0 ± 4.3	76.3 ± 9.4	78.9 ± 2.1	71.4 ± 4.9
	\emptyset 1-WL Iterations:	3.0 ± 1.7	1.1 ± 0.7	4.2 ± 1.6	3.9 ± 1.7	3.0 ± 1.7
Parametrized	Test Accuracy:	48.3 ± 8.1	72.4 ± 4.1	85.1 ± 8.6	83.6 ± 2.2	75.2 ± 3.9
	# 1-WL Iterations:	1	1	1	3	1

Building on these insights, investigating the optimal number of 1-WL iterations that lead

to the best-performing models is another interesting observation. To illustrate its effect, we created ??, which showcases the performance achieved by all models and configurations grouped by the number of 1-WL iterations for each dataset in a violin graph. Note that both classification and regression datasets are included in this figure. Depending on the dataset type, the y-axis represents either classification accuracy or test error, where higher accuracies imply better performance for classification tasks, and higher test errors imply poorer performance for regression tasks (and vice versa).

In general, the results indicate that the performance of 1-WL+NN models tends to improve as the number of 1-WL iterations decreases, except for the NCI1 and ZINC datasets.

Notably, for NCI1, the optimal number of iterations coincides with the number of 1-WL iterations needed for the maximum achievable accuracy (refer to ??). This correlation holds for all other classification datasets, except for MUTAG. However, the behavior observed in MUTAG can potentially be attributed to the fact that according to the introduced taxonomy (?) in Section ??, MUTAG can be effectively solved by simply replacing its node features with an encoding of the node degree. Since the first iteration of the 1-WL algorithm is efficiently an encoding of the node degree, this might explain the observed pattern.

Interpreting the results of the ZINC dataset is more complex and somewhat peculiar. While the smaller subset of the dataset, ZINC10K, clearly indicates that fewer 1-WL iterations yield better results, the overall ZINC dataset suggests the opposite. This discrepancy could be attributed to the limited number of runs conducted on the entire dataset, which were restricted due to scalability and time constraints. Therefore, these results should be interpreted with caution.

In conclusion, reducing the number of 1-WL iterations leads to improved generability and overall better performance of the models, with the clear exception of NCI1. However, as our results indicate, most 1-WL+NN models achieve optimal performance when employing only a single iteration of the 1-WL algorithm, which is essentially equivalent to encoding the node degree. This observation demonstrates that 1-WL+NN models do not require the full

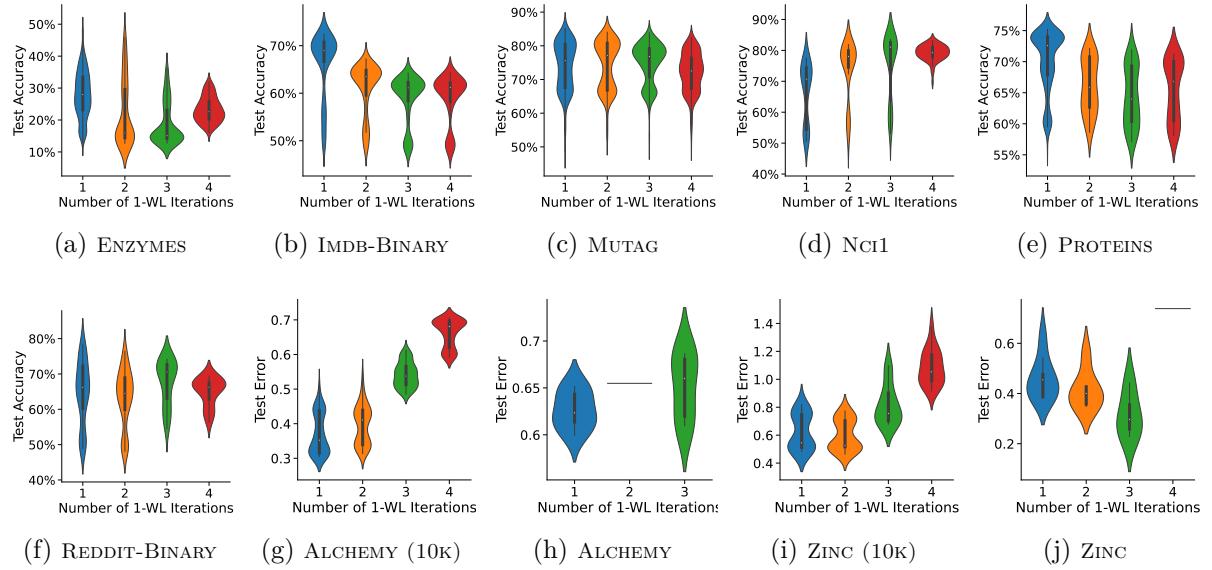


Figure 7: Impact on performance by the number of iterations of the 1-WL algorithm.

expressiveness of the 1-WL algorithm to achieve comparable performance to GNN models, which raises the question of whether GNN models, despite their theoretical ability to be highly expressive, also tend to rely primarily on node degree information for computation. We will explore this question in the following section.

6.2 Difference in Learning Behavior between GNN and 1-WL+NN Models

The learning behavior of 1-WL+NN models presents a unique challenge due to the expressiveness of the 1-WL algorithm, which directly impacts their performance. Compared to GNN models, 1-WL+NN models exhibit a significant discrepancy between their training and test performance. While it is common to observe higher training performance than testing performance in various machine learning applications, this discrepancy is particularly pronounced in our evaluation of the 1-WL+NN models.

?? visually represents the difference in performance between the training and testing performance for both 1-WL+NN and GNN models across all classification datasets. Different quantiles from the best-performing models were considered, ranging from the top 1% to the top 100%. Each bar in the graph represents the mean difference between training accuracy and test accuracy for the respective quantile of runs. 1-WL+NN models are depicted in blue, while GNN models are represented in orange. In comparison, shorter or negative bars indicate better generalization where the performance on the training set aligns more closely with the testing set. The black line in each bar represents the standard error.

Upon analyzing ??, it reveals that 1-WL+NN models exhibit more significant overfitting than

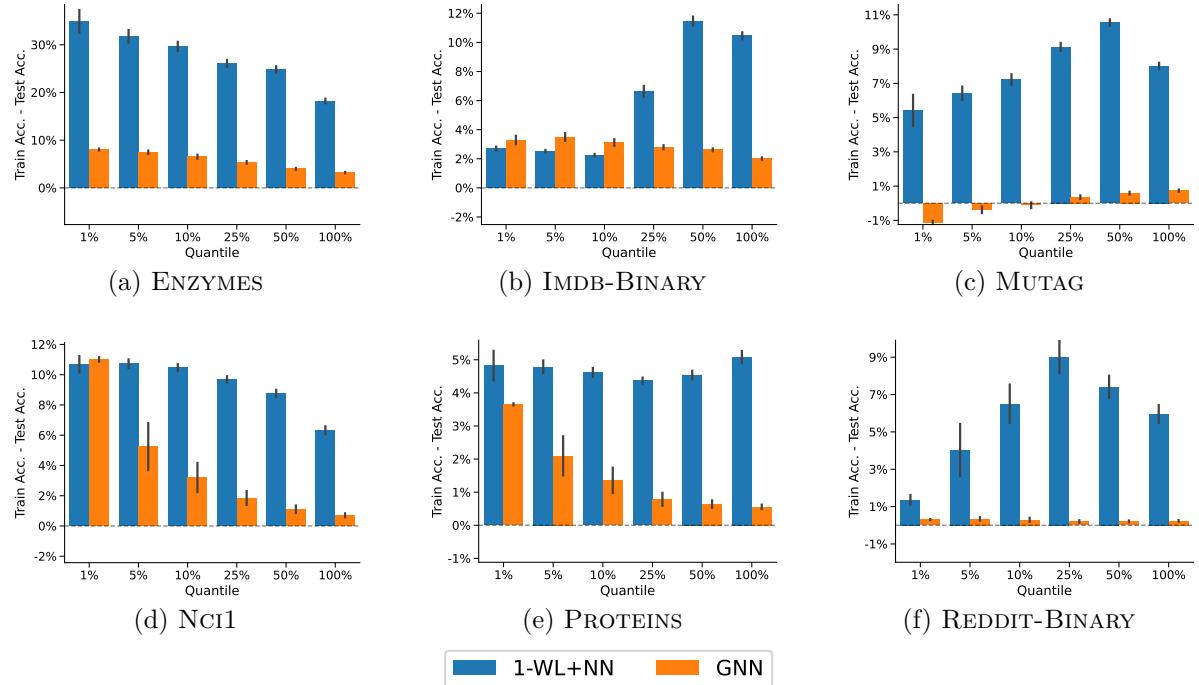


Figure 8: Mean difference of the classification accuracies of the training and testing set for each dataset. In detail, we grouped by different quantiles of the best performing models and the type of the model.

GNN models, especially on datasets such as ENZYMES, MUTAG, PROTEINS, and REDDIT-BINARY. For instance, in the case of ENZYMES, the top 1% of best-performing 1-WL+NN models display, on average, a 27% higher training accuracy than their test accuracy, highlighting the extreme extent of overfitting in 1-WL+NN models.

While GNN models also experience some overfitting, primarily observed in the NCI1 dataset, it is not as prevalent across all datasets and not as dramatic as in the case of 1-WL+NN models. Moreover, in datasets like MUTAG or REDDIT-BINARY, GNN models demonstrate superior generalization capability. In these cases, the mean difference in performance is close to 0%, and in the case of Mutag, it is even negative, which indicates that the tested GNN models were able to learn general patterns in these datasets rather than simply memorizing the data.

However, this raises the question as to why 1-WL+NN models exhibit such dramatic overfitting. We hypothesize that the expressiveness of the 1-WL algorithm leads to the computation of highly detailed colorings of the 1-WL+NN’s input graphs, such that the models simply memorizes these.

To gain insight into the algorithm’s expressiveness, we calculated the ratio of unique colors to the total number of possible colors for each classification dataset. For example, the ENZYMES dataset consists of 600 graphs with a total of 19 580 nodes. After a single iteration of the 1-WL algorithm, all colorings consist of only 231 unique colors, accounting for less than 0.01% of the total number of possible unique colors (the total number of nodes). However, after two iterations, this number has already increased to 10 416, comprising 53.2% of the total available colors. This example demonstrates the significant expressiveness of the colorings after just a few iterations.

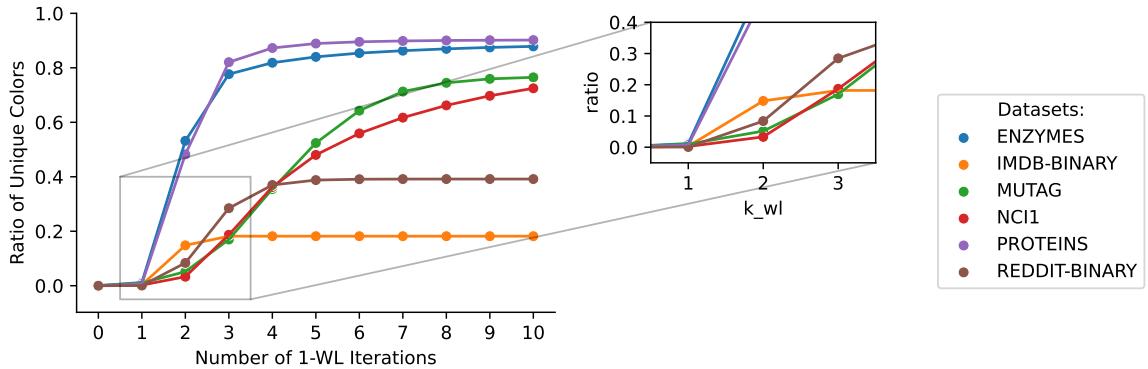


Figure 9: Overview of the ratio of unique colors used by the 1-WL algorithm when applied for a specified number of iterations for each dataset. The ratios are computed using each dataset’s total number of nodes as the maximum number of unique colors that could appear in all colorings.

?? illustrates the ratio of unique colors used compared to the total possible number of unique colors (total number of nodes in each dataset). As expected due to the convergence behavior of the 1-WL algorithm, each ratio converges at some point. However, the ratios rise rapidly and dramatically for all datasets initially, indicating the strong expressiveness of the algorithm for already small numbers of iterations. Even a relatively small ratio, such as approximately 10%, signifies a significant number of unique colors in the colorings computed by the 1-WL algorithm. It is important to consider the scale of the datasets when interpreting these ratios. For example, the MUTAG dataset consists of approximately 3 000 nodes, while the ENZYMES

and IMDB-BINARY datasets contain around 20 000 nodes each. The PROTEINS dataset reaches 43 000 nodes, followed by the NCI1 dataset with 122 000 nodes, and finally, the REDDIT-BINARY dataset with a staggering 859 000 nodes. For a comprehensive overview of the datasets and their sizes, please refer to ??, and for detailed information on the unique color count, also including the regression datasets and their different splits, consult ?? in the Appendix.

Furthermore, it is important to note that the colors assigned by the 1-WL algorithm have no inherent structural connection between them. Even if two nodes are colored with two distinct integers that are close together, it does not imply any similarity between them. While our experiments use an implementation of the 1-WL algorithm that assigns each node the smallest unused color, a meaningful connection about the distance of colors is still absent due to shuffling the dataset and applying the 1-WL algorithm for a fixed number of iterations.

To investigate the impact of the number of 1-WL iterations on overfitting, we plotted ??, which focuses solely on 1-WL+NN models and compares the different numbers of 1-WL iterations. Across all classification datasets, it is evident that the overfitting behavior becomes more substantial when increasing the number of 1-WL iterations. This finding supports our belief that the explosive increase in the number of unique colors is the primary reason for the observed overfitting behavior. In the case of ENZYMES, with an increased number of 1-WL iterations, the average training accuracy is over 60 % higher than the test accuracy, demonstrating the worsening of the overfitting behavior.

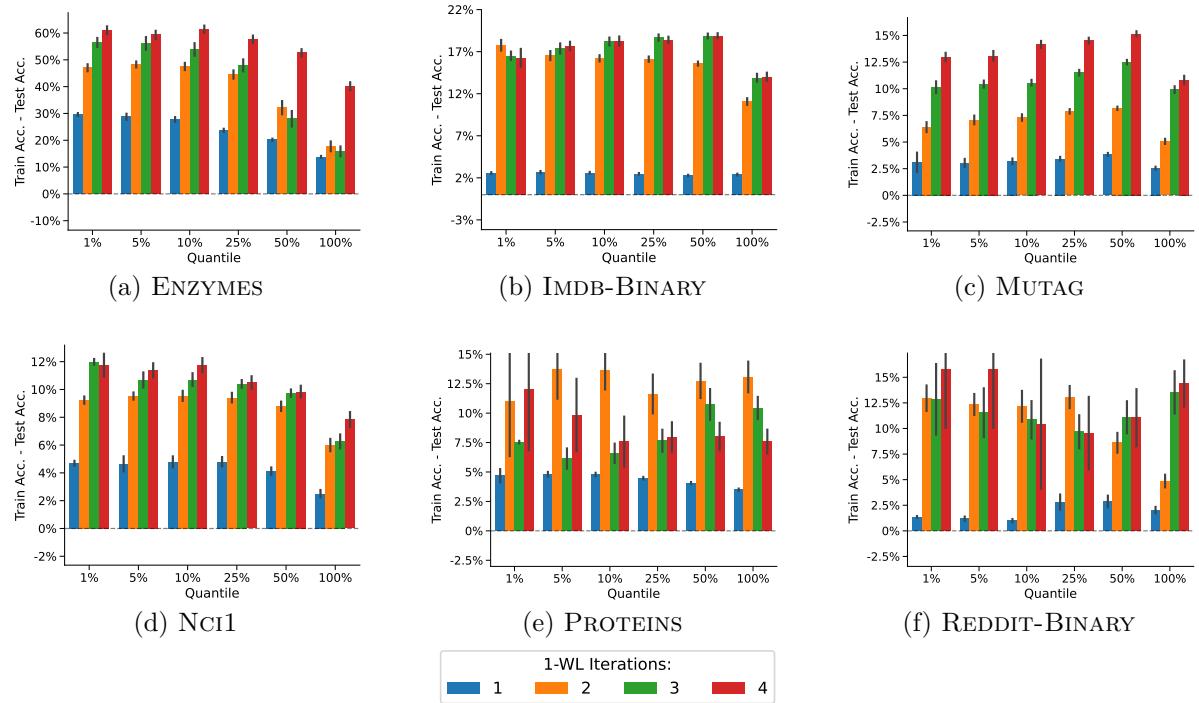


Figure 10: Mean difference in the classification accuracy of the training and test sets achieved by all 1-WL+NN models for each dataset. In detail, we grouped the best models according to different quantiles and the number of iterations of the 1-WL algorithm used by each model.

In conclusion, our analysis reveals that 1-WL+NN models exhibit optimal generalization when limited to one iteration of the 1-WL algorithm. Increasing the number of iterations leads

to a significant overfitting behavior. This insight indicates that the expressiveness of the 1-WL algorithm surpasses the requirements for effective learning. In comparison, GNN models excel in learning and representing graph structures in a more generalized manner. The observation that a single iteration of the 1-WL algorithm leads to the best generalization also aligns with the results from the previous section, where the models achieving the highest accuracy on the test set also utilized only a single iteration of the 1-WL algorithm (except for the NCI1 dataset). This observation raises an intriguing question: Do GNNs also attempt to compute a similarly expressive representation of all graphs, akin to a single iteration of the 1-WL algorithm? We will explore this question in the following section.

6.3 Approximating 1-WL Coloring: Investigating GNN Node Representations

This section will explore the node features computed by a GNN. Specifically, we will analyze the ability of GNNs to approximate the coloring computed by a single iteration of the 1-WL algorithm.

To visualize the approximation capabilities of the best-performing GNN models on various classification datasets, refer to Figure ???. The visualization consists of a pair of color-coded matrices for each dataset, where each pair represent a single randomly drawn graph from the test set of the respective GNN model. The left matrix illustrates the distance matrix of the node representations computed by the GNN model for each pair of nodes i and j . In contrast, the right matrix represents the distances of the colorings generated by a single iteration of the 1-WL algorithm. The distances are measured using the Euclidean distance and then normalized between 0 and 1 for the GNN matrix. While the distances between the 1-WL coloring are either 0 (for nodes with the same color) or 1 (for nodes with different colors) since the colorings produced by the 1-WL algorithm are discrete and do not encode any information in their distances. Further, we computed the similarity between both matrices by using their absolute difference and average these, resulting in a single value represented as the Mean Absolute Error (MAE). The MAE, along with its standard deviation and the index of the graph in the datasets, is included on the right side of the 1-WL distance matrix.

The visualization shows that the GNN representations already provide convincing approximations of the patterns observed in the 1-WL algorithm's matrix. However, it is important to note that these results are based on a single randomly selected graph from each model's test set. To obtain a more comprehensive understanding of the approximation capabilities, we repeated this process for each dataset using ten randomly selected graphs from the test set of the respective GNN model, refer to ?????????????? in the Appendix.

Before diving deeper into analyzing the approximation capabilities, it is crucial to highlight the significant difference between the node representations computed by a GNN and the colors computed by the 1-WL algorithm. Unlike the discrete colors with no order relation calculated by the 1-WL algorithm, the representations computed by each GNN model are multidimensional and continuous. This difference makes a perfect approximation of the 1-WL colorings challenging, as determining when two node representations are similar is not as straightforward as in the discrete 1-WL case. In the visualization, we avoided making this decision by uniformly normalizing the distances between 0 and 1 and using a continuous color bar for color coding similar nodes. However, to comprehensively evaluate each dataset entirely, we employed two approaches for tackling this problem: 1) evaluating the approximation using a threshold value for similarity (continuous approximation) and 2) translating the values into discrete distances and using the F1 score for assessing similarity (discrete approximation).

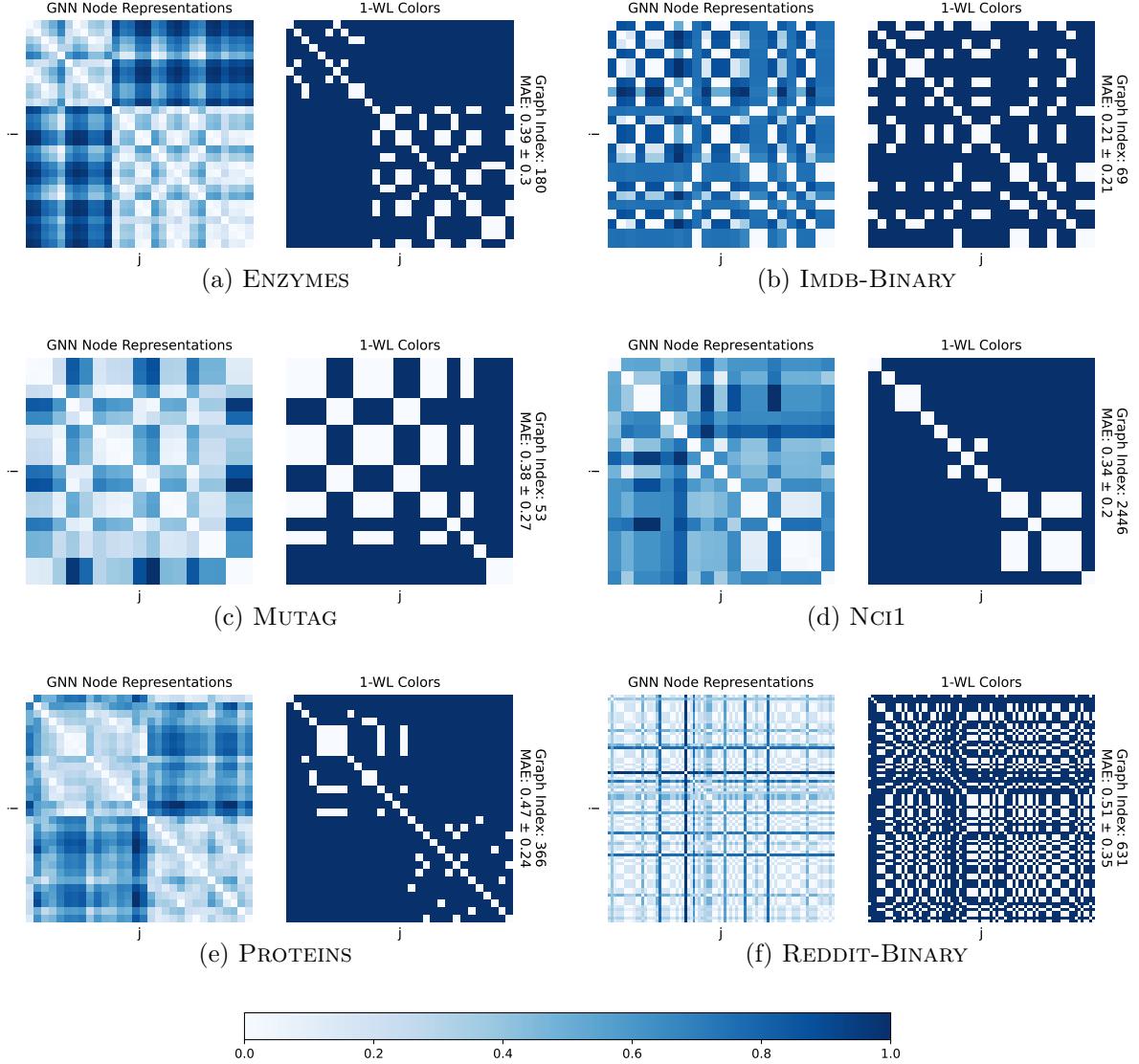


Figure 11: Visualizing the performance of the best-performing GNN models of each dataset in approximating node colors computed by the 1-WL algorithm after running for one iteration. Here we randomly sampled a single graph for each dataset and visualized the approximation performance.

In the first procedure, two node representations are considered similar if their Euclidean distance is less than a threshold value, ϵ . Additionally, we consider the distances perfectly separable if their distance is greater or equal to $1 - \epsilon$. In detail, we modify the GNN distance matrix for every graph such that all distances less than ϵ are set to 0, all distances greater or equal to $1 - \epsilon$ to 1, otherwise the original distance is preserved. Afterward, we compute, as before, the MAE between each GNN matrix and the 1-WL color matrix and average all values for the entire dataset in a single MAE value in the end. We tested this procedure with various threshold values $\epsilon \in [0, 0.5]$. See Figure ?? for an illustration of the MAE for all classification datasets. Since the distances are normalized, a higher ϵ value indicates classifying the majority

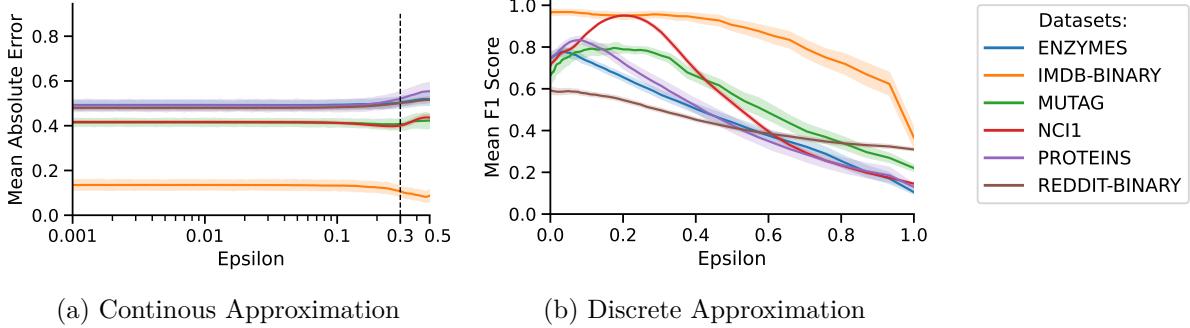


Figure 12: Results obtained by using the proposed evaluation procedures to assess the ability of GNNs to approximate 1-WL colorings.

of distances as negligible. As a result, the values of interest are small, which is why a logarithmic scale is used for the x-axis.

The second procedure aims to transform the regression-like values into a classification task by introducing a threshold value, ϵ , which maps distances between two node representations to a discrete value. Distances less than ϵ are classified as 0, while distances greater than or equal to ϵ are classified as 1. This process generates a distance matrix with discrete values, and the F1 score is used to evaluate the approximation capability of this newly devised matrix to the 1-WL coloring distance matrix. The process is repeated for each graph in a dataset, and the average F1 score is calculated as the final evaluation metric. Since the number of node pairs distinguished by the 1-WL algorithm differs from the number of nodes assigned the same color, an imbalance between the two labels exists. To counter any biases due to this imbalance, the F1 score is computed with macro weighting. Figure ?? provides a visualization of the results for each classification dataset.

Analyzing the figures, it becomes apparent that the approximation of the 1-WL coloring is not perfect, as indicated by the relatively high MAE and the varying F1 scores for different datasets, with the clear exception of the IMDB-BINARY dataset. Since the IMDB-BINARY and REDDIT-BINARY datasets lack node features, we initialized their features using the common procedure of encoding their node degrees. Since the coloring computed by a single iteration of the 1-WL algorithm is an encoding of the node degree, this explains the good performance of the IMDB-BINARY dataset. However, due to the large size of the REDDIT-BINARY dataset and its graphs, a one-hot encoding of the node degree is not practical. Instead, a continuous, one-dimensional node degree encoding is employed that uniformly maps a node degree into the range $[-1, 1]$. This difference in encoding is evident in the significant performance gap between these datasets in the approximation evaluation.

Interestingly, the MAE remains constant for ϵ values up to 0.3, as depicted in Figure ??, suggesting that the GNN models tend to map nodes with different values far away from each other and vice versa for small values. This observation aligns with the low standard deviation values, indicating consistent behavior across all graphs. Moreover, to strengthen this insight, we plotted the frequency of distances that appears in the GNN distance matrices for each dataset. We calculated the frequency by creating ten uniformly, non-overlapping intervals between 0 and 1 and counted the number of distances that fall in this interval for each. Interestingly, we do not see an intense concentration of any distances in any of the intervals such that we can out rule any bias when analyzing the threshold values. Since 0.3 is already a considerable threshold

for removing distances, we infer that the GNNs compute a coloring that incorporates essential information from the 1-WL coloring while leaving out less important information. The visual representations show that the distances between nodes exhibit similar patterns as the coloring produced by the 1-WL algorithm, although not as perfect. Since the GNN models perform nearly as well as most of the 1-WL+NN models in terms of their accuracy performance, we conclude that GNNs are able to learn a more efficient encoding of the graph structure that holds the most important structural information of the 1-WL coloring.

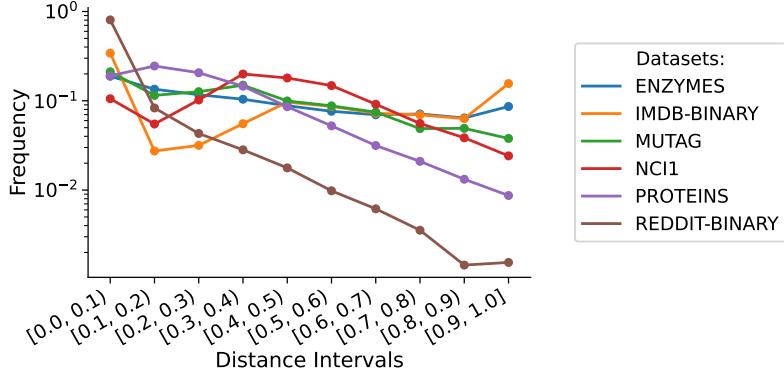


Figure 13: Visualization of the frequency of the distance between node pairs in the representation computed by the best-performing GNN model. In detail, we divided the distance space into ten non-overlapping intervals and counted the number of occurrences that fall into this interval for each dataset.

Analyzing the other procedure, we observe that each dataset has its own optimal threshold value ϵ for the F1 score. Interestingly, these optima consistently fall within a narrow range between 0.0 and 0.2. Moreover, most datasets achieve a high F1 score, providing further evidence for our hypothesis regarding the encoding of essential structural information. With the exception of REDDIT-BINARY, almost all datasets achieve a score higher than 0.77. Notably, the NCI1 dataset stands out with an exceptional Mean F1 score of 0.95 for $\epsilon = 0.2$. This result indicates that the node representations computed by the best GNN model for NCI1 effectively encode the structural information captured by a single iteration of the 1-WL algorithm. Furthermore, considering the dataset taxonomy introduced in ??, NCI1 belongs to the dataset category that encodes a substantial amount of information in their graph structures, which aligns with our findings. It is also worth noting that NCI1 is the only dataset that demonstrates improved performance, in terms of accuracy, with an increased number of 1-WL iterations in the testing of the 1-WL+NN models (refer to Figure ??). To further investigate this behavior, we explored the approximation capability of the GNN model for the coloring computed by three iterations of the 1-WL algorithm specifically for the NCI1 dataset. The results are visualized in Figure ?? in the Appendix. However, as explained in the previous section, the number of unique colors increases significantly with an increased number of 1-WL iterations. This has the effect that almost all nodes in a graph need to be mapped to separable node representation, which is not the case, as we see in ???. This limitation applies to all datasets, leading us to conclude that GNNs cannot effectively approximate a higher number of 1-WL iterations.

Although the REDDIT-BINARY dataset exhibits visually good approximation, as displayed in ?? and ?? in the Appendix, it performs poorly in the approximation evaluation for the entire dataset. We believe that the reason for this is due to the sheer size of the dataset, particularly

the size of its graphs. While all other classification datasets have an average of approximately 30 nodes per graph (including the regression datasets), REDDIT-BINARY averages 429.6 nodes per graph. The significantly larger number of nodes causes challenges in efficiently propagating information across the graph for GNN models. With a fixed number of message-passing layers, set to 5 for scalability and comparability, it is likely that some information is not shared with the entire graph. This issue is also present in other datasets but is more pronounced in REDDIT-BINARY, as indicated by the average number of nodes.

In conclusion, considering the similar performance of GNN and 1-WL+NN models and the relatively robust approximation capabilities of GNNs compared to the 1-WL coloring after a single iteration, we argue that GNNs can learn a more efficient encoding of this information. Given that both frameworks achieve comparable performance on all datasets in terms of accuracy, combined with the fact that most models utilize only a single iteration of the 1-WL algorithm or an approximation, the question arises as to whether this difference remains when pooling this information using the same pooling function. This question will be investigated in the next section.

6.4 Comparing Pooled Graph Representations: GNN vs. 1-WL+NN Models

After the previous section, we observed that GNNs do not perfectly approximate the coloring calculated by the 1-WL algorithm. However, we hypothesized that GNNs might employ a more efficient encoding containing only essential information. To delve deeper into this, we compared the different graph representations derived after applying the respective pooling functions. To do this, we selected the best-performing GNN and 1-WL+NN models in terms of classification accuracy for each dataset and replaced the final MLP with various algorithms.

Table 8: Overview of the classification accuracies achieved by the best model configuration for each dataset in percent and standard deviation. Additionally, the performance of each configuration was further evaluated by substituting the final MLP with either a SVM utilizing a linear kernel (SVM Linear) or the Radial Basis Function (SVM RBF), as well as the k -NN classifier with different values for k .

Method	Dataset					
	ENZYMEs	IMDB-BINARY	MUTAG	Nci1	PROTEINS	REDDIT-BINARY
1-WL+NN	MLP	48.3 \pm 8.1	72.4 \pm 4.1	85.1 \pm 8.6	83.6 \pm 4.1	75.2 \pm 3.9
	SVM Linear	34.4 \pm 5.5	71.2 \pm 3.9	86.4 \pm 8.9	83.4 \pm 2.1	73.9 \pm 4.1
	SVM RBF	45.0 \pm 7.0	72.8 \pm 4.3	83.2 \pm 7.5	83.6 \pm 1.9	75.2 \pm 4.0
	k -NN	56.3 \pm 5.8 ($k=1$)	72.3 \pm 4.1 ($k=11$)	86.7 \pm 7.7 ($k=10$)	83.9 \pm 1.8 ($k=5$)	73.9 \pm 4.1 ($k=19$)
GNN	MLP	34.4 \pm 7.0	74.7 \pm 3.8	84.6 \pm 8.7	79.9 \pm 2.2	74.3 \pm 5.1
	SVM Linear	33.2 \pm 5.9	73.9 \pm 4.2	87.4 \pm 6.8	67.4 \pm 2.2	74.7 \pm 4.2
	SVM RBF	35.9 \pm 6.0	74.1 \pm 3.9	86.0 \pm 7.4	73.0 \pm 1.9	74.6 \pm 4.6
	k -NN	51.6 \pm 7.0 ($k=1$)	74.3 \pm 4.0 ($k=132$)	88.3 \pm 6.5 ($k=38$)	77.5 \pm 1.7 ($k=2$)	74.9 \pm 4.3 ($k=27$)
						90.8 \pm 1.8 ($k=15$)

Specifically, we used two different configurations of Support Vector Machines (SVM): one with a linear kernel (SVM Linear) and another with a radial basis function (SVM RBF). Additionally, we employed the k -Nearest-Neighbor (k -NN) classifier for different values of k . Each algorithm was trained exclusively on the fixed output of the pooling function. To ensure fair comparisons, we froze the weights of the GNN models and the Look-Up Table of the 1-WL+NN models during this process to ensure no optimization of underlying parameters before pooling. This approach guarantees more comparable results across algorithms. Moreover, we maintained

the same training, validation, and test split used for training and evaluating the MLP. For a comprehensive evaluation, we summarized the accuracies and standard deviations achieved by each method, comparing them to the MLP’s performance in ???. Furthermore, in the case of the k -NN algorithm, we provided the achieved accuracy for the optimal k value, including the k value in the brackets. This analysis will allow us to make statements about the linear separability of the graph representations as well as how well these cluster with respect to their class labels.

Starting with the investigation of linear separability, we used the performance of SVM Linear as our basis of evaluation since this method learns a linear high-dimensional hyperplane to separate the graph representations optimally. The results of the SVM RBF serve as a baseline that shows how effectively the data will be linear separable after applying the RBF kernel. Therefore, if both SVM methods achieve similar performance, we can conclude that the graph representations are already relatively linearly separable, and there is no need for applying an additional kernel function.

Interestingly, for both GNN and 1-WL+NN models, the linear SVM performed comparably to the MLP, with the exceptions of ENZYMES for the 1-WL+NN model and NCI1 for the GNN model. In fact, SVM Linear even outperformed the MLP of the GNN model on MUTAG, indicating the graph representations strong linear separability. Furthermore, since the insights were similar for both GNN and 1-WL+NN models, we concluded that both methods map their graph representations in such a way as to facilitate easy linear separation. Additionally, we observed that the accuracy achieved by SVM RBF is higher than the one of SVM Linear; however, this difference is negligible and insignificant (except for the ENZYMES dataset). This insight, again, demonstrates how well the graph representations are linear separability.

For a visualization of the decision boundary calculated by SVM Linear, see ???. We used the t-distributed stochastic neighbor embedding (t-SNE) to reduce the dimensions of the graph representation for visualization purposes. However, note that the t-SNE method distorts the actual distances between data points, such that these visualizations only aid the imagination of the actual relation between graph representations.

The insight that the GNN and the 1-WL+NN model compute node representations that, when pooled into a single graph representation, allow for good linear separability further support our hypothesis that the extra expressiveness contained in the 1-WL colorings is unnecessary, and GNNs indeed employ a more efficient encoding, as suggested in the previous section.

Another interesting property we also want to investigate is how well these graph representations cluster in their high dimensional space with respect to their class label. To do so, we utilized the k -NN classifier for different values of k with k ranging from 1 to 200 (except for the MUTAG dataset, which only contains 188 graphs, k was limited to a range of 1 to 150). Our primary interest was to observe if any range of k values exists where the accuracy consistently remained high. The existence of such a range indicates the presence of meaningful clusters among all graph representations. In order to observe such patterns, we plotted the accuracy achieved on each dataset against the corresponding k values, as illustrated in ??.

Interestingly we can see for all datasets that for some range of k values, there exists a sort of plateau of the accuracy curve, except for the ENZYMES and NCI1 datasets. These plateaus corresponded to the highest accuracy achieved among all k values, indicating that both graph representations computed by 1-WL+NN and GNN models indeed form meaningful clusters with respect to their class labels. Furthermore, as the GNN model outperformed the 1-WL+NN model for most k values, we can conclude that the higher expressiveness of the node representation computed by the 1-WL algorithm is obsolete and the GNN models indeed compute a more

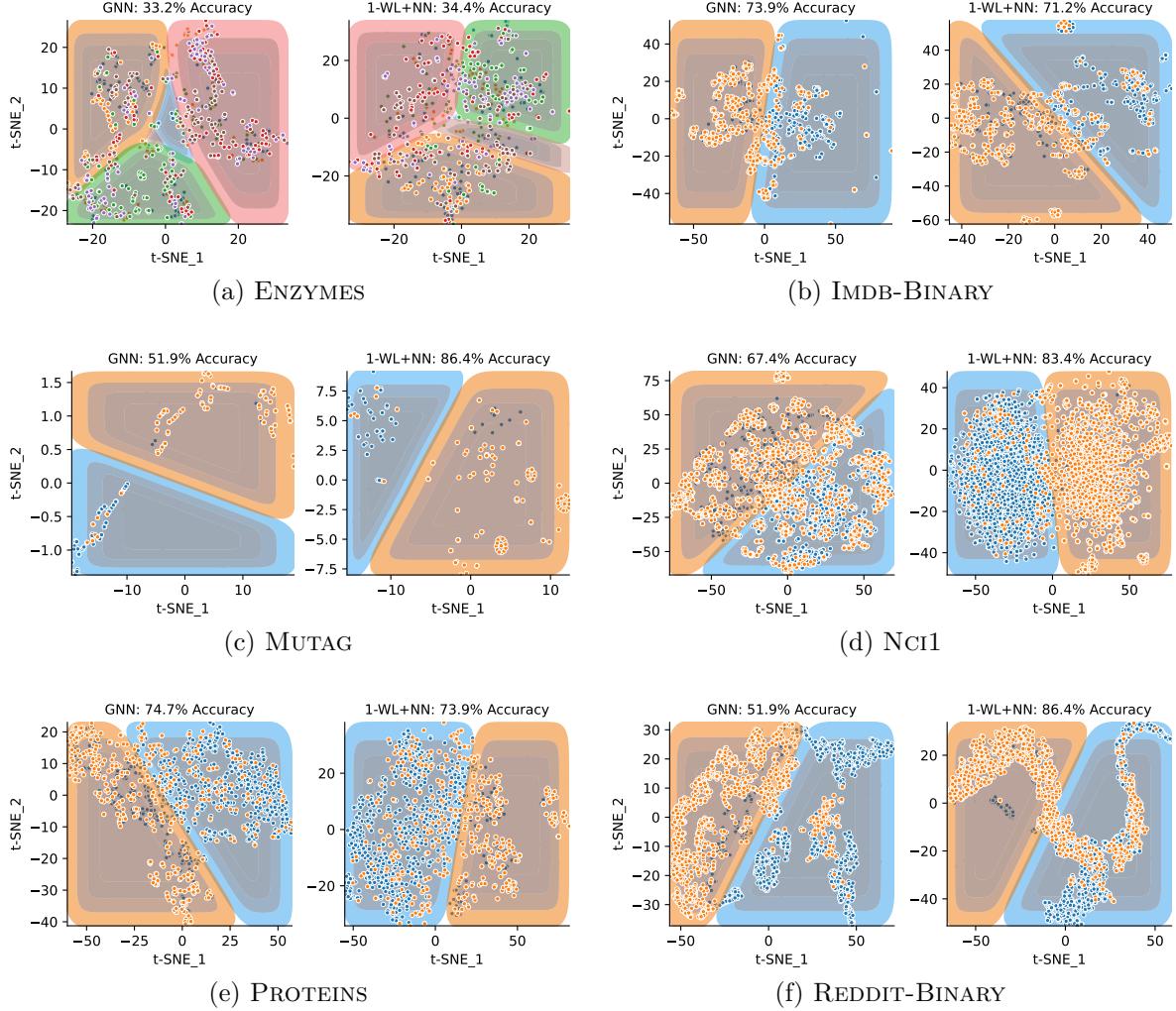


Figure 14: Visualization of the decision boundary of each SVM Linear using t-SNE for the reduction of the dimensionality to two dimensions.

efficient encoding of its nodes, which leads to pooled graph representations that achieved the same or even better results in terms of clustering. This insight is even more pronounced in the example of the REDDIT-BINARY dataset, as here, a significant difference exists in its clustering behavior compared to the 1-WL+NN models.

In the case of the ENZYMES dataset, although there is no such a plateau, the accuracy achievable on the graph representations is for both types of models very similar, such that the lack of indication for clustering can probably be attributed to the dataset itself.

In the case of the ENZYMES dataset, while there is no clear plateau in the curve, the accuracy achievable on the graph representations is very similar for both types of models, indicating that the lack of a distinct clustering indication may be attributed to the characteristics of the dataset.

The real exception, however, lies in the behavior of the k -NN when applied to the NCI1 dataset. Here, only the graph representations computed by the 1-WL+NN model exhibit strong clustering performance that remains consistently high across all values of k . In contrast, the

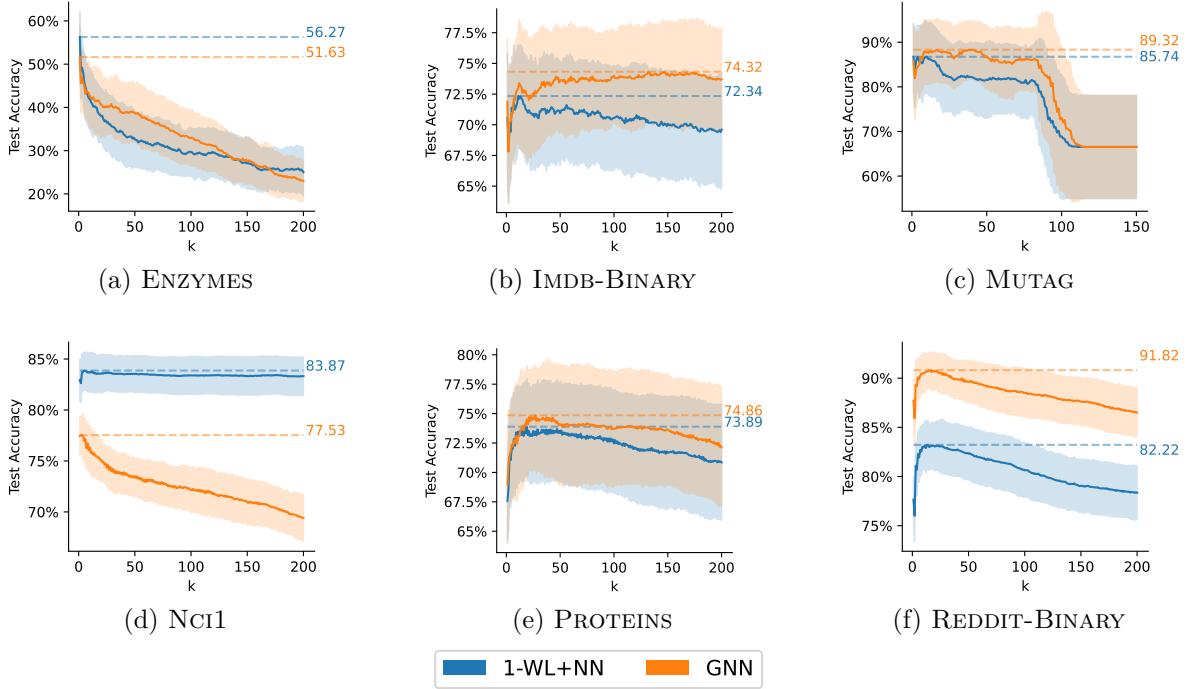


Figure 15: Average classification accuracy achieved on each dataset by replacing the multilayer perceptron of the best-performing 1-WL+NN and GNN model with a classifier based on the k -nearest neighbors algorithm. We tested for different values of k .

graph representations of the GNN model do not seem to cluster at all, and the maximum achievable accuracy is significantly lower than the average accuracy achieved by the 1-WL+NN model representations. We attribute this effect to the 1-WL+NN model’s use of three iterations of the 1-WL algorithm, compared to just one iteration used by all other models of the other datasets. Since the NCI1 dataset encodes essential information for solving its respective task structurally, this pattern might only be specific to this dataset. Moreover, we observed that no other dataset showed better performance of its 1-WL+NN models with an increased number of 1-WL iterations, as evident in ??.

In conclusion, our findings indicate that the pooled graph representations computed by both 1-WL+NN and GNN models are highly linearly separable in their high-dimensional space while also forming meaningful clusters with respect to their class labels. Notably, the GNN models demonstrated superior clustering performance, providing further support for our hypothesis that the message-passing layers of a GNN encode essential information without overly emphasizing expressiveness. This suggests that the GNN models strike a balance between efficient encoding and required expressiveness depending on their application.

6.5 Preprocessing Data for GNNs

As already outlined at the beginning of this section, is that in comparison to all configurations we tested for 1-WL+NN and GNN models, the best accuracy achieved on all classification datasets is by a 1-WL+NN model, with the clear exception for the IMDB-BINARY and REDDIT-BINARY dataset. What makes this observation so interesting is that both datasets are the only ones

lacking node features, such that we initialize all GNN models working on them with an encoding of their node degrees. This difference raises the question of whether GNN models generally perform better when their input graphs undergo preprocessing. Specifically, we aimed to explore if GNN models achieve improved performance when their graphs are preprocessed using the 1-WL algorithm for a single iteration. We will refer to such GNN models as 1-WL:GNN.

This idea seems promising as most of the best-performing 1-WL+NN models also utilize a single iteration of the 1-WL algorithm as their basis for computation. Further, we know that a 1-WL:GNN model can already achieve the same accuracy as each 1-WL+NN model by simply forwarding the preprocessed input graph to the pooling function. However, we hope that 1-WL:GNN models leverage their message-passing layers to generate an improved node representation, ultimately leading to enhanced performance in the end.

We conducted the same hyperparameter optimization routines as for the normal GNN models and listed the classification accuracy achieved by the best-performing configuration of a 1-WL:GNN model for each dataset in Table 8. We also included the best performances of each 1-WL+NN and GNN models for comparison. However, it is essential to note that due to the time constraints of this work and the prioritization of hyperparameter optimization for GNN and 1-WL+NN models, we tested fewer configurations for 1-WL:GNN models. Consequently, these results should be interpreted with caution.

Table 9: Overview of the classification accuracies achieved by the best model configuration for each dataset in percent and standard deviation. Additionally, the performance of each configuration was further evaluated by substituting the final MLP with either a SVM utilizing a linear kernel (**SVM Linear**) or the Radial Basis Function (**SVM RBF**), as well as the k -NN classifier with different values for k .

Model	Dataset					
	ENZYMES	IMDB-BINARY	MUTAG	NCI1	PROTEINS	REDDIT-BINARY
1-WL+NN	48.3 \pm 8.1	72.4 \pm 4.1	85.1 \pm 8.6	83.6 \pm 2.2	75.2 \pm 3.9	78.4 \pm 2.7
1-WL:GNN	35.9 \pm 6.9	72.0 \pm 4.1	83.9 \pm 11.0	NaN	74.5 \pm 4.1	84.3 \pm 2.8
GNN	34.4 \pm 7.0	74.7 \pm 3.8	84.6 \pm 8.7	79.9 \pm 2.2	74.3 \pm 5.1	86.9 \pm 3.2

Upon analyzing the best accuracies achieved by each 1-WL:GNN model in comparison to the other model types, we observed that its performance is not significantly better than any of the GNN or 1-WL+NN models. While it does show minor improvements compared to the GNN models, such as in the ENZYMES and PROTEINS datasets, these improvements are not significant. However, it is important to note that the number of runs conducted for 1-WL:GNN models, 143 runs in total, is considerably smaller compared to the 5 422 runs conducted for 1-WL+NN models and 1 043 runs for GNN models. As a result, the limited number of runs might have impacted the statistical robustness of the results. For a full list of the number of runs conducted, refer to ?? in the Appendix. Further investigation with a larger number of runs could provide more conclusive insights into the performance of 1-WL:GNN models.

6.6 Impact of the Size of Datasets on the Performance of 1-WL+NN

At the beginning of our empirical investigation, an initial idea emerged when observing the first results achieved by 1-WL+NN models on the classification datasets that indicated good

performance. We speculated that these results could be attributed to the small sizes of the datasets, making them particularly suitable for 1-WL+NN models. This idea also raised a question about the scalability of 1-WL+NN models on larger datasets, potentially limiting their performance compared to GNN models, which might scale more effectively. To explore this hypothesis, we sought to investigate significantly larger datasets and selected the well-known regression datasets ALCHEMY and ZINC. Additionally, we utilized a fixed split, significantly smaller than the original dataset, for both ALCHEMY and ZINC. Consequently, we tested 1-WL+NN models on both the entire dataset and the smaller split to examine potential differences in performance. An overview of the results is provided in ?? at the beginning of this section.

Similar to the classification datasets, we conducted a hyperparameter optimization for both 1-WL+NN and GNN models. The primary question of interest here was whether we would observe performance differences between the various splits for each dataset and if there would be discrepancies between the two types of models.

Upon analyzing these results, the first notable observation is that the best performances were achieved entirely by GNN models, specifically the GIN:Sum model. However, in comparison to the GINE- ε model by ?, which we included as a reference to showcase the performance achieved by a different GNN model than ours, both our GNN and 1-WL+NN models demonstrated inferior performance. We conjecture that this discrepancy can be attributed to GINE- ε utilizing the edge features of its input graph in its computation, while our models do not.

Regarding the differences in performance between the various splits for each dataset, the evidence is not straightforward. While 1-WL+NN models performed better on the smaller split ALCHEMY(10K) than the full split, GNN models demonstrated a similar margin of improvement. Moreover, we observed the opposite phenomenon, with 1-WL+NN and GNN models achieving better performance on the larger ZINC split than the smaller ZINC(10K) split.

One reason why such differences were not evident for the 1-WL+NN models may be because, although the number of graphs in each dataset is significantly larger than in the classification datasets, the best performing configuration still utilized only a single iteration of the 1-WL algorithm for both ALCHEMY splits, two for ZINC(10K), and three for ZINC. Consequently, the number of unique colors was limited. See ?? for an overview of the number of unique colors for various numbers of 1-WL iterations. In the case of the models for ALCHEMY, the number of unique colors remained constant at 70 for both splits, indicating that the larger number of graphs between the splits does not have an impact when employing only a single iteration of the 1-WL algorithm. However, the number of unique colors varied more dramatically for the models on the ZINC datasets, with the best model for ZINC(10K) dealing with 9 818 unique colors and the best model on ZINC encountering 290 473 unique colors.

This discrepancy can also be attributed to the splits used for ZINC and ZINC(10K), as opposed to the alchemy split, where the number of unique colors after a single iteration of the 1-WL algorithm is not equal. Specifically, for ZINC, there are 773 unique colors, and for ZINC(10K), there are only 392 unique colors. Further, it is important to note that due to time constraints and the immense time required, especially for training models on the entire ALCHEMY or ZINC dataset, the number of configurations we tested is relatively low compared to ZINC(10K) and ALCHEMY(10K). For a complete overview of the number of runs for each dataset and each model, refer to ?? in the Appendix.

In conclusion, the impact of the size of a dataset on the performance of 1-WL+NN models must be investigated more thoroughly. In detail, further research needs to investigate datasets of large sizes as well as consisting of large graphs, such that the number of unique colors is already relatively high even after a single iteration of the 1-WL algorithm. A good example of

such a dataset is the REDDIT-BINARY dataset, where the number of unique colors after a single iteration of the 1-WL algorithm is 566. For this dataset, a performance gap between 1-WL+NN and GNN models can be observed, with the best GNN model achieving up to an 8 % higher accuracy than the best 1-WL+NN model. However, it is also essential to acknowledge that the reasons for such a performance gap are not solely attributable to the dataset's size, as other factors are at play, as discussed in the preceding sections.

7 Discussion

Introdcution for this section!

7.1 Learned Lessons

- 1-WL+NN is a very good tool for investigating GNNs
 - many imperical simalarities:
 - * Similiar accuracy performance
 - * approximation of coloring / node representations
 - * linear separability and clustering
 - * equivalent expressiveness
 - However point out that this is simply a lab tool -> not suited for real world applications due to a absence of a global 1-WL alogirhtm -> GNNs are in this regard way more flexible
- Summarize all GNN insights:
 -

7.2 Future Work

- other GNN architectures
- more datasets -> In particular larger dataset for classficaiton like MalNet
- Noising von datasets schauen wie 1-WL+NN and GNN damit copen
- More investigation in 1-WL:GNN, seems promising!

7.3 Conclusion

Conclude the Thesis!

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A Appendix Part I

1 Theoretical Connection

1.1 ??: Composition Lemma for 1-WL+NN

Before we begin with the actual composition proof, we give a formal definition and notation for multilayer perceptrons.

Definition 27 (Multilayer Perceptron). Multilayer perceptrons are a class of functions from \mathbb{R}^n to \mathbb{R}^m , with $n, m \in \mathbb{N}$. In this thesis, we define a multilayer perceptron as a finite sequence, such that a multilayer perceptron MLP is defined as $\text{MLP} := (\text{MLP})_{i \in [T]}$ where T is the number of layers. For every $i \in [T]$, the i .th layer of the MLP is the i .th item in the finite sequence $(\text{MLP})_i$. Further, all layers are recursively defined on any input v as:

$$\begin{aligned} (\text{MLP})_0(v) &:= v \\ (\text{MLP})_{i+1}(v) &:= \sigma_i(W_i \cdot (\text{MLP})_i(v) + b_i), \quad \forall i \in [k-1] \end{aligned}$$

where σ_i is an element wise activation function, W_i is the weight matrix and b_i the bias vector of layer i . Note, that for each W_i , the succeeding W_{i+1} must have the same number of columns as W_i has rows, in order to be well-defined. Similarly, for every layer i , W_i and b_i have to have the same number of rows. Following this definition, when applying a MLP on an input $v \in \mathbb{R}^n$ it is defined as $\text{MLP}(v) := (\text{MLP})_k(v)$.

Having established a formal definition and notation, we will now proof the 1-WL+NN composition lemma.

Proof of ??. Let \mathcal{C} be a collection of functions computed by 1-WL+NN, $h_1, \dots, h_n \in \mathcal{C}$, and MLP^\bullet a multilayer perceptron. Further, let f_1, \dots, f_n be the encoding functions, as well as $\text{MLP}_1, \dots, \text{MLP}_n$ be the multilayer perceptrons used by h_1, \dots, h_n respectively. As outlined in ??, we will now construct f^* and MLP^* , such that for all graphs $G \in \mathcal{X}$:

$$\text{MLP}^\bullet(h_1(G), \dots, h_n(G)) = \text{MLP}^* \circ f^*(\{\{1\text{-WL}(G)(v) \mid v \in V(G)\}\}),$$

with which we can conclude that the composition of multiple functions computable by 1-WL+NN, is also 1-WL+NN computable.

We define the new encoding function f^* to work as follows on an arbitrary input multiset M :

$$f^*(M) := \text{concat}\left(\begin{bmatrix} f_1(M) \\ \vdots \\ f_n(M) \end{bmatrix}\right),$$

where concat is the concatenation function, concatenating all encoding vectors to one single vector.

Using the decomposition introduced in ??, we can decompose each MLP_i for $i \in [n]$ at layer $j > 0$ as follows: $(\text{MLP}_i)_j(v) := \sigma_{i,j}(W_j^i \cdot (\text{MLP}_i)_{j-1}(v) + b_j^i)$. Using this notation we construct MLP^* as follows:

$$\begin{aligned} (\text{MLP}^*)_0(v) &:= v \\ (\text{MLP}^*)_{j+1}(v) &:= \sigma_j^*(W_j^* \cdot (\text{MLP}^*)_j(v) + \text{concat}\left(\begin{bmatrix} b_j^1 \\ \vdots \\ b_j^n \end{bmatrix}\right)) \quad , \forall j \in [T-1] \\ (\text{MLP}^*)_{j+T+1}(v) &:= (\text{MLP}^\bullet)_{j+1}(v) \quad , \forall j \in [T^\bullet - 1] \end{aligned}$$

where T is the maximum number of layers of the set of MLP_i 's, and T^\bullet is the number of layers of the given MLP^\bullet . Thereby, we define in the first equation the start of the sequence as the input; with the second line, we construct the “simultaneous” execution of the MLP_i 's, and in the last equation line, we add the layers of the given MLP^\bullet to the end. Further, we define the weight matrix W_j^* as follows:

$$W_j^* := \begin{bmatrix} W_j^1 & 0 & \dots & 0 \\ 0 & W_j^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & W_j^n \end{bmatrix},$$

such that we build a new matrix where each individual weight matrix is placed along the diagonal. Here we denote with “0” zero matrices with the correct dimensions, such that W_j^* is well-defined. Important to note, should for an MLP_i , W_j^i not exist, because it has less than j layers, we use for W_j^i the identity matrix I_m where m is the dimension of the output computed by MLP_i . And finally, we define the overall activation function σ_j^* as following:

$$\sigma_j^*(v) := \begin{bmatrix} \sigma_{1,j}(v[1]) \\ \vdots \\ \sigma_{1,j}(v[d_1]) \\ \vdots \\ \vdots \\ \sigma_{n,j}(v[d_1 + \dots + d_{n-1} + 1]) \\ \vdots \\ \sigma_{n,j}(v[d_1 + \dots + d_n]) \end{bmatrix},$$

where d_i is the dimension of the output of MLP_i at layer j , and for the ease of readability we denote the i .th component of vector v here with $v[i]$. Thereby, we construct an activation function that applies each respective activation function of the MLP_i 's individually to their respective computation. \square

1.2 ???: Continuous Case: “GNN \subseteq 1-WL+NN”

We will prove ?? by introducing a definition and then two lemmas using that definition, collectively proving the entire theorem. In particular, we define the property that a collection

of functions is able to find any \simeq_{1WL} equivalence class. In the subsequent ??, we will show that the quality of 1-WL-Discriminating of a collection implies the ability to locate any \simeq_{1WL} -equivalence class. Finally, with ??, that this quality further implies the capability of being GNN-Approximating. The overall proof is inspired by the work of (?).

For this proof, we make the major assumption that a continuous 1-WL algorithm exists that operates over continuous values.

Definition 28 (Locating \simeq_{1WL} -Equivalence Classes). Let \mathcal{C} be a collection of continuous functions from \mathcal{X} to \mathbb{R} . If for all $\epsilon \in \mathbb{R}$ with $\epsilon > 0$ and for every graph $G^* \in \mathcal{X}$ the function h_{G^*} exists in \mathcal{C} , with the following properties:

1. for all $G \in \mathcal{X} : h_{G^*}(G) \geq 0$,
2. for all $G \in \mathcal{X}$ with $G \simeq_{1WL} G^* : h_{G^*}(G) = 0$, and
3. there exists a constant $\delta_{G^*} > 0$, such that for all $G \in \mathcal{X}$ with $h_{G^*}(G) < \delta_{G^*}$ there exists a graph $G' \in \mathcal{X}/\simeq_{1WL}(G)$ in the equivalence class of G such that $\|G' - G^*\|_2 < \epsilon$

we say \mathcal{C} is able to locate every \simeq_{1WL} equivalence class.

One can interpret this function h_{G^*} as a kind of loss function that measures the similarity between its input graph to G^* . It yields no loss for the input G , if G is indistinguishable from G^* by the 1-WL algorithm ($G \simeq_{1WL} G^*$), only a small loss if G is close to a graph in the \simeq_{1WL} equivalence class of G^* (the loss is upper bounded by δ_{G^*}), and an arbitrary loss otherwise.

Lemma 29. Let \mathcal{C} be a collection of continuous functions from \mathcal{X} to \mathbb{R} computable by 1-WL+NN. If \mathcal{C} is 1-WL-Discriminating, then there exists a collection of functions \mathcal{C}' computable by 1-WL+NN that is able to locate every \simeq_{1WL} equivalence class on \mathcal{X} .

Proof. Let $G^* \in \mathcal{X}$ be fixed and $\epsilon > 0$ be given. Since \mathcal{C} is 1-WL-Discriminating, we know that for every $G \in \mathcal{X}$ with $G \not\simeq_{1WL} G^*$, there exists a function $h_{G,G^*} \in \mathcal{C}$ such that $h_{G,G^*}(G) \neq h_{G,G^*}(G^*)$. We use this property to construct for each $G \in \mathcal{X}$ with $G \not\simeq_{1WL} G^*$ a set A_{G,G^*} as follows:

$$A_{G,G^*} := \{G' \in \mathcal{X} \mid h_{G,G^*}(G') \in (h_{G,G^*}(G) \pm \frac{|h_{G,G^*}(G) - h_{G,G^*}(G^*)|}{2})\},$$

where $(a \pm b)$ is the open set $(a-b, a+b)$ over \mathbb{R} . One can use ?? below for a better understanding, when a graph G' is contained in A_{G,G^*} and when not. With the illustration one can easily see, that $G \in A_{G,G^*}$ and $G^* \notin A_{G,G^*}$.

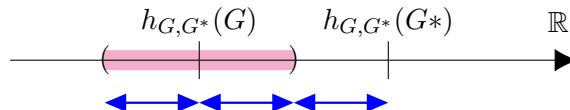


Figure A.1: An illustration to better understand the proof. The set A_{G,G^*} consists of all graphs G that are mapped by h_{G,G^*} into the pink area, which size depends on the distance between the image of G and G^* under h_{G,G^*} . Moreover, the blue distances all have the same length.

Furthermore, by assumption h_{G,G^*} is continuous, such that A_{G,G^*} is an open set. For every $G \in \mathcal{X}$ with $G \simeq_{1WL} G^*$ we define A_{G,G^*} as follows:

$$A_{G,G^*} := \{G' \in \mathcal{X} \mid \|G' - G\|_2 < \epsilon\},$$

where $\|\cdot\|_2$ is the l_2 norm.

Thus, $\{A_{G,G^*}\}_{G \in \mathcal{X}}$ is an open cover of \mathcal{X} . Since \mathcal{X} is compact, there exists a finite subset \mathcal{X}_0 such that $\{A_{G,G^*}\}_{G \in \mathcal{X}_0}$ also covers \mathcal{X} . Hence, $\forall G \in \mathcal{X} \exists G_0 \in \mathcal{X}_0 : G \in A_{G_0,G^*}$.

We define the desired function h_{G^*} as follows:

$$h_{G^*}(\cdot) = \sum_{\substack{G_0 \in \mathcal{X}_0 \\ G_0 \not\simeq_{WL} G^*}} \bar{h}_{G_0,G^*}(\cdot), \quad (\text{A.1})$$

where we define \bar{h}_{G_0,G^*} almost exactly the same as in a previous proof:

$$\bar{h}_{G_0,G^*}(\cdot) = |h_{G_0,G^*}(\cdot) - h_{G_0,G^*}(G^*)| \quad (\text{A.2})$$

$$= \max(h_{G_0,G^*}(\cdot) - h_{G_0,G^*}(G^*)) + \max(h_{G_0,G^*}(G^*) - h_{G_0,G^*}(\cdot)) \quad (\text{A.3})$$

Note that, “ $h_{G_0,G^*}(G^*)$ ” is a constant in the definition of $\bar{h}_{G_0,G^*}(\cdot)$. We will shortly proof, that $h_{G^*}(\cdot)$ fulfills the desired properties on input $G \in \mathcal{X}$:

1. By construction, any \bar{h}_{G_0,G^*} is non-negative, such that the sum over these functions is also non-negative.
2. If $G \simeq_{WL} G^*$, using ?? we know that $h_{G^*}(G) = h_{G^*}(G^*)$, and by definition $h_{G^*}(G^*) = 0$, such that we can conclude $h_{G^*}(G) = 0$.
3. Let δ_{G^*} be:

$$\delta_{G^*} := \frac{1}{2} \min_{\substack{G_0 \in \mathcal{X} \\ G_0 \not\simeq_{WL} G^*}} |h_{G_0,G^*}(G_0) - h_{G_0,G^*}(G^*)|.$$

Prove by contraposition: Assume that for every graph $G' \in \mathcal{X} / \simeq_{WL}(G)$: $\|G' - G^*\|_2 \geq \epsilon$. Hence, $G \notin \bigcup_{G' \in \mathcal{X} / \simeq_{WL}(G)} A_{G',G^*}$ (not in the union of l_2 balls of size ϵ around all graphs of the equivalence class of G^*). However, since $\{A_{G,G^*}\}_{G \in \mathcal{X}_0}$ is a cover of \mathcal{X} , there must exist a $G_0 \in \mathcal{X}_0$ with $G_0 \not\simeq_{WL} G^*$ such that $G \in A_{G_0,G^*}$. Thus, by definition of A_{G_0,G^*} we know that $h_{G_0,G^*}(G) \in (h_{G_0,G^*}(G_0) \pm \frac{|h_{G_0,G^*}(G_0) - h_{G_0,G^*}(G^*)|}{2})$, which when reformulated states:

$$|h_{G_0,G^*}(G) - h_{G_0,G^*}(G_0)| < \frac{1}{2} |h_{G_0,G^*}(G_0) - h_{G_0,G^*}(G^*)|. \quad (\text{A.4})$$

Using this, we can prove $\bar{h}_{G_0,G^*}(G) \geq \delta_{G^*}$, which implies $h_{G^*}(G) \geq \delta_{G^*}$ and concludes the proof:

$$\begin{aligned} \bar{h}_{G_0,G^*}(G) &= |h_{G_0,G^*}(G) - h_{G_0,G^*}(G^*)| \\ &\geq |h_{G_0,G^*}(G_0) - h_{G_0,G^*}(G^*)| - |h_{G_0,G^*}(G) - h_{G_0,G^*}(G_0)| \end{aligned} \quad (\text{A.5})$$

$$\geq \frac{1}{2} |h_{G_0,G^*}(G_0) - h_{G_0,G^*}(G^*)| \quad (\text{A.6})$$

$$\geq \frac{1}{2} \min_{\substack{G_0 \in \mathcal{X} \\ G_0 \not\simeq_{WL} G^*}} |h_{G_0,G^*}(G_0) - h_{G_0,G^*}(G^*)| =: \delta_{G^*}$$

To understand these inequalities, it helps to visualize them, hence see ???. We will try to give an explanation now, using the colored distances depicted in ???. In ???, we use the fact that the red distance is always greater than the green minus the blue distance. Further, using ??, we know that the blue distance is always smaller than half of than

the green distance. Using this fact, it is easy to see that in ?? the green minus the blue distance is always greater than or equal to the half of the green distance.

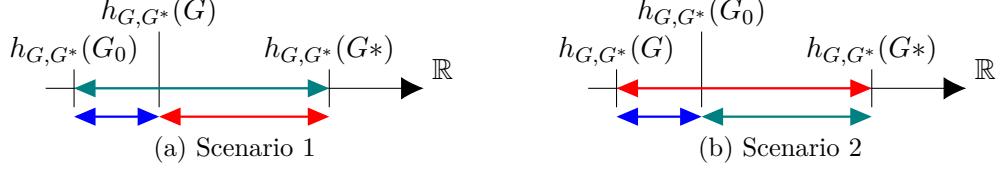


Figure A.2: An illustration of two general scenarios to better understand the above used inequalities. Note that, there exists two more general scenarios, with $h_{G_0,G^*}(G^*) < h_{G_0,G^*}(G_0)$, but since we use the absolute function as our measure of distance, these scenarios are equivalent to the ones depicted due to the symmetry of the absolute function. In ?? below, we listed all colors used to decode the distances in the figures with their corresponding term in the inequalities.

Color	Term
red	$ h_{G_0,G^*}(G) - h_{G_0,G^*}(G^*) $
green	$ h_{G_0,G^*}(G_0) - h_{G_0,G^*}(G^*) $
blue	$ h_{G_0,G^*}(G) - h_{G_0,G^*}(G_0) $

Table A.1: Color legend for the proof ?? and ??.

Consequently, we know that if $h_{G^*}(G) < \delta_G$ it means that $G \in \bigcup_{G' \in \mathcal{X}/\simeq_{1WL}(G^*)} A_{G',G^*}$, which implies that there exists $G' \in \mathcal{X}/\simeq_{1WL}(G)$ with $\|G' - G^*\|_2 < \epsilon$.

As a final note, we can construct a multilayer perceptron $\text{MLP}_{h_{G^*}}$ computing the function $h_{G^*}(\cdot)$ as in ?. The $\text{MLP}_{h_{G^*}}$ gets as input a vector of the output of the finite set of functions $\{\bar{h}_{G_0,G^*}\}_{G_0 \in \mathcal{X}_0, G_0 \not\simeq_{1WL} G^*}$ applied on the input graph. Further, ?? can be encoded by replacing the max operator by the non-linear activation function ReLU. Using ??, we can conclude that $h_{G^*}(\cdot)$ is computable by 1-WL+NN. \square

Lemma 30. Let \mathcal{C} be a collection of continuous functions from \mathcal{X} to \mathbb{R} computable by 1-WL+NN. If \mathcal{C} is able to locate every \simeq_{1WL} equivalence class on \mathcal{X} , then there exists a collection of functions \mathcal{C}' computable by 1-WL+NN that is **GNN-Approximating**.

Proof. Let \mathcal{A} be a continuous function from \mathcal{X} to \mathbb{R} computable by a GNN. Since \mathcal{X} is compact, this implies that \mathcal{A} is uniformly continuous on \mathcal{X} , which further implies that $\forall \epsilon > 0 \exists r > 0$ such that $\forall G_1, G_2 \in \mathcal{X}$, if $\|G_1 - G_2\|_2 < r$, then $|\mathcal{A}(G_1) - \mathcal{A}(G_2)| < \epsilon$. Further, since \mathcal{A} is GNN computable, we know that $\forall G_1, G_2 \in \mathcal{X}$, if $G_1 \simeq_{1WL} G_2 : \mathcal{A}(G_1) = \mathcal{A}(G_2)$, hence if $G' \in \mathcal{X}/\simeq_{1WL}(G_1)$ with $\|G' - G_2\|_2 < r$ exists, than $|\mathcal{A}(G_1) - \mathcal{A}(G_2)| < \epsilon$.

Throughout the rest of the proof, let $\epsilon > 0$ be fixed. Using the property described above, we know that for this ϵ there exists a constant r , which we will fix for the remainder of the proof as well. Further, by the assumptions of the lemma we try to prove, we know that for any $\epsilon' > 0$ there exists $h_G \in \mathcal{C}$ for any $G \in \mathcal{X}$ with the above described properties. We choose $\epsilon' := r$ for

all $h_G \in \mathcal{C}$ throughout the proof.

For any $G \in \mathcal{X}$, we define the set $h_G^{-1}(a) := \{G' \in \mathcal{X} \mid h_G(G') \in [0, a]\}$, as the set of graphs that are mapped into the open interval $[0, a)$ by h_G . We illustrated this set in ?? for a better understanding.

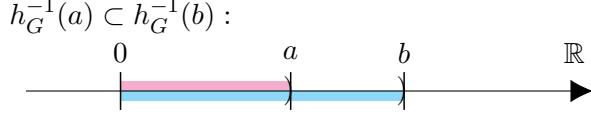


Figure A.3: Illustration of the set $h_G^{-1}(\cdot)$ for two arbitrary values a, b with $a < b$. The pink area visualizes the open interval $[0, a)$, and similarly in blue for $[0, b)$.

Then, by definition of h_G , there exists a constant δ_G such that:

$$h_G^{-1}(\delta_G) \subseteq \bigcup_{G' \in \mathcal{X} / \approx_{WL}(G)} \{G'' \in \mathcal{X} \mid \|G'' - G'\|_2 < r\}.$$

Since h_G is continuous by definition, $h_G^{-1}(\delta_G)$ is an open set. Hence, $\{h_G^{-1}(\delta_G)\}_{G \in \mathcal{X}}$ is an open cover of \mathcal{X} , and further as \mathcal{X} is compact, there exists a finite subset $\mathcal{X}_0 \subseteq \mathcal{X}$ such that $\{h_{G_0}^{-1}(\delta_{G_0})\}_{G_0 \in \mathcal{X}_0}$ is also a cover of \mathcal{X} . For each $G_0 \in \mathcal{X}_0$, we construct the function φ_{G_0} from \mathcal{X} to \mathbb{R} as follows:

$$\varphi_{G_0}(\cdot) := \max(\delta_{G_0} - h_{G_0}(\cdot), 0).$$

The function has two important properties, for once it is non-negative and for any $G \in \mathcal{X} : \varphi_{G_0}(G) > 0$, if and only if $G \in h_{G_0}^{-1}(\delta_{G_0})$, thereby acting as a sort of weak indicator function. Building up on this property, we construct for each $G_0 \in \mathcal{X}_0$ the function ψ_{G_0} from \mathcal{X} to \mathbb{R} as follows:

$$\psi_{G_0}(\cdot) := \frac{\varphi_{G_0}(\cdot)}{\sum_{G' \in \mathcal{X}_0} \varphi_{G'}(\cdot)},$$

which is well-defined, because $\{h_{G_0}^{-1}(\delta_{G_0})\}_{G_0 \in \mathcal{X}_0}$ is a cover of \mathcal{X} , such that we can conclude that for any input graph G on $\psi_{G_0}(\cdot)$ there exists a set $h_{G_0}^{-1}(\delta_{G_0})$ with $G \in h_{G_0}^{-1}(\delta_{G_0})$ implying $\varphi_{G_0}(G) > 0$, thus making the denominator not 0. The function ψ_{G_0} has two important properties, for once it is non-negative, because φ_G for all $G \in \mathcal{X}$ is non-negative, and for any $G \in \mathcal{X} : \psi_{G_0}(G) > 0$, if and only if $G \in h_{G_0}^{-1}(\delta_{G_0})$.

Further, we can observe that the set of functions $\{\psi_{G_0}\}_{G_0 \in \mathcal{X}_0}$ is a partition of unity on \mathcal{X} with respect to the open cover $\{h_{G_0}^{-1}(\delta_{G_0})\}_{G_0 \in \mathcal{X}_0}$, because:

1. For any $G \in \mathcal{X}$ the set of functions mapping G not to 0 is finite, as the set of all functions $\{\psi_{G_0}\}_{G_0 \in \mathcal{X}_0}$ is finite, since \mathcal{X}_0 is finite.
2. For any $G \in \mathcal{X} : \sum_{G_0 \in \mathcal{X}_0} \psi_{G_0}(G) = 1$, since:

$$\sum_{G_0 \in \mathcal{X}_0} \psi_{G_0}(G) = \sum_{G_0 \in \mathcal{X}_0} \frac{\varphi_{G_0}(G)}{\sum_{G' \in \mathcal{X}_0} \varphi_{G'}(G)} = \frac{\sum_{G_0 \in \mathcal{X}_0} \varphi_{G_0}(G)}{\sum_{G' \in \mathcal{X}_0} \varphi_{G'}(G)} = 1.$$

We can use this property to decompose the given function \mathcal{A} as follows on any input $G \in \mathcal{X}$:

$$\mathcal{A}(G) = \mathcal{A}(G) \cdot \left(\sum_{G_0 \in \mathcal{X}_0} \psi_{G_0}(G) \right) = \sum_{G_0 \in \mathcal{X}_0} \mathcal{A}(G) \cdot \psi_{G_0}(G).$$

Recall the property from the beginning of the proof that for any $G \in \mathcal{X}$ if $G \in h_{G_0}^{-1}(\delta_{G_0})$, then there exists a $G' \in \mathcal{X}/\simeq_{1WL}(G)$ with $\|G' - G_0\|_2 < r$, which implies that $|\mathcal{A}(G) - \mathcal{A}(G_0)| < \epsilon$. With this, we can construct a function $\hat{\mathcal{A}}$ on \mathcal{X} that approximates \mathcal{A} within accuracy ϵ :

$$\hat{\mathcal{A}}(\cdot) = \sum_{G_0 \in \mathcal{X}_0} \mathcal{A}(G_0) \cdot \psi_{G_0}(\cdot).$$

Note that, “ $\mathcal{A}(G_0)$ ” is a constant here. To prove that $\hat{\mathcal{A}}$ approximates \mathcal{A} within accuracy ϵ we need to show that $\sup_{G \in \mathcal{X}} |\mathcal{A}(G) - \hat{\mathcal{A}}(G)| < \epsilon$. Let $G \in \mathcal{X}$ be arbitrary, then:

$$\begin{aligned} |\mathcal{A}(G) - \hat{\mathcal{A}}(G)| &= \left| \mathcal{A}(G) \cdot \left(\sum_{G_0 \in \mathcal{X}_0} \psi_{G_0}(G) \right) - \sum_{G_0 \in \mathcal{X}_0} \mathcal{A}(G_0) \cdot \psi_{G_0}(G) \right| \\ &= \left| \sum_{G_0 \in \mathcal{X}_0} \mathcal{A}(G) \cdot \psi_{G_0}(G) - \sum_{G_0 \in \mathcal{X}_0} \mathcal{A}(G_0) \cdot \psi_{G_0}(G) \right| \\ &= \sum_{G_0 \in \mathcal{X}_0} |\mathcal{A}(G) - \mathcal{A}(G_0)| \cdot \psi_{G_0}(G) \\ &< \sum_{G_0 \in \mathcal{X}_0} \epsilon \cdot \psi_{G_0}(G) \\ &= \epsilon \cdot \sum_{G_0 \in \mathcal{X}_0} \psi_{G_0}(G) \\ &= \epsilon \cdot 1 \end{aligned} \tag{A.7}$$

In ??, we use the fact that applies to all $G_0 \in \mathcal{X}_0$ on input G :

- If $\psi_{G_0}(G) > 0$, than we know that $G \in h_{G_0}^{-1}(\delta_{G_0})$, such that we can upper bound $|\mathcal{A}(G) - \mathcal{A}(G_0)| < \epsilon$.
- If $\psi_{G_0}(G) = 0$, we know that the no matter what $|\mathcal{A}(G) - \mathcal{A}(G_0)|$ is, the summand is 0, such that we can just assume $|\mathcal{A}(G) - \mathcal{A}(G_0)| < \epsilon$ without loss of generality.

In the end, we give a short explanation, that $\hat{\mathcal{A}}$ is computable by 1-WL+NN. For this we construct a multilayer perceptron with three layers and then conclude with ?? the computability. Let us therefore break down the whole construction of $\hat{\mathcal{A}}$:

$$\hat{\mathcal{A}}(\cdot) = \sum_{G_0 \in \mathcal{X}_0} \mathcal{A}(G_0) \cdot \frac{1}{\sum_{G' \in \mathcal{X}_0} \max(\delta_{G'} - h_{G'}(\cdot), 0)} \cdot \max(\delta_{G_0} - h_{G_0}(\cdot), 0).$$

We construct a multilayer perceptron $\text{MLP}_{\hat{\mathcal{A}}}$ that takes in as input a vector of the output of the finite set of functions $\{h_{G_0}\}_{G_0 \in \mathcal{X}_0}$ applied on the input graph. In the first layer we compute each “ $\max(\delta_{G_0} - h_{G_0}(\cdot))$ ” term, where δ_{G_0} is a constant, in particular the bias, and the max operator is replaced by the activation function ReLU. In the second layer, we compute the sum of the denominator of the fraction, to which we apply the activation function $f(x) := \frac{1}{x}$. In the last layer we compute the overall sum where $\mathcal{A}(G_0)$ is a constant. \square

B Appendix Part II

1 Testing Configuration

1.1 Definition of the Normalized Shannon-Index

We used the following definition of the Normalized Shannon-Index.

Definition 31 (Normalized Shannon-Index). The metric is computed as follows:

$$-\frac{1}{\log_2(|C|)} \cdot \sum_{i \in C} \frac{n_i}{n} \cdot \log_2\left(\frac{n_i}{n}\right) \quad (\text{B.1})$$

where n is the total number of samples of the dataset, C is the set of all classes, and n_i is the number of samples of the class $i \in C$.

As an example, for the dataset PROTEINS the variables are set to be the following: $C = \{0, 1\}$ with $n_0 = 663$ and $n_1 = 450$, so that $n = 1113$, yielding a rounded value of 0.973.

1.2 Theoretical Maximum Accuracy Analysis

Table B.1: An overview of the maximum theoretical classification accuracy achievable for each dataset based on the number of 1-WL iterations in percent. A hyphen “-” indicates that the maximum accuracy has converged with fewer iterations, implying that further iterations do not improve the accuracy. “OOM” denotes out of memory error.

Datasets	Iterations of the 1-WL algorithm					
	0	1	2	3	4	5
Bioinformatics	DD	1.00	-	-	-	-
	ENZYMES	0.81	1.00	-	-	-
	KKI	1.00	-	-	-	-
	OHSU	1.00	-	-	-	-
	Peking_1	1.00	-	-	-	-
	PROTEINS	0.92	1.00	-	-	-
Small molecules	AIDS	1.00	1.00	-	-	-
	BZR	0.96	0.99	1.00	-	-
	COX2	0.93	0.96	0.99	1.00	-
	DHFR	0.92	0.95	1.00	1.00	-
	FRANKENSTEIN	0.63	0.77	0.88	0.89	0.89
	MUTAG	0.93	0.96	0.99	1.00	-
	NCI1	0.91	1.00	1.00	1.00	-
	NCI109	0.92	1.00	1.00	1.00	-
	PTC_MR	0.92	0.98	0.99	-	-
Social networks	COLLAB	0.61	0.98	-	-	-
	IMDB-BINARY	0.61	0.89	-	-	-
	IMDB-MULTI	0.44	0.63	-	-	-
	REDDIT-BINARY	0.84	1.00	-	-	-
	REDDIT-MULTI-5K	0.55	1.00	-	-	-
	REDDIT-MULTI-12K	0.36	OOM	OOM	OOM	OOM

1.3 Hyperparameter Configuration and Optimization

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Table B.2: Listing of the hyperparameter with which we configured the 1-WL+NN models for each classification dataset. In terms of accuracy, the choice of hyperparameters that led to the best-performing configuration is highlighted in boldface if there are numerous options for a hyperparameter.

Hyperparameter	Dataset					
	ENZYMES	IMDB-BINARY	MUTAG	NCI1	PROTEINS	REDDIT-BINARY
Batch Size	32	32	32	33	32	32
Learning Rate	$X \sim U(0.0001, 0.1)$					
Max Epochs	200	200	200	200	200	200
Optimizer	Adam	Adam	Adam	Adam	Adam	Adam
Scheduler	ReduceLROnPlateau	ReduceLROnPlateau	ReduceLROnPlateau	ReduceLROnPlateau	ReduceLROnPlateau	ReduceLROnPlateau
Number of 1-WL iterations	{1, 2, 3}	{1, 2, 3, 4}	{1, 2, 3, 4}	{1, 2, 3}	{1, 2, 3, 4}	{1, 2}
Use 1-WL-Convergence	False	False	False	False	False	False
MLP Activation Function	ReLU	ReLU	ReLU	ReLU	ReLU	ReLU
MLP Normalization	BatchNorm	BatchNorm	BatchNorm	BatchNorm	BatchNorm	BatchNorm
MLP Number of Layers	{2, 3, 4, 5}	{2, 3, 4, 5}	{2, 3, 4, 5}	{2, 3, 4, 5}	{2, 3, 4, 5}	{2, 3, 4, 5}
MLP Dropout	$X \sim U(0, 0.2)$					
Embedding Dimension	{None, 16, 32, 64, 128}					
Pooling function	{Max, Mean, Sum}					

Table B.3: Listing of the hyperparameter with which we configured the GNN models for each classification dataset. In terms of accuracy, the choice of hyperparameters that led to the best-performing configuration is highlighted in boldface if there are numerous options for a hyperparameter.

Hyperparameter	Dataset					
	ENZYMES	IMDB-BINARY	MUTAG	NCI1	PROTEINS	REDDIT-BINARY
Batch Size	32	32	32	{33, 129}	32	32
Learning Rate	$X \sim U(0.0001, 0.1)$					
Max Epochs	200	200	200	200	200	200
Optimizer	Adam	Adam	Adam	Adam	Adam	Adam
Scheduler	ReduceLROnPlateau	ReduceLROnPlateau	ReduceLROnPlateau	ReduceLROnPlateau	ReduceLROnPlateau	ReduceLROnPlateau
GNN Architecture	{GAT, GCN, GIN}					
GNN Activation Function	ReLU	ReLU	ReLU	ReLU	ReLU	ReLU
GNN Dropout	$X \sim U(0, 0.2)$					
GNN Hidden Dimension	{16, 32, 64, 128}	{16, 32, 64, 128}	{16, 32, 64, 128}	{16, 32, 64, 128}	{16, 32, 64, 128}	{16, 32, 64, 128}
GNN Jumping-Knowledge	cat	cat	cat	cat	cat	cat
GNN Number of Layers	5	5	5	5	5	5
MLP Activation Function	ReLU	ReLU	ReLU	ReLU	ReLU	ReLU
MLP Normalization	BatchNorm	BatchNorm	BatchNorm	BatchNorm	BatchNorm	BatchNorm
MLP Number of Layers	{2, 3, 4, 5}	{2, 3, 4, 5}	{2, 3, 4, 5}	{2, 3, 4, 5}	{2, 3, 4, 5}	{2, 3, 4, 5}
MLP Dropout	$X \sim U(0, 0.2)$					
Pooling function	{Max, Mean, Sum}					

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Table B.4: Listing of the hyperparameter with which we configured the 1-WL+NN models for each regression dataset. In terms of accuracy, the choice of hyperparameters that led to the best-performing configuration is highlighted in boldface if there are numerous options for a hyperparameter.

Hyperparameter	Dataset			
	ALCHEMY	ALCHEMY(10K)	ZINC	ZINC(10K)
Batch Size	25	25	25	25
Learning Rate	0.001	0.001	0.001	0.001
Max Epochs	1000	1000	1000	1000
Optimizer	Adam	Adam	Adam	Adam
Scheduler	ReduceLROnPlateau	ReduceLROnPlateau	ReduceLROnPlateau	ReduceLROnPlateau
Number of 1-WL iterations	{1, 2, 3}	{1, 2, 3, 4}	{1, 2, 3}	{1, 2, 3, 4}
Use 1-WL-Convergence	False	False	False	False
MLP Activation Function	ReLU	ReLU	ReLU	ReLU
MLP Normalization	BatchNorm	BatchNorm	BatchNorm	BatchNorm
MLP Number of Layers	{2, 3, 4, 5}	{2, 3, 4, 5}	{2, 3, 4, 5}	{2, 3, 4, 5}
MLP Dropout	$X \sim U(0, 0.2)$			
Embedding Dimension	{None, 16, 32, 64, 128}			
Pooling function	{Max, Mean, Sum}	{Max, Mean, Sum}	{Max, Mean, Sum}	{Max, Mean, Sum}

Table B.5: Overview of the number of different configurations tested for each model type and each dataset. TODO: Format once all experiments are finished!

model_type	dataset_pool_type	Alchemy	Alchemy10K	ENZYMEs	IMDB-BINARY	MUTAG	NCI1	PROTEINS	REDDIT-BINARY	test_accuracy	
										Zinc	Zinc 10k
1-WL+NN	Embedding:Max	3	95	338	282	290	79	245	45	6	33
	Embedding:Mean	6	77	288	271	299	109	216	50	8	24
	Embedding:Sum	5	273	296	293	302	79	215	48	7	18
	Max	0	0	86	70	150	26	35	40	0	0
	Mean	0	0	76	67	120	19	27	40	0	0
	Sum	0	0	85	67	130	14	29	41	0	0
1-WL+NN:GNN	Max	0	0	4	10	8	0	13	5	0	0
	Mean	0	0	5	11	11	0	19	9	0	0
	Sum	0	0	10	8	10	0	11	9	0	0
GNN	Max	1	1	69	56	101	53	40	18	1	1
	Mean	2	2	78	50	86	49	52	20	2	2
	Sum	1	1	82	55	100	52	41	25	1	1

2 Empirical Testing

Table B.6: Overview of the number of unique colors in the colorings computed by the 1-WL algorithm when applied to each dataset. Specifically, we specified the number of iterations of the 1-WL algorithm. Additionally, the “# Nodes” column showcases the maximum number of unique colors that can appear in the colorings.

Dataset	Number of 1-WL Iterations										
	0	1	2	3	4	5	6	7	8	9	10 # Nodes
Classification	ENZYMEs	2	231	10416	15208	16029	16450	16722	16895	17026	17130 19580
	IMDB-BINARY	1	65	2931	3595	3595	3595	3595	3595	3595	3595 19773
	MUTAG	2	33	174	572	1197	1766	2167	2403	2511	2560 3371
	NCI1	2	292	4058	22948	44508	58948	68632	75754	81263	85590 122747
	PROTEINS	2	297	20962	35676	37940	38653	38926	39064	39141	39180 43471
	REDDIT-BINARY	1	566	71893	244529	317728	333258	335961	336412	336490	336506 859254
Regres.	ALCHEMY	2	70	4782	164224	620332	995264	1166951	1216094	1225861	1227632 2046329
	ALCHEMY(10k)	2	70	2764	33903	76822	98394	104687	105907	106109	106149 121422
	ZINC	2	773	28874	290473	1000917	1977437	2921087	3690341	4270959	4681881 4945363
	ZINC(10k)	2	392	9818	61198	132862	185699	216210	233484	242866	247688 249971

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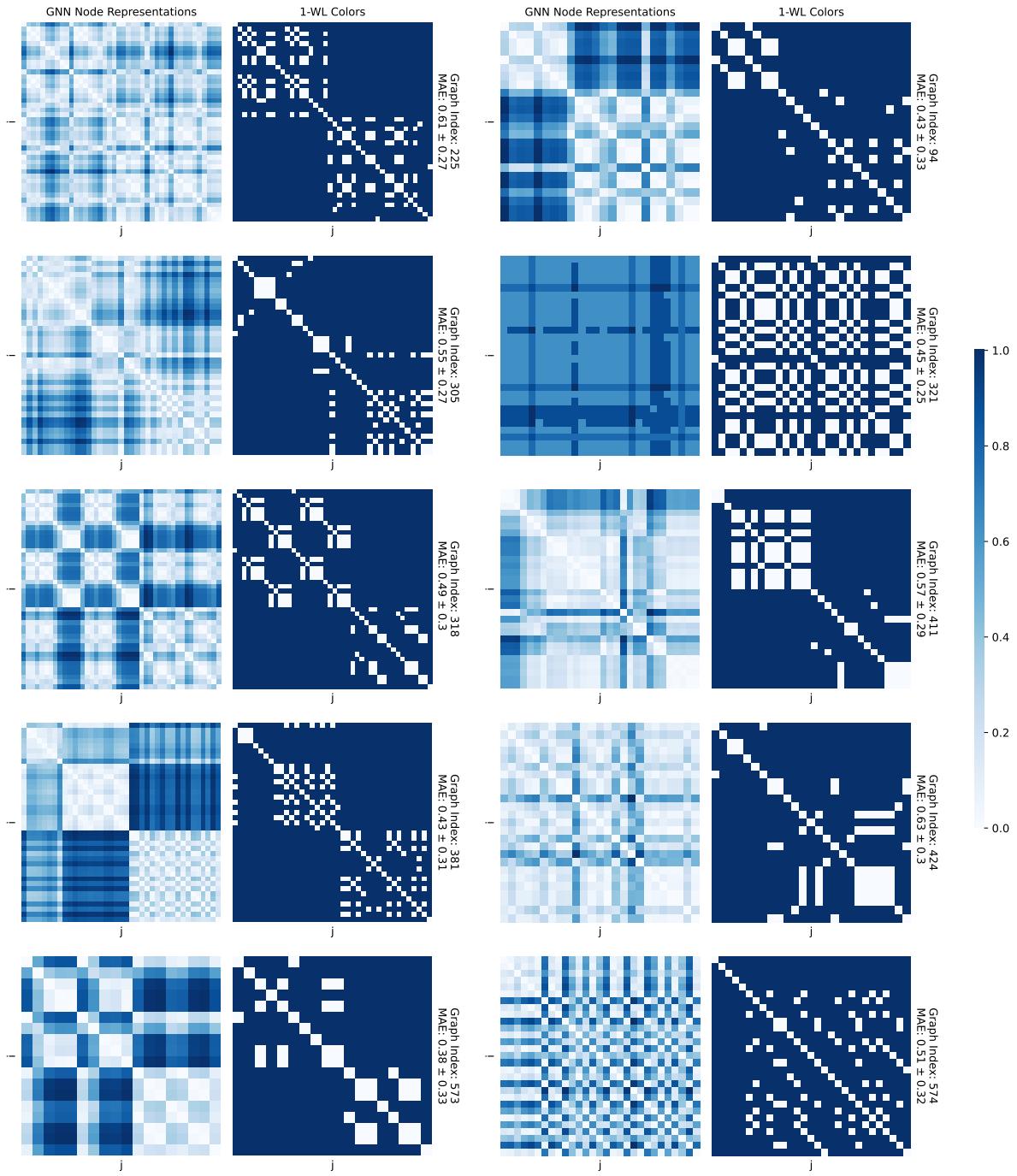


Figure B.1: Visualizing the performance of the best performing GNN on the ENZYMES dataset in approximating node colors computed by the 1-WL algorithm. The ten graphs shown are randomly sampled from the GNN’s test set, and the 1-WL algorithm ran only for one iteration. The average error for the entire test set is 0.49 ± 0.3 .

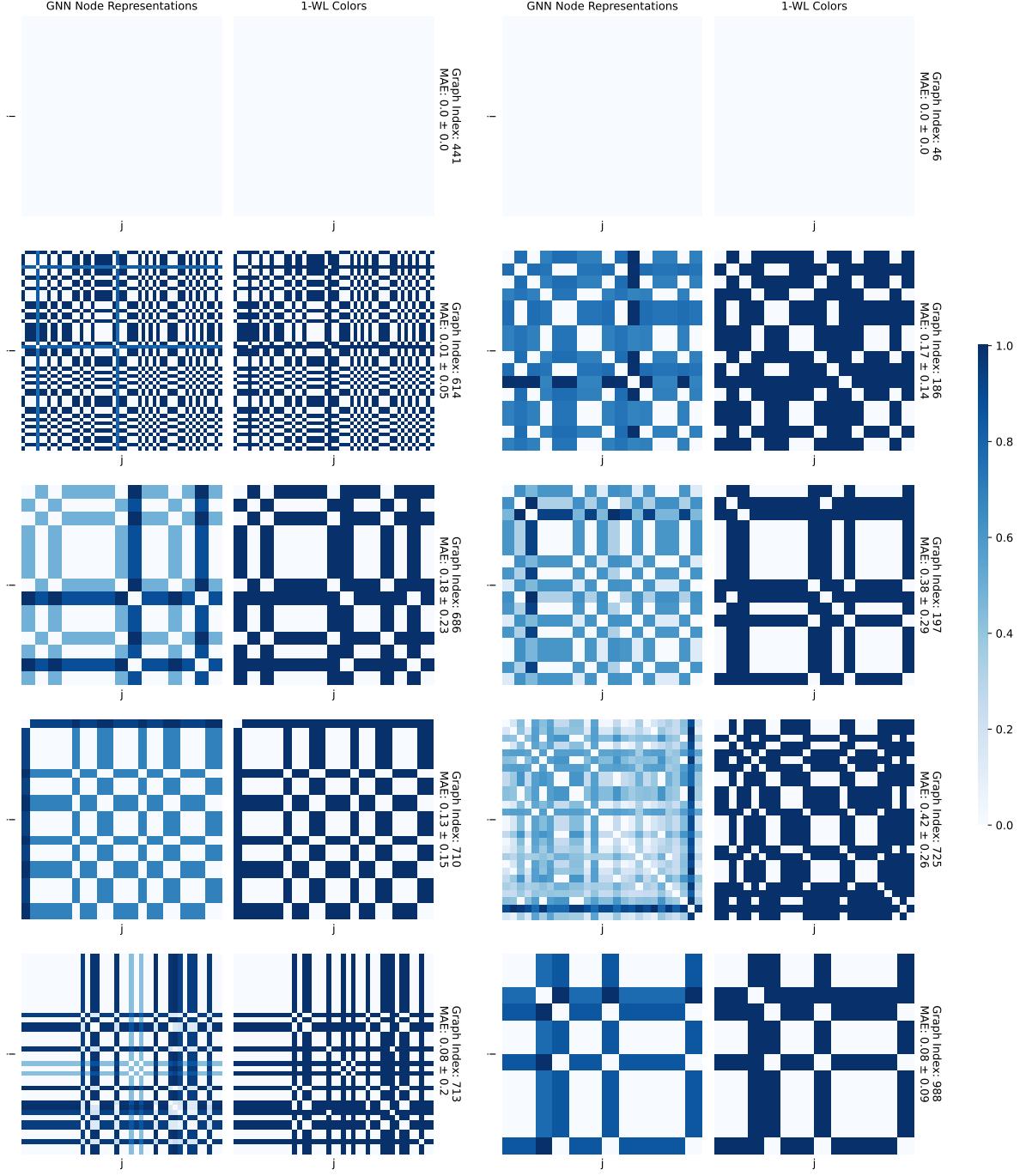


Figure B.2: Visualizing the performance of the best performing GNN on the IMDB-BINARY dataset in approximating node colors computed by the 1-WL algorithm. The ten graphs shown are randomly sampled from the GNN’s test set, and the 1-WL algorithm ran only for one iteration. The average error for the entire test set is 0.14 ± 0.15 . Note that IMDB BINARY does not contain any node features, so we artificially initialize each node feature with a one-hot encoding of its degree.

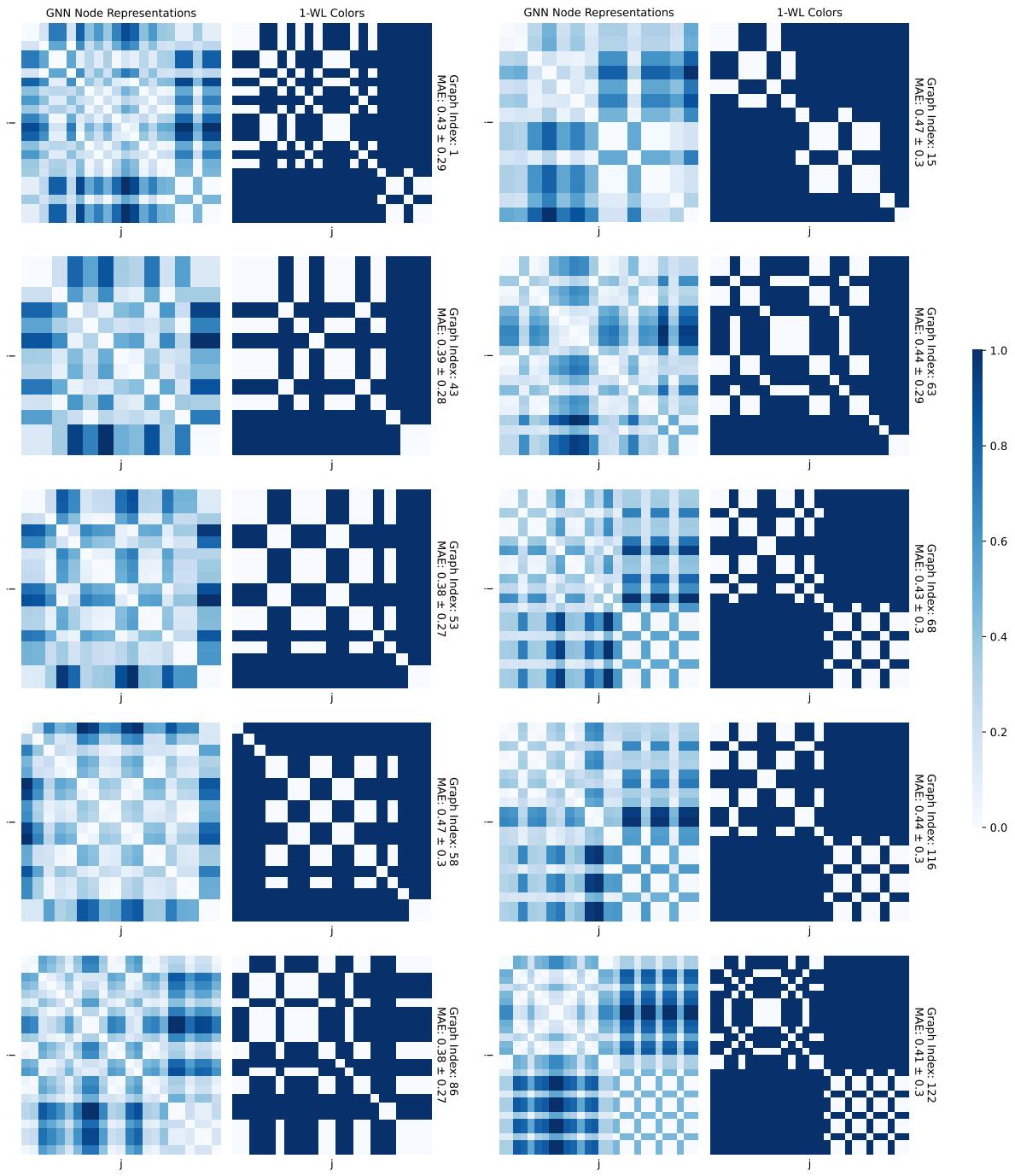


Figure B.3: Visualizing the performance of the best performing GNN on the MUTAG dataset in approximating node colors computed by the 1-WL algorithm. The ten graphs shown are randomly sampled from the GNN’s test set, and the 1-WL algorithm ran only for three iteration. The average error for the entire test set is 0.42 ± 0.29 .

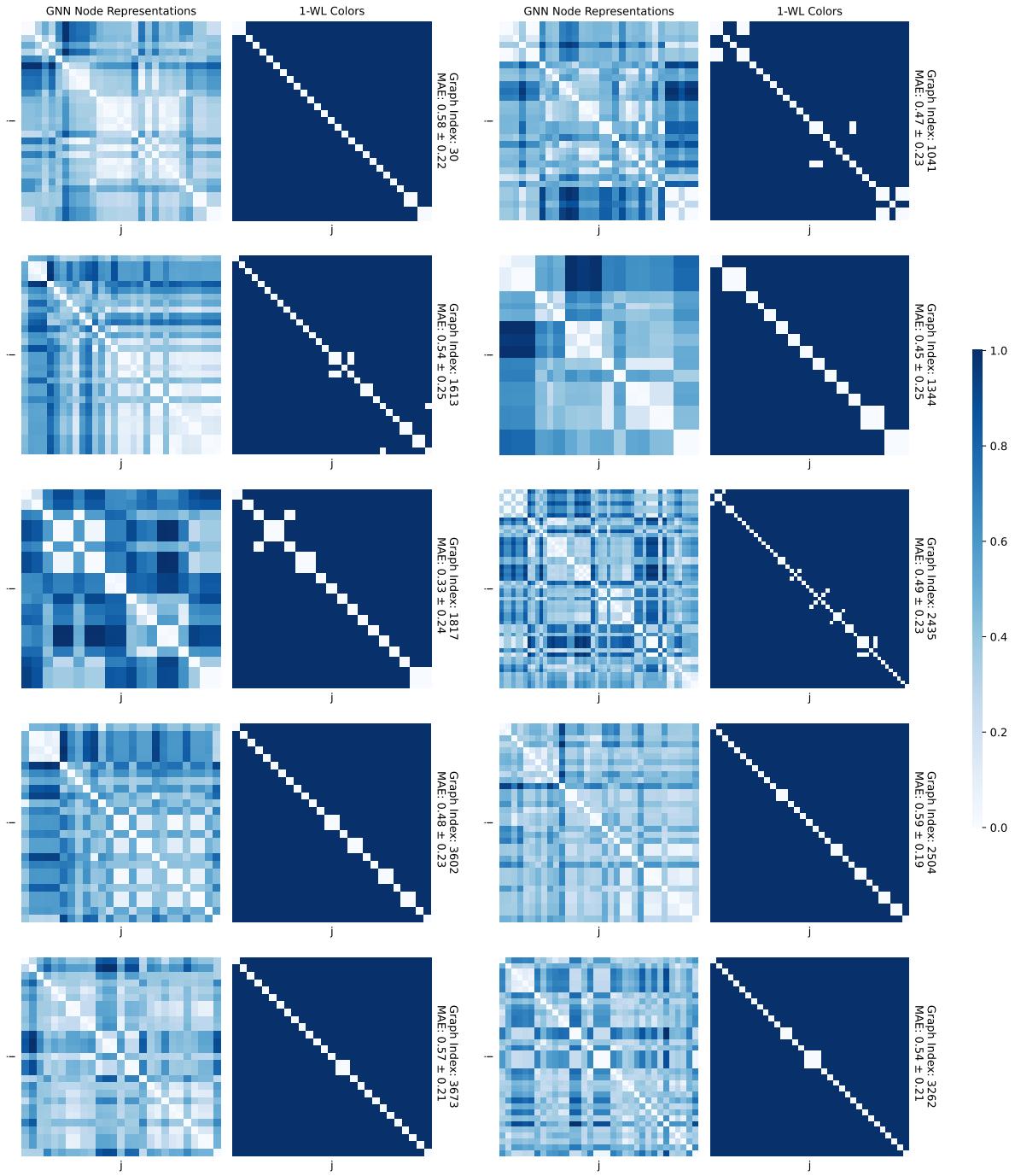


Figure B.4: Visualizing the performance of the best performing GNN on the NCI1 dataset in approximating node colors computed by the 1-WL algorithm. The ten graphs shown are randomly sampled from the GNN’s test set, and the 1-WL algorithm ran only for three iteration. The average error for the entire test set is 0.50 ± 0.24 .

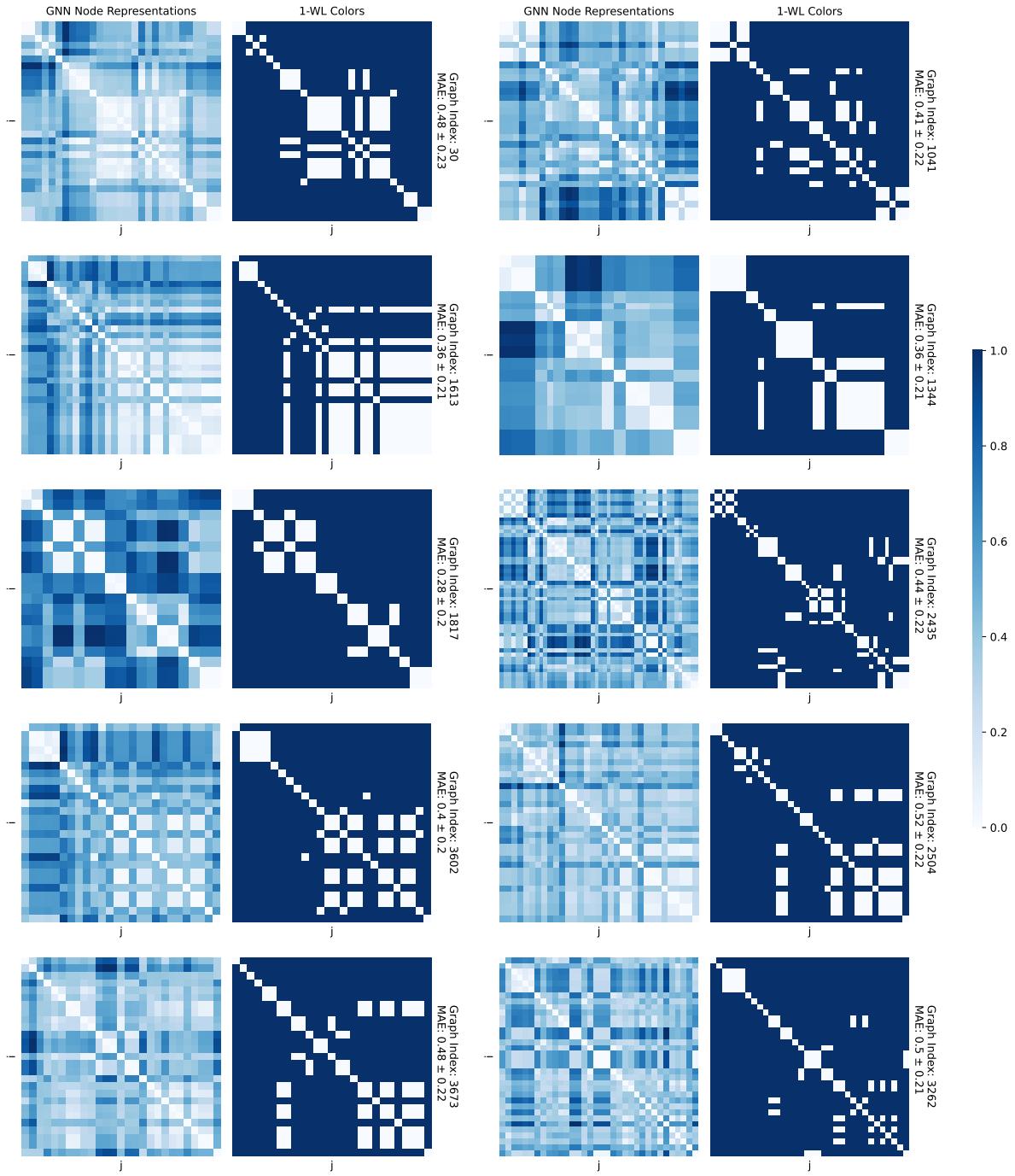


Figure B.5: Visualizing the performance of the best performing GNN on the Nci1 dataset in approximating node colors computed by the 1-WL algorithm. The ten graphs shown are randomly sampled from the GNN’s test set, and the 1-WL algorithm ran only for one iteration. The average error for the entire test set is 0.42 ± 0.22 .

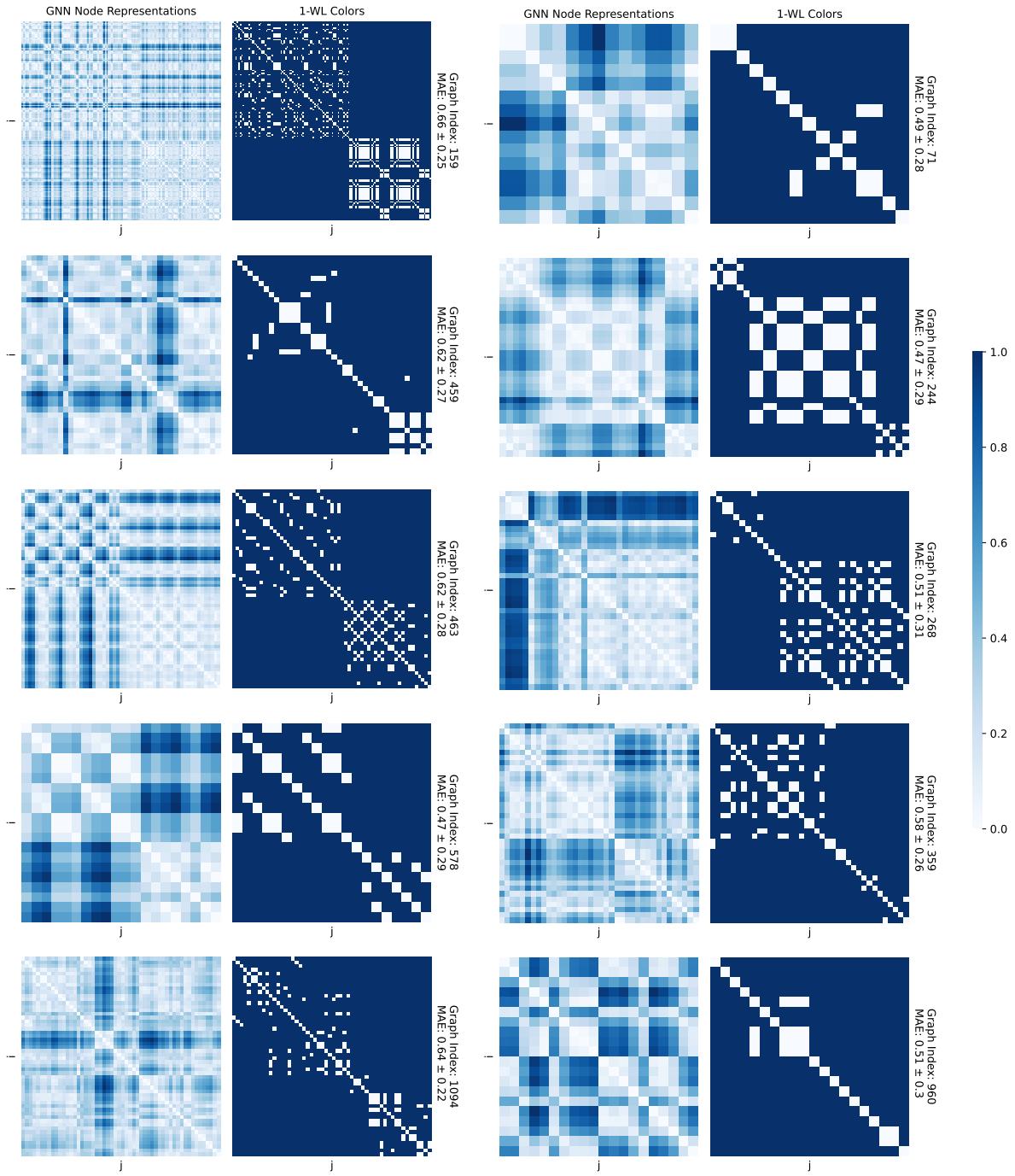


Figure B.6: Visualizing the performance of the best performing GNN on the PROTEINS dataset in approximating node colors computed by the 1-WL algorithm. The ten graphs shown are randomly sampled from the GNN's test set, and the 1-WL algorithm ran only for one iteration. The average error for the entire test set is 0.49 ± 0.26 .

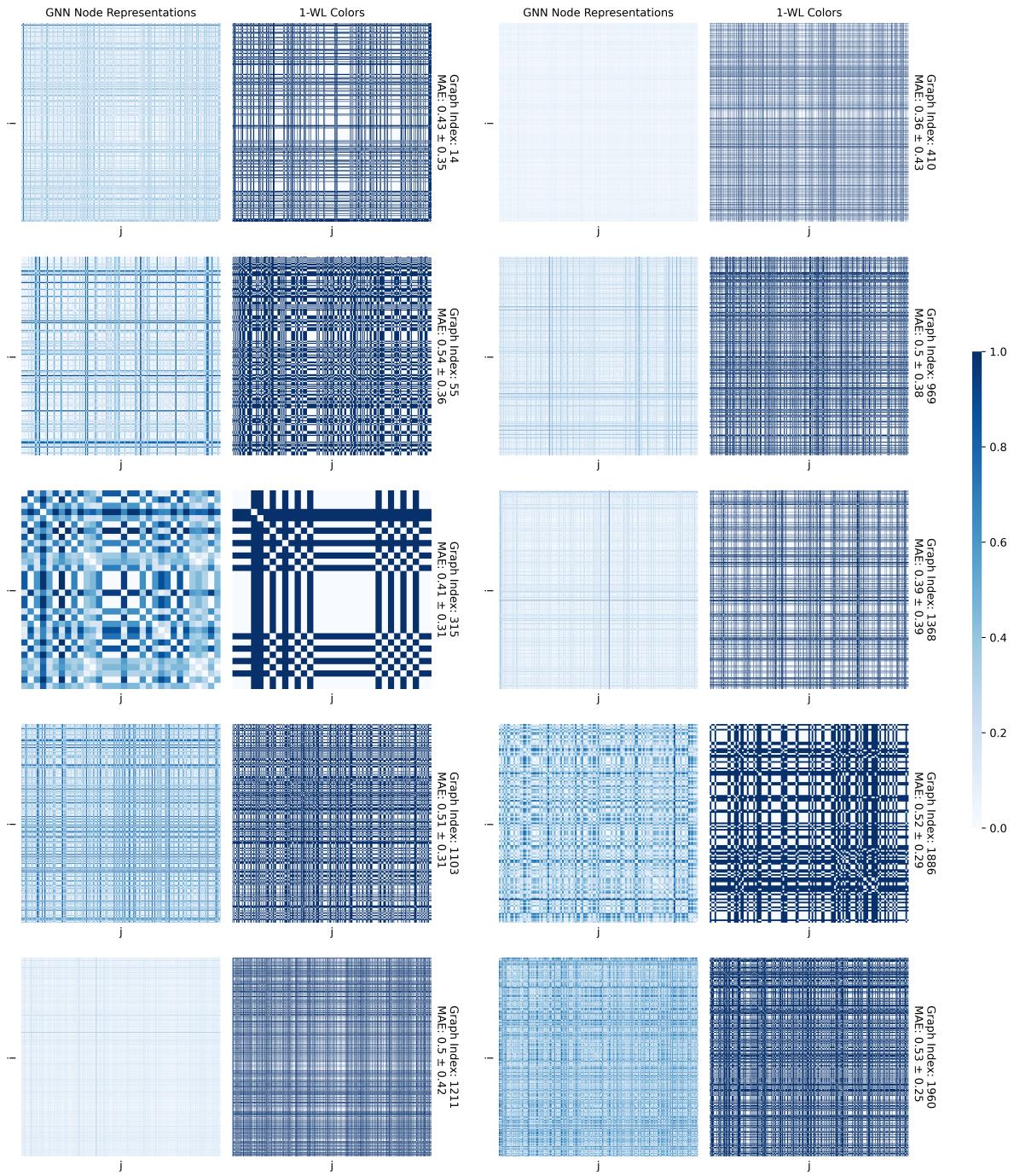


Figure B.7: Visualizing the performance of the best performing GNN on the REDDIT-BINARY dataset in approximating node colors computed by the 1-WL algorithm. The ten graphs shown are randomly sampled from the GNN’s test set, and the 1-WL algorithm ran only for one iteration. The average error for the entire test set is 0.48 ± 0.32 .